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Chapter 0

Introduction

This Master's thesis studies the dynamics of the N-Body Problem. But first we will need to define what this problem is, and set it in its historical context.

0.1 Historical overview

At the beginnings of history in Europe, people believed that the Earth was flat, and all celestial bodies were held in the sky. The first mathematician in Europe who suggested that the Earth was spherical, was the Greek mathematician Pythagoras, around year 500 before Christ, and this was accepted by most Greek philosophers at the time. But the first Universe model was not suggested until year 410 BC, by Eudoxus of Cnidus, Greek astronomer, who set the Earth at the centre of the Universe, and the Sun and the planets in giant transparent spheres which orbited the Earth, in such a way that the Sun turned around the Earth every 24 hours and the rest of planets had orbits beyond the Sun's one. Then, Aristotle, another Greek philosopher and scientist, suggested that objects in a space which was finite in space but eternal in time, were unchanging and moved in perfect circles. Then about a century later, another Greek astronomer, Aristarchus of Samos, suggested that it was the Sun which was at the centre of the Universe. It was Eratosthenes who calculated the radius of the Earth. Ptolemy, a Roman astronomer, suggested in year 150 after Christ that planets like Mars moved in circles as they orbited the Earth, where the circles were called epicycles. And this was another geocentric model.

But more than a millenium had to pass until the next Universe model, because of the obscurantism of Medieval Age. The first model in Modern Age, was from a Polish astronomer called Copernicus, who introduced again in 1543 the heliocentric model suggested by Ptolemy, because he believed that planets had to move in perfect circles. But he would not realise that the Sun was just another star until almost 60 years later, when Italian philosopher Giordano Bruno suggested so. And Tycho Brahe finally proved in 1572 with several naked-eye observations that observed objects in the Universe were not moving staticly in circles, as Aristotle suggested. The astronomer who would suggest that celestial bodies do not move in circles, but in ellipses, was the German Astronomer Kepler. He stated these following three laws:

1. The orbit of every planet is an ellipse with the Sun at one of the two foci.
2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

But these laws and the fact that the sky is always changing was proven by Italian natural astronomer Galileo, with several observations which contradicted the initial idea from Aristotle. These observations were verified three times in 1610.

The English physician Newton stated by his own in 1687 the three laws of motion, which we call Newton's Laws of Motion:

1. In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
2. In an inertial frame of reference, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration a of the object: $F = ma$.
3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

With the help of astronomer John Flamsteed, he studied Kepler's laws and made the Law of Universal Gravitation, which states that the gravitational force between two bodies is an attraction force, directly proportional to the product of their masses, and inversely proportional to the square of their distance. By using his third law of motion, he suggested in Kepler's Laws that there was also interaction among planets, not just between a planet and the Sun, and in the same way, among satellites, not just between a satellite and its planet, between satellites and the Sun, and so on. And this is how the N-body problem arised from this: it is the problem of predicting the individual motions of N celestial objects interacting with each other gravitationally.

0.2 Aims of this project

As we have recently stated, the N-Body Problem is the problem arised from the Newton's Laws of Motion and the Newton's Universal Gravitational Law which studies the prediction of the individual motions of N celestial objects interacting with each other gravitationally. We will see that the equations of motion of this problem become singular when two or more celestial objects coincide in a same position, and we will call these situations *collisions*. Moreover, we will be able to talk about the 2-Body Problem, which is the problem for $N = 2$, the 3-Body Problem, which is the problem for $N = 3$, and so on.

This problem is usually defined in the $\mathbb{R}^3$ vector space. But we can anyway state particular cases where we restrict to a 2-dimensional affine space (which we will call coplanar N-Body Problem, so we will consider it in $\mathbb{R}^2$), or even to a 1-dimensional affine space (which we will call collinear N-Body Problem, so we will consider it in $\mathbb{R}$).

We will focus on the collinear 3-Body Problem in particular. Our goal is to regularise the equations of motion defined by the collinear 3-Body Problem (a particular case of the general N-Body Problem, in the way which Newton stated them, where $N = 3$ and the space is a straight line). To regularise a system of equations is to make changes of variables and time transformations in order to obtain a new system of equations of motion which is well-defined where the origianal one did not use to. These regularisations will be useful in order to be able to see how the flow of the original equations of motion behaves near the singularities. In principle, we will assume first that singularities happen at collisions, because Newton's Universal Gravitational Law states that the force applied on one celestial body is a linear combination of the inverses of the squares of the distances with the rest of celestial bodies; later, we will see in a very summarised way that these situations are not the only ones where singularities happen with more than 3 bodies. With this, we will be able to see that in the new coordinates and in the new time variable, the particles do not actually collide, but they have a special behaviour near collisions. This property will be highlighted along the fourth chapter of this project and somehow in the third chapter. We will assume first that singularities just happen in collisions, in order to make calculations easier. When we arrive to the conclusions, we will finally cite some references which state that indeed, there are also some singularities not due to collisions for $N \geq 4$, i.e, for problems with more than 3 bodies, for instance, situations where all celestial bodies go to infinity in finite time.

In the first chapter, we will start stating the main features of the problems as they appear at the very beginning of McGehee's paper [4], before it starts focusing on the 3-Body Problem. But then, immediately, we will follow the book from Meyer-Hall [5] in order to show general theory of the N-Body Problem, and we will define what a point mass is, and state some definitions, what collisions are from a mathematical point of view, and which are the collision set, the linear and angular momenta, the centre of masses, the central configurations and the moment of inertia. In the same way, we will state the main properties about the N-Body Problem, which are the fact that the linear and angular momenta and the centre of masses are first integrals of the problem, as long as the properties of the central configurations. Also, in the same way, we will state some results, such as the non-existence of fixed points, and that the Lagrange central configurations for the planar 3-Body Problem always form equilateral triangles with the point masses as vertices. All these results will be useful to manage further calculations.

In the second chapter, still by following [5], we will state Sundman's Theorem, which tells us that total collision happens in finite time and when total angular momentum is 0, so the particles collide by moving in a straight line. In fact, we will see that the total collision is the collision where the N bodies coincide in a same position. Then, we will prove two previous lemmas which will make our calculations simpler, which are Lagrange-Jacobi formula and Sundman's inequality.

In the third chapter, still by following [5], we will talk about the coplanar 2-Body Problem, which is the problem restricted to a 2-dimensional space when $N = 2$. To make things easier, we will consider this problem as the planar Kepler problem, by fixing one of the masses at the point $(0, 0)$ and considering its dynamics as static (i.e., the position does not change along the time). Then, we will perform the Levi-Civita regularisation, based on the Poincaré factorisation. When we sketch this regularisation, we will profit its idea, which will be useful along the fourth chapter. For this, we will first have to identify the $\mathbb{R}^2$ plane with the complex plane $\mathbb{C}$, in order to make calculations easier. We will see indeed that collisions in this case can be regularised, and in fact, that they behave like a double harmonic oscillator.

In the fourth chapter, we will finally focus on the collinear 3-Body Problem, as aimed at the beginning, by following the paper from McGehee [4]. Our aim here will be to show that triple collision for the collinear 3-Body Problem can be regularised, in the same way than the coplanar 2-Body Problem from the previous chapter. For this, we will perform several changes of coordinates and time transformations in order to be able to see what happens near the triple collision and double collisions, which will be also defined at the beginning of the chapter. Here, we will do these transformations: splitting the positions and the momenta by radial and tangential variables; removing the triple collision; and putting the tangential positions variable as a function of a scalar; removing double collisions. After all this, we will ask ourselves for which values of the masses orbits can be extended through triple collision, so we will define what an isolationg block is. After a deep analysis of the transformed problem, we will see a particular case with masses 1 and 3 big enough and equal one to the other, and mass 2 arbitrarly small, where we will be able to simplify more the problem and prove that the dynamics are close to circles when the value of the second mass tends to 0. All the details of what we will do in this chapter will be explained at the starting abstract of the chapter.

Finally, in the last chapter, we will state the conclusions and some further theorems, which were just conjectures which McGehee stated at his paper [4] at the time he wrote it, but have been proved along the years since it was first published in 1974. For this, we will state references which prove these theorems which in McGehee's epoque were just conjectures, and detail some things which he did not state, by using them. In the same way, we will cite some references which show that for the collinear N-Body Problem for $N \geq 4$, there are singularities not due to collisions, for instance, when all particles go to infinity in finite time, or when some particles make infinite oscillations between two subsystems of particles in finite time.

Chapter 1

The N-body problem

In celestial mechanics, one of the most important problems to study is the N-body problem. In this chapter, we describe the main features of the problem and its behaviour in special cases which we can calculate in an easier way. We put the general formulation appearing at the beginning of the article [4], and then we focus on the most important things from the book [5] with some modifications in order to be able to see what we will need in next chapters. Consider an inertial reference system. Consider N punctual bodies with masses $m_i > 0$ at positions $q_i \in \mathbb{R}^k$, $i = 1, ..., N$, where $k = 1, 2$ or 3, in the chosen reference system. By using the standard Euclidean norm in $\mathbb{R}^k$:

$$\|q\| := \sqrt{\sum_{i=1}^{k} |q_i|^2}$$

we define the negative potential energy as:

$$U := \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{\|q_i - q_j\|}. \tag{1.1}$$

The motion of the bodies shall be described by the system of ODEs given by the formula:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3}. \tag{1.2}$$

Note that as the equations become singular wherever $q_i = q_j$ for certain $i \neq j$, this ODE is not well defined everywhere. So we shall define the collision set:

$$\Delta := \{q \in \mathbb{R}^{kN} : q_i = q_j \text{ for some } i \neq j\}. \tag{1.3}$$

1.1 The equations of the N-Body Problem

We are interested in transforming this second-order ODEs system into a first-order system in such a way that it can be written as a Hamiltonian system. A Hamiltonian function is a $\mathcal{C}^{r+1}$ function ($r \geq 1$) $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (set with variables (q, p)) with associated ODEs system:

$$\begin{aligned} \dot{q} =& \frac{\partial H}{\partial p} \\ \dot{p} =& -\frac{\partial H}{\partial q}. \end{aligned} \tag{1.4}$$

We define the matrix of masses $\mathbf{M}$ as:

$$\mathbf{M} := \begin{pmatrix} m_1\mathbf{1}_k & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & m_n\mathbf{1}_k \end{pmatrix} \in \mathbb{R}^{kN\times kN}$$

where $\mathbf{1}_k$ are the identity matrices of dimension $k \times k$, and we can rewrite the system (1.2) as:

$$\mathbf{M}\ddot{q} = \frac{\partial U}{\partial q}. \tag{1.5}$$

We define the momenta as:

$$p_i := m_i \dot{q}_i \tag{1.6}$$

or what is the same, $p = \mathbf{M}\dot{q}$. We define the kinetic energy as:

$$T(p) := \frac{1}{2} p^\top \mathbf{M}^{-1} p. \tag{1.7}$$

Finally, we define the Hamiltonian function as:

$$H(q,p) := T(p) - U(q). \tag{1.8}$$

By using the definitions (1.1) and (1.6) in order to get derivatives, we define the Hamiltonian system from (1.4) as:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \mathbf{M}^{-1}p \\ \dot{p} &= -\frac{\partial H}{\partial q} = \nabla U. \end{aligned} \tag{1.9}$$

1.2 The classical integrals

The N-body problem is an ODEs first-order system with $6N$ equations. Since it is Hamiltonian, it is a Hamiltonian system with $3N$ degrees of freedom (known from now on as DOF). Several first integrals are known for the N-Body Problem which allows us to reduce the number of DOF. In this section, we collect them, in the form of propositions.

Proposition 1.2.1 *The linear momentum:*

$$L := \sum_{i=1}^{N} p_i$$

is a first integral of the system (1.9). Therefore, the linear momentum corresponds to k constants of motion of equation (1.9). The subspace:

$$P := \{p \in \mathbb{R}^{kN} : L = 0\} \tag{1.10}$$

is invariant under the flow of equation (1.9).

Proof: L is a first integral of the system, since when taking derivatives with respect to time, we have that if we define:

$$a_{ij} := \frac{G m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3}$$

then we have:

$$\dot{L} = \sum_{\substack{1 \le i,j \le N \\ i \ne j}} \frac{G m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3} = \sum_{\substack{1 \le i,j \le N \\ i \ne j}} a_{ij}$$

which gives 0, since $a_{ij} = -a_{ji}$. □

Proposition 1.2.2 *The centre of masses:*

$$C := \sum_{i=1}^{N} m_i q_i$$

is such that $C - Lt$ is a first integral of the system (1.9). Therefore, the centre of masses corresponds to k constants of motion of equation (1.9). If we restrict to P defined in (1.10), then the subspace:

$$Q := \{q \in \mathbb{R}^{kN} : C = 0\} \tag{1.11}$$

is invariant under the flow of equation (1.9).

Proof: Let us take time derivatives of C, and we have that $\dot{C} = L$, and since L is a first integral, as proven before, we get that $\ddot{C} = 0$, and thus we have that $C = Lt + C_0$, and then C_0 is another first integral of the problem. □

Proposition 1.2.3 *The total angular momentum:*

$$A := \sum_{i=1}^{N} q_i \wedge p_i$$

is a first integral of the system (1.9). Therefore, the angular momentum corresponds to $\binom{k}{2}$ constants of motion of equation (1.9).

Remark: $\binom{k}{2}$ is 0 when $k = 1$, because exterior product with scalars has no sense, 1 when $k = 2$, because it is a determinant, and 3 when $k = 3$, because of the nature of $\mathbb{R}^3$.

Proof: A is a first integral of the problem, since when taking again the time derivative, we get that:

$$\dot{A} = \sum_{i=1}^{N} (\dot{q}_i \wedge p_i + q_i \wedge \dot{p}_i) = \sum_{i=1}^{N} \frac{1}{m_i} p_i \wedge p_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \ne i}}^{N} \frac{G m_i m_j q_i \wedge q_j}{\|q_i - q_j\|^3} - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \ne i}}^{N} \frac{G m_i m_j q_i \wedge q_i}{\|q_i - q_j\|^3} \tag{1.12}$$

and if we define:

$$b_{ij} := \frac{G m_i m_j q_i \wedge q_j}{\|q_i - q_j\|^3}$$

we can see that the expression (1.11) vanishes, since $b_{ij} = -b_{ji}$ and the skew-symmetry of the exterior product cancels out products of a vector with itself. □

Finally, we have the energy as another first integral, which is the Hamiltonian function defined in (1.8). By definition of any autonomous Hamiltonian system, it is a constant of motion. We have found one more constant of motion. Suming up all the constants of motion the system (1.9) has, there are a total of $2k + \binom{k}{2} + 1$ constants of motion, that is, 3 for $k = 1$, 6 for $k = 2$, and 10 for $k = 3$. So now we can define the constant energy levels as:

$$M(h) := \{(q, p) \in (Q \backslash \Delta) \times P : H(q, p) = h\}, \quad h \in \mathbb{R}.$$

1.3 Equilibrium solutions

Proposition 1.3.1 *The N-body problem defined at (1.9) has no fixed points.*

Proof: We want to find the equilibrium points of the system (1.9), that is:

$$\begin{aligned} \frac{\partial U}{\partial q_i} =&0, \quad i = 1 : N \\ p_i =&0, \quad i = 1 : N. \end{aligned} \tag{1.13}$$

We check that U is homogeneous of degree -1:

$$\sum_{i=1}^{N} q_i^\top \frac{\partial U}{\partial q_i} = \sum_{1\le i<j\le N} \frac{Gm_im_j(q_i-q_j)^\top(q_j-q_i)}{\|q_i-q_j\|^3} = -\sum_{1\le i<j\le N} \frac{Gm_im_j}{\|q_i-q_j\|} = -U. \tag{1.14}$$

U is a strictly positive function, since it is sum of strictly positive terms. But if (1.13) were true, then (1.14) would be 0, while it is actually negative, so we arrive to a contradiction. Thus there are no fixed points of the N-body problem. □

1.4 Central configurations

Central configurations (for short, CC), are a special type of solutions of the N-body problem characterised by maintaining the relative positions among the bodies, defined as dilations/contractions and rotations from a configuration of N bodies with fixed time, with respect to the origin. There are many solutions of (1.9) using the property of CCs, but for simplicity, let us find those of the kind $q_i(t) = \phi(t)a_i$, it means, products of scalar-valued functions and constant vectors. Let us plug it into the system which we already defined in (1.9):

$$m_i\ddot{\phi}a_i = \sum_{j=1,\, j\ne i}^{N} \frac{Gm_im_j\phi(a_j-a_i)}{|\phi|^3\|a_i-a_j\|^3}.$$

We can rearrange this equation as:

$$m_i|\phi|^3\phi^{-1}\ddot{\phi}a_i = \sum_{j=1,\, j\ne i}^{N} \frac{Gm_im_j(a_j-a_i)}{\|a_i-a_j\|^3}. \tag{1.15}$$

The right-hand side is constant, so the left-hand side must too. Let it be called $-\lambda$:

$$\begin{aligned} \ddot{\phi} =& -\frac{\lambda\phi}{|\phi|^3} \\ -\lambda m_i a_i =& \sum_{j=1,\, j\ne i}^{N} \frac{Gm_im_j(a_j-a_i)}{\|a_i-a_j\|^3}. \end{aligned} \tag{1.16}$$

There are many solutions for ϕ of (1.16). For example, one solution is $\alpha t^{2/3}$, where $\alpha^3 = 9\lambda/2$. For this solution, $t = 0$ corresponds to total collision, and $t = \infty$ to the invariant set where all bodies reach infinity. In chapter 2 we will define more precisely what a total collision is. Whereas equation (1.16) for the a_i's is a nontrivial algebraic system of equations, whose solutions are only known for $N = 2, 3$ but not for greater N, according to [5].

Now consider the planar N-body problem, where all vectors lie in $\mathbb{R}^2$. We will identify this space with the complex plane $\mathbb{C}$ by considering q_j, p_j as complex numbers. We shall look for an homographic solution of (1.2), i.e., a solution made just by rotations and dilations/contractions with respect to the origin of $\mathbb{C}$, by letting $q_j(t) = \phi(t)a_j$, where now both quantities lie in $\mathbb{C}$. Thus we get the same equations as in (1.15) and (1.16), and $\phi(t)$ is the solution of the planar Kepler problem, e.g., circular, elliptic, parabolic, or hyperbolic. We can state the decomposition of $\phi(t)$ and the a_j's by the radial and the angular components:

$$q_j(t) = \phi(t)a_j = R(t)r_j e^{i(\Theta(t)+\theta_j)}$$

so $R(t)$ will be an homothety and $\Theta(t)$ will be a rotation of the system.

Note that any uniform scaling of a CC, i.e., a multiplication of all positions by a same constant, is also a CC. In order to measure the size of the system, we define the moment of inertia of the system as:

$$I := \frac{1}{2}\sum_{i=1}^{N} m_i q_i^\top q_i. \tag{1.17}$$

Then if we take the vectors

$$q := \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}, \quad a := \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$$

we will be able to rewrite the second equation from (1.14) as:

$$\frac{\partial U}{\partial q}(a) + \lambda \frac{\partial I}{\partial q}(a) = 0. \tag{1.18}$$

So the constant λ can be interpreted as a Lagrange multiplier. If we fix a CC a and we make the scalar product with (1.18), we get:

$$\frac{\partial U}{\partial q}(a)^\top a + \lambda \frac{\partial I}{\partial q}(a)^\top a = 0. \tag{1.19}$$

As U and I are homogeneous functions of degree -1 and 2, respectively, we can rewrite the equation (1.19) as:

$$-U(a) + 2\lambda I(a) = 0.$$

So we get:

$$\lambda = \frac{U(a)}{2I(a)} > 0.$$

Moreover, summing up (1.16) $\forall i$ gives this equality:

$$\sum_{i=1}^{N} m_i a_i = 0$$

so the centre of mass of a CC is at the origin.

Proposition 1.4.1 *Central configurations of the N-body problem can be counted modulo orthogonal changes and dilations/contractions.*

Proof: If $\mathbf{A}$ is an orthogonal matrix, then $\mathbf{A}a_1, ..., \mathbf{A}a_n$ is a CC with the same λ:

$$-\lambda m_i \mathbf{A}a_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_i m_j \mathbf{A}(a_j - a_i)}{\|\mathbf{A}a_i - \mathbf{A}a_j\|^3} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_i m_j \mathbf{A}(a_j - a_i)}{\|a_i - a_j\|^3}$$

where we have used that $\|\mathbf{A}\| = 1$, given that $\forall x$:

$$\|\mathbf{A}x\|^2 = x^\top \mathbf{A}^\top \mathbf{A} x = x^\top x = \|x\|^2$$

since $\mathbf{A}$ is orthogonal, and we can take the square root at the equation because norms are positive. Now, if $\tau \neq 0$, then τa is a CC with $\tilde{\lambda} := \lambda \tau^3$:

$$-\lambda m_i \tau a_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{G m_i m_j \tau (a_j - a_i)}{\|\tau a_i - \tau a_j\|^3} \implies -\lambda \tau^3 m_i a_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{G m_i m_j (a_j - a_i)}{\|a_i - a_j\|^3}. \square$$

1.5 The Lagrange central configurations in the planar 3-Body Problem

Now consider the second formula from (1.16) for the planar 3-Body Problem. Then we look for $a_1, a_2, a_3 \in \mathbb{R}^2$. By setting the center of mass at the origin, we can delete two dimensions. If we fix the moment of inertia I, we can reduce another dimension. And if we identify two different bodies configuration which just differ by a rotation, we can reduce another dimension. So we look for critical points in a 2-dimensional manifold. However, finding it is quite difficult, so we will use another trick for the planar 3-Body Problem.

Theorem 1.5.1 *For any values of the masses, there are two and only two noncollinear central configurations for the 3-body problem, namely, the three particles are at the vertices of an equilateral triangle centered at the origin. The two solutions correspond to the two orientations of the triangle when labelled by the masses.*

Proof: Let $\rho_{ij} := \|q_i - q_j\|$ be the distance between the i-th particle and the j-th one. Given that the centre of mass is fixed at the origin, two rotationally equivalent configurations are identified. So we can rewrite U from formula (1.1) as:

$$U = G\left(\frac{m_1 m_2}{\rho_{12}} + \frac{m_2 m_3}{\rho_{23}} + \frac{m_3 m_1}{\rho_{31}}\right).$$

Let $M := m_1 + m_2 + m_3$ be the total mass, and assume it is at the origin, so:

$$\sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \rho_{ij}^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \|q_i - q_j\|^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \|q_i\|^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \|q_j\|^2$$
$$-2\sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j q_i^\top q_j = 2MI + 2MI - \sum_{i=1}^{3} m_i q_i^\top \left(\sum_{j=1}^{3} m_j q_j\right) = 4MI$$

and the last term vanishes because we have fixed the centre of mass at the origin. Thus we can isolate the moment of intertia I:

$$I = \frac{1}{4M}\sum_{i=1}^{3}\sum_{j=1}^{3} m_i m_j \rho_{ij}^2.$$

So I can also be written in terms of the mutual distances. Fixing I is the same as imposing:

$$I^* = \frac{1}{2}(m_1 m_2 \rho_{12}^2 + m_2 m_3 \rho_{23}^2 + m_3 m_1 \rho_{31}^2).$$

Indeed, we apply Lagrangian equations of the kind $-U + \lambda I^* = 0$, that is, by taking partial derivatives respect to ρ_{ij}:

$$-G\frac{m_i m_j}{\rho_{ij}^2} + \lambda m_i m_j \rho_{ij} = 0, \quad (i,j) = (1,2),(2,3),(3,1)$$

which clearly has an only solution $\rho_{ij} = (G/\lambda)^{-1/3}, \quad \forall (i,j)$. λ is a scale parameter, and the solutions are equilateral triangles, as we wanted to prove. $\square$

Chapter 2

Total collision: Sundman's Theorem

In the N-body problem, a total collision happens when all the masses collide at the same time in the same point. Sundman's Theorem claims that total collision is only possible if the total angular momentum is zero.

Before we state Sundman's Theorem, first we will need some technical lemmas.

Lemma 2.0.1 ***(Lagrange-Jacobi formula):*** *Let I be the moment of inertia defined at (1.17), T be the kinetic energy defined at (1.7), U be the potential energy defined at (1.1), and h the total energy of the system of N-bodies. Then:*

$$\ddot{I} = 2T - U = T + h. \tag{2.1}$$

Proof: Starting with (1.13) we differentiate I twice with respect to time, using the ODEs system (1.4). Let us compute the first derivative:

$$\dot{I} = \sum_{i=1}^{N} m_i q_i^\top \dot{q}_i = \sum_{i=1}^{N} q_i^\top p_i.$$

If we take the derivative of this expression, we get:

$$\ddot{I} = \sum_{i=1}^{N} (\dot{q}_i{}^\top p_i + q_i^\top \dot{p}_i) = \sum_{i=1}^{N} \frac{1}{m_i} p_i^\top p_i + \sum_{i=1}^{N} q_i^\top \frac{\partial U}{\partial q_i} = 2T - U$$

which is $T + h$, because $h = T - U$, by definition of the Hamiltonian we defined at (1.8). $\square$

Lemma 2.0.2 ***(Sundman's inequality):*** *Let $c := \|A\|$ be the norm of the angular momentum and $h = T - U$ the total energy of the system. Then we have that:*

$$c^2 \leq 4I(\ddot{I} - h). \tag{2.2}$$

Proof: Note that:

$$c = \|A\| = \left\| \sum_{i=1}^{N} m_i q_i \wedge \dot{q}_i \right\| \leq \sum_{i=1}^{N} m_i \|q_i\| \|\dot{q}_i\| = \sum_{i=1}^{N} (\sqrt{m_i}\|q_i\|)(\sqrt{m_i}\|\dot{q}_i\|).$$

Now we raise this inequality to the square power and we apply Cauchy-Schwarz's inequality, so we get that:

$$c^2 \leq \left(\sum_{i=1}^{N} m_i \|q_i\|^2 \right) \left(\sum_{i=1}^{N} m_i \|\dot{q}_i\|^2 \right) = 2I2T = 4IT$$

which is $4I(\ddot{I} - h)$, because of the Lemma 2.0.1. $\square$

Now we state the main theorem of this chapter:

Theorem 2.0.3 ***(Sundman's Theorem on total collapse):*** *If total collapse occurs, then angular momentum is zero, and total collapse is reached in a finite amount of time. That is, if $I(t) \to 0$ as $t \to t_1$, then $t_1 < \infty$ and $A = 0$.*

Proof: Consider an orbit of the system (1.8) leading to total collapse, with energy level h, so by Lemma 2.0.1 we have that $\ddot{I} = T + h$. Assume $I(t)$ is defined $\forall t \geq 0$ and:

$$I \to 0 \text{ as } t \to \infty. \tag{2.3}$$

Then this must happen:

$$q \to 0; \quad U \to \infty \tag{2.4}$$

because it is homogeneous of degree -1. As h is constant, and by (1.8) we know that $h = T - U$, then $T \to \infty$ must happen. So $\exists t^* > 0$ such that $\ddot{I} > 1 \, \forall t \geq t^*$. Now we integrate this inequality to get $I(t) \geq (1/2)t^2 + at + b$ for some $a, b \in \mathbb{R}$ constants. But this and (2.3) cannot happen at the same time, so total collapse can only happen in a finite amount of time.

Now assume that $\exists t_1 < \infty$ such that $I \to 0$ as $t \to t_1^-$ and so as in (2.4) $U \to \infty$ and $\ddot{I} \to \infty$. Thus $\exists t_2 < t_1$ such that $\ddot{I}(t) > 0 \quad \forall t_2 \leq t < t_1$. Because $I > 0$, $\ddot{I} > 0$ on $t_2 \leq t < t_1$ and $I(t) \to 0$ as $t \to t_1$ it follows that $\dot{I} < 0$ on $t_2 \leq t < t_1$ because of convexity.

Now let us multiply both sides of Sundman's inequality (2.2) by $-(1/4)\dot{I}I^{-1} > 0$ to get:

$$-\frac{1}{4}c^2 \dot{I} I^{-1} \leq h\dot{I} - \dot{I}\ddot{I}.$$

Now we integrate this inequality to get:

$$\frac{1}{4}c^2 \log(I^{-1}) \leq hI - \frac{1}{2}\dot{I}^2 + K \leq hI + K$$

where K is an integration constant and we have used the positivity of the squares. Thus,

$$\frac{1}{4}c^2 \leq \frac{hI + K}{\log(I^{-1})}.$$

As $t \to t_1$, $I \to 0$ and so the right-hand side tends to 0. But by definition, we have that $c = \|A\|$, and A is a first integral of our system, so it is constant, and as the right-hand side tends to 0, then c must be 0, and as it is $\|A\|$, so we have that $A = 0$, by definition of norms. □

Chapter 3

Double collisions: Levi-Civita regularisation

In this chapter, we will focus in the planar Kepler problem (the 2-Body Problem with one mass centered at the position $(0,0)$), and we will call *singularities* to the configurations where both bodies are at the point $(0,0)$. Recall that we can do so thanks to Proposition 1.2.2 and with a translation of variables. We will identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We will follow section 7.6 from [5]. Then, with a suitable change of variables, we will regularise the problem near $0 \in \mathbb{C}$, in order to see how the dynamics work.

3.1 Complex coordinates

As we can see the planar Kepler problem as one mass fixed at the origin and the other one with coordinates $x \in \mathbb{C}$, we can call $y \in \mathbb{C}$ its conjugate variable in the sense of Hamiltonian mechanics, and see both of them in the complex plane. Here we have a 2 DOF problem. Now we can consider the change of variables from x and y to z and w, defined as:

$$\begin{aligned} z =& x - iy, \qquad w = x + iy, \\ x =& \frac{z+w}{2}, \qquad y = \frac{w-z}{2i} \end{aligned} \tag{3.1}$$

We may call this linear change $\alpha(z, w)$. We know that $x, y \in \mathbb{C}$. So if we make zero their imaginary part, it means, $x, y \in \mathbb{R}$, we have that z and w are conjugate complex numbers one with respect to the other. Now we consider a formal or convergent series:

$$f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i y^j.$$

Now we can consider g the composition of f with α:

$$g(z, w) = f \circ \alpha(z, w) = f\left(\frac{z+w}{2}, \frac{w-z}{2i}\right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} z^m w^n.$$

We can see that f is a real series when $x, y \in \mathbb{R}$ (i.e., all a_{ij} are real) if and only if A_{mn} and A_{nm} are conjugate numbers. These restrictions will be called reality conditions on g. We can also rewrite this in another way:

$$\overline{f(x, y)} = f(\bar{x}, \bar{y}) \iff \overline{g(z, w)} = \bar{g}(w, z).$$

Note that the bar over all the function corresponds to conjugate both coefficients and variables, whereas the bar just on the name of the function means just conjugating coefficients.

The change of coordinates α defined on (3.1) is symplectic with multiplier $2i$ so while making the composition of the Hamiltonian H, we will have to change it to $2i\tilde{H}$.

For example, we can use the case of the harmonic oscillator, defined by:

$$H(x,y) = \frac{1}{2}(x^2 + y^2)$$

which is transformed by (3.1) into:

$$H(z,w) = izw$$

and the equations of motion become:

$$\begin{aligned}\dot{z} =& iz;\\ \dot{w} =& -iw.\end{aligned}$$

As in this problem x and y are real, we can consider z and w as conjugate variables, and the only motion equation we will be interested in, is $\dot{z} = iz$.

Another way to do change of coordinates is when we have a 2 DOF problem in $\mathbb{R}^4$ with coordinates $x = (x_1, x_2)$ and $y = (y_1, y_2)$, which are conjugate in the Hamiltonian sense. If we consider the changes of coordinates from x and y to z and w defined by:

$$\begin{aligned}z =& x_1 + ix_2,\\ w =& \frac{1}{2}(y_1 - iy_2)\end{aligned} \tag{3.2}$$

and defining in an analogous way the conjugates $\bar{z}$ and $\bar{w}$, then we will have that:

$$dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dz \wedge dw + d\bar{z} \wedge d\bar{w}$$

We will use this for the planar Kepler problem.

3.2 Levi-Civita regularisation

From now on, we will work on the planar Kepler problem, and we will call the Levi-Civita regularisation to the changes of coordinates and time transformations made to remove the singularity at 0. Namely, the problem has Hamiltonian formulation:

$$H_d = \frac{1}{2}|y|^2 - \frac{1}{|x|} + \frac{d^2}{2} \tag{3.3}$$

where x and y are complex numbers. We will use the constant $d^2/2$ in order to make things more clear, using that Hamiltonian functions plus a constant have the same dynamics, because the general Levi-Civita regularisations always take place at the level of energy 0. Note that the original Hamiltonian H had this formula:

$$H = \frac{1}{2}|y|^2 - \frac{1}{|x|}. \tag{3.4}$$

Note that taking $H_d = 0$ is the same that taking $H = -d^2/2$, as we can check on the formula (3.4). So to make things more clear with Levi-Civita's statement, we will use the Hamiltonian H_d defined at (3.3). First, we change the variables from x and y to z and w using the generating function:

$$S(y,z) = yz^2.$$

Now we take the variables x and w as conjugates respectively of y and z, this means:

$$\begin{aligned} x =& \frac{\partial S}{\partial y} = z^2, \\ w =& \frac{\partial S}{\partial z} = 2yz. \end{aligned} \tag{3.5}$$

First of all, we shall check where this change is injective, by taking partial derivatives:

$$\begin{pmatrix} 0 & 2z \\ 2z & 2y \end{pmatrix}.$$

As this matrix has determinant $-4z^2$, then the change of coordinates is injective everywhere except at $z = 0$. Now we can compute the Hamiltonian:

$$H_d = \frac{1}{8|z|^2}(|w|^2 + 4d^2|z|^2 - 8). \tag{3.6}$$

We can observe that between the parentheses, we have an expression of a Hamiltonian of 2 harmonic oscillators with the same frequencies. Poincaré used a trick to remove the factor, which will be useful for us to remove the singularity and still keep the Hamiltonian character of our problem.

This trick consists in the following: if we have a Hamiltonian:

$$H(x) = \phi(x)L(x) \tag{3.7}$$

where $\phi(x)$ is a positive function. Then the equations of motion are:

$$\dot{x} = \phi J \nabla L + L J \nabla \phi.$$

Then we fix the level set $L = 0$, and thus the equations of motion become $\dot{x} = \phi J \nabla L$, so we can reparametrise the time by $d\tau = \phi(x)dt$ and with the nomenclature $' = d/d\tau$, we obtain $x' = J\nabla L$. Thus the flow defined by L on $L = 0$ is a reparameterisation of the flow defined by H on $H = 0$.

Now we turn back to the planar Kepler problem. The function L defined at (3.7), applied to H_d defined at (3.6), corresponds to the expression between the brackets. We change time by:

$$4|z|^2 d\tau = dt$$

and consider the new Hamiltonian:

$$L = \frac{1}{2}(|w|^2 + 4d^2|z|^2) - 4.$$

Now the flow defined by H_d on the set $H_d = 0$ has been reparametrised to the flow defined by L on the set $L = 0$. Anyway, we can define a new Hamiltonian $L_4 := L + 4$. Now we can see from the first equation from (3.5) that z and $-z$ are brought to the same x, and in order to have an injective change out of $z = 0$, with the help of the second equation from (3.5), we shall identify (z, w) with $(-z, -w)$. With $L_4 = 4$, we have a set homeomorphic to $\mathbb{S}^3$, and with this identification, we transform it into $\mathbb{P}^3$. The flow defined by L_4 at $L_4 = 4$ has no singularities, so we have regularised the dynamics.

The flow defined by L consists on two harmonic oscillators. We can see this Hamiltonian as the one of a particle with mass $m = 1$ moving in the w-plane subjected to the force of a linear spring fixed to the origin with elasticity constant $4d^2$. This system can admit up to 3 functionally independent first integrals:

$$E_1 = \frac{1}{2}(w_1^2 + 4d^2 z_1^2), \quad E_2 = \frac{1}{2}(w_2^2 + 4d^2 z_2^2), \quad A = z_1 w_2 - z_2 w_1 \tag{3.8}$$

which E_i are the energies in the direction of the z_i planes, and A is the angular momentum.

Now we can describe the dynamics of L_4:

$$\begin{aligned}\dot{z}_1 =& w_1;\\ \dot{w}_1 =& 4d^2 z_1;\\ \dot{z}_2 =& w_2;\\ \dot{w}_2 =& 4d^2 z_2.\end{aligned} \tag{3.9}$$

Indeed, as claimed in (3.8), we can used the equations of motion (3.9) to check that E_1, E_2 and A are first integrals of the problem defined by Hamiltonian L_4.

Therefore, the conclusion of this chapter is that the planar Kepler problem near the collision behaves like two coupled harmonic oscillators of a same frequency.

We can see the correspondence of the flow given by equations (3.9) with the trajectories of the flow given by the original planar Kepler problem when $|x|$ is constant along time. In polar coordinates, $r^2\dot{\theta} = c$ is constant, and:

$$r = \frac{c^2/\mu}{1 + e\cos(f)}$$

where $e = \|\mathbf{e}\|$ is said to be the excentricity of the ellipse, $f = \theta - g$, g is the angle from the positive x_1 axis to $\mathbf{e}$, and $\mathbf{e} \in \mathbb{R}^3$ is a constant vector given by integrating the expression:

$$-\mu\frac{d}{dt}\left(\frac{q}{\|q\|}\right) = A \wedge \dot{p}$$

where $q = (x_1, x_2, 0)$ and $p = (y_1, y_2, 0)$, and $A = (0, 0, c)$ is the angular momentum vector.

Chapter 4

Triple collision in the collinear three body problem

For this chapter, we follow the article [4]. In the previous chapters, we considered the general N-body problem in any dimension. Now, what we want to explain in this chapter is the collinear case of the 3-Body Problem, that is to say, we fix $N = 3$ and we restrict to the case where all point masses move in a fixed straight line. Then we transform the dynamics in such a way we delete the singularities due to collisions.

Here, a triple collision is the total collision defined in Chapter 2 with Sundman's Theorem, as we are in the 3-Body Problem. A double collision, or binary collision, is a configuration of the 3 point masses where two coincide in a same position at a same time but the other one is in a different place.

The collinear N-Body Problem is a problem where all particles move in a same straight line, and it corresponds to equations (1.9) with $k = 1$. We want to see the triple collision (total collapse) in the collinear three body problem ($N = 3$). So we will take a position (q_1, q_2, q_3) of the particles as a point in $\mathbb{R}^3$. Fixing the centre of mass at the origin as in Proposition 1.2.2 will reduce the space of positions to the plane Q determined by the equations from (1.11). The momenta will be also reduced to the plane defined by (1.10), and the equations (1.2) will be transformed into a 4-dimensional first-order ODEs system. Conservation of energy defined by the Hamiltonian (1.8) will further reduce the system to a 3-dimensional system defined on a constant energy manifold.

Triple collision, as defined above, will coincide with the origin in Q. We will make a transformation in such a way that we remove the origin from the position coordinates, and we restrict the positions in Q to a circle which we will call S. With a suitable change of coordinates in the variables which define the momenta, this transformation will let us extend the system to a 2-dimensional boundary, which we will call the "triple collision manifold", onto the constant energy manifold. A time transformation will scale the vector field in a way that it will be extended to the boundary. The boundary will be invariant for the extended vector field. All points in this boundary will correspond to triple collision in the coordinates we had first. The flow on the boundary will be completely ficticious, since orbits on it are defined in points which in the original coordinates are singular. However, we will see that the flow on the whole constant energy surface, including the ficticious boundary, is continuous. Hence the flow close to the boundary will behave as the flow on the boundary for an arbitrarily long time. Therefore, the behaviour of orbits on the triple collision manifold will be used to understand what is supposed to happen with orbits close to the triple collision.

To exemplify how the properties of the triple collision manifold can be exploited, we will remind the proof of Sundman's Theorem (2.0.3), and we will also see a new statement, due to Siegel according to McGehee [4], in which the set of orbits ending in triple collision forms a smooth submanifold of the constant energy surface. For this, we will show for the collinear case that this result follows from the Stable Manifold Theorem applied to two fixed points which we will state in Section 4.4.

We will also use the properties of the triple collision manifold to ask ourselves if orbits can be extended through triple collision. By examining the flow on the triple collision manifold, we will see that triple collision cannot be "regularised" at least for some values of the masses. In section 4.7, we will define what an isolating block is, and when we say that it is regularisable.

Finally, the flow on the triple collision manifold will be used to show that for some values of the masses, the kinetic energy of the system is arbitrarily high after going close to triple collision. That is to say, one of the particles gains arbitrarily its speed in one direction, while the other two particles are close together and moving in the opposite direction with a large speed. At the conclusions (Chapter 5), we will ask ourselves if there exist singularities which do not correspond to singularities in the N-Body Problem, which will be determined by Painlevé's Theorem.

From now on, all the images which are present in this chapter are extracted from McGehee's paper [4]. This image shows us the position coordinates:

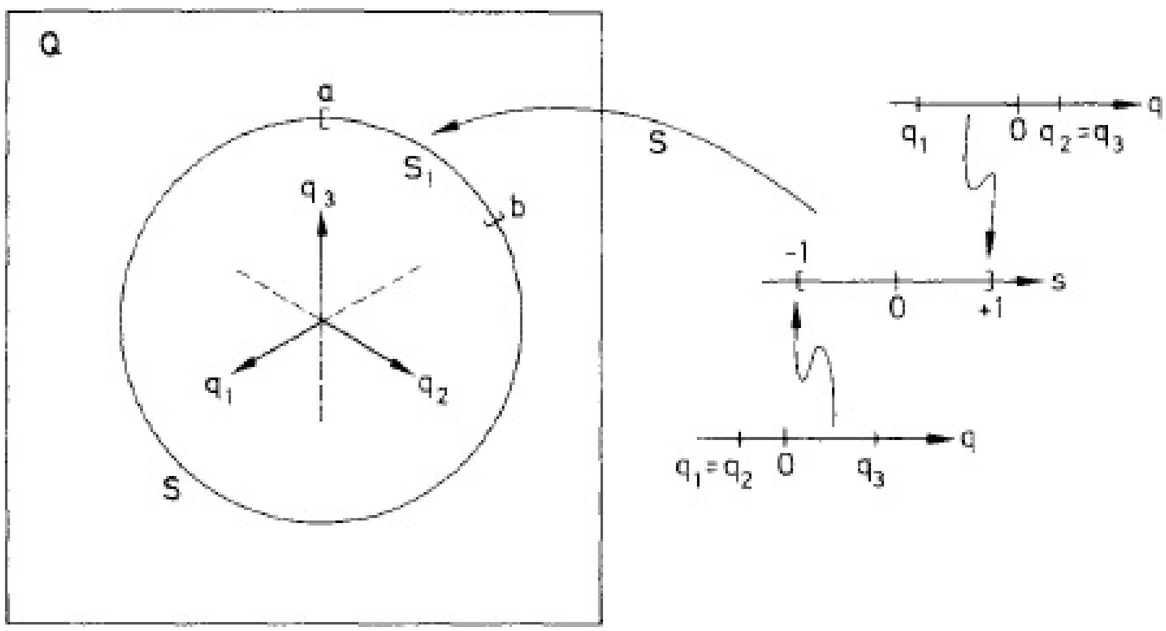

Fig. 1. Position coordinates

We begin by writing the equations of motion in Hamiltonian form. We define $q = (q_1, q_2, q_3)^\top \in \mathbb{R}^3$ and the mass matrix as:

$$\mathbf{M} := \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

So the negative potential energy defined in (1.1) will be redefined as:

$$U(q) := \frac{Gm_1m_2}{|q_1 - q_2|} + \frac{Gm_1m_3}{|q_1 - q_3|} + \frac{Gm_2m_3}{|q_2 - q_3|}$$

and thus we can define the momenta and the Hamiltonian function as in (1.7) and (1.8). As we have seen before, the function U is not defined everywhere, but outside the set of collision points, which we will call:

$$\Delta := \{q \in \mathbb{R}^3 : q_1 = q_2,\ q_1 = q_3 \text{ or } q_2 = q_3\}.$$

The Hamiltonian system defines a vector field on $(\mathbb{R}^3\backslash\Delta)\times\mathbb{R}^3$, which we will call a vector field with singularities on $\mathbb{R}^6$. As we have proved in Propositions 1.2.1 and 1.2.2, we can reduce the dimension of the Hamiltonian system to the subspace $Q\times P$, where

$$Q := \{q\in\mathbb{R}^3 : m_1q_1+m_2q_2+m_3q_3=0\}$$

are the points with centre of mass at the origin, and

$$P := \{p\in\mathbb{R}^3 : p_1+p_2+p_3=0\}$$

are the momenta with zero total linear momentum. Note that one can restrict Hamiltonian system (1.8) to $(Q\backslash\Delta)\times P$. This manifold has dimension 4, since we have reduced 2 dimensions from the 6 we had at first. Since the Hamiltonian function is constant along solutions, it defines this invariant set:

$$M(h) := \{(q,p)\in(Q\backslash\Delta)\times P : H(q,p)=h\},\quad h\in\mathbb{R}. \tag{4.1}$$

However, the flow generated by this vector field is not complete, that is to say, solutions do not exist for all time. In finite time, as stated in Theorem 2.0.3, some solutions tend to collision and hence leave $M(h)$. Solutions which end in triple collision will be slowed down so that they approach collision in infinite time.

4.1 Singularities due to triple collision

We want to examine the singularities $q=0$. To this end, we define the radius of q induced by $\mathbf{M}$ as:

$$r := \sqrt{q^\top\mathbf{M}q}$$

and so we define the unit circle in Q with the norm given by the moment of inertia r^2 as:

$$S := \{q\in Q : r^2=1\}.$$

We will call a point on S a configuration for the system of particles. So we will study the configuration space in polar coordinates: $(0,\infty)\times S\to Q\backslash\{0\}$ as $(r,\mathbf{s})\mapsto r\mathbf{s}$. We define the variables:

$$\begin{aligned} r =&\sqrt{q^\top\mathbf{M}q}\\ \mathbf{s} =&r^{-1}q\\ y =&p^\top\mathbf{s}\\ \mathbf{x} =&p-y\mathbf{M}\mathbf{s}. \end{aligned} \tag{4.2}$$

We can easily see that $\mathbf{s}\in S$. It satisfies $\mathbf{x}^\top\mathbf{s}=0$. Indeed:

$$\mathbf{x}^\top\mathbf{s}=p^\top\mathbf{s}-y\mathbf{s}^\top\mathbf{M}\mathbf{s}=p^\top\mathbf{s}-yr^{-2}q^\top\mathbf{M}q=p^\top\mathbf{s}-p^\top\mathbf{s}r^{-2}r^2=0.$$

Thus we have split the momentum p into a radial coordinate y and a tangential coordinate $\mathbf{x}$. Now we define the tangent bundle of S as:

$$T := \{(q,p)\in S\times P : p^\top q=0\} \tag{4.3}$$

so we can see that $(\mathbf{s},\mathbf{x})\in T$. The old variables can be written in terms of the new ones:

$$\begin{aligned} q =&r\mathbf{s};\\ p =&\mathbf{x}+y\mathbf{M}\mathbf{s} \end{aligned}$$

which defines a real analytic diffeomorphism:

$$\begin{aligned}(0,+\infty)\times\mathbb{R}\times T \to&(Q\backslash\{0\})\times P\\ (r,y,(\mathbf{s},\mathbf{x}))\mapsto&(r\mathbf{s},\mathbf{x}+y\mathbf{M}\mathbf{s}).\end{aligned}\tag{4.4}$$

Now by taking the orthogonality of $\mathbf{x}$ and $\mathbf{s}$, we can see that the kinetic energy (1.8) can be written as:

$$T(p)=\frac{1}{2}(\mathbf{x}^\top\mathbf{M}\mathbf{x}+y^2)$$

while we can use the homogeneity of the negative potential energy to rewrite it as:

$$U(q)=r^{-1}U(\mathbf{s})$$

so the Hamiltonian relation $M(h)$ defined in (4.1) can be written as:

$$\frac{1}{2}(\mathbf{x}^\top\mathbf{M}\mathbf{x}+y^2)-r^{-1}U(\mathbf{s})=h.\tag{4.5}$$

So the equations of motion (4.2) become:

$$\begin{aligned}\dot r=&y\\ \dot y=&r^{-1}\mathbf{x}^\top\mathbf{M}^{-1}\mathbf{x}-r^{-2}U(\mathbf{s})\\ \dot{\mathbf{s}}=&r^{-1}\mathbf{M}^{-1}\mathbf{x}\\ \dot{\mathbf{x}}=&-r^{-1}y\mathbf{x}-r^{-1}(\mathbf{x}^\top\mathbf{M}^{-1}\mathbf{x})\mathbf{M}\mathbf{s}+r^{-2}U(\mathbf{s})\mathbf{M}\mathbf{s}+r^{-2}\nabla U(\mathbf{s}).\end{aligned}\tag{4.6}$$

We can check it differentiating directly from the definition of the new variables:

$$\begin{aligned}\dot r=&\frac{q^\top\mathbf{M}\dot q}{r}=\frac{q^\top p}{r}=\mathbf{s}^\top p=y\\ \dot{\mathbf{s}}=&\frac{\dot r}{r^2}q-r^{-1}\mathbf{M}^{-1}p=yr^{-1}\mathbf{s}-r^{-1}\mathbf{M}^{-1}\mathbf{x}-r^{-1}y\mathbf{s}=-r^{-1}\mathbf{M}^{-1}\mathbf{x}\\ \dot y=&r^{-2}\nabla U(\mathbf{s})\mathbf{s}+(\mathbf{x}+y\mathbf{M}\mathbf{s})^\top(r^{-1}\mathbf{M}^{-1}\mathbf{x})=-r^{-2}U(\mathbf{s})+r^{-1}\mathbf{x}^\top\mathbf{M}^{-1}\mathbf{x}\\ \dot{\mathbf{x}}=&r^{-2}\nabla U(\mathbf{s})-\dot y\mathbf{M}\mathbf{s}-y\mathbf{M}\dot{\mathbf{s}}=r^{-2}\nabla U(\mathbf{s})-r^{-1}(\mathbf{x}^\top\mathbf{M}^{-1}\mathbf{x})\mathbf{M}\mathbf{s}-r^{-2}U(\mathbf{s})\mathbf{M}\mathbf{s}-r^{-1}y\mathbf{x}\end{aligned}$$

where in the third equation we have used the orthogonality of $\mathbf{x}$ and $\mathbf{s}$ and that U is homogeneous of degree -1.

Equations (4.6) define a vector field with singularities on $[0,+\infty)\times\mathbb{R}\times T$. We have now expanded the singularities due to triple collision. Whereas for equations (1.9), the set of triple collision points was $\{0\}\times P$, the set of triple collision points for equations (4.6) is now $\{0\}\times\mathbb{R}\times T$, that is to say, the set where $r=0$. The set of double collision points is $(0,+\infty)\times\mathbb{R}\times(T\cap(\Delta\backslash P))$, i.e., the set where $\mathbf{s}$ satisfies $s_i=s_j$ for some $i\neq j$, where $i,j=1,2,3$. We will deal with double collisions in section 4.3, but now we want to remove the singularity at $r=0$. Here we have a proposition:

Proposition 4.1.1 *The change of variables:*

$$\begin{aligned}\mathbf{u}=&\sqrt{r}\mathbf{x}\\ v=&\sqrt{r}y\end{aligned}\tag{4.7}$$

which defines the real analytic diffeomorphism:

$$\begin{aligned}(0,+\infty)\times\mathbb{R}\times T\to&(0,+\infty)\times\mathbb{R}\times T\\ (r,v,(\mathbf{s},\mathbf{u}))\mapsto&(r,r^{-1/2}v,(\mathbf{s},r^{-1/2}\mathbf{u}))\end{aligned}\tag{4.8}$$

and the time transformation:

$$dt = r^{3/2} d\tau \tag{4.9}$$

transform system (4.6) into:

$$\begin{aligned} r' =& rv \\ v' =& \frac{1}{2}v^2 + \mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u} - U(\mathbf{s}) \\ \mathbf{s}' =& \mathbf{M}^{-1}\mathbf{u} \\ \mathbf{u}' =& -\frac{1}{2}v\mathbf{u} - (\mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u})\mathbf{M}\mathbf{s} + U(\mathbf{s})\mathbf{M}\mathbf{s} + \nabla U(\mathbf{s}) \end{aligned} \tag{4.10}$$

and thus remove the singularities of it due to total collision, i.e., at $r = 0$. The energy relation (4.5) becomes:

$$\frac{1}{2}(\mathbf{u}^\top \mathbf{M}\mathbf{u} + v^2) - U(\mathbf{s}) = rh. \tag{4.11}$$

Proof: First, we can see directly that the change of variables (4.7) transforms the Hamiltonian relation (4.5) into (4.11). Secondly, the equations of motion (4.6) become:

$$\begin{aligned} \dot{r} =& r^{-1/2}v \\ \dot{v} =& r^{-3/2}\left(\frac{1}{2}v^2 + \mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u} - U(\mathbf{s})\right) \\ \dot{\mathbf{s}} =& r^{-3/2}\mathbf{M}^{-1}\mathbf{u} \\ \dot{\mathbf{u}} =& r^{-3/2}\left(-\frac{1}{2}v\mathbf{u} - (\mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u})\mathbf{M}\mathbf{s} + U(\mathbf{s})\mathbf{M}\mathbf{s} + \nabla U(\mathbf{s})\right). \end{aligned}$$

We can see that $\dot{r}$ and $\dot{\mathbf{s}}$ expressions can be directly taken from the definition of the change, while with $\dot{v}$ and $\dot{\mathbf{u}}$ we will have to take derivatives. This is easier to perform by deriving directly in the old variables taken from (4.7):

$$\begin{aligned} -\frac{1}{2}r^{-3/2}\dot{r}v + r^{-1/2}\dot{v} =& -\frac{1}{2}r^{-2}v + r^{-1/2}\dot{v} = r^{-2}\mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u} - r^{-2}U(\mathbf{s}) \\ \implies \dot{v} =& r^{-3/2}\left(\frac{1}{2}v^2 + \mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u} - U(\mathbf{s})\right); \\ -\frac{1}{2}r^{-3/2}\dot{r}\mathbf{u} + r^{-1/2}\dot{\mathbf{u}} =& -\frac{1}{2}r^{-2}\mathbf{u} + r^{-1/2}\dot{\mathbf{u}} = -r^{-2}v\mathbf{u} - r^{-2}(\mathbf{u}\mathbf{M}^{-1}\mathbf{u})\mathbf{M}\mathbf{s} + r^{-2}U(\mathbf{s})\mathbf{M}\mathbf{s} + r^{-2}\nabla U(\mathbf{s}) \\ \implies \dot{\mathbf{u}} =& r^{-3/2}\left(-\frac{1}{2}v\mathbf{u} - (\mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u})\mathbf{M}\mathbf{s} + U(\mathbf{s})\mathbf{M}\mathbf{s} + \nabla U(\mathbf{s})\right). \end{aligned}$$

This new change of variables has still a singularity at $r = 0$, so now the next step is to use the time transformation defined in (4.9) and we shall define the new derivative sign $'$ as $d/d\tau$. Thus the equations become, as one can prove directly, the ones shown in (4.10), so we have removed successfully the singularities at $r = 0$, as we wanted. □

Since the equations defined at (4.10) do not have singularities at $r = 0$, we have extended the equations of motion to include triple collision. Note that $\{r = 0\}$ is invariant for equations (4.10). Time transformation (4.9) slows down the orbits for small r in the way that to reach a solution ending in triple collision, we will need infinite time. The set of orbits ending in triple collision is now the set of orbits asymptotically close to the invariant set $\{r = 0\}$. Orbits in here can also be used to describe orbits of (4.10) for small r, i.e., orbits going close to triple collision.

4.2 Orthonormal coordinates

We would want to define a new energy surface $M(h)$, but first we will have to delete the singularities due to double collisions. In next section we will extend orbits through double collision by an analogue change of coordinates and a time transformation which lets us regularise double collision. To make computations easier, we will have to build first a new coordinates system, in which equations (4.10) transform to a vector field in $\mathbb{R}^4$.

We shall order the positions as $q_1 < q_2 < q_3$, and define the sets:

$$\begin{aligned} S_0 &:= \{\mathbf{s} \in S : s_1 < s_2 < s_3\}, \\ S_1 &:= \{\mathbf{s} \in S : s_1 \leq s_2 \leq s_3\}, \\ T_0 &:= \{(\mathbf{s}, \mathbf{u}) \in T : \mathbf{s} \in S_0\}, \\ T_1 &:= \{(\mathbf{s}, \mathbf{u}) \in T : \mathbf{s} \in S_1\}. \end{aligned}$$

First when defining Q and P, we reduced the configuration space from 6 dimensions to 4. Then, when defining S and T, we split its dimensions in 2 of radial components induced by $\mathbf{M}$ and 2 of unit vectors. So S has dimension 1, thus do S_0 and S_1. So we can interpretate them respectly as an open segment and its corresponding closed one. So we can define $\mathbf{a}$ and $\mathbf{b}$ as two points on S with $a_1 = a_2 < a_3$ and $b_1 < b_2 = b_3$. We know they are the two points of S_1 which do not belong to S_0, and by definition, they are double collisions, see fig. 1. All together, we will work with a subset of $\mathbb{R}^2$ in variables $\mathbf{s}$ and $\mathbf{u}$ instead of with one of $\mathbb{R}^6$, so we want to define a diffeomorphism between $[-1, 1] \times \mathbb{R}$ and T_1. We define the matrices:

$$\mathbf{A}_1 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; \quad \mathbf{A}_2 := \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

and now we define the rotation matrix:

$$\mathbf{A} := \frac{1}{m_1 + m_2 + m_3} \mathbf{A}_1 \mathbf{M} + \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \mathbf{M}^{-1} \mathbf{A}_2. \tag{4.12}$$

Proposition 4.2.1 *The matrix $\mathbf{A}$ defined in (4.12) is indeed a rotation matrix, with the following properties:*

- $\mathbf{A}^\top \mathbf{M} \mathbf{A} = \mathbf{M}$;
- $\mathbf{A} : Q \to Q$;
- $q^\top \mathbf{M} \mathbf{A} q = 0$ *if* $q \in Q$;
- $\mathbf{A}^2 q = -q$ *if* $q \in Q$;
- $\mathbf{a}^\top \mathbf{A}^\top \mathbf{M} \mathbf{b} > 0$.

Proof: For the first property, first we use that $\mathbf{M}$ is diagonal, $\mathbf{A}_1$ is symmetric and $\mathbf{A}_2$ is skew-symmetric:

$$\begin{aligned} \mathbf{A}^\top \mathbf{M} \mathbf{A} = \frac{1}{(m_1 + m_2 + m_3)^2} \mathbf{M} \mathbf{A}_1 \mathbf{M} \mathbf{A}_1 \mathbf{M} + \sqrt{\frac{m_1 m_2 m_3}{(m_1 + m_2 + m_3)^3}} (\mathbf{M} \mathbf{A}_1 \mathbf{A}_2 - \mathbf{A}_2 \mathbf{A}_1 \mathbf{M}) \\ - \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3} \mathbf{A}_2 \mathbf{M}^{-1} \mathbf{A}_2. \end{aligned}$$

Now we take each of the three summands, and for the first one, we have that:

$$\begin{gathered} \mathbf{A}_1 \mathbf{M} = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 \end{pmatrix}; \quad \frac{1}{m_1 + m_2 + m_3} (\mathbf{A}_1 \mathbf{M})^2 = \mathbf{A}_1 \mathbf{M}; \\ \frac{1}{(m_1 + m_2 + m_3)^2} \mathbf{M} \mathbf{A}_1 \mathbf{M} \mathbf{A}_1 \mathbf{M} = \frac{1}{m_1 + m_2 + m_3} \begin{pmatrix} m_1^2 & m_1 m_2 & m_1 m_3 \\ m_1 m_2 & m_2^2 & m_2 m_3 \\ m_1 m_3 & m_2 m_3 & m_3^2 \end{pmatrix}. \end{gathered} \tag{4.13}$$

For the second one, we have that it cancels out, since:

$$\mathbf{A}_1\mathbf{A}_2 = -(\mathbf{A}_2\mathbf{A}_1)^\top = \mathbf{0}. \tag{4.14}$$

And for the third one, we have that:

$$\mathbf{M}^{-1}\mathbf{A}_2 = \begin{pmatrix} 0 & m_1^{-1} & -m_1^{-1} \\ -m_2^{-1} & 0 & m_2^{-1} \\ m_3^{-1} & -m_3^{-1} & 0 \end{pmatrix}; \tag{4.15}$$

$$-\frac{m_1m_2m_3}{m_1+m_2+m_3}\mathbf{A}_2\mathbf{M}^{-1}\mathbf{A}_2 = -\frac{1}{m_1+m_2+m_3}\begin{pmatrix} -m_1m_2-m_1m_3 & m_1m_2 & m_1m_3 \\ m_1m_2 & -m_1m_2-m_2m_3 & m_2m_3 \\ m_1m_3 & m_2m_3 & -m_1m_3-m_2m_3 \end{pmatrix}. \tag{4.16}$$

Finally, by summing up, we have that the total result gives $\mathbf{M}$.

Now we take the second property: $\mathbf{A}_1\mathbf{M}q = 0$, since $q \in Q$ and $\mathbf{A}_1$ is filled by ones. And:

$$\mathbf{M}\mathbf{M}^{-1}\mathbf{A}_2 q = \mathbf{A}_2 q = \begin{pmatrix} q_2 - q_3 \\ -q_1 + q_3 \\ q_1 - q_2 \end{pmatrix}$$

and summing up, we obtain 0.

Now we prove the third property:

$$q^\top\mathbf{M}\mathbf{A}q = \frac{1}{m_1+m_2+m_3}q^\top\mathbf{M}\mathbf{A}_1\mathbf{M}q + \sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}q^\top\mathbf{A}_2 q = 0$$

where we have used the property of $\mathbf{A}_1\mathbf{M}$ applied to $q \in Q$ and the skew-symmetry of $\mathbf{A}_2$.

Now to prove the fourth property, we compute $\mathbf{A}_2$, using the properties (4.13), (4.14), (4.15) and (4.16):

$$\begin{aligned}\mathbf{A}^2 =& \frac{1}{(m_1+m_2+m_3)^2}(\mathbf{A}_1\mathbf{M})^2 + \frac{m_1m_2m_3}{m_1+m_2+m_3}(\mathbf{M}^{-1}\mathbf{A}_2)^2 \\ =& \frac{1}{m_1+m_2+m_3}\begin{pmatrix} m_1-m_2-m_3 & 2m_2 & 2m_3 \\ 2m_1 & m_2-m_1-m_3 & 2m_3 \\ 2m_1 & 2m_2 & m_3-m_1-m_2 \end{pmatrix}\end{aligned}$$

which applied to $q \in Q$, by taking into account its properties, gives $-q$.

Finally, we prove the last property: let us consider first the matrices product:

$$\mathbf{A}^\top\mathbf{M} = \frac{1}{m_1+m_2+m_3}\mathbf{M}\mathbf{A}_1\mathbf{M} - \mu\mathbf{A}_2, \quad \mu := \sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}.$$

By multiplying this expression by $\mathbf{b}$, we get that the first summand vanishes, and by performing $b_2 = b_3$, we get that:

$$\mathbf{A}^\top\mathbf{M}\mathbf{b} = -\mu\begin{pmatrix} b_2-b_3 \\ -b_1+b_3 \\ b_1-b_2 \end{pmatrix} = \mu(b_2-b_1)\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Now by multiplying scalarly this equality by $\mathbf{a}$, we obtain:

$$\mathbf{a}^\top\mathbf{A}^\top\mathbf{M}\mathbf{b} = \mu(b_2-b_1)(a_3-a_2) > 0$$

because of the definition of $\mathbf{a}$ and $\mathbf{b}$ as extremes of S_1. $\square$

We can consider Q having a scalar product induced by $\mathbf{M}$:

$$\langle q_1, q_2 \rangle := q_1^\top \mathbf{M} q_2, \quad q_1, q_2 \in Q.$$

Thus $\mathbf{A}$ is a rotation by $\pi/2$ in Q, $\mathbf{a}, \mathbf{b}$ have unit length, and $\{\mathbf{a}, \mathbf{A}\mathbf{a}\}$ is an orthonormal basis for Q, by property 3 from Theorem 4.2.1. Therefore, we have that:

$$\mathbf{b} = (\mathbf{a}^\top \mathbf{M}\mathbf{b})\mathbf{a} + (\mathbf{a}^\top \mathbf{A}^\top \mathbf{M}\mathbf{b})\mathbf{A}\mathbf{a}. \tag{4.17}$$

We can check this fact by multiplying scalarly by $\mathbf{a}$ on the left that this is true:

$$\mathbf{a}^\top \mathbf{M}\mathbf{b} = (\mathbf{a}^\top \mathbf{M}\mathbf{b})(\mathbf{a}^\top \mathbf{M}\mathbf{a}) + (\mathbf{a}^\top \mathbf{A}^\top \mathbf{M}\mathbf{b})(\mathbf{a}^\top \mathbf{M}\mathbf{A}\mathbf{a}).$$

Since $\mathbf{a}^\top \mathbf{M}\mathbf{a} = 1$ and $\mathbf{a}^\top \mathbf{M}\mathbf{A}\mathbf{a} = 0$, then (4.17) holds. Note that $\mathbf{a}^\top \mathbf{M}\mathbf{b}$ is a constant just depending on the masses, and:

$$0 < \mathbf{a}^\top \mathbf{M}\mathbf{b} < 1.$$

Now we choose $\lambda \in (0, \pi/4)$ such that:

$$\cos(2\lambda) = \mathbf{a}^\top \mathbf{M}\mathbf{b}.$$

Note that λ only depends on the masses. Now we plug this on (4.17) and we have that:

$$\mathbf{b} = \cos(2\lambda)\mathbf{a} + \sin(2\lambda)\mathbf{A}\mathbf{a}. \tag{4.18}$$

Now $\forall s \in \mathbb{R}$ we define:

$$\Sigma(s) := \sin(2\lambda)^{-1}(\sin(\lambda(1-s))\mathbf{a} + \sin(\lambda(1+s))\mathbf{b}). \tag{4.19}$$

Proposition 4.2.2 $\Sigma : [-1, 1] \to S_1$ *is a real analytic diffeomorphism such that:*

$$\Sigma'(s) = \lambda \mathbf{A}\Sigma(s). \tag{4.20}$$

Proof: Σ is clearly analytic, $\Sigma(-1) = \mathbf{a}$ and $\Sigma(1) = \mathbf{b}$. Since $\mathbf{a}, \mathbf{b} \in Q$, $\Sigma(s) \in Q \quad \forall s \in \mathbb{R}$. First we have to compute the value of $\sin(\lambda(1-s)) + \sin(\lambda(1+s))\cos(2\lambda)$ in order to simplify expressions. By using in $\sin(\lambda(1-s))$ the subtraction of angles 2λ and $\lambda(1+s)$, we have indeed:

$$\begin{aligned} \sin(\lambda(1-s)) + \sin(\lambda(1+s))\cos(2\lambda) &= \cos(\lambda(1+s))\sin(2\lambda) \\ - \sin(\lambda(1+s))\cos(2\lambda) + \sin(\lambda(1+s))\cos(2\lambda) &= \cos(\lambda(1+s))\sin(2\lambda). \end{aligned} \tag{4.21}$$

Equation (4.18) gives:

$$\begin{aligned} \Sigma(s) :=& \sin(2\lambda)^{-1}(\sin(\lambda(1-s))\mathbf{a} + \sin(\lambda(1+s))\mathbf{b}) \\ =& \sin(2\lambda)^{-1}(\sin(\lambda(1-s))\mathbf{a} + \sin(\lambda(1+s))\cos(2\lambda)\mathbf{a} + \sin(\lambda(1+s))\sin(2\lambda)\mathbf{A}\mathbf{a}) \end{aligned}$$

which by using (4.21) gives:

$$\cos(\lambda(1+s))\mathbf{a} + \sin(\lambda(1+s))\mathbf{A}\mathbf{a}.$$

Thus we have proven that:

$$\Sigma(s) = \cos(\lambda(1+s))\mathbf{a} + \sin(\lambda(1+s))\mathbf{A}\mathbf{a}. \tag{4.22}$$

Now we take the derivative using the expression from (4.22):

$$\Sigma'(s) = -\lambda\sin(\lambda(1+s))\mathbf{a} + \lambda\cos(\lambda(1+s))\mathbf{A}\mathbf{a} = \lambda\sin(\lambda(1+s))\mathbf{A}^2\mathbf{a} + \lambda\cos(\lambda(1+s))\mathbf{A}\mathbf{a} = \lambda\mathbf{A}\Sigma(s)$$

where we have used the fourth property of $\mathbf{A}$ in Proposition 4.2.1. Now we have to prove that the inner product of Σ is 1:

$$\begin{aligned}\Sigma(s)^\top \mathbf{M}\Sigma(s) =& \cos(\lambda(1+s))^2\mathbf{a}^\top\mathbf{M}\mathbf{a} + 2\cos(\lambda(1+s))\sin(\lambda(1+s))\mathbf{a}^\top\mathbf{M}\mathbf{A}\mathbf{a} + \sin(\lambda(1+s))^2\mathbf{a}^\top\mathbf{A}^\top\mathbf{M}\mathbf{A}\mathbf{a}\\ =& \cos(\lambda(1+s))^2 + \sin(\lambda(1+s))^2 = 1.\end{aligned} \tag{4.23}$$

Now we have to prove that this function is one-to-one. If we separate Σ in its three coordinates, we get the following formulas, by using $a_1 = a_2$ and $b_2 = b_3$, which can be easily checked:

$$\begin{aligned}\Sigma_2(s) - \Sigma_1(s) =& \sin(2\lambda)^{-1}(b_2 - b_1)\sin(\lambda(1+s))\\ \Sigma_3(s) - \Sigma_2(s) =& \sin(2\lambda)^{-1}(a_3 - a_2)\sin(\lambda(1-s))\\ \Sigma_3(s) - \Sigma_1(s) =& \sin(2\lambda)^{-1}((b_2 - b_1)\sin(\lambda(1+s)) + (a_3 - a_2)\sin(\lambda(1-s)))\end{aligned}$$

Since $0 < \lambda < \pi/4$ by the choice we did of λ, and $s \in [-1,1]$, $\sin(\lambda(1 \pm s))$ have inverses and are positive. Therefore Σ is one-to-one and $\Sigma(s) \in S_1\, \forall s \in [-1,1]$. □

Proposition 4.2.3 *The function Σ defined in (4.19) defines a change of coordinates:*

$$\begin{aligned}s =& \Sigma^{-1}(\mathbf{s})\\ u =& \mathbf{s}^\top\mathbf{A}^\top\mathbf{u}\end{aligned} \tag{4.24}$$

where $(\mathbf{s}, \mathbf{u}) \in T_1$. Here, the real analytic diffeomorphism:

$$\begin{aligned}[0,+\infty) \times \mathbb{R} \times [-1,1] \times \mathbb{R} \to& [0,+\infty) \times \mathbb{R} \times T_1\\ (r, v, s, u) \mapsto& (r, v, (\Sigma(s), u\mathbf{M}\mathbf{A}\Sigma(s)))\end{aligned} \tag{4.25}$$

is defined. The change of coordinates (4.24) transforms the system (4.10) into:

$$\begin{aligned}r' =& rv\\ v' =& \frac{1}{2}v^2 + u^2 - V(s)\\ s' =& \lambda^{-1}u\\ u' =& -\frac{1}{2}vu + \lambda^{-1}V'(s)\end{aligned} \tag{4.26}$$

where:

$$\begin{aligned}V : (-1,1) \to& \mathbb{R}\\ s \mapsto& U(\Sigma(s)).\end{aligned} \tag{4.27}$$

The energy relation (4.11) becomes:

$$\frac{1}{2}(u^2 + v^2) - V(s) = rh. \tag{4.28}$$

Proof: We define the function V in (4.27) in an explicit way:

$$\begin{aligned}&V(s) := G\sin(2\lambda)\\ &\times \left(\frac{m_1 m_2}{(b_2 - b_1)\sin(\lambda(1+s))} + \frac{m_2 m_3}{(a_3 - a_2)\sin(\lambda(1-s))} + \frac{m_1 m_3}{(b_2 - b_1)\sin(\lambda(1+s)) + (a_3 - a_2)\sin(\lambda(1-s))}\right).\end{aligned}$$

We can compute its derivative as:

$$V'(s) = \lambda \nabla U(\Sigma(s))^\top \mathbf{A}\Sigma(s). \tag{4.29}$$

Now we check that the energy relation (4.11) becomes (4.28):

$$\frac{1}{2}(\mathbf{u}^\top \mathbf{M}^{-1}\mathbf{u} + v^2) - U(\mathbf{s}) = \frac{1}{2}(u\mathbf{s}^\top \mathbf{A}^\top \mathbf{M}\mathbf{M}^{-1}\mathbf{M}\mathbf{A}\mathbf{s}u + v^2) - V(s) = \frac{1}{2}(u^2 + v^2) - V(s) = rh.$$

The first equation has not changed. The second one is obtained through the same matrix computation as for the energy. From the third equation from (4.10) we have that $\mathbf{s}' = \mathbf{M}^{-1}\mathbf{u}$, where $\mathbf{s} \in S$. Then when doing the first change of coordinates from (4.24), and knowing the property (4.20), we take time derivatives on the first change of (4.24):

$$\mathbf{s}' = \Sigma'(s)s' = \lambda \mathbf{A}\Sigma(s)s'.$$

Now we equate this expression to (4.10):

$$\mathbf{M}^{-1}\mathbf{u} = \lambda \mathbf{A}\Sigma(s)s'$$

and we multiply by $\Sigma(s)^\top \mathbf{A}^\top$ at the left, applying the first property of Proposition 4.2.1:

$$\Sigma(s)^\top \mathbf{A}^\top \mathbf{u} = \lambda \Sigma(s)^\top \mathbf{A}^\top \mathbf{M}\mathbf{A}\Sigma(s)s' = \lambda \Sigma(s)^\top \mathbf{M}\Sigma(s)s'.$$

The left-hand side of this equality is u, by the second change of coordinates from (4.20), and the right-hand side is $\lambda s'$, because by (4.23), we know that $\Sigma(s)^\top \mathbf{M}\Sigma(s) = 1 \quad \forall s \in [-1,1]$, so we finally have that $s' = \lambda^{-1}u$.

Finally, we have the fourth equation to prove:

$$u' = (\mathbf{s}^\top)'\mathbf{A}^\top \mathbf{u} + \mathbf{s}^\top \mathbf{A}^\top \mathbf{u}' = \mathbf{u}^\top \mathbf{M}^{-1}\mathbf{A}^\top \mathbf{u} - \frac{1}{2}\mathbf{s}^\top \mathbf{A}^\top \mathbf{u}v$$
$$-u^2\mathbf{s}^\top \mathbf{A}^\top \mathbf{M}\mathbf{s} + V(s)\mathbf{s}^\top \mathbf{A}^\top \mathbf{M}\mathbf{s} + \mathbf{s}^\top \mathbf{A}^\top \nabla U(\mathbf{s}) = -\frac{1}{2}uv + \lambda^{-1}V'(s).$$

This happens because the first, the third and the fourth summands vanish because of orthogonality. Thus we have proved that the new system we get is (4.26). □

The equations (4.26) define a vector field with singularities on $[0,+\infty) \times \mathbb{R} \times [-1,1] \times \mathbb{R}$. The singularities happen when $s = \pm 1$. The vector field defined by equations (4.26) and the vector field (4.10) are diffeomorphically equivalent by transformation (4.25) by restricting variables $r, v, \mathbf{s}, \mathbf{u}$ to $[0,+\infty) \times \mathbb{R} \times T_1$. Thus $\{r = 0\}$ corresponds to triple collision while $\{s = \pm 1\}$ corresponds to double collision. Notice that we have restricted our problem to the ordered case $q_1 \le q_2 \le q_3$. Therefore, the only double collisions can just happen between bodies 1 and 2 ($s = -1$) or between bodies 2 and 3 ($s = +1$). In next section we will delete singularities due to double collision by transforming equations (4.26) to a vector field without singularities on $[0,+\infty) \times \mathbb{R} \times [-1,1] \times \mathbb{R}$.

4.3 Regularisation of double collisions

It is well-known by Easton and Sundman that orbits can be extended through double collisions even for the 3-body problem in 3 dimensions. In this section, we will use a transformation similar to Sundman's to globally regularise all double collisons on an energy manifold. The regularisation corresponds physically to an elastic bounce.

Now for $s \in (-1,1)$ we define the function:

$$W(s) := 2(1 - s^2)V(s) \tag{4.30}$$

and by using the definition of V from (4.27), we can write it as:

$$W(s) = 2G\lambda^{-1}\sin(2\lambda)(W_1(s) + W_2(s) + W_3(s))$$

where

$$\begin{aligned} W_1(s) :=& \frac{m_1 m_2(1-s)}{(b_2 - b_1)\mathrm{sinc}(\lambda(1+s))} \\ W_2(s) :=& \frac{m_2 m_3(1+s)}{(a_3 - a_2)\mathrm{sinc}(\lambda(1-s))} \\ W_3(s) :=& \frac{\lambda m_1 m_3(1-s^2)}{(b_2 - b_1)\sin(\lambda(1+s)) + (a_3 - a_2)\sin(\lambda(1-s))} \end{aligned}$$

where we have defined:

$$\mathrm{sinc}(x) := \frac{\sin(x)}{x}.$$

Since $\mathrm{sinc}(x)$ can be extended to a positive real analytic function on $[0, \pi/2]$, and since $0 < \lambda < \pi/4$, W_1 and W_2 become real analytic functions on $[-1, 1]$. So does W_3 because it is a linear combination of W_1 and W_2. Thus W can be extended to a positive real analytic function on $[-1, 1]$, which we keep denoting by W.

In equation (4.25), the binary collisions correspond to $s = \pm 1$, because of the definition of V in (4.27) and W in (4.30).

Proposition 4.3.1 *The change of coordinates:*

$$w := (1-s^2)W(s)^{-1/2}u \tag{4.31}$$

which defines the real analytic diffeomorphism:

$$\begin{aligned} [0, +\infty) \times \mathbb{R} \times (-1, 1) \times \mathbb{R} \to& [0, +\infty) \times \mathbb{R} \times (-1, 1) \times \mathbb{R} \\ (r, v, s, w) \mapsto& (r, v, s, (1-s^2)^{-1}\sqrt{W(s)}w) \end{aligned} \tag{4.32}$$

and the time transformation:

$$d\tau = \lambda(1-s^2)W(s)^{-1/2}d\sigma \tag{4.33}$$

transform the system (4.26) into:

$$\begin{aligned} \frac{dr}{d\sigma} =& \frac{\lambda(1-s^2)}{\sqrt{W(s)}}rv \\ \frac{dv}{d\sigma} =& \frac{\lambda\sqrt{W(s)}}{2}\left(1 - \frac{1-s^2}{W(s)}(v^2 - 4rh)\right) \\ \frac{ds}{d\sigma} =& w \\ \frac{dw}{d\sigma} =& -s + \frac{2s(1-s^2)}{W(s)}(v^2 - 2rh) + \frac{1}{2}\frac{W'(s)}{W(s)}(1 - s^2 - w^2) - \frac{\lambda(1-s^2)}{2\sqrt{W(s)}}vw. \end{aligned} \tag{4.34}$$

So these transformations remove the singularities due to binary collisions. The energy relation (4.28) becomes:

$$w^2 + s^2 - 1 + (1-s^2)^2 W(s)^{-1}(v^2 - 2rh) = 0. \tag{4.35}$$

Proof: With the change of coordinates defined in (4.31), we transform the left-hand side from (4.28):

$$\frac{1}{2}(u^2+v^2)-V(s)=\frac{1}{2}(w^2(1-s^2)^{-2}W(s)+v^2)-\frac{W(s)}{2(1-s^2)}.$$

Now we multiply this equality by $2(1-s^2)^2$ and bring all things to the left-hand side:

$$w^2W(s)+v^2(1-s^2)^2-2rh(1-s^2)^2-W(s)(1-s^2)=0.$$

Finally, by dividing this expression by $W(s)$, we get the result on (4.35). We will need the expression (4.35) in this form: we multiply all the expression by 2, then we put at the right-hand side the terms with $W(s)$, and we get:

$$2(1-s^2)-2w^2=\frac{2(v^2-2rh)(1-s^2)^2}{W(s)}$$

and by dividing by $1-s^2$ and subtracting 1 to both sides, we get:

$$1-\frac{2w^2}{1-s^2}=\frac{2(1-s^2)}{W(s)}(v^2-2rh)-1. \tag{4.36}$$

Now we want to find the expression of equations of motion (4.26) in terms of the change of variables (4.31). The first and the third equations come directly from applying the relation between w and u. The second one has some manipulation:

$$v'=\frac{1}{2}v^2+(1-s^2)^{-2}W(s)w^2-\frac{W(s)}{2(1-s^2)}=\frac{1}{2}v^2-\frac{W(s)}{2(1-s^2)}\left(1-\frac{2w^2}{1-s^2}\right).$$

In the fourth equation, we apply the rule of the derivative of the quotient of two functions, in this case, of V in terms of W defined in (4.30):

$$V(s)=\frac{W(s)}{2(1-s^2)} \implies V'(s)=\frac{W'(s)(1-s^2)+2sW(s)}{2(1-s^2)^2}.$$

We also apply the several product derivative. That is to say, if we want to take the derivative of the product of n functions, it is:

$$\left(\prod_{i=1}^{n}f_i\right)'=\sum_{i=1}^{n}f_i'\prod_{j\neq i}^{1:n}f_j.$$

We can prove this because we can take the induction by n. We will take it for $n=3$. We will plug both statements on the proof of the fourth equation wherever $V'(s)$ appears from (4.26) to obtain:

$$\begin{aligned}
w' &= -\frac{2ss'w}{1-s^2}-\frac{W'(s)s'w}{2}+(1-s^2)W(s)^{-1/2}u' \\
&= -w^2\left(\frac{2s}{1-s^2}+\frac{W'(s)}{2}\right)\frac{\sqrt{W(s)}}{\lambda(1-s^2)}+(1-s^2)W(s)^{-1/2}\left(-\frac{1}{2}v\frac{\sqrt{W(s)}w}{1-s^2}+\frac{1}{2\lambda}\frac{W'(s)(1-s^2)+2sW(s)}{(1-s^2)^2}\right) \\
&= -\frac{1}{2}vw+\frac{\sqrt{W(s)}}{\lambda(1-s^2)}\left(s\left(1-\frac{2w^2}{1-s^2}\right)+\frac{W'(s)}{2W(s)}(1-s^2-w^2)\right).
\end{aligned}$$

So the equations of motion (4.26) become:

$$\begin{aligned}
r' &= rv \\
v' &= \frac{1}{2}v^2 - \frac{W(s)}{2(1-s^2)}\left(1 - \frac{2w^2}{1-s^2}\right) \\
s' &= \frac{\sqrt{W(s)}}{\lambda(1-s^2)}w \\
w' &= -\frac{1}{2}vw + \frac{\sqrt{W(s)}}{\lambda(1-s^2)}\left(s\left(1-\frac{2w^2}{1-s^2}\right) + \frac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2)\right).
\end{aligned} \tag{4.37}$$

Finally, to remove singularities due to double collisions, we take the new time variable defined in (4.33), so we have that we check that the equations (4.37) become:

$$\begin{aligned}
\frac{dr}{d\sigma} &= \frac{\lambda(1-s^2)}{\sqrt{W(s)}}rv \\
\frac{dv}{d\sigma} &= \frac{\lambda}{2}\left(\frac{1-s^2}{\sqrt{W(s)}}v^2 - \sqrt{W(s)}\left(1-\frac{2w^2}{1-s^2}\right)\right) \\
\frac{ds}{d\sigma} &= w \\
\frac{dw}{d\sigma} &= s\left(1-\frac{2w^2}{1-s^2}\right) + \frac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2) - \frac{\lambda(1-s^2)}{2\sqrt{W(s)}}vw.
\end{aligned} \tag{4.38}$$

Now we need to apply the energy relation (4.31) in order to get the equations defined in (4.34). The equation for $dv/d\sigma$ has this computation:

$$\frac{dv}{d\sigma} = \frac{\lambda\sqrt{W(s)}}{2}\left(\frac{1-s^2}{W(s)}v^2 - 1 + \frac{2w^2}{1-s^2}\right) = \frac{\lambda\sqrt{W(s)}}{2}\left(1 - \frac{1-s^2}{W(s)}(v^2-4rh)\right).$$

And the equation for $dw/d\sigma$ can be obtained automatically since the first term is automatically transformed with the energy relation (4.36). □

These equations define a real analytic vector field on $[0,+\infty)\times\mathbb{R}\times[-1,1]\times\mathbb{R}$. Now for $h\in\mathbb{R}$ we define the set:

$$N(h) := \{(r,v,s,w)\in[0,+\infty)\times\mathbb{R}\times[-1,1]\times\mathbb{R} : \text{(4.34) holds}\} \tag{4.39}$$

which is invariant under the equations of motion (4.34). Since the gradient of expression (4.35) is never zero on $N(h)$, this set defines a 3-dimensional real analytic submanifold on $[0,+\infty)\times\mathbb{R}\times[-1,1]\times\mathbb{R}$. Since (4.35) is the transformed Hamiltonian from $M(h)$, $N(h)$ is invariant under the vector field (4.38). Therefore, the equations (4.38) define a vector field on the part of $N(h)$ where there are no singularities, and (4.34) is the extension of equations (4.38) to all the set $N(h)$.

We define the subsets:

$$\begin{aligned}
N_3(h) &:= \{(0,v,s,w)\in N(h)\} \\
N_2(h) &:= \{(r,v,\pm 1,w)\in N(h)\} \\
N_1(h) &:= N(h)\setminus(N_3(h)\cup N_2(h)).
\end{aligned}$$

We started defining a vector field (1.9) on the constant energy set $M(h)$ for every fixed $h \in \mathbb{R}$. We used the transformations (4.4), (4.8), (4.25) and (4.32) in order to remove singularities. If we compose all them, we define the embedding $M(h) \to N(h)$. In fact, this embedding is a real analytic diffeomorphism onto $N_1(h)$, because it does not cover singularities due to collisions. After transforming the time with the rules (4.9) and (4.33), the equations from (1.9) on $M(h)$ are transformed into the equations from (4.34) on $N_1(h)$. However, this new vector field can be extended to a real analytic vector field on $N(h)$. The extension to points in $N_2(h)$ shows that we have regularised double collisions. The extension to points in $N_3(h)$ shows that we have included the points of the triple collision manifold. Any orbit on $M(h)$ is brought to an orbit on $N_1(h)$. However, an orbit which ends in a double collision on $M(h)$ can be now represented with a point in $N_2(h)$. An orbit which ended in triple collision on $M(h)$ now approaches slowly and asymptotically to a point in $N_3(h)$.

Solutions which end in triple collision are now defined for all time. The vector field (4.34) is not complete at all, because the time transformation (4.9) increases the speed of orbits for large r, so some solutions which used to reach ∞ in infinite time reach now that point in finite time. To avoid this problem, we could have used the transformation:

$$dt = \frac{r^{3/2}}{1 + r^{3/2}} d\tau$$

instead of (4.9), so we could have divided vector field (4.34) by the scalar $1 + r^{3/2}$ so it would be complete. However, since in this chapter our interest is to see just orbits near $r = 0$, we have chosen the easier transformation (4.9), which lets us understand better the problem.

For any point $x = (r, v, s, w) \in N(h)$ we will denote $\varphi(x, t)$ the solution of our equations of motion beginning at x when $t = 0$. So we will call φ to the flow determined by the vector field (4.33).

4.4 The triple collision manifold

Here we will restrict to the invariant set of total collision points, $N_3(h)$, which is defined when $r = 0$, and we will see how particles behave at this submanifold, and thus, how particles in the original coordinates behave near triple collision. We have a 3-dimensional manifold $N(h)$, a vector field defined by equations (4.34) on $N(h)$, and a flow φ given by this vector field and given initial conditions. As $r = 0$, the energy relation (4.35) becomes:

$$w^2 + s^2 - 1 + (1 - s^2)^2 W(s)^{-1} v^2 = 0. \tag{4.40}$$

We will call "the triple collision manifold" to $N_3(h)$. Notice that the expression (4.40) does not depend on h, as in the formula, we can see that the term which depends on h in (4.35) is billinear on h and r, so when $r = 0$ it cancels out. So $N_3(h)$ does not depend on h, and we can write $C := N_3(h)$ for ease of notation. By the shape of the equation, we can deduce that it is homeomorphic to $\mathbb{S}^2$ minus 4 points: if $s = \pm 1$, then $v \to \pm\infty$, as the dependence on v of the formula (4.40) is quadratic, so we have 4 singular points. However, the continuity of the flow φ allows us to use the flow φ on C to describe what happens with flows on $N_1(h)$ near C, and thus to describe how the solutions of the original equations near triple collision look like. In the following image we can see the shape of the manifold:

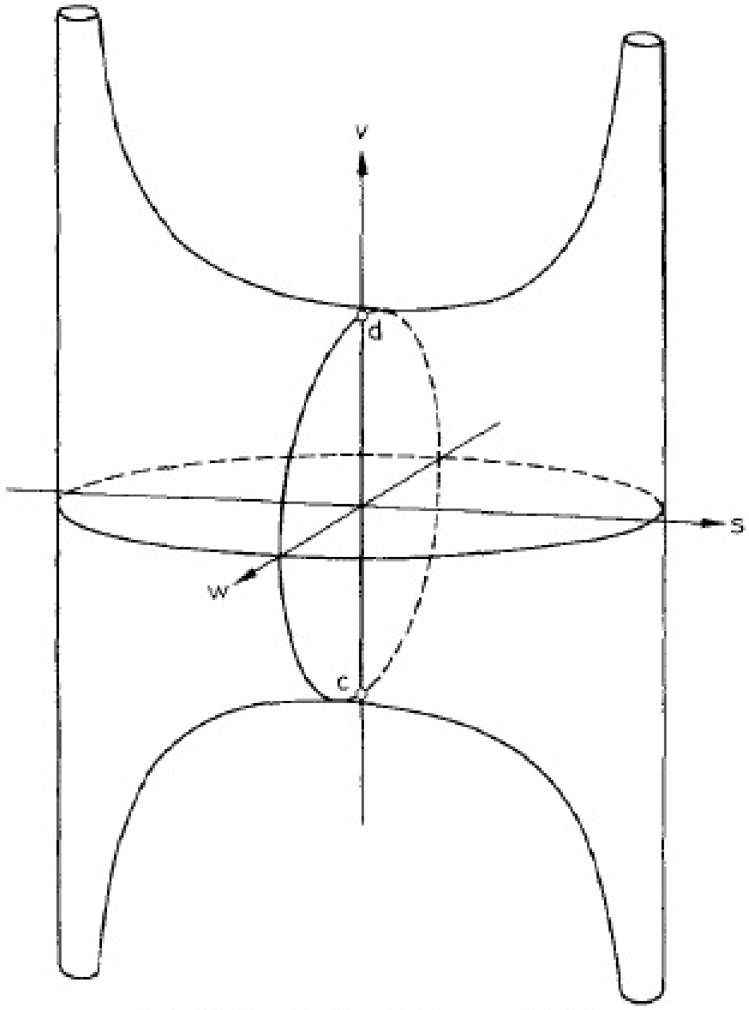

Fig. 2. The triple collision manifold

So the vector field defined in (4.34) becomes:

$$\begin{aligned}\frac{dv}{d\sigma} &= \frac{\lambda\sqrt{W(s)}}{2}\left(1 - \frac{1-s^2}{W(s)}v^2\right)\\ \frac{ds}{d\sigma} &= w \\ \frac{dw}{d\sigma} &= -s + \frac{2s(1-s^2)}{W(s)}(v^2 - 2rh) + \frac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-w^2) - \frac{\lambda(1-s^2)}{2\sqrt{W(s)}}vw.\end{aligned} \tag{4.41}$$

But before working with it, we should recall from section 1.4 the notion of central configuration:

Definition 4.4.1 *A point* $\mathbf{s}_0 \in S$ *is called a central configuration if* $\exists\mu \in \mathbb{R}$, $\mu > 0$ *such that:*

$$\nabla U(\mathbf{s}_0) = \mu \mathbf{M}\mathbf{s}_0.$$

Recall from the first equation from (1.16) that if $\rho(t)$ is a solution of the equation $\ddot{\rho} = -\mu\rho|\rho|^{-3}$. Moreover, $q(t) = \rho(t)\mathbf{s}_0$ is a solution of $\mathbf{M}\ddot{q} = \nabla U(q)$, and as we stated in Sundman's Theorem 2.0.3, solutions of ρ tend to 0 in finite time, so homographic solutions of q, i.e., those which keep the central configurations except for dilations/contractions (we cannot talk about rotations here because we are in the collinear case) either start or end in triple collision. Thus triple collision orbits exist. In the coordinates we have introduced, central configurations correspond to critical points of V. And also recall that $V(s) = U(\Sigma(s))$.

Proposition 4.4.2 $\mathbf{s}_0 = \Sigma(s_0)$ *is a central configuration* $\iff V'(s_0) = 0$.

Proof: Step 1: Assume $\mathbf{s}_0$ is a central configuration. Equation (4.29) implies:

$$V'(s_0) = \lambda \nabla U(\mathbf{s}_0)^\top \mathbf{A}\mathbf{s}_0 = \lambda\mu \mathbf{s}_0^\top \mathbf{M}\mathbf{A}\mathbf{s}_0 = 0$$

because of the orthogonality of $\mathbf{A}$.

Step 2: Now assume that $V'(s_0) = 0$. Then:

$$\nabla U(\mathbf{s}_0)^\top \mathbf{A}\mathbf{s}_0 = 0.$$

Since $\nabla U(\mathbf{s}_0) \in Q$, we can write:

$$\nabla U(\mathbf{s}_0) = \alpha \mathbf{M}\mathbf{s}_0 + \beta \mathbf{M}\mathbf{A}\mathbf{s}_0$$

for some $\alpha, \beta \in \mathbb{R}$. Therefore, by properties 1 and 3 stated in Proposition 4.2.1, we have that:

$$0 = \alpha \mathbf{s}_0^\top \mathbf{A}^\top \mathbf{M}\mathbf{s}_0 + \beta \mathbf{s}_0^\top \mathbf{A}^\top \mathbf{M}\mathbf{A}\mathbf{s}_0 = \beta \implies \beta = 0$$

hence $\nabla U(\mathbf{s}_0) = \alpha \mathbf{M}\mathbf{s}_0$. Thus $\mathbf{s}_0$ is a central configuration. □

Of all possible collinear central configurations, there is only one such that $s_{1c} < s_{2c} < s_{3c}$.

Proposition 4.4.3 *The collinear 3-body problem has a unique central configuration* $\mathbf{s}_c \in S_1$.

Proof: By the previous proposition we must prove that V has exactly one critical point on $(-1, 1)$. Since $V(s) \to \infty$ as $s \to \pm 1$, V has a critical point. Using equations (4.20), (4.29) and property 4 from the Proposition 4.2.1, we compute:

$$V''(s) = \lambda^2(\Sigma(s)\mathbf{A}^\top \nabla^2 U(\Sigma(s))\mathbf{A}\Sigma(s) - \nabla U(\Sigma(s))\mathbf{A}^2\Sigma(s)) = \lambda^2(\Sigma(s)\mathbf{A}^\top \nabla^2 U(\Sigma(s))\mathbf{A}\Sigma(s) + V(s)). \quad (4.42)$$

Now we can see by the definition of U that $\nabla^2 U$ is positive definite:

$$\xi^\top \nabla^2 U(q)\eta = \sum_{1 \le i < j \le 3} \frac{2m_i m_j(\xi_i - \xi_j)(\eta_i - \eta_j)}{|q_i - q_j|^3}.$$

This proof and the fact that $V(s) > 0$ imply that $V''(s) > 0 \quad \forall s \in (-1, 1)$. Thus V has an only minimum point (and no maximum). □

Now we can compute the fixed points for the flow on C. From $dv/d\sigma$ we can compute $\mathbf{s}_c = \Sigma(s_c)$, and:

$$v_c := \sqrt{2V(s_c)} = \sqrt{\frac{W(s_c)}{1 - s_c^2}}. \quad (4.43)$$

Now we define two points on C:

$$\mathbf{c} := (0, -v_c, s_c, 0)^\top, \quad \mathbf{d} := (0, v_c, s_c, 0)^\top.$$

Proposition 4.4.4 *The flow* φ *restricted to* C *has exactly two equilibrium points,* $\mathbf{c}$ *and* $\mathbf{d}$.

Proof: The point $x = (0, v, s, w) \in C$ is a fixed point of $\varphi \iff$ it is a zero of the vector field (4.41) if $w = 0$ and:

$$v^2 = \frac{W(s)}{1-s^2}. \tag{4.44}$$

Thus by performing:

$$W(s) = 2(1-s^2)V(s) \implies W'(s) = -4sV(s) + 2(1-s^2)V'(s) \implies \frac{W'(s)}{W(s)} = -\frac{2s}{1-s^2} + \frac{V'(s)}{V(s)} \tag{4.45}$$

we get from the third equation from (4.41) that:

$$s - s + \frac{1}{2}\frac{V'(s)}{V(s)}(1-s^2) = 0 \implies s = s_c$$

thus we plug this value to (4.44) to obtain that $v = \pm v_c$. And fixed vectors are **c** and **d**. □

Now we want to show that the coordinate v increases along solutions of equations (4.41). For this, we must define what a gradient-like flow is, and then show for which function φ is gradient-like on C.

Definition 4.4.5 *Let ψ be a flow on a complete metric space X. Assume $\exists g : X \to \mathbb{R}$ a continuous function such that if $t > 0$, then:*

$$g(\psi(x,t)) < g(x)$$

unless x is a fixed point. Also assume that all fixed points of ψ are isolated. Then ψ is gradient-like with respect to g.

Note that this property is a similar definition to what a Lyapunov function is with respect to a given flow.

Proposition 4.4.6 *The flow φ restricted to C is gradient-like with respect to $g(0, v, s, w) := -v$*

Proof: The first equation from (4.41) implies that $dv/d\sigma > 0$ if $s = \pm 1$. Then we transform the relation (4.36) into:

$$\frac{w^2}{1-s^2} = 1 - \frac{1-s^2}{W(s)}v^2. \tag{4.46}$$

For $s = \pm 1$, we get that:

$$\frac{dv}{d\sigma} = \frac{\lambda}{2}\sqrt{W(s)} > 0$$

and for $s \neq \pm 1$ we combine (4.16) with the first equation from (4.41) to get:

$$\frac{dv}{d\sigma} = \frac{\lambda}{2}\sqrt{W(s)}\frac{w^2}{1-s^2}$$

which is clearly positive, so we get the same for the case that $w \neq 0$ or $s = \pm 1$. In the other case, by taking $w = 0$ and that $ds/d\sigma = w = 0$. Now we get the expression of the second derivative:

$$\frac{d^2v}{d\sigma^2} = \lambda\sqrt{W(s)}\frac{w}{1-s^2}\frac{dw}{d\sigma} + O(w^3)$$

which is 0 when $w = 0$. And now we take the third derivative, so we use the definition of W in (4.30), the third equation from (4.41) and the derivative relation (4.15), so we get that:

$$\begin{aligned}\frac{d^3v}{d\sigma^3} =&\lambda\frac{\sqrt{W(s)}}{1-s^2}\left(-s+\frac{2s(1-s^2)}{W(s)}v^2+\frac{W'(s)}{2W(s)}(1-s^2)\right)^2+O(w)\\ =&\lambda\frac{\sqrt{W(s)}}{1-s^2}\left(-s+\frac{2s(1-s^2)}{W(s)}v^2-s+\frac{V'(s)}{2V(s)}(1-s^2)\right)^2+O(w)\\ =&\lambda\frac{\sqrt{W(s)}}{1-s^2}\left(-2s+\frac{2s(1-s^2)}{W(s)}v^2+\frac{V'(s)}{W(s)}(1-s^2)^2\right)^2+O(w)\end{aligned}$$

which is positive when $w = 0$, except when $s = s_c$, which is 0, using the definition of v_c from (4.43), and that $V'(s_c) = 0$ as stated in proposition 4.4.2. Thus v grows everywhere except at the two fixed points. □

Propositions 4.4.4 and 4.4.6 describe how the flow behaves on the triple collision manifold C. In section 4.8 we will detail for some values of the masses how this flow behaves, by using tools of derivatives respect to the initial conditions, but first let us talk about two theorems concerning the set of orbits ending in triple collision.

4.5 Asymptotic behaviour of triple collision orbits

According to McGehee [4], Sundman proved for the 3-Body Problem in 3 dimensions that an orbit ending in triple collision asymptotically approaches a central configuration. In this section, our aim is to give a different proof, more specific for the collinear 3-Body Problem.

A solution $(q(t), p(t)) \in Q \times P$ of equations (1.9) will be called a triple collision orbit if $\exists t_1 \in \mathbb{R}$ such that $q(t) \to 0$ as $t \to t_1$. If the limit is reached from the left, it will be said to end in triple collision, whereas if the limit is reached from the right, it will be said to begin in triple collision.

Theorem 4.5.1 *(Sundman) Let $(q(t), p(t)) \in Q \times P$ from equations (1.9) be a triple collision orbit such that $\exists t_1 > 0$ such that $q(t) \to 0$ as $t \to t_1$. Then as $t \to t_1$:*

$$\frac{q(t)}{r(t)} \to \mathbf{s}_c$$

and

$$r(t) \sim \kappa(t_1 - t)^{2/3}.$$

Remind from (1.17) that $(1/2)r(t)^2$ is the moment of inertia of the system. Thus not only did Sundman prove that triple collision orbits approach a central configuration, but that the moment of inertia goes to 0 with the same convergency speed as $(t_1 - t)^{4/3}$. This is equivalent to say that:

$$q(t) \sim \kappa(t_1 - t)^{2/3}\mathbf{s}_c.$$

The value of the constant κ will be given in the proof of Theorem 4.5.1. It will use the transformations which we have defined in previous sections, and will have its basis in the theory of flows on metric spaces. Before proving Theorem 4.5.1, we shall define what an ω-limit set is and give some statements. For a flow ψ in a complete metric space X, we define the ω-limit set of a point $x_0 \in X$ as:

$$\omega(x_0) := \bigcap_{t>0} \overline{\psi(x, [t, +\infty))}.$$

where the bar represents the topological closure. Theorem 4.5.1 will be considered kind of a corollary of the following lemma:

Lemma 4.5.2 *Let ψ be a flow on a locally compact metric space X. Let $x_0 \in X$ be a point such that $\omega(x_0)$ is a non-empty compact set. Assume ψ restricted to $\omega(x_0)$ is gradient-like. Then $\omega(x_0)$ is an alone point.*

To prove this lemma, we will have to characterise what an ω-limit set of a point is, which we will give at Theorem 4.5.4. A method which we will use here, is to introduce the concept of a "chain-minimal" flow and to prove that the ω-limit set of a point is chain-minimal. Then we will show that a gradient-like chain-minimal flow is an alone point, proving with this Lemma 4.5.2.

Definition 4.5.3 *Let φ be a flow on a complete metric space X with metric d. Let $x, y \in X$ be two different points, and let ε and T be two positive numbers. We say that a collection $(x_1, ..., x_{n+1}, t_1, ..., t_n)$ is a $(\varepsilon, T, \varphi)$-chain from x to y provided the following conditions are accomplished for $i = 1, ..., n$:*

- $x_i \in X$, $x = x_1$, $y = x_{n+1}$,
- $t_i \geq T$,
- $d(\varphi(x_i, t_i), x_{i+1}) \leq \varepsilon$.

Notation: Let $x, y \in X$. We will write $x > y$ whenever $\exists$ a $(\varepsilon, T, \varphi)$-chain from x to y $\forall \varepsilon, T > 0$.

One can see that $>$ is a transitive relation on X but it is not reflexive or skew-symmetric. We may have to refer to $>$ as an order relation. For a gradient-like flow, this relation matches with the gradient-like function in the following sense.

Lemma 4.5.4 *Let φ be a flow on a compact metric space X. Assume φ is gradient-like with respect to a continuous function g, and let $x, y \in X$. Then:*

$$x > y \implies g(x) \geq g(y).$$

Proof: Assume $x > y$ and $g(x) < g(y)$. Choose $a_1, a_2 \in \mathbb{R}$ in such a way that:

$$g(x) < a_1 < a_2 < g(y)$$

and $K := g^{-1}([a_1, a_2])$ has no fixed points inside. Let:

$$K_1 := g^{-1}((-\infty, a_1]), \quad K_2 := g^{-1}([a_2, +\infty)).$$

Then K_1 and K_2 are disjoint compact sets. Let $\delta := d(K_1, K_2)$. Since K is compact and no fixed points belong to it, $\exists T > 0$ such that:

$$x \in K, \quad t \geq T \implies \varphi(x, t) \in K_1.$$

Thus, for $\varepsilon < \delta$ and $T' > T$, there is no $(\varepsilon, T', \varphi)$-chain from x to $y \implies$ contradiction!!! $\square$

Now we generalise a minimal flow in the following manner:

Definition 4.5.5 *Let φ be a flow on a complete metric space X. We will say φ is chain-minimal if $x > y \quad \forall x, y \in X$.*

Now the following property of ω-limit sets is accomplished:

Theorem 4.5.6 *Let φ be a flow on a locally compact metric space X. Let $x_0 \in X$ be such that $\omega(x_0)$ is a non-empty compact set. Then $\varphi|_{\omega(x_0)}$ is chain-minimal.*

To understand this, we shall take into account that we are considering the flow restricted to the ω-limit set. We can trivially prove that $x, y \in \omega(x_0) \implies x > y$ for the order relation given by φ on X. However, the order relation on $\omega(x_0)$ given by the restricted flow does not coincide with the restricted order relation. We can discern so if we consider non-wandering points. All points in $\omega(x_0)$ are non-wandering for the flow φ, because $\omega(x_0)$ is a limit set for the flow φ. However, they may wander in the flow restricted to $\omega(x_0)$. For instance, if we take the flow in $\mathbb{R}^2$ in polar coordinates:

$$\begin{aligned}\dot{r} =& -r(r+1)(r-1)\\ \dot{\theta} =& 1\end{aligned}$$

we can see that in the ω-limit cycle $\{r = 1\}$ we cannot define when $(1, \theta_1) > (1, \theta_2)$ for $\theta_1 \neq \theta_2$, because this limit cycle is an orbit of period 2π.

We should also be aware of that if X is compact, $\omega(x_0)$ is automatically non-empty and compact. Thus $\omega(x_0)$ is always chain-minimal for flows on compact metric spaces. To prove Theorem 4.5.6, we will need the next proposition:

Proposition 4.5.7 *Let φ be a flow on a locally compact metric space X. Let $x_0 \in X$ be such that $\omega(x_0)$ is a non-empty compact set. Then $\omega(x_0)$ is connected. Moreover, $\forall U \supset \omega(x_0)$ open set, $\exists t' > 0$ such that $\varphi(x_0, t) \in U$ for $t \geq t'$.*

Proof: When X is compact, we already know which statement Proposition 4.5.7 is. It is enough to assume that $\omega(x_0)$ is non-empty to ensure that we can use the usual proof when X is locally compact.□

Proof of Theorem 4.5.6: Let $\Omega := \omega(x_0)$ and let ψ denote the flow φ restricted to Ω. Let $y, y' \in \Omega$, and let ε and T be given. We shall build a $(\varepsilon, T, \varphi)$-chain from y to y'. Let $U \subset X$ be an open subset, $U \supset \Omega$ relatively compact, i.e., with compact closure. Let us set $\delta < \varepsilon/2$ in such a way that:

$$u_1, u_2 \in U, \quad d(u_1, u_2) < \delta \implies d(\varphi(u_1, t), \varphi(u_2, t)) < \frac{\varepsilon}{2} \quad \forall t \in [0, 2T]. \tag{4.47}$$

Let $V \subset U$ be an open subset such that $\Omega \subset V$ and $d(x, \Omega) < \delta \quad \forall x \in V$. By Proposition 4.5.7, $\exists t'$ such that $\varphi(x_0, [t', +\infty)) \subset V$. Let $x_1 \in \varphi(x_0, [t', +\infty))$ be chosen in such a way that $d(x, 1, y) < \delta$. Let $T' > T$ be chosen in such a way that $d(\varphi(x_1, T'), y') < \varepsilon/2$. Let $n := \text{ceil}(T'/T)$ and let $x_i := \varphi(x_{i-1}, T)$ for $i = 2, ..., n$. Let $x_{n+1} := \varphi(x_n, T' - (n-1)T) = \varphi(x_1, T')$. For $i = 2, ..., n$, let $y_i \in \Omega$ be chosen in such a way that $d(y_i, x_i) < \delta$. Let $y_1 = y$ and $y_{n+1} = y'$. Let $t_i := T$ for $i = 1, ..., n-1$, and let $t_n := T' - (n-1)T$.

We want to prove that $(y_1, ..., y_{n+1}, t_1, ..., t_n)$ is a $(\varepsilon, T, \varphi)$-chain from y to y'. Since points 1 and 2 from Definition 4.5.3 yield by construction, the third one is the only one which rests us to prove, i.e.:

$$d(\psi(y_i, t_i), y_{i+1}) \leq \varepsilon, \quad i = 1, ..., n. \tag{4.48}$$

But $d(y_i, x_i) < \delta$ and $t_i < 2T$, so by (4.47):

$$d(\psi(y_i, t_i), x_{i+1}) = d(\varphi(y_i, t_i), \varphi(x_i, t_i)) < \frac{\varepsilon}{2}.$$

Since $d(y_{i+1}, x_{i+1}) < \delta < \varepsilon/2$, triangle inequality automatically proves (4.48). Thus $(y_1, ..., y_{n+1}, t_1, ..., t_n)$ is a $(\varepsilon, T, \varphi)$-chain from y to y'.□

Theorem 4.5.6, Proposition 4.5.7 and the following proposition prove Lemma 4.5.2 as a corollary:

Proposition 4.5.8 *Let φ be a chain-minimal gradient-like flow on a compact connected non-empty metric space X. Then X is an alone point.*

Proof: Let $x, y \in X$. Since $x > y$, we have that $g(x) \geq g(y)$. Since $y > x$, we have that $g(y) \geq g(x)$. Thus g is constant on X, so X just contains fixed points. Since the fixed points are alone and X is compact, X is just made by a finite number of alone points. Since X is connected and non-empty, it is just made of an alone point.□

So now Lemma 4.5.2 is now proved. □

To prove Theorem 4.5.1 it will be enough to show that the ω-limit set of a triple collision orbit is non-empty and compact. Now we state the following notation, which will be useful from here on.

For $\nu > \alpha > 0$ let us define the following subsets of the transformed constant energy manifold $N(h)$:

$$\begin{aligned}
B(h,\alpha) :=& \{(r,v,s,w) \in N(h) : r \leq \alpha\},\\
B_0(h,\alpha,\nu) :=& \{(r,v,s,w) \in B(h,\alpha) : |v| \leq \nu - r\},\\
B^-(h,\alpha,\nu) :=& \{(r,v,s,w) \in B(h,\alpha) : v \geq \nu - r\},\\
B^+(h,\alpha,\nu) :=& \{(r,v,s,w) \in B(h,\alpha) : -v \geq \nu - r\},\\
b^\pm(h,\alpha,\nu) :=& \{(\alpha,v,s,w) \in B^\pm(h,\alpha,\nu)\},\\
\beta^\pm(h,\alpha,\nu) :=& B^\pm(h,\alpha,\nu) \cap B_0(h,\alpha,\nu).
\end{aligned}$$

This image describes well these sets defined just above:

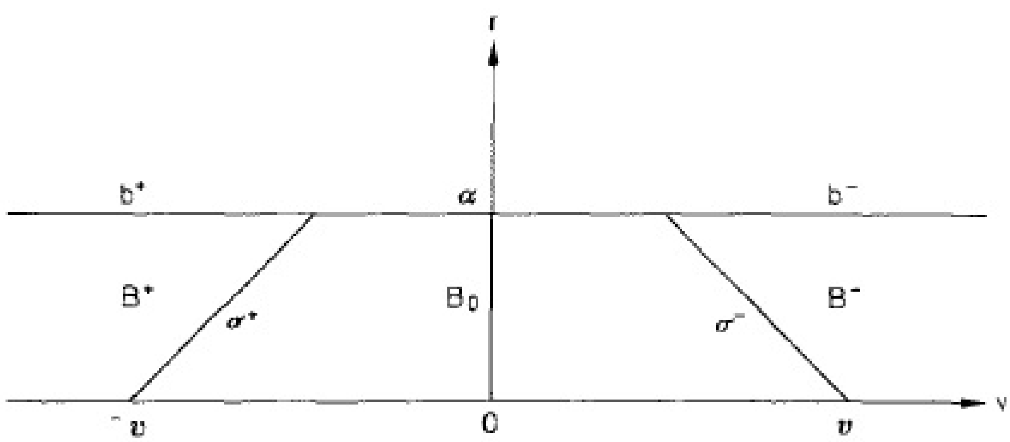

Fig. 3. The isolating block **B**(h, α). Note that the coordinates s and w are not shown

Notice that as we have used the time variables t, τ, σ instead of t, t', τ, respectively, used in McGehee [4], the $\sigma^\pm$ sets drawed in this image correspond to $\beta^\pm$ in this chapter.

We say that an orbit segment $\varphi(x_0, [\sigma, \sigma'])$ is maximal in a closed set K if it is a subset of K but $\varphi(x_0, I) \not\subset K \quad \forall I \supset [\sigma, \sigma']$.

Proposition 4.5.9 *Let $\nu > \alpha - h$. If $\varphi(x_0, [\sigma, \sigma'])$ is a maximal orbit segment in $B^-(h,\alpha,\nu)$, then $\varphi(x_0, \sigma) \in \beta^-(h,\alpha,\nu)$ and $\varphi(x_0, \sigma') \in b^-(h,\alpha,\nu)$.*

Proof: First we add the two equations from (4.34):

$$\begin{aligned}\frac{d}{d\sigma}(r+v) =&\frac{\lambda(1-s^2)}{\sqrt{W(s)}}rv + \frac{\lambda\sqrt{W(s)}}{2}\left(1-\frac{1-s^2}{W(s)}(v^2-4rh)\right)\\ =&\lambda\left(\frac{\sqrt{W(s)}}{2}-\frac{1-s^2}{2\sqrt{W(s)}}v^2+r\frac{1-s^2}{\sqrt{W(s)}}(v+h)+\frac{1-s^2}{\sqrt{W(s)}}rh\right)\end{aligned} \tag{4.49}$$

and by plugging the energy relation (4.35) by isolating w^2, this expression is equal to:

$$\lambda\left(\frac{\sqrt{W(s)}}{2(1-s^2)}w^2+r\frac{1-s^2}{\sqrt{W(s)}}(v+h)\right)$$

which is greater than 0 for $s \neq \pm 1$, given that $v = \nu - r$, $r \leq \alpha \implies v \geq \nu - \alpha$, and $\nu > \alpha - h \implies v + h > 0$, for $(r, v, s, w) \in \beta^-$. Whereas for $s = \pm 1$, the equation (4.49) is equal to:

$$\frac{\lambda}{2}\sqrt{W(s)}.$$

Therefore, points on β^- are going inside B^-, so $\varphi(x_0, \sigma') \in b^-$. Since $(r, v, s, w) \in b^-$ implies $v > 0$, the first equation from (4.34) implies that points of b^- are going out of B^-, so $\varphi(x_0, \sigma) \in \beta^-$. □

Proposition 4.5.10 *Let $\nu > \alpha - h$. If $\varphi(x_0, [\sigma, \sigma'])$ is a maximal orbit segment in $B^+(h, \alpha, \nu)$, then $\varphi(x_0, \sigma) \in b^+(h, \alpha, \nu)$ and $\varphi(x_0, \sigma') \in \beta^+(h, \alpha, \nu)$.*

Proof: It is similar to proposition 4.5.9, only with the two changes given at this proposition, and that $d/d\sigma(v - r) > 0$, changing at the given conditions v by $-v$. □

Proof of Theorem 4.5.1: We just consider the case of orbits ending in triple collision; the case of orbits starting from collision is analogous.

Let $x_0 = (r_0, v_0, s_0, w_0) \in N(h)$ be the image of (q_0, p_0) under the changes of variables (4.2), (4.7), (4.24) and (4.31). Therefore, we bring the orbit $\{(q(t), p(t)) : t \in [0, t_1)\}$ towards the orbit $\varphi(x_0, [0, \sigma_1))$. Since we defined a triple collision orbit $r(\sigma) \to 0$ as $\sigma \to \sigma_1$, and since $\{r = 0\}$ is an invariant set for the flow, we obtain that $\sigma_1 = \infty$.

Fix $\alpha > 0$. Since $r(\sigma) \to 0$, $\exists \sigma_2 > 0$ such that $r(\sigma) < \alpha$ and thus $\varphi(x_0, \sigma) \in B(h, \alpha) \quad \forall \sigma \geq \sigma_2$. Now we set ν in a certain way that:

$$\nu > \alpha + \max(-h, |v(\sigma_2)|).$$

Then $\varphi(x_0, \sigma_2) \in B_0(h, \alpha, \nu)$. We will show by contradiction that:

$$\varphi(x_0, \sigma) \in B_0(h, \alpha, \nu) \quad \forall \sigma \geq \sigma_2. \tag{4.50}$$

Assume $\varphi(x_0, \sigma_3) \in B^-(h, \alpha, \nu)$ for a certain $\sigma_3 \geq \sigma_2$. Then Proposition 4.5.9 and $r(\sigma) < \alpha$ imply that $\varphi(x_0, \sigma) \in B^- \quad \forall \sigma \geq \sigma_3$. But $v > 0$ for points in B^-, so the first equation from (4.34) implies that $dr/d\sigma \geq 0$ for $\sigma \geq \sigma_3$. But this contradicts $r(\sigma) \to 0$. Therefore, $\varphi(x_0, \sigma) \notin B^-$ for $\sigma \geq \sigma_2$. Proposition 4.5.10 implies that $\varphi(x_0, \sigma) \notin B^+$ for $\sigma \geq \sigma_2$. Thus (4.50) is proved.

Since B_0 is compact, $\omega(x_0)$ is a non-empty compact set. Since $r(\sigma) \to 0$, $\omega(x_0) \in C$. By Proposition 4.4.6, the flow restricted to $\omega(x_0)$ is gradient-like. Therefore, by Lemma 4.5.2, $\omega(x_0)$ is exactly one point, and we can see that it must be a fixed point. By Proposition 4.4.4, the only two fixed points are **c** and **d**, so:

$$(r, v, s, w) \xrightarrow{\sigma \to \infty} (0, \pm v_c, s_c, 0).$$

Thus $r^{-1}q = \mathbf{s} \to \mathbf{s}_c$ as $\sigma \to \infty$, and we have proved the first equation from the statement of theorem 4.5.1.

The first equation from (4.34) implies that $v(\sigma) \not\to v_c$ since $dr/d\sigma > 0$. Therefore, $\varphi(x_0, \sigma) \to \mathbf{c}$ as $\sigma \to \infty$, and we have:

$$\frac{dr}{d\sigma} \sim \frac{\lambda(1-s_c^2)}{\sqrt{W(s_c)}} v_c r \text{ as } \sigma \to \infty.$$

Time transformations (4.9) and (4.33) imply that:

$$\dot{r} \sim -v_c r^{-1/2}$$

and thus:

$$r(t) \sim \left(\frac{3}{2} v_c\right)^{2/3} (t_1 - t)^{2/3} \text{ as } t \to t_1^-.$$

Therefore, the second equation from the statement of theorem 4.5.1 is now proven, and then we have completed its proof. □

We should notice that orbits starting in triple collision satisfy that $\varphi(x_0, \sigma) \to \mathbf{d}$ as $\sigma \to \infty$.

4.6 The set of triple collision orbits

McGehee [4] followed in this section what Siegel did in one of his references: to see what happens in the set of all triple collision orbits for the 3-Body Problem in 3 dimensions. He asked himself the main question whether triple collision can be regularised. We will talk about this issue in section 4.7, but now, our aim is to show a corollary from what Siegel did according to what McGehee [4] followed, where he proved that the set of orbits ending in triple collision forms a smooth submanifold of the constant energy manifold. In this section, we will give the proof which McGehee [4] followed from Siegel's corollary in the collinear case.

The Stable Manifold Theorem applied to the critical point $\mathbf{c}$ on C proves this corollary. By Theorem 4.5.1, the set of orbits ending in triple collision is exactly $W^S(\mathbf{c})$. We will compute the Jacobian from the vector field defined by equations (4.34) at this point, and show that it has two negative eigenvalues and thus $W^S(\mathbf{c})$ is 2-dimensional.

Now we are looking for the stability of points in triple collision manifold given at equations (4.41). And it is given in this following theorem:

Theorem 4.6.1 *(Siegel) The set of orbits ending in triple collision forms a real-analytic 2-dimensional submanifold contained in the 3-dimensional constant energy manifold.*

Proof: First we will need to compute the Jacobian of the system defined at (4.34), let this vector field be called X:

$$\mathbf{D}X = \begin{pmatrix} \frac{\lambda(1-s^2)}{\sqrt{W(s)}} v & \frac{\lambda(1-s^2)}{\sqrt{W(s)}} r & -\lambda r v \frac{2s\sqrt{W(s)} + (1-s^2)W'(s)(1/2)W(s)^{-1/2}}{W(s)} & 0 \\ 2\frac{\lambda(1-s^2)}{\sqrt{W(s)}} h & -\frac{\lambda(1-s^2)}{\sqrt{W(s)}} v & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{4hs(1-s^2)}{W(s)} & \frac{4sv(1-s^2)}{W(s)} & a_{43} & -\frac{wW'(s)}{W(s)} - \frac{\lambda v(1-s^2)}{2\sqrt{W(s)}} \end{pmatrix}$$

where

$$a_{23} := \frac{\lambda W'(s)}{4\sqrt{W(s)}} \left(1 - \frac{1-s^2}{W(s)}(v^2 - 4rh)\right) + \frac{\lambda}{2}\sqrt{W(s)}(v^2 - 4rh)\frac{2sW(s) + (1-s^2)W'(s)}{W(s)^2}$$

and

$$a_{43} := -1 + (v^2 - 2rh)\frac{(2-6s^2)W(s) - 2s(1-s^2)W'(s)}{W(s)^2} - \frac{sW'(s)}{W(s)}$$
$$+ \frac{(W''(s)W(s) - W'(s)^2)(1-s^2-w^2)}{2W(s)^2} + \frac{\lambda vw}{2}\frac{2s\sqrt{W(s) - (1-s^2)(1/2)W'(s)W(s)^{-1/2}}}{W(s)}$$

and by evaluating at $\mathbf{c}$, taking into account the relation (4.30) defined at the beginning of section 4.3, and using (4.43), we get that this matrix is:

$$\mathbf{D}X(\mathbf{c}) = \begin{pmatrix} -\lambda\sqrt{1-s_c^2} & 0 & 0 & 0 \\ \frac{2\lambda\sqrt{1-s_c^2}}{v_c}h & \lambda\sqrt{1-s_c^2} & a_{23}(\mathbf{c}) & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{4s_c h}{v_c^2} & -\frac{4s_c}{v_c} & a_{43}(\mathbf{c}) & \frac{\lambda}{2}\sqrt{1-s_c^2} \end{pmatrix}.$$

This image illustrates what happens with the dynamics in the manifold C:

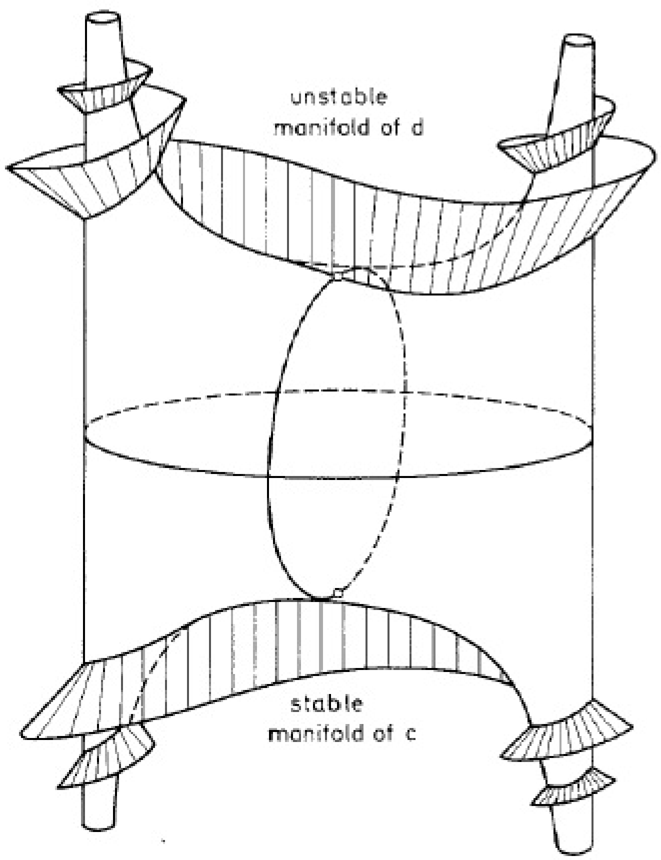

Fig. 4. The upper (lower) shaded surface is the set of orbits beginning (ending) in triple collision

Now we have to check the values of a_{23} and a_{43} at $\mathbf{c}$. First we check that $a_{23}(\mathbf{c}) = 0$:

$$a_{23}(\mathbf{c}) = -\frac{\lambda s_c\sqrt{W(s_c)}}{2(1-s_c^2)} \cdot 0 + \frac{\lambda}{2}v_c^3\sqrt{1-s_c^2}\left(\frac{2s_c}{W(s_c)} - \frac{2s_c}{W(s_c)}\right) = 0.$$

Now to find $a_{43}(\mathbf{c})$, by using from (4.45) that:

$$\begin{aligned} W'(s) =& W(s)\left(-\frac{2s}{1-s^2}+\frac{V'(s)}{V(s)}\right) \\ \implies W''(s) =& W(s)\left(-\frac{2s}{1-s^2}+\frac{V'(s)}{V(s)}\right)^2 + W(s)\left(-\frac{2(1-s^2)-4s^2}{(1-s^2)^2}+\frac{V''(s)V(s)-V'(s)^2}{V(s)^2}\right) \end{aligned}$$

and then evaluating at $s = s_c$ and using (4.43) and $V'(s_c) = 0$ from Proposition 4.4.2:

$$\begin{aligned} W''(s_c) =& \frac{4s_c^2v_c^2}{1-s_c^2} + W(s_c)\left(\frac{-2-2s_c^2}{(1-s_c^2)^2}+\frac{2(1-s_c^2)V''(s_c)}{W(s_c)}\right) = \frac{4s_c^2v_c^2}{1-s_c^2} - 2\frac{(1+s_c^2)v_c^2}{1-s_c^2} + 2(1-s_c^2)V''(s_c) \\ =& -2+2(1-s_c^2)V''(s_c) \end{aligned}$$

then by using again (4.43) and (4.45) we get that:

$$\begin{aligned} a_{43}(\vec{c}) =& -1+v_c^2\left(\frac{2-6s_c^2}{v_c^2(1-s_c^2)}+\frac{4s_c^2}{v_c^2(1-s_c^2)}\right)+\frac{2s_c}{1-s_c^2}+\frac{-2v_c^2-2(1-s_c^2)V''(s_c)}{2v_c^2}-\frac{2s_c}{1-s_c^2} \\ =& -1+2-1+\frac{(1-s_c^2)V''(s_c)}{v_c^2} = \frac{(1-s_c^2)V''(s_c)}{v_c^2}. \end{aligned}$$

Now by multiplying $\mathbf{D}X(\mathbf{c})$ by a vector $y \in \mathbb{R}^4$ we get that:

$$\mathbf{D}X(\mathbf{c})y = \begin{pmatrix} -\lambda\sqrt{1-s_c^2}y_1 \\ \frac{\lambda\sqrt{1-s_c^2}}{v_c}(2hy_1+v_cy_2) \\ y_4 \\ -\frac{4s_c}{v_c^2}(hy_1+v_cy_2)+\frac{(1-s_c^2)V''(s_c)}{v_c^2}y_3+\frac{\lambda}{2}\sqrt{1-s_c^2}y_4 \end{pmatrix}.$$

From the definition of $N(h)$, we know that its tangent subspace $T_{\mathbf{c}}N(h)$ must be all vectors from $\mathbb{R}^4$ which are orthogonal to the gradient of the expression of $N(h)$, so we take the derivatives gradient of (4.35):

$$\begin{pmatrix} -2h(1-s^2)^2W(s)^{-1} \\ 2v(1-s^2)W(s)^{-1} \\ 2s-(4s(1-s^2)W(s)^{-1}+(1-s^2)^2W'(s)W(s)^{-2})(v^2-2rh) \\ 2w \end{pmatrix}.$$

By evaluating on $\mathbf{c}$ and using again (4.43) and (4.45) we get the vector:

$$(-2h(1-s_c^2)v_c^{-2}, -2(1-s_c^2)v_c^{-1}, 0, 0)^\top.$$

To prove that the third component is zero, we have used that:

$$2s_c - (4s_cv_c^{-2} - 2s_cv_c^{-2})v_c^2 = 0$$

so as we want to state the tangent subspace to the manifold defined by (4.35), first we divide the vector by $-2(1-s_c^2)v_c^{-2}$, and we see that $T_{\mathbf{c}}N(h)$ is:

$$T_{\mathbf{c}}N(h) = \{y \in \mathbb{R}^4 : hy_1 + v_cy_2 = 0\}$$

so if we apply this, we have that for $y \in T_{\mathbf{c}}N(h)$:

$$\mathbf{D}X(\mathbf{c})y = \left(-\lambda\sqrt{1-s_c^2}y_1, \frac{\lambda\sqrt{1-s_c^2}}{v_c}hy_1, y_4, \frac{V''(s_c)}{v_c^2}(1-s_c^2)y_3+\frac{\lambda}{2}\sqrt{1-s_c^2}y_4\right)^\top.$$

Now we choose a basis $\{\xi_1, \xi_2, \xi_3\}$ as follows:

$$\xi_1 =(-v_c, h, 0, 0)^\top,$$
$$\xi_2 =(0, 0, 1, 0)^\top,$$
$$\xi_3 =(0, 0, 0, 1)^\top.$$

Note that $\{\xi_2, \xi_3\}$ is a basis for $T_\mathbf{c}C$. The matrix for $\mathbf{D}X(\mathbf{c})$ in this basis is:

$$\begin{pmatrix} -\lambda\sqrt{1-s_c^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & (1-s_c^2)\frac{V''(s_c)}{v_c^2} & \frac{\lambda}{2}\sqrt{1-s_c^2} \end{pmatrix}.$$

Thus ξ_1 is an eigenvector with eigenvalue $-\lambda\sqrt{1-s_c^2}$. The characteristic polynomial for this matrix restricted to $T_\mathbf{c}C$ is:

$$x^2 - \frac{\lambda}{2}\sqrt{1-s_c^2}x - (1-s_c^2)\frac{V''(s_c)}{v_c^2} = 0.$$

In the proof of Proposition 4.4.3 we proved that $V''(s_c) > 0$. Thus this quadratic polynomic equation has a positive and a negative root. Thus $\mathbf{c}$ has a 1-dimensional stable manifold and a one-dimensional unstable manifold in C. On $N(h)$ we also take into account the negative eigenvalue we have obtained for the eigenvector ξ_1 and therefore, we have a 2-dimensional stable manifold. □

The orbits starting at triple collision are asymptoticly close to the point $\mathbf{d}$ as $\sigma \to -\infty$, and we shall replace v_c by $-v_c$ everywhere, obtaining a one-dimensional stable manifold, and a two-dimensional unstable manifold.

4.7 The isolating block about triple collision

In this section, our question will be whether orbits can be extended through triple collision. Indeed, an orbit ending in triple collision and one beginning in triple collision can be connected in any way. The main problem is if any of these connections we make has any sense.

According to McGehee [4], Siegel researched this issue from an analytic point of view. He focused in a single orbit ending in triple collision and his question was if that orbit, as a function of time, could continue in a meaningful way. The answer he found about this is that not in general.

However, in the context of flows on manifolds, it could be more natural to give another perspective to this issue. For instance, we may ask if given an orbit ending in triple collision and one beginning in triple collision, there is a well-defined flow. To ensure this, it is necessary that every two orbits starting close the one to the other and close to a triple collision orbit keep in the same neighbourhood even when time becomes large.

McGehee [4] recovered a definition of regularisation from Easton which makes more precise what we have explained above. He defined a different concept to describe the integral surfaces of the 3-Body Problem after extending to double collisions. In this section, this main idea will be described in a summarised way, and we will use it to prove that there are some values of the masses such that orbits cannot be extended through triple collision.

We introduce first some notation. Let ψ be a flow on a manifold M, and let B be a submanifold of the same dimension. Let $\mathbf{b} = \partial B$ and define:

$$\mathbf{b}^+ :=\{x \in \mathbf{b} : \exists\varepsilon > 0 : \psi(x, (-\varepsilon, 0)) \cap B = \emptyset\}$$
$$\mathbf{b}^- :=\{x \in \mathbf{b} : \exists\varepsilon > 0 : \psi(x, (0, \varepsilon)) \cap B = \emptyset\}.$$

Definition 4.7.1 *We will call B an isolating block as $\mathbf{b}^+ \cup \mathbf{b}^- = \mathbf{b}$.*

Now let:

$$\mathbf{a}^+ := \{x \in \mathbf{b}^+ : \psi(x,t) \in B \quad \forall t \geq 0\}$$
$$\mathbf{a}^- := \{x \in \mathbf{b}^- : \psi(x,t) \in B \quad \forall t \leq 0\}$$

and define $\Psi : \mathbf{b}^+ \backslash \mathbf{a}^+ \to \mathbf{b}^- \backslash \mathbf{a}^-$ as such a Poincaré map which by following the flow brings a particle from the preimage to the image. We will call "map across the block B" to Ψ. We get the property that if B is an isolating block, then Ψ is an homeomorphism.

Definition 4.7.2 *B is regularisable if we can extend Ψ to an homeomorphism between the whole b^+ and the whole b^-; otherwise it is non-regularisable.*

Now let φ be the flow of equations (4.34), and $B(h,\alpha)$ defined at section 4.5. We will see that $\forall h$, $B(h,\alpha)$ is an isolating block for α small enough. In fact:

$$\mathbf{b}^{\pm}(h,\alpha) = \{(\alpha, v, s, w) \in B(h,\alpha) : \pm v \leq 0\}.$$

To understand this idea more clearly, remind the image described at fig. 3.

If we call $W^S(x)$ and $W^U(x)$ the stable and the unstabe manifolds from a point x, we can see that $\mathbf{a}^+(h,\alpha) = \mathbf{b}^+(h,\alpha) \cap W^S(\mathbf{c})$, and $\mathbf{a}^-(h,\alpha) = \mathbf{b}^-(h,\alpha) \cap W^U(\mathbf{d})$. If $B(h,\alpha)$ is non-regularisable, neither will be $B(h,\gamma) \quad \forall \gamma \leq \alpha$. Thus we will say that triple collision is non-regularisable for energy level h if $\exists \alpha : B(h,\alpha)$ is non-regularisable. We want to prove the following theorem.

Theorem 4.7.3 *$\exists$ masses m_1, m_2, m_3 such that triple collision is non-regularisable $\forall h$.*

Before proving this theorem, we have to examine the flow on C, and see what happens with the stable and unstable manifolds of C.

Definition 4.7.4 *The flow φ on C is totally degenerate if $W^U(\mathbf{c}) = W^S(\mathbf{d})$.*

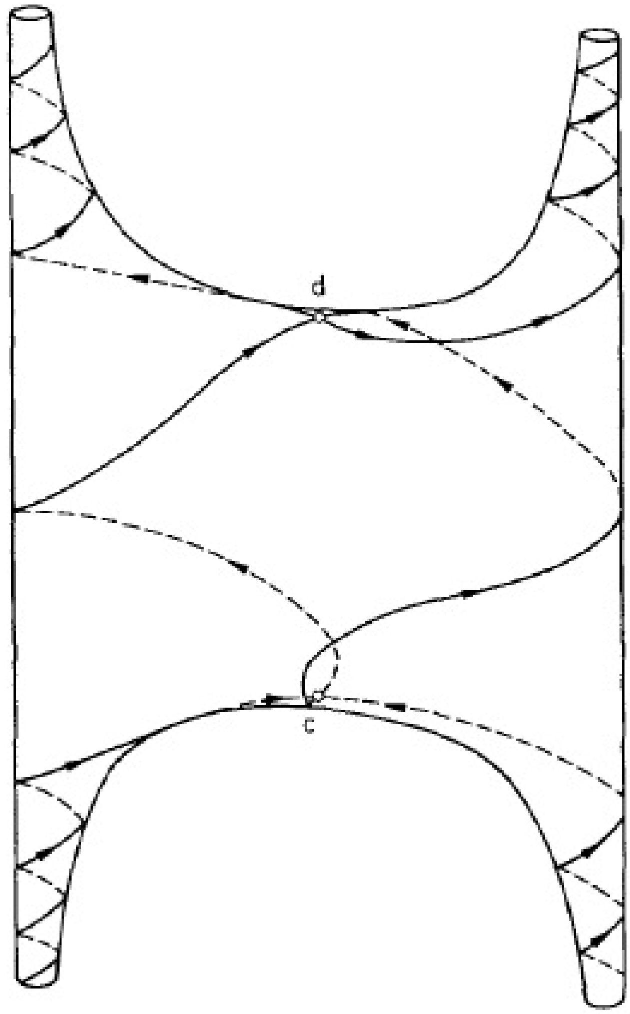

Fig. 5. Flow on the triple collision manifold in the totally degenerate case

We will see in section 4.8 that there are some values of the masses such that φ is not totally degenerate. In what remains from this section, we will see that triple collision is non-regularisable if φ is not totally degenerate. Now we have to remind from section 4.5 the definitions of B^-, b^- and β^-, fix an energy level h and $\alpha > 0$, and define the isolating block $B = B(h, \alpha)$. We also write $B^-(\nu) = B^-(h, \alpha, \nu)$ and so on. Let Φ be the map across the block B.

Proposition 4.7.5 *Assume φ on C is not totally degenerate. Let $a_0 \in \mathbf{a}^+$, let $U \subset \mathbf{b}^+$ open such that $a_0 \in U$, and let $\nu > v_c$. Then $\exists x_\nu \in U$ such that $\Phi(x_\nu) \in b^-(\nu)$.*

Proof: As φ is gradient-like and non-degenerate, we know that $W^U(\mathbf{c}) \cap \beta^-(\nu) \neq \emptyset$. Moreover, from Proposition 4.5.9 we have that $\beta^-(\nu)$ is a section for the flow. Also does $\mathbf{b}^+$ by definition. Since φ can be approximated by its linear part near $\mathbf{c}$, $\exists x_\nu \in U, y_\nu \in \beta^-(\nu)$ and $\sigma > 0$ such that $\varphi(x_\nu, \sigma) = y_\nu$. Thus the orbit through x_ν crosses $\beta^-(\nu)$ and goes into $B^-(\nu)$. By Proposition 4.5.9 the orbit just leaves $B^-(\nu)$ on $b^-(\nu)$. Hence $\Phi(x_\nu) \in b^-(\nu)$. □

Proposition 4.7.6 *If φ on C is not totally degenerate, then triple collision is non-regularisable.*

Proof: Let $a_0 \in \mathbf{a}^+$, $\nu > v_c$. $\exists x$ quite close to a_0 such that $\Phi(x) \in b^-(\nu)$. Since:

$$\bigcap_{\nu > v_c} b^-(\nu) = \emptyset$$

Φ cannot be extended to a_0. $\square$

We should distinguish at first sight the regularisation defined by what McGehee [4] followed from Easton and from Siegel. As stated in the abstract of the paper from Marchal [6], regularisation by Siegel is the analytical regularisation, whereas Easton's regularisation is the topological regularisation, or what is the same, the regularisation by continuity. Proposition 4.7.6 shows that the homographic orbit cannot be extended through triple collision. Remember that an homography is defined by dilations/contractions and rotations. However, if we see what $\ddot{\rho} = -\mu\rho|\rho^{-3}|$ describes as an homographic solution, we see that its behaviour is exactly the same as the double collision. Thus the homographic orbit can be extended as a function of time, but the extension does not seem to be continuous with respect to orbits close to it.

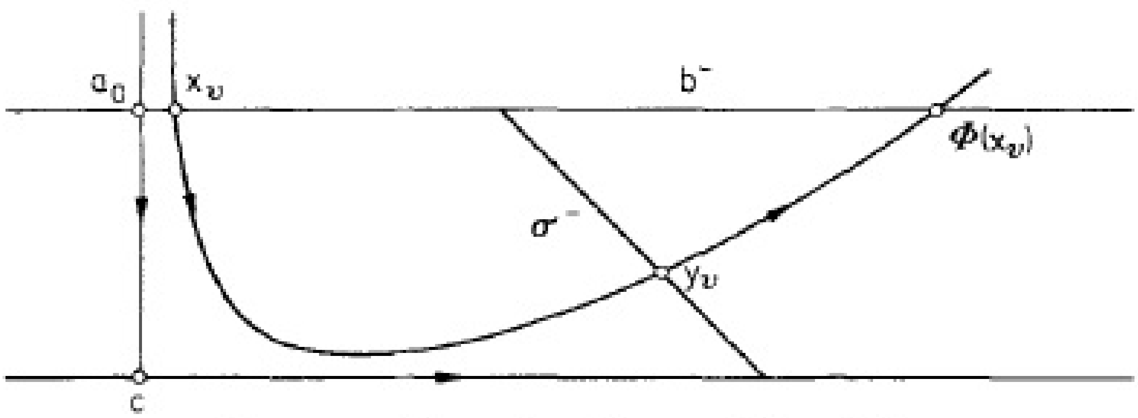

Fig. 6. An orbit passing close to triple collision

As stated in section 4.5 with fig. 3, we have again in this image that σ^- corresponds to our β^-.

At first sight, it may seem that we are not probably referring to three particles moving along a line. But if we relate r, v, s, w with q, p, we see an implication of Proposition 4.7.5 which may seem unusual.

Proof of Theorem 4.7.3: Remember from (1.17) that $(1/2)r^2$ is the moment of inertia of the system of point masses. The isolating block B accomplishes $r \leq \alpha$, and $\mathbf{b}$ accomplishes $r = \alpha$. On $\mathbf{b}^+$ we have the 1-dimensional set $\mathbf{a}^+$ of points whose orbits end in triple collision. According to proposition 4.5.8, orbits starting close to $\mathbf{a}^+$ leave from B with v as large as we want.

Consider the energy relation (4.35). Since B has the property that $r \leq \alpha$, we can note that $|w| \leq 1$ for large v, and thus the value of $1 - s^2$ becomes small. So, for large values of ν, $B^-(\nu)$ has two components, one containing points with $s \sim +1$, and the other one with $s \sim -1$. Therefore, orbits going close to triple collision leave from B with $r = \alpha$, $|w| \leq 1$ and s close to ± 1. Remember that $s \sim +1$ corresponds to a configuration with particles 2 and 3 as a binary, while $s \sim -1$ corresponds to particles 1 and 2 as a binary. Now we rewrite the momenta vector with transformations (4.2), (4.7), (4.24) and (4.31) as:

$$p = \mathbf{x} + y\mathbf{Ms} = \frac{1}{\sqrt{r}}(\mathbf{u} + v\mathbf{Ms}) = \frac{1}{\sqrt{r}}(u\mathbf{MA}\Sigma(s) + v\mathbf{M}\Sigma(s)) = \frac{1}{\sqrt{r}}\left(v\mathbf{M}\Sigma(s) + w\frac{\sqrt{W(s)}}{1-s^2}\mathbf{MA}\Sigma(s)\right).$$

Using equation (4.21) we compute the momentum of the particle 3:

$$p_3 = \frac{m_3 a_3}{\sqrt{r}}\left(v\cos(\lambda(1+s)) + w\frac{\sqrt{W(s)}}{1-s^2}\sin(\lambda(1+s))\right).$$

Now consider an orbit going close to triple collision and leaving from B with s near -1. Since $r = \alpha$, $|w| \leq 1$ and v is large, p_3 must be large, and particles 1 and 2 must form a binary. We have the same with particle 1 and s close to $+1$. Thus it is now proven that after going close to triple collision, one of the particles' speed becomes arbitrarily high in one direction, while the other two particles form a binary which moves fastly in the opposite direction.

Now consider $\{(r, v, s, w) \in C : v > \nu\}$, and large ν, so this set has a component $C_+(\nu)$ with $s \sim +1$ and another component $C_-(\nu)$ with $s \sim -1$. When the flow on C is non-degenerate, as we will see in next section, it can happen that one branch of $W^U(\mathbf{c})$ intersects $C_+(\nu)$ while the other branch intersects $C_-(\nu)$, and we will see that it happens for some values of the masses. In this case, Proposition 4.7.5 implies that orbits going close to triple collision emerge from B with $s \sim \pm 1$ depending on the side of $W^S(\mathbf{c})$ they begin. Thus orbits beginning with nearby trajectories and close to a triple collision orbit can leave with totally different configurations, and in the same way with high speeds.

Theorem 4.7.3 follows from Proposition 4.7.6 if we are able to prove that $\exists m_1, m_2, m_3$ such that φ on C is not totally degenerate, thus as it is a continuous function respect to the masses, it remains so under small perturbations. Therefore, the set of masses such that φ is not totally degenerate, is open, thus if it is non-empty, then Theorem 4.7.3 will be proved. □

In fact, in next section we will prove that indeed, this set is non-empty.

4.8 A special case

Now we are interested in setting a special case which proves the Theorem 4.7.3. And this one will be the case where $m := m_1 = m_3$, $m_2 = \varepsilon m$. We want to set a manifold C' and a vector field in such a way that when $\varepsilon \to 0$, the flow makes cricles on C'.

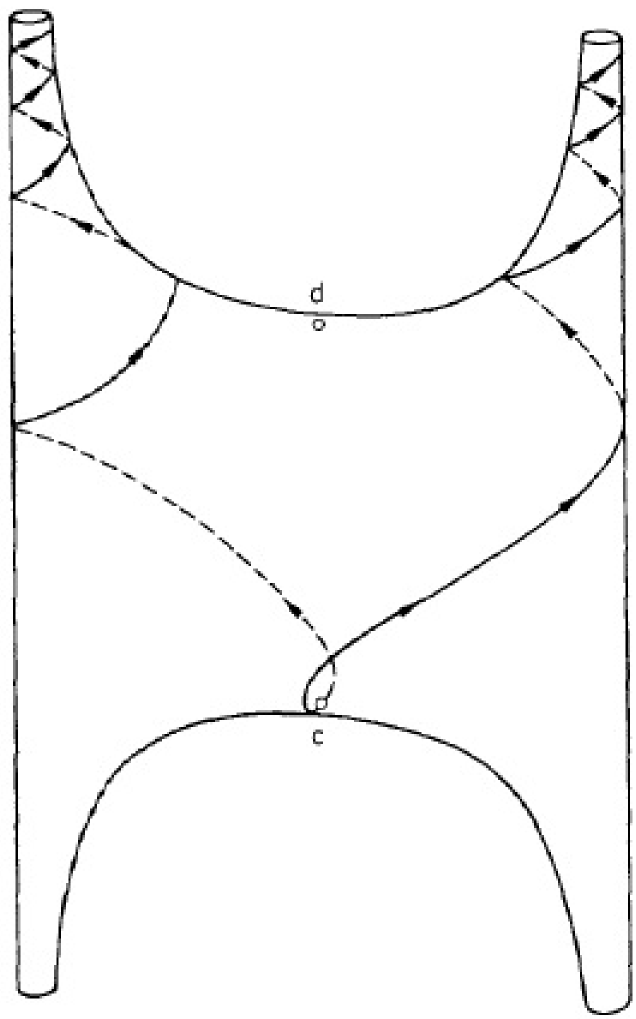

Fig. 7. Flow on the triple collision manifold in a non-degenerate case

We need to recall the definition of **a** and **b** explained at the very beginning of section 4.2, and the definition of λ above (4.20). Here we can write:

$$\mathbf{a} = \sqrt{\alpha}(-1, -1, 1+\varepsilon).$$

As the scalar product of **a** by itself induced by **M** must be 1, we have that:

$$\alpha(1+\varepsilon+(1+\varepsilon)^2)m = 1 \implies \alpha = (m(1+\varepsilon)(2+\varepsilon))^{-1}.$$

So **a** is defined by:

$$\mathbf{a} = (m(1+\varepsilon)(2+\varepsilon))^{-1/2}(-1, -1, 1+\varepsilon).$$

As **a** and **b** are equivalent by a rotation, we have that:

$$\mathbf{b} = (m(1+\varepsilon)(2+\varepsilon))^{-1/2}(-1-\varepsilon, 1, 1).$$

Then we compute that:

$$\mathbf{a}^\top \mathbf{M} \mathbf{b} = ((1+\varepsilon)(2+\varepsilon))^{-1}(1+\varepsilon-\varepsilon+1+\varepsilon) = (1+\varepsilon)^{-1}.$$

So $\lambda \in (0, \pi/4)$ satisfies:

$$\cos(2\lambda) = \frac{1}{1+\varepsilon}.$$

Thus $\lambda \to 0$ as $\varepsilon \to 0$. Now we rewrite the explicit equation from (4.27) written just below as:

$$\begin{aligned} V(s) =& Gm^{5/2}\sin(2\lambda)\sqrt{(1+\varepsilon)(2+\varepsilon)} \\ &\cdot\left(\frac{\varepsilon}{(2+\varepsilon)\sin(\lambda(1+s))}+\frac{\varepsilon}{(2+\varepsilon)\sin(\lambda(1-s))}+\frac{1}{(2+\varepsilon)(\sin(\lambda(1+s))+\sin(\lambda(1-s)))}\right) \\ =& G\sqrt{\frac{m^5(1+\varepsilon)}{2+\varepsilon}}\sin(2\lambda)\left(\frac{\varepsilon}{\sin(\lambda(1+s))}+\frac{\varepsilon}{\sin(\lambda(1-s))}+\frac{1}{\sin(\lambda(1+s))+\sin(\lambda(1-s))}\right). \end{aligned} \tag{4.51}$$

From this expression and $W(s)$ defined at (4.30), we see that V and W are both even functions.

Proposition 4.8.1 *$\forall m$, $\exists\varepsilon$ for which the flow φ on C is not totally degenerate.*

Proof: A branch of $W^U(\mathbf{c})$ is a single orbit $\varphi(x_0,(-\infty,+\infty))$ such that $\varphi(x_0,\sigma) \to \mathbf{c}$ as $\sigma \to -\infty$. Let $\sigma_1 \in (-\infty,+\infty]$ be such that $\varphi_2(x_0,\sigma_1) = +v_c$ (the v value). We include $\sigma_1 = +\infty$ in case $\varphi(x_0,\sigma) \to \mathbf{d}$ as $\sigma \to +\infty$. From the gradient-like property of φ, σ_1 is unique. By the third equation of (4.41), the line $\{s = +1\}$ is a section of the flow on C. Let $Z(\varepsilon) := \#(\varphi(x_0,(-\infty,\sigma_1)) \cap \{s = +1\})$, which belongs to $\mathbb{N}$. Again from the gradient-like property of φ, $Z(\varepsilon) \neq 0$. Now assume φ is totally degenerate $\forall\varepsilon$. Then $Z(\varepsilon) =$ const. Then by the next proposition we will have that:

$$\lim_{\varepsilon\to 0} Z(\varepsilon) = +\infty$$

so the assumption is contradicted. □

Recall from Proposition 4.4.3 that we saw that V has a unique minimum on $[-1,1]$, and now as V is even, then $s_c = 0$, and thus $v_c = \sqrt{2V(0)} = \sqrt{W(0)}$. From equation (4.49) we see that:

$$\lim_{\varepsilon\to 0} V(0) = \lim_{\varepsilon\to 0} G\sqrt{\frac{m^5(1+\varepsilon)}{2+\varepsilon}}\frac{\sin(2\lambda)}{\sin(\lambda)}\left(2\varepsilon+\frac{1}{2}\right) = G\sqrt{\frac{m^5}{2}}.$$

Now let $\mu := v_c^{-2}$, which just depends on the masses and approaches a positive limit as $\varepsilon \to 0$. Now consider:

$$g(v,s) := \frac{1-(1-s^2)v^2W(s)^{-1}}{1-\mu v^2}. \tag{4.52}$$

Then g is a positive continuous function on $(-v_c,v_c)\times[-1,1]$.

Proposition 4.8.2 *The new variable:*

$$\eta := \frac{w}{\sqrt{g(v,s)}}, \tag{4.53}$$

the time transformation:

$$d\sigma = \sqrt{g(v,s)}d\gamma, \tag{4.54}$$

and the limit when $\lambda \to 0$ transform the system (4.41) into:

$$\begin{aligned} \frac{dv}{d\gamma} =& 0 \\ \frac{ds}{d\gamma} =& \eta \\ \frac{d\eta}{d\gamma} =& -s(1-\mu v^2) \end{aligned} \tag{4.55}$$

thus v tends to constant. The energy relation (4.53) is transformed into:

$$\eta^2 - (1-s^2)(1-\mu v^2) = 0. \tag{4.56}$$

Proof: The transformation (4.53) transforms the energy relation (4.40) into:

$$g(v,s)\eta^2 + s^2 - 1 + (1-s^2)^2 W(s)^{-1} v^2 = 0$$

and by using the definition of $g(v,s)$ at (4.52) it is:

$$g(v,s)\eta^2 + s^2 - 1 + (1-s^2)(1-g(v,s)(1-\mu v^2)) = 0$$

which by simplyfing $g(v,s)$ yields (4.56). Then the change of coordinates (4.53) defines the transformation:

$$\begin{aligned}(-v_c,v_c)\times[-1,1]\times\mathbb{R} \to& (-v_c,v_c)\times[-1,1]\times\mathbb{R}\\ (v,s,w) \mapsto& (v,s,g(v,s)^{-1/2}w)\end{aligned}$$

which brings:

$$C_0 := \{(v,s,w)\in C : -v_c < v < v_c\}$$

to:

$$C' := \{(v,s,\eta)\in(-v_c,v_c)\times[-1,1]\times\mathbb{R} : \text{(4.56) holds}\}.$$

Now we want to find the expression of $d\eta/d\sigma$. For this, we need some previous steps. First, we can set the definition of $g(v,s)$ in (4.52) as:

$$W(s) = \frac{v^2(1-s^2)}{1-(1-\mu v^2)g(v,s)} \tag{4.57}$$

On the one hand, by using (4.52), (4.53), (4.56) and (4.57), we transform the equations from (4.41) into:

$$\begin{aligned}\frac{dv}{d\sigma} =& \frac{\lambda}{2}\sqrt{W(s)}g(v,s)(1-\mu v^2);\\ \frac{ds}{d\sigma} =& \sqrt{g(v,s)}\eta;\\ \frac{dw}{d\sigma} =& -s + \frac{2s(1-s^2)}{W(s)}v^2 + \frac{1}{2}\frac{W'(s)}{W(s)}(1-s^2-g(v,s)(1-s^2)(1-\mu v^2)) - \frac{\lambda}{2}\frac{1-s^2}{\sqrt{W(s)}}\sqrt{g(v,s)}v\eta\\ =& -s + \frac{2s(1-s^2)}{W(s)}v^2 + \frac{1}{2}\frac{W'(s)}{W(s)}\frac{v^2(1-s^2)^2}{W(s)} - \frac{\lambda}{2}\frac{1-s^2}{\sqrt{W(s)}}\sqrt{g(v,s)}v\eta.\end{aligned} \tag{4.58}$$

On the other hand, we need the partial derivatives of $g(v,s)$. By using again (4.52):

$$\begin{aligned}\frac{\partial g}{\partial v}(v,s) =& \frac{-2v(1-s^2)W(s)^{-1}(1-\mu v^2) + 2\mu v(1-(1-s^2)v^2W(s)^{-1})}{(1-\mu v^2)^2} = -\frac{2v(1-s^2)}{W(s)(1-\mu v^2)} + \frac{2\mu v}{1-\mu v^2}g(v,s);\\ \frac{\partial g}{\partial s}(v,s) =& \frac{2sv^2W(s)^{-1} + (1-s^2)v^2W'(s)W(s)^{-2}}{1-\mu v^2}.\end{aligned} \tag{4.59}$$

Now by using (4.52), (4.53), (4.56), (4.57), (4.58) and (4.59) we can compute the expression of $d\eta/d\sigma$:

$$
\begin{aligned}
\frac{d\eta}{d\sigma} =& -\frac{1}{2}\frac{\partial g}{\partial v}(v,s)g(v,s)^{-3/2}w\frac{dv}{d\sigma} - \frac{1}{2}\frac{\partial g}{\partial s}(v,s)g(v,s)^{-3/2}w^2 + g(v,s)^{-1/2}\frac{dw}{d\sigma} \\
=& -\frac{1}{2}\frac{\partial g}{\partial v}(v,s)g(v,s)^{-1}\eta\frac{dv}{d\sigma} - \frac{1}{2}\frac{\partial g}{\partial s}(v,s)g(v,s)^{-1/2}(1-s^2)(1-\mu v^2) + g(v,s)^{-1/2}\frac{dw}{d\sigma} \\
=& -\frac{\lambda}{4}\sqrt{W(s)}(1-\mu v^2)\left(-\frac{2v(1-s^2)}{W(s)(1-\mu v^2)} + \frac{2\mu v}{1-\mu v^2}g(v,s)\right)\eta \\
& -\frac{1}{2}g(v,s)^{-1/2}(1-s^2)(2sv^2W(s)^{-1} + (1-s^2)v^2W'(s)W(s)^{-2}) \\
& -sg(v,s)^{-1/2} + 2s(1-s^2)v^2W(s)^{-1}g(v,s)^{-1/2} + \frac{1}{2}\frac{W'(s)}{W(s)}v^2(1-s^2)^2g(v,s)^{-1/2} - \frac{\lambda}{2}\frac{1-s^2}{\sqrt{W(s)}}v\eta \\
=& -sg(v,s)^{-1/2}(v^2(1-s^2)W(s)^{-1} + 1 - 2v^2(1-s^2)W(s)^{-1}) \\
& -\frac{\lambda}{2}\sqrt{W(s)}(-v^{-1} + v^{-1}(1-\mu v^2)g(v,s) + \mu v g(v,s) + v^{-1} - v^{-1}(1-\mu v^2)g(v,s))\eta \\
=& -sg(v,s)^{-1/2}(1-(1-s^2)v^2W(s)^{-1}) - \frac{\lambda}{2}\mu\sqrt{W(s)}g(v,s)v\eta \\
=& -\sqrt{g(v,s)}s(1-\mu v^2) - \frac{\lambda}{2}\mu\sqrt{W(s)}g(v,s)v\eta.
\end{aligned}
$$

So the system (4.41) when transformed from C_0 to C' becomes:

$$
\begin{aligned}
\frac{dv}{d\sigma} =& \frac{\lambda}{2}\sqrt{W(s)}g(v,s)(1-\mu v^2) \\
\frac{ds}{d\sigma} =& \sqrt{g(v,s)}\eta \\
\frac{d\eta}{d\sigma} =& -\sqrt{g(v,s)}s(1-\mu v^2) - \frac{\lambda}{2}\mu\sqrt{W(s)}g(v,s)v\eta.
\end{aligned}
\tag{4.60}
$$

If we make the time transformation (4.54), the vector field on C' becomes:

$$
\begin{aligned}
\frac{dv}{d\gamma} =& \frac{\lambda}{2}\sqrt{W(s)}\sqrt{g(v,s)}(1-\mu v^2) \\
\frac{ds}{d\gamma} =& \eta \\
\frac{d\eta}{d\gamma} =& -s(1-\mu v^2) - \frac{\lambda}{2}\mu\sqrt{W(s)}\sqrt{g(v,s)}v\eta.
\end{aligned}
\tag{4.61}
$$

Now from the definition of W in (4.30) and equation (4.51) we see that:

$$\lim_{\lambda\to 0} W(s) = G\sqrt{2m^5}(1-s^2)$$

uniformly on $[-1,1]$. Thus equation (4.52) gives us:

$$\lim_{\lambda\to 0} W(s)g(v,s) = G\sqrt{2m^5}(1-s^2)$$

uniformly on compact subsets of $(-v_c, v_c)\times[-1,1]$. Therefore:

$$\lim_{\lambda\to 0} \lambda\sqrt{W(s)}\sqrt{g(v,s)} = 0$$

uniformly on compact subsets. Hence the orbits of the vector field (4.61) converge to the ones of the vector field defined at (4.55).□

We see that the proof of Proposition 4.8.1 is accomplished because the vector field (4.55) draws circles on C' given by the constant variable v, so the vector field (4.61) must make a spiral around C' several times, an amount which tends to ∞ as $\lambda \to 0$, which is also accomplished for the vector field (4.60), since its flow on C' is homeomorphic to the flow on C_0 given by the system (4.41), so as we can see, $Z(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Then, as equations (4.41) are invariant under the transformation $(v, s, w) \mapsto (v, -s, -w)$, the two branches of $W^U(\mathbf{c})$ are one reflection of the other. By recalling the definitions of $C_+(\nu)$ and $C_-(\nu)$ at the end of section 4.7, and because of the symmetry, if $C_+(\nu) \cap W^U(\mathbf{c}) \neq \emptyset$ then $C_-(\nu) \cap W^U(\mathbf{c}) \neq \emptyset$, thus as Proposition 4.8.1 is proved, then we have proved Theorem 4.7.3 and Proposition 4.8.1.

The vector field (4.61) can be interpretated in the following way. The mass values of the point masses are $m_1 = m_3 = m$ and $m_2 = \varepsilon m$. As $\lambda \to 0$, $\varepsilon \to 0$ and $m_2 \to 0$. Therefore, the vector field defined by equations (4.55) represents triple collision when the central point mass has a mass value of zero. To interpretate $\varepsilon = 0$ in a physical meaning is difficult, since all transformations we have done before become singular. However, for small ε, the fast spiralling around C' represents the central particle oscillating between another particle and the other several times while going close to triple collision. The number of oscillations tends to ∞ as $\varepsilon \to 0$. In the case we have shown in this section, where $m_1 = m_3$ the flow on C is symmetric. Since W is even, equations (4.41) do not change under the transformation $(v, s, w) \mapsto (v, -s, -w)$. Therefore, one of the two branches of $W^U(\mathbf{c})$ is a reflection of the other. If we remember the definitions of $C_+(\nu)$ and $C_-(\nu)$ given at the end of section 4.7, we can use the symmetry to show that if one branch of $W^U(\mathbf{c})$ intersects $C_+(\nu)$, then the other brach must intersect $C_-(\nu)$. Thus the existence of flows as they appear in fig. 7 is given by Proposition 4.8.1.

Chapter 5

Conclusions

After all this work done, many questions keep unanswered by this work. We may state two important ones from them:

- Can one describe the set of values of the three masses for which triple collision is non-regularisable?
- Do the results and methods apply to situations other than triple collision in the collinear 3-body problem?

For the first question, we may note that the equations $\mathbf{M}\ddot{q} = \nabla U(q)$ remain invariant by rescaling the mass matrix and the positions vector as:

$$\begin{aligned}\mathbf{M} =& \mu^{-1}\mathbf{M}' \\ q =& \sqrt[3]{\mu} q'\end{aligned}$$

for $\mu > 0$. Thus triple collision is non-regularisable for masses m_1, m_2, m_3 iff it is non-regularisable for their uniform scaling by μ. Thus we can just consider masses on the simplex:

$$\mathcal{M} := \{m_1, m_2, m_3 \in (0,1) : m_1 + m_2 + m_3 = 1\}.$$

In section 4.7 we saw that triple collision is non-regularisable if the flow on C is not totally degenerate. We also saw that the set of masses which accomplishes so is open. Recall from the explanation just below Proposition 4.7.6 the definitions of Siegel and Easton regularisations.

Theorem 5.0.1 *The set of masses for which triple collision is Siegel regularisable is a countable dense set of curves in $\mathcal{M}$ (level curves of k) both in the Lagrangian and Eulerian cases, where $k = k(m)$ is a continuous function of the masses, $k : \mathcal{M} \to [0,1)$ in the Lagrangian case, and $k : \mathcal{M} \to (0,7)$ in the Eulerian case.*

To make things more clear, an Eulerian CC is the collinear configuration, whereas the Lagrangian CC is the configuration explained at section 1.5.

We can see a very detailed proof in the paper by Carles Simó [7], following these references from McGehee: the same reference in which we have been working on, [4], and "Dynamical Systems, Theory and Applications", J. Moser (ed.), Springer (1975). Its idea of proof in the collinear case is very similar to what we have done in the equations of motion (4.41) for necessary conditions, and an integral argument for sufficient conditions.

For the second question, we can note that the flow for the collinear problem is contained in the flow for the problem in higher dimensions as an invariant set, thus we can see first an implication: if the problem is not regularisable in the line, then neither in the plane. And if not in the plane, then neither in the space. But it could be regularisable on the line but not in the plane, and in the plane but not in the space.

The transformations used in section 4.1 can be extended to a higher number of particles, i.e., we can transform N-tuple collision into an invariant manifold by deleting the origin, and we can transform the time variable in such a way that there is no orbit reaching total collision in finite time. However, McGehee did not set the equations about this generalisation.

The question about the existence of singularities other than collisions has been studied by Zhihong Xia. Painlevé proved that all singularities of the 3-body problem are due to collisions. The other singularities are found by Xia, Moser, Mather and McGehee. Xia found these singularities for $N \geq 5$, and the proof is explained with some figures in the reference [8], which builds a construction made of two rotating binaries happening in different planes, and an extra mass in the middle of them, between the centres of mass of the binaries, and going from one binary to the other. Whereas Moser, Mather and McGehee proved so for $N = 4$, and one can check the results in [9]. Their main idea is that the first mass goes to $-\infty$, the third and the fourth masses go to $+\infty$, and the second mass oscillates. One can also look for further details and theorems about this issue done by Jinxin Xue at [10], which not only does it talk about the 4-Body Problem, but also about the Planar 2-Centre-2-Body Problem.

If we consider the special collinear 5-body problem, with $m_1 = m_3 = m_5 = m$ and $m_2 = m_4 = \varepsilon m$, we will be able to take arbitrarily high kinetic energy whenever particles 1, 2 and 3, or 3, 4 and 5 pass close to triple collision. Let us recall that a binary is a subsystem made of two masses which are close together, so in the whole system, they behave in a similar way than a normal point mass. In this case, we consider masses 1, 2, and 4, 5 in respective double collisions, mass 3 will oscillate between these two binary systems, reaching one of the systems twice as fast as every time it reaches the other one, and thus make infinite oscillations in finite time. For instance, let us take particles 1 and 2 as system A, and particles 4 and 5 as system B. At time $t = 0$, particle 3 is moving so that it overtakes binary system A in time $t_1 < 1/2$, and then it takes back and forward in times $1/4, 1/8, 1/16$, and so on. So in less than in unit time, $A \to -\infty$ and $B \to +\infty$, and particle 3 oscillates between A and B an infinite number of times.

Theorem 5.0.2 *The orbit described just above exists.*

Indeed, in the proof of the paper from Moser, Mather and McGehee [9], we have an introduction which talks about this behaviour for the case of the collinear 4-Body Problem, where the two last masses form a binary, and then proves that for $N = 4$ there are singularities not due to collisions. Anyway, the results explained there can be brought to the collinear 5-Body Problem case, by replacing the first mass by a binary.

However, we should note that this last theorem contradicts an old conjecture from Saari, stated in McGehee's paper [4], which proved that all singularities in the collinear N-body problem were due to collisions, but Saari did not extend orbits through double collisions. So the orbit described in this last conjecture has an infinite number of binary collisions.

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