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Title: Invariant Curves For Area Preserving Maps

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INVARIANT CURVES FOR AREA PRESERVING MAPS

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Introduction

The study of existence of quasi-periodic motion back to the time mathematicians and astronomers started to study the motion of celestial bodies through newtonian laws. In some sense, existence of quasi-periodic motion in the Solar System is a trace of its stability. The Solar System can be viewed as a small perturbation of a direct product of Kepler problems, which are integrable. Then, at the end of 19th century, Poincaré considered the study of small perturbations of completely integrable Hamiltonian systems (and, in particular, of the persistence of quasi-periodic motion) as the most important in Dynamics.

The breakthrough in the problem did not arrived until the mid 50's and early 60's, with the works of A.N. Kolmogorov [1], V.I. Arnold [2] and J. Moser [3], in which they proved that, under suitable non-degeneracy conditions of the integrable system, most of the tori carrying quasi-periodic motion of the integrable system do persist slightly deformed in the perturbed system. Arnold himself asked what would happen when pertubations are not small, i.e. in the far from integrable regime.

We will consider these problems for discrete versions of Hamiltonian systems, that are exact symplectic maps. We will first illustrate these problems of persistence and existence of invariant tori for the famous standard map F_ε.

Given the standard map F_ε is an area preserved map with dependence on (x, y). When $\varepsilon = 0$ we call F_0 unperturbed system .

$$\begin{aligned} F_\varepsilon : \mathbb{T} \times \mathbb{R} &\to \mathbb{T} \times \mathbb{R} \\ (x, y, \varepsilon) &\to (x + y - \frac{\varepsilon}{2\pi}\sin(2\pi x), y - \frac{\varepsilon}{2\pi}\sin(2\pi x)) \end{aligned} \tag{1}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

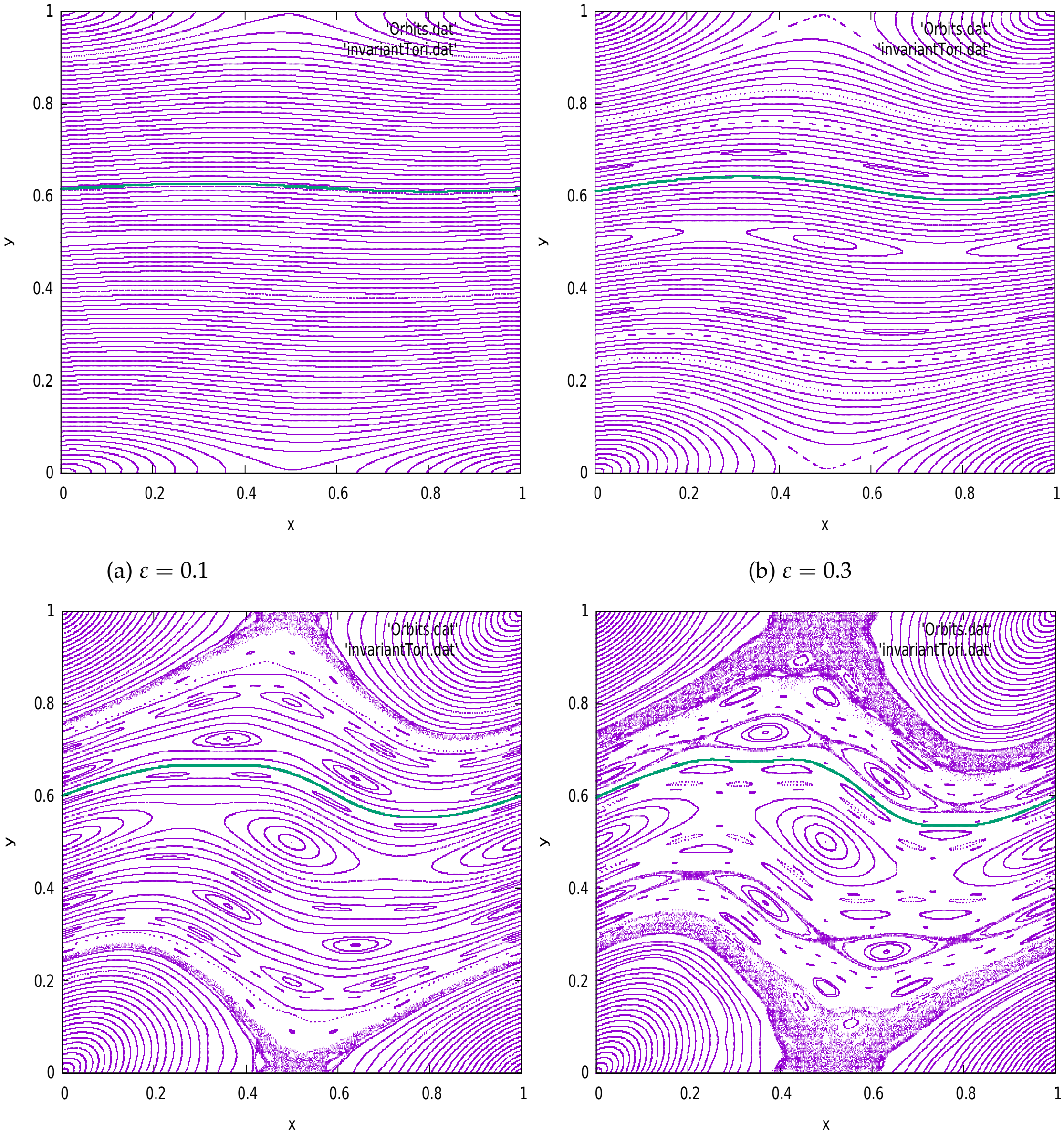
For $\varepsilon = 0$ the dynamics is simple: For any $(x_0, y_0) \in \mathbb{T} \times \mathbb{R}$ the orbit is given by the rigid rotation $F_0^n(x_0, y_0) = (x_0 + ny_0, y_0)$ with rotation number y_0.

$$\begin{cases} \text{if } y_0 = p/q \in \mathbb{Q} & \text{The orbit is periodic in the invariant circle } \mathbb{T} \times \{y_0\}, \\ \text{if } y_0 \in \mathbb{R}/\mathbb{Q} & \text{The orbit is dense in the invariant curve } \mathbb{T} \times \{y_0\}\ . \end{cases} \tag{2}$$

For a given initial point (x_0, y_0) where $y_0 = p/q$ with $p \neq 0$, then $\mathbb{T} \times \{y_0\}$ is a invariant curve. Do these invariant curves persist after the perturbation?

One can use KAM methods (in spirit of Kolmogorov) to answer the previous question. KAM methods typically deal with a perturbative setting in such a way that the problem is written as perturbation of an integrable system and take advantage of the existence of action angle-like coordinates for the unperturbed system. Where the variable y is called action and the variable x is the angle. In particular, it consists in applying a sequence of canonical transformations such that in step k we eliminate the dependence on the variable action at order k. In that way we can have $\{y = y_0\} \times \mathbb{T}$ is a invariant tori for the perturbed system and (y_0, x_0) is an initial point.

For $\varepsilon = 0$, we have explained in (2) and for $\varepsilon > 0$, KAM theory concludes that "most" of the previous invariant curves mentioned in (2) persist, even they are slightly deformed. Moreover these curves are successively destroyed as ε increased. A standard procedure in KAM theory (in the spirit of Kolmogorov) is first to fix the rotation number and then study the persistence of the curves. Then, varying the rotation number for a higher values of ε.



(a) $\varepsilon = 0.1$

(b) $\varepsilon = 0.3$

(c) $\varepsilon = 0.7$

(d) $\varepsilon = 0.9$

Figure 1: Phase space of the standard map for different values of the parameter ε and in green we plot the invariant curve.

In the Figure 1, we see the dynamic of Standard map with different values of parameter

ε. As we said before even the parameters are slightly increased these curves are successively destroyed. The invariant curves exist in unperturbed system and as ε increased they persist we call them primary tori. For instance, in Figure 1 in green is primary tori. On the other cases, the new invariant curve are called secondary tori because they are not present for $\varepsilon = 0$. The new invariant curves are born from an elliptic fixed point, notice that in Figure 1 the point $(0, 0)$ are foliated by invariant tori and they are secondary tori.

But this method is often difficult to be applicable to the many problems and applications are non-perturbative systems. In addition, sometimes it is very difficult to establish action-angle variables for the unperturbed system.

In this work we will consider this circle of ideas and results for the case of families of exact symplectic maps. We will state KAM theorems for existence and persistence of invariant tori in the so-called a posteriori format: given an approximate invariant torus, and under suitable non-degeneracy conditions, there is a true invariant torus nearby. The proof is designed in a way one can produce a numerical algorithm to compute invariant tori. This has been implemented for the numerically computation of primary and secondary invariant circles.

Essentially, this method is carried out by adding a small function to the previous approximations of torus and this small function is obtained solving a linearized equation around the approximated torus. Thus, we obtain a Newton-like iterative scheme to solve the invariance equation. In particular, do not require the Hamiltonian system either to be written in action-angle variables or to be a perturbation of an integrable one.

The main tool of this approach is the so-called parameterization method, that produces a Newton-like iterative method to solve the invariance equation for an invariant torus, and the KAM theorem is a result on the convergence of the method. The method was introduced in this setting by A. González, À. Jorba, R. de la Llave and J. Villanueva [4] (see [DlL01] for preliminary a version). In this work we have followed the review of the method exposed in [HCF$^+$16].

The main achievements of this work are:

- We have adapted the KAM theorem in [HCF$^+$16] to the case the ambient space have extra geometrical structures apart from an exact symplectic structure: we assume there is a compatible triple (a simplectic form, an almost complex structure and a Riemannian structure);
- We have adapted the theorem to consider invariant tori with different topologies: tori that are homotopic to the zero-section of a cylinder (primary tori), and tori that are homotopically trivial (secondary tori);
- We have included analytical dependence on parameters;
- We have designed algorithms of computation and continuation of invariant circles (primary and secondary) in families of area preserving maps, and implemented them with programming language C, for the standard map.

Finally, we briefly describe the organization of this work: in Chapter 1 we introduce the setting, including the key geometrical structures we will use; in Chapter 2 we state and proof the main theorem of this work: a KAM theorem in a posteriori format; in Chapter 3 we describe several algorithms of computation of invariant tori and apply them to compute primary and secondary tori for the standard map: the C programs are included in the two final appendices.

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Chapter 1

The Parameterization Method in KAM Theory

1.1 Geometric Properties of Invariant Tori

This section is devoted to describe the relevant geometric features and to present the construction of a suitable adapted frame that allows us to study the linearized dynamics around and an invariant torus. Certainly, in chapter 4 of [HCF+16] there are 2 more adapted frames are that also interesting choices.

Generalizing the discussion in this section, throughout this chapter we consider $F : \mathcal{A} \times U \to \mathcal{A}$, with F is a family of exact symplectic maps and we look for a family of parameterizations $K : \mathbb{T}^n \times U \to \mathcal{A}$ of invariant tori. Specially, we are interested in two kinds of ambient space $\mathcal{A}$.

On the one hand, in the "primary context" we work in $\mathcal{A} = \mathbb{T}^n \times \mathbb{R}^n$, F is homotopic to the identity (i.e $F(x, y, \varepsilon) - (x, 0)$ is 1- periodic in x) and K is homotopic to the zero section (i.e $K(\theta, \varepsilon) - (\theta, 0)$ is 1-periodic in θ).

$$
\begin{aligned}
K : \mathbb{T}^n \times U &\longrightarrow \mathcal{A} \times U, \qquad \mathcal{A} = \mathbb{T}^n \times \mathbb{R}^n \\
(\theta, \varepsilon) &\longrightarrow \begin{bmatrix} \theta + K_p^x(\theta, \varepsilon) \\ K_p^y(\theta, \varepsilon) \end{bmatrix}
\end{aligned} \tag{1.1}
$$

where K_p^x, K_p^y, the projection on the angular and action variable, are the periodic functions.

On the other hand, in the "secondary context" we work in $\mathcal{A} = \mathbb{R}^n \times \mathbb{R}^n$, F a family of exact symplectomorphisms and K is homotopically trivial (a point).

$$
\begin{aligned}
K : \mathbb{T}^n \times U &\longrightarrow \mathcal{A} \times U, \qquad \mathcal{A} = \mathbb{R}^n \times \mathbb{R}^n \\
(\theta, \varepsilon) &\longrightarrow \begin{bmatrix} K_p^x(\theta, \varepsilon) \\ K_p^y(\theta, \varepsilon) \end{bmatrix}
\end{aligned} \tag{1.2}
$$

1.1.1 Symplectic Structures

We will recall some concepts in symplectic geometry. In ambient space $\mathcal{A}$ is either an annulus or a star sharped. We consider $U \subset \mathbb{R}^p$ is the parameter space and we assume that $\mathcal{A}$ is endowed with a closed non-degenerate symplectic form ω_ε whose matrix representation is given by $\Omega : \mathcal{A} \times U \to \mathbb{R}^{2n\times 2n}$ that is an antisymmetric, non-degenerate matrix, that means $\Omega(z,\varepsilon)^\top = -\Omega(z,\varepsilon)$, $\det\ \Omega(z,\varepsilon) \neq 0$ respectively. In such a way

$$z \in \mathcal{A},\ u, v \in \mathbb{R}^{2n} \quad \omega_\varepsilon(z)(u,v) = u^\top \Omega(z,\varepsilon) v. \tag{1.3}$$

Furthermore, the closed condition is equivalent to

$$\frac{\partial(\Omega(z,\varepsilon))_{r,s}}{\partial z_t} + \frac{\partial(\Omega(z,\varepsilon))_{s,t}}{\partial z_r} + \frac{\partial(\Omega(z,\varepsilon))_{t,r}}{\partial z_s} = 0, \tag{1.4}$$

where we use the notation $(\Omega(z,\varepsilon))_{i,j}$ for the $(i,j)-$ component of $\Omega(z,\varepsilon)$. The symplectic form is exact if, moreover, there exists 1-form α such that $\omega = d\alpha$. In matrix representation, that means, there exists $a : \mathcal{A} \times U \to \mathbb{R}^{2n}$ of the form $a(z,\varepsilon) = (a_1(z,\varepsilon), \cdots, a_{2n}(z,\varepsilon))^\top$ such that

$$\Omega(z,\varepsilon) = D_z a(z,\varepsilon)^\top - D_z a(z,\varepsilon). \tag{1.5}$$

Remark 1.1. Two main examples of exact symplectic form ω in matrix representation in this work are the following

(a) $\mathcal{A} = \mathbb{T}^n \times \mathbb{R}^n$

$$a(z,\varepsilon) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}, \qquad D_z a(z,\varepsilon) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{1.6}$$

(b) $\mathcal{A} = \mathbb{R}^n \times \mathbb{R}^n$

$$a(z,\varepsilon) = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} y \\ -x \end{bmatrix}, \qquad D_z a(z,\varepsilon) = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.7}$$

In both cases we get $\Omega(z,\varepsilon) = \Omega_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We want to emphasize that in $2D$ standard case, we use Ω_0 that does not depend on the point z.

A family of diffeomorphisms $F : \mathcal{A} \times U \longrightarrow \mathcal{A}$ is symplectic if

$$D_z F(z,\varepsilon)^\top \Omega(F(z,\varepsilon),\varepsilon) D_z F(z,\varepsilon) = \Omega(z,\varepsilon), \quad \forall z \in \mathcal{A}, \varepsilon \in U \tag{1.8}$$

Moreover, F is exact symplectomorphism if there exists a differentiable function $S : \mathcal{A} \times U \to \mathbb{R}$ (primitive function) such that

$$D_z S(z,\varepsilon)^\top = a(F(z,\varepsilon),\varepsilon)^\top D_z F(z,\varepsilon) - a(z,\epsilon)^\top, \quad \forall z \in \mathcal{A}, \varepsilon \in U. \tag{1.9}$$

Remark 1.2.

(a) When $\mathcal{A} = \mathbb{T}^n \times \mathbb{R}^n$, we have F is 1-periodic in x. Then, the compatibility condition for

$$\frac{\partial S}{\partial x} = (F^y)^\top \frac{\partial F^x}{\partial y} - y^\top, \tag{1.10}$$

$$\frac{\partial S}{\partial y} = (F^y)^\top \frac{\partial F^y}{\partial x}, \tag{1.11}$$

which is the symplectic condition, is a necessary condition to find the primitive function S, but not sufficient. From Poincaré's Lemma we know there is a primitive function S in $\mathbb{R}^{2n}$, but since the ambient space $\mathcal{A} = \mathbb{T}^n \times \mathbb{R}^n$ we need to check the primitive function is also 1-periodic in x.

(b) In case of $\mathcal{A} = \mathbb{R}^n \times \mathbb{R}^n$, the compatibility condition for

$$\frac{\partial S}{\partial x} = \frac{1}{2}\left((F^y)^\top \frac{\partial F^x}{\partial x} - (F^x)^\top \frac{\partial F^y}{\partial x} - y^\top\right), \tag{1.12}$$

$$\frac{\partial S}{\partial y} = \frac{1}{2}\left((F^y)^\top \frac{\partial F^x}{\partial y} - (F^x)^\top \frac{\partial F^y}{\partial y} + x^\top\right), \tag{1.13}$$

is sufficient from Poincaré 's Lemma and then there exists the primitive function S.

Definition 1.3. *Given such a family of parameterizations $K : \mathbb{T}^n \times U \to \mathcal{A}$ and $\mathcal{K} = K(\mathbb{T}^n \times U)$ is a family of Lagrangian tori if for all $\theta \in \mathbb{T}^n, \varepsilon \in U$,*

$$\Omega_K(\theta,\varepsilon) = D_\theta K(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon) = O_n.$$

Definition 1.4. *For a family K of parameterizations of quasi-periodic invariant tori with frequency $\omega \in \mathbb{R}^n$, $\mathcal{K} = K(\mathbb{T}^n \times U)$ is $F-$invariant, and*

$$F(K(\theta,\varepsilon),\varepsilon) = K(R_\omega(\theta),\varepsilon). \tag{1.14}$$

with $R_\omega(\theta) = \theta + \omega$ where $\forall k \in \mathbb{Z}^n \setminus \{0\}$, $k \cdot \omega \notin \mathbb{Z}$($\omega$ non-resonant).

Lemma 1.5. *Let $F : \mathcal{A} \times U \to \mathcal{A}$ be a family of exact symplectomorphisms and $K : \mathbb{T}^n \times U \to \mathcal{A}$ be a family of parameterizations of a quasi-periodic invariant tori. Then, $\mathcal{K}$ is a family of Lagrangian tori.*

Proof. Derivating the invariance equation (1.14) in both sides we get

$$D_z F(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon) = D_\theta K(\theta+\omega,\varepsilon). \tag{1.15}$$

Moreover,

$$\begin{aligned}\Omega_K(\theta,\varepsilon) &= D_\theta K(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon) \\ &= D_\theta K(\theta,\varepsilon)^\top D_z F(K(\theta,\varepsilon),\varepsilon)^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) D_z F(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon) \\ &= D_\theta K(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon) D_\theta K(\theta+\omega,\varepsilon) = \Omega_K(\theta+\omega,\varepsilon)\end{aligned}$$

where we used the invariance equation (1.14), (1.15) and (1.9). We get $\Omega_K(\theta,\varepsilon) - \Omega_K(\theta+\omega,\varepsilon) = O_n$, then $\Omega_K(\theta,\varepsilon)$ is constant, because ω is non-resonant. Moreover, the components of $\Omega_K(\theta,\varepsilon)$ are sums of derivatives of periodic functions:

$$\begin{aligned}(\Omega_K(\theta,\varepsilon))_{i,j} &= \sum_{m=1}^{2n}\left(\frac{\partial}{\partial\theta_i}\left(a_m(K(\theta,\varepsilon),\varepsilon)\frac{\partial K_m(\theta,\varepsilon)}{\partial\theta_j}\right) - a_m(K(\theta,\varepsilon),\varepsilon)\frac{\partial^2 K_m(\theta,\varepsilon)}{\partial\theta_j\,\partial\theta_i}\right) \\ &- \sum_{m=1}^{2n}\left(\frac{\partial}{\partial\theta_i}\left(a_m(K(\theta,\varepsilon),\varepsilon)\frac{\partial K_m(\theta,\varepsilon)}{\partial\theta_i}\right) - a_m(K(\theta,\varepsilon),\varepsilon)\frac{\partial^2 K_m(\theta,\varepsilon)}{\partial\theta_i\,\partial\theta_j}\right) \\ &= \sum_{m=1}^{2n}\left(\frac{\partial a_m(K(\theta,\varepsilon))}{\partial\theta_i}\frac{\partial K_m(\theta,\varepsilon)}{\partial\theta_j} - \frac{\partial a_m(K(\theta,\varepsilon))}{\partial\theta_j}\frac{\partial K_m(\theta,\varepsilon)}{\partial\theta_i}\right),\end{aligned} \tag{1.16}$$

from which we can see $\Omega_K = O_n$. □

Remark 1.6. In $2D$ standard case,

$$D_\theta K(\theta,\varepsilon)^\top \Omega_0 D_\theta K(\theta,\varepsilon) = O_2$$

It leads us to $\Omega_K = O_2$ in $2D$, that means, any $K : \mathbb{T} \times U \to \mathcal{A}$, $\mathcal{K} = K(\mathbb{T} \times U)$ is a family of Lagrangian tori.

1.1.2 Construction of Symplectic Adapted Frame

Our next goal is to construct a suitable expression for the tangent map of a family of exact symplectomorphisms F_ε around a family of Lagrangian invariant tori. Consider the family of Lagrangian tori parameterized by K, the adapted frame should be partially defined by the tangent vector of K. Then, consider the map $L : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times n}$ given by $L(\theta, \varepsilon) = D_\theta K(\theta, \varepsilon)$. By Lemma 1.5, L defines a family of Lagrangian frames, i.e,

$$L(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon) = O_n$$

Then, our goal is to construct a complementary subspace, given by $N : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times n}$ such that $P : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times 2n}$ the juxtaposed matrix

$$P(\theta,\varepsilon) = \begin{pmatrix} L(\theta,\varepsilon) & N(\theta,\varepsilon) \end{pmatrix} \tag{1.17}$$

satisfies the symplectic condition

$$P(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon) = \Omega_0. \tag{1.18}$$

In this way, we say that $P : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times 2n}$ is a symplectic frame. In particular, the symplectic form in the adapted frame reduces to the standard symplectic form.

In order to construct adapted frame we will assume $\mathcal{A}$ is endowed with a family of compatible triples $(\Omega, \mathcal{J}, G)$ meaning,

(i) a family of symplectic forms ω_ε whose matrix representations are given by $\Omega : \mathcal{A} \times U \to \mathbb{R}^{2n\times 2n}$ such that

$$z \in \mathcal{A},\ u, v \in \mathbb{R}^{2n} \quad \omega_\varepsilon(z)(u, v) = u^\top \Omega(z,\varepsilon)v$$

and $\Omega(z, \varepsilon)$ satisfies

- antisymmetric: $\Omega(z,\varepsilon)^\top = -\Omega(z,\varepsilon)$,
- non-degenerate: $\det \Omega(z,\varepsilon) \neq 0$,

(ii) a family of almost complex structures $\mathcal{J}$ whose matrix representations are given by $J : \mathcal{A} \times U \to \mathbb{R}^{2n\times 2n}$ such that

$$J(z,\varepsilon)^2 = -Id_{2n}.$$

(iii) a family of Riemannian structures g whose matrix representations are given by $G : \mathcal{A} \times U \to \mathbb{R}^{2n\times 2n}$ such that

- symmetric: $G(z,\varepsilon)^\top = G(z,\varepsilon)$, $\quad \det G(z,\varepsilon) \neq 0$,
- positive definite: $u^\top G(z,\varepsilon)u > 0 \quad \forall u \in \mathbb{R}^{2n} \setminus \{0\}$

where the compatibility conditions reads: $\omega_\varepsilon(z)(u, v) = g_\varepsilon(z)(u, Jv) \quad \forall z \in \mathcal{A}\ \forall u, v \in T_z\mathcal{A},\ \forall \varepsilon \in U$. It leads to

$$\Omega(z,\varepsilon) = G(z,\varepsilon)J(z,\varepsilon). \tag{1.19}$$

Remark 1.7. This is not strictly necessary for the construction of adapted frames, but make simpler some computations.

Since K is an embedding it follows that $L(\theta,\varepsilon)$ has the rank n. Then, the map $N^0 : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times n}$ defined by

$$N^0(\theta,\varepsilon) = J(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon) \tag{1.20}$$

such that the matrix $P^0(\theta,\varepsilon) = \begin{pmatrix} L(\theta,\varepsilon) & N^0(\theta,\varepsilon)\end{pmatrix}$ has non-vanishing determinant that is the non-degenerate condition.

Under these assumptions, we complement the Lagrangian subspace generated by $L(\theta,\varepsilon)$ as follows

$$N(\theta,\varepsilon) = L(\theta,\varepsilon)A(\theta,\varepsilon) + N^0(\theta,\varepsilon)B(\theta,\varepsilon),$$

in such a way that $P(\theta,\varepsilon)$ satisfies (1.18), it leads to

$$\begin{aligned} L^\top \Omega L = O_n, \quad L^\top \Omega N = -Id_n, \\ N^\top \Omega L = Id_n, \quad N^\top \Omega N = O_n. \end{aligned} \tag{1.21}$$

After the computations we get $A(\theta,\varepsilon) = O_n$ and

$$\begin{aligned} B(\theta,\varepsilon) &= (L(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon)J(K(\theta,\varepsilon),\varepsilon)\ L(\theta,\varepsilon))^{-1} \\ &= (L(\theta,\varepsilon)^\top G(K(\theta,\varepsilon),\varepsilon)\ L(\theta,\varepsilon))^{-1} \\ &= (G_k(\theta,\varepsilon))^{-1}, \end{aligned} \tag{1.22}$$

such that

$$N(\theta,\varepsilon) = N^0(\theta,\varepsilon)(G_k(\theta,\varepsilon))^{-1}.$$

We will often split the matrix P in four blocks as follow

$$P(\theta,\varepsilon) = \begin{bmatrix} L^x(\theta,\varepsilon) & N^x(\theta,\varepsilon) \\ L^y(\theta,\varepsilon) & N^y(\theta,\varepsilon) \end{bmatrix}. \tag{1.23}$$

and from the symplectic condition of P we get

$$P(\theta,\varepsilon)^{-1} = -\Omega_0 P(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon) \tag{1.24}$$

for K a family of parameterization of invariant tori.

Remark 1.8. In the standard case, we have

$$\Omega_0 = \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}, \quad J_0 = \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}, \quad G = \begin{bmatrix} Id & 0 \\ 0 & Id \end{bmatrix} \tag{1.25}$$

An important consequence of the previous construction is the reducibility of the linear dynamics around a family of tori. That is, the following lemma states that the linearized dynamics around a family of invariant tori is upper-triangular and symplectic in the adapted frame.

Lemma 1.9. *Let $F : \mathcal{A} \times U \to \mathcal{A}$ be a family of symplectomorphisms and $K : \mathbb{T}^n \times U \to \mathcal{A}$ be a family of parameterizations. Assume that $\mathcal{K} = K(\mathbb{T}^n \times U)$ is F-invariant and that the dynamics on $\mathcal{K}$ is conjugate with the ergodic rotation $R_\omega(\theta) = \theta + \omega$:*

$$F(K(\theta,\varepsilon),\varepsilon) = K(R_\omega(\theta,\varepsilon),\varepsilon),$$

Assume the matrix map $P : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times 2n}$*, defined in* (1.17)*, satisfies* $P(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon) = \Omega_0$*, Then, the frame* P *reduces the linearized dynamics* $D_zF \circ K$ *to a block-triangular matrix:*

$$P(\theta+\omega,\varepsilon)^{-1}D_zF(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon) = \Lambda(\theta,\varepsilon), \quad \Lambda(\theta,\varepsilon) = \begin{bmatrix} Id_n & T(\theta,\varepsilon) \\ 0 & Id_n \end{bmatrix} \tag{1.26}$$

where $T : \mathbb{T}^n \to \mathbb{R}^{n\times n}$ *is defined by the symmetric matrix*

$$T(\theta,\varepsilon) = N(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon). \tag{1.27}$$

The matrix $T(\theta,\varepsilon)$ *is called torsion matrix.*

Proof. From $\mathcal{K} = K(\mathbb{T}^n \times U)$ is $F-$ invariant, $F(K(\theta,\varepsilon),\varepsilon) = K(\theta+\omega,\varepsilon)$, we differentiate the invariance equation with respect to θ. Then, we get

$$D_zF(K(\theta,\varepsilon),\varepsilon)D_\theta K(\theta,\varepsilon) = D_\theta K(\theta+\omega,\varepsilon). \tag{1.28}$$

Notice that

$$P(\theta+\omega,\varepsilon)^{-1}D_zF(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon) = P(\theta+\omega,\varepsilon)^{-1}L(\theta+\omega,\varepsilon) = \begin{bmatrix} Id_n \\ 0 \end{bmatrix} \tag{1.29}$$

where the last equality is hold by the definition of $P(\theta+\omega,\varepsilon)$ that the first first column is $L(\theta+\omega,\varepsilon)$.

Then, $\Lambda(\theta,\varepsilon) = P(\theta+\omega,\varepsilon)^{-1}D_zF(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon)$ is symplectic due to the fact that the multiplication of symplectic matrices is symplectic.

The necessary symplectic condition and (1.29) lead us to the following matrix

$$P(\theta+\omega,\varepsilon)^{-1}D_zF(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon) = \begin{bmatrix} Id_n & T(\theta,\varepsilon) \\ 0 & Id_n \end{bmatrix}.$$

Then,

$$P(\theta+\omega,\varepsilon)^{-1}D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) = \begin{bmatrix} T(\theta,\varepsilon) \\ Id_n \end{bmatrix}.$$

with

$$T(\theta,\varepsilon) = N(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon).$$

and

$$\begin{aligned} &-L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) \\ &\quad = -L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)N(\theta+\omega,\varepsilon) = Id_n. \end{aligned}$$

□

1.2 Cohomological Equations

KAM method explained here consists on adding iteratively a small function to a given family of approximation of the tori. This correction is obtained by solving (approximately) the linearized equation around the family of approximated tori. It leads us to an important study of the so-called cohomological equation that will be presented in the section (1.2).

Then, we describe the one step of this procedure in order to clarify the ideas and also to emphasize that in the section (1.3) process is expected to converge quadratically to a family of invariant tori. That means, we will see the error when applying once the KAM step is quadratically small with respect to initial error.

Definition 1.10. *Given a non-resonant $\omega \in \mathbb{R}^n$, we define the cohomological operator $\mathcal{L}$ on functions $u : \mathbb{T}^n \times U \to \mathbb{R}$ as follows:*

$$\mathcal{L}u(\theta, \varepsilon) := u(\theta, \varepsilon) - u(\theta + \omega, \varepsilon). \tag{1.30}$$

Definition 1.11. *Given a continuous function $u : \mathbb{T}^n \times U \to \mathbb{R}$ that is periodic in θ, we denote the average of u as $\langle u \rangle(\varepsilon) = \int_{\mathbb{T}^n} u(\theta, \varepsilon) d\theta$.*

In the KAM theory we can find the following one-bite cohomological equation:

$$\mathcal{L}u = v - \langle v \rangle, \tag{1.31}$$

where $v : \mathbb{T}^n \times U \to \mathbb{R}$ is know and u has to be determined.

Let $v : \mathbb{T}^n \times U \to \mathbb{R}$ to be a continuous function. Using the Fourier expansions

$$v(\theta, \varepsilon) = \sum_{k \in \mathbb{Z}^n, k \neq 0} \hat{v}(\varepsilon)_k e^{2\pi i k \theta}, \quad u(\theta, \varepsilon) = \sum_{k \in \mathbb{Z}^n, k \neq 0} \hat{u}(\varepsilon)_k e^{2\pi i k \theta}.$$

Note the formal solution is immediately obtained by matching the Fourier series terms.

$$\Re v(\theta, \varepsilon) = \sum_{k \in \mathbb{Z}^n, k \neq 0} \hat{u}_k(\varepsilon) e^{2\pi i k \theta}, \qquad \widetilde{u}_k = \frac{\widetilde{v}_k(\varepsilon)}{1 - e^{2\pi i k \omega}} \tag{1.32}$$

$\Re$ is the solution of the one-bite cohomological equation (1.32). Moreover, all the solutions of (1.31) differ by a constant.

Remark 1.12. The ergocity does not provide the regularity of the solution. Moreover, notice that $1 - e^{2\pi i k \theta}$ can be very small even ω appears in (1.32) is rationally independent. It is the so-called small divisor problem an it requires non-resonance condition on ω.

It is well-known that a sufficient condition for the solvability of the small divisors equation (1.31) is that ω satisfies a Diophantine property as defined below.

Definition 1.13. *Given $\gamma > 0$ and $\tau > n$, $\omega \in \mathbb{R}^n$ is Diophantine frequency vector of type (γ, τ) if and only if*

$$|k \cdot \omega - m| \geq \frac{\gamma}{|k|_1^{\tau}}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\},\ m \in \mathbb{Z}^n, \tag{1.33}$$

where $|k|_1 = \sum_{j=1}^{n} |k_i|$.

Remark 1.14. Later on, we will state a theorem of existence of solution of (1.31) under much stronger hypotheses such as analycity.

1.3 Approximate Invariant Tori and Approximate Reducibility

The aim of this section is to see how the error affects to the geometrical properties discussed in (1.1.2). Given a family of parameterizations $K : \mathbb{T}^n \times U \to \mathcal{A}$ of approximate invariant tori, we consider its error function $E : \mathbb{T}^n \times U \to \mathbb{R}^{2n}$

$$E(\theta, \varepsilon) = F(K(\theta, \varepsilon), \varepsilon) - K(\theta + \omega, \varepsilon). \tag{1.34}$$

Remark 1.15. It is important to emphasize that in the two contexts (primary and secondary context), the error function $E(\theta, \varepsilon)$ is 1-periodic funtcion in the $\theta-$variable.

In the "secondary context" this is easier, since K is 1-periodic in the θ-variable. In the "primary context", however, the periodicity of E comes from the fact that

$$F_p(z, \varepsilon) = F(z, \varepsilon) - \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad K_p(\theta, \varepsilon) = K(\theta, \varepsilon) - \begin{pmatrix} \theta \\ 0 \end{pmatrix} \tag{1.35}$$

The two contexts can be mixed. The point is that one has to adapt the topological properties of the tori to the family of exact symplectomorphisms so that the function E given in (1.34) is 1-periodic in the θ-variable.

As we mentioned before KAM theorem consists on adding a small function in order refine $K(\theta, \varepsilon)$ by means a Newton-like method. We add a correction $\triangle K(\theta, \varepsilon)$ to $K(\theta, \varepsilon)$ to get a new parameterization $\overline{K}(\theta, \varepsilon) = K(\theta, \varepsilon) + \triangle K(\theta, \varepsilon)$ and we expect the new error $\bar{E}(\theta, \varepsilon) = F(\bar{K}(\theta, \varepsilon), \varepsilon) - \bar{K}(\theta + \omega, \varepsilon)$ is quadractically smaller than E.

With the taylor expansion

$$\begin{aligned} \bar{E}(\theta, \varepsilon) &= F(\bar{K}(\theta, \varepsilon), \varepsilon) - \bar{K}(\theta + \omega, \varepsilon) \\ &= F(K(\theta, \varepsilon), \varepsilon) + D_z F(K(\theta, \varepsilon), \varepsilon) \triangle K(\theta, \varepsilon) \\ &+ \int_0^1 (1-t) D^2 F(K(\theta, \varepsilon) + t \triangle K(\theta, \varepsilon)) \triangle K(\theta, \varepsilon)^{\otimes 2} dt \\ &= E(\theta, \varepsilon) + D_z F(K(\theta, \varepsilon), \varepsilon) \triangle K(\theta, \varepsilon) - \triangle K(\theta + \omega, \varepsilon) \\ &+ \int_0^1 (1-t) D^2 F(K(\theta, \varepsilon) + t \triangle K(\theta, \varepsilon)) \triangle K(\theta, \varepsilon)^{\otimes 2} dt. \end{aligned} \tag{1.36}$$

Since the correction term $\triangle K$ is expected to be of the same order of the error term E, the choice for $\triangle K$ is to be a solution of the so-called linearized equation,

$$D_z F(K(\theta, \varepsilon), \varepsilon) \triangle K(\theta, \varepsilon) - \triangle K(\theta + \omega, \varepsilon) = -E(\theta, \varepsilon). \tag{1.37}$$

We emphasize, however, in spirit of the Newton-like method. We do not need to solve (1.36) exactly, but with an error. In order to do so, we will use the adapted frame introduced in subsection (1.1.2)

Since the tori are not truly invariant, then the adapted frames are only approximately symplectic and reducibility properties are also verified approximately. The error affects in symplectic character of the frame we call $E_{sym}(\theta, \varepsilon)$ and in the reducibility of the tangent map, $E_{red}(\theta)$. Fortunately, both of them are controlled by the error $E(\theta, \varepsilon)$ and later we see how they are controlled.

To face the equation (1.37) we consider the frame described in the subsection (1.1.2). Given the map $N^0 : \mathbb{T}^n \times U \to \mathbb{R}^{2n \times n}$, defined in the previous section, satisfies the non-degenerate condition, $\det(L(\theta,\varepsilon)\ N^0(\theta,\varepsilon)) \neq 0$. We consider $P : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times 2n}$ obtained by juxtaposing the two $2n \times n$ matrices $L(\theta,\varepsilon)$ and $N(\theta,\varepsilon)$ where

$$L(\theta,\varepsilon) = D_z K(\theta,\varepsilon), \tag{1.38}$$

$$N(\theta,\varepsilon) = N^0(\theta,\varepsilon)B(\theta,\varepsilon), \tag{1.39}$$

$$B(\theta,\varepsilon) = (L(\theta,\varepsilon)^\top G(K(\theta,\varepsilon),\varepsilon)\ L(\theta,\varepsilon))^{-1} \tag{1.40}$$

First, we start by controlling Ω_K. From the equation (1.16) we can deduce that $\langle \Omega_K \rangle = O_n$. Taking the derivates in both sides of the equation (1.34), we get

$$D_\theta E(\theta,\varepsilon) = D_z F(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon) - L(\theta+\omega,\varepsilon). \tag{1.41}$$

using the definition of $\mathcal{L}\Omega_K$, (1.34), (1.41) and symplectic conditions for F and P we get

$$\begin{aligned}
\mathcal{L}_\varepsilon \Omega_K(\theta,\varepsilon) &= D_\theta K(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon) - D_\theta K(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon) D_\theta K(\theta+\omega,\varepsilon)\\
&= D_\theta K(\theta,\varepsilon)^\top D_z F(K(\theta,\varepsilon),\varepsilon)^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) D_z F(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon)\\
&- D_\theta K(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon) D_\theta K(\theta+\omega,\varepsilon)\\
&= (D_\theta E(\theta,\varepsilon) + D_\theta K(\theta+\omega,\varepsilon))^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon)(D_\theta E(\theta,\varepsilon) + D_\theta K(\theta+\omega,\varepsilon))\\
&- D_\theta K(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon) D_\theta K(\theta+\omega,\varepsilon)\\
&= D_\theta K(\theta+\omega,\varepsilon)^\top \triangle\, \Omega(\theta,\varepsilon) D_\theta K(\theta+\omega,\varepsilon)\\
&+ D_\theta K(\theta+\omega)^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) D_\theta E(\theta,\varepsilon)\\
&+ D_\theta E(\theta,\varepsilon)^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) D_z F(K(\theta,\varepsilon),\varepsilon) D_\theta K(\theta,\varepsilon),
\end{aligned} \tag{1.42}$$

where using the Taylor's formula

$$\triangle\Omega(\theta,\varepsilon) = \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) - \Omega(K(\theta+\omega,\varepsilon),\varepsilon) = \int_0^1 D\Omega(K(\theta+\omega,\varepsilon) + tE(\theta,\varepsilon))E(\theta,\varepsilon)\, dt. \tag{1.43}$$

It is clear from (1.43) that (1.42) is controlled by $E(\theta,\varepsilon)$. Furthermore, Ω_K, that describes the error in the Lagrangian character of the tangent bundle, is also controlled since it is the (formal) solution of cohomological equation in (1.42).

The error in the symplectic character of the frame is

$$E_{sym}(\theta,\varepsilon) = P(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon) P(\theta,\varepsilon) - \Omega_0 \tag{1.44}$$

in matrix representation

$$\begin{aligned}
E_{sym}(\theta,\varepsilon) &= \begin{bmatrix} L(\theta,\varepsilon)^T \Omega(K(\theta,\varepsilon),\varepsilon) L(\theta,\varepsilon) & L(\theta,\varepsilon)^T \Omega(K(\theta,\varepsilon),\varepsilon) N(\theta,\varepsilon) + Id_n \\ N(\theta,\varepsilon)^T \Omega(K(\theta,\varepsilon),\varepsilon) L(\theta,\varepsilon) - Id_n & N(\theta,\varepsilon)^T \Omega(K(\theta,\varepsilon),\varepsilon) N(\theta,\varepsilon) \end{bmatrix}\\
&= \begin{bmatrix} \Omega_K(\theta,\varepsilon) & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{1.45}$$

it is also controlled by $E(\theta,\varepsilon)$.

We need to compute $P(\theta,\varepsilon)^{-1}$ in order to use later to be applicable in computation of reducibility error. The $P(\theta,\varepsilon)$ is invertible if the norm of error $E(\theta,\varepsilon)$ is sufficient small using the Neumann series,

$$\begin{aligned}
P(\theta,\varepsilon)^{-1} &= (\Omega_0 + E_{sym}(\theta,\varepsilon))^{-1} P(\theta,\varepsilon)^T \Omega(K(\theta,\varepsilon),\varepsilon)\\
&= -(Id - \Omega_0 E_{sym}(\theta,\varepsilon))^{-1} \Omega_0 P(\theta,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon).
\end{aligned} \tag{1.46}$$

Notice that is different from (1.24), since the parameterization is approximate. Next, we have to control the error in these properties. We define the reducibility error as following,

$$E_{red} = -(Id - \Omega_0 E_{sym}(\theta,\varepsilon))^{-1}\Omega_0 P(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon) - \Lambda(\theta,\varepsilon). \tag{1.47}$$

These block components of matrix $E_{red}(\theta,\omega)$, denoted by $E^{i,j}_{red}(\theta,\varepsilon)$, are

$$\begin{aligned}
E^{1,1}_{red} &= N(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon) - Id_n \\
&= N(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)(D_\theta E(\theta,\varepsilon) + L(\theta+\omega,\varepsilon)) - Id_n \\
&= N(\theta+\omega,\varepsilon)^\top \Omega_0 D_\theta E(\theta,\varepsilon), \\
E^{1,2}_{red} &= N(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) - T(\theta,\varepsilon) = O_n \\
E^{2,1}_{red} &= -L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon) \\
&= -L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)(D_zE(\theta,\varepsilon) + L(\theta+\omega,\varepsilon)) \\
&= -\Omega_K(\theta+\omega,\varepsilon) - L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zE(\theta,\varepsilon), \\
E^{2,2}_{red} &= -L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) - Id_n \\
&= L(\theta+\omega,\varepsilon) \triangle \Omega(\theta,\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) \\
&+ (D_\theta E(\theta,\varepsilon) - D_zF(K(\theta,\varepsilon),\varepsilon)L(\theta,\varepsilon))^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) - Id_n \\
&= L(\theta+\omega,\varepsilon)^\top \triangle \Omega(\theta,\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon) \\
&+ D_\theta E(\theta,\varepsilon)^\top \Omega(F(K(\theta,\varepsilon),\varepsilon),\varepsilon)D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon).
\end{aligned} \tag{1.48}$$

We observe that E_{red} is also depends on $E(\theta,\varepsilon)$.

Remark 1.16. From Remark (1.6), we know that in standard sysmplectic $\Omega_K = O_{2n}$ then we do not need to control $\mathcal{L}\Omega_K$. Even more, $E_{sym} = O_{2n}$ due to the fact that $P(\theta,\varepsilon)^\top \Omega_0 P(\theta,\varepsilon) = \Omega_0$.

The matrix E_{red} is defined in following

$$\begin{aligned}
E^{1,1}_{red} &= N(\theta+\omega,\varepsilon)^\top \Omega_0(D_zE(\theta,\varepsilon) + L(\theta+\omega,\varepsilon)) - Id_n \\
&= N(\theta+\omega,\varepsilon)^\top \Omega_0 D_zE(\theta,\varepsilon) \\
E^{1,2}_{red} &= 0 \\
E^{2,1}_{red} &= -L(\theta+\omega,\varepsilon)^\top \Omega_0 D_zE(\theta,\varepsilon) \\
E^{2,2}_{red} &= D_\theta E(\theta,\varepsilon)^\top \Omega_0 D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon)
\end{aligned} \tag{1.49}$$

Now we study the linearized equation (1.37) by using the constructed frame. We introduce $\triangle K(\theta,\varepsilon) = P(\theta,\varepsilon)\xi(\theta,\varepsilon)$ so that (1.37) becomes to

$$D_zF(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon)\xi(\theta,\varepsilon) - P(\theta+\omega,\varepsilon)\xi(\theta+\omega,\varepsilon) = -E(\theta,\varepsilon). \tag{1.50}$$

Then, we multiply both sides by $-\Omega_0 P(\theta+\omega)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)$ and we get

$$\begin{aligned}
&(\Lambda(\theta,\varepsilon) + E_{red}(\theta,\varepsilon))\xi(\theta,\varepsilon) - (Id - \Omega_0 E_{sym})\xi(\theta+\omega,\varepsilon) \\
&\qquad = \Omega_0 P(\theta+\omega)^\top \Omega(K(\theta+\omega,\varepsilon))E(\theta,\varepsilon)
\end{aligned}. \tag{1.51}$$

Skipping second order error terms, we get

$$\Lambda(\theta,\varepsilon)\xi(\theta,\varepsilon) - \xi(\theta+\omega,\varepsilon) = \eta(\theta,\varepsilon), \tag{1.52}$$

where

$$\begin{aligned}\eta(\theta,\varepsilon) &= \Omega_0 P(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon) \\ &= (\Lambda(\theta,\varepsilon)+E_{red}(\theta,\varepsilon))\,\xi(\theta,\varepsilon) - (Id_n - \Omega_0 E_{sym}(\theta+\omega,\varepsilon))\xi(\theta+\omega,\varepsilon).\end{aligned} \tag{1.53}$$

with

$$\eta_0 = \langle \eta \rangle = \begin{bmatrix} \langle -N(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon)\rangle \\ \langle L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon)\rangle \end{bmatrix}$$

In order to get an solution of the (approximately) linearized equation, we use the following Lemma. Notice in the Lemma is solving different equation, but we emphasize when we solving (1.54) we are neglecting average $\langle \eta \rangle$. Since a key result is that this average is quadratically smaller respect to the error E and it is provided by the Lemma (1.18). As we want to find the parameterization of the same order as E so that we can add a quadratically smaller term .

Lemma 1.17. *Let us consider vector-valued maps $\eta = (\eta^L, \eta^N) : \mathbb{T}^n \times U \to \mathbb{R}^{2n\times n}$ and a matrix-valued map $T : \mathbb{T}^n \times U \to \mathbb{R}^{n\times n}$. Assume that T satisfies the non-degeneracy condition* $\det\langle T(\theta,\varepsilon)\rangle \neq 0$*, $\forall \varepsilon \in U$. Then, for any $\xi_0^L : U \to \mathbb{R}^n$, the system of equations*

$$\begin{bmatrix} Id_n & T(\theta,\varepsilon) \\ 0 & Id_n \end{bmatrix} \begin{bmatrix} \xi^L(\theta,\varepsilon) \\ \xi^N(\theta,\varepsilon) \end{bmatrix} - \begin{bmatrix} \xi^L(\theta+\omega,\varepsilon) \\ \xi^N(\theta+\omega,\varepsilon) \end{bmatrix} = \begin{bmatrix} \eta^L(\theta,\varepsilon) \\ \eta^N(\theta,\varepsilon) - \langle \eta^N \rangle \end{bmatrix} \tag{1.54}$$

has a (formal) solution given by

$$\xi^N(\theta,\varepsilon) = \Re_\omega\left(\eta^N(\theta,\varepsilon)\right) + \xi_0^N(\varepsilon), \tag{1.55}$$

$$\xi^L(\theta,\varepsilon) = \Re_\omega\left(\eta^L(\theta,\varepsilon) - T(\theta,\varepsilon)\xi^N(\theta,\varepsilon)\right) + \xi_0^L(\varepsilon), \tag{1.56}$$

where

$$\xi_0^N(\epsilon) = \langle T\rangle^{-1}\langle \eta^L - T\Re_\omega(\eta^N)\rangle \tag{1.57}$$

and $\Re_\omega$ is the solution of the one-bite cohomological equation (1.32)*. Note that $\langle \xi^N \rangle = \xi_0^N$ and we have the freedom of choosing any value for $\xi_0^L \in \mathbb{R}$.*

Proof. The triangular form of the system (1.54) allows us to face the second equation $\mathcal{L}_\omega \xi^N(\theta,\varepsilon) = \eta^N(\theta,\varepsilon) - \langle \eta^N \rangle$, where $\mathcal{L}$ is given by (1.31). The right-hand side has zero average, so we can obtain the solution (1.55) with $\xi_0^N := \langle \xi \rangle$ can be any arbitrary function in $\mathbb{R}^n$. Then, the left equation is $\mathcal{L}_\omega \xi^L(\theta,\varepsilon) = \eta^L(\theta,\varepsilon) - T(\theta,\varepsilon)\xi^N(\theta,\varepsilon)$ and we choose ξ_0^N in such a way that $\langle \eta^L(\theta,\varepsilon) - T(\theta,\varepsilon)\xi^N(\theta,\varepsilon)\rangle = 0$.

It leads us to the definition of ξ_0^N defined in (1.57). Moreover, this can be determined since $\det\langle T(\theta,\varepsilon)\rangle \neq 0$ and we get the solution (1.55) with $\xi_0^L := \langle \xi^L \rangle$ can be any function with dependence on ε . □

Lemma 1.18. *If $K(\theta,\varepsilon)$ is a family of approximately invariant tori with error $E(\theta,\varepsilon)$, then*

$$\eta_0^N = \langle DE(\theta,\varepsilon) \triangle a(\theta,\varepsilon) + L(\theta+\omega,\varepsilon)^\top \triangle^2 a(\theta,\varepsilon)\rangle$$

where

$$\triangle a(\theta,\varepsilon) = a(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) - a(K(\theta+\omega,\varepsilon),\varepsilon) = \int_0^1 (Da(K(\theta+\omega)+tE(\theta,\varepsilon),\varepsilon))E(\theta,\varepsilon)\,dt,$$

$$\begin{aligned}\triangle^2 a(\theta,\varepsilon) &= a(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) - a(K(\theta+\omega,\varepsilon),\varepsilon) - Da(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon) \\ &= \int_0^1 (1-t)D^2 a(K(\theta+\omega,\varepsilon)+tE(\theta,\varepsilon),\varepsilon)E(\theta,\varepsilon)^{\otimes 2}\,dt.\end{aligned}$$

Proof. We prove for a general Ω defined in (1.5) instead of Ω_0

$$\begin{aligned}
&L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta,\varepsilon),\varepsilon)E(\theta,\varepsilon)\\
&= L(\theta+\omega,\varepsilon)^\top D_\theta a(K(\theta,\varepsilon),\varepsilon)^\top E(\theta,\varepsilon) - L(\theta+\varepsilon,\omega)^\top D_\theta a(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon)\\
&= (D_\theta(a(K(\theta+\omega,\varepsilon),\varepsilon)))^\top E(\theta,\varepsilon) + L(\theta+\omega,\varepsilon)^\top(\triangle^2 a(\theta,\varepsilon) - a(F(K(\theta,\varepsilon),\varepsilon),\varepsilon) + a(K(\theta+\omega,\varepsilon),\varepsilon))\\
&= (D_\theta(a(K(\theta+\omega),\varepsilon)^\top E(\theta,\varepsilon)))^\top - (D_\theta E(\theta,\varepsilon))^\top a(K(\theta+\omega,\varepsilon),\varepsilon) + L(\theta+\omega)^\top \triangle^2 a(\theta,\varepsilon)\\
&- (D_z F(K(\theta,\varepsilon),\varepsilon))L(\theta,\varepsilon) - D_\theta E(\theta,\varepsilon)^\top a(F(K(\theta),\varepsilon),\varepsilon) + L(\theta+\omega,\varepsilon)a(K(\theta+\omega,\varepsilon),\varepsilon)\\
&= \Big(D_\theta(a(K(\theta+\omega,\varepsilon),\varepsilon))^\top E(\theta,\varepsilon)\Big)^\top + (D_\theta E(\theta))^\top \triangle a(\theta,\varepsilon) + L(\theta+\omega,\varepsilon)\triangle^2 a(\theta,\varepsilon)\\
&- D_\theta(S(K(\theta,\varepsilon)))^\top - L(\theta,\varepsilon)a(K(\theta,\varepsilon),\varepsilon) + L(\theta+\omega)^\top a(K(\theta+\omega,\varepsilon),\varepsilon),
\end{aligned}$$

where in the last identity we use that S is the primitive function of F with

$$DS(z,\varepsilon) = a(F(z,\varepsilon),\varepsilon)^\top D_z F(z,\varepsilon) - a(z,\varepsilon)^\top, \forall z\in\mathcal{A}, \varepsilon\in U.$$

Furthermore, in the next-to-last equality we used

$$(D_\theta(a(K(\theta+\omega,\varepsilon),\varepsilon)^\top E(\theta,\varepsilon)))^\top = (D_\theta(a(K(\theta+\omega,\varepsilon),\varepsilon)))^\top E(\theta,\varepsilon) + (D_\theta E(\theta,\varepsilon))^\top a(K(\theta+\omega,\varepsilon),\varepsilon).$$

The Lemma is proved since we take the average $\langle\ \rangle$ in both sides. Moreover the average of $D_\theta((a(K(\theta,\varepsilon),\varepsilon))^\top E(\theta,\varepsilon))$ and $D_\theta(S(K(\theta,\varepsilon),\varepsilon))$ is zero due to the fact that the partial derivative of periodic function has zero average. In addition, the average of $a(K(\theta,\varepsilon),\varepsilon)^\top L(\theta+\omega,\varepsilon) - a(K(\theta,\varepsilon),\varepsilon)^\top L(\theta,\varepsilon)$ is zero. □

Now, we can apply the Lemma (1.17) to the equation (1.51) with $\eta(\theta,\varepsilon)$ and T defined in (1.53) and (1.27) respectively. Moreover, throughout this work we choose the solution satisfying $\xi_0^L = 0$. Then, we obtain the solution $\xi(\theta,\varepsilon) = (\xi^L(\theta,\varepsilon), \xi^N(\theta,\varepsilon))$ and so does $\triangle K(\theta,\varepsilon)$. They can be controlled in terms of $E(\theta,\varepsilon)$.

Once we obtain the correction by solving the linearized equation, we need to check the speed of the convergence. Note that the solution ξ, and so $\triangle K$, can be controlled in terms of $E(\theta,\varepsilon)$ as $E_{sym}(\theta,\varepsilon)$ and $E_{red}(\theta,\varepsilon)$. In addition, the terms $E_{red}(\theta,\varepsilon)\xi(\theta,\varepsilon)$ and $\Omega_0 E_{sym}(\theta+\omega,\varepsilon)\xi(\theta+\omega,\varepsilon)$ in equation (1.53) are quadratic in $E(\theta,\varepsilon)$. Let us see the following results in order to easily compute the new error associated to the new parameterization $\bar{K}$ and $\bar{E}$.

Lemma 1.19. *Let us assume that the twist condition* $\det\langle T(\theta,\varepsilon)\rangle \neq 0$ *is hold. Let us consider the solution* $\xi(\theta,\varepsilon)$ *obtained in Lemma* (1.17) *with* $\eta(\theta,\varepsilon) = \Omega_0 P(\theta+\omega)^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon)$ *and* T *given by* (1.27)*, that satisfying* $\xi_0^L = 0$*. Then, if we take* $\triangle K(\theta,\varepsilon) = P(\theta,\varepsilon)\xi(\theta,\varepsilon)$ *we have the new error of applying one step of Newton-like can be expressed in the following form*

$$\begin{aligned}
\bar{E}(\theta,\varepsilon) &= DF(K(\theta,\varepsilon))\triangle K(\theta,\varepsilon) - \triangle K(\theta+\omega,\varepsilon) + E(\theta,\varepsilon) + \triangle^2 F(\theta,\varepsilon)\\
&= P(\theta+\omega,\varepsilon)(Id - \Omega E_{sym}(\theta,\varepsilon))^{-1}E_{lin}(\theta,\varepsilon) + \triangle^2 F(\theta,\varepsilon)
\end{aligned} \tag{1.58}$$

where

$$\begin{aligned}
\triangle^2 F(\theta,\varepsilon) &= F(K(\theta,\varepsilon)+\triangle K(\theta,\varepsilon)) - F(K(\theta,\varepsilon)) - DF(K(\theta,\varepsilon))\triangle K(\theta,\varepsilon)\\
&= \int_0^1 (1-t)D^2F(K(\theta,\varepsilon)+t\triangle K(\theta,\varepsilon))\triangle K(\theta,\varepsilon)^{\otimes 2}dt
\end{aligned}$$

and

$$E_{lin}(\theta,\varepsilon) = E_{red}(\theta,\varepsilon)\xi(\theta,\varepsilon) + \Omega_0 E_{sym}(\theta+\omega)\xi(\theta+\omega) - \begin{bmatrix} 0 \\ \langle \eta^N \rangle \end{bmatrix} \tag{1.59}$$

is the error of solving linearized equation and with $\langle \eta^N \rangle = \eta_0^N = \langle L(\theta+\omega,\varepsilon)\rangle^\top \Omega(K(\theta+\omega,\varepsilon),\varepsilon)E(\theta,\varepsilon)\rangle$.

Proof. We use Lemma (1.17) to get ξ. Then, using

$$(-\Omega_0 P(\theta+\omega)^\top \Omega(K(\theta,\varepsilon)))^{-1} = P(\theta+\omega)(Id_n - \Omega_0 E_{sym}(\theta+\omega)),$$

we get

$$\begin{aligned}
&D_z F(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon)\xi(\theta,\varepsilon) - P(\theta+\omega,\varepsilon)\xi(\theta+\omega,\varepsilon) + E(\theta,\varepsilon) = \\
&\quad = P(\theta+\omega)(Id_n - \Omega_0 E_{sym}(\theta+\omega))(-\Omega_0 P(\theta+\omega)^\top \Omega(K(\theta,\varepsilon)))\big[D_z F(K(\theta,\varepsilon),\varepsilon)P(\theta,\varepsilon)\xi(\theta,\varepsilon) \\
&\quad - P(\theta+\omega,\varepsilon)\xi(\theta+\omega,\varepsilon) + E(\theta,\varepsilon)\big] \\
&\quad = P(\theta+\omega)(Id_n - \Omega_0 E_{sym}(\theta+\omega))\big[(\Lambda(\theta,\varepsilon) + E_{red}(\theta,\varepsilon))\xi(\theta,\varepsilon) \\
&\quad - (Id - \Omega_0 E_{sym}(\theta+\omega,\varepsilon))\xi(\theta+\omega,\varepsilon) - \eta(\theta,\varepsilon) + \begin{pmatrix} o \\ \langle \eta^N \rangle \end{pmatrix} - \begin{pmatrix} o \\ \langle \eta^N \rangle \end{pmatrix}\big] \\
&\quad = P(\theta+\omega)(Id_n - \Omega_0 E_{sym}(\theta+\omega))\big[E_{red}\xi(\theta,\varepsilon) + \Omega_0 E_{sym}(\theta+\omega,\varepsilon) - \begin{pmatrix} o \\ \langle \eta^N \rangle \end{pmatrix}\big]
\end{aligned}$$

□

We have already seen that the terms $E_{red}(\theta,\varepsilon)\xi(\theta,\varepsilon)$ and $\Omega_0 E_{sym}(\theta+\omega,\varepsilon)\xi(\theta+\omega)$ are quadratic in E. To conclude the right-hand side of (1.59) is quadratic and so is the average of $L(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))E(\theta,\varepsilon)$. To sum up, the correction that we adding is the same order of E and the errors as E_{sym}, E_{red} and E_{lin} are quadratically small in E.

Remark 1.20. In particular we apply the Lemma (1.18) to the two main cases.

(a) "Primary context": $\mathcal{A} = \mathbb{T}^n \times \mathbb{R}^n$

$$\begin{aligned}
\triangle a(\theta,\varepsilon) &= \begin{bmatrix} E^y(\theta,\varepsilon) \\ 0 \end{bmatrix} \\
\triangle^2 a(\theta,\varepsilon) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\langle L(\theta+\omega,\varepsilon)^\top \Omega_0 \rangle &= \langle D_\theta E^x(\theta,\varepsilon)E^y(\theta,\varepsilon)\rangle
\end{aligned}$$

(b) "Secondary context": $\mathcal{A} = \mathbb{R}^n \times \mathbb{R}^n$

$$\begin{aligned}
\triangle a(\theta,\varepsilon) &= \frac{1}{2}\begin{bmatrix} E^y(\theta,\varepsilon) \\ -E^x(\theta,\varepsilon) \end{bmatrix} \\
\triangle^2 a(\theta,\varepsilon) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\langle L(\theta+\omega,\varepsilon)^\top \Omega_0 E(\theta,\varepsilon)\rangle &= \langle DE(\theta,\varepsilon)\Omega_0 E(\theta,\varepsilon)\rangle
\end{aligned}$$

We have performed one step of Newton-like method to correct the invariance equation.

In summary, we have obtained the following iteration process from starting $K_0(\theta,\varepsilon) = K(\theta,\varepsilon)$ and $E_0(\theta,\varepsilon) = E(\theta,\varepsilon)$ and the new error $\bar{E}(\theta,\varepsilon)$ is quadratic in $E(\theta,\varepsilon)$.

1. Given an approximately invariant torus K_n starting $K_0(\theta,\varepsilon) = K(\theta,\varepsilon)$ and $E_0(\theta,\varepsilon) = E(\theta,\varepsilon)$. Evaluate the error $E_n(\theta,\varepsilon) := F(K_n(\theta,\varepsilon)) - K_n(\theta+\omega,\varepsilon)$.

2. Construct the frame P_n, for the approximately invariant torus $\triangle K_n$, using the formulas (1.38),(1.39) and (1.40).

3. Obtain $\triangle K_n(\theta,\varepsilon) = P_n(\theta,\varepsilon)\xi_n(\theta,\varepsilon)$ where we can compute $\xi_n(\theta,\varepsilon)$ from Lemma (1.17).

4. Correct the parameterization $K_{n+1} = K_n + \triangle K_n$. If we compute E_{n+1} that is small enough we stop with the parameterization K_{n+1}, otherwise we go to step 1 with K_{n+1}.

The rest of the section we will see that K_{n+1} converges to the invariant tori of frequency ω. In the following chapter we will explain how this iterative process can be implemented in a computer.

Chapter 2

A KAM Theorem For Family of Exact Symplectic maps

In this chapter we present a KAM theorem in a posteriori format for family of exact symplectic maps to a family of Lagrangian invariant tori. Roughly speaking, a KAM theorem states that if we have a good approximation of a family of invariant tori with frequency ω, then under some conditions of non-degeneracy and non-resonance one can find a family of true invariant tori nearby.

Throughout this chapter, $F : \mathcal{A} \times U \to \mathcal{A}$ is a family of exact symplectomorphisms y and $K : \mathbb{T}^n \times U \to \mathcal{A}$ is a family of parameterization of invariant tori. In the "primary context", the maps F_ε are also assumed to be homotopic to the identity, and the tori are assumed to be homotopic to the zeros section. On the other hand, in "secondary context", sysmplectic maps F_ε in a star sharped open set are exact symplectic for free, and the tori are assumed to be contractible to a point (homotopically trivial).

In the section (2.1) we recall some notions in order to define the norms that we will use to prove the KAM theorem in analytic contest. The main statement will be presented in the section (2.2). The rest of the chapter will focus on the proof.

2.1 Preliminaries in Analysis

We establish rigorous bounds for the norms of the objects, in $2n-$dimensional ambient spaces presented in the previous section. To deal with small divisor equations, we work with Banach spaces of real-analytic functions in complex neighborhoods of real domains.

2.1.1 Analytic Norms

A complex strip of $\mathbb{T}^n$ of width $\rho \geq 0$ is defined by

$$\mathbb{T}^n_\rho = \{\theta \in \mathbb{C}^n \setminus \mathbb{Z}^n : |Im\ \theta_i| < \rho,\ i = 1, \cdots, n\}.$$

Moreover, a complex strip of $\mathcal{A}$ is a complex connected open neighborhood $\mathcal{B}$ of $\mathcal{A}$.

If the ambient space $\mathcal{A} \subset \mathbb{T}^n_\rho \times \mathbb{R}^n$ is connected and homotopically non trivial (i.e $\mathcal{A}$ is an annulus), then we will consider a complex connected neighborhood $\mathcal{B} \subset (\mathbb{C}^n \setminus \mathbb{Z}^n) \times \mathbb{C}^n$ covering

$\mathcal{A}$. If the ambient space $\mathcal{A} \subset \mathbb{R}^{n\times n}$ is homotopically trivial (for instance, $\mathcal{A}$ is a ball) then we will consider a complex connected neighborhood $\mathcal{B} \subset \mathbb{C}^{n\times n}$.

We will also consider a complex set $V \subset \mathbb{C}^p$ of parameters covering the real set $U \subset \mathbb{R}^p$ of parameters.

Definition 2.1.

- *A function defined on $\mathbb{T}^n \times U$ (resp. on $\mathcal{A} \times U$), U is a open neighborhood of $\mathbb{R}^d$, is real-analytic if it can be extended to a complex strip $\mathbb{T}^n_\rho \times V$ (resp. on $\mathcal{A} \times V$) with $V \subset \mathbb{C}^d$ a complex neighborhood of real domain U.*

- *Given $\rho > 0$ we consider analytic functions $u : \bar{\mathbb{T}}^n_\rho \times \bar{V} \to \mathbb{C}$ such that $u(\mathbb{T}^n \times U) \subset \mathbb{R}$ and such that they can be continuously extended up to the boundary $\mathbb{T}^n \times U$, that means, they are real analytic in $\mathbb{T}^n_\rho \times V$. We endow these functions with the norm*

$$\|u\|_{\mathbb{T}^n_\rho \times V} = \sup_{\theta\in\mathbb{T}_\rho, \varepsilon\in V} |u(\theta,\varepsilon)| . \tag{2.1}$$

A similar norm for functions $u : \mathcal{B} \times V \to \mathbb{C}$ such that $u(\mathcal{A} \times U) \subset \mathbb{R}$, given by

$$\|u\|_{\mathcal{B}\times V} = \sup_{z\in\mathcal{B}, \varepsilon\in V} |u(z,\varepsilon)| . \tag{2.2}$$

These sets of functions, endowed with the corresponding norms (2.1) and (2.2), are Banach spaces. On the other hand, the set of matrix of analytic functions with the norm (2.3) is the induced norm on vector of matrix valued functions.

Definition 2.2. *If A is an $n_1 \times n_2$ matrix of analytic functions on $\mathbb{T}^n_\rho \times V$ (resp. on $\mathcal{B} \times V$), we extended the norm* (2.1) *as follows:*

$$\|A\|_{\mathbb{T}^n_\rho \times V} = \max_{i=1,\cdots,n_1} \sum_{j=1}^{n_2} \|A_{i,j}\|_{\mathbb{T}^n_\rho \times V} \tag{2.3}$$

Along the proof of the KAM theorem, we need to control the norms (2.3) for the objects involved in the Newton correction described in the section (1.3). For instance,

$$\|D_z F\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \frac{\partial F_i}{\partial z_j} \right\|_{\mathcal{B}\times V}, \quad \left\|D_z^2 F\right\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j,k=1}^{2} \left\| \frac{\partial F_i}{\partial z_j \partial z_k} \right\|_{\mathcal{B}\times V} \tag{2.4}$$

$$\|D_z a\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \frac{\partial a_i}{\partial z_j} \right\|_{\mathcal{B}\times V}, \quad \left\|D_z^2 a\right\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \frac{\partial^2 a_i}{\partial z_j \partial z_k} \right\|_{\mathcal{B}\times V} \tag{2.5}$$

$$\|\Omega\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \|\Omega_{i,j}\|_{\mathcal{B}\times V}, \quad \|D_z \Omega\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \frac{\partial \Omega_{i,j}}{\partial z_k} \right\|_{\mathcal{B}\times V} \tag{2.6}$$

$$\|J\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \|J_{i,j}\|_{\mathcal{B}\times V}, \quad \|D_z J\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \frac{\partial J_{i,j}}{\partial z_k} \right\|_{\mathcal{B}\times V}, \tag{2.7}$$

$$\|G\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \|G_{i,j}\|_{\mathcal{B}\times V}, \quad \|D_z G\|_{\mathcal{B}\times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \frac{\partial G_{i,j}}{\partial z_k} \right\|_{\mathcal{B}\times V} \tag{2.8}$$

where $F : \mathcal{A} \times U \to \mathcal{A}$.

2.1.2 Cauchy and Rüssmann Estimates

In order to control functions on the complex torus $\mathbb{T}^n_\rho \times V$ that are modified along the iteration, we will use the following Lemmas.

Lemma 2.3. ***(Cauchy estimates)*** *Let $u : \mathbb{T}^n_\rho \times V \to \mathbb{C}$ be an analytic functions, with $\rho > 0$, and continuous up to the boundary. Then, for any $0 < \delta < \rho$ we obtain that partial derivative $\frac{\partial u}{\partial \theta_j}$ is analytic in $\mathbb{T}^n_{\rho-\delta}$ and continuous up to the boundary with following inequality*

$$\left\| \frac{\partial u}{\partial \theta_j} \right\|_{\mathbb{T}^n_{\rho-\delta}\times V} \leq \frac{1}{\delta} \left\| u \right\|_{\mathbb{T}^n_\rho \times V} .$$

This estimate is extended to vector functions $u : \mathbb{T}^n_\rho \times V \to \mathbb{C}^m$ as follows:

$$\| Du \|_{\mathbb{T}^n_{\rho-\delta}\times V} = \max_{i=1,\cdots,m} \sum_{j=1}^{n} \left\| \frac{\partial u_i}{\partial \theta_j} \right\|_{\mathbb{T}^n_{\rho-\delta}\times V} \leq \frac{n}{\delta} \| u \|_{\mathbb{T}^n_\rho \times V}$$

$$\left\| Du^\top \right\|_{\mathbb{T}^n_{\rho-\delta}\times V} \leq \frac{m}{\delta} \| u \|_{\mathbb{T}^n_\rho \times V}$$

Once we assume the Diophantine properties we find that equation (1.31) can be solved in analytic way using the following well-known result. The proof is a standard argument that can be found in [DlL01],[Rüs75] and [Rüs76].

Lemma 2.4. ***(Rüssman estimates)*** *Let $\omega \in \mathbb{R}^n$ be Diophantine of type (γ, τ), for some $\gamma > 0$ and $\tau \geq n$. Then, for any analytic function $v : \mathbb{T}^n_\rho \times V \to \mathbb{C}$, with $\rho > 0$ and $\|v\|_{\mathbb{T}^n_\rho \times V} < \infty$, there exists a unique zero-average analytic solution $u : \mathbb{T}^n_\rho \times V \to \mathbb{C}$ of $\mathcal{L}_\omega u = v - \langle v \rangle$. Moreover, there exists a constant c_R such that for any $0 < \delta < \rho$,*

$$\|u\|_{\mathbb{T}^n_{\rho-\delta}\times V} \leq \frac{c_R}{\gamma \delta^\tau} \|v\|_{\mathbb{T}^n_\rho \times V} \tag{2.9}$$

The constant c_R(depending only on τ and n) is given by

$$c_R = \frac{\sqrt{2^{n-3}\xi(2, 2\tau)\Gamma(2\tau + 1)}}{(2\pi)^\tau}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is the gamma function and $\xi = (s, q) = \sum_{m=0}^\infty \frac{1}{(q+m)^s}$ is the Hurwitz zeta function.

2.1.3 Neumann Series

In order to control the invertibility of matrices and the non-degeneracy condition, we will use Neumann series. A Neumann series is series of the form

$$\sum_{k=0}^{\infty} Y^k, \tag{2.10}$$

where Y is a square matrix with real or complex coefficients.

If for a matrix norm $\|\cdot\|$, we have $\|Y\| < 1$ then $(Id - Y)$ is invertible and

$$(Id - Y)^{-1} = \sum_{k=0}^{\infty} Y^k, \tag{2.11}$$

where Id is the identity matrix and

$$\left\|(Id - Y)^{-1}\right\| \leq \frac{1}{1 - \|Y\|}.$$

Example 2.5. This result on operator is analogous to geometric series in $\mathbb{R}$, in which we find

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k. \tag{2.12}$$

Lemma 2.6. *Let X and Y to be two square matrices of same size. If X is invertible and $\|X - Y\| < \left\|X^{-1}\right\|^{-1}$, then Y is also invertible and*

$$\left\|Y^{-1}\right\| \leq \frac{\|X^{-1}\|}{1 - \|X - Y\| \, \|Y^{-1}\|}. \tag{2.13}$$

Proof. Since $\left\|X^{-1}(Y - X)\right\| < 1$, then $X^{-1}Y = (Id + X^{-1}(Y - X))$ is invertible. Therefore, Y is invertible and

$$Y^{-1} = (Id + X^{-1}(Y - X))^{-1} X^{-1}. \tag{2.14}$$

The bound of $\left\|Y^{-1}\right\|$ follows the Neumann series. □

2.2 The KAM Theorem

In this section we will state sufficient conditions to guarantee the existence of a family of F-invariant tori with fixed Diophantine frequency close to a family of approximately invariant tori. The following theorem is the main result in this work.

Theorem 2.7. *Consider an exact symplectic structure $\omega = d\alpha$ on the ambient space $\mathcal{A}$, an exact symplectic map $F : \mathcal{A} \times U \to \mathcal{A}$ and a frequency vector $\omega \in \mathbb{R}^n$. Let us assume that the following hypotheses hold:*

H1 The family of compatible triples $(\Omega,\ J,\ G)$ of the parameter depending exact symplectic form given by Ω, a the matrix representation of the action form and the map F are real analytic and can be analytically extended to $\mathcal{B} \times V$. That is:

- *(i) $a : \mathcal{B} \times V \to \mathbb{C}^{2n}$ real-analytic.*
- *(ii) $\Omega,\ J,\ G : \mathcal{B} \times V \to \mathbb{C}^{2n \times 2n}$ real-analytic such that $\Omega(z, \varepsilon) = (D_z a(z, \varepsilon))^\top - D_z a(z, \varepsilon)$, $J^2 = -Id_{2n}$ and $\Omega(z, \varepsilon) = G(z, \varepsilon) J(z, \varepsilon)$ with G positive definite.*
- *(iii) $F : \mathcal{B} \times V \to \mathcal{B}$ real analytic.*

Moreover, that there are constants $c_{F,1}, c_{F,2}, c_{\Omega,0}, c_{\Omega,1}, , c_{G,0}, c_{G,1}, c_{J,0}, c^*_{J,0}, c_{J,1}, c^*_{J,1}, c_{a,1}$ and $c_{a,2}$ such that
$\|D_z F\|_{\mathcal{B} \times V} \leq c_{F,1}$, $\left\|D_z^2 F\right\|_{\mathcal{B} \times V} \leq c_{F,2}$, $\|\Omega\|_{\mathcal{B} \times V} \leq c_{\Omega,0}$, $\|D_z \Omega\|_{\mathcal{B} \times V} \leq c_{\Omega,1}$, $\|D_z a\|_{\mathcal{B} \times V} \leq c_{a,1}$, $\|G\|_{\mathcal{B} \times V} \leq c_{G,0}$, $\|D_z G\|_{\mathcal{B} \times V} \leq c_{G,1}$, $\|J\|_{\mathcal{B} \times V} \leq c_{J,0}$, $\left\|J^\top\right\|_{\mathcal{B} \times V} \leq c^*_{J,0}$, $\|D_z J\|_{\mathcal{B} \times V} \leq c_{J,1}$, $\left\|D_z J^\top\right\|_{\mathcal{B} \times V} \leq c^*_{J,1}$ and $\left\|D_z^2 a\right\|_{\mathcal{B} \times V} \leq c_{a,2}$.

H2 *There exists* $K : \hat{\mathbb{T}}^n \to \mathcal{B}$, *with* $\rho > 0$ *and there exist constants* σ_L *and*σ_L^* *such that*

$$\|D_\theta K\|_{\mathbb{T}^n_\rho \times V} < \sigma_L, \quad \left\|D_\theta K^\top\right\|_{\mathbb{T}^n_\rho \times V} < \sigma_L^*, \quad dist(K(\mathbb{T}^n_\rho \times V), \partial\mathcal{B}) > 0. \tag{2.15}$$

where we define the distance, given two subsets $X, Y \in \mathbb{C}^2$*, as following*

$$dist(X,Y) = \inf\{|x-y|, x \in X, y \in Y\}, \tag{2.16}$$

with $|\cdot|$ *is the maximum norm.*

H3 *There exists a map* $G_k : \mathbb{T}^n_\rho \times V \to \mathbb{C}^{2n\times 2n}$ *defined in* (1.22) *satisfies the invertibility condition*

$$\|B\|_{\mathbb{T}^n_\rho \times V} = \left\|G_K^{-1}\right\|_{\mathbb{T}^n_\rho \times V} < \sigma_B.$$

H4 *There exists a constant* σ_T *such that the matrix-valued map* T (1.27) *satisfies the dynamical non-degeneracy condition* $\left|\langle T\rangle^{-1}\right| < \sigma_T$.

H5 *The frequency vector* ω *satisfies Diophantine conditions of type* (γ, τ).

Then, for every $0 < \rho_\infty < \rho$ *there exists a constant* $\hat{C}_*$ *such that if*

$$\frac{\hat{C}_* \|E\|_{\mathbb{T}^n_\rho \times V}}{\gamma^4 \rho^{4\tau}} < 1 \tag{2.17}$$

then there exists a family of $F-$*invariant tori* $K_\infty(\mathbb{T}^n \times U)$*, with the same frequency* ω*, analytic in* $\mathbb{T}^n_{\rho_\infty} \times V$*, that satisfies*

$$\|D_\theta K_\infty\|_{\mathbb{T}^n_{\rho_\infty} \times V} < \sigma_L, \quad \left\|D_\theta K_\infty^\top\right\|_{\mathbb{T}^n_{\rho_\infty} \times V} < \sigma_L^*, \quad dist(K_\infty(\mathbb{T}^n_{\rho_\infty} \times V), \partial\mathcal{B}) > 0. \tag{2.18}$$

Furthermore, the tori $K_\infty(\mathbb{T}^n_{\rho_\infty} \times V)$ *is close to the original approximation, in the sense that there exists a constant* $\hat{C}_{**}$ *such that*

$$\|K_\infty - K\|_{\mathbb{T}^n_{\rho_\infty} \times V} < \frac{\hat{C}_{**}}{\gamma^2 \rho^{2\tau}} \|E\|_{\mathbb{T}^n_\rho \times V}. \tag{2.19}$$

Remark 2.8. Notice that $B(\theta,\varepsilon) = G_K^{-1}(\theta,\varepsilon)$ such that $B(\theta,\varepsilon) = B(\theta,\varepsilon)^\top$. Then $\|B\|_{\mathbb{T}^n_\rho \times V} = \left\|G_K^{-1}\right\|_{\mathbb{T}^n_\rho \times V} < \sigma_B^* = \sigma_B$. We left the the notations σ_B^* in order to compare with mamotreto. In addition, the expressions of $\hat{C}_*$ and $\hat{C}_{**}$ will be explicitly given in the proof and they depend explicitly on the initial data.

Remark 2.9. The hypothesis $H1$ requires the conditions on the dynamics system, $H2$ requires the sufficient conditions on the parameterization K. Moreover, $H3$ is asking for geometrical non-degeneracy condition on adapted frame. The last hypothesis $H4$ is asking the twist condition on the mean value of T.

2.3 One Step of the Newton-Like Method

In this section we aim to prove the theorem (2.7). The following Lemma is the so-called the iterative Lemma that will play an important role in the proof. Roughly speaking the ideas of the proof consists in performing one iteration of the Newton-like method to correct the parameterization of the tori by means of the construction presented in (2.3).

Concretely, after applying the Newton-like method using the following Lemma to control the analytic norms of the involved objects and then see the convergence in the next section.

Lemma 2.10. ***(The Iterative Lemma)*** *Consider the same setting and hypotheses of the Theorem* (2.7)*. Then, there exists constants $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_3^*, \hat{C}_4$ and $\hat{C}_5$ (depending explicitly on the constants defined in the hypotheses) such that if*

$$\frac{\hat{C}_1 \|E\|_{\mathbb{T}^n_\rho} \times V}{\gamma^2 \delta^{2\tau+1}} < 1 \tag{2.20}$$

for some $0 < \delta < \rho$, then we have a family of approximately invariant tori of the same frequency ω given by $\overline{K} = K + \triangle K$, that defines new objects $\overline{B}$ and $\overline{T}$ (obtained replacing K by $\overline{K}$) satisfying

$$\|D\overline{K}\|_{\mathbb{T}^n_{\rho-3\delta}\times V} < \sigma_L, \quad \left\|D\overline{K}^\top\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V} < \sigma_L^*, \quad dist(\overline{K}(\mathbb{T}^n_{\rho-3\delta} \times V), \partial\mathcal{B}) > 0 \tag{2.21}$$

$$\|\overline{B}\|_{\mathbb{T}_{\rho-3\delta}\times V} < \sigma_B, \quad \left|\langle\overline{T}\rangle^{-1}\right| < \sigma_T, \tag{2.22}$$

and

$$\|\overline{K} - K\|_{\mathbb{T}^n_{\rho-2\delta}\times V} < \frac{\hat{C}_2}{\gamma^2\delta^{2\tau}} \|E\|_{\mathbb{T}^n_\rho\times V}, \tag{2.23}$$

$$\|\overline{B} - B\|_{\mathbb{T}^n_{\rho-3\delta}\times V} < \frac{\hat{C}_3}{\gamma^2\delta^{2\tau+1}} \|E\|_{\mathbb{T}^n_\rho\times V}, \tag{2.24}$$

$$\left|\langle\overline{T}\rangle^{-1} - \langle T\rangle^{-1}\right| < \frac{\hat{C}_4}{\gamma^2\delta^{2\tau+1}} \|E\|_{\mathbb{T}^n_\rho\times V}. \tag{2.25}$$

The new error of the invariance is given by

$$\overline{E}(\theta,\varepsilon) = F(\overline{K}(\theta,\varepsilon),\varepsilon) - \overline{K}(\theta+\omega,\varepsilon), \tag{2.26}$$

and satisfies

$$\|\overline{E}\|_{\mathbb{T}^n_{\rho-3\delta}\times V} < \frac{\hat{C}_5}{\gamma^2\delta^{4\tau}} \|E\|_{\mathbb{T}^n_\rho\times V}. \tag{2.27}$$

Remark 2.11. The constants $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_3^*, \hat{C}_4$ and $\hat{C}_5$ are explicitly given in the proof, since it is a constructive proof.

Proof. Essentially we have to estimate the norms of the objects involved in the one step of the Newton-like method to correct the invariance of the torus. We will use quiet often Lemma (2.3) and (2.4) and the properties of the Banach algebras defined by the norms (2.1) and (2.2).

We follow the computations and notation in [HCF+16]. We will even keep some notation for making easier the comparison.

We start by controlling the objects $L(\theta,\varepsilon), N(\theta,\varepsilon)$ and $B(\theta,\varepsilon)$. By the hypothesis, we have $\|L\|_{\mathbb{T}^n_\rho\times V} < \sigma_L$ and $\|B\|_{\mathbb{T}^n_\rho\times V} < \sigma_B$ and $\sigma_B = \sigma_B^*$. Then

$$\left\|N^0\right\|_{\mathbb{T}^n_\rho\times V} \leq \|J\|_{\mathcal{B}\times V} \|D_z K\|_{\mathbb{T}^n_\rho\times V} \leq c_{J,0}\sigma_L =: c_{N,0},$$

$$\left\|N_0^\top\right\|_{\mathbb{T}^n_\rho\times V} \leq \left\|D_z K^\top\right\|_{\mathbb{T}^n_\rho\times V} \left\|J^\top\right\|_{\mathcal{B}\times V} \leq \sigma_L^* c_{J,0}^* =: c_{N,0},$$

Then, we also need to control $\left\|\bar{N}^0 - N^0\right\|_{\mathbb{T}^n_\rho \times V}$ later and

$$\|N\|_{\mathbb{T}^n_\rho \times V} = \left\|N^0\right\|_{\mathbb{T}^n_\rho \times V} \|B\|_{\mathbb{T}^n_\rho \times V} \leq c_{N^0}\sigma_B =: c_N, \quad \left\|N^\top\right\|_{\mathbb{T}^n_\rho \times V} \leq \sigma_B^* c_{N^0}^* =: c_N^* \tag{2.28}$$

Then, the matrix $P(\theta, \varepsilon)$:

$$\|P\|_{\mathbb{T}^n_\rho \times V} \leq \|L\|_{\mathbb{T}^n_\rho \times V} + \|N\|_{\mathbb{T}^n_\rho \times V} \leq \sigma_L + c_N =: c_P, \tag{2.29}$$

and the torsion matrix $T(\theta, \varepsilon)$

$$\|T\|_{\mathbb{T}^n_\rho \times V} \leq \left\|N^\top\right\|_{\mathbb{T}^n_\rho \times V} \|\Omega\|_{\mathcal{B} \times V} \|D_z F\|_{\mathcal{B} \times V} \|N\|_{\mathbb{T}^n_\rho \times V} \leq c_N^* c_{\Omega,0} c_{F,1} c_N =: c_T. \tag{2.30}$$

Now we estimate the norm of Ω_K. First of all, we get

$$\|\triangle\Omega\|_{\mathbb{T}^n_\rho \times V} = \max_{i=1,\cdots,2n} \sum_{j=1}^{2n} \left\| \int_0^1 \sum_{k=1}^{2n} \frac{\partial \Omega_{i,j}}{\partial z_k} (K(\theta+\omega, \varepsilon), \varepsilon + tE(\theta,\varepsilon) E(\theta,\varepsilon)\, dt \right\|_{\mathbb{T}^n_\rho \times V} \tag{2.31}$$

$$\leq \max_{i=1,\cdots,2n} \sum_{j,k} \left\| \frac{\partial \Omega_{i,j}}{\partial z_k} \right\|_{\mathcal{B}} \|E\|_{\mathbb{T}^n_\rho \times V} \leq c_{\Omega,1} \|E\|_{\mathbb{T}^n_\rho \times V} \tag{2.32}$$

and using (1.42) we get

$$\|\mathcal{L}\Omega_K\|_{\mathbb{T}^n_{\rho-\delta} \times V} \leq (\sigma_L^* \sigma_L c_{\Omega,1} \delta + n\sigma_L^* c_{\Omega,0} + 2nc_{\Omega,0} c_{F,1} \sigma_L) \frac{\|E\|_{\mathbb{T}^n_\rho \times V}}{\delta} =: \frac{C_1}{\delta} \|E\|_{\mathbb{T}^n_\rho \times V} \tag{2.33}$$

Then, applying these estimates to get the norm of Ω_K using Rüssmann estimates we end up with

$$\|\Omega_K\|_{\mathbb{T}^n_{\rho-2\delta} \times V} \leq \frac{c_R C_1}{\gamma \delta^{\tau+1}} \|E\|_{\mathbb{T}^n_\rho \times V} =: \frac{C_2}{\gamma \delta^{\tau+1}} \|E\|_{\mathbb{T}^n_\rho \times V}. \tag{2.34}$$

Notice that after applying Cauchy's estimates and Rüssman estimates the analytic bound has been reduced form $\mathbb{T}^n_\rho \times V$ to $\mathbb{T}^n_{\rho-2\delta} \times V$.

Then, the error in the symplectic character of the frame is as following

$$\|E_{sym}\|_{\mathbb{T}^n_{\rho-2\delta} \times V} \leq \frac{C_2}{\gamma \delta^{\tau+1}} \|E\|_{\mathbb{T}^n_\rho \times V} =: \frac{C_3}{\gamma \delta^{\tau+1}} \|E\|_{\mathbb{T}^n_\rho \times V} \tag{2.35}$$

We keep the notation to compare with [HCF⁺16].

Remark 2.12. In $2D$ case $C_1 = C_2 = C_3 = 0$

Next we control the norm of the error reducibility E_{red} defined in (1.47)

$$\|E_{red}\|_{\mathbb{T}^n_{\rho-2\delta} \times V} = \max_{i=1,2} \sum_{j=1}^{2} \left\|E_{red}^{i,j}\right\|_{\mathbb{T}^n_{\rho-2\delta} \times V}$$

we used the Lemma (2.3) in order to get the estimates

$$C_4 := nc_N^* c_{\Omega,0} \gamma \delta^\tau + c_A C_2, \tag{2.36}$$

$$C_5 := C_2 + n\sigma_L^* c_{\Omega,0} \gamma \delta^\tau, \tag{2.37}$$

$$C_6 := c_A C_2 + \sigma_L^* c_{\Omega,1} c_{F,1} c_N \gamma \delta^{\tau+1} + 2nc_{\Omega,0} c_{F,1} c_N \gamma \delta^\tau, \tag{2.38}$$

Thus,

$$\|E_{red}\|_{\mathbb{T}^n_{\rho-2\delta}\times V} \leq \frac{\max\{C_4, C_5+C_6\}}{\gamma\delta^\tau}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{C_7}{\gamma\delta^{\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V}.$$

Now we need to estimate on the correction $\triangle K(\theta,\varepsilon) = P(\theta,\varepsilon)\xi(\theta,\varepsilon)$. To this end, we use the Lemma (1.17) with $\eta(\theta,\varepsilon) = \Omega_0 P(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega))E(\theta,\varepsilon)$ and the torsion matrix $T(\theta,\varepsilon)$, recalling that we select

$$\xi_0^N = \langle T\rangle^{-1}\langle \eta^L - T\Re(\eta^N)\rangle, \quad \xi_0^L = 0. \tag{2.39}$$

Then, we need to compute

$$\left\|\eta^L\right\|_{\mathbb{T}^n_\rho\times V} = \left\|N(\theta+\omega,\varepsilon)^\top\Omega(K(\theta,\varepsilon),\varepsilon)E(\theta,\varepsilon)\right\|_{\mathbb{T}^n_\rho\times V} \leq c_N^* c_{\Omega,0}\|E\|_{\mathbb{T}^n_\rho\times V}, \tag{2.40}$$

$$\left\|\eta^N\right\|_{\mathbb{T}^n_\rho\times V} = \left\|L(\theta+\omega,\varepsilon)^\top\Omega(K(\theta,\varepsilon),\varepsilon)E(\theta,\varepsilon)\right\|_{\mathbb{T}^n_\rho\times V} \leq \sigma_L^* c_{\Omega,0}\|E\|_{\mathbb{T}^n_\rho\times V}. \tag{2.41}$$

Then, using Rüssman in (2.41), we get

$$\left\|\Re(\eta^N)\right\|_{\mathbb{T}^n_{\rho-\delta}\times V} \leq \frac{c_R\sigma_L^* c_{\Omega,0}}{\gamma\delta^\tau}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{C_8}{\gamma\delta^\tau}\|E\|_{\mathbb{T}^n_\rho\times V}. \tag{2.42}$$

From the defition (1.57)of ξ_0^N, we know

$$\begin{aligned}\left\|\xi_0^N\right\|_V &\leq \sigma_T\left(\left\|\eta^L\right\|_{\mathbb{T}^n_\rho\times V} + c_T\frac{C_8}{\gamma\delta^{tau}}\|E\|_{\mathbb{T}^n_\rho\times V}\right)\\ &\leq \frac{\sigma_T\left(c_N^* c_{\Omega,0}\gamma\delta^\tau + c_T C_8\right)}{\gamma\delta^\tau}\|E\|_{\mathbb{T}^n_\rho\times V},\end{aligned}$$

then, use (2.39), hypothesis $H4$ and again Rüssman estimates in (2.40) thus getting ξ^N

$$\left\|\xi^N\right\|_{\mathbb{T}^n_{\rho-\delta}\times V} \leq \frac{C_8+\sigma_T(c_N^* c_{\Omega,0}\gamma\delta^\tau + c_T C_8)}{\gamma\delta^{\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{C_9}{\gamma\delta^{\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V}, \tag{2.43}$$

Analogous to ξ^L, defined in (1.56),

$$\left\|\xi^L\right\|_{\mathbb{T}^n_{\rho-2\delta}\times V} \leq \frac{c_R(c_N^* c_{\Omega,0}\gamma\delta^\tau + c_T C_9)}{\gamma^2\delta^{2\tau}}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{C_{10}}{\gamma^2\delta^{2\tau}}\|E\|_{\mathbb{T}^n_\rho\times V}.$$

Now, let control the norm of t new parameterization and the related objects.

$$\|\overline{K}-K\|_{\mathbb{T}^n_{\rho-2\delta}\times V} = \|\triangle K\|_{\mathbb{T}^n_{\rho-2\delta}\times V} \leq \frac{\sigma_L C_{10} + c_N C_9\gamma\delta^\tau}{\gamma^2\delta^{2\tau}}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{\hat{C}_2}{\gamma^2\delta^{2\tau}}\|E\|_{\mathbb{T}^n_\rho\times V} \tag{2.44}$$

using this expression and Cauchy estimates (Lemma (2.3)), we obtain the first estimate in (2.21)

$$\|D_\theta\overline{K}\|_{\mathbb{T}^n_{\rho-3\delta}\times V} \leq \|D_\theta K\|_{\mathbb{T}^n_\rho\times V} + \|D\triangle K\|_{\mathbb{T}^n_{\rho-3\delta}\times V} \leq \|D_\theta K\|_{\mathbb{T}^n_\rho\times V} + \frac{n\hat{C}_2}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} < \sigma_L \tag{2.45}$$

The last inequality in the previous computation is obtained by including this condition in hypothesis (2.20) (see the explicit form of $\hat{C}_1$ in (2.64)). The control of the transposed object in (2.21),

$$\left\|D\overline{K}^\top\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V} \leq \left\|D_\theta K^\top\right\|_{\mathbb{T}^n_\rho\times V} + \frac{2n\hat{C}_2}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} < \sigma_L^*. \tag{2.46}$$

Once we control the partial derivative of $\bar{K}$, we can control $\left\|\bar{N}^0 - N^0\right\|_{\mathbb{T}^n_\rho \times V}$

$$\begin{aligned}\left\|\bar{N}^0 - N^0\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V} &\leq \|J(\bar{K})D_\theta\bar{K} - J(K)D_\theta K\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &\|(J(\bar{K}) - J(K))D_\theta\bar{K}\|_{\mathbb{T}^n_{\rho-3\delta}\times V} + \|J(\bar{K})(D_\theta\bar{K} - D_\theta K)\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &\leq c_{J,1}\|\triangle K\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\|D_\theta\bar{K}\|_{\mathbb{T}^n_{\rho-3\delta}\times V} + c_{J,0}\|D_\theta\triangle K\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &\leq \frac{\hat{C}_2(c_{J,1}\sigma_L\delta + nc_{J,0})}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{c_{N^0}}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V}.\end{aligned} \tag{2.47}$$

Doing similar operations, we obtain

$$\left\|\bar{N}^0 - N^0\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V} \leq \frac{2n\hat{C}_2(\sigma_L^* c_{J,1}^*\delta + c_{J,0}^*)}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{c^*_{N^0}}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V}. \tag{2.48}$$

In order to control $\overline{B}$ and $\langle\overline{T}\rangle^{-1}$ we need to use that for matrices X and Y such that X is invertible and Y close enough to X, then Y is invertible defined in (2.14) and the bound norm is given by (2.13).

First, we compute we use expression (2.13) to $X = G_{\bar{K}}$ and $Y = G_K$ in order to get the norm of B and $\overline{B}$.

$$\begin{aligned}\|G_{\bar{K}} - G_K\|_{\mathbb{T}^n_{\rho-3\delta}\times V} &= \left\|D_\theta\bar{K}^\top G(\bar{K})D_\theta\bar{K} - D_\theta K^\top G(K)D_\theta K\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &\leq \left\|D_\theta\bar{K}^\top G(\bar{K})D_\theta\bar{K} - D_\theta\bar{K}^\top G(\bar{K})D_\theta K\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &+ \left\|D_\theta\bar{K}^\top G(\bar{K})D_\theta K - D_\theta\bar{K}^\top G(K)D_\theta K\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &+ \left\|D_\theta\bar{K}^\top G(K)D_\theta K - D_\theta K^\top G(K)D_\theta K\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\\ &\leq \sigma_L^* c_{G,0}\|\triangle D_\theta K\|_{\mathbb{T}^n_{\rho-3\delta}\times V} + \sigma_L^* c_{G,1}\|\triangle K\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\sigma_L + \left\|D_\theta\triangle K^\top\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V} c_{G,0}\sigma_L\\ &\leq \sigma_L^* c_{G,0}\frac{n\hat{C}_2}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} + \sigma_L^* c_{G,1}\frac{\hat{C}_2}{\gamma^2\delta^{2\tau}}\|E\|_{\mathbb{T}^n_\rho\times V}\sigma_L + \frac{2n\hat{C}_2}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} c_{G,0}\sigma_L\\ &\leq \frac{\hat{C}_2(n\sigma_L^* c_{G,1} + \sigma_L^* c_{G,1}\sigma_L\delta + 2nc_{G,0}\sigma_L)}{\gamma^2\delta^{2\tau+1}} =: \frac{C_{11}}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V}.\end{aligned} \tag{2.49}$$

Since G is symmetric

$$\left\|G^\top_{\bar{K}} - G^\top_K\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V} \leq \frac{C_{11}}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} =: \frac{C^*_{11}}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V}, \tag{2.50}$$

We keep the notation in order to compare with [HCF+16]. Moreover, we ask $2\left\|X^{-1}\right\|\|Y - X\| < 1$ and $2\left\|X^{-T}\right\|\left\|Y^\top - X^\top\right\| < 1$, that means, we ask the following conditions to be included in (2.20),

$$2\frac{\sigma_B C_{11}}{\gamma^2\delta^{2\tau+1}}\|E\|_{\mathbb{T}^n_\rho\times V} < 1, \tag{2.51}$$

in order to get $\left\|Y^{-1} - X^{-1}\right\| \leq \frac{\left\|X^{-1}\right\|^2\|Y-X\|}{1-\left\|X^{-1}\right\|\|Y-X\|} \leq 2\left\|X^{-1}\right\|^2\|Y - X\|$.

$$\left\|\overline{B}-B\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}=\left\|Y^{-1}-X^{-1}\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\leq\frac{2\sigma_B^2C_{11}}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V}=:\frac{\hat{C}_3}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V} \tag{2.52}$$

Since $B(\theta,\varepsilon)=B^\top(\theta,\varepsilon)$ then we have $\hat{C}_3=\hat{C}_3^*$. Having all the objects in control, we want to get the second estimate in (2.21)

$$\left\|\overline{B}\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\leq\left\|B\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}+\left\|\overline{B}-B\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\leq\left\|B\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}+\frac{\hat{C}_3}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V}<\sigma_B, \tag{2.53}$$

where the last inequality is included in (2.20).

Secondly, we present $X=T$ and $Y=\overline{T}$ in order to hold the equation (2.25). As consequence, we need to control the new matrices $\overline{N}(\theta,\varepsilon)$.

$$\left\|\overline{N}-N\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\leq\frac{c_{N^0}\hat{C}_3}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V}=:\frac{C_{12}}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V} \tag{2.54}$$

and

$$\left\|\overline{N}^\top-N^\top\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\leq\frac{c^*_{N^0}\hat{C}^*_3}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V}=:\frac{C^*_{12}}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V} \tag{2.55}$$

Then,

$$\begin{aligned}\overline{T}-T=&\overline{N}^\top(\theta+\omega,\varepsilon)\Omega(\overline{K}(\theta+\omega,\varepsilon))D_zF(\overline{K}(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&-N^\top(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon)\\=&\overline{N}^\top(\theta+\omega,\varepsilon)\Omega(\overline{K}(\theta+\omega,\varepsilon))D_zF(\overline{K}(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&-N^\top(\theta+\omega,\varepsilon)\Omega(\overline{K}(\theta+\omega,\varepsilon))D_zF(\overline{K}(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&+N^\top(\theta+\omega,\varepsilon)\Omega(\overline{K}(\theta+\omega,\varepsilon))D_zF(\overline{K}(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&-N^\top(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))D_zF(\overline{K}(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&+N^\top(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))D_zF(\overline{K}(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&-N^\top(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))D_zF(K(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&+N^\top(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))D_zF(K(\theta,\varepsilon),\varepsilon)\overline{N}(\theta,\varepsilon)\\&-N^\top(\theta+\omega,\varepsilon)\Omega(K(\theta+\omega,\varepsilon))D_zF(K(\theta,\varepsilon),\varepsilon)N(\theta,\varepsilon)\end{aligned}$$

we get

$$\left\|\overline{T}-T\right\|_{\mathbb{T}^n_{\rho-3\delta}\times V}\leq\frac{C_{13}}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V} \tag{2.56}$$

with

$$C_{13}:=c_N^*c_N\hat{C}_2(c_{\Omega,0}c_{F,2}+c_{\Omega,1}c_{F,1})\delta+c_{\Omega,0}c_{F,1}(c_N^*C_{12}+c_NC_{12}^*).$$

Using the equation (2.13) and (2.56), we obtain the estimate in (2.25)

$$\left|\overline{T}^{-1}-T^{-1}\right|\leq 2\frac{\sigma_T^2C_{13}}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V} \tag{2.57}$$

Computations are analogous to those performed to control the object $\overline{B}$ and we have to ask the last inequality

$$\left|\overline{T}^{-1}\right|\leq\left|T^{-1}\right|+2\frac{\sigma_T^2C_{13}}{\gamma^2\delta^{2\tau+1}}\left\|E\right\|_{\mathbb{T}^n_\rho\times V}<\sigma_T \tag{2.58}$$

Note that the closure of $\overline{K}(\mathbb{T}^n_{\rho-2\delta} \times V)$ lies in $\mathcal{B}$, since

$$\begin{aligned} dist(\overline{K}(\mathbb{T}^n_{\rho-2\delta} \times V), \partial\mathcal{B}) &\geq dist(\overline{K}(\mathbb{T}^n_{\rho} \times V), \partial\mathcal{B}) - \|\triangle K\|_{\mathbb{T}^n_{\rho-2\delta}\times V} \\ &\geq dist(\overline{K}(\mathbb{T}^n_{\rho} \times V), \partial\mathcal{B}) - \frac{\hat{C}_2}{\gamma^2\delta^{2\tau}} \|E\|_{\mathbb{T}^n_{\rho}\times V} > 0, \end{aligned} \tag{2.59}$$

where the last inequality is also included in (2.20).

At this point we have bounded the norms (2.21)-(2.25) and we have to estimate the error. Now, we use the expression in Lemma (1.19) to control the modulus of the average

$$\left|\langle \eta^N \rangle\right|_V \leq \left|\langle L(\theta+\omega,\varepsilon)^\top \Omega(K(\theta+\omega,\varepsilon))E(\theta,\varepsilon)\rangle\right|_V \leq \left(\frac{2nc_{a,1}}{\delta} + \frac{c_{a,2}}{2}\right) \|E\|^2_{\mathbb{T}^n_{\rho}\times V}$$

and then we control the norm of $E_{lin}(\theta,\varepsilon)$ in Lemma (1.18),

$$\|E_{lin}\|_{\mathbb{T}^n_{\rho-2\delta}\times V} \leq \left(\frac{(C_3+C_7)\max\{C_9\delta\gamma^\tau, C_{10}\}}{\gamma^3\delta^{3\tau+1}} + \frac{2nc_{a,1}}{\delta} + \frac{c_{a,2}}{2}\right) \|E\|^2_{\mathbb{T}^n_{\rho}\times V} =: \frac{C_{14}}{\gamma^3\delta^{3\tau+1}} \|E\|^2_{\mathbb{T}^n_{\rho}\times V}. \tag{2.60}$$

The last object to bound is the new error defined in (1.36), before that we need to bound $(Id - \Omega_0 E_{sym})^{-1}$.

This is done by the Neumann series

$$\left\|(Id - X)^{-1}\right\| \leq \sum_{k=0}^{\infty} \left\|X^k\right\| \leq \frac{1}{1-\|X\|}, \tag{2.61}$$

using $X = \Omega_0 E_{sym}$ and we ask $\|E_{sym}\| < \frac{1}{2}$ to be included in (2.20), that means,

$$\frac{2C_3}{\gamma\delta^{\tau+1}} \|E\|_{\mathbb{T}^n_{\rho}\times V} < 1. \tag{2.62}$$

Then, we have $\|(Id - \Omega E_{sym})^{-1}\| < 2$ and the new error of invariance, the last estimate (2.26),

$$\|\overline{E}\|_{\mathbb{T}^n_{\rho-2\delta}\times V} < \left(2c_P C_{14}\gamma\delta^{\tau-1} + \frac{1}{2}c_{F,2}\hat{C}_2^2\right) \frac{\|E\|^2_{\mathbb{T}^n_{\rho}\times V}}{\gamma^4\delta^{4\tau}} =: \frac{\hat{C}_5}{\gamma^4\delta^{4\tau}} \|E\|^2_{\mathbb{T}^n_{\rho}\times V}. \tag{2.63}$$

Finally, we complete the proof of the Lemma by merging conditions (2.45), (2.46), (2.51), (2.53), (**??**), (2.58), (2.59) and (2.62) presenting the constant $\hat{C}_1$ as

$$\begin{aligned} \hat{C}_1 := \max\{2C_3\gamma\delta^\tau, \frac{n\hat{C}_2}{\sigma_L - \|D_\theta K\|_{\mathbb{T}^n_{\rho}\times V}}, \frac{2n\hat{C}_2}{\sigma_L^* - \|D_\theta K^\top\|_{\mathbb{T}^n_{\rho}\times V}}, \frac{\hat{C}_3}{\sigma_B - \|B\|_{\mathbb{T}^n_{\rho}\times V}}, \\ \frac{\hat{C}_3^*}{\sigma_B^* - \|B^\top\|_{\mathbb{T}^n_{\rho}\times V}}, \frac{\hat{C}_4}{\sigma_T - \langle T\rangle^{-1}}, \frac{\hat{C}_2\delta}{dist(K(\mathbb{T}^n_{\rho}\times V), \partial\mathcal{B})}\} \end{aligned} \tag{2.64}$$

that appears in (2.20). □

Remark 2.13. The estimates in the proof can be adapted to the 2−dimensional case of primary and secondary tori. For instance, $\|\Omega_0\|_{\mathcal{B}\times V} = 1$, $\|D_z\Omega_0\|_{\mathcal{B}\times V} = 0$, there are not E_{sym} and other simplifications.

2.4 Convergence of the KAM Process

Once we prove the Iteration Lemma, we will apply a sequence of Iteration Lemma in order to prove the Theorem (2.7). Consider the family of approximately invariant tori $K_0 := K$ with the initial error $E_0 := E$. Moreover, we introduce $B_0 := B$ and $T_0 := T$ associated with the initial approximation.

In every iteration we reduce the domain of analycity of the objects. Assuming that the initial object is defined in an analytic band of size ρ_0 and we want the final object to be defined in an analytic band of size ρ_∞. We want to select all the intermediate ρ_s of analytict bands. Tod do so we select the parameters $a_1 > 1,\ a_2 > 1,\ a_3 = a_3(a_1, a_2)$ and define

$$\rho_0 = \rho, \quad \delta_0 = \frac{\rho_0}{a_3}, \quad \rho_s = \rho_{s-1} - 3\delta_{s-1}, \quad \delta_s = \frac{\delta_0}{a_1^s},\ \rho_\infty = \lim_{s\to\infty} \rho_s = \frac{\rho_0}{a_2}$$

where the explicit form of a_3 is $a_3 = 3\frac{a_1}{a_1-1}\frac{a_2}{a_2-1}$.

We select the parameters a_1 the size of reduction in each step and a_2 is size of reduction between the initial and final band. From that we can deduce the parameter a_3 and also is possible to select hte parameters a_1 and a_3 first, then let a_2 depends on a_1 and a_2.

Then, we consider the objects K_s, E_s, B_s and T_s at $s-$step. Notice that the condition (2.20) is required in order the following objects can be controlled uniformly with with respect to s-step,

$$\|D_\theta K_s\|_{\mathbb{T}^n_{\rho_s}\times V},\ \left\|D_\theta K_s^\top\right\|_{\mathbb{T}^n_{\rho_s}\times V},\ \|B_s\|_{\mathbb{T}^n_{\rho_s}\times V},\ dist(K_s(\mathbb{T}^n_{\rho_s}\times V),\ \partial\mathcal{B}),\ \left|\langle T\rangle^{-1}\right|_V, \tag{2.65}$$

As a consequence, the constants that appear in the Iterative Lemma (2.10) can be the same for all steps by taking the worst value of δ_s, i.e, $\delta_0 = \rho_0/a_3$.

Now we proceed by induction. We assume that the iterative Lemma have applied s times the Iterative Lemma and prove that it can be applied again. In other words the assumptions of (2.7) are hold at $s-$step with the bound are the worst value $\delta_0 = \rho_0/a_3$. To this end, we first compute the error E_s in terms of E_0 as follows:

$$\|E_s\|_{\mathbb{T}^n_{\rho_s}\times V} < \frac{\hat{C}_5}{\gamma^4\delta_{s-1}^{4\tau}}\|E_{s-1}\|^2_{\mathbb{T}_{\rho^n_{s-1}}\times V} = \frac{\hat{C}_5 a_1^{4\tau(s-1)}}{\gamma^4\delta_0^{4\tau}}\|E_{s-1}\|^2_{\mathbb{T}^n_{\rho_{s-1}}\times V}, \tag{2.66}$$

and iterating this sequence backwards

$$\|E_s\|_{\mathbb{T}^n_{\rho_s}\times V} < \frac{\hat{C}_5 a_1^{4\tau(s-1)}}{\gamma^4\delta_0^{4\tau}}\|E_{s-1}\|^2_{\mathbb{T}_{\rho_{s-1}}\times V} \le \frac{\hat{C}_5 a_1^{4\tau(s-1)}}{\gamma^4\delta_0^{4\tau}}\left(\frac{\hat{C}_5 a_1^{4\tau(s-2)}}{\gamma^4\delta_0^{4\tau}}\right)^2\cdots\left(\frac{\hat{C}_5 a_1^{4\tau\cdot 1}}{\gamma^4\delta_0^{4\tau}}\right)^{2^{s-1}}\|E_0\|^{2+2+\cdots+2^{s-1}}_{\mathbb{T}^n_{\rho_0}\times V}$$

and using

$$\begin{aligned}&1+2+\cdots+2^{s-1} = 2^s-1,\\ &1(s-1)+2(s-2)+2^2(s-3)+\cdots+2^{s-2}1 = 2^s-s-1.\end{aligned}$$

We get

$$\|E_s\|_{\mathbb{T}^n_{\rho_s}\times V} < \left(\frac{a_1^{4\tau}\hat{C}_5\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}}{\gamma^4\delta_0^{4\tau}}\right)^{2^s-1} a_1^{-4\tau s}\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}. \tag{2.67}$$

We use this expression in order to verify condition (2.20) so that we can perform the step $s+1$. But, before that we need to consider a decreasing sequence of errors , we assume that

$$\frac{a^{4\tau}\hat{C}_5\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}}{\gamma^4\delta_0^{4\tau}} < 1, \tag{2.68}$$

Then, it holds because we can include it in the hypothesis (2.17).

Secondly, we need to verify the conditions included in (2.20) are hold with the expression for $\hat{C}_1$ is given in (2.64). Essentially, in the expression there are two kind of conditions. On the one hand, we have conditions like (2.62). On the other hand, we have conditions like (2.45). depending also on other objects at the $s-$step.

Let start with the first kind of the condition that is direct using (2.67) and $\tau \geq n$:

$$\begin{aligned}\frac{2C_3 \|E_s\|_{\mathbb{T}^n_{\rho_s}\times V}}{\gamma\delta_s^{\tau+1}} &< \frac{2C_3 a_1^{(\tau+1)s}}{\gamma\delta_0^{\tau+1}}\left(\frac{a_1^{4\tau}\hat{C}_5 \|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}}{\gamma^4\delta_0^{4\tau}}\right)^{2^s-1} a_1^{-4\tau s}\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}\\ &< \frac{2C_3}{\gamma\delta_0^{\tau+1}}\|E_0\|_{\mathbb{T}^n_{\rho_s}\times V} < 1,\end{aligned}$$

where the last inequality is included in (2.17). Then, other kind of the condition that depends on other objects. We should relate with the initial one, for instance

$$\begin{aligned}\|D_\theta K_s\|_{\mathbb{T}^n_{\rho_s}\times V} + \frac{n\hat{C}_2\|E_s\|_{\mathbb{T}^n_{\rho_s}\times V}}{\gamma^2\delta_s^{2\tau+1}} &< \|D_\theta K_0\|_{\mathbb{T}^n_{\rho_0}\times V} + \sum_{j=0}^{s}\frac{n\hat{C}_2\|E_j\|_{\mathbb{T}^n_{\rho_j}\times V}}{\gamma^2\delta_j^{2\tau+1}}\\ &< \|D_\theta K_0\|_{\mathbb{T}^n_{\rho_0}\times V} + \sum_{j=0}^{\infty}\frac{n\hat{C}_2 a_1^{(2\tau+1)j}}{\gamma^2\delta_0^{2\tau+1}}\left(\frac{a_1^{4\tau}\hat{C}_5\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}}{\gamma^4\delta_0^{4\tau}}\right)^{2^j-1} a_1^{-4\tau j}\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}\\ &< \|D_\theta K_0\|_{\mathbb{T}^n_{\rho_0}\times V} + \frac{n\hat{C}_2}{\gamma^2\delta_0^{2\tau+1}}\sum_{j=0}^{\infty} a_1^{(1-2\tau)j}\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}\\ &= \|D_\theta K_0\|_{\mathbb{T}^n_{\rho_0}\times V} + \frac{n\hat{C}_2}{\gamma^2\delta_0^{2\tau+1}}\left(\frac{1}{1-a_1^{1-2\tau}}\right)\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V} < \sigma_L\end{aligned} \tag{2.69}$$

As usual, the last inequality is included in (2.17) and we proceed similarly with $\left\|D_\theta K_s^\top\right\|_{\mathbb{T}^n_{\rho_s}\times V}$.

$$\begin{aligned}dist(K_s(\mathbb{T}^n_{\rho_s}\times V),\partial\mathcal{B}) &\geq dist(K(\mathbb{T}^n_{\rho_{s-1}}\times V),\partial\mathcal{B}) - \|\triangle K_s\|_{\mathbb{T}^n_{\rho_s}\times V}\\ &\geq dist(K(\mathbb{T}^n_{\rho_{s-1}}\times V),\partial\mathcal{B}) - \frac{\hat{C}_2}{\gamma^2\delta_s^{2\tau}}\|E\|_{\mathbb{T}^n_{\rho_s}\times V}\\ &\geq dist(K(\mathbb{T}^n_{\rho_{s-2}}\times V),\partial\mathcal{B}) - \left(\frac{\hat{C}_2}{\gamma^2\delta_{s-1}^{2\tau}}\|E\|_{\mathbb{T}^n_{\rho_{s-1}}\times V} + \frac{\hat{C}_2}{\gamma^2\delta_s^{2\tau}}\|E\|_{\mathbb{T}^n_{\rho_s}\times V}\right)\\ &\geq \cdots \geq dist(K(\mathbb{T}^n_{\rho_0}\times V),\partial\mathcal{B}) - \sum_{j=0}^{s}\frac{\hat{C}_2}{\gamma^2\delta_j^{2\tau}}\|E\|_{\mathbb{T}^n_{\rho_j}\times V}\\ &\geq dist(K(\mathbb{T}^n_{\rho_0}\times V),\partial\mathcal{B}) - \frac{\hat{C}_2}{\gamma^2\delta_0^{2\tau}}\left(\frac{1}{1-a_1^{-2\tau}}\right)\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V} > 0,\end{aligned} \tag{2.70}$$

where the last inequality is included in (2.17). Now we fact to B,

$$\begin{aligned}\|B\|_{\mathbb{T}^n_{\rho_s}\times V} &\leq \frac{\hat{C}_3}{\gamma^2\delta_s^{2\tau+1}}\|E\|_{\mathbb{T}^n_{\rho_s}\times V} \leq \sum_{j=0}^{\infty}\frac{\hat{C}_3}{\gamma^2\delta_j^{2\tau+1}}\|E\|_{\mathbb{T}^n_{\rho_j}\times V}\\ &\leq \frac{\hat{C}_3}{\gamma^2\delta_0^{2\tau+1}}\left(\frac{1}{1-a_1^{1-2\tau}}\right)\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V} \leq \sigma_B,\end{aligned} \tag{2.71}$$

where the last inequality is included in (2.17). The same strategy to T and we obtain the condition that we have to ask to E_0

$$\frac{\hat{C}_8 \, \|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}}{\gamma^2\delta_0^{2\tau+1}} < 1 \tag{2.72}$$

where

$$\hat{C}_8 := \max\{\frac{\hat{C}_6}{1-a_1^{1-2\tau}}, \frac{\hat{C}_7}{1-a_1^{-2\tau}}\} \tag{2.73}$$

$$\hat{C}_6 := \max\{\frac{n\hat{C}_2}{\sigma_L - \|D_\theta K_0\|_{\mathbb{T}^n_{\rho_0}\times V}}, \frac{2n\hat{C}_2}{\sigma_L^* - \|D_\theta K_0^\top\|_{\mathbb{T}^n_{\rho_0}\times V}}, \frac{\hat{C}_3}{\sigma_B - \|B_0\|_{\mathbb{T}^n_{\rho_0}\times V}}, \tag{2.74}$$

$$\frac{\hat{C}_3^*}{\sigma_B^* - \|B_0^\top\|_{\mathbb{T}^n_{\rho_0}\times V}}, \frac{\hat{C}_4}{\sigma_T - \langle T_0\rangle^{-1}}\}, \tag{2.75}$$

$$\hat{C}_7 := \frac{\hat{C}_2\delta_0}{dist(K_0(\mathbb{T}^n_{\rho_0}\times V), \partial\mathcal{B})}. \tag{2.76}$$

The conditions of the second king that have to ask are included in (2.72).

Finally, we give the expression of $\hat{C}_*$ the condition (2.17) on $\|E_0\|_{\mathbb{T}^n_{\rho_0}\times V}$,

$$\hat{C}_* = \max\{(a_1a_3)^{4\tau}\hat{C}_5,\ (a_3)^{2\tau+1}\hat{C}_8\gamma^2\rho_0^{2\tau-1}\}$$

so that the hypotheses $H1 - H4$ are satisfied and also the condition (2.20). Then, we are able to apply the Iterative Lemma again. Moreover, note that the sequence of error is a decreasing sequence such that $\|E_s\|_{\mathbb{T}_{\rho_s}\times V} \to 0$ when $s \to \infty$ and as a consequence the iterative scheme converges to a true quasi-periodic family tori K_∞ defined in $\mathbb{T}_{\rho^n_\infty \times V}$.

Remark 2.14. Notice in the proof we can see the KAM theorem is applicable to either primary tori or secondary tori. In the next chapter we present an algorithm in order to find both kind of tori.

We have been keeping the dependence of the parameter ε and we proved that the parameterization K is analytic in $\mathbb{T}^n_{\rho_\infty} \times V$. Therefore, K is analytic respect to ε we can differentiate respect to ε. Furthermore, in the next chapter we will introduce the continuation method suh a way we need the analyticity of parameter ε.

2.5 Continuation of Derivatives with respect to Parameter

A main motivation to develop the theory with parameters is that one easily obtains analytic dependence on them. Much more, we can also compute derivative with respect to parameters, which can be very useful when implementing continuation methods.

For a given family of initial parameterizations K and applying the KAM method we get a family of F-invariant parameterizations $\bar{K}$

$$F(\bar{K}(\theta,\varepsilon),\varepsilon) - \bar{K}(\theta+\omega,\varepsilon) = 0. \tag{2.77}$$

If we differentiate (2.77) respect to the variable ε, we obtain

$$D_zF(\bar{K}(\theta,\varepsilon),\varepsilon)\partial_\varepsilon\bar{K}(\theta,\varepsilon) + \partial_\varepsilon F(\bar{K}(\theta,\varepsilon),\varepsilon) - \partial_\varepsilon\bar{K}(\theta+\omega,\varepsilon) = 0$$

and the previous equation can be rewritten in following form

$$D_z F(\bar{K}(\theta,\varepsilon),\varepsilon)\partial_\varepsilon \bar{K}(\theta,\varepsilon) - \partial_\varepsilon \bar{K}(\theta+\omega,\varepsilon) = -\partial_\varepsilon F(\bar{K}(\theta,\varepsilon),\varepsilon). \tag{2.78}$$

Notice that the structure of the equation (2.78) is similar to the equation (1.51). That means we can apply the same adapted frame $P(\theta,\varepsilon)$ and use Lemma (1.17) in order to get $\partial_\varepsilon \bar{K}$. Then, we can take $K_{\varepsilon_0+\triangle\varepsilon} \approx \bar{K}(\theta,\varepsilon) + \partial_\varepsilon \bar{K}(\theta,\varepsilon) \triangle \varepsilon$ as initial approximation to find the true invariant circles with the parameter $\varepsilon + \triangle\varepsilon$ using the KAM method.

Chapter 3

Algorithms

In this chapter we present algorithm to compute invariant circles in (families of area) preserving maps. Consider invariant circles with two homotopy types:

- Primary context: Circles on the cylinder $\mathbb{T} \times \mathbb{R}$ that are homotopically non trivial.
- Secondary context: Circles on the plane $\mathbb{R}^2$ or on the cylinder $\mathbb{T} \times \mathbb{R}$ that are homotopically trivial.

The algorithms are based on the contrustive proof of the KAM Theorem in a posteriori format described in the previous chapter. Moreover, we have been keeping the dependence of the parameter ε in the proof of Lemma (2.7) and we proved that the parameterization K is analytic in $\mathbb{T}^n_{\rho_\infty} \times V$. As a consequence, K depends also analytically on parameters and we can perform a continuation method with respect to the parameter ε.

We aim to find the approximate true invariant circles by solving computationally the invariance equation. To this end, we present two different spaces of discretization of functions: Grid space (The values of the periodic functions on a regular grid of points) and Fourier space (The coefficients of a trigonometric polynomial interpolation) to do the operations computationally. Sometimes is easier in Grid space and sometimes in Fourier space. For instance, differentiating a map is more comfortable in Fourier meanwhile the product of the functions is easier in Grid space. We will keep same time two space and combine with with the use of FFT, a algorithm transforms functions in Grid space to Fourier space and vice versa.

All operations require $O(N)$ storage in the corresponding space, except the DFT method needs $O(N^2)$ operation to perform. In order to implement an effective algorithm we use FFT that require $O(N \log N)$, then in each step of Newton like require $N \log N$ storage.

3.1 An Algorithm to Compute Invariant Curves

The algorithm is derived from the parameterization method is based on running an efficient Newton-like method to solve the invariance equation. Essentially, this algorithm consists in implementing the proof of the iterative procedure defined in section (1.3).

In particular, for a family of exact symplectic maps of form We will geive the details of the algorithm to compute primary tori in a one-parameter family of area preserving maps. We will assume that the ambient space is $\mathcal{A} \times \mathbb{R}$, endowed with the standard compatible triple $(\Omega_0,\ J_0,\ G_0)$.

The family is writen as

$$F(x,y,\varepsilon) = \begin{pmatrix} x \\ 0 \end{pmatrix} + F_p(x,y,\varepsilon) = \begin{pmatrix} x + F_p^x(x,y,\varepsilon) \\ F_p^y(x,y,\varepsilon), \end{pmatrix}$$

where F_p is one-parameter in the x-variable. We assume that $\det D_z F(x,y,\varepsilon) = 1$.

we want to find two kinds of family of parameterization of invariant tori

- Primary context:
$$K(\theta,\varepsilon) = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + K_p(x,y,\varepsilon) = \begin{pmatrix} \theta + K_p^x(\theta,\varepsilon) \\ K_p^y(\theta,\varepsilon) \end{pmatrix}$$
where K_p is one-periodic in the θ-variable.

- Secondary context:
$$K(\theta,\varepsilon) = \begin{pmatrix} K_p^x(\theta,\varepsilon) \\ K_p^y(\theta,\varepsilon) \end{pmatrix}$$

Notice that, in the implementation, we represent K_p either in Grid space pr Fourier space. Let us describe the algorithm of the parameterization method for a invariant circle of a family of exact symplectic maps. In particular, we present it in $2D$ ambient space such a way it will be coherent with the example: Standard map. Consider a periodic function f on $\mathbb{T}_\rho \times V$ and sample of points on the regular grid of size N

$$\theta_j = \frac{j}{N} \quad \text{for j =0, } \cdots \text{,N-1.}$$

It lets us define the discretization of f in the real space as $1-$ dimensional array $\{f_j\}_{j=0}^{N-1}$ with $f_j := f(\theta_j)$. Meanwhile, the Fourier discretization is represented by $\{\hat{f}_k\}_{k=0}^{N-1}$ where $\hat{f}_k$ are the coefficients of Fourier series.

Now, we denote the discrete Fourier transform as DFT and it allows us to transform the grid discretization to Fourier discretization in following form:

$$\{\hat{f}_k\}_{k=0}^{N-1} = DFT(\{f_j\}_{j=0}^{N-1}) \quad \text{with} \quad \hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-2\pi i k \theta_j}.$$

Specially, the average is given by

$$\hat{f}_0 = \langle \{f_j\} \rangle = \frac{1}{N}\sum_{j=0}^{N} f_j. \tag{3.1}$$

The DFT produces the interpolating trigonometric polynomial on the grid, that is,

$$f_j = f(\theta_j) = \sum_k \hat{f}_k e^{2\pi i k \theta_j},$$

and we denote $\{f_j\}_{j=0}^{N-1} = DFT^{-1}(\{\hat{f}_k\}_{j=0}^{N-1})$.

Notice that for real-valued functions, the following symmetry holds:

$$\hat{f}_k = \hat{f}^*_{N_F-k}$$

where $*$ stands for the complex conjugate, and $\hat{f}_0$ is real.

We emphasize that the right way to approximate functions in our context is by means of the truncated Fourier series

$$f(\theta) \approx \sum_{k=0}^{N-1} \hat{f}_k e^{2\pi i \hat{k}\theta}$$

using the symmetry we define

$$\hat{k} = \begin{cases} k & \text{if } 0 \le k \le N/2, \\ k - N & \text{if } N/2 \le k < N. \end{cases}$$

Given f a periodic function discretized as $\{\hat{f}_k\}_{k=0}^{N-1}$. Everything above can be extended to vector or matrix functions and as we said before, some manipulations of periodic functions are easier discretized in te Fourier space.

(i) The derivative $\partial_\theta f$:

$$\{\widehat{(\partial_\theta f)}_k\}_{k=0}^{N-1} \quad \text{where} \quad \widehat{(\partial_\theta f)}_k = 2\pi i \hat{k} \hat{f}_k, \tag{3.2}$$

(ii) The composition $f \circ R_\omega$:

$$\{\widehat{(f \circ R_\omega)}_k\}_{k=0}^{N-1} \quad \text{where} \quad \widehat{(f \circ R_\omega)}_k = e^{2\pi i \hat{k}\omega} \hat{f}_k, \tag{3.3}$$

(iii) The soution of one-bite cohomological equation $\Re(f)$:

$$\{\widehat{(\Re(f))}_k\}_{k=0}^{N-1}, \quad \text{where} \quad \widehat{(\Re(f))}_k = \begin{cases} (1 - e^{2\pi \hat{k}\omega})^{-1} \hat{f}_k & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases} \tag{3.4}$$

On the other hand, as operations of product of two functions, product by a constant and etc... are manipulations of periodic function in Grid space.

We are working with periodic functions and we use the notations of parameterization K presented in (1.1) and (1.2). Then, the approximation (truncated Fourier series) is coded by a sample of points $\{K_{p,j}\}_{j=0}^{N-1}$, p stands for periodic, where $K_{p,j} = K_p(\theta_j)$ of Fourier coefficients $\{\hat{K}_{p,k}\}_{j=0}^{N-1}$ that are connected by DFT or DFT^{-1}. We describe the implementation of the iteration of Newton-like method.

We first present the algorithm to primary context and then we will adapt the methodology to secondary context, but they have major part in common. Notice that we will skip the notation of dependence in parameter ε, that is, $F = F_\varepsilon$.

Algorithm 3.1. ***(Newton step)*** *Let us consider an exact symplectic map on $\mathcal{A} \times U$, with $\mathcal{A} \subset \mathbb{T} \times \mathbb{R}$ is the 2D annulus, $F(x, y, \varepsilon) = (x, 0) + F_p(x, y, \varepsilon)$.*

Input 1: *Non-reasonant frequency $\omega \in \mathbb{R}$ and a given ε.*

Input 2: *Sampling of an approximately invariant torus K of the form* (1.1) *with the periodic parts are coded as $\{\hat{K}_{p,k}\}_{k=0}^{N-1}$ and $\{K_{p,j}\}_{j=0}^{N-1}$ on a regular grid on $\mathbb{T}$ of size N.*

Then, we proceed as follows:

***Step 1**: To evaluate the error $E(\theta,\varepsilon) = F(K(\theta,\varepsilon),\varepsilon) - K(\theta+\omega,\varepsilon)$ using the explicit form of F and K, we get*

$$E(\theta,\varepsilon) = \begin{pmatrix} K_p^x(\theta,\varepsilon) + F_p^x(K(\theta,\varepsilon),\varepsilon) - K_p^x(\theta+\omega,\varepsilon) - \omega \\ F_p^y(K(\theta,\varepsilon),\varepsilon) - K_p^y(\theta+\omega,\varepsilon) \end{pmatrix}. \tag{3.5}$$

To evaluate this formula at grid points we first compute $F_p(K(\theta,\varepsilon),\varepsilon)$ at grid points since it is computed directly from the grid. Thus we get $\{(F_p \circ K)_j\}_{j=0}^{N-1}$ with

$$(F_p \circ K)_j = F_p(\theta_j + K_{p,j}^x, K_{p,j}^y, \varepsilon).$$

Then, $K_p(\theta+\omega,\varepsilon)$ is easier obtained in Fourier space using (3.3) *and use DFT^{-1} in order to get it in Grid space.*

$$\{(K_p \circ R_\omega)_j\}_{j=0}^{N-1} = DFT^{-1}\left(\{(\widehat{K_p \circ R_\omega})_k\}_{k=0}^{N-1}\right).$$

Hence, the computation of the error (3.5) *at grid*

$$error = \sqrt{\left\|\{E_j^x\}_{j=0}^{N-1}\right\|_\infty^2 + \left\|\{E_j^y\}_{j0}^{N-1}\right\|_\infty^2}. \tag{3.6}$$

where $\left\|\{E_j^x\}_{j=0}^{N-1}\right\|_\infty = \max_{j=0,\cdots,N-1} |E_j|.$

Then, given a tolerance tol the next decision will be

$$\begin{cases} \textit{Stop the computation} & \textit{if error} < tol \\ \textit{Go back to Step 1} & \textit{Otherwise} \end{cases} \tag{3.7}$$

***Step 2** To construct the frame $P(\theta,\varepsilon)$ we compute the tangent vectors $L(\theta,\varepsilon)$ in Fourier space $\{\hat{L}_k\}_{k=0}^{N-1}$ and Grid space $\{L_j\}_{j=0}^{N-1}$ where*

$$\hat{L}_k = \begin{pmatrix}1\\0\end{pmatrix}\delta_{k,0} + \widehat{D_\theta K}_{p,k} = \begin{pmatrix}1\\0\end{pmatrix}\delta_{k,0} + \begin{pmatrix}\widehat{\partial_\theta K}_{p,k}^x \\ \widehat{\partial_\theta K}_{p,k}^y\end{pmatrix} \qquad L_j = \begin{pmatrix}1\\0\end{pmatrix} + D_\theta K_{p,j}, \tag{3.8}$$

where $\delta_{k,0}$ is Kronecker's delta that is 1 if $k=0$ otherwise equals to 0, $\widehat{D_\theta K}_{p,k}$ is computed using (3.2) *and $\{D_\theta K_{p,j}\}_{j=0}^{N-1} = DFT^{-1}\left(\{\widehat{D_\theta K}_{p,k}\}_{k=0}^{N-1}\right)$. Then,*

$$B(\theta,\varepsilon) = \left(L(\theta,\varepsilon)^\top L(\theta,\varepsilon)\right)^{-1}.$$

We complement $L(\theta,\varepsilon)$ by computing $N(\theta,\varepsilon) = N^0 B$ as $\{N_j\}_{j=0}^{N-1}$ with

$$N_j = N_j^0 B_j = J L_j (L_j^\top L_j)^{-1} = \begin{pmatrix} -L_j^y (L_j^T L_j)^{-1} \\ -L_j^x (L_j^T L_j)^{-1} \end{pmatrix}. \tag{3.9}$$

These end up with

$$\{P_j\}_{j=0}^{N-1} = \{L_j \;\; N_j\}, \qquad \{P_k\}_{k=0}^{N-1} = DFT\left(\{P_j\}_{j=0}^{N-1}\right) = \{(\hat{L}_k \;\; \hat{N}_k)\}_{k=0}^{N-1}. \tag{3.10}$$

Step 3: *To obtain the correction* $\xi(\theta,\varepsilon)$ *of* $K(\theta,\varepsilon)$ *on the adapted frame, we first compute* $\eta(\theta,\varepsilon) = -\Omega_0 P(\theta+\omega,\varepsilon)\Omega_0 E(\theta,\varepsilon)$ *in Grid as* $\{\eta_j^L\}_{j=0}^{N-1}$ *and* $\{\eta_j^N\}_{j=0}^{N-1}$ *with*

$$\eta_j^L = \{-(N \circ R_\omega)_j^\top \Omega_0 E_j\}$$
$$\eta_j^N = \{(L \circ R_\omega)_j^\top \Omega_0 E_j\}$$

where $P(\theta+\omega,\varepsilon)_j$ *is obtained using* (3.3) *in Fourier space and applying* DFT^{-1}*. Then,* $\{\hat{\eta}_k^L\}_{k=0}^{N-1}$ *and* $\{\hat{\eta}_k^N\}_{k=0}^{N-1}$ *are obtained using DFT.*

We compute the torsion matrix $T(\theta,\varepsilon)$ *in Grid as* $\{T_j\}_{j=0}^{N-1}$ *with*

$$T_j = (N \circ R_\omega)_j^\top \Omega_0 (DF \circ K)_j N_j.$$

In order to obtain $\xi(\theta,\varepsilon)$ *the solutions of the one-bite cohomological equation, we first compute* $\{\widehat{\Re(\eta^N)}\}_{k=0}^{N-1}$ *using* (3.4)*, then* $\{\Re(\eta_j^N)\}_{j=0}^{N-1}$ *using* DFT^{-1} *and*

$$\hat{\xi}_0^N = \frac{1}{N}\sum_{j=0}^{N-1} T_j^{-1}(\eta_j^L - T_j \Re(\eta^N)_j) \tag{3.11}$$

Then, we compute ξ^N *in Fourier space as*

$$\{\hat{\xi}_K^N\}_{k=0}^{N-1} = \{\widehat{\Re(\eta^N)}_k\}_{k=0}^{N-1} + \xi_0^N. \tag{3.12}$$

Then, using DFT^{-1} *we get the expression of* ξ_N *in Grid. Moreover, it can be used to get the expression of* $\{\eta_j^L - T_j\widehat{\Re(\eta^N)}_j\}_{j=0}^{N-1}$ *on Grid space, Then, we fix* ξ_0^N *in order to satisfy the compatibility condition for the equation for* ξ^L *.*

Hence, we compute $\{\hat{\eta}_k^L - (\widehat{T\Re(\eta^N)})_k\}_{k=0}^{N-1}$ *using DFT and we obtain* $\{\hat{\xi}_k^L\}_{k=0}^{N-1}$ *with*

$$\hat{\xi}_k^L = \hat{\eta}_k^L - (\widehat{T\Re(\xi^N)})_k = \hat{\eta}_k^L - (\widehat{T\Re\eta^N})_k - (T\hat{\xi}_0^N)_k \quad k = 0,\cdots,N-1.$$

Finally, we compute

$$\{\xi_j^L\}_{j=0}^{N-1} = DFT^{-1}(\{\hat{\xi}_k^L\}), \quad \{\xi_j^N\}_{j=0}^{N-1} = \{\xi_0^N + (\Re(\eta^N))_j\}.$$

Step 4: *To obtain the new parameterization*

$$\bar{K}_p(\theta,\varepsilon) = K_p(\theta,\varepsilon) + \triangle K(\theta,\varepsilon) \tag{3.13}$$

where we compute the expression of the period part of the parameterization $\{K_{p,j}\}_{j=0}^{N-1}$ *in Grid with*

$$\bar{K}_{p,j} = K_{p,j} + \xi_j P(\theta,\varepsilon)_j = K_{p,j} + L_j\xi_j^L + N_j\xi_j^N \tag{3.14}$$

and the Fourier coefficients $\{\hat{K}_{p,k}\}_{k=0}^{N-1} = DFT\left(\{K_{p,j}\}_{j=0}^{N-1}\right)$*.*

Remark 3.2. Notice that changing few lines of equation, the algorithm is applicable to secondary tori. The changes will be

$$K(\theta,\varepsilon) = (K^x(\theta,\varepsilon), K^y(\theta,\varepsilon)) = K_p(\theta,\varepsilon), \tag{3.15}$$
$$E^x(\theta,\varepsilon) = K^x(\theta,\varepsilon) + F_p^x(K(\theta,\varepsilon),\varepsilon) - K^x(\theta+\omega,\varepsilon) \tag{3.16}$$
$$L_j = D_\theta K_j, \quad \hat{L}_k = \widehat{D_\theta K}_k. \tag{3.17}$$

3.2 Newton Iteration and Continuation Method of the Invariant Circle

In this section we aim to give an algorithm, given the initial parameterization K_{ε_0} and the parameter ε_0, is able to compute the invariant circles increasing (or decreasing) the parameter ε until reaching the breakdown of the invariant circle. Remember that the parameterization K_ε depends on the parameter ε analytically and that is why we can perform the continuation method.

The main idea is to assume for a given initial parameterization K and we applying Algorithm (3.1) to it. Then, we get $\bar{K}$ satisfies the invariance equation up to a given tolerance and if we differentiate respect to the variable ε with the refined parameterization. We get

$$D_z F(\bar{K}(\theta,\varepsilon),\varepsilon)\partial_\varepsilon \bar{K}(\theta,\varepsilon) - \partial_\varepsilon \bar{K}(\theta+\omega,\varepsilon) = -\partial_\varepsilon F(\bar{K}(\theta,\varepsilon),\varepsilon). \tag{3.18}$$

Then, we can give an initial approximation $K_{\varepsilon_0+\triangle\varepsilon} \approx \bar{K}(\theta,\varepsilon) + \partial_\varepsilon \bar{K}(\theta,\varepsilon) \triangle \varepsilon$ to find the invariant circle with the parameter $\varepsilon + \triangle\varepsilon$ and we are able to apply Algorithm (3.1) In other words, the algorithm of continuation method is applying iteratively of the algorithm (3.1).

Algorithm 3.3. *Let us consider an exact symplectic map on $\mathcal{A} \times U$, with $\mathcal{A}$ is the $2-$ dimensional annulus, $F(x,y,\varepsilon) = (x,0) + F_p(x,y,\varepsilon)$ is endowed with a closed non-degenerate symplectic product that given by the antisymmetric matrix Ω_0. Given the initial increment $\triangle\varepsilon = 0.1$ and tol.*

> ***Input 1:*** *Non-reasonant frequency $\omega \in \mathbb{R}$ and an initial ε_0.*
>
> ***Input 2:*** *Sampling of an approximately invariant torus K_{ε_0} of the form (1.1) with parameter ε_0 on a regular grid on $\mathbb{T}$ of size N.*

Then, the continuation method is presented by the following steps.

> ***Step 0: (Newton Step)*** *To compute the convergent parameterization $\bar{K}_{\varepsilon_0}$ for the initial parameterization K_{ε_0} with the parameter $\varepsilon = \varepsilon_0$ applying the algorithm (3.1). Moreover, the conditions of stopping the iterations are error $<$ tol and the number of total iterations smaller than a finite number.*
>
> $$\begin{cases} \textit{Stop the program} & \textit{if it does not converge} \\ \textit{Continue the program} & \textit{Otherwise} \end{cases} \tag{3.19}$$
>
> ***Step 1:*** *To compute $\partial_\varepsilon \bar{K}_{\varepsilon_0}$ where we use the same strategy as step 2 and 3 in Algorithm (3.1) to the equation (2.78). We first compute the adapted frame $P(\theta,\varepsilon)$ and then $\xi(\theta,\varepsilon)$ modifying $E(\theta,\varepsilon) = \partial_\varepsilon F(\bar{K}_{\varepsilon_0}(\theta,\varepsilon),\varepsilon)$ in order to get $\partial_\varepsilon \bar{K}_{\varepsilon_0}(\theta,\varepsilon)$.*
>
> ***Step 2:*** *To compute the convergent parameterization $\bar{K}_{\varepsilon_0+\triangle\varepsilon}$ applying (3.1) to the initial approximation $\bar{K}(\theta,\varepsilon) + \partial_\varepsilon \bar{K}(\theta,\varepsilon) \triangle \varepsilon$.*
>
> *If the Newton-like method does not converge, we double the number N and recompute the discretizations in Grid and Fourier spaces that appeared in Algorithm. And then restart with step 2 using $K_{\varepsilon_0+\triangle\varepsilon}$ that is double in the memory and same $\varepsilon = \varepsilon_0 + \triangle\varepsilon$.*
>
> *Furthermore, if it does not convergence again, we reduce the size of $\triangle\varepsilon$. For instance, we use the increment is $\triangle\varepsilon/10$ instead of ε and restart the step 2 with $K_{\varepsilon_0+\frac{\triangle\varepsilon}{10}}$.*

We will keep reducing the size of the increment $\triangle\varepsilon$ in case of divergence of Newton-like method until the size step is smaller than say, 10^{-6} and we stop the program. If eventually it converges, we will go back to step 1 with the approximation $K_{\varepsilon_0+\triangle\varepsilon}$ and $\varepsilon = \varepsilon_0 + \triangle\varepsilon$. .

Remark 3.4. Notice the construction of the algorithm allows us to apply it to secondary tori changing the Newton-like method suitable for the secondary tori.

Remark 3.5. Given an initial approximately invariant tori we can apply the continuation method forward that is to sum the $\triangle\varepsilon$ in each step of iteration. But also we can dot the method backward, that means, we reduce the size step in each step of iteration.

Once we implement it, we can find the lower bound of the breakdown (using the method iterating forward) and the upper bound of the breakdown(iterating the method backward).

Notice that in case of iterating backward, the invariant circles will be small near the elliptic points and the tangent maps $L(\theta, \varepsilon)$ will be big ,that is , the first column of the adapted frame $P(\theta, \varepsilon)$. In order to eliminate this affect we suggest to scale the adapted frame using the following strategy:

First, for the invariant circle K we introduce the notion of invariant of Calabi

$$c = \frac{1}{2}\int_0^1 \left(\frac{\partial K^x}{\partial\theta}K^y - K^x\frac{\partial K^y}{\partial\theta}\right)\, d\theta \tag{3.20}$$

stands for the area below the curve K.

Then, since the secondary tori in $2D$ are similar to circles we can approximate c to the area of secondary tori.

$$c = \pm\pi r^2 \quad \rightarrow \quad r = \sqrt{\frac{|c|}{\pi}}$$

Finally, we scale the frame

$$\begin{aligned} L(\theta, \varepsilon) &\rightarrow \frac{1}{r}L(\theta, \varepsilon) \\ N(\theta, \varepsilon) &\rightarrow rN(\theta, \varepsilon) \end{aligned} \tag{3.21}$$

3.3 Application of Algorithms: Standard Map

In this section we apply the Algorithm (3.1) and (3.3) to see that the invariant curve of primary and secondary tori of the Standard map persist for $\varepsilon \neq 0$. Moreover, the hypotheses of the Theorem (2.7) are satisfied such that the existence of these invariant curves are proved. Our aim is to implement algorithms in Language C in order to see for which ε the persistence of two kind of the invariant tori breakdowns.

On the one hand, the persistence of the primary tori is seen using the continuation method given the initial approximation and implemented Newton iteration in `kam_con_eps_primary.c` found in appendix A. The program does not need any input and they are written inside the program. Once run the program, there are two kinds of outputs. The first one, the outputs on the screen where we can see the different quantities associated to the parameterization in each step. For instance, the error, norm of the torsion matrix T, the adapted frame P, etc... The second one is a file called `parameterization` where are the refined parameterizations in Grid space.

On the other hand, the persistence of secondary tori is seen in different way. We first perform `secondary_rn.c` to get the sampling of an approximately invariant circle K_{ε_0} on the regular grid. We can obtain them running the program `secondary_rn.c`, it does not need any input and returns a file called `initial_K_secondary.dat`. Then, we implemented the Newton iteration and Continuation method in `kam_cont_eps_secondary.c`. The inputs and outputs are same to primary case only the output file calle `Kparameterization.dat`.

To run the program using the following commands

- `name_of_program.c -lm`
- `./a.out`

3.3.1 Primary Tori in Standard Map

The simplest approximation of an Primary tori for the Standard map (1), with the initial parameter $\varepsilon = 0$, is of the form:

$$K(\theta,\varepsilon) = \begin{pmatrix} \theta \\ \omega \end{pmatrix} \quad D_\theta K(\theta,\varepsilon) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.22}$$

where $\omega = (\sqrt{5}-1)/2$ is the golden mean.

We get the outputs displayed in Figure (3.1) and table (**??**)

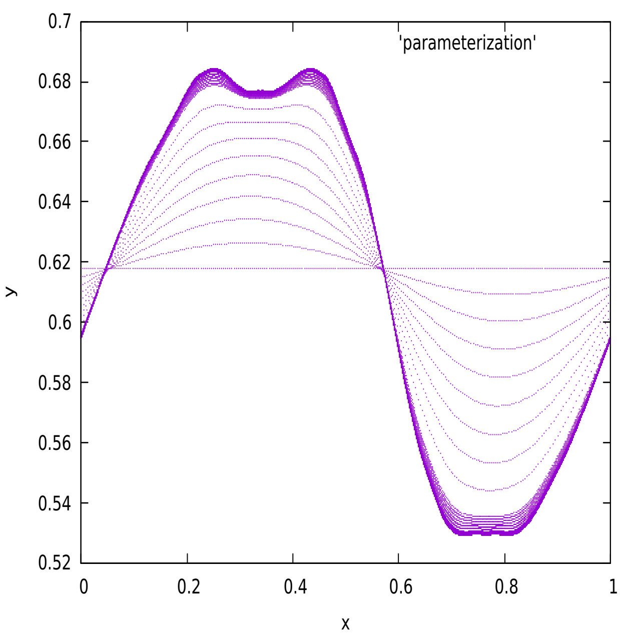


Figure 3.1: The invariant curves obtained from the continuation method respect to ε given the initial parameterization (3.22) and $\varepsilon = 0$.

ε	N/2	E
0.1000	128	4.5e-13
0.2000	128	2.1e-12
0.3000	128	5.6e-12
0.4000	128	1.5e-11
0.5000	128	4.4e-11
0.6000	128	5.2e-17
0.7000	128	1.1e-14
0.8000	128	6.3e-11
0.9000	256	9.9e-11
0.9100	512	2.1e-13

ε	N/2	E
0.9200	512	1.4e-12
0.9300	512	1.8e-11
0.9400	1024	7.7e-12
0.9500	1024	4.8e-11
0.9600	2048	8.6e-13
0.9610	4096	8.9e-11
0.9620	4096	8.7e-15
0.9630	4096	9.0e-15
0.9640	4096	9.8e-15
0.9650	4096	1.4e-14

ε	N/2	E
0.9660	4096	8.3e-14
0.9670	4096	8.4e-13
0.9680	4096	9.0e-12
0.9690	4096	9.9e-11
0.9700	8192	4.4e-11
0.9710	16384	2.0e-11
0.9711	32768	4.2e-11
0.9712	32768	6.6e-11
0.9713	32768	4.4e-12
0.9714	32768	2.9e-11

Table 3.1: Continuation with respect to parameter ε with the quantities the number $N/2$ of significant Fourier coefficient required and the error of invariance.

Remark 3.6. The lower bound we get for the breakdown is $\varepsilon = 0.9714$ and has been numerically observed that this golden mean curve persists up to $\varepsilon_c \approx 0.971635$ (see [Gre79],[Mac93]). Using theorem similar ro rhe one exposed in this work, with the use of computer, it has been rigously proved that invariant circle persist for $\varepsilon = 0.9716$.

3.3.2 Secondary Tori in Standard Map

The case of the secondary tori is different from primary tori. They do not exist in an unperturbed systems and they appear in the perturbed system. Therefore, we can not use the initial approximation found in the unperturbed system, we need to find an initial approximate invariant librettional circle secondary tori for $\varepsilon \neq 0$. In order to look for it, the linearized dynamics will give information to us.

We know that the linerized dynamics around the elliptic point $(0,0)$ is given by the matrix A.

$$A = D_z F(0,0,\varepsilon) = \begin{pmatrix} 1-\varepsilon & 1 \\ -\varepsilon & 1 \end{pmatrix}$$

whose eigenvalues are $\lambda_\pm = e^{\pm 2\pi i \rho} = \cos(2\pi\rho) + i\sin(2\pi\rho) = \frac{2-\varepsilon}{2} + i(\pm\frac{\sqrt{\varepsilon(4-\varepsilon)}}{2})$ and the eigenvalues $\lambda_\pm =$ (notice that $\lambda_+\lambda_- = 1$). Hence, for $|\varepsilon| < 2$ the origin is linear center and around the origin there will be 'circles' with the corresponding rotation number ω. On the other side, given a parameter $\varepsilon_0 > 0$, we can select an orbit whose rotation number ρ is close to ω.

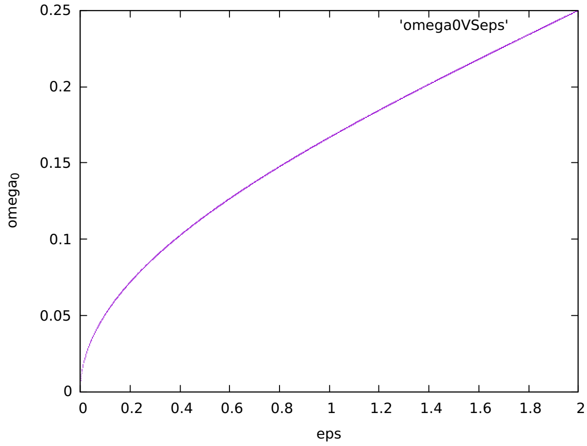


Figure 3.2: From the equation $\lambda_+ = e^{2\pi i\omega}$ we get $\frac{2-\epsilon}{2} = \cos(2\pi\omega)$ and we compute this plot allows us to choose a ε with non- reasonant frequency. In particular, we choose $\varepsilon_0 = 0.4$ and $\omega_0 = (\sqrt{5}-1)/15 \approx 8.240453e-2$

Once we chosen the ε_0 and ω_0 we need to compute the initial parameterization. The strategy we used is to take the initial points on the axis x and compute the rotation number approximate by $rn(z,\varepsilon)$. Where $rn(z,\varepsilon)$ (see function `secondary_rn.c` in appendix II) is the average of the the angles between the points z_n and $z_{n+1} = F(z_n,\varepsilon)$ with the initial point is $z_0 = (x_0, 0)$ and $n = 0, \cdots, 100000$.

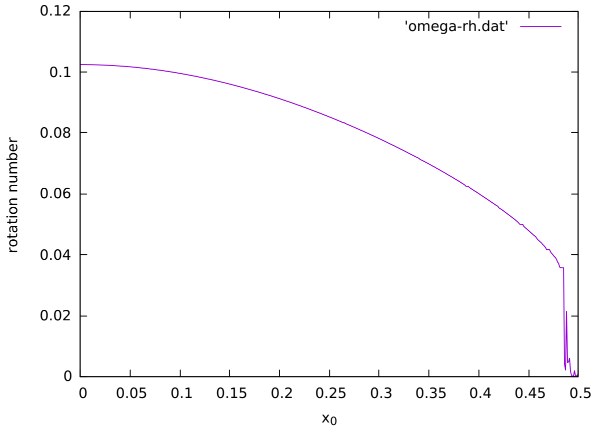


Figure 3.3: Using the input as $\varepsilon = 0.4$

T0 find x_0 corresponding to $rn_0 = rn(x_0, 0, 0.4) \approx \omega_0$ and we use the secant method with the initial guess rn_0 in order to refine the rotation number. Then, we get $\bar{x}_0$ and rn_0 such that $|\omega - rn_0| < 10-^{-12}$.

At this points, we have the parameter ε_0, the non-reasonant frequency ω_0 and a sample of points, that are not in order, of an approximately parameterization K_{ε_0} 0n a non-regular grid. But we can order them and use cubic interpolation so that get a sample of points of K_{ε_0} on a regular grid.

Given the initial parameter ε_0, the non-reasonant frequency $\omega \in \mathbb{R}$ and the initial parameterization. If we continue the method forward and backward , we get the results displyed in Figure (3.5) and Table (3.2).

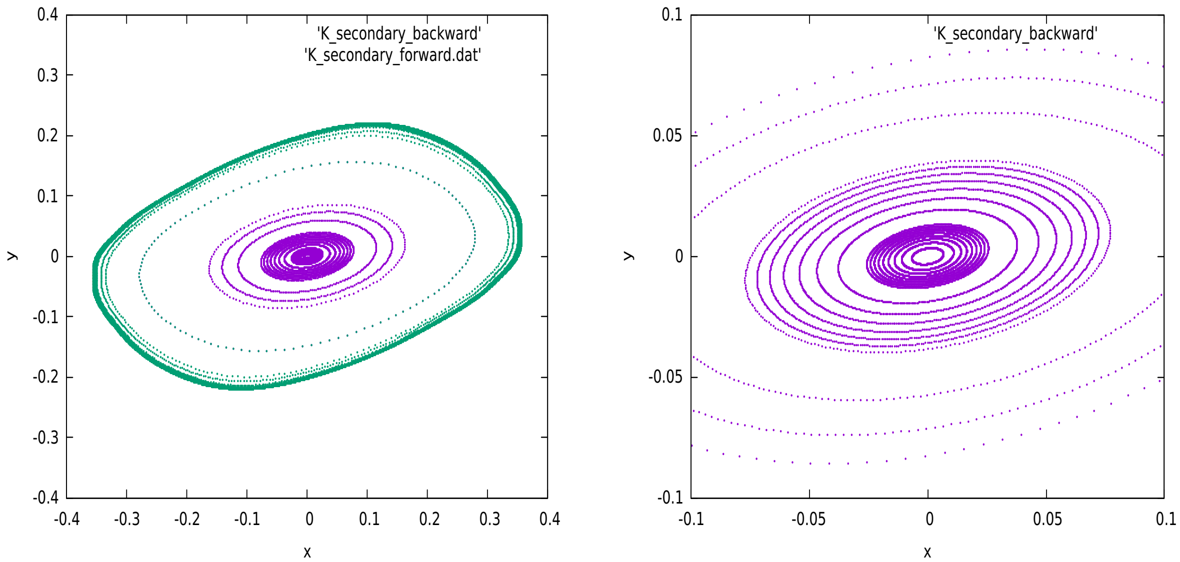
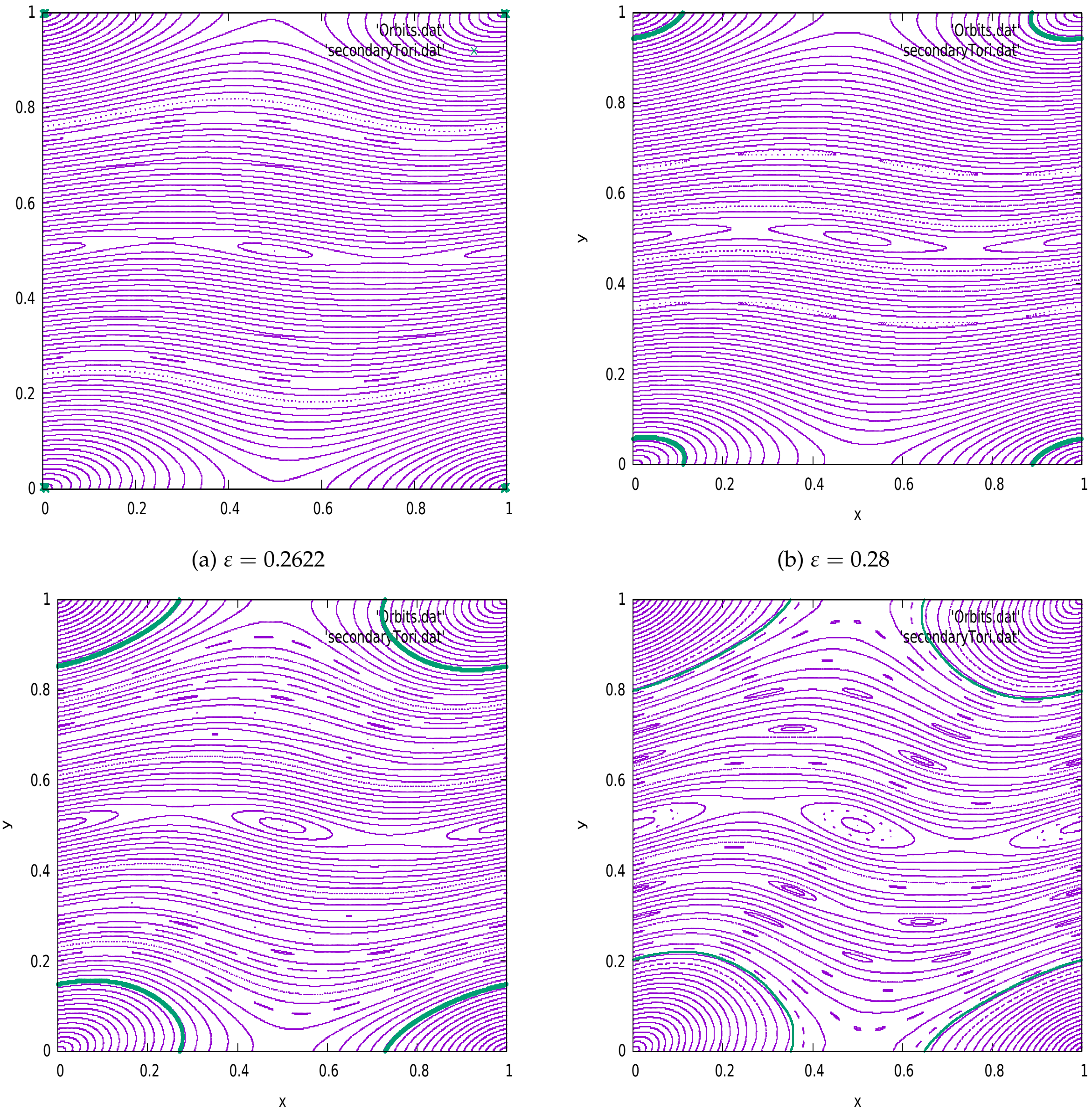


Figure 3.4: Left: The invariant curves obtained from the Continuation method iterating forward and backward. Right: The invariant curve obtained from the Continuation method iterating backward.



(a) $\varepsilon = 0.2622$ (b) $\varepsilon = 0.28$

(c) $\varepsilon = 0.4$ (d) $\varepsilon = 0.56059$

Figure 3.5: The green curves are secondary tori, obtained form Continuation Method with $\omega = (\sqrt{5} - 1)/15$, presented in phase space of Standard map.

ε	N/2	E
0.2622	512	3.2e-9
0.2623	512	1.7e-9
0.2624	512	8.9e-9
0.2625	512	4.4e-9
0.2626	512	2.6e-9
0.2627	512	1.7e-9
0.2628	512	1.1e-9
0.2629	512	8.2e-9
0.2630	256	1.6e-10
0.2640	256	7.6e-10
0.2650	256	3.2e-10
0.2660	256	3.8e-10
0.2670	256	4.3e-10
0.2680	256	4.8e-10
0.2690	256	5.2e-10
0.2700	128	7.3e-10
0.2800	128	9.4e-10
0.2900	128	1.3e-10
0.3000	64	1.5e-9

ε	N/2	E
0.4000	**64**	**3.8e-9**
0.5000	128	5.8e-9
0.5100	256	6.2e-9
0.5200	256	6.7e-9
0.5300	256	7.3e-9
0.5400	256	8.1e-9
0.5500	256	9.1e-9
0.5510	512	9.2e-9
0.5520	512	9.3e-9
0.5530	512	9.4e-9
0.5540	512	9.5e-9
0.5550	512	9.6e-9
0.5560	512	9.7e-9
0.5570	512	9.8e-9
0.5580	512	9.9e-9
0.5581	1024	9.9e-9
0.5582	1024	9.9e-9
0.5583	1024	9.9e-9
0.5584	1024	9.9e-9

Table 3.2: Continuation with respect to parameter ε with the outputs given the number $N/2$ of Fourier coefficients required and the error of invariance. Moreover, from $\varepsilon = 0.4$ upward the quantities are obtained the continuation with respect to ε iterating backward and the rest are obtained iterating the method forward.

Remark 3.7. The upper bound of the breakdown is for $\varepsilon = 0.2622$, see the following table, and it is very near to the parameter corresponding to the elliptic point is $\varepsilon = 0.26214$.

Appendix A

Codes: Primary Tori

Programmed by Chanyan Wang and Alex Haro.
Name of Program: `kam_cont_eps_primary.c`

```
// KAM method in parameterization method with example Standard Map
//it is of form F=(x+Fp, Fp) wihich Fp is a periodic function
//Specially to find Primary tori
// Includes mini-library for complex periodic functions, using dft


#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>
#include <complex.h>

typedef long double real;
typedef long double complex cmplx;


/*---------------CREAT-MEMORY-AND-FREE-MEMORY
    ------------------------------------------------*/
cmplx *allocpf(int N){
cmplx *pf;

if (!(pf= (cmplx *) calloc(N, sizeof(cmplx))))
exit(-1);
return pf;
}

cmplx *doubleF(cmplx *pf, int N){
pf= (cmplx *)realloc(pf, 2*N*sizeof(cmplx));
if (pf==NULL) {
puts("doubling Fourier series fails!\n");
exit(-1);
}
for (int k= N/2; k<N; k++) {
pf[k+N]= pf[k];
pf[k]= 0;
pf[k+N/2]=0;
}
```

```
return pf;

}


/*CREATE MEMORY FOR 2-VECTOR*/
void allocvpf(cmplx *pf[2], int N)
{
pf[0]= allocpf(N); pf[1]= allocpf(N);
}

/*CREATE MEMORY FOR MATRIX 2X2*/
void allocmpf(cmplx *pf[2][2], int N)
{
pf[0][0]= allocpf(N);  pf[0][1]= allocpf(N);
pf[1][0]= allocpf(N);  pf[1][1]= allocpf(N);
}

/*FREE MEMORY FOR 2-VECTOR*/
void freevpf(cmplx *pf[2])
{
free(pf[0]); free(pf[1]);
}

/*FREE MEMORY FOR MATRIX 2X2*/
void freempf(cmplx *pf[2][2])
{
free(pf[0][0]); free(pf[0][1]);
free(pf[1][0]); free(pf[1][1]);
}

/* CLEAN TAIL OF FOURIER SERIES */
void cleanF(cmplx *f, int N)
{
for (int k= N/4; k<3*N/4; k++)
f[k]= 0;
}

/*-------------------FDFT-AND-BDFT
    ----------------------------------------------------------*/
/*GRID TO FOURIER SPACE: FOWARD DISCRETE FOURIER TRANSFORM*/
void fdft(cmplx *fF, int N, cmplx *fG){
int j, k;

for (k= 0; k<N; k++) {
for (j= 0, fF[k]= 0; j<N; j++)
fF[k]+= fG[j]*cexp(-2*M_PI*j*k*I/N);
fF[k]/= N;
}

}

/*FOURIER TO GRID SPACE: BACKWARD DISCRETE FOURIER TRANSFORM*/
void bdft(cmplx *fG, int N, cmplx *fF){
int j, k;
for (j= 0; j<N; j++) {
```

```
for (k= 0, fG[j]= 0; k<N; k++)
fG[j]+= fF[k]*cexp(2*M_PI*j*k*I/N);
}
}

/*------------ This is FFT version----------------*/

void separate (cmplx *a, int n) {
cmplx b[n/2];  // get temp heap storage
int i;
for(i=0; i<n/2; i++)    // copy all odd elements to heap storage
b[i] = a[i*2+1];
for(i=0; i<n/2; i++)    // copy all even elements to lower-half of a[]
a[i] = a[i*2];
for(i=0; i<n/2; i++)    // copy all odd (from heap) to upper-half of a[]
a[i+n/2] = b[i];
}

// N must be a power-of-2, or bad things will happen.
// Currently no check for this condition.
//
// N input samples in X[] are FFT'd and results left in X[].
// Because of Nyquist theorem, N samples means
// only first N/2 FFT results in X[] are the answer.
// (upper half of X[] is a reflection with no new information).

void _ffft(cmplx *X, int N) {
int i, k;
cmplx e, o, w;
if(N < 2) {
// bottom of recursion.
// Do nothing here, because already X[0] = x[0]
} else {
separate(X,N);      // all evens to lower half, all odds to upper half
_ffft(X,     N/2);   // recurse even items
_ffft(X+N/2, N/2);   // recurse odd  items
// combine results of two half recursions
for( k=0; k<N/2; k++) {
e = X[k    ];   // even
o = X[k+N/2];   // odd
// w is the "twiddle-factor"
w = cexpl( -2.*M_PI*I*k/N );
X[k    ] = e/2. + w * o/2.;
X[k+N/2] = e/2. - w * o/2.;
}
}
}

void ffft(cmplx *fF, int N, cmplx *fG){
int k;
for (k=0; k<N; k++){
fF[k]= fG[k];
}
_ffft(fF, N);
}
```

```
void _bfft(cmplx *X, int N) {
int i, k;
cmplx e, o, w;
if(N < 2) {
// bottom of recursion.
// Do nothing here, because already X[0] = x[0]
} else {
separate(X,N);       // all evens to lower half, all odds to upper half
_bfft(X,      N/2);   // recurse even items
_bfft(X+N/2, N/2);   // recurse odd  items
// combine results of two half recursions
for(k=0; k<N/2; k++) {
e = X[k     ];    // even
o = X[k+N/2];    // odd
// w is the "twiddle-factor"
w = cexpl( 2.*M_PI*I*k/N );
X[k     ] = e + w * o;
X[k+N/2] = e - w * o;
}
}
}

void bfft(cmplx *fG, int N, cmplx *fF){
int k;
for (k=0; k<N; k++){
fG[k]= fF[k];
}
_bfft(fG, N);
}



/*---------------OPERATIONS-IN-FOURIER-AND-GRID-SPACES
    -------------------------------------------------------*/
/*VARIABLE i CAN BE USED IN TWO SPACES WHILE j FOR GRID AND k FOR FOURIER*/
void sumcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int i; for (i= 0; i<N; g[i]= f1[i]+f2[i], i++);
}

void difcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int i; for (i= 0; i<N; g[i]= f1[i]-f2[i], i++);
}

void mulcpf(cmplx *g, int N, cmplx c, cmplx *f){
int i; for (i= 0; i<N; g[i]= c*f[i], i++);
}

void prdcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int j; for (j= 0; j<N; g[j]= f1[j]*f2[j], j++);
}

void divcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int j; for (j= 0; j<N; g[j]= f1[j]/f2[j], j++);
}

void invcpf(cmplx *g, int N, cmplx *f){
```

```
int j; for (j= 0; j<N; g[j]= 1./f[j], j++);
}

void copcpf(cmplx *g, int N, cmplx *f){
int j; for (j= 0; j<N; g[j]= f[j], j++);
}

void dercpf(cmplx *g, int N, cmplx *f){
int k;
for (k= 0; k<N/2; k++) g[k]= 2*M_PI*I*k*f[k];
for (k= N/2; k<N; k++) g[k]= 2*M_PI*I*(k-N)*f[k];
}

void rotcpf(cmplx *g, int N, real omega, cmplx *f){
int k;
for (k= 0; k<N/2; k++) g[k]= cexpl(2*M_PI*I*k*omega)*f[k];
for (k= N/2; k<N; k++) g[k]= cexpl(2*M_PI*I*(k-N)*omega)*f[k];
}


void Lcpf(cmplx *g, int N, real omega, cmplx *f){
/* given \xi(x) - \xi(x+\omega) = \eta(x)
Lf(x) = f(x)-f(x+omega) */
int k;
for (k= 0; k<N/2; k++) g[k]= (1-cexpl(2*M_PI*I*k*omega))*f[k];
for (k= N/2; k<N; k++) g[k]= (1-cexpl(2*M_PI*I*(k-N)*omega))*f[k];
}

void Rcpf(cmplx *g, int N, real omega, cmplx *f){
/*Given f(x)-f(x+omega) = g(x)
*Rg(x) = \sum{k \in Z menos 0} \frac{g_{k}}{1-e^{2\pi I k\omega}} */
//note that k empieza des de 1
int k;
g[0]= 0;
for (k= 1; k<N/2; k++) g[k]= f[k]/(1-cexpl(2*M_PI*I*k*omega));
for (k= N/2; k<N; k++) g[k]= f[k]/(1-cexpl(2*M_PI*I*(k-N)*omega));
}

cmplx avgG(cmplx *f, int N){
int j; real avg;
for (j= 0, avg= 0; j<N; avg+= f[j], j++);
return avg/N;
}

cmplx avgF(cmplx *f, int N){
return f[0];
}

real supnorm(cmplx *f, int N){
int j; real max= 0;
for(j= 0; j< N; j++){
if (cabsl(f[j]) > max) max= cabsl(f[j]);
}
return max;
}
```

```
real l1norm(cmplx *f, int N){
int k; real sum= 0;
for(k= 0; k< N; sum+= cabsl(f[k]), k++);
return sum;
}

real taill1norm(cmplx *f, int N)
{
int k; real sum= 0;
for(k= N/4; k< 3*N/4; sum+= cabsl(f[k]), k++);
return sum;
}

real tailsupnorm(cmplx *f, int N)
{
int k; real max= 0;
for(k= N/4; k< 3*N/4; k++){
if (cabsl(f[k]) > max) max= cabsl(f[k]);
}
return max;
}


// KAM
real eps= 0, omega= 0;

/*----------------EVALUATION-IN-THE-PERIODIC-FUNCTIONS-OF-STANDARD-MAP
    ------------------------------------------------------------*/
//THE PERIODIC FUNCTIONS
void Fp(cmplx z[2], cmplx Fp[2]){
Fp[0]= Fp[1]= z[1]-eps/(2*M_PI)*csinl(2*M_PI*z[0]);
}

//THE DERIVATIVE OF THE PERIODIC FUNCTIONS
void DFp(cmplx z[2], cmplx DFp[2][2]){
DFp[0][0]= DFp[1][0]= -eps*ccosl(2*M_PI*z[0]);
DFp[0][1]= DFp[1][1] = 1;
}


/*
    -----------------------------------------------------------------------------------------
    */
/*MAIN*/

int main(int argc, char *argv[])
{
int N, k, i, j, l, count= 1, max_iter =35; //count para contar los paso de
    convergencia
real tolerance = 1.e-10,e =0, correction =0, deltaEps= 0.1, eps0=0;
int ok= -1, first= 1;

eps= 0.0;
N =2*128;
omega= (sqrtl(5)-1)*.5;

```

```
FILE *file;
file = fopen("parameterization","w");
if(file == NULL){
printf("there a problems in opening file\n");
exit(-1);
}

cmplx z[2], Fpz[2], DFpz[2][2], DFN0, DFN1, b, T0, dFeps;
cmplx *TG,
*hetaG0,
*hetaF0;
cmplx *zF[2],
*zG[2],
*rzF[2], *etaG[2],
*rzG[2], *EG[2], *etaF[2], *xiF[2], *deltazG[2],
*xiG[2],
*z0G[2],
*deps0zG[2];
cmplx *PF[2][2], *rPF[2][2],
*PG[2][2],
*rPG[2][2];

allocvpf(zG, N);
allocvpf(rzF, N);
allocvpf(rzG, N);
allocvpf(xiG, N);
allocvpf(zF,N);
allocvpf(z0G,N);
allocvpf(deps0zG,N);

allocmpf(PF, N);
allocmpf(PG, N);
allocmpf(rPG, N);

TG= allocpf(N);
hetaG0= allocpf(N);
hetaF0= allocpf(N);

zF[1][0]= omega;

/*------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------*/
/*CONTINUATION METHOD RESPECT TO EPSILON*/
do{
printf("################################################################################\n");
printf("eps=% LE\n",eps);

count= 0;
do{
count++;
cleanF(zF[0],N); cleanF(zF[1],N);

/*GIVEN THE INITIAL PARAMETERIZATION K REPRESENTED BY zF, REFINE WITH NEWTON
```

```
    STPE */
/*STEP 1: CALCULATIONS OF ERROR*/
bfft(zG[0], N, zF[0]); bfft(zG[1], N, zF[1]);
rotcpf(rzF[0], N, omega, zF[0]); rotcpf(rzF[1], N, omega, zF[1]);
bfft(rzG[0], N, rzF[0]);  bfft(rzG[1], N, rzF[1]);

/*CALCULATIONS OF ERROR IN GRID*/
EG[0]= rzG[0]; EG[1]= rzG[1];
for (j= 0; j<N; j++){
z[0]= (real) j/N + zG[0][j]; z[1]=  zG[1][j];
Fp(z,Fpz);
EG[0][j]= zG[0][j] + Fpz[0] - rzG[0][j] - omega; EG[1][j]= Fpz[1] - rzG[1][j];
}

real err_x = supnorm(EG[0], N);
real err_y = supnorm(EG[1], N);
e = sqrt(err_x*err_x + err_y*err_y);
printf("
    ------------------------------------------------------------------------------------\
    n");
printf("count = %d, error = % Le\n",count, e);
printf("error_x = % Le \n error_y = % Le \n", err_x,err_y);

if(!finite(e)) break;

if (e < tolerance){
ok=1;
printf("######---------refined K for eps=% Le and N= %d -------#####\n",eps,N)
    ;
fprintf(file, "# ----refined K for eps= %Le--------\n",eps);
for(j= 0; j<N; j++){
//printf("% LE % LE\n", (real)j/N +creall(zG[0][j]), creall(zG[1][j]));
fprintf(file, "% LE % LE\n", (real)j/N +creall(zG[0][j]), creall(zG[1][j]));
}
for(j =0; j<N; j++){
EG[0][j] = EG[1][j]=-csin(2*M_PI*((real)j/N +zG[0][j]))/(2*M_PI);
z0G[0][j]= zG[0][j];
z0G[1][j]= zG[1][j];
}
}


/*STEP 2: CONSTRUCT THE FRAME P CONSISTS OF L,N*/
//L = (P[0][0], P[1][0])
dercpf(PF[0][0], N, zF[0]); dercpf(PF[1][0], N, zF[1]);
PF[0][0][0]= 1; // rotating curve
bfft(PG[0][0], N, PF[0][0]); bfft(PG[1][0], N, PF[1][0]);
printf("supnorm PG= % LE % LE\n", supnorm(PG[0][0], N),  supnorm(PG[0][1], N))
    ;

//N = -J L / (LT L)
for (j= 0; j<N; j++){
b= PG[0][0][j]*PG[0][0][j] +  PG[1][0][j]*PG[1][0][j];
PG[0][1][j]= -PG[1][0][j]/b;
PG[1][1][j]=  PG[0][0][j]/b;
}
```

```
printf("              % LE % LE\n", supnorm(PG[1][0], N),  supnorm(PG[1][1], N));
ffft(PF[0][1], N, PG[0][1]); ffft(PF[1][1], N, PG[1][1]);

/*AT THIS POINTS, WE HAVE THE FRAME P IN FOURIER AND GRID*/

/*ROTATED FREM IN FOURIER*/
rPF[0][0]= PF[0][0]; rPF[1][0]= PF[1][0]; rPF[0][1]= PF[0][1]; rPF[1][1]= PF[1][1];
rotcpf(rPF[0][0], N, omega, PF[0][0]); rotcpf(rPF[0][1], N, omega, PF[0][1]);
rotcpf(rPF[1][0], N, omega, PF[1][0]); rotcpf(rPF[1][1], N, omega, PF[1][1]);

/*ROTATED FRAME IN GRID*/
bfft(rPG[0][0], N, rPF[0][0]); bfft(rPG[0][1], N, rPF[0][1]);
bfft(rPG[1][0], N, rPF[1][0]); bfft(rPG[1][1], N, rPF[1][1]);

/*STEP 3: APPLING LEMMA TO FIND xi*/

/*COMPUTATIONS OF TORSION*/
for (j= 0; j<N; j++){
z[0]= (real) j/N + zG[0][j]; z[1]=  zG[1][j];
DFp(z,DFpz);
DFN0= (1 + DFpz[0][0])*PG[0][1][j] + DFpz[0][1]*PG[1][1][j];
DFN1= DFpz[1][0]*PG[0][1][j] + DFpz[1][1]*PG[1][1][j];
TG[j]= rPG[1][1][j]*DFN0 - rPG[0][1][j]*DFN1;
}

/*AVERAGE OF TORSION*/
T0= avgG(TG,N);
printf("torsion T0= % LE % LE\n", creall(T0), cimagl(T0));

/*etaG; eta IN GRID*/
etaG[0]= rzF[0]; etaG[1]= rzF[1]; // To save memory
for (j= 0; j<N; j++){
etaG[0][j]= -(rPG[1][1][j]*EG[0][j] - rPG[0][1][j]*EG[1][j]);
etaG[1][j]=  (rPG[1][0][j]*EG[0][j] - rPG[0][0][j]*EG[1][j]);
}

etaF[0]= EG[0]; etaF[1]= EG[1]; // To save memory
ffft(etaF[0], N, etaG[0]); ffft(etaF[1], N, etaG[1]);

// Xi1
etaF[1]= EG[1]; // To save memory
ffft(etaF[1], N, etaG[1]);
xiF[1]= etaF[1]; // to save space

Rcpf(xiF[1], N, omega, etaF[1]);  // xiF[1] with zero average!
bfft(xiG[1], N, xiF[1]); // with zero average

for(j= 0; j<N; j++)
etaG[0][j]-= TG[j]*xiG[1][j];
printf("#supnorm hetaG0= % Le\n",
supnorm(etaG[0], N));

xiF[1][0] = avgG(etaG[0],N)/T0;
// FALTABA EL BUCLE SIGUIENTE!
```

```
for(j= 0; j<N; j++)
xiG[1][j]+=  xiF[1][0];

for(j= 0; j<N; j++)
etaG[0][j]-= TG[j]*xiF[1][0];
printf("#supnorm hetaG0= % Le\n",
supnorm(etaG[0], N));

etaF[0]= EG[0]; // To save memory
ffft(etaF[0], N, etaG[0]);
xiF[0]= etaF[0];
Rcpf(xiF[0], N, omega, etaF[0]);  // xiF[0][0]= 0;
bfft(xiG[0], N, xiF[0]);


/*AT THIS POINTS, WE HAVE COMPUTED xi IN FOURIER AND GRID*/
deltazG[0]= rzG[0];  deltazG[1]= rzG[1];
for (j= 0; j<N; j++) {
deltazG[0][j]= PG[0][0][j]*xiG[0][j]+ PG[0][1][j]*xiG[1][j];
deltazG[1][j]= PG[1][0][j]*xiG[0][j]+ PG[1][1][j]*xiG[1][j];
}
printf("supnorm xiG= % LE % LE\n", supnorm(xiG[0], N),  supnorm(xiG[1], N));

real error_x = supnorm(deltazG[0], N);
real error_y = supnorm(deltazG[1], N);
correction = sqrt(error_x* error_x + error_y *error_y);

/*THE PERIODIC FUNCTIONS OF THE PARAMETERIZATION K*/
if(e>tolerance){
printf("correc_x = % Le \n correc_y = % Le \n", error_x,error_y);
sumcpf(zG[0], N, zG[0], deltazG[0]);  sumcpf(zG[1], N, zG[1], deltazG[1]);
ffft(zF[0], N, zG[0]); ffft(zF[1], N, zG[1]);
}
}while(ok!=1 && count<6);

if (ok==-1) {
cmplx *z0F[2], *deps0zF[2];

puts("Doubling Fourier Series!!\n");
if (first) goto freedom;
ok= 0;

allocvpf(z0F,N);
ffft(z0F[0],N,z0G[0]); ffft(z0F[1],N,z0G[1]);
z0F[0]= doubleF(z0F[0],N); z0F[1]= doubleF(z0F[1],N);

allocvpf(deps0zF,N);
ffft(deps0zF[0],N,deps0zG[0]); ffft(deps0zF[1],N,deps0zG[1]);
deps0zF[0]= doubleF(deps0zF[0],N); deps0zF[1]= doubleF(deps0zF[1],N);


free(TG); free(hetaG0); free(hetaF0);
freevpf(zG); freevpf(zF); freevpf(rzF); freevpf(rzG); freevpf(xiG);
freempf(PF); freempf(PG); freempf(rPG);

freevpf(z0G); freevpf(deps0zG);
```

```
N= 2*N;
printf("------------------N=%d--------------\n",N);

TG= allocpf(N); hetaG0= allocpf(N); hetaF0= allocpf(N);
allocvpf(zG, N); allocvpf(zF, N); allocvpf(rzF, N); allocvpf(rzG, N); allocvpf
    (xiG, N);
allocmpf(PF, N); allocmpf(PG, N); allocmpf(rPG, N);

allocvpf(z0G,N); allocvpf(deps0zG,N);
bfft(z0G[0],N,z0F[0]);  bfft(z0G[1],N,z0F[1]);
bfft(deps0zG[0],N,deps0zF[0]);  bfft(deps0zG[1],N,deps0zF[1]);
freevpf(z0F); freevpf(deps0zF);
}
else {
if(ok== 0){

deltaEps= deltaEps /10;
eps = eps0 + deltaEps;
}else{
ok=-1;
first= 0;

eps0=eps;
for(j =0; j<N; j++){
deps0zG[0][j]= deltazG[0][j];
deps0zG[1][j]= deltazG[1][j];
}
eps+= deltaEps;
}
/*K_(eps+deltaEps) = K_eps + (DK/d eps)*deltaEps IT IS AN APPROXIMATION */
}
for(k=0; k<N; k++){
zG[0][k]=z0G[0][k] + deps0zG[0][k]*deltaEps;
zG[1][k]=z0G[1][k] + deps0zG[1][k]*deltaEps;
}
ffft(zF[0], N, zG[0]); ffft(zF[1], N, zG[1]);
}while(deltaEps >10e-6);


fclose(file);

freedom:
free(TG); free(hetaG0); free(hetaF0);

freevpf(zG); freevpf(zF); freevpf(rzF); freevpf(rzG); freevpf(xiG);
freevpf(z0G); freevpf(deps0zG);

freempf(PF);
freempf(PG);
freempf(rPG);
}
```

Appendix B

Codes: Secondary Tori

Programmed by Chanyan Wang and Alex Haro
Name of Program: `secondary_rn.c`

```
//Compute the approximate parameterization K
//Given omega,eps find the point (x0,0) has a rotation number
//is different un tolerance 1.e-12 with omega and
// interpolates the initial parameterization in uniform space.

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>
#include <complex.h>

typedef long double real;
typedef long double complex cmplx;


real eps=0.4;

void F(real z[2], real Fz[2]){
//THE STANDARD MAP
Fz[1]= z[1]-eps/(2*M_PI)*sinl(2*M_PI*z[0]);
Fz[0]= z[0] + Fz[1];
}

real secondary_rotation_number(real z0[2], int n)
{
//GIVEN THE INITIAL POINTS AND NUMBER OF ITERATIONS RETURN THE ROTAITON
//NUMBER ASSSOCIATED TO THE POINT
real z[2], Fz[2], rn, ang;
int i;

z[0]= z0[0]; z[1]= z0[1];

for (i= 0, rn= 0; i<n; i++){
F(z, Fz);
rn+= asinl((Fz[0]*z[1]-Fz[1]*z[0])/hypotl(z[0],z[1])/hypotl(Fz[0],Fz[1]));
z[0]= Fz[0]; z[1]= Fz[1];
}
```

```
return rn/(n*2*M_PI);
}

real secant(real z[2], real omega){
real f;

f= omega -secondary_rotation_number(z,10000);
return f;

}
real findz(real omega){
real z[2], xN_1[2], xN[2],deltax=0.0001;
real fN,fN_1,aux,rn;

z[1]= 0; z[0]= 0;
xN_1[1]= xN[1]= 0;

do{
z[0]+= deltax;
rn= secondary_rotation_number(z, 100000);
}while(fabsl(omega -rn)>1.e-3 && z[0] <0.5);

xN_1[0]= z[0];
xN[0] = xN_1[0] + deltax;
do{
fN=secant(xN,omega);
fN_1= secant(xN_1,omega);
aux= xN[0];
xN[0]= xN[0] - fN*(xN[0]-xN_1[0])/(fN-fN_1);
xN_1[0]= aux;
}while(fabsl(secant(xN,omega))>1.e-12);

return xN[0];

}
int compare(const void *pa, const void *pb){
real *a = ( void *)pa;
real *b = ( void *)pb;
int diff = (a[0] > b[0]) -(a[0] < b[0]);
if(diff != 0) return diff;

return (a[1] > b[1]) - (a[1] < b[1]);
}

int main(int argc, char *argv[]){
//WRITE ON A FILE THE SAMPLE OF POINTS OF PARAMETERIZATION ON A REGULAR GRID
    OF SIZE N
int N= 128, NP= 1000;
real z[2], theta_K[NP+3][3],K[N][2];
FILE *file;
file = fopen("initial_K_secondary.dat","w");
if(file == NULL){
printf("there a problems in opening file\n");
exit(-1);
}

```

```
real omega= (sqrtl(5)-1)/15;
z[0]= findz(omega);
z[1]= 0;

real aux1[2],aux2[2];
theta_K[1][0]= 0;
theta_K[1][1]= z[0];
theta_K[1][2]= z[1];
printf("rn= % .20LE\n", omega);
printf("    % .20LE\n",  secondary_rotation_number(z,100000));

//printf("#% LE  % LE  % LE \n",theta_K[0][0], theta_K[0][1], theta_K[0][2]);
fprintf(file,"# % .20LE % .20LE %d\n",eps, omega, N);

for(int i= 2; i<=NP; i++){
aux1[0]= theta_K[i-1][1];
aux1[1]= theta_K[i-1][2] ;
F(aux1, aux2);
theta_K[i][1]= aux2[0];
theta_K[i][2]= aux2[1];
theta_K[i][0]= (theta_K[i-1][0] + omega) - floor(theta_K[i-1][0] + omega);
}

qsort(theta_K+1, NP, 3*sizeof(real), compare);

theta_K[0][0]= theta_K[NP][0]-1;
theta_K[0][1]= theta_K[NP][1];
theta_K[0][2]= theta_K[NP][2];

theta_K[NP+1][0]= theta_K[0][0]+1;
theta_K[NP+1][1]= theta_K[0][1];
theta_K[NP+1][2]= theta_K[0][2];
theta_K[NP+2][0]= theta_K[1][0]+1;
theta_K[NP+2][1]= theta_K[1][1];
theta_K[NP+2][2]= theta_K[1][2];

printf("#--------------AFTER SORTING QSORT--------\n");

for(int i=0; i<= (NP+1); i++)
printf("% LE  % LE  % LE \n",
theta_K[i][0], theta_K[i][1], theta_K[i][2]);


printf("------------interpolation ---------- \n");
real x=0;
int j0, j1=1, j2, j3;
real L0, L1, L2, L3;

K[0][0]= z[0];
K[0][1]= z[1];
fprintf(file,"% .20LE % .20LE % .20LE \n",0.L,K[0][0],K[0][1]);

for(int i=1; i< N; i++){
x= (real) i/N;

while((x-theta_K[j1][0])*(theta_K[j1+1][0]-x)<0 && j1 <NP) j1++;
```

```
j0= j1-1;
j2= j1+1;
j3= j2+1;
//Cubic Interpolation
L0=
(x- theta_K[j1][0])/(theta_K[j0][0]- theta_K[j1][0])*
(x- theta_K[j2][0])/(theta_K[j0][0]- theta_K[j2][0])*
(x- theta_K[j3][0])/(theta_K[j0][0]- theta_K[j3][0]);
L1=
(x- theta_K[j0][0])/(theta_K[j1][0]- theta_K[j0][0])*
(x- theta_K[j2][0])/(theta_K[j1][0]- theta_K[j2][0])*
(x- theta_K[j3][0])/(theta_K[j1][0]- theta_K[j3][0]);
L2=
(x- theta_K[j0][0])/(theta_K[j2][0]- theta_K[j0][0])*
(x- theta_K[j1][0])/(theta_K[j2][0]- theta_K[j1][0])*
(x- theta_K[j3][0])/(theta_K[j2][0]- theta_K[j3][0]);
L3=
(x- theta_K[j0][0])/(theta_K[j3][0]- theta_K[j0][0])*
(x- theta_K[j1][0])/(theta_K[j3][0]- theta_K[j1][0])*
(x- theta_K[j2][0])/(theta_K[j3][0]- theta_K[j2][0]);

K[i][0]= theta_K[j0][1]*L0 + theta_K[j1][1]*L1 +
theta_K[j2][1]*L2 + theta_K[j3][1]*L3;
K[i][1]= theta_K[j0][2]*L0 + theta_K[j1][2]*L1 +
theta_K[j2][2]*L2 + theta_K[j3][2]*L3;
//TO CHECK THERE ARE NO MISTAKES IN ORDERING
//printf("% LE < % LE < % LE \n", theta_K[j1][0], x, theta_K[j2][0]);
fprintf(file, "% .20LE % .20LE % .20LE\n",x, K[i][0],K[i][1]);
}
fprintf(file,"% .20LE % .20LE % .20LE \n",1.L,K[0][0],K[0][1]);
fclose(file);
return 1;
}
```

The one is used to compute the invariant tori using the Algorith (3.3). Name of Program: kam_cont_eps_secondary.c

```
// KAM method in parameterization method with example Standard Map
//Specially to find the Secondary tori
// Includes mini-library for complex periodic functions, using dft


#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <assert.h>
#include <complex.h>

typedef long double real;
typedef long double complex cmplx;


/*---------------CREAT-MEMORY-AND-FREE-MEMORY
    ---------------------------------------------*/
cmplx *allocpf(int N){
cmplx *pf;
```

```
if (!(pf= (cmplx *) calloc(N, sizeof(cmplx))))
exit(-1);
return pf;
}

cmplx *doubleF(cmplx *pf, int N){
pf= (cmplx *)realloc(pf, 2*N*sizeof(cmplx));
if (pf==NULL) {
puts("doubling Fourier series fails!\n");
exit(-1);
}
for (int k= N/2; k<N; k++) {
pf[k+N]= pf[k];
pf[k]= 0;
pf[k+N/2]=0;
}
return pf;

}


/*CREATE MEMORY FOR 2-VECTOR*/
void allocvpf(cmplx *pf[2], int N)
{
pf[0]= allocpf(N); pf[1]= allocpf(N);
}

/*CREATE MEMORY FOR MATRIX 2X2*/
void allocmpf(cmplx *pf[2][2], int N)
{
pf[0][0]= allocpf(N);  pf[0][1]= allocpf(N);
pf[1][0]= allocpf(N);  pf[1][1]= allocpf(N);
}

/*FREE MEMORY FOR 2-VECTOR*/
void freevpf(cmplx *pf[2])
{
free(pf[0]); free(pf[1]);
}

/*FREE MEMORY FOR MATRIX 2X2*/
void freempf(cmplx *pf[2][2])
{
free(pf[0][0]); free(pf[0][1]);
free(pf[1][0]); free(pf[1][1]);
}

/* CLEAN TAIL OF FOURIER SERIES */
void cleanF(cmplx *f, int N)
{
for (int k= N/4; k<3*N/4; k++)
f[k]= 0;
}
```

```
/*------------ This is FFT version----------------*/

void separate (cmplx *a, int n) {
cmplx b[n/2];  // get temp heap storage
int i;
for(i=0; i<n/2; i++)     // copy all odd elements to heap storage
b[i] = a[i*2+1];
for(i=0; i<n/2; i++)     // copy all even elements to lower-half of a[]
a[i] = a[i*2];
for(i=0; i<n/2; i++)     // copy all odd (from heap) to upper-half of a[]
a[i+n/2] = b[i];
}

// N must be a power-of-2, or bad things will happen.
// Currently no check for this condition.
//
// N input samples in X[] are FFT'd and results left in X[].
// Because of Nyquist theorem, N samples means
// only first N/2 FFT results in X[] are the answer.
// (upper half of X[] is a reflection with no new information).

void _ffft(cmplx *X, int N) {
int i, k;
cmplx e, o, w;
if(N < 2) {
// bottom of recursion.
// Do nothing here, because already X[0] = x[0]
} else {
separate(X,N);       // all evens to lower half, all odds to upper half
_ffft(X,     N/2);    // recurse even items
_ffft(X+N/2, N/2);    // recurse odd  items
// combine results of two half recursions
for( k=0; k<N/2; k++) {
e = X[k     ];    // even
o = X[k+N/2];    // odd
// w is the "twiddle-factor"
w = cexpl( -2.*M_PI*I*k/N );
X[k     ] = e/2. + w * o/2.;
X[k+N/2] = e/2. - w * o/2.;
}
}
}

void ffft(cmplx *fF, int N, cmplx *fG){
int k;
for (k=0; k<N; k++){
fF[k]= fG[k];
}
_ffft(fF, N);
}

void _bfft(cmplx *X, int N) {
int i, k;
cmplx e, o, w;
if(N < 2) {
// bottom of recursion.
```

```
// Do nothing here, because already X[0] = x[0]
} else {
separate(X,N);      // all evens to lower half, all odds to upper half
_bfft(X,     N/2);   // recurse even items
_bfft(X+N/2, N/2);   // recurse odd  items
// combine results of two half recursions
for(k=0; k<N/2; k++) {
e = X[k    ];   // even
o = X[k+N/2];   // odd
// w is the "twiddle-factor"
w = cexpl( 2.*M_PI*I*k/N );
X[k    ] = e + w * o;
X[k+N/2] = e - w * o;
}
}
}

void bfft(cmplx *fG, int N, cmplx *fF){
int k;
for (k=0; k<N; k++){
fG[k]= fF[k];
}
_bfft(fG, N);
}



/*---------------OPERATIONS-IN-FOURIER-AND-GRID-SPACES
    -----------------------------------------------------*/
/*VARIABLE i CAN BE USED IN TWO SPACES WHILE j FOR GRID AND k FOR FOURIER*/
void sumcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int i; for (i= 0; i<N; g[i]= f1[i]+f2[i], i++);
}

void difcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int i; for (i= 0; i<N; g[i]= f1[i]-f2[i], i++);
}

void mulcpf(cmplx *g, int N, cmplx c, cmplx *f){
int i; for (i= 0; i<N; g[i]= c*f[i], i++);
}

void prdcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int j; for (j= 0; j<N; g[j]= f1[j]*f2[j], j++);
}

void divcpf(cmplx *g, int N, cmplx *f1, cmplx *f2){
int j; for (j= 0; j<N; g[j]= f1[j]/f2[j], j++);
}

void invcpf(cmplx *g, int N, cmplx *f){
int j; for (j= 0; j<N; g[j]= 1./f[j], j++);
}

void copcpf(cmplx *g, int N, cmplx *f){
int j; for (j= 0; j<N; g[j]= f[j], j++);
```

```
}

void dercpf(cmplx *g, int N, cmplx *f){
int k;
for (k= 0; k<N/2; k++) g[k]= 2*M_PI*I*k*f[k];
for (k= N/2; k<N; k++) g[k]= 2*M_PI*I*(k-N)*f[k];
}

void rotcpf(cmplx *g, int N, real omega, cmplx *f){
int k;
for (k= 0; k<N/2; k++) g[k]= cexpl(2*M_PI*I*k*omega)*f[k];
for (k= N/2; k<N; k++) g[k]= cexpl(2*M_PI*I*(k-N)*omega)*f[k];
}


void Lcpf(cmplx *g, int N, real omega, cmplx *f){
/* given \xi(x) - \xi(x+\omega) = \eta(x)
Lf(x) = f(x)-f(x+omega) */
int k;
for (k= 0; k<N/2; k++) g[k]= (1-cexpl(2*M_PI*I*k*omega))*f[k];
for (k= N/2; k<N; k++) g[k]= (1-cexpl(2*M_PI*I*(k-N)*omega))*f[k];
}

void Rcpf(cmplx *g, int N, real omega, cmplx *f){
/*Given f(x)-f(x+omega) = g(x)
*Rg(x) = \sum{k \in Z menos 0} \frac{g_{k}}{1-e^{2\pi I k\omega}} */
//note that k empieza des de 1
int k;
g[0]= 0;
for (k= 1; k<N/2; k++) g[k]= f[k]/(1-cexpl(2*M_PI*I*k*omega));
for (k= N/2; k<N; k++) g[k]= f[k]/(1-cexpl(2*M_PI*I*(k-N)*omega));
}

cmplx avgG(cmplx *f, int N){
int j; real avg;
for (j= 0, avg= 0; j<N; avg+= f[j], j++);
return avg/N;
}

cmplx avgF(cmplx *f, int N){
return f[0];
}

real supnorm(cmplx *f, int N){
int j; real max= 0;
for(j= 0; j< N; j++){
if (cabsl(f[j]) > max) max= cabsl(f[j]);
}
return max;
}

real l1norm(cmplx *f, int N){
int k; real sum= 0;
for(k= 0; k< N; sum+= cabsl(f[k]), k++);
return sum;
}
```

```
real taill1norm(cmplx *f, int N)
{
int k; real sum= 0;
for(k= N/4; k< 3*N/4; sum+= cabsl(f[k]), k++);
return sum;
}

real tailsupnorm(cmplx *f, int N)
{
int k; real max= 0;
for(k= N/4; k< 3*N/4; k++){
if (cabsl(f[k]) > max) max= cabsl(f[k]);
}
return max;
}


// KAM
real eps= 0, omega= 0;

/*----------------EVALUATION-IN-THE-PERIODIC-FUNCTIONS-OF-STANDARD-MAP
    -------------------------------------------------------------*/
//THE PERIODIC FUNCTIONS
void Fp(cmplx z[2], cmplx Fp[2]){
Fp[0]= Fp[1]= z[1]-eps/(2*M_PI)*csinl(2*M_PI*z[0]);
}

//THE DERIVATIVE OF THE PERIODIC FUNCTIONS
void DFp(cmplx z[2], cmplx DFp[2][2]){
DFp[0][0]= DFp[1][0]= -eps*ccosl(2*M_PI*z[0]);
DFp[0][1]= DFp[1][1] = 1;
}


/*
    ------------------------------------------------------------------------------------------
    */
/*MAIN*/

int main(int argc, char *argv[])
{
int N, k, i, j, l, count= 1;
//deltaEps=0.1: the continuation method iterating forward and deltaEps=0.1
    iterating backward
real tolerance = 1.e-8,e =0, correction =0, deltaEps= -0.1, eps0=0;
int ok= -1, first= 1;
real a,c,aux;


FILE *file,*inputs;
file = fopen("Kparameterization.dat","w");
if(file == NULL){
printf("there a problems in opening file\n");
exit(-1);
}
```

```
if ((inputs = fopen("initial_K_secondary.dat", "r")) == NULL)
return EXIT_FAILURE;
char sost;
fscanf(inputs, "%c %LE %LE %d", &sost, &eps, &omega, &N);
printf("eps = %LE omega=% LE N=%d\n",eps, omega, N);

cmplx z[2], Fpz[2], DFpz[2][2], DFN0, DFN1, b, T0, dFeps;
cmplx *TG,
*hetaG0,
*hetaF0;
cmplx *zF[2],
*zG[2],
*rzF[2], *etaG[2],
*rzG[2], *EG[2], *etaF[2], *xiF[2], *deltazG[2],
*xiG[2],
*z0G[2],
*deps0zG[2];
cmplx *PF[2][2], *rPF[2][2],
*PG[2][2],
*rPG[2][2];


allocvpf(zG,N);
allocvpf(zF,N);
for (j = 0; j < N; j++){
fscanf(inputs, " %LE %LE  %LE ",&aux, &a, &c);
zG[0][j]= a + 0*I;
zG[1][j]= c + 0*I;
}

fclose(inputs);
ffft(zF[0], N, zG[0]); ffft(zF[1], N, zG[1]);

allocvpf(rzF, N);
allocvpf(rzG, N);
allocvpf(xiG, N);

allocvpf(z0G,N);
allocvpf(deps0zG,N);


allocmpf(PF, N);
allocmpf(PG, N);
allocmpf(rPG, N);

TG= allocpf(N);
hetaG0= allocpf(N);
hetaF0= allocpf(N);



/*
    ---------------------------------------------------------------------------------------
    */
/*CONTINUATION METHOD RESPECT TO EPSILON*/
do{
```

```
printf("
    #############################################################################\
    n");
printf("eps=% LE\n",eps);

count= 0;
do{

count++;
printf("tail K: % LE %LE\n", tailsupnorm(zF[0],N),
tailsupnorm(zF[1],N));
cleanF(zF[0],N); cleanF(zF[1],N);

/*GIVEN THE INITIAL PARAMETERIZATION K REPRESENTED BY zF, REFINE WITH NEWTON
    STPE */
/*STEP 1: CALCULATIONS OF ERROR*/
bfft(zG[0], N, zF[0]); bfft(zG[1], N, zF[1]);
rotcpf(rzF[0], N, omega, zF[0]); rotcpf(rzF[1], N, omega, zF[1]);
bfft(rzG[0], N, rzF[0]);  bfft(rzG[1], N, rzF[1]);

/*CALCULATIONS OF ERROR IN GRID*/
EG[0]= rzG[0]; EG[1]= rzG[1];
for (j= 0; j<N; j++){
z[0]= zG[0][j]; z[1]=  zG[1][j]; //modified respecto to primary tori
Fp(z,Fpz);
EG[0][j]= zG[0][j] + Fpz[0] - rzG[0][j];
EG[1][j]= Fpz[1] - rzG[1][j];  //modified
}

real err_x = supnorm(EG[0], N);
real err_y = supnorm(EG[1], N);
e = sqrt(err_x*err_x + err_y*err_y);
printf("
    --------------------------------------------------------------------------------\
    n");
printf("count = %d, error = % Le\n",count, e);
printf("error_x = % Le \n error_y = % Le \n", err_x,err_y);

if(!finite(e)) break;

if (e < tolerance){
ok=1;
printf("######---------refined K for eps=% Le and N=%d-------####\n",eps,N);
fprintf(file,"#---------refined K for eps=% Le-------\n",eps);
for(j= 0; j<N; j++){
fprintf(file, "% LE % LE\n", creall(zG[0][j]), creall(zG[1][j]));
}
for(j =0; j<N; j++){
EG[0][j] = EG[1][j]=-csin(2*M_PI*(zG[0][j]))/(2*M_PI);  //modified
z0G[0][j]= zG[0][j];
z0G[1][j]= zG[1][j];
}
}

cmplx calabi=0+I*0;

```

```
/*STEP 2: CONSTRUCT THE FRAME P CONSISTS OF L,N*/
//L = (P[0][0], P[1][0])
dercpf(PF[0][0], N, zF[0]); dercpf(PF[1][0], N, zF[1]);
// PF[0][0][0]= 0; It is a libration curve. Modified
bfft(PG[0][0], N, PF[0][0]); bfft(PG[1][0], N, PF[1][0]);
printf("supnorm PG= % LE % LE\n",
supnorm(PG[0][0], N),  supnorm(PG[1][0], N));

//N = -J L / (LT L)
for (j= 0; j<N; j++){
b= PG[0][0][j]*PG[0][0][j] +  PG[1][0][j]*PG[1][0][j];
PG[0][1][j]= -PG[1][0][j]/b;
PG[1][1][j]=  PG[0][0][j]/b;
}
printf("            % LE % LE\n",
supnorm(PG[0][1], N),  supnorm(PG[1][1], N));
ffft(PF[0][1], N, PG[0][1]); ffft(PF[1][1], N, PG[1][1]);

/*AT THIS POINTS, WE HAVE THE FRAME P IN FOURIER AND GRID*/
/*STEP 2.5: a scaled frame */
for(j= 1; j<N-1; j++){
c+=(PG[0][0][j]*zG[1][j]-PG[1][0][j]*zG[0][j]);
}
c/=(2*N) ;
printf("invariant calabi = % LE\n",creall(c));
cmplx r;
r= sqrtl(fabsl(c)/M_PI);
printf("radi = %LE\n",creall(r));
for(j= 0; j<N; j++){
PG[0][0][j] /=r;
PG[1][0][j] /=r;
PG[0][1][j] *=r;
PG[1][1][j] *=r;

PF[0][0][j] /=r;
PF[1][0][j] /=r;
PF[0][1][j] *=r;
PF[1][1][j] *=r;

}


printf("supnorm  scaled PG= % LE % LE\n",
supnorm(PG[0][0], N),  supnorm(PG[1][0], N));
printf("            % LE % LE\n",
supnorm(PG[0][1], N),  supnorm(PG[1][1], N));

/*ROTATED FRAME IN FOURIER*/
rPF[0][0]= PF[0][0]; rPF[1][0]= PF[1][0];
rPF[0][1]= PF[0][1]; rPF[1][1]= PF[1][1];
rotcpf(rPF[0][0], N, omega, PF[0][0]);
rotcpf(rPF[0][1], N, omega, PF[0][1]);
rotcpf(rPF[1][0], N, omega, PF[1][0]);
rotcpf(rPF[1][1], N, omega, PF[1][1]);

/*ROTATED FRAME IN GRID*/
```

```
bfft(rPG[0][0], N, rPF[0][0]); bfft(rPG[0][1], N, rPF[0][1]);
bfft(rPG[1][0], N, rPF[1][0]); bfft(rPG[1][1], N, rPF[1][1]);

/*STEP 3: APPLING LEMMA TO FIND xi*/

/*COMPUTATIONS OF TORSION*/
for (j= 0; j<N; j++){
z[0]= zG[0][j]; z[1]=  zG[1][j];  //modified
DFp(z,DFpz);
DFN0= (1+DFpz[0][0])*PG[0][1][j] + DFpz[0][1]*PG[1][1][j];
DFN1= DFpz[1][0]*PG[0][1][j] + DFpz[1][1]*PG[1][1][j];
TG[j]= rPG[1][1][j]*DFN0 - rPG[0][1][j]*DFN1;
}

/*AVERAGE OF TORSION*/
T0= avgG(TG,N);
printf("torsion T0= % LE % LE\n", creall(T0), cimagl(T0));

/*etaG; eta IN GRID*/
etaG[0]= rzF[0]; etaG[1]= rzF[1]; // To save memory
for (j= 0; j<N; j++){
etaG[0][j]= -(rPG[1][1][j]*EG[0][j]-rPG[0][1][j]*EG[1][j]);
etaG[1][j]=  (rPG[1][0][j]*EG[0][j]-rPG[0][0][j]*EG[1][j]);
}

etaF[0]= EG[0]; etaF[1]= EG[1]; // To save memory
ffft(etaF[0], N, etaG[0]); ffft(etaF[1], N, etaG[1]);

printf(" average etaN= % LE + I % LE\n",
creall(etaF[1][0]), cimagl(etaF[1][0]));
// Xi1
etaF[1]= EG[1]; // To save memory
ffft(etaF[1], N, etaG[1]);
xiF[1]= etaF[1]; // to save space

Rcpf(xiF[1], N, omega, etaF[1]);  // xiF[1] with zero average!
bfft(xiG[1], N, xiF[1]); // with zero average

for(j= 0; j<N; j++)
etaG[0][j]-= TG[j]*xiG[1][j];
printf("#supnorm hetaG0= % Le\n",
supnorm(etaG[0], N));

xiF[1][0] = avgG(etaG[0],N)/T0;
// FALTABA EL BUCLE SIGUIENTE!
for(j= 0; j<N; j++)
xiG[1][j]+=  xiF[1][0];

for(j= 0; j<N; j++)
etaG[0][j]-= TG[j]*xiF[1][0];
printf("#supnorm hetaG0= % Le\n",
supnorm(etaG[0], N));

etaF[0]= EG[0]; // To save memory
ffft(etaF[0], N, etaG[0]);
xiF[0]= etaF[0];
```

```
Rcpf(xiF[0], N, omega, etaF[0]);  // xiF[0][0]= 0;
bfft(xiG[0], N, xiF[0]);


/*AT THIS POINTS, WE HAVE COMPUTED xi IN FOURIER AND GRID*/
deltazG[0]= rzG[0];  deltazG[1]= rzG[1];
for (j= 0; j<N; j++) {
deltazG[0][j]= PG[0][0][j]*xiG[0][j]+PG[0][1][j]*xiG[1][j];
deltazG[1][j]= PG[1][0][j]*xiG[0][j]+PG[1][1][j]*xiG[1][j];
}
printf("supnorm xiG= % LE % LE\n",
supnorm(xiG[0], N),  supnorm(xiG[1], N));

real error_x = supnorm(deltazG[0], N);
real error_y = supnorm(deltazG[1], N);
correction = sqrt(error_x* error_x + error_y *error_y);
/*THE PERIODIC FUNCTIONS OF THE PARAMETERIZATION K*/
if(e>tolerance){
printf("correc_x = % Le \n correc_y = % Le \n", error_x,error_y);
sumcpf(zG[0], N, zG[0], deltazG[0]);  sumcpf(zG[1], N, zG[1], deltazG[1]);
ffft(zF[0], N, zG[0]); ffft(zF[1], N, zG[1]);
}
}while(ok!=1 && count<6);

if (ok==-1) {
cmplx *z0F[2], *deps0zF[2];

puts("Doubling Fourier Series!!\n");
if (first) goto freedom;
ok= 0;

allocvpf(z0F,N);
ffft(z0F[0],N,z0G[0]); ffft(z0F[1],N,z0G[1]);
z0F[0]= doubleF(z0F[0],N); z0F[1]= doubleF(z0F[1],N);

allocvpf(deps0zF,N);
ffft(deps0zF[0],N,deps0zG[0]); ffft(deps0zF[1],N,deps0zG[1]);
deps0zF[0]= doubleF(deps0zF[0],N); deps0zF[1]= doubleF(deps0zF[1],N);


free(TG); free(hetaG0); free(hetaF0);
freevpf(zG); freevpf(zF); freevpf(rzF); freevpf(rzG); freevpf(xiG);
freempf(PF); freempf(PG); freempf(rPG);

freevpf(z0G); freevpf(deps0zG);

N= 2*N;
printf("------------------N=%d--------------\n",N);

TG= allocpf(N); hetaG0= allocpf(N); hetaF0= allocpf(N);
allocvpf(zG, N); allocvpf(zF, N); allocvpf(rzF, N); allocvpf(rzG, N); allocvpf
    (xiG, N);
allocmpf(PF, N); allocmpf(PG, N); allocmpf(rPG, N);

allocvpf(z0G,N); allocvpf(deps0zG,N);
bfft(z0G[0],N,z0F[0]);  bfft(z0G[1],N,z0F[1]);
```

```
bfft(deps0zG[0],N,deps0zF[0]);  bfft(deps0zG[1],N,deps0zF[1]);

freevpf(z0F); freevpf(deps0zF);
}
else {
if(ok== 0){

deltaEps= deltaEps /10;
eps = eps0 + deltaEps;
}else{
ok=-1;
first= 0;

eps0=eps;
for(j =0; j<N; j++){
deps0zG[0][j]= deltazG[0][j];
deps0zG[1][j]= deltazG[1][j];
}
eps+= deltaEps;
}
/*K_(eps+deltaEps) = K_eps + (dK/d eps)*deltaEps IT IS AN APPROXIMATION */
}
for(k=0; k<N; k++){
zG[0][k]=z0G[0][k] + deps0zG[0][k]*deltaEps;
zG[1][k]=z0G[1][k] + deps0zG[1][k]*deltaEps;
}
ffft(zF[0], N, zG[0]); ffft(zF[1], N, zG[1]);
}while(fabsl(deltaEps) >10e-6);

fclose(file);

freedom:
free(TG); free(hetaG0); free(hetaF0);

freevpf(zG); freevpf(zF); freevpf(rzF); freevpf(rzG); freevpf(xiG);
freevpf(z0G); freevpf(deps0zG);

freempf(PF);
freempf(PG);
freempf(rPG);
}
```

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