

# Solving Parametric Wave Propagation Models with Domain Decomposed Reduced Order Methods

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## I. EXTENDED ABSTRACT

In the field of geophysics seismic imaging is performed to understand the sub-surface layers of the earth for commercial and research purposes. A typical seismic image is developed by solving the Helmholtz model iteratively over a wide range of frequencies. The seismic image consists of solving the Helmholtz model in a 10-15 km long spatial dimensions for a wide range of parameters as a forward modelling. Thus, extensive computations are needed. Reduced order techniques especially proper generalized decomposition ( PGD ) could be employed to accelerate this iterative models using surrogate models with real-time online computation phase. However, PGD computations require an offline phase which is highly penalized for large frequencies. For high frequencies this results in difficulty in implementing realistic geophysical models by the PGD strategy. To circumvent this issue a domain decomposition (DDM) technique combined with the PGD is proposed. In the present work DDM-PGD strategy is imposed on the spatial dimension of the model. PGD-DDM strategy computes local spatial surrogate models which result in convergences locally. Global convergence is ensured by choice of the DDM strategy.

The model is defined in a 2D heterogeneous velocity medium. The DDM-PGD has transmission conditions on the boundaries of the sub-domains based on perfectly matched layers (PML) which avoid artificial reflections and match incoming and outgoing waves. Specifically, transmission conditions are defined by overlapping layers of PML. Overlapping PML type transmission conditions are chosen because they are known to have better convergences in the global residual updates for full order models, which are also independent of the number of sub-domains used. PML transmission conditions absorb artificial reflections completely and have better conformity with the incoming and the outgoing waves. Global surrogate model is built by adding all the spatial local surrogate models. This process can be ensembled either in serial or parallel approach if HPC resources are available.

## A. Problem Statement

The parametric Helmholtz model ( 2D in space ) is defined as;

$$\nabla \cdot (\mathbf{P} \cdot \nabla u) + k^2 S_x S_y u = f \quad (1)$$

where,

$$u = u(\mathbf{x}, \omega, s, c), f = f(\mathbf{x}, \omega, s, c)$$

Here,  $s$  is the source position,  $c$  is the speed of propagation in the medium and  $\omega$  is the frequency.  $f$  is the source term and  $k$  is the wave number defined as  $k = \frac{\omega}{c}$ .  $\mathbf{P}$  is the anisotropy coefficient matrix which controls the absorption of the outgoing rays and signify the PML layer. In the present work, the computational domain is assumed to be a rectangle which is truncated by the PML. The sub-domains are of the identical shape of the full domain. The boundary conditions of the sub-domains are of PML type as well which are overlapped and act as transmission conditions.

## B. PGD-DDM

To generate the PGD, the Helmholtz potential function is defined in terms of product of explicit separable functions wherein the parameters are also considered as a separable function. The separated representation form for the potential function  $u$  is defined as follows;

$$u(\mathbf{x}, \omega) \approx \sum_{i=1}^N F_1^i(\mathbf{x}) F_2^i(\omega) \quad (2)$$

The above equation has been reduced to multiplication of lower dimensional functions. It is noted that adding extra parameters do not increase the dimensions to the powers of  $N$ , but the further dimensions are added to the final tensor in the form of extended product of low dimensional functions. Solving (1) with the approximation of (2) requires a greedy strategy, which is;

$$u \approx u_{PGD} = \underbrace{F_1(\mathbf{x}) F_2(\omega)}_{\text{Nth mode}} + \sum_j^{N-1} \left( F_1^j(\mathbf{x}) F_2^j(\omega) \right)$$

It is assumed that the function  $u$  is defined by  $N$  modes out of which  $N-1$  modes are known and our objective is to find the  $N^{\text{th}}$  to completely define  $u_{PGD}$ . To solve the above equation, previous terms are assumed to be known and the rest of the new terms are calculated iteratively enriching the final solution ( this method is also called as fixed-point or alternating directions algorithm ) until a global convergence is obtained, which is defined as:

$$\text{Convergence} = \frac{\left\| \prod_{i=1}^2 F_i^m - \prod_{i=1}^2 F_i^{m-1} \right\|_{\mathcal{L}^2(\Omega \times I_\omega)}^2}{\left\| \prod_{i=1}^2 F_i^{m-1} \right\|_{\mathcal{L}^2(\Omega \times I_\omega)}^2},$$

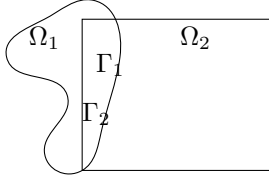
in an appropriate norm. With the above PGD strategy we incorporate the DDM overlapping strategy. In the case of a Helmholtz model with first order absorbing boundary conditions, following scheme is used as DDM. Assuming appropriate overlap, the sub-domain equations are solved until the resulting full domain solution have the desired error wrt. original solution,

$$-\Delta u^{j,i} - k_i^2 u^{j,i} = f(\mathbf{x}), \quad \forall(\mathbf{x}) \in \Omega_{\mathbf{x}}^i$$

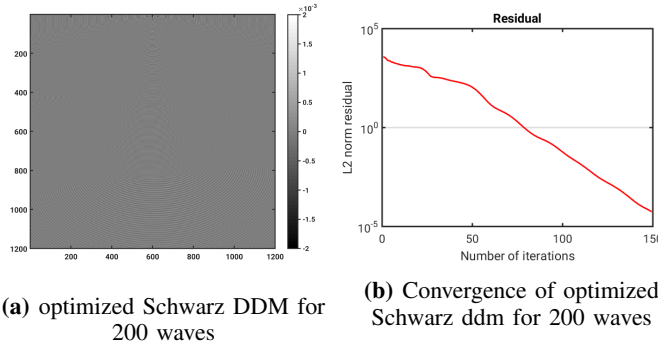
$$(\nabla \cdot \vec{n}) = ik u^{j,i}, \quad \forall(\mathbf{x}) \in \partial\Omega_{\mathbf{x}}^i / \Gamma^i$$

$$u^{j,i} = u^{j-1,i-1} \in \partial\Omega_{\mathbf{x}}^{i-1} \cap \partial\Omega_{\mathbf{x}}^i \quad \text{compatibility condition}$$

$$u^{j,i} = 0 \in \Gamma^{j,i} \cap \partial\Omega_{\mathbf{x}}^i \quad \text{a homogenous dirichlet b.c is imposed}$$



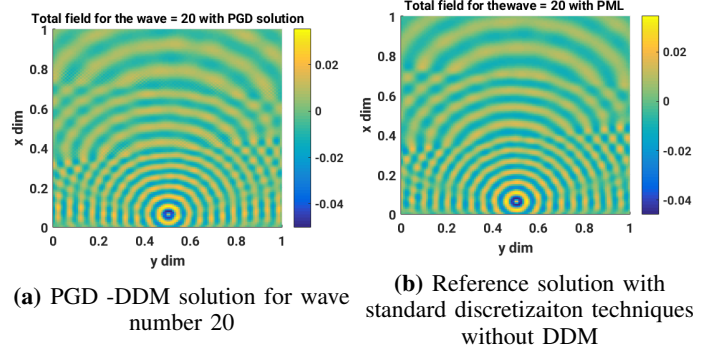
To increase convergences optimized Schwarz methods are used in place of regular Schwarz methods. [1][2] have shown much improved convergences when using optimized Schwarz. The following figures show the convergence for the multiplicative optimized Schwarz. It shows its utility to be used for DDM-PGD



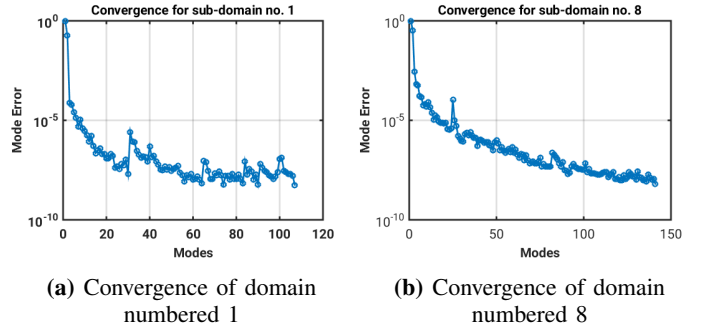
**Fig. 1:** Full order Helmholtz solved by Schwarz DDM

### C. Numerical Experiments

In this section, examples of 2D spatial Helmholtz model is shown. The discrete formulation is solved using 9-point rotated finite difference scheme. The results are shown for a with the proposed strategy. For the PGD formulation the domain  $\mathbf{x} \in [0, 1]^2$  and  $\omega \in [6, 20]$ . The model shows good convergence of PGD algorithm in each sub-domains. Fig(2) shows the expected DDM-PGD result is good representation of true solution. The model has transverse layers of heterogeneity.



**Fig. 2:** Reference and PGD-DDM figures



**Fig. 3:** Convergences for sub-domains chosen at random out of 16

The convergences are good and in line with the expectations as shown in fig(3). The convergences of each sub-domain takes less outer iteration of modes of PGD than the PGD over the complete domain.

### D. Conclusion

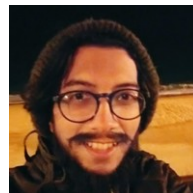
In this study, we have presented a novel method to incorporate a DDM strategy to PGD algorithm for a wave problem like Helmholtz model. We believe this work will help in developing faster solution for efficient scalable algorithms for PGD of large scale wave problems

## II. ACKNOWLEDGMENT

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**Prattya Datta** received his Bachelor of Engineering degree from Jadavpur University, India. He subsequently received his Master in Technology from Indian Institute of Technology Madras and Msc from University Paris Diderot 7. He is currently employed as a PhD in Barcelona Supercomputing Center with Department of CASE.