MASTER'S THESIS

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Title: Combinatorial properties of convex polygons in point sets

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Abstract

The Erdős-Szekeres theorem is a famous result in Discrete geometry that inspired a lot of research and motivated new problems. The theorem states that for every integer $n \geq 3$ there is another integer N_0 such that any set of $N \geq N_0$ points in general position in the plane contains the vertex set of a convex $n-$gon. Related is the question on the number of empty (without interior points) convex $k-$gons, X_k, in a set of n points, for $k = 3, 4, 5, \ldots$. A known result states that the alternating sum of the X_k's only depends on the number n of points, but not on the precise positions of the points. A proof was given by Pinchasi, Radoičić, and Sharir in 2006. In this thesis we extend this result to the numbers of convex $k-$gons with ℓ interior points, $X_{k,\ell}$, and provide several formulas involving these numbers. All these formulas only depend on the number n of points of the set. The proofs are based on a continuous motion argument. We further show that with this proof technique at most $n-2$ linearly independent equations for the $X_{k,\ell}$'s can be obtained and we provide $n-2$ such equations. We also obtain several other formulas, building upon a work by Huemer, Oliveros, Pérez-Lantero, and Vogtenhuber.

The obtained formulas could further be useful to solve some open problems related to the Erdős-Szekeres theorem. This thesis also surveys several known results and questions related to this classical problem for point sets in the plane.

Resum

El teorema d'Erdős-Szekeres és un famós resultat en geometria discreta que ha inspirat molts investigadors i ha motivat nous problemes. El teorema enuncia que per un enter $n \geq 3$ existeix un N_0 tal que, per qualsevol conjunt de $N \geq N_0$ punts en posició general en el pla, hi ha un conjunt de vèrtexs d'un $n-$gon convex. Relacionada amb aquest resultat tenim la qüestió sobre el número de $k-$gons convexos buits (sense punts interiors), X_k, en un conjunt de n punts, per $k = 3, 4, 5, \ldots$. Un conegut resultat enuncia que la suma alternada de X_k's només depen del número n de punts, però no de la posició dels punts, una demostració del qual va ser donada per Pinchasi, Radoičić, i Sharir el 2006. En aquesta tesi estenem aquest resultat a números de $k-$gons convexos amb ℓ punts interiors, $X_{k,\ell}$, i aportem diverses fórmules que involucren aquests números. Totes aquestes fórmules només depenen del número n de punts del conjunt i les demostracions estan basades amb l'argument del moviment continu. A més a més, mostrem que amb aquesta tècnica de demostració com a molt podem obtenir $n-2$ equacions linealment independents pels $X_{k,\ell}$'s, i nosaltres n'aportem $n-2$. També obtenim diverses altres fórmules, continuant el treball de Huemer, Oliveros, Pérez-Lantero, i Vogtenhuber.

Les fórmules obtingudes podrien ser usades per resoldre alguns problemes oberts relacionats amb el teorema d'Erdős-Szekeres. Aquesta tesi també resumeix diversos resultats coneguts i qüestions relacionades amb aquest problema clàssic per conjunts de punts en el pla.

Acknowledgments

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Contents

Chapter 1

Introduction

In 1935 Paul Erdős and George Szekeres formulated their classical problem on the existence of convex polygons in a planar point set. The publication [8] was the first joint paper of these mathematicians who also were part of a group of young Jewish Hungarian mathematicians in Budapest.

For many years, many researchers worked in problems motivated by this first publication and there are still open questions to resolve, described in different surveys such as [4, 17, 23]. The most notorious ones are to to solve the conjecture by Erdős and Szekeres on the existence on convex polygons and to determine the smallest integer N such that every set of N points contains an empty convex hexagon.

The aim of this thesis is to derive some relationships on the number of polygons on a point set with a given number of points in their interior and to connect them to theorems related with the Erdős-Szekeres Theorem and its generalizations.

More specifically, we focus on problems on the existence of empty convex polygons and on the existence of convex polygons with at least a given number of points in their interior, for which the developments have been started more recently.

In particular, during the work carried out in this thesis, we have analyzed the paper of Pinchasi et al. [27] which provides several equalities and inequalities involving the number of empty polygons, and several related quantities. In this work, the authors also present some implication of these relations and discuss their connection to some open problems.

Furthermore, Huemer et al. [13] extended these results to polygons having a fixed number of points in their interior. In consequence, they deduce formulas for sums of numbers of polygons with a given number of points in their interior.

In this thesis, we continuous the work started in [13] and provide news formulas that depend only on the size of the planar point set but not on the precise position of the points, and we also show new inequalities derived from the previous work. These inequalities extend previous results from [27] and related papers [1, 5, 6, 16].Moreover, these new expressions might be used to prove some of the results related to the Erdős-Szekeres

Theorem using adequate algorithms and computer implementation.

Let us briefly describe the kind of formulas we obtained in this work.

Let S be a set of n points in the plane in general position (that is, no three points of S are collinear) and let $X_{k,l}(S)$ be the number of convex $k-$gons (a $k-$gon is a simple polygon that is spanned by exactly k points) in S that have exactly ℓ points of S in their interior. An example of a set of 6 points is shown in Fig. 1.1 and its values $X_{k,\ell}$ are described in Table 1.1.

Figure 1.1: Set of 6 points.

k \ ℓ	0	1	2	3
3	17	3	0	0
4	9	3	0	0
5	2	1	0	0
6	0	0	0	0

Table 1.1: The values $X_{k,\ell}$ for the set described in Fig. 1.1.

With this notation, one of the interesting expressions proved in [27] and also knwon from other papers [1, 5, 6, 16] is

$$\sum_{k\geq 3}(-1)^{k+1}X_{k,0}(S) = \binom{n}{2} - n + 1. \tag{1.1}$$

This expression, the so-called zero-th alternating moment of S for the convex $k-$gons that do not contain any point of S in the interior, was a starting point to derive more formulas in this thesis. Specifically, we consider, in the same direction as [13], different weighted functions $f(k,\ell)$ and general weighted sums $\sum_{k\geq 3}\sum_{\ell\geq 0} f(k,\ell)X_{k,\ell}$ which only depend on the number n of points. For instance we prove that for $f(k,\ell) = x^k(1+x)^\ell$, where x is any real number, the following equality holds:

$$\sum_{k=3}^{n}\sum_{\ell=0}^{n-k} x^k\,(1+x)^\ell\,X_{k,\ell} = (1+x)^n - 1 - x\cdot n - x^2\binom{n}{2}, \quad \text{for any } x \in \mathbb{R}. \tag{1.2}$$

Therefore, in this thesis we obtain new expressions with the property that the value of the sum is invariant over all point sets S of same cardinality.

This work is organized in three different parts.

First, in Section 1.1, we present the necessary notation. Then, in Section 1.2, the introduction contains an overview of the results related to Erdős-Szekeres Theorem and the progress to this interesting problem. In Section 1.3 of this introduction, we present the proof technique of the continuous motion argument

In Chapter 2, we describe the results proved in [27] and [13], which are the starting points of this thesis. In this chapter, we then extend some of these results and find new formulas for the numbers of convex polygons using similar techniques as in [13, 27].

In Chapter 3, we show the results obtained for specific configurations, the so- called double chain in Section 3.1, and the so-called double circle in Section 3.2.

1.1 Notation

In this work we consider a set S of n points in the plane in general position. A set S is in *general position* if no three points of S lie on a line.

A $k-$*gon* is a simple polygon spanned by k points of S. Moreover, a $k-$gon is *convex* if every segment connecting two points of the $k-$gon lies entirely inside the $k-$gon. A $k-$gon is *empty* if it does not contain points of S in its interior.

The *convex hull* of a set S is the smallest convex set that contains S and we will denote with h the number of points in the boundary of the convex hull, which are also called *extreme points* of S. In consequence, the other points $n-h$ points of S are denoted as *interior points*.

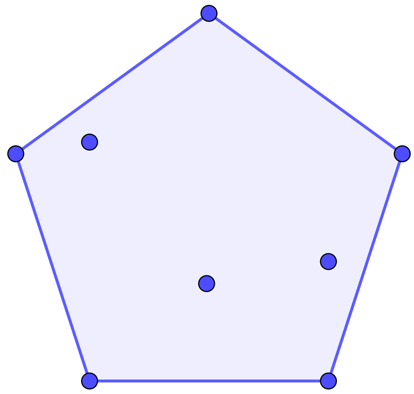


Figure 1.2: A planar point set S of $n = 8$ points in general position, with $h = 5$ points in the boundary of the convex hull and with 3 interior points.

1.2 Previous results

Many results on this topic can be found in the surveys [23, 17]. In the following we also give an overview.

The origin of the classical Erdős-Szekeres problem started around 1932 when the Hungarian Esther Klein observed that any set of 5 points in general position in the plane contains a convex quadrilateral.

Proposition 1.2.1 (Klein)**.** *Any set of at least 5 points in general position in the plane contains 4 points that are the vertices of a convex quadrilateral.*

Proof. Consider the convex hull of the set of five points.

If it is a pentagon or a quadrilateral, then we are done.

Otherwise, it is a triangle and two points must lie in its interior. Therefore, the line determined by these two points divides the triangle into two parts such that, by the pigeonhole principle, two vertices of the triangle must lie on the same side of the line. Thus, these two vertices of the triangle and the two points inside the triangle form a convex quadrilateral. □

Observation 1.2.2. *A visual proof of Proposition 1.2.1 is shown in Fig. 1.3.*

Moreover, she noticed that this result has a generalization.

Problem 1.2.3. *Given an integer $k \geq 3$, does there exist a positive integer $N(k)$ such that any set of at least $N(k)$ points in general position in the plane, contains at least one convex $k-$gon?*

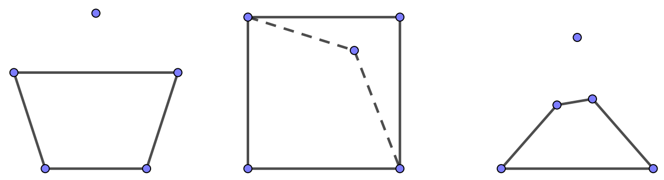


Figure 1.3: Three different configurations of a set of 5 points according to the size of the convex hull.

This problem was answered in 1935 by her friends Erdős and Szekeres, see [8]. Furthermore, this problem was named *The Happy End Problem* by Erdős because it led to the marriage of the mathematicians George Szekeres and Esther Klein. More details of the history behind the problem can be found in [30].

Note that, this problem can be understood as a geometric version of Ramsey's Theorem.

Theorem 1.2.4 (Ramsey). *Given integers $k > 0$ and $\ell_1, \ldots, \ell_r \geq k$, there is a positive integer $R(k, \ell_1, \ldots, \ell_r)$ such that for all $n \geq R(k, \ell_1, \ldots, \ell_r)$ the following holds. For each $r-$coloring of the $k-$subsets of $[n]$, there is $i \in [r]$ and a l_i-subset such that all its $k-$subsets are $i-$colored.*

In fact, the first proof of the existence of $N(k)$ was based on a rediscovering of Ramsey's theorem by the authors of [8], independent of Ramsey's publication [28].

To solve the problem, they also used the following result.

Lemma 1.2.5. *If every four points in a set of k points form a convex quadrilateral, then the k points are in convex position.*

Proof. Suppose that the boundary of the convex hull of these k points is not a $k-$gon. Then there exists some point P in its interior.

Triangulate the convex hull such that P lies in the interior of one of these triangles, which contradicts the convexity hypothesis. □

Now we can prove the classical Erdős-Szekeres theorem.

Theorem 1.2.6 (Erdős-Szekeres). *For every $k \geq 3$, there is a positive integer $N(k)$ such that every set of $n \geq N(k)$ points in general position in the plane contains at least one convex $k-$gon.*

Proof. Consider Ramsey's Theorem with $r = 2$.

Let $n \geq R(4, k, 5)$ points in general position, and for every subset of 4 points consider their convex hull. If it is a quadrilateral, color it with 1; if it is a triangle, color it with 2.

By Ramsey's theorem, one of the following holds:

(i) there is a $k-$subset for which every subset of 4 points is colored with 1,

(ii) there is a $5-$subset for which every subset of 4 points is colored with 2.

By Proposition 1.2.1, the case (ii) is not possible. Therefore, the case (i) must hold which, by Lemma 1.2.5, gives the result. □

In the same paper [8], they described another method based on consideration of convex and concave sequences of points that produces a better upper bound $N(k) \leq \binom{2k-4}{k-2} + 1$. Moreover, Erdős and Szekeres posed the following conjecture.

Conjecture 1.2.7. *For any $k \geq 3$, $N(k) = 2^{k-2} + 1$.*

This conjectured exact result is only solved for the values $k = 3, 4, 5$ and 6.

Clearly, the case $k = 3$ is trivial and the case $k = 4$ is due to Klein as shown in Proposition 1.2.1. In addition, Erdős and Szekeres mentioned in their paper that their friends Makai and Turan proved that $N(5) = 9$ but the first published proof of this result is due to Kalbfleisch et al. [15].

Proposition 1.2.8. *Any set of at least 9 points in general position in the plane contains 5 points that are the vertices of a convex pentagon, i.e., $N(5) = 9$.*

More recently Szekeres and Peters established that $N(6) = 17$ with the assistance of Brendan McKay and heavy computing (see [32]).

For the best known lower bound, Erdős [9] showed that $N(k) \geq 2^{k-2} + 1$.

For the upper bound, after some other developments of many researchers, Suk [31] showed recently that $N(k) \leq 2^{k+6k^{2/3}\log k}$.

Furthermore, this problem was the origin that that led Erdős to define a new problem on convex polygons in 1978 (see [7]).

Problem 1.2.9. *Given an integer $k \geq 3$, does there exist a positive integer $H(k)$ such that any set S of at least $H(k)$ points in general position in the plane, contains at least one empty convex $k-$gon, i.e., a polygon that does not contain any point of S in its interior?*

If we analyze the first values of $H(k)$, we observe trivially that $H(3) = 3$ and it is not difficult to prove that $H(4) = 5$ based on the the argument of Proposition 1.2.1.

Proposition 1.2.10. *Any set of at least 5 points in general position in the plane contains 4 points that are the vertices of an empty convex quadrilateral, i.e.,* $H(4) = 5$*.*

Proof. Consider the convex hull of the subset of the first five points, when ordering all the points of the set by $x-$coordinate. This subset is separated from the remaining points of the set.

If it is a pentagon, then we are done.

If it is a triangle, we can apply the same argument as in Proposition 1.2.1 and we find a convex empty quadrilateral.

If it is a quadrilateral and it is non-empty, consider a diagonal with two of the extreme points a, c of the set, see Fig. 1.4. Let e be the interior point and d the extreme point in the opposite side of e respect to the diagonal ac, then the convex quadrilateral $aecd$ is empty. □

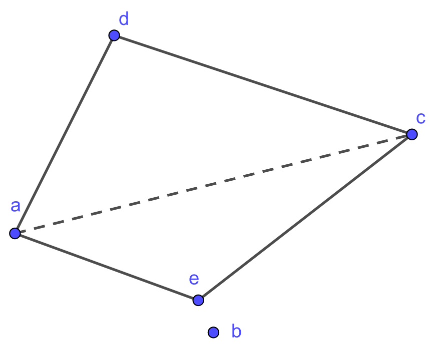


Figure 1.4: Configuration of a set of 5 points with 4 extreme points, the diagonal ac is represented with a dashed line and the edges of the empty quadrilateral with straight lines.

The next value of the Problem 1.2.9 was determined by Harborth [12].

Theorem 1.2.11. *Any set of at least 10 points in general position in the plane contain 5 points that are the vertices of an empty convex pentagon, i.e.,* $H(5) = 10$*.*

On the other hand, Horton [14] proved that $H(k)$ does not exist for all $k \geq 7$.

However, the existence of $H(6)$ was a major open problem for many years.

In [26], Overmars established the lower bound $H(6) \geq 30$ with a computer program that produced a set of 29 points in the plane with no empty convex hexagons.

For the upper bound, simultaneously in 2005, Gerken [11] and Nicolás [24] proved that $H(6) \leq N(9)$ and $H(6) \leq N(25)$ respectively. From these results, Valtr [35] simplified Gerken's proof based on a key lemma that describes point sets with no empty hexagons.

This result, combined with the bound on $N(8)$ in [33], establishes the bound $H(6) \leq 463$. Moreover, Koshelev [18] improved upon this result and gave the inequality $H(6) \leq \max\{N(8), 400\}$.

An interesting point regarding this result is that Koshelev [19] also followed the result of Szekeres and Peters [32] to establish that $N(6) = 17$. More specifically, he showed that every set S of 17 points contains an hexagon with at most 2 points of S in its interior and provides an example of a set S of 17 points that does not have an hexagon with only 1 point of S in its interior. Moreover, he proves that every set S of 18 points contains a convex hexagon with at most 1 point of S in its interior.

On this direction, it is possible to think about the following problem described in [4, 17].

Problem 1.2.12. *Given integers $k \geq 3$ and $\ell \geq 0$, does there exist a positive integer $H(k, \ell)$ such that any set S of at least $H(k, \ell)$ points in general position in the plane, contains at least one convex $k-$gon with at most ℓ points of S in its interior?*

For this problem, we clearly have the inequalities $N(k) \leq H(k, \ell) \leq H(k)$ if the expressions exist. In addition, we have the chain

$$H(k) = H(k, 0) \geq H(k, 1) \geq H(k, 2) \geq \ldots \geq H(k, \ell'), \tag{1.3}$$

for a value ℓ' such that $H(k, \ell') = N(k)$.

From the previous results on empty convex polygons, some small values are trivial: $H(3, 0) = 3$, $H(4, 0) = 5$ and $H(5, 0) = 10$.

In order to show the case $H(5, 1)$, we detail the following property.

Lemma 1.2.13. *Any convex pentagon with two or more interior points always contains a convex pentagon with a smaller number of interior points.*

Proof. The line determined by two of the interior points divides the extreme points of the convex pentagon into two parts such that, by the pigeonhole principle, at least three vertices of the pentagon must lie on the same side of the line. Thus, these three vertices of the convex pentagon and these other two selected points inside the pentagon form a convex pentagon with fewer interior points. □

Proposition 1.2.14. *Any set of at least 9 points in general position in the plane contains 5 points that are the vertices of a convex pentagon with at most one point inside, i.e., $H(5, 1) = 9$.*

Proof. From Proposition 1.2.8, we have that there exits $\ell' \geq 0$ such that $9 = N(5) = H(5, \ell')$.

If $\ell' \leq 1$ then we are done. Otherwise, applying Lemma 1.2.13, we can reduce the number of interior points in the convex pentagon until we get $\ell' = 1$. □

Observation 1.2.15. *Since we have obtained that $N(5) = H(5,1)$, it implies that $H(5,\ell) = 9$ for all $\ell \geq 1$ so we denote it as $H(5, \geq 1) = 9$.*

Other results related to this problem were obtained by Sendov [29] proving the non-existence of $H(k,\ell)$ for specific values of ℓ where $k > 7$ with the use of the Horton set defined in [14] for the proof of the non-existence of $H(7)$. More specifically, $H(k,\ell)$ does not exist for $\ell \leq (r+4)2^{m-1} - 4m - r - 1$ where $k + 2 = 4m + r$ with m an integer and $r \in \{0,1,2,3\}$. Besides, Nyklova [25] showed that $H(6, \geq 6) = N(6)$.

Note that, Sendov [29] provides an asymptotic estimate of the form $(\sqrt[4]{2} + o(1))^k$ for the maximal value of ℓ such that $H(k,\ell)$ does not exist.

Also interesting are the results commented above of Koshelev [19] which give an upper bound for $H(6,1)$.

Theorem 1.2.16. *$H(6,1) \leq N(7) \leq 127$.*

Observation 1.2.17. *If Conjecture 1.2.7 is true, then the upper bound of Theorem 1.2.16 can be written as $H(6,1) \leq 33$.*

Koshelev in [20, 21] also derived a result on the existence of $H(k,\ell)$.

Theorem 1.2.18. *Given integers $k \geq 3$ and $\ell \geq 0$,*

- *if $\ell = 2\binom{k-8}{(k-8)/2} - 1$, the value $H(k,\ell)$ does not exist,*
- *if $\ell = \binom{k-7}{(k-7)/2} - 1$, the value $H(k,\ell)$ does not exist.*

This result improves the values given previously by Sendov and Nyklova. His estimations for the maximal value of ℓ such that $H(k,\ell)$ does not exist are asymptotically equal to $(2 + o(1))^k$.

Another point of interest about this problem is to find values of ℓ such that $H(k,\ell) = N(k)$ or $H(k,\ell) > N(k)$. Since for the exact values of $N(k)$ we only know Conjecture 1.2.7, Koshelev [20] estimated the maximum value of ℓ for which $H(k,\ell) > 2^{k-2}+1$.

Theorem 1.2.19. *Given an integer $k \geq 6$, then it holds*

$$H\left(k, \binom{k-3}{\lceil (k-3)/2 \rceil} - \left\lceil \frac{k}{2} \right\rceil\right) > 2^{k-2} + 1. \tag{1.4}$$

Some more related results, mainly from the work by Pinchasi et al. [27] are shown in the next Chapter 2, and are used to derive new formulas.

1.3 Continuous motion argument

An important argument that will be used in the proofs of this work is the continuous motion argument. In this section, we explain the basic idea of continuous motion proofs in the plane (but this continuous motion argument can also be done in $\mathbb{R}^d$ for $d > 2$). Of course, this argument is not new for the analysis of configurations in combinatorial geometry (see for example [3, 34]) because it is very useful to prove properties that hold for any point set.

In many of the proofs of this work, the goal is to demonstrate a property for all point sets in the plane which first is shown to hold for some particular point set (usually a set of points in convex position). Then, we can just try to move the points from the point set for which the property holds into any particular point configuration and check the property throughout. The changes on the combinatorial structure of the point set often only appear on few discrete and specific positions, so it is sufficient to analyze these changes.

More formally, let $S(0) := S$ be a set of n points $\{p_1, \ldots, p_n\}$ and consider the changes under a generic continuous motion $S(t) = \{p_1(t), \ldots, p_n(t)\}$. That is, each $p_i(t)$ depends continuously on time t, and $S(t)$ is in a sufficiently generic position except for a finite number of critical instants $t_1, t_2, t_3, \ldots$ at which only one degeneracy appears.

In the plane, these degeneracies appear when one triple of points becomes collinear. Therefore the idea of the argument is to prove that a property does not change when during the movement one point crosses the segment defined by two other points, see Fig 1.5. In that sense, we consider that a continuous motion of the points of the set is sufficiently generic if the points of the set remain distinct and in general position during the motion, except for a finite number of critical instants where exactly three points are collinear.

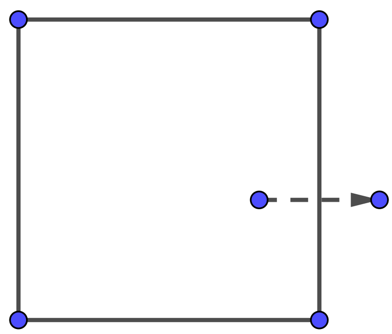


Figure 1.5: The point in the interior of the convex quadrilateral is moving towards the exterior. A degeneracy appears when the point reaches a side of the quadrilateral.

Chapter 2

Sums of numbers of polygons

This chapter contains a variety of results concerning the number of convex polygons with a given number of interior points in planar point sets.

In Section 2.1 we consider alternating sums of the numbers of convex polygons based on the results of [27] and [13].

Then, in Section 2.2 we describe some inequalities derived from the previous alternating sums.

Other interesting equations are the weighted sums that depend only on the size of the point set which are shown in Section 2.3. We find new equations that allow to deduce some of the theorems related to the Erdős-Szekeres Theorem.

Finally, in Section 2.4, we also introduce moment sums as a combination of the previous sums that gives us some ideas on the possibility to extend the expressions of the alternating sums.

2.1 Alternating sums

This section describes a set of results of equalities involving the numbers of convex polygons based on alternating sums. The results related with empty convex polygons in this section, was showed in [27], and an extension to convex polygons with one interior point was first proved in [13].

In order to show the results, we first introduce the necessary definitions. Many of them are already given in [13, 27].

Definition 2.1.1. *For any point set S, the number of convex $k-$gons in S that have exactly ℓ points of S in their interior is $X_{k,\ell}(S)$.*

Definition 2.1.2. *The $r-th$ alternating moment of $\{X_{k,\ell}(S)\}$ is*

$$M_{r,\ell}(S) = \sum_{k\geq 3}(-1)^{k+1} m_r(k) X_{k,\ell}(S), \tag{2.1}$$

where the multiplicative factor $m_r(k)$ *is*

$$m_r(k) = \begin{cases} 1 & r = 0, \\ \frac{k}{r}\binom{k-r-1}{r-1} & r \geq 1 \text{ and } k \geq 2r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Note that the condition $k \geq 2r$ of the factor $m_r(k)$ comes from the fact that the binomial coefficient is indexed by a pair $k-r-1 \geq r-1$ because the expression $m_r(k)$ corresponds to the number of ways to choose r elements from a circular list of k elements, which is known as Cayley's problem (see [22]).

The previous definitions count all the convex polygons in S. A similar definition for counting the convex polygons with specified fixed points in their interior is the following.

Definition 2.1.3. *Let* $p_1, \ldots, p_m \in S$ *be fixed points, the number of convex* $k-$*gons in* S *that have exactly* $p_1, \ldots, p_m$ *and* $\ell - m$ *other points of* S *in their interior is* $X_{k,\ell}^{p_1,\ldots,p_m}(S)$.

Definition 2.1.4. *Let* $p_1, \ldots, p_m \in S$ *be fixed points, the* $r-$*th alternating moment of* $\left\{X_{k,\ell}^{p_1,\ldots,p_m}(S)\right\}$ *is*

$$M_{r,\ell}^{p_1,\ldots,p_m}(S) = \sum_{k\geq 3}(-1)^{k+1} m_r(k) X_{k,\ell}^{p_1,\ldots,p_m}(S). \tag{2.3}$$

Further, we can also consider this count taken into account only the convex polygons that are incident to a directed edge.

Definition 2.1.5. *Let* e *be a directed edge spanned by two points of* S*, the number of convex* $k-$*gons in* S *with* ℓ *interior points, that are incident to* e *and lie to the left of* e*, is* $X_{k,\ell}(S;e)$.

Definition 2.1.6. *Let* e *be a directed edge spanned by two points of* S*, the* $r-$*th alternating moment of* $\{X_{k,\ell}(S;e)\}$ *is*

$$M_{r,\ell}(S;e) = \sum_{k\geq 3}(-1)^{k+1} m_r(k) X_{k,\ell}(S;e). \tag{2.4}$$

Note that the previous definitions may be extended to an arbitrary number of edges that are in convex position.

Definition 2.1.7. *Let* $e_1, \ldots, e_r$ *be* r *fixed edges spanned by distinct points of* S *such that these* r *edges are in convex position, the number of convex* $k-$*gons (with* $k \geq 2r$*) in* S *with* ℓ *interior points, with* $e_1, \ldots, e_r$ *on their boundary, is* $X_{k,\ell}(S; e_1, \ldots, e_r)$.

Definition 2.1.8. *Let* $e_1, \ldots, e_r$ *be* r *fixed edges spanned by distinct points of* S *and are in convex position, the* $r-$*th alternating moment of* $\{X_{k,\ell}(S; e_1 \ldots, e_r)\}$ *is*

$$M_{r,\ell}(S; e_1, \ldots, e_r) = \sum_{k\geq 3}(-1)^{k+1} m_r(k) X_{k,\ell}(S; e_1 \ldots, e_r). \tag{2.5}$$

From now on, we will usually omit S from the previous expressions to simplify our notation.

The first theorem shows that the alternating sum of the number of empty convex polygons only depends on the size of the point set. We follow the proof of Pinchasi et al. [27], but this theorem is already known from a survey on convex geometries by [5].

Theorem 2.1.9. *For any point set S of n points in general position, it holds that $M_{0,0} = \binom{n}{2} - n + 1$.*

Proof. We first prove that $M_{0,0}$ only depends on n using a continuous motion argument.

Thereto, we consider a sufficiently generic movement of the points of S. Let r be a point of S that is about to cross the edge $e = pq$ spanned by two points p, q of S.

Note that, the only convex polygons spanned by S that may change are those that have e as an edge or p, q and r as vertices. Thus, let Q be a convex polygon with e as an edge (see Fig. 2.1). If r is about to enter Q, then the polygon Q stops being empty and the $k+1$-gon Q' (defined replacing e by the path prq) stops being convex. In consequence, their combined contribution to $M_{0,0}$ is zero because they differ by 1 in size, so they do not change the value of $M_{0,0}$.

An analogous argument applies if r is about to exit Q because then Q starts being empty and Q' starts being convex, and again it does not change the value of $M_{0,0}$.

Note that, these events described above are the only possible events that may change the value of $M_{0,0}$.

Now, we can obtain the value of $M_{0,0}$ with the assumption that the n points of S are in convex position. In this case, we clearly have $X_{k,0} = \binom{n}{k}$ for all $k \geq 3$ and we obtain

$$M_{0,0} = \binom{n}{3} - \binom{n}{4} + \binom{n}{5} - \cdots = \binom{n}{2} - n + 1. \qquad (2.6)$$

□

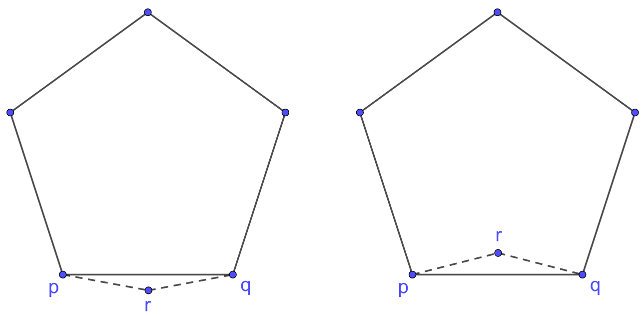


Figure 2.1: The empty convex $k - gon$ Q of the proof of Theorem 2.1.9 with r in its exterior (left) and with r in its interior (right).

The situation is not as simple for the other $r-$th alternating moment of empty convex polygons and depends on the shape of the point set. So we need some previous lemmas to derive their relation with geometric parameters of the point set.

To prove the relation for $M_{1,0}$, we need the following Lemma, proved in [27], that gives a relation of the alternating sum of the empty convex polygons incident to a directed edge.

Lemma 2.1.10. *For any point set S of n points in general position and a directed edge e spanned by two points of S, it holds that*

$$M_{0,0}(e) = \begin{cases} 1 & \text{if } e \text{ has at least one point of } S \text{ to its left,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Proof. We first prove that $M_{0,0}(e)$ only depends on n using a continuous motion argument, similarly to the proof of Theorem 2.1.9.

Thereto, we consider a sufficiently generic movement of the points to the left of e (without crossing the line supporting e) while the endpoints of e and the points on the other side of e remain fixed. Let r be a point to the left of e that is about to cross an edge $f = pq$ spanned by two points p, q of S.

Note that, the only convex polygons that are incident to e that may change are those that have f as an edge or p, q and r as vertices. Thus, let Q be a convex polygon with f as an edge (see Fig. 2.1, because the argument is analogous to the proof of Theorem 2.1.9), if r is about to enter Q, then the polygon Q stops being empty and the $k+1$-gon Q' (defined replacing f by the path prq) stops being convex. In consequence, their combined contribution to $M_{0,0}(e)$ is zero because they differ by 1 in size, so they do not change the value of $M_{0,0}$.

An analogous argument applies if r is about to exit Q because then Q starts being empty and Q' starts being convex, and again it does not change the value of $M_{0,0}(e)$.

Note that, these events described above are the only possible events that may change the value of $M_{0,0}(e)$.

Now, we can obtain the value of $M_{0,0}(e)$ with the assumption that the m points to the left of e are in convex position. In this case, we clearly have $X_{k,0}(e) = \binom{m}{k-2}$ for all $k \geq 3$ and we obtain

$$M_{0,0}(e) = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots \tag{2.8}$$

which is 1 if $m > 0$ and 0 if $m = 0$. □

Therefore, using the previous lemma, it is not difficult to shown the result for $M_{1,0}$, proved in [27]. This equality was also obatined by Ahrens et al. [1] using tools from matroid theory.

Theorem 2.1.11. *For any point set S of n points in general position, h of them in the convex hull, it holds that*

$$M_{1,0} = 2\binom{n}{2} - h. \tag{2.9}$$

Proof. We first observe that $\sum_e M_{0,0}(e) = 3X_{3,0} - 4X_{4,0} + 5X_{5,0} - \ldots = M_{1,0}$ because each empty convex $k-$gon Q is counted exactly k times in $\sum_e M_{1,0}(e)$, once for each of its edges.

Now, using Lemma 2.1.10 and since the total number of directed edges spanned by S which are not in the convex hull is $2\binom{n}{2} - h$, we obtain

$$M_{1,0} = \sum_e M_{0,0}(e) = 2\binom{n}{2} - h. \tag{2.10}$$

□

In order to extend the previous expressions of $M_{0,0}$ and $M_{1,0}$, we need to introduce two new geometric definitions given in [27].

Definition 2.1.12. *Let $e_1, \ldots, e_r$ be edges spanned by S that lie in convex position, the region $\tau(e_1, \ldots, e_r)$ is formed by the intersection of the r halfplanes that are bounded by the lines supporting $e_1, \ldots, e_r$ and containing the other edges.*

Definition 2.1.13. *$T_r(S)$, for $r \geq 2$ is the number of $r-$tuples of vertex-disjoint edges $e_1, \ldots, e_r$ spanned by S that lie in convex position and $\tau(e_1, \ldots, e_r)$ has no point of S in its interior. Moreover, we denote $T_0(S) = 0$ and $T_1(S) = h$.*

Observation 2.1.14. *A pair of edges p_1p_2, p_3p_4 that is counted in T_2 has to lie in convex position and the wedge bounded by their supporting lines is empty, as shown in Fig. 2.2.*

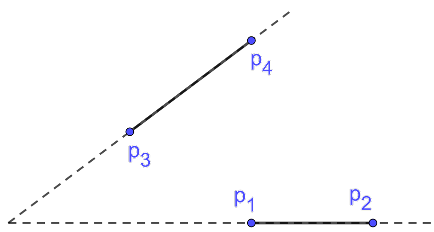


Figure 2.2: A pair of edges p_1p_2 and p_3p_4 that is counted in T_2.

Similar to the case of Theorem 2.1.11, we first show the expression for the alternating sum of the empty convex polygons that contain r edges. This Lemma was proved in [27].

Lemma 2.1.15. *For any point set S of n points in general position and $e_1, \dots, e_r$ a set of vertex-disjoint edges spanned by S that lie in convex position, let m be the number of points of S inside the region $\tau(e_1, \dots, e_r)$. It holds that*

$$M_{0,0}(e_1, \dots e_r) = \begin{cases} 0 & \text{if } m \neq 0, \\ -1 & \text{if } m = 0. \end{cases} \tag{2.11}$$

Proof. If $m = 0$ the only non-zero element of the set $\{X_{k,0}(e_1, \dots e_r)\}_{k \geq 3}$ is the value $X_{2r,0}(e_1, \dots e_r) = 1$. Therefore, if $m = 0$,

$$M_{0,0}(e_1, \dots e_r) = (-1)^{2r+1} = -1. \tag{2.12}$$

Now we claim that the value of $M_{0,0}(e_1, \dots, e_r)$ depends only on the number of points of S that lie in the region $\tau(e_1, \dots, e_r)$.

Consider a continuous motion of the points of S inside the region $\tau(e_1, \dots, e_r)$, which is sufficiently generic, without crossing any of the lines bounding this region, and while the endpoints of $e_1, \dots, e_r$ as well as the points of S outside this region remain fixed.

Similar to the proof of Theorem 2.1.9, let us consider one of the m points, s, and two distinct points p, q inside $\tau(e_1, \dots, e_r)$. Note that the only polygons spanned by S (including the edges $e_1, \dots e_r$) whose convexity may change are those that have the edge pq or the vertex s. Thus, let Q be a convex polygon containing the edge pq (as well as the edges $e_1, \dots e_r$) and suppose that s is outside Q. Then there exists a convex $(k+1)-$gon Q' obtained replacing the edge pq by the path psq (see Fig 2.1). If s is about to cross the edge pq then Q and Q' stop being convex and, since their sizes differ by 1, their combined contribution to $M_{0,0}$ is 0 and the movement does not affect its value. On the other hand, a symmetric argument applies if s is inside Q.

Therefore, if $m \neq 0$, we can obtain $M_{0,0}(e_1, \dots e_r)$ by placing the m points together with the endpoints of $e_1, \dots e_r$, in convex position. Hence,

$$M_{0,0}(e_1, \dots e_r) = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \cdots = 0. \tag{2.13}$$

□

Now we can extend the previous results to $M_{r,0}$ for any $r \geq 2$, as shown in [27].

Theorem 2.1.16. *For any point set S of n points in general position and for any $r \geq 2$, it holds that*

$$M_{r,0} = -T_r. \tag{2.14}$$

Proof. First, observe that each empty convex $k-$gon Q is counted exactly $\frac{k}{r}\binom{k-r-1}{r-1}$ times in $\sum_{\{e_1,\dots,e_r\}} M_{0,0}(e_1, \dots, e_r)$, once for each $r-$tuple of vertex-disjoint edges in convex position, because it is the number of ways of selecting r objects, no two consecutive, from k objects arranged in a circle. Therefore, using an analogous argument to Theorem 2.1.11, we obtain that $\sum_{\{e_1,\dots,e_r\}} M_{0,0}(e_1,\dots,e_r) = M_{r,0}$ (where the sum is over all unordered $r-$tuples of edges in convex position).

Now, using Lemma 2.1.15 and since the total number of $r-$tuples of edges which are empty convex polygons is T_r, we obtain

$$M_{r,0} = \sum_{\{e_1,\dots,e_r\}} M_{0,0}(e_1,\dots,e_r) = -T_r. \tag{2.15}$$

□

A possible extension of these results is on alternating sums of numbers of polygons with $\ell > 0$ interior points. An interesting result on this direction is given in [13] that relates the moment $M_{0,1}$ with the number of points in the boundary of the convex hull. To prove the Theorem 2.1.18, they also show as a side result, an expression for the alternating sum of the empty convex polygons with a fixed point in their interior.

Lemma 2.1.17. *For any point set S of n points in general position and any point $p \in S$ it holds that*

$$M^p_{0,1} = \begin{cases} 0 & \text{if } p \text{ is a point of the convex hull of } S, \\ 1 & \text{otherwise.} \end{cases} \tag{2.16}$$

Proof. On the one hand, if p is a point of the convex hull of S, clearly it cannot be in the interior of any polygon spanned by points of S, thus $M^p_{1,0} = 0$.

On the other hand, if p is an interior point of S then let us consider that p is very close to an edge e of the convex hull of S. In this case, p is contained in exactly all polygons that contain e as an edge, therefore $M^p_{0,1}(S) = M_{0,0}(S \setminus \{p\}\,;e) = 1$ by Lemma 2.1.10. Otherwise, if p is located arbitrarily inside S (not close to the convex hull), we consider a continuous motion of p which is sufficiently generic. It means that during the motion $M^p_{0,1}(S)$ can only change if a collinearity occurs.

Therefore, let qr be an edge spanned by two points of S and suppose that p is about to cross the edge qr, from its left to its right side. Then, before the collinearity the contribution of the polygons that contain p before the movement is $M_{0,0}(S \setminus \{p\}\,;qr) = 1$ and after the cross we have that the polygons that contain p after the movement is $M_{0,0}(S \setminus \{p\}\,;rq) = 1$ applying Lemma 2.1.10 because qr is not a convex hull edge of S.

In conclusion, since the alternating sums of the polygons that change (before the movement and afterwards) are the same, they do not affect the value of $M^p_{0,1}(S)$ and the value may be obtained for a point p that lies very close to a convex hull edge. □

Theorem 2.1.18. *For any set S of n points in general position, h of them in the convex hull, it holds that $M_{0,1}(S) = n - h$.*

Proof. Using Lemma 2.1.17 and, since the number of interior points of S is $n - h$ we get

$$M_{0,1}(S) = \sum_{p \in S} M^p_{0,1}(S) = n - h. \tag{2.17}$$

□

2.2 Inequalities

From the results presented in Section 2.1, in [27] the authors also derive inequalities that involve the parameters $X_{k,0}$. To obtain these results, they first state theorems related with the moments $M_{r,0}$ which will let us deduce the inequalities for the numbers of convex polygons.

In this section, we first introduce the necessary notation to prove the inequalities for the moments $M_{0,0}$ and $M_{1,0}$ from [27]. We then apply this proof technique to obtain a new inequality for the moment $M_{1,1}$. Using these results, we provide some inequalities that involve the parameters $X_{k,\ell}$ deduced from that.

Definition 2.2.1. *Let e be an edge spanned by two points $p, q \in S$. $S^+_{pq} = \{y_1, \ldots, y_m\}$ is the set of all points $y_i \in S$ that lie to the left of e and are such that the triangle pqy_i is empty.*

Observation 2.2.2. *By definition, the number of points of S^+_{pq} is $\left|S^+_{pq}\right| = X_{3,0}(e)$.*

Observation 2.2.3. *If Q is an empty convex $k-$gon that lies to the left of e and it is incident to e, then the other vertices of Q must belong to the set S^+_{pq}.*

Moreover, to derive some results, we consider an order for the points of the set S^+_{pq} stated in [27].

Proposition 2.2.4. *The set S^+_{pq} is totally ordered with the following order: $y_i \prec y_j$ if y_j lies to the right of the directed lines $\vec{py_i}$ and $\vec{qy_i}$.*

We will assume that the points of the set S^+_{pq} are enumerated in the order of Proposition 2.2.4, and using the notation of the previous definitions, we introduce the following properties and notation already given in [27].

Observation 2.2.5. *For any $i < m$, $py_iy_{i+1}q$ is a convex empty quadrilateral. In consequence, we also have $X_{4,0}(e) \geq X_{3,0}(e) - 1$.*

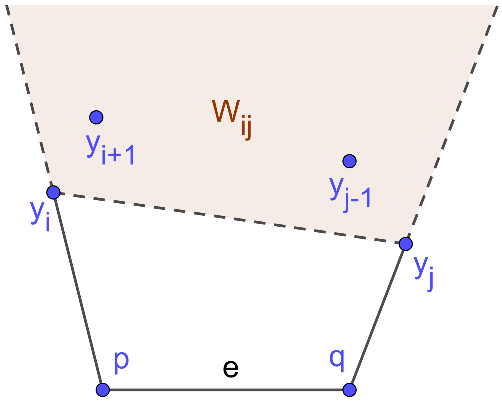


Figure 2.3: The region W_{ij}.

Definition 2.2.6. *For each $1 \leq i < j \leq m$, W_{ij} is the open region formed by the intersection of the three halfplanes lying to the right of $p\vec{y}_i$, to the left of $y_i\vec{y}_j$ and to the left of $q\vec{y}_j$.*

Definition 2.2.7. *For each $1 \leq i < j \leq m$, $X_{k,0}^{(i,j)}$ is the number of empty convex $k-$gons whose vertices belong to $S_{pq}^+ \cap W_{ij}$ and that have y_i, y_j as vertices.*

These definitions are interesting in order to find the following relation, given in [27], between empty convex polygons.

Lemma 2.2.8. *If we denote $F_{ij} = X_{3,0}^{(i,j)} - X_{4,0}^{(i,j)} + \ldots + (-1)^{t-1}X_{t-2,0}^{(i,j)}$, then*

$$X_{5,0}(e) - X_{6,0}(e) + \ldots + (-1)^{t+1}X_{t,0}(e) = \sum_{i,j} F_{ij}, \tag{2.18}$$

where the sum extends over all $i < j$ such that the quadrilateral py_iy_jq is empty.

Proof. Let K be any empty convex $k-$gon which lies to the left of e and it is incident to e. If K is no a triangle, let y_i (resp. y_j) be the vertex adjacent to p (resp. to q). Then py_iy_jq forms an empty convex quadrilateral and the other vertices of K belong to $S_{pq}^+ \cap W_{ij}$ forming an empty convex $(k-2)-$gon together with the vertices y_i, y_j.

The converse is also true and we obtain (if the quadrilateral py_iy_jq is empty) a one-to-one correspondence between empty convex $k-$gons with y_ipqy_j consecutive vertices, and empty convex $(k-2)-$gons formed by points of $S_{pq}^+ \cap W_{ij}$ and with y_iy_j consecutive vertices. Then, by the one-to-one correspondence, we have the sum of (2.18). □

Using this notation, another result of [27] is the following inequality based on Lemma 2.1.10.

Lemma 2.2.9. *For any point set S of n points in general position and a directed edge e spanned by two points of S, if there is at least one point of S to the left of e, then it holds that*

- *for each $t \geq 3$ odd,*

$$X_{3,0}(e) - X_{4,0}(e) + X_{5,0}(e) - \ldots + X_{t,0}(e) \geq 1, \tag{2.19}$$

- *for each $t \geq 4$ even,*

$$X_{3,0}(e) - X_{4,0}(e) + X_{5,0}(e) - \ldots + X_{t,0}(e) \leq 1, \tag{2.20}$$

and equalities hold if and only if $X_{t+1,0}(e) = 0$.

Proof. We observe that in Lemma 2.1.10 we have the result including all the terms, $X_{3,0}(e) - X_{4,0}(e) + X_{5,0} - \ldots = 1$. Therefore, if $X_{t+1,0}(e) = 0$, then $X_{u,0}(e) = 0$ for all $u > t$ and the equality in the lemma is satisfied.

Now we will prove the statement by induction on t.

For the base case $t = 3$, we have $X_{3,0}(e) \geq 1$ because we are assuming that there is at least one point of S to the left of e. For the equality, if $X_{3,0}(e) = 1$ then $X_{4,0}(e) = 0$ because otherwise the two vertices of an empty convex quadrilateral incident to e would give two empty triangles incident to e. On the other hand, if $X_{4,0}(e) = 0$ from Observation 2.2.5 we have $0 \geq X_{3,0}(e) - 1$. Using both, $0 \geq X_{3,0}(e) - 1 \geq 1 - 1 = 0$ thus $X_{3,0}(e) = 0$.

For the base case $t = 4$, again using Observation 2.2.5 we have $X_{3,0}(e) - X_{4,0}(e) \leq 1$. For the equality, if $X_{4,0}(e) = X_{3,0}(e) - 1$ then $X_{5,0}(e) = 0$ due to the same observation. For the converse, if $X_{5,0}(e) = 0$ then $X_{3,0}^{(i,j)} = 0$ for every $i < j$ with py_iy_jq empty, which implies that the only empty quadrilaterals are those with $j = i + 1$ and, by Observation 2.2.5, $X_{4,0}(e) = X_{3,0}(e) - 1$.

For the inductive step, let $t \geq 4$ be given and suppose that the statement holds for all $t' < t$.

We first consider the case that t is even.

By inductive hypothesis, if $S^+_{pq} \cap W_{ij}$ is non-empty then $F_{ij} \leq 1$ and, by definition, if $S^+_{pq} \cap W_{ij}$ is empty then $F_{ij} = 0$.

Note that there are $X_{4,0}(e)$ pairs y_iy_j such that the quadrilateral py_iy_jq is empty and convex, and among these, by Observation 2.2.5, exactly $X_{4,0}(e) - (X_{3,0}(e) - 1)$ have $j > i + 1$.

Hence, $\sum_{i,j} F_{ij} \leq X_{4,0}(e) - (X_{3,0}(e) - 1)$ and, by Lemma 2.2.8, we have

$$X_{3,0}(e) - X_{4,0}(e) + X_{5,0}(e) - \ldots + X_{t,0}(e) \leq X_{3,0}(e) - X_{4,0}(e) + X_{4,0}(e) - (X_{3,0}(e) - 1) = 1. \tag{2.21}$$

If the equality holds, $F_{ij} = 1$ if $S^+_{pq} \cap W_{ij}$ is non-empty. Then, by induction, $X_{t-1,0}^{(i,j)} = 0$ for every i, j with py_iy_jq empty. Since the existence of an empty convex $(t+1)-$gon incident to e would imply that exists some i, j such that $X_{t-1,0}^{(i,j)} > 0$, it implies that $X_{t+1,0}(e) = 0$.

The case with t odd, uses that, by inductive hypothesis if $S^+_{pq} \cap W_{ij}$ is non-empty then $F_{ij} \geq 1$ and we proceed in completely analogy the the case with t even,

$$X_{3,0}(e) - X_{4,0}(E) + X_{5,0}(e) - \ldots + X_{t,0}(e) \geq X_{3,0}(e) - X_{4,0}(e) + X_{4,0}(e) - (X_{3,0}(e) - 1) = 1. \quad (2.22)$$

For the equality, it is also analogous to the case mentioned above with t even. □

Definition 2.2.10. *Let Q be an empty convex $k-$gon, let p be the lowest vertex of Q and let a, b be the two vertices of Q adjecent to p,*

- *W_{pab} is the wedge delimited by the rays $\vec{pa}$ and $\vec{pb}$,*
- *$X^{pab}_{k,0}$ is the number of empty convex $k-$gons contained in W_{pab} having the edge ab as an edge and separated from p by that edge.*

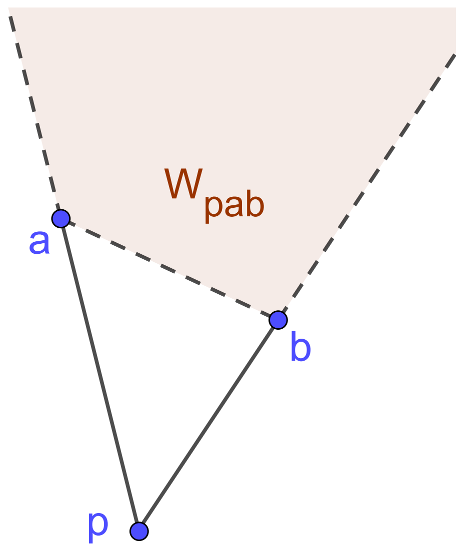


Figure 2.4: The region W_{pab}.

These definitions are interesting in order to find the following relation between empty convex polygons, already given in [27].

Lemma 2.2.11. *If we denote $F_{pab} = X^{pab}_{3,0} - X^{pab}_{4,0} + \ldots + (-1)^t X^{pab}_{t-1,0}$, then*

$$-X_{4,0} + X_{5,0} + \ldots + (-1)^{t+1} X_{t,0} = -\sum_{pab} F_{pab}, \quad (2.23)$$

Proof. Let K be any empty convex $k-$gon with p its lowest vertex, and $a,\ b$ the adjacent vertices to p. Then, the triangle pab is empty and the $(k-1)-$gon formed by the other vertices of K together with a and b, is contained in the region W_{pab}.

The converse is also true and we obtain (if the triangle pab is empty) a one-to-one correspondence between empty convex $k-$gons whose lowest vertex is p, and empty convex $(k-1)-$gons formed by points of W_{pab}, with $a,\ b$ consecutive vertices and the triangle pab empty.

Then, by the one-to-one correspondence, we have the sum of (2.23). □

Now, we can introduce the inequalities related to $M_{0,0}$ and $M_{1,0}$ and their proofs, obtained in [27].

Theorem 2.2.12. *For any point set S in general position in the plane, we have*

- *for each $t \geq 3$ odd,*

$$X_{3,0} - X_{4,0} + X_{5,0} - \ldots + X_{t,0} \geq \binom{n}{2} - n + 1, \tag{2.24}$$

- *for each $t \geq 4$ even,*

$$X_{3,0} - X_{4,0} + X_{5,0} - \ldots + X_{t,0} \leq \binom{n}{2} - n + 1, \tag{2.25}$$

and the equalities hold if and only if $X_{t+1,0} = 0$.

Proof. Observe that, by Theorem 2.1.9, if $X_{t+1,0} = 0$ then clearly equality holds.

Now, we consider the case t odd.

Note that, by Lemma 2.2.9, $F_{pab} \leq 1$ for any empty triangle pab with at least one point of the set in the interior of W_{pab}.

Now, we claim that there are exactly $\binom{n}{2} - n + 1$ empty triangles pab such that W_{pab} does not contain any additional point of the set. Indeed, we sort the points of the set in decreasing $y-$order: $p_1, \ldots, p_n$. For a point p_i we sort the points $p_1, \ldots, p_{i-1}$ in angular order about p_i. Then, the empty triangles with p_i as their lower vertex and with no other points in the interior of the wedge are precisely those whose other two vertices are consecutive points in this angular order, so the total is $i-2$. Summing for all $i = 3, \ldots, n$ we get $1 + 2 + \ldots + (n-2) = \binom{n}{2} - n + 1$.

Then we have,

$$\begin{aligned} X_{3,0} - X_{4,0} + X_{5,0} - \ldots + X_{t,0} &= X_{3,0} - \sum_{pab} F_{pab} \geq \\ &\geq X_{3,0} - |\{pab|\ pab \text{ empty and } W_{pab} \text{ non-empty}\}| = \\ &= \binom{n}{2} - n + 1. \end{aligned} \tag{2.26}$$

For the equality, we have that $F_{pab} = 1$ for every empty triangle pab with W_{pab} non-empty. Using Lemma 2.2.9, we observe that in this case $X_{t,0}^{pab} = 0$ for any empty triangle pab. Since every empty convex $(t+1)-$gon would imply that there exists an empty triangle pab with $X_{t,0}^{pab} > 0$, it implies that $X_{t+1,0} = 0$.

The case with t even is analogous to the previous one with the observation that, by Lemma 2.2.9, $F_{pab} \geq 1$ for any empty triangle pab with at least one point of the set in the interior of W_{pab}. Then,

$$\begin{aligned} X_{3,0} - X_{4,0} + X_{5,0} - \ldots + X_{t,0} &= X_{3,0} - \sum_{pab} F_{pab} \leq \\ &\leq X_{3,0} - |\{pab|\ pab \text{ empty and } W_{pab} \text{ non-empty}\}| = \\ &= \binom{n}{2} - n + 1. \end{aligned} \tag{2.27}$$

The equality is proven with the same argument applied in the case with t odd. □

Theorem 2.2.13. *For any point set S in general position in the plane, we have*

- *for each $t \geq 3$ odd,*

$$3X_{3,0} - 4X_{4,0} + X_{5,0} - \ldots + tX_{t,0} \geq 2\binom{n}{2} - h, \tag{2.28}$$

- *for each $t \geq 4$ even,*

$$3X_{3,0} - 4X_{4,0} + X_{5,0} - \ldots + tX_{t,0} \leq 2\binom{n}{2} - h, \tag{2.29}$$

  and the equalities hold if and only if $X_{t+1,0} = 0$.

Proof. We first observe that, by Theorem 2.1.11, if $X_{t+1,0} = 0$ then the equality holds.

Note that, with the same argument as in Theorem 2.1.11, it is clear that

$$3X_{3,0} - 4X_{4,0} + \ldots + (-1)^{t+1} tX_{t,0} = \sum_{e} \left(X_{3,0}(e) - X_{4,0}(e) + \ldots (-1)^{t+1} tX_{t,0}(e)\right). \tag{2.30}$$

If t is odd, by Lemma 2.2.9, $X_{3,0}(e) - X_{4,0}(e) + \ldots + (-1)^{t+1} tX_{t,0}(e) \geq 1$ if e is not an edge of the convex hull of S. Therefore,

$$\begin{aligned} 3X_{3,0} - 4X_{4,0} + \ldots + (-1)^{t+1} tX_{t,0} &= \sum_{e} \left(X_{3,0}(e) - X_{4,0}(e) + \ldots (-1)^{t+1} tX_{t,0}(e)\right) \geq \\ &\geq 2\binom{n}{2} - h. \end{aligned} \tag{2.31}$$

If equality holds, $X_{3,0}(e) - X_{4,0}(e) + \dots (-1)^{t+1} t X_{t,0}(e) = 1$ if e is not an edge of the convex hull of S and, by Lemma 2.2.9, $X_{t+1,0}(e) = 0$ for these edges. Therefore, $X_{t+1,0}(e) = 0$ for all the edges and, in consequence, $X_{t+1,0} = 0$.

If t is even, the proof is analogous with the unique difference that the inequality is $X_{3,0}(e) - X_{4,0}(e) + \dots (-1)^{t+1} t X_{t,0}(e) \leq 1$ if e is not an edge of the convex hull of S. Therefore, the proof proceeds in the same manner but the direction of the inequalities is reversed. □

We next proceed to extend these results to the moment $M_{0,1}$, which is a new result of the thesis.

Lemma 2.2.14. *For any point set S of n points in general position and a point $p \in S$ in the interior of the convex hull, it holds that*

- *for each $t \geq 3$ odd,*
$$X^p_{3,1} - X^p_{4,1} + \dots + X^p_{t,1} \geq 1, \tag{2.32}$$
- *for each $t \geq 4$ even,*
$$X^p_{3,1} - X^p_{4,1} + \dots + X^p_{t,1} \leq 1, \tag{2.33}$$

and the equalities hold if and only if $X^p_{t+1,1} = 0$.

Proof. First of all, by Lemma 2.1.17, we have that if $X^p_{t+1,1} = 0$ then the equality holds.

Note that, with the same argument as in Lemma 2.1.17, let us consider that p is very close to an edge e of the convex hull of S, and then

$$X^p_{3,1} - X^p_{4,1} + \dots + X^p_{t,1} = X_{3,0}(S \setminus \{p\}\,; e) - X_{4,0}(S \setminus \{p\}\,; e) + \dots + X_{t,0}(S \setminus \{p\}\,; e). \tag{2.34}$$

Then the results of Lemma 2.2.9 applies.

Moreover, if we consider a continuous motion of p which is sufficiently generic proceding exaclty in the same manner as in the proof of Lemma 2.1.17 the value of (2.34) does not change if p is located arbitrarily inside S. □

Theorem 2.2.15. *For any point set S in general position in the plane, we have*

- *for each $t \geq 3$ odd,*
$$X_{3,1} - X_{4,1} + X_{5,1} - \dots + X_{t,1} \geq n - h, \tag{2.35}$$
- *for each $t \geq 4$ even,*
$$X_{3,1} - X_{4,1} + X_{5,1} - \dots + X_{t,1} \leq n - h, \tag{2.36}$$

and the equalities hold if and only if $X_{t+1,1} = 0$.

Proof. Observe that, by Theorem 2.1.18, if $X_{t+1,1} = 0$ then the equality holds.

Note that, using the same argument as in Theorem 2.1.18, it is clear that

$$X_{3,1} - X_{4,1} + X_{5,1} - \ldots + X_{t,1} = \sum_{p \in S} \left(X^p_{3,1} - X^p_{4,1} + X^p_{5,1} - \ldots + X^p_{t,1} \right). \tag{2.37}$$

If t is odd, by Lemma 2.2.14, $X^p_{3,1} - X^p_{4,1} + \ldots + X^p_{t,1} \geq 1$ if p is an interior point of S. Therefore,

$$\begin{aligned} X_{3,1} - X_{4,1} + X_{5,1} - \ldots + X_{t,1} &= \sum_{p \in S} \left(X^p_{3,1} - X^p_{4,1} + \ldots + X^p_{t,1} \right) \geq \\ &\geq n - h. \end{aligned} \tag{2.38}$$

If equality holds, then $X^p_{3,1} - X^p_{4,1} + X^p_{5,1} - \ldots + X^p_{t,1} = 1$ if p is an interior point of S and, by Lemma 2.2.14, $X^p_{t+1,1} = 0$ for these points. Therefore, $X^p_{t+1,1} = 0$ for all the points and, in consequence, $X_{t+1,1} = 0$.

If t is even, the proof is analogous with the unique difference that the inequality is $X^p_{3,1} - X^p_{4,1} + X^p_{5,1} - \ldots + X^p_{t,1} \leq 1$ if p is an interior point of S. Therefore, the proof proceeds in the same manner but the direction of the inequalities is reversed. □

Furthermore, Pinchasi et al. [27] extended these theorems to the sums $M_{r,0}$.

Theorem 2.2.16. *For any point set S in general position in the plane and for any $r \geq 2$, we have*

- *for each $t \geq 2r + 1$ odd,*

$$\sum_{k=2r}^{t} (-1)^k m_r(k) X_{k,0} \leq T_r, \tag{2.39}$$

- *for each $t \geq 2r$ even,*

$$\sum_{k=2r}^{t} (-1)^{k+t} m_r(k) X_{k,0} \geq T_r, \tag{2.40}$$

and the equalities hold if and only if $X_{t+1,0} = 0$.

As mentioned above, these results imply a new variety of inequalities related with the numbers of convex polygons. A first important result is for the lower bound of the number of empty convex quadrilaterals, already given in [27].

Corollary 2.2.17.

$$X_{4,0} \geq \max \left\{ X_{3,0} - \binom{n}{2} + n - 1, \frac{3}{4} X_{3,0} - \frac{n(n-1) - h}{4} \right\}. \tag{2.41}$$

Proof. It is enough to observe that from Theorems 2.2.12 and 2.2.13 for $t = 4$, we obtain the following inequalities:

$$X_{3,0} - X_{4,0} \leq \binom{n}{2} - n + 1, \tag{2.42}$$

$$3X_{3,0} - 4X_{4,0} \leq 2\binom{n}{2} - h. \tag{2.43}$$

□

In the same way, another result of [27] is for the lower bound of the number of empty convex pentagons.

Corollary 2.2.18. $X_{5,0} \geq \frac{2}{5}X_{4,0} - \frac{1}{5}T_2$.

Proof. The result is immediate from Theorem 2.2.16 for $r = 2$ and $t = 5$. □

A similar new result may be deduced from Theorem 2.2.15 with $t = 4$.

Corollary 2.2.19. $X_{4,1} \geq X_{3,1} - n + h$.

2.3 Weighted sums

In this section, we analyze other sums of numbers of polygons with interior points. Basically, we consider sums of the form $F(S) = \sum_{\ell \geq 0} \sum_{k \geq 3} f(k,\ell) X_{k,\ell}$ following the results initiated by Huemer et al. [13].

In [13], a relation that gives the sum $F(S)$ only dependent on n is proven.

Theorem 2.3.1. *For any point set S of n points in general position and for any function $f(k,\ell)$ that fulfills the recurrence relation*

$$f(k,\ell) = f(k+1,\ell-1) + f(k,\ell-1), \tag{2.44}$$

the sum $F(S) = \sum_{\ell \geq 0} \sum_{k \geq 3} f(k,\ell) X_{k,\ell}(S)$ is invariant over all point sets S of same cardinality, that is, $F(S)$ only depends on n.

Proof. We claim that any continuous motion of the points of S which is sufficiently generic does not change the value of $F(S)$.

Consider that $p, q, r \in S$ become collinear, with r lying between p and q, thus the only convex polygons spanned by S that may change are those that have pq as an edge (and r in its interior) or p, q, r as vertices.

Let Q be a convex $k - gon$ with ℓ interior points that contains pq as an edge and r in its interior, see Fig 2.5. If r moves outside of Q, then the polygon Q has $\ell - 1$ points in its interior and the $(k+1)-$gon Q' obtained by replacing the edge pq of Q by the polygonal path prq, starts being convex with $\ell - 1$ points in its interior. Since $f(k,\ell) = f(k+1,\ell-1) + f(k,\ell-1)$, the movement of the point does not change the value of $F(S)$.

Symmetrically, if r moves inside Q (with ℓ points in its interior), then Q has $\ell + 1$ points in its interior and the $(k+1)-$gon Q', with also ℓ points in its interior, stops being convex. Again, this does not change the value of $F(S)$. □

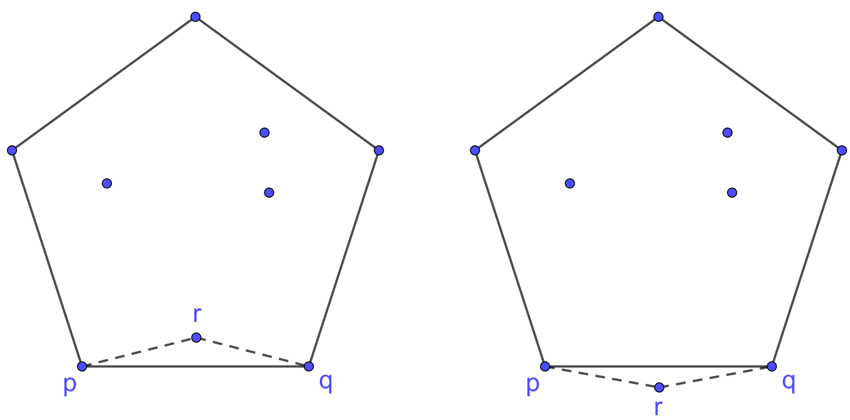


Figure 2.5: The convex $k - gon$ with ℓ interior points Q of the proof of Theorem 2.3.1 with r in its interior (left) and with r outside (right).

Moreover, they show some functions that fulfill the recurrence relation (2.44).

Corollary 2.3.2. *For any point set S of n points in general position, it holds that*

$$\sum_{k=3}^{n}\sum_{\ell=0}^{n-3} 2^{\ell} X_{k,\ell}(S) = 2^n - \frac{n^2}{2} - \frac{n}{2} - 1. \tag{2.45}$$

Corollary 2.3.3. *For any point set S of n points in general position and every integer $3 \leq m \leq n$, it holds that*

$$\sum_{k=3}^{m}\sum_{\ell=m-k}^{n-k} \binom{\ell}{m-k} X_{k,\ell}(S) = \binom{n}{m}. \tag{2.46}$$

Corollary 2.3.4. *For any point set S of n points in general position, it holds that*

$$\sum_{k=3}^{n}\sum_{\ell=0}^{n-3} 2\cos\left(\frac{(2k+\ell)\,\pi}{3}\right) X_{k,\ell}(S) \;=\; \binom{n}{2} + n - 2 + 2\cos\left(\frac{n\pi}{3}\right), \tag{2.47}$$

$$\sum_{k=3}^{n}\sum_{\ell=0}^{n-3} \frac{2}{\sqrt{3}}\sin\left(\frac{(2k+\ell)\,\pi}{3}\right) X_{k,\ell}(S) \;=\; \binom{n}{2} - n + \frac{2}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right). \tag{2.48}$$

Corollary 2.3.5. *Let* $\{\mathrm{Fib}(n)\}$ *be the sequence of Fibonacci numbers, satisfying* $\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$ *with* $\mathrm{Fib}(1) = \mathrm{Fib}(2) = 1$*. For any point set S of n points in general position, it holds that*

$$\sum_{k=3}^{n}\sum_{\ell=0}^{n-3} \mathrm{Fib}(k+2\ell) X_{k,\ell}(S) \;=\; \mathrm{Fib}(2n) - n - \binom{n}{2}, \tag{2.49}$$

$$\sum_{k=3}^{n}\sum_{\ell=0}^{n-3} (-1)^{k+l}\,\mathrm{Fib}(k-\ell) X_{k,\ell}(S) \;=\; -\mathrm{Fib}(n) + n - \binom{n}{2}. \tag{2.50}$$

Note that the Fibonacci numbers are also defined for negative integers.

In this work, we derive new functions $f(k,\ell)$ in terms of the functions $f(k,0)$.

Theorem 2.3.6. *The general solution of the recurrence relation (2.44) is given by the equation*

$$f(k,\ell) = \sum_{i=0}^{l} \binom{\ell}{i}\, f(k+i,0). \tag{2.51}$$

Proof. We will prove the statement by induction on ℓ.

Obviously, the case $\ell = 0$ is immediate because the sum only includes one term, $f(k,0) = f(k,0)$.

For the base case, we have $\ell = 1$, then the result is also clear because the equality corresponds to (2.44) for $\ell = 1$,

$$f(k,1) = f(k+1,0) + f(k,0). \tag{2.52}$$

For the inductive step, let $\lambda \geq 0$ be given and suppose (2.51) holds for $\ell = \lambda$. Then

$$\begin{aligned} f(k,\lambda+1) &= f(k+1,\lambda)+f(k,\lambda) = \sum_{i=0}^{\lambda}\binom{\lambda}{i}f(k+1+i,0)+\sum_{i=0}^{\lambda}\binom{\lambda}{i}f(k+i,0) = \\ &= \sum_{i=1}^{\lambda+1}\binom{\lambda}{i-1}f(k+i,0)+\sum_{i=0}^{\lambda}\binom{\lambda}{i}f(k+i,0) = \\ &= f(k,0)+\sum_{i=1}^{\lambda}\left[\binom{\lambda}{i}+\binom{\lambda}{i-1}\right]f(k+i,0)+f(k+i,0) = \\ &= \sum_{i=0}^{\lambda+1}\binom{\lambda+1}{i}\ f(k+i,0). \end{aligned} \tag{2.53}$$

□

Note that, there may be many functions $f(k,\ell)$ that fulfill (2.51), thus many different sums $F(S)$ that only depend on n. An interesting point of view is to analyze how many of them are independent.

Proposition 2.3.7. *The maximum number of linear independent equations $F(S) = \sum_{\ell\geq 0}\sum_{k\geq 3} f(k,\ell)X_{k,\ell}$, where $f(k,\ell)$ satisfies (2.51), in terms of the variables $X_{k,\ell}$ is $n-2$.*

Proof. Let $F_j(S) = \sum_{\ell\geq 0}\sum_{k\geq 3} f_j(k,\ell)X_{k,\ell}$ (for $1 \leq j \leq n-1$) be sums that only depend on n and $f_j(k,\ell)$ satisfies (2.51) for all j. Let us consider the matrix

$$\begin{pmatrix} f_1(3,0) & f_1(3,1) & \cdots & f_1(k,\ell) & \cdots & f_1(n-1,1) & f_1(n,0) \\ f_2(3,0) & f_2(3,1) & \cdots & f_2(k,\ell) & \cdots & f_2(n-1,1) & f_2(n,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_j(3,0) & f_j(3,1) & \cdots & f_j(k,\ell) & \cdots & f_j(n-1,1) & f_j(n,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1}(3,0) & f_{n-1}(3,1) & \cdots & f_{n-1}(k,\ell) & \cdots & f_{n-1}(n-1,1) & f_{n-1}(n,0) \end{pmatrix}. \tag{2.54}$$

It is sufficient to show that the matrix (2.54) has rank $n-2$.

Note that, the first column contains the function with $k = 3$ and $\ell = 0$, then the following columns contain the value $k = 3$ and the values of ℓ in ascending order, then there are the functions with $k = 4$ and so on. This order implies that the coefficient terms of the pair (k,ℓ) are in the column C_h such that $h = \sum_{i=3}^{k-1}(n-i+1)+\ell+1$.

Since $f_j(k,\ell)$ satisfies (2.51), we can express the matrix (2.54) as follows,

$$\begin{pmatrix} f_1(3,0) & \cdots & \sum_{i=0}^{\ell}\binom{\ell}{i} f_1(k+i,0) & \cdots & f_1(n,0) \\ f_2(3,0) & \cdots & \sum_{i=0}^{\ell}\binom{\ell}{i} f_2(k+i,0) & \cdots & f_2(n,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_j(3,0) & \cdots & \sum_{i=0}^{\ell}\binom{\ell}{i} f_j(k+i,0) & \cdots & f_j(n,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1}(3,0) & \cdots & \sum_{i=0}^{\ell}\binom{\ell}{i} f_{n-1}(k+i,0) & \cdots & f_{n-1}(n,0) \end{pmatrix}. \tag{2.55}$$

We can observe that the columns corresponding to the functions with $\ell > 0$ depend on the $n-2$ columns with $\ell = 0$, so with the adequate operations these columns can be transformed to columns of zeros. That is, if we change each column corresponding to the functions with $\ell > 0$ in the following manner:

$$C_{\sum_{s=3}^{k-1}(n-s+1)+\ell+1} \to C_{\sum_{s=3}^{k-1}(n-s+1)+\ell+1} - \sum_{i=0}^{\ell}\binom{\ell}{i} C_{\sum_{s=3}^{k+i-1}(n-s+1)+1},$$

we obtain

$$\begin{pmatrix} f_1(3,0) & 0 & \cdots & f_1(k,0) & \cdots & 0 & f_1(n,0) \\ f_2(3,0) & 0 & \cdots & f_2(k,0) & \cdots & 0 & f_2(n,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_j(3,0) & 0 & \cdots & f_j(k,0) & \cdots & 0 & f_j(n,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1}(3,0) & 0 & \cdots & f_{n-1}(k,0) & \cdots & 0 & f_{n-1}(n,0) \end{pmatrix}. \tag{2.56}$$

Clearly, there are only $n-2$ non-zero columns corresponding to the functions $f_j(k,0)$ and the maximum possible rank is $n-2$. □

Using the general solution of Theorem 2.3.6, we derive other relations for sums over all convex polygons.

Corollary 2.3.8. *For any point set S of n points in general position and for any $x \in \mathbb{R}$, it holds that*

$$P_x(S) := \sum_{k=3}^{n}\sum_{\ell=0}^{n-k} x^k (1+x)^{\ell} X_{k,\ell} = (1+x)^n - 1 - x \cdot n - x^2\binom{n}{2}. \tag{2.57}$$

Proof. Define $f(k+i,0) = x^{k+i}$, then

$$f(k,\ell) = \sum_{i=0}^{\ell}\binom{\ell}{i} f(k+i,0) = x^k \sum_{i=0}^{\ell}\binom{\ell}{i} x^i = x^k \cdot (1+x)^{\ell}. \tag{2.58}$$

The result then follows by considering a set of n points in convex position and we have

$$\sum_{k=3}^{n} x^k \binom{n}{k} = (1+x)^n - 1 - x \cdot n - x^2 \cdot \binom{n}{2}. \tag{2.59}$$

□

Observation 2.3.9. *The relation from Corollary 2.3.2 is a particular case of the relations of Corollary 2.3.8 for $x = 1$.*

As shown in Proposition 2.3.7, the maximum number of possible linear independent equations is $n-2$. From Corollary 2.3.8, we can obtain multiple equations of this form. Therefore, we evaluate the independence of these equations.

Proposition 2.3.10. *Let $x_j \in \mathbb{R}_{\neq 0}$ (distinct) for $1 \leq j \leq n-2$ and consider the $n-2$ equations $P_j = \sum_{k=3}^{n} \sum_{\ell=0}^{n-k} x_j^k (1+x_j)^\ell X_{k,\ell}$. Then, these equations are independent.*

Proof. Using the argument of Proposition 2.3.7, it is sufficient to analyze the columns corresponding to the variables $X_{k,0}$. Thus, we consider the matrix

$$\begin{pmatrix} x_1^3 & x_1^4 & \cdots & x_1^k & \cdots & x_1^n \\ x_2^3 & x_2^4 & \cdots & x_2^k & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ x_j^3 & x_j^4 & \cdots & x_j^k & \cdots & x_j^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ x_{n-2}^3 & x_{n-2}^4 & \cdots & x_{n-2}^k & \cdots & x_{n-2}^n \end{pmatrix}. \tag{2.60}$$

Now, we can divide each row j by x_j^3 and we obtain

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^{k-3} & \cdots & x_1^{n-3} \\ 1 & x_2 & \cdots & x_2^{k-3} & \cdots & x_2^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & x_j & \cdots & x_j^{k-3} & \cdots & x_j^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & x_{n-2} & \cdots & x_{n-2}^{k-3} & \cdots & x_{n-2}^{n-3} \end{pmatrix}. \tag{2.61}$$

It is a $(n-2) \times (n-2)$ Vandermonde matrix and all x_i are distinct, therefore the matrix rank is $n-2$ and there are $n-2$ linear independent equations. □

Clearly, from Corollary 2.3.8 we can derive an infinite number of equations and we can reach the maximum possible rank as shown in Proposition 2.3.10. Therefore, the following result is immediate.

Corollary 2.3.11. *Any sum $F(S)$ where the functions $f(k,\ell)$ fulfill (2.51), can be expressed in terms of $n-2$ equations of the form (2.58) with distinct values $x \in \mathbb{R}$.*

This result gives a new way to prove theorems related with the Erdős-Szekeres theorem. That is, we propose to create a system of equations with specific characteristic and analyze if the system is compatible or not.

In particular, an immediate result is Proposition 1.2.8, i.e., $N(4) = 5$. In this case, let $n = 5$ and we suppose, for the sake of a contradiction, that $X_{4,0} = X_{4,1} = X_{5,0} = 0$. Now, we consider the system given by the functions

$$P_i = \sum_{k=3}^{n} \sum_{\ell=0}^{n-k} i^k (1+i)^\ell X_{k,\ell} = (1+i)^n - 1 - i \cdot n - i^2 \binom{n}{2}, \qquad \text{for } i = 1,2,3. \qquad (2.62)$$

It means that the coefficient terms are $f_i(k,\ell) = i^k(1+i)^\ell$ and the constant terms are $(1+i)^n - 1 - i \cdot n - i^2 \cdot \binom{n}{2}$ for the values $i = 1,2,3$.

Therefore, for the variables $X_{3,0}$, $X_{3,1}$ and $X_{3,2}$ we obtain the following system.

$$\begin{pmatrix} 1 & 2 & 4 \\ 8 & 24 & 72 \\ 27 & 108 & 432 \end{pmatrix} \begin{pmatrix} X_{3,0} \\ X_{3,1} \\ X_{3,2} \end{pmatrix} = \begin{pmatrix} 16 \\ 192 \\ 918 \end{pmatrix}. \qquad (2.63)$$

It is not difficult to check that the solution of system (2.63) is $X_{3,0} = 6$, $X_{3,1} = 3$ and $X_{3,2} = 1$.

However, by Theorem 2.1.18, in this case we have

$$X_{3,1} - X_{4,1} = n - h. \qquad (2.64)$$

Since clearly $h = 3$, the equality does not hold: $3 - 0 \neq 5 - 3$. Thus, the system does not have a geometrical solution and the statement has been proved.

Note that, the number of variables increases much faster than the number of independent equations according to Corollary 2.3.11. Therefore, for a greater value of n, this argument needs the derivation of other types of equations and, probably, other geometrical parameters as well. For example, if we consider a set of $n = 9$ points and we suppose that $X_{k,\ell} = 0$ for every pair $k \geq 5$, $\ell \geq 0$ (in order to check that $N(5) = 9$), the number of variables with $k \leq 4$ is 13 whereas the number of independent equations is 7.

2.4 Moment sums

In Section 2.1, we derived expressions for the moments $M_{r,0}$ and $M_{0,1}$ but there are still other $r-$th moments to describe geometric parameters of the point set. In this section, we show that combinations of the moments described in Section 2.1 only depend on the size of the point set and we analyze the possible extensions of the expressions in Section 2.1 using the results of the combinations of moments.

We first observe that combining the expressions of Theorems 2.1.11 and 2.1.18, we obtain that

$$M_{1,0} - M_{0,1} = 2\binom{n}{2} - h - (n-h) = 2\binom{n}{2} - n. \tag{2.65}$$

This equation depends only on n and it was the base case for Huemer et al. [13] to extend it to arbitrary moments.

Theorem 2.4.1. *For any point set S of n points in general position and any $r \geq 0$, it holds that*

$$F_r(S) := \sum_{\ell=0}^{r} \sum_{k\geq 3} (-1)^{k-\ell+1} m_{r-\ell}(k-\ell) X_{k,\ell}(S) = \begin{cases} \binom{n}{2} - n + 1 & r = 0, \\ 2\binom{n}{2} - n & r = 1, \\ -m_r(n) & r \geq 2, \end{cases} \tag{2.66}$$

where $m_r(n)$ is defined as in Definition 2.1.2.

Proof. For $r = 0$, we get

$$F_0(S) = M_{0,0}(S) = \binom{n}{2} - n + 1, \tag{2.67}$$

as shown in Theorem 2.1.9

For $r = 1$, we get

$$F_1(S) = M_{1,0}(S) - M_{0,1}(S) = 2\binom{n}{2} - h - (n-h) = 2\binom{n}{2} - n, \tag{2.68}$$

as shown in Theorems 2.1.11 and 2.1.18.

For the general case $r \geq 2$, we denote

$$f_r(k,\ell) := \begin{cases} (-1)^{k-\ell+1} m_{r-\ell}(k-\ell) & 0 \leq \ell \leq r, \\ 0 & r < \ell. \end{cases} \tag{2.69}$$

Now we prove that the terms $f_r(k,\ell)$ satisfy the recurrence relation from Theorem 2.3.1.

For $\ell > r+1$, the result is clear since $f_r(k,\ell) = f_r(k+1,\ell-1) + f_r(k,\ell-1) = 0$ by definition.

For $\ell = r+1$, clearly $f_r(k,\ell) = 0$ and we also have

$$\begin{aligned} f_r(k+1,\ell-1) + f_r(k,\ell-1) &= (-1)^{(k+1)-(\ell-1)+1} m_{r-(\ell-1)}\left((k+1)-(\ell-1)\right) + \\ &\quad + (-1)^{k-(\ell-1)+1} m_{r-(\ell-1)}\left(k-(\ell-1)\right) = \\ &= (-1)^{k-\ell+3} m_0\left(k-\ell+2\right) + (-1)^{k-\ell+2} m_0\left(k-\ell+1\right) = \\ &= (-1)^{k-\ell+3} + (-1)^{k-\ell+2} = 0. \end{aligned} \tag{2.70}$$

For $\ell = r$, clearly $f_r(k,\ell) = (-1)^{k-\ell+1}$ and we also obtain

$$\begin{aligned} f_r(k+1,\ell-1) + f_r(k,\ell-1) &= (-1)^{(k+1)-(\ell-1)+1} m_{r-(\ell-1)}\left((k+1)-(\ell-1)\right) + \\ &\quad + (-1)^{k-(\ell-1)+1} m_{r-(\ell-1)}\left(k-(\ell-1)\right) = \\ &= (-1)^{k-\ell+3} m_1\left(k-\ell+2\right) + (-1)^{k-\ell+2} m_1\left(k-\ell+1\right) = \\ &= (-1)^{k-\ell+1}\left[(k-\ell+2)-(k-\ell+1)\right] = (-1)^{k-\ell+1} \end{aligned} \tag{2.71}$$

For $0 < \ell < r$, we consider the equality in terms of the functions $m_r(k)$,

$$\begin{aligned} f_r(k,\ell) &= f_r(k+1,\ell-1) + f_r(k,\ell-1) \\ (-1)^{k-\ell+1} m_{r-\ell}(k-\ell) &= (-1)^{(k-\ell+3} m_{r-\ell+1}\left(k-\ell+2\right) + (-1)^{k-\ell+2} m_{r-\ell+1}\left(k-\ell+1\right) \\ m_{r-\ell}(k-\ell) &= m_{r-\ell+1}\left(k-\ell+2\right) - m_{r-\ell+1}\left(k-\ell+1\right). \end{aligned} \tag{2.72}$$

From the previous equality, if we consider the change of variables $r' = r - \ell + 1$ and $k' = k - \ell + 1$, we obtain

$$\begin{aligned} m_{r'-1}(k'-1) &= m_{r'}(k'+1) - m_{r'}(k') &\iff \\ \frac{k'-1}{r'-1}\binom{k'-r'-1}{r'-2} &= \frac{k'+1}{r'}\binom{k'-r'}{r'-1} - \frac{k'}{r'}\binom{k'-r'-1}{r'-1} &\iff \\ \frac{(k'-1)(k'-r'-1)!}{(r'-1)(r'-2)!(k'-2r'+1)!} &= \frac{(k'+1)(k'-r')!}{r'(r'-1)!(k'-2r'+1)!} - \frac{k'(k'-r'-1)!}{r'(r'-1)!(k'-2r')!}. \end{aligned} \tag{2.73}$$

Now, we multiply both sides by $r'! \cdot (k'-2r'+1)!$, divide over $(k'-r'-1)!$, and we have

$$\begin{aligned} r'(k'-1) &= (k'+1)(k'-r') - k'(k'-2r'+1) \iff \\ r'k' - r' &= k'^2 - r'k' + k' - r' - k'^2 + 2r'k' - k'. \end{aligned} \tag{2.74}$$

Therefore, the sums $F_r(S) = \sum_{k\geq 3}\sum_{\ell\geq 0} f_r(k,\ell)X_{k,\ell}(S)$ only depend on n.

It remains to show that the value is $-m_r(n)$. Consider a set of n points S_C in convex position and let $p_1,\ldots,p_n$ be the points of S_C as they appear in counterclockwise order. Then $X_{k,\ell}(S_C) = 0$ for all $\ell > 0$ and we can rewrite the sum as

$$\begin{aligned} F_r(S_C) &= \sum_{k\geq 3}(-1)^{k+1} f_r(k,0)X_{k,0}(S_C) = \sum_{k\geq 3}(-1)^{k+1} m_r(k)X_{k,0}(S_C) = \\ &= M_{r,0}(S_C) = -T_r(S_C). \end{aligned} \tag{2.75}$$

Therefore, we have to show that $T_r(S_C) = m_r(n)$.

To prove that, we first observe that any $r-$tuple $\{e_1,\ldots,e_r\}$ of the convex set S_C can be seen as cyclically sorted. Moreover, if the $r-$tuple contributes to $T_r(S_C)$, the endpoint of any edge in this sorting is a neighbor to the starting point of the next edge, i.e., if $e_i = p_xp_y$ then $e_{i+1} = p_{y+1}p_z$ becuase, otherwise, the point in between has to be contained in the region $\tau(e_1,\ldots,e_r)$.

Let $\sigma_r(n)$ be the number of $r-$tuples that contribute to $T_r(S_C)$ and for which e_1 starts at p_1. Then, we will prove by induction on r that $\sigma_r(n) = \binom{n-r-1}{r-1}$.

For the base case, we have $r = 2$ and the sum is over all possibilities to choose the endpoint of e_1 (notice that the starting point of e_2 is fixed by the endpoint of e_1 and the endpoint of e_2 is fixed by the starting point of e_1), thus

$$\sigma_2(n) = \sum_{i=2}^{n-2} 1 = n - 3 = \binom{n-2-1}{2-1}. \tag{2.76}$$

For the inductive step, suppose that the statement holds for $r - 1$. Then the sum is over all possibilities to choose the endpoint of e_1 and gives

$$\sigma_r(n) = \sum_{i=2}^{n-2(r-1)} \sigma_{r-1}(n-i) = \sum_{i=2}^{n-2r+2} \binom{n-i-(r-1)-1}{(r-1)-1} = \sum_{i=2}^{n-2r+2} \binom{n-r-i}{r-2}. \tag{2.77}$$

If we consider the change $i' = n - r - i$ we obtain a finite telescoping series which gives the following result,

$$\sigma_r(n) = \sum_{i'=r-2}^{n-r-2} \binom{i'}{r-2} = \sum_{i'=r-2}^{n-r-2} \left(\binom{i'+1}{r-1} - \binom{i'}{r-1} \right) = \binom{n-r-1}{r-1}. \tag{2.78}$$

To finish the proof, we observe that when starting e_1 at an arbitrary point, every $r-$tuple is counted exactly r times (once for the starting point of every edge).

Hence, we obtain $T_r(S_C) = \frac{n}{r}\sigma_r(n) = \frac{n}{r}\binom{n-r-1}{r-1} = m_r(n)$. □

An interesting case of these combinations is for $r = 2$ because we know two of the four terms involved from Theorems 2.1.16 and 2.1.18,

$$\begin{aligned} F_2(S) &= M_{2,0}(S) - M_{1,1}(S) + M_{0,1}(S) + M_{0,2}(S) = \\ &= -T_2(S) - M_{1,1}(S) + (n-h) + M_{0,2}(S) = -\frac{n(n-3)}{2}. \end{aligned} \tag{2.79}$$

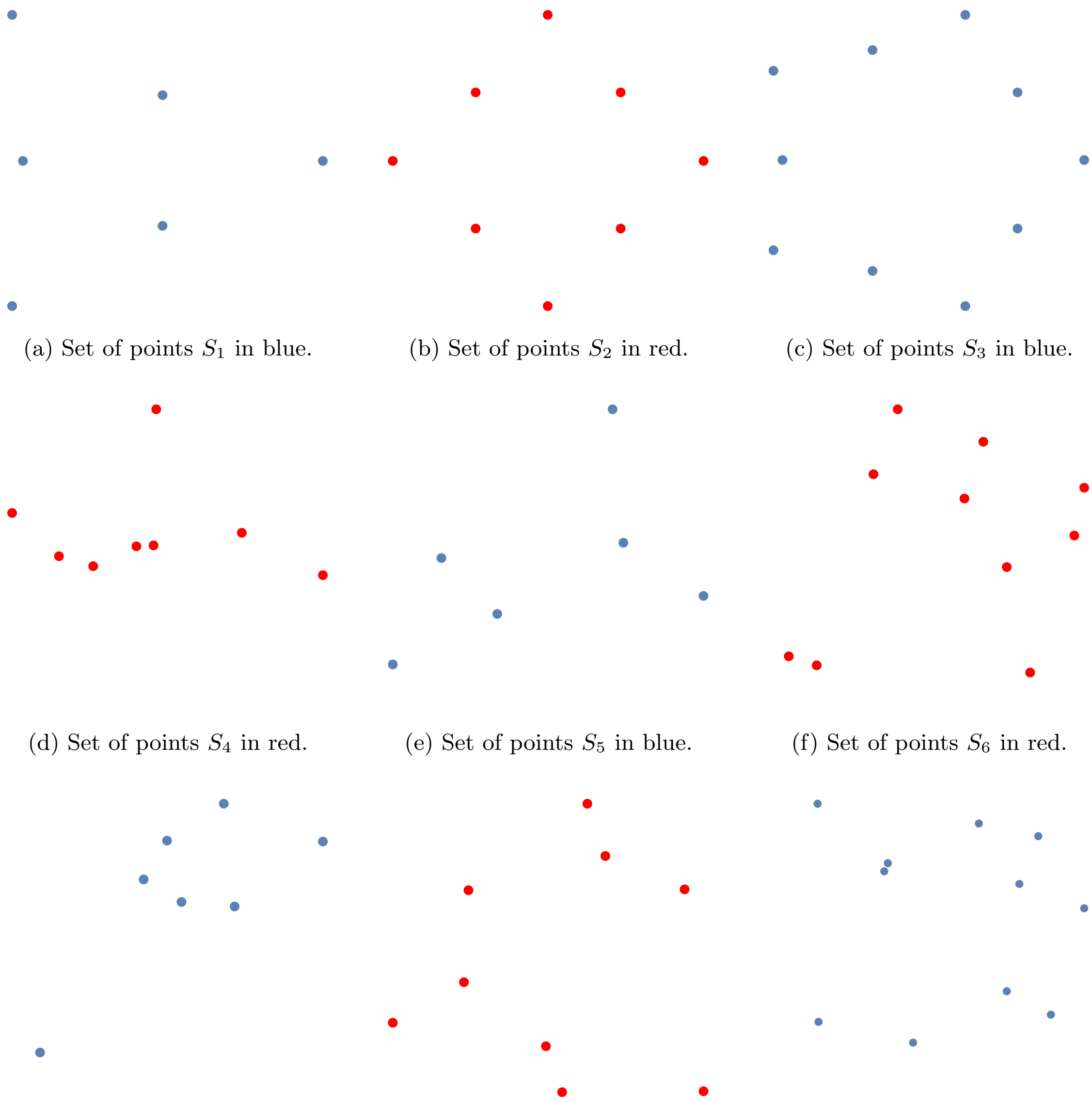
Therefore, from (2.79) we immediately get the following result.

Corollary 2.4.2. *For any point set S in general position, it holds that*

$$M_{0,2}(S) - M_{1,1}(S) = T_2(S) + h - \frac{n(n-1)}{2}. \tag{2.80}$$

From Corollary 2.4.2, we may think that $M_{0,2}(S)$ or $M_{1,1}(S)$ have also the dependence of $T_2,\ h,\ n^2, n$ individually. However, we can select a counterexample with different point sets in order to check that it is not true. Moreover, we use this counterexample to analyze if other dependencies could be analyzed later on.

Consider the sets of points described in Fig. 2.6, and their characteristic values detailed in Table 2.1.



(a) Set of points S_1 in blue. (b) Set of points S_2 in red. (c) Set of points S_3 in blue.

(d) Set of points S_4 in red. (e) Set of points S_5 in blue. (f) Set of points S_6 in red.

(g) Set of points S_7 in blue. (h) Set of points S_8 in red. (i) Set of points S_9 in blue.

Figure 2.6: 9 sets of points in the plane.

Let α_i, β_i for $1 \leq i \leq 7$ be variables such that,

$$M_{0,2}(S) = \alpha_1 + \alpha_2 n + \alpha_3 n^2 + \alpha_4 n^3 + \alpha_5 h + \alpha_6 T_2 + \alpha_7 T_3, \quad (2.81)$$

$$M_{1,1}(S) = \beta_1 + \beta_2 n + \beta_3 n^2 + \beta_4 n^3 + \beta_5 h + \beta_6 T_2 + \beta_7 T_3. \quad (2.82)$$

Set	n	h	T_2	T_3	$M_{0,2}(S)$	$M_{1,1}(S)$
S_1	6	3	6	0	0	6
S_2	8	4	16	4	0	8
S_3	10	5	30	25	0	10
S_4	8	5	20	0	1	4
S_5	6	4	7	0	0	4
S_6	10	7	36	4	0	2
S_7	7	5	12	2	1	5
S_8	9	6	30	2	-2	-2
S_9	11	7	45	21	0	3

Table 2.1: The values n, h, T_2 and T_3 for the sets described in Fig. 2.6.

Using the values of Table 2.1, from (2.81), we obtain the following system.

$$\begin{pmatrix} 1 & 6 & 36 & 216 & 3 & 6 & 0 \\ 1 & 8 & 64 & 512 & 4 & 16 & 4 \\ 1 & 10 & 100 & 1000 & 5 & 30 & 25 \\ 1 & 8 & 64 & 512 & 5 & 20 & 0 \\ 1 & 6 & 36 & 216 & 4 & 7 & 0 \\ 1 & 10 & 100 & 1000 & 7 & 36 & 4 \\ 1 & 7 & 49 & 343 & 5 & 12 & 2 \\ 1 & 9 & 81 & 729 & 6 & 30 & 2 \\ 1 & 11 & 121 & 1331 & 7 & 45 & 21 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}. \tag{2.83}$$

Similarly, from (2.82), we obtain the following system.

$$\begin{pmatrix} 1 & 6 & 36 & 216 & 3 & 6 & 0 \\ 1 & 8 & 64 & 512 & 4 & 16 & 4 \\ 1 & 10 & 100 & 1000 & 5 & 30 & 25 \\ 1 & 8 & 64 & 512 & 5 & 20 & 0 \\ 1 & 6 & 36 & 216 & 4 & 7 & 0 \\ 1 & 10 & 100 & 1000 & 7 & 36 & 4 \\ 1 & 7 & 49 & 343 & 5 & 12 & 2 \\ 1 & 9 & 81 & 729 & 6 & 30 & 2 \\ 1 & 11 & 121 & 1331 & 7 & 45 & 21 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 10 \\ 4 \\ 4 \\ 2 \\ 5 \\ -2 \\ 3 \end{pmatrix}. \tag{2.84}$$

It is not difficult to check that the coefficient matrix[1] of the systems (2.83) and (2.84) has rank 7. However, the corresponding augmented matrices[2] have rank 8 which implies

[1] The *coefficient matrix* is the matrix of the system that contains just the coefficient of the variables.

[2] The *augmented matrix* is the coefficient matrix augmented with an additional column consisting of the vector of constant entries, i.e., the column on the right-side of the system.

that, by the Rouché-Fröbenius Theorem, the systems of equations have no solutions in both cases.

Since the systems (2.83) and (2.84) do not have solution, we obtain the following result.

Proposition 2.4.3. *It is not possible to express $M_{0,2}(S)$ neither $M_{1,1}(S)$ in terms of* $1,\ n,\ n^2,\ n^3,\ h,\ T_2$ *and* T_3.

This result shows that is not possible to derive the moment sums $M_{r,\ell>0}$ in terms of the values defined previously.

On the other hand, since the moments $M_{r,0}$ of Theorem 2.1.16 were derived from results on $M_{0,0}(e_1,\ldots,e_r)$, one may think to apply the same strategy in order to obtain similar expressions for $M_{1,1}$ or $M_{0,2}$, but in [13] the authors derive that there is no exact value for $M_{0,1}(e)$.

Theorem 2.4.4. *For any point set S of n points in general position in the plane and any edge e of S, it holds that* $\max\{h,4\}-n \leq M_{0,1}(e) \leq 1$.

To ensure that these bounds are tight, they also detail the following example.

Consider a quadrilateral with the points denoted as $p_1,\ldots,p_4$ with a concave chain of $n-4$ interior points denoted as $q_1,\ldots q_{n-4}$, as shown in Fig. 2.7.

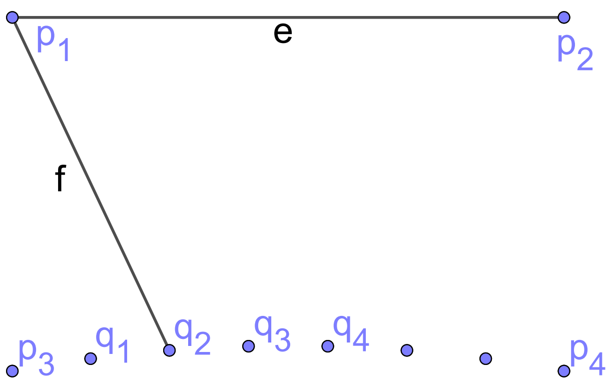


Figure 2.7: Set of points that is an example for the bounds of Theorem 2.4.4.

For the lower bound, we observe that the convex polygons with one interior point that contain the edge e, $X_{k,1}(e)$, can only be quadrilaterals where the interior point is a point of the concave chain. Since there are $n-4$ interior points of the concave chains, we have $X_{4,1}(e)=n-4$ and in consequence $M_{0,1}(S;e)=4-n$.

For the upper bound, we observe that the convex polygons with one interior point that contain the edge f, $X_{k,1}(f)$ can only be the triangle $p_1q_2p_3$, the triangle $p_1q_2q_4$ and the quadrilateral $p_1q_2q_4p_2$. Thus, $M_{0,1}(S;f) = 1 + 1 - 1 = 1$.

Therefore, Proposition 2.4.3 goes in the same direction as Theorem 2.4.4 in order to obtain expressions for the moments $M_{r,\ell>0}$, other strategies and the definition of other characteristic values of the point sets are necessary.

Chapter 3

Special configurations

In this chapter, we consider two well known configurations, namely the double circle [2] and the double chain [10]. We will apply the results of Chapter 2 to these two special configurations.

3.1 Double Chain

Definition 3.1.1. *The double chain of* $2n$ *points consists of two sets of* n *points each, one forming a convex chain and one forming a concave chain. We denote them by the upper and the lower chain. The two chains are sufficiently far apart from each other.*

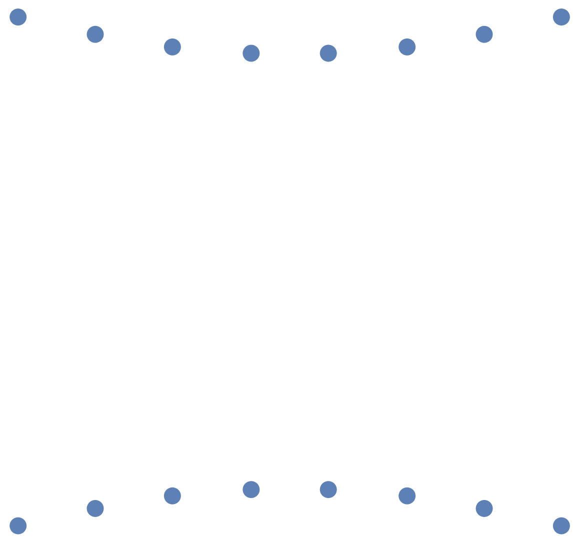


Figure 3.1: Double chain of 16 points.

Note that also Fig. 2.7 resembles the double chain

From this definition, it is easy to observe the following properties.

Observation 3.1.2. *The number of points in the boundary of the convex hull is* $h = 4$.

Observation 3.1.3. *Any non-empty convex polygon must contain points of the two chains.*

Observation 3.1.4. *Any non-empty convex polygon has at most 4 points in the boundary.*

In order to derive the $r-$th moments with $\ell \geq 1$ for the double chain, we first count the number of polygons with interior points.

Proposition 3.1.5. *The number of triangles with* $\ell \geq 0$ *interior points of the double chain of* $2n$ *points is* $X_{3,\ell} = 2n(n-1-\ell)$.

Proof. Without loss of generality, consider that the boundary of the triangle contains two points of the lower chain and one of the upper chain. Note that the interior points have to be contained in the lower chain, and if the points of the lower chain are sorted from left to right (enumerated as $y_1, \ldots, y_n$) then they have to be consecutive. It means that a triangle with ℓ interior points that contains y_i as the left-extreme point, the interior points have to be $y_{i+1}, \ldots, y_{i+\ell}$ and the other extreme point is $y_{i+\ell+1}$.

Therefore, for the lower chain we have $n-1-\ell$ possibilities to take 2 points in the boundary and ℓ in the interior whereas we have n possibilities to take 1 point for the upper chain. Since we can argue in the same manner for triangles with two points on the upper chain, we multiply the value by 2 and we obtain $X_{3,\ell} = 2n(n-1-\ell)$, as claimed. □

Proposition 3.1.6. *The number of convex quadrilaterals with* $\ell \geq 1$ *interior points of the double chain of* $2n$ *points is*

$$X_{4,\ell} = \sum_{i=0}^{\ell} (n-1-i)\,(n-1-\ell+i)\,. \tag{3.1}$$

Proof. We first observe that the boundary of the quadrilateral has to contain two points of the lower chain and two points of the upper chain.

Using the same argument as in the proof of Proposition 3.1.5, the interior points are i consecutive points of the lower chain and $\ell - i$ points of the upper chain for any $i = 0, \ldots, \ell$. Therefore, for every i there are $n-1-i$ possibilities for the lower chain and $n-1-(\ell-i)$ possibilities for the upper chain. Hence,

$$X_{4,\ell} = \sum_{i=0}^{\ell} (n-1-i)\,(n-1-\ell+i)\,. \tag{3.2}$$

□

Now, we can obtain an expression for the $r-$th moments with $\ell \geq 1$ interior points.

Theorem 3.1.7.

$$M_{0,\ell} = 2n(n-1-\ell) - \sum_{i=0}^{\ell} (n-1-i)(n-1-\ell+i), \quad (3.3)$$

$$M_{1,\ell} = 6n(n-1-\ell) - 4\sum_{i=0}^{\ell} (n-1-i)(n-1-\ell+i), \quad (3.4)$$

$$M_{2,\ell} = -2\sum_{i=0}^{\ell} (n-1-i)(n-1-\ell+i), \quad (3.5)$$

$$M_{r,\ell} = 0 \qquad \textit{for all } r \geq 3. \quad (3.6)$$

Proof. We first observe that $m_r(4) = m_r(3) = 0$ for all $r \geq 3$. Then, by Observation 3.1.4, clearly $M_{r,\ell\geq 1} = 0$ for all $r \geq 3$.

In addition $m_2(3) = 0$ which implies that for any $\ell \geq 1$,

$$M_{2,\ell} = -\frac{4}{2}X_{4,\ell} = -2\sum_{i=0}^{\ell} (n-1-i)(n-1-\ell+i). \quad (3.7)$$

For the case $r = 0$ and any $\ell \geq 1$,

$$M_{0,\ell} = X_{3,\ell} - X_{4,\ell} = 2n(n-1-\ell) - \sum_{i=0}^{\ell} (n-1-i)(n-1-\ell+i). \quad (3.8)$$

For the case $r = 1$ and any $\ell \geq 1$,

$$M_{1,\ell} = 3X_{3,\ell} - 4X_{4,\ell} = 6n(n-1-\ell) - 4\sum_{i=0}^{\ell} (n-1-i)(n-1-\ell+i). \quad (3.9)$$

□

3.2 Double circle

Definition 3.2.1. *The double circle DC_{2n} of $2n$ points contains n extreme points forming a regular $n-$gon. The remaining n interior points are placed sufficiently close to the edges of the $n-$gon, such that the set of interior edges, that are not crossed by any other edge, forms a star shaped region.*

To deduce some properties of the double circle, it is interesting to mention that in Lemma 2.1.15 we have defined the $r-$th moment for a set of vertex-disjoint edges. Now, we observe that the same argument of that proof is valid avoiding the condition that the edges are vertex-disjoint, which gives the following result.

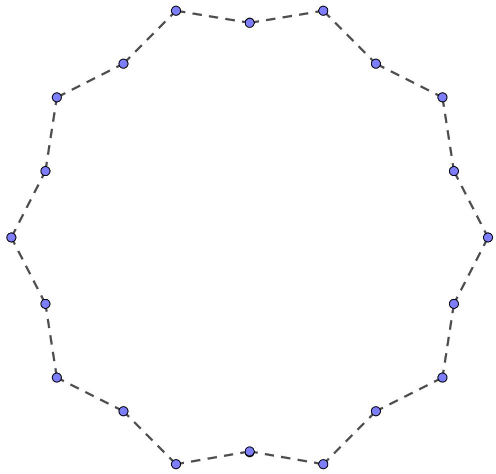


Figure 3.2: Double circle of 20 points.

Lemma 3.2.2. *Let $e_1, \ldots, e_\ell$ be a set of edges of the boundary of the convex hull of DC_{2n} and let κ be the number of endpoints of the edges. Let p_i be the point of DC_{2n} closest to e_i and let m be the number of points of $DC_{2n} \setminus \{p_1, \ldots, p_\ell\}$ inside the region $\tau(e_1, \ldots, e_\ell)$. Then,*

$$M_{0,0}(DC_{2n} \setminus \{p_1, \ldots, p_\ell\}; e_1, \ldots e_\ell) = \begin{cases} 0 & \text{if } m \neq 0, \\ (-1)^\kappa & \text{if } m = 0. \end{cases} \tag{3.10}$$

Proof. This proof is analogous to the proof of Lemma 2.1.15. The unique difference is for the case $m = 0$ where the only non-zero element of the set $\{X_{k,0}(DC_{2n} \setminus \{p_1, \ldots, p_\ell\}; e_1, \ldots e_\ell)\}_{k \geq 3}$ is $X_{\kappa,0}(DC_{2n} \setminus \{p_1, \ldots, p_\ell\}; e_1, \ldots e_\ell) = 1$. Therefore, if $m = 0$,

$$M_{0,0}(DC_{2n} \setminus \{p_1, \ldots, p_\ell\}; e_1, \ldots e_\ell) = (-1)^{\kappa+1} = -1. \tag{3.11}$$

□

An example of the notation used in this lemma is shown in Fig. 3.3.

Lemma 3.2.2 is necessary for the proof of the following result.

Lemma 3.2.3. *Let $p_1, \ldots, p_\ell \in DC_{2n}$ be $2 \leq \ell \leq n-1$ points of the double circle DC_{2n} of $2n$ points, then it holds that $M_{0,\ell}^{p_1,\ldots,p_\ell} = 0$.*

Proof. Obviously, if exists i such that p_i is an extreme point of DC_{2n}, then p_i cannot be in the interior of any polygon spanned by points of DC_{2n}, thus $M_{0,\ell}^{p_1,\ldots,p_\ell} = 0$.

Therefore, consider that all the points p_i are interior points of DC_{2n} and then every convex polygon that contains exactly $p_1, \ldots, p_\ell$ in its interior is an empty convex polygon

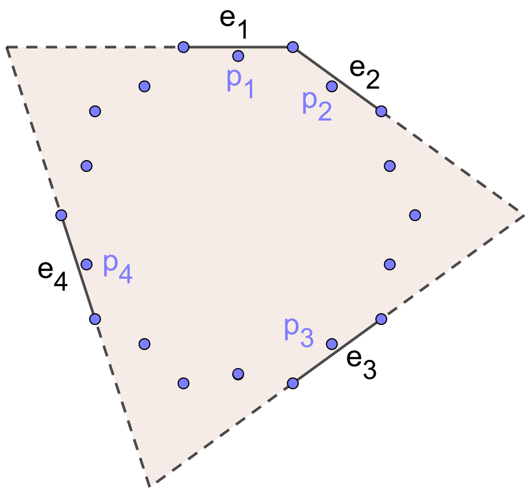


Figure 3.3: A region $\tau(e_1, e_2, e_3, e_4)$ of the double circle with $m = 9$ and $\kappa = 7$.

in $DC_{2n} \setminus \{p_1, \ldots, p_\ell\}$. Moreover, by definition, the interior points p_i lie very close to convex hull edges e_i of DC_{2n}, so the points p_i are contained in exactly all polygons that contains the edges e_i. Thus, $M_{0,\ell}^{p_1,\ldots,p_\ell} = M_{0,0}(DC_{2n} \setminus \{p_1, \ldots, p_\ell\}; e_1, \ldots e_\ell) = 0$ by Lemma 3.2.2. □

These lemmas allow to deduce expressions involving convex polygons of the double circle with the same techniques used in the general case and we obtain the following result.

Theorem 3.2.4. *$M_{0,\ell}(DC_{2n}) = 0$ for $2 \leq \ell \leq n-1$ and $M_{0,n}(DC_{2n}) = 1$.*

Proof. Clearly, any $k-$gon counted in $M_{0,\ell}(DC_{2n})$ contains exactly ℓ points in its interior, thus we can sum over the possible $\ell-$subsets of points and obtain that

$$M_{0,\ell}(DC_{2n}) = \sum_{\{p_1,\ldots,p_\ell\} \subset DC_{2n}} M_{0,\ell}^{p_1,\ldots,p_\ell}, \tag{3.12}$$

which, by Lemma 3.2.3, is 0 for $2 \leq \ell \leq n-1$ and 1 for $\ell = n$ because the alternating sum $M_{0,\ell}^{p_1,\ldots,p_\ell}$ is zero for all $n-$subsets except for the one which contains exactly the n interior points. □

This result together with Corollary 2.4.2 provides the immediate following result.

Corollary 3.2.5. *$M_{1,1}(DC_{2n}) = -T_2(DC_{2n}) - h + \frac{2n(2n-1)}{2} = 2n(n-1) - T_2(DC_{2n})$.*

Observation 3.2.6. *The argument used in the proof of Lemma 3.2.3 is based on the fact that any interior point lies very close to a convex hull edge. Therefore, any planar point set that fulfills this condition also satisfy the results of Theorem 3.2.4 and Corollary 3.2.5.*

Conclusions

The famous paper [8] by Erdős and Szekeres influenced and motivated new proof techniques and several new other problems, some of them still unsolved. In this thesis, we presented an updated account of results immediately related to the Erdős-Szekeres Theorem. We first introduced the context and outlined the main proof techniques, Based on continous motion arguments, we then developed several new results.

In this project, we have analyzed alternating sums of numbers of convex polygons with interior points connecting them to known geometrical properties of a planar point set. These expressions allowed us to derive inequalities involving the quantities $X_{k,\ell}$, some of them completely new. In this direction, we also have shown that similar techniques, as the continuous motion argument, allow us to deduce weighted sums that only depend on the cardinality of the point set. Specifically, we have obtained $n-2$ new independent formulas that only depend on the size n of the point set and we observe that it is the maximum possible number of independent equations that can be obtained by continuous motion arguments (that is, when Equation (2.44) is satisfied).

Finally, we have applied the expressions of the alternating sums to two families of special point configurations, the double chain and the double circle. For these two families we obtained closed expressions for the moment sums. In the general case of arbitrary point sets, we have not been able to deduce equivalent formula for higher moment sums (which possibly might not only depend on the number n of points).

Our obtained $n-2$ (independent) formulas relating the numbers of convex polygons may open a way to deduce further results related to the Erdős-Szekeres Theorem. For example, we can form a system of equations with variables $X_{k,\ell}$ by using the known relations among the $X_{k,\ell}$. To determine whether any sufficiently large set of points contains a convex k-gon, we can - by argument of contradiction - assume that there is a point set without a convex k-gon; then many of the variables $X_{k,\ell}$ can be set to 0. If the system of equation then has no solution, we get a contradiction. However, we have seen that the number of equations is still insufficient to have a determinate system. Therefore, other possible options to analyze could be to find new types of equations or to evaluate all the possible values for some of the variables $X_{k,\ell}$. For this last option, it is necessary to use adequate algorithms and computer implementation because it implies an exponential number of systems of equations and possible solutions. Thus, it is also necessary to obtain tight bounds for several values of $X_{k,\ell}$ and more properties to check the geometry of the solution. This approach has been undertaken in this thesis, but the program has not been optimized and the computational result has not been sufficiently satisfactory to obtain a conclusive result.

Note that, most of the results obtained during this thesis are also valid in higher dimensions, although we did not state these formulas explicitly. Specifically, the results over the weighted sums and the new formulas obtained that only depend on the size of the point set also hold in higher dimension. The reason of that is the fact that most of the results are deduced from the continuous motion argument which can also be applied in $\mathbb{R}^d$ for $d > 2$.

In conclusion, we have derived several expressions involving the numbers of convex polygons with interior points, that is, the quantities $X_{k,\ell}$, some of them new. These relationships might have implications to long-standing open problems related to the Erdős-Szekeres Theorem.

References

[1] Ahrens, C., Gordon, G., McMahon, E.W.: Convexity and the beta invariant. *Discrete Comput. Geom.*, **22**, 411-424 (1999).

[2] Aichholzer, O., Hurtado, F., Noy, M.: A lower bound on the number of triangulations of planar point sets. *Computational Geometry*, **29**, 135-145 (2004).

[3] Andrzejak, A., Aronov, B., Har-Peled, S., *et al.*: Results on k-sets and j-facets via Continuous Motion. *Proceedings of the Fourteenth Annual Symposium on Computational Geometry*, 192-199 (1998).

[4] Bárány, I., Károlyi, G.: Problems and results around the Erdős-Szekeres convex polygon theorem. *Japanese Conference on Discrete and Computational Geometry* (2000), *Lecture Notes in Comput. Sci.*, **2098**, 91-105 (2001).

[5] Edelman, P.H., Jamison, R.: The theory of convex geometries. *Geometriae Dedicata*, **19**, 247-270 (1987).

[6] Edelman, P.H., Reiner, V.: Counting the interior of a point configuration. *Discrete Comput. Geom.*, **23**, 1-13 (2000).

[7] Erdős, P.: Some more problems on elementary geometry. *Austral. Math. Soc. Gaz.*, **5**, 52-54 (1978).

[8] Erdős, P., Szekeres, G.: A combinatorial problem in geometry. *Compositio Mathematica*, **2**, 463-470 (1935).

[9] Erdős, P., Szekeres, G.: On some extremum problems in elementary geometry. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **3-4**, 53-62 (1961).

[10] García, A., Noy, M., Tejel, J.: Lower bounds on the number of crossing-free subgraphs of K_n. *Comput. Geom.*, **16**, 211-221 (2000).

[11] Gerken, T.: Empty convex hexagons in planar point sets. *Discrete Comput. Geom.*, **39**, 239-272 (2008).

[12] Harborth, H.: Konvexe Fünfecke in ebenen Punktmengen, *Elem. Math.*, **33**, 116-118 (1978).

[13] Huemer, C., Oliveros, D., Pérez-Lantero, P., Vogtenhuber, B.: Alternating sums of numbers of polygons with interior points. Preprint (2017).

[14] Horton, J. D.: Sets with no empty convex 7-gon. *Canad. Math. Bull.*, **26**, 482-484 (1983).

[15] Kalbfleisch, J. D.; Kalbfleisch, J. G.; Stanton, R. G.: A combinatorial problem on convex n-gons. *Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing*, **1**, 180-188 (1970).

[16] Klain, D.: An Euler relation for valuations of polytopes. *Adv. Math.*, **147**, 1-34 (1999).

[17] Koshelev, V. A.: On Erdős-Szekeres problem and related problems. In ArXiv: `arXiv:0910.2700v1` (2009).

[18] Koshelev, V. A.: The Erdős-Szekeres problem. *Dokl. Math.*, **76**, 603-605 (2007).

[19] Koshelev, V. A.: Computer solution of the almost empty hexagon problem. *Math. Notes*, **89**, 455-458 (2011).

[20] Koshelev, V. A.: On Erdős-Szekeres-type problems. *Electronic Notes in Discrete Mathematics*, **34**, 447-451 (2009).

[21] Koshelev, V. A.: Interior Points in the Erdős-Szekeres Theorems. *Math. Notes.* **91**, 542-557 (2012).

[22] Matoušek, J., Nešetšil, J.: Invitation to Discrete Mathematics, Clarendon Press, Oxford (1998).

[23] Morris, W., Soltan, V.: The Erdős-Szekeres problem. In: Nash J., Rassias, M. (eds), Open problems in mathematics, Springer, New York, 351-375 (2016).

[24] Nicolás, C.: The empty hexagon theorem. *Discrete Comput. Geom.*, **38**, 389-397 (2007).

[25] Nyklova, H.: Almost empty polygons. *Studia Scientiarum Mathematicarum Hungarica*, **40**, 269-286, (2003).

[26] Overmars, M.: Finding sets of points without empty convex 6-gons. *Discrete Comput. Geom.*, **29**, 153-158 (2002).

[27] Pinchasi, R., Radoičić, R., Sharir, M.: On empty convex polygons in a planar point set. *Journal of Comb. Theory*, **113**, 385-419 (2006).

[28] Ramsey, F. P.: On a problem of formal logic. *Proceedings of the London Mathematical Society*, **30**, 338-384 (1930).

[29] Sendov, B.: Compulsory configurations in the plane. *Fundam. and Prikl. Math.*, **1**, 491-516 (1995).

[30] Soifer, A.: The Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of Its Creators. Springer, New York (2009).

[31] Suk, A.: On the Erdős-Szekeres convex polygon problem. *Journal of the American Mathematical Society*, **30**, 1047-1053 (2017).

[32] Szekeres, G., Peters, L.: Computer solution to the 17-point Erdős-Szekeres problem. *The ANZIAM Journal*, **48(2)**, 151-164 (2006).

[33] Tóth, G., Valtr, P.: The Erdős-Szekeres theorem: upper bounds and related results. *Combinatorial and Computational Geometry*, **52**, 557-568 (2005).

[34] Tverberg, H.: A generalization of Radon's Theorem. *Journal of the London Mathematical Society*, **41**, 123-128 (1966).

[35] Valtr, P.: On empty hexagons. *Contemp. Math.*, **453**, 433-441 (2008).