A Journey to Understanding:

Developing Computational Fluency in Multi-digit Multiplication

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This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Kristen Tripet

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ABSTRACT

This thesis presents a domain-specific instructional theory for multi-digit multiplication, focused on building students' conceptual understanding and computational fluency.

Multiplicative thinking is crucial to students' mathematical development. Without a sound understanding of multiplicative structure students' capacity to develop deep understanding and fluency in fractions, decimals, proportional reasoning and ratios is severely limited. Understanding multiplicative structure enables students to move beyond additive strategies such as skip counting and repeated addition, to efficient and sophisticated strategies that grow from their ability to think simultaneously about composite units. While there has been substantial research into early stages of students' understanding in multiplication, there has been limited work exploring the more complex domain of multi-digit multiplication.

The domain-specific instructional theory developed through the research described in this thesis includes a number of key features that build on and expand existing research. It draws on social constructivist perspectives to document the social and cognitive development of learners by uncovering, examining and analysing the mathematical practices that emerge through the collective learning in the classroom. Students' invented strategies for multi-digit multiplication problems are connected to key developmental understandings of commutativity, associativity and distributivity. The crucial role of the array is explored in the process of students' sense-making and reasoning. The theory developed in the research proposes an instructional sequence from a model of specific, contextualised situations, through to a model for more generalised mathematical reasoning in the domain of multi-digit multiplication.

Design Research methods were used to inform the development of the domain-specific instructional theory. A hypothetical learning trajectory was constructed based on a review of relevant research-based literature into students' use of the array and on curriculum documentation guiding teachers' practices. The learning trajectory was tested in two separate teaching

experiments, each involving ten teaching episodes conducted over a two-week period. Work samples, video and interview responses from a total of 55 Year 5 students from two classes were analysed and used to inform evidence-based refinement and modification of the learning trajectory. Ways in which learning could be supported through the implementation of the learning trajectory were also documented.

Several key findings emerged through the design research. Students used a variety of invented strategies that drew on additive and multiplicative thinking, in some cases exclusively and in others, in combination. Students' reasoning and justification relied predominately on the array, enabling them to make sense of the multiplicative structure in a way that symbolic recording alone did not. A number of different forms of the array were used in the study, with students electing to use different forms of the array based on the function they needed the array to serve. As students' appreciation of and confidence with the multiplicative structure increased, their reliance on the array decreased, allowing them to move to more numerical notation underpinned by the sense-making developed through use of the array.

Four mathematical practices relating to the social development of students' understanding of the multiplicative structure were identified. Two of these: partitioning based on place value, and; using factors to manipulate the array, were based on students' use of the array as a tool for sense-making. The other two, thinking multiplicatively, and; looking for friendly numbers, were based on ways that the students worked mathematically. Additionally, a set of five mathematical norms was identified as central to each students' development of these mathematical practices. These were: looking for similarity and difference; making inferences; using representations; justification, and: forming generalisations.

The research highlights some crucial aspects of teaching practice that are essential if students are to develop a sound understanding of multi-digit multiplication. First, instruction needs to focus on multiplicative structure built on representations that highlight fundamental mathematical properties. Second, computational fluency grows from an understanding of structure but needs to be explicitly developed through focused discussion of the mathematical features of particular strategies and representations. Third, mathematics classrooms must be focused on sense-making through carefully orchestrated discussion of students' invented strategies and representations. Finally, the affordances and constraints of different forms of representation must be recognised in order to make clear the function that each of the possible forms might perform.

The research therefore adds to the existing literature relating to multi-digit multiplication and to that relating to the development of sociomathematical norms. It brings cognitive and social perspectives of learning together in a new way, proposing a focus on form and function in multiplicative thinking and a set of transcendent mathematical norms that underpin students' reasoning in mathematics.

CHAPTER 1

INTRODUCTION

The general principles that lie behind what we know as 'mathematics' never change: when one is added to one, the answer will always be two. However, the environment in which mathematics is employed has undergone radical and rapid change. While we now have the tools to process massive amounts of mathematical data at extraordinary speed (Foster, 2018; Seeley, 2009), these tools have paradoxically mandated the need for a proficient understanding and utilisation of mathematics in our daily lives.

This need has presented enormous challenges to the teaching of mathematics. Where once mathematical fluency was defined in terms of a student's facility with arithmetic procedures (National Research Council, 2001), today's students need a strong grasp of mathematical principles and deep number sense (Australian Curriculum Assessment and Reporting Authority (ACARA), 2017b; Dole, Carmichael, Thiele, Simpson, & Toole, 2018) as a springboard into flexible, efficient and accurate mental calculations for solving mathematical problems (National Council of Teachers of Mathematics, 2014a). Much of the mathematical focus in classrooms has traditionally been driven by the teaching and subsequent practice of procedural computational methods (Clarke, 2005; Fuson, 2003), placing a low ceiling on mathematical proficiency—one that will see students flailing and failing in the later years of schooling due to unchallenged misconceptions and unresolved misunderstandings. Seeley (2009) stressed that we need to "go beyond teaching basic skills, beyond requiring students to know how to perform procedures, and beyond offering recipes for solving problems that look alike" (p. 4). In our contemporary context in which machines are able to perform complex algorithmic calculations efficiently and quickly, the focus has shifted towards increased flexibility in, and strategic approaches to, completing arithmetic calculations (Cartwright, 2018; Dole et al., 2018; Foster, 2018).

The definition of 'computational fluency' has shifted from pure recall of taught methods to a fundamental understanding of how mathematics works. The National Research Council (2014a) states that 'fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem solving'. This notion of computational fluency is supported by Cartwright (2018), who defines fluency as the intersection of conceptual understanding, strategic competence and adaptive reasoning. Cartwright (2018) states that "mathematical fluency is the result when students' strategies and ability to reason are concurrent with their conceptual understanding" (p. 208).

Computational fluency in multiplication stems from an ability to think multiplicatively. Multiplicative thinking is the ability to *think* and *reason* using a deep conceptual understanding of the multiplicative structure and is substantially more complex than additive thinking (Siemon, 2013). Whereas addition is concerned with the joining of sets, multiplication is about replication (Barmby, Harries, Higgins, & Suggate, 2009). Students need to appreciate the binary nature of multiplication, where each number involved represents a distinct input in the multiplication process (Barmby et al., 2009; Steffe, 1994). The complexity of multiplicative thinking "poses a real and persistent barrier to many students' mathematical progress in the middle years of schooling" (Siemon, 2013, p. 43).

The research reported in this thesis explored the development of students' computational fluency in multi-digit multiplication. This introductory chapter presents the context of the study, the specific aims and an overview of the full thesis. The context of the study stemmed from two key incidents in my professional life, both connected to my role as an Education Consultant in Mathematics for the Association of Independent Schools of NSW (AISNSW). These two events, described as *inciting incidents*, are detailed in the following section, presenting the context for this study.

Inciting Incident 1—A Gap in the Research

In my role as an Education Consultant for the AISNSW, I worked alongside schools in the implementation of the professional learning program *Learning in Early Numeracy (LIEN)*. LIEN was developed by the AISNSW and based on the Early Numeracy Research Project (ENRP), a Victorian Department of Education initiative, which sought to identify processes for supporting and enhancing numeracy learning in the early years of school (Clarke et al., 2002; Clarke, Sullivan, Cheeseman, & Clarke, 2000). The LIEN project provided teachers with a framework describing students' development of understanding in number, specifically in the domains of counting, place value, addition and subtraction, and multiplication and division. For each domain a sequence of growth points were presented as key stepping stones along a path that students travel in their development of mathematical understanding. The growth points served as a tool to support teachers to structure their teaching and assess their students' understanding of mathematics.

As part of the LIEN project, teachers participated in a series of professional learning workshops exploring students' progression between growth points in each domain. I presented the multiplication and division workshop at many schools. On numerous occasions, teachers commented on students' difficulty transitioning from single-digit multiplication to calculations with multi-digits. There was a steady progression evident in the growth points for single-digit multiplication, but a broad jump in understanding was made in transitioning straight to multi-digit multiplication, which was only included in an aside comment (Figure 1.1).

I started my own research into multi-digit multiplication to further support teachers and improve the professional learning module. Through my research, two things became apparent. First, there was limited research in the domain of multi-digit multiplication. Secondly, multiplicative thinking was an area of weakness for many middle years students (Clark & Kamii, 1996; Hurst & Hurrell, 2016; Siemon, 2013; Siemon, Izard, Breed, & Virgona, 2006).

Growth Point 0	***Not apparent*** Not yet able to create and count the total of several small groups.
Growth Point 1	***Counting group items as ones*** To find the total in a multiple group situation, refers to individual items only.
Growth Point 2	***Modelling multiplication and division (all objects perceived)*** Models all objects to solve multiplicative and sharing situations.
Growth Point 3	***Abstracting multiplication and division*** Solves multiplication and division problems where objects are not all modelled or perceived.
Growth Point 4	***Basic derived and intuitive strategies for multiplication*** Can solve a range of multiplication problems using strategies such as commutativity, skip counting and building up from known facts.
Growth Point 5	***Basic, derived and intuitive strategies for division*** Can solve a range of division problems using strategies such as fact families and building up from known facts.
Growth Point 6	***Extending multiplication and division*** Can solve a range of multiplication and division problems (including multi-digit numbers) in practical contexts

Figure 1.1—The LIEN Growth Points for multiplication and division

Teachers (with whom I worked) perceived that students' difficulties lay with the procedural calculation relating to multi-digit multiplication. However, many student errors and misunderstandings in multi-digit multiplication stem directly from the use of additive strategies in the multiplicative context, including counting, repeated addition, repeated doubling and forms of re-grouping (Ambrose, Baek, & Carpenter, 2003; Barmby et al., 2009; Izsak, 2004; Young-Loveridge & Mills, 2009). Errors based on the use of additive strategies in the multiplicative context speaks to student difficulty with *thinking multiplicatively* as a whole process, rather than a simple struggle with procedure.

When my study began, research had already shown the array to be a powerful tool for multiplication (Barmby et al., 2009; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Davis, 2008; Fosnot & Dolk, 2001), particularly due to its ability to highlight the multiplicative properties of commutativity, associativity and distributivity. Use of multiplicative procedures, and an understanding of how they work, grows from students' appreciation of these three properties

(Ambrose et al., 2003; Larsson, 2016). The professional learning workshop on multiplication and division introduced teachers to the area model of multiplication, or open array (the term used in this thesis, from Fosnot & Dolk, 2001). Teachers found this representation to be a helpful model for illustrating multi-digit multiplication, yet there was little research to show how students' understanding of the use of the array for multi-digit computation developed.

Through my preliminary research associated with the LIEN project, it became apparent that a significant contribution could be made to the field of multi-digit multiplication. There was scope to explore students developed understanding in the domain and how teachers could support this development. Of particular interest were the changing and developing ways that students interacted with the array.

Inciting Incident 2—The Freudenthal Institute

The second inciting incident was the award of an overseas study scholarship in 2010 with the AISNSW. As part of the scholarship tour, I chose to spend three days at the Freudenthal Institute in the Netherlands. I was particularly interested in learning more about the application of *Realistic Mathematics Education* (RME) to classroom teaching, the basis of the work at the Freudenthal Institute. I recognised many parallels between the philosophy of RME and my own beliefs about what constitutes quality teaching in mathematics.

RME is based on the ideals for mathematics learning presented by Hans Freudenthal (1905 – 1990), who believed mathematics to be a human endeavour. Freudenthal argued that mathematics should not be learnt as a closed field of knowledge, rather, learning should occur through a process of *mathematisation* (Gravemeijer, 2004; Van den Heuvel-Panhuizen & Drijvers, 2014), that is, making sense of a situation mathematically. This process of mathematisation in the classroom context resonated with me. In RME, mathematical learning is a process where students reinvent conventional mathematics for themselves (Gravemeijer, 2004) as they develop ways to solve problems in the social setting of the classroom (Gravemeijer, Bowers, & Stephan, 2003a). The use of contexts is a key feature. Freudenthal believed that mathematics needed to be connected to reality so that it remained relevant to students (Van den Heuvel-Panhuizen, 2003). The elements of RME are discussed in more detail in Chapters 4 and 5.

The Freudenthal Institute was involved in the research and development of mathematics instructional sequences for classrooms using the philosophy of RME. According to an RME approach, the focus of design is on harnessing students' constructions rather than on tasks. The learning sequence is designed based on how the thinking and learning of students might unfold as they participate in tasks and activities (Gravemeijer, 2004). The anticipated learning sequences are tested through classroom implementation and refined according to whether the actual constructions of the students align with the constructions that were hypothesised. The end result is a viable, localised instructional sequence that can be adapted by teachers (Gravemeijer, Bowers, & Stephan, 2003b). Three design heuristics guide the RME designer and are described in detail in Chapter 5 of this thesis: experientially real contexts, guided reinvention and emergent modelling.

Through my scholarship visit to the Freudenthal Institute, I recognised RME as a valuable theory on designing mathematics instructional sequences that could be used by teachers in their classrooms.

Aims of the Study

This thesis presents a viable instructional theory for multi-digit multiplication (Gravemeijer et al., 2003b). The research was focused by two specific aims, each of which relate to the two inciting incidents. The first aim was to explore how students construct understanding in multi-digit multiplication. Students' understanding was explored from two perspectives: the social development of understanding and individual students' constructions of understanding. Social learning is categorised as *mathematical practices*, that is, the taken-as-shared ways that the class community reasons and argues mathematically (Cobb & Yackel, 1995). Individual students' understanding is related to the heuristics of RME; the pathways students follow as they reinvent conventional mathematics and the changing and developing ways that the students make use of tools.

The second aim of the research was to examine how students' development of multi-digit multiplication understanding could be represented in a learning trajectory. The objective of the learning trajectory was to document the expected pathway of student learning and the means by which learning could be supported. The trajectory was designed using the design heuristics of RME. Particular attention was given to the tasks that afford the reinvention of conventional mathematics, the focus of mathematical discourse and the use of representations, or models, and how these models help students reorganise their reasoning (Cobb, 2003).

The aims of the research were supported by five specific research questions:

1. How does the array develop from a *model of* a contextual situation to a *model for* more generalised mathematical reasoning in multi-digit multiplication?
2. What key developmental understandings (KDUs) in multi-digit multiplication are reflected in a hypothesised learning trajectory?
3. What strategies do students reinvent through the implementation of the learning trajectory?
4. What social mathematical practices emerge in the classroom through the implementation of the hypothesised learning trajectory?
5. How can students' learning be supported through the implementation of such a learning trajectory?

Structure of the Thesis

This thesis comprises eight chapters. Following this introductory chapter, the second and third chapters provide comprehensive literature reviews which informed the study. The content of Chapter 2 focuses on computational fluency and mathematical understanding. The chapter opens by emphasising the changing definition for fluency and presents fluency as comprising both conceptual and procedural knowledge, the basis of mathematical understanding. As students create connections between their internal representations of mathematical concepts and procedures, they build mathematical understanding. The nature of these connections and the role of representations in understanding is discussed as part of the literature review.

The literature presented in Chapter 3 focuses on the domain of multiplication. The broad idea of multiplicative thinking is discussed, and then key research work specific to multi-digit multiplication is explored. Findings into developing understanding of multiplicative properties and the use of the array as tool for multiplication are reviewed. The chapter also presents the strategies students use to solve multi-digit multiplication problems. In both of the literature review chapters, the implications for my own research are described, with the specific research questions emerging through these descriptions.

Chapter 4 is the methodology chapter. The research was grounded by a theoretical framework built on constructivism (the theory for learning) and RME (the theory for design). The methodology used for the study was Design Research. Much of the chapter is devoted to a discussion of the three research phases of Design Research: the preparatory thought experiments, the teaching experiment and the retrospective analysis. The specific methods that were used in the three phases conclude the chapter.

The enactment of these three phases are used to structure Chapters 5, 6 and 7. The preparatory thought experiments that occurred at the start of the research are the focus of Chapter 5. Chapter 6 presents the results of the two teaching experiments. The results report the social mathematical practices that emerged through the teaching sequence. The mathematical reasoning of individual students and how they participated in, and contributed to, the emergence of the mathematical practice is described.

Chapter 7 is devoted to the retrospective analysis phase of the Design Research process. The key findings of the research are presented as three *themes* in the chapter. The three themes are discussed, and the research questions are answered. This chapter also documents the revised learning trajectory as a domain-specific instructional theory for multi-digit multiplication.

The final chapter highlights the major findings of the research and presents the significant contributions that this thesis makes to theory and practice.

CHAPTER 2

LITERATURE REVIEW

COMPUTATIONAL FLUENCY & UNDERSTANDING

Introduction

This thesis is concerned with building students' fluency with, and understanding of, multi-digit multiplication. This chapter, divided into two main sections, presents a review of literature on computational fluency and mathematical understanding. The first section defines computational fluency and explains that fluency is built on a deep number sense. The second part of the chapter explores mathematical understanding. It examines research on understanding as the product of students' learning and as the process of how students come to understand new content. The role of representations is highlighted as crucial in the development of understanding. Each section concludes by detailing implications for this specific study.

Computational Fluency

Achieving fluency is a key goal of the Australian Curriculum: Mathematics (ACARA, 2017a, 2017b) and curriculum documentation internationally (Department for Education, 2013; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The Australian Curriculum: Mathematics (ACARA, 2017a) defines fluency as "choosing appropriate procedures; carrying out procedures flexibly, accurately, efficiently and appropriately; and recalling factual knowledge and concepts readily". The curriculum states that students demonstrate fluency when they "calculate answers efficiently, when they recognise robust ways of answering questions, when they choose appropriate methods and approximations…". Fluency conceptualised in this manner reaches beyond rote learning and automatic recall of facts and procedures (Dole et al., 2018). Instead, conceptual understanding, reasoning and problem solving are integral to the development and demonstration of fluency (Cartwright, 2018; National Research Council, 2001). The National Council for Teachers of Mathematics (National Council of Teachers of Mathematics, 2014b) recommends that teaching focused on conceptual knowledge should 'precede and coincide' with instruction on procedures. Such instruction will enable all students to have 'a deep and flexible knowledge of a variety of procedures, along with an ability to make critical judgments about which procedures or strategies are appropriate for use in particular situations.'

In many mathematics classrooms the importance of efficiency and accuracy are readily acknowledged by teachers and developed through practice. However, flexibility with procedures is too often disregarded and unobserved in students' working (Keiser, 2010), which presents real concerns for developing students' fluency. Much of the emphasis in classroom practice today is still on the use and mastery of procedures for standard algorithms over the flexible selection of appropriate methods (Bobis, Mulligan, & Lowrie, 2013).

The Place of Formal Algorithms

There has been considerable research into the limitations and obstacles of teaching operational understanding solely through an algorithm (Clarke, 2005; Kamii & Dominick, 1998; Narode, Board, & Davenport, 1993). Formal vertical algorithms have been found to compromise students' place value understanding (Kamii & Dominick, 1998) and conceptual knowledge (Narode et al., 1993). These algorithms do not correspond to the way we naturally read numbers (Clarke, 2005; Kamii & Dominick, 1998): an algorithm requires the user to work vertically from right to left rather than horizontally from left to right. Madell (1985) found that when children were asked to invent their own methods to solve problems, they worked with numbers from left to right, in accordance with how we ask children to read and interpret numbers.

Kamii et al. (1998) demonstrated that when students attempted to solve calculations, they would try to remember the steps of the vertical algorithm and inadvertently make mistakes, rather than rationally thinking through the problem. When asked to solve 7 + 52 + 186, the group that had not been taught the vertical written algorithm answered with 50% accuracy. They used their own solution methods to produce answers ranging from 221 to 284. The algorithm group had only 26% accuracy and produced answers over a range of 29 to 838. This discrepancy suggests a lack of number sense and an underdeveloped appreciation of what a reasonable answer should be. It also suggests that the use of algorithms can encourage students to blindly accept answers given and to surrender their own thinking (Kamii & Dominick, 1998; Plunkett, 1979). Fuson et al. (1997) concludes that it is through the *invention* of procedures that children gain strong understanding of operations. Studies have shown that children who used their own methods for problems, rather than relying on the algorithm, were more likely to produce the correct answer (Carraher, Carraher, & Schliemann, 1985).

As an overemphasis on formal computational methods limits students' mathematical thinking and inhibits the development of number sense (Vincent, 2013; Yang & Wu, 2010), it can also be concluded that the development of computational fluency is hindered. Indeed, it can be argued that such an approach is an oversimplification of what it means to be computationally fluent (Baroody, Feil, & Johnson, 2007). In these circumstances efficiency and accuracy may be evident, but flexibility is not (Dunphy, 2007; Yang & Li, 2008; Yang & Wu, 2010).

Fluency through Number Sense

Flexibility to choose appropriate strategies given a problem is a key indicator of procedural knowledge and is a "nontrivial and often overlooked competency" (Star, 2005, p. 409). Having multiple methods available to solve calculations requires students to have a deep understanding of the meaning of and relationship between operations and numbers (Russell, 2000). The importance of flexibility to be able to choose between multiple methods is reinforced by Ma (1999) who says:

Being able to calculate in multiple ways means that one has transcended the formality of the algorithm and reached the essence of the numerical operations—the underlying mathematical ideas and principles. The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules, but connected ideas. Being able to and tending to solve a problem in more than one way, therefore, reveals the ability and the predilection to make connections between and among mathematical areas and topics (p. 112).

Using numbers and operations in flexible ways and to make connections is described as 'number sense' (McIntosh, Reys, & Reys, 1992). McIntosh et al. (1992) define number sense as:

…a person's understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and develop useful strategies for handling numbers and operations. It reflects an inclination to use numbers and quantitative methods as a mean of communicating, processing and interpreting information (p. 3).

The notion of flexibility as an important component of number sense is iterated by Yang and Wu (2010):

…[number sense] comprises a deep understanding of numbers, operations, and their relationships; a high degree of flexibility and efficiency with numbers and operations; and an application of knowledge to numerical situations (p. 380).

Askew's (2008) study with Year 4 students illustrates the importance of flexibility through number sense in operational work. Students were asked to find the number of tickets remaining for a concert with 5003 seats, after 4997 tickets had already been sold. The majority of students relied on a formal vertical algorithm, most of whom calculated incorrectly. A smaller group used a number line to inefficiently jump back 4997 from 5003. Only one pair of students used the number line to efficiently calculate the difference between the two numbers as six. Askew (2008) stated that the lack of flexibility to be able to choose between different strategies highlights the difference between "working with numbers strategically and working with numbers procedurally… Working strategically means starting with a more holistic view—thinking about the numbers involved as well as the operation" (p. 33).

Developing fluency requires deep number sense: a balance and connection between conceptual understanding and procedural proficiency. On the one hand, computational methods that are over-practised without understanding are often forgotten or remembered incorrectly (Russell, 2000); on the other hand, understanding without basic facts and computational methods will limit problem solving ability (Hiebert, 1999; National Council of Teachers of Mathematics, 2014b). A balanced and iterative relationship between conceptual and computational understanding is required for real computational fluency (Rittle-Johnson, Siegler, & Alibali, 2001).

Procedural and Conceptual Knowledge

Mathematical understanding comprises both procedural and conceptual knowledge (Larsson, 2016). Students need to know and think of mathematics as both conceptual and procedural (Sfard, 1991; Skemp, 1976). Sfard (1991) states that working with mathematical concepts "both as a process and as an object is indispensable for a deep understanding of mathematics" (p. 5).

Procedural knowledge is displayed through proficiency with procedures, where procedures are defined as a series of steps or actions to accomplish a goal (Rittle-Johnson, Schneider, & Star, 2015). Conceptual knowledge is "comprehension of mathematical concepts, operations and relations" (National Research Council, 2001, p. 5).

Rittle-Johnson et al. (2015) build on this definition, claiming that conceptual knowledge is beyond just comprehension and encompasses both the ability to abstract concepts and knowing the general principles behind them. There is debate over the relationship between these two forms of knowledge in the development of understanding. On the one hand, some argue that students develop a knowledge of concepts through the use of procedures, particularly in problem contexts (Siegler & Stern, 1998). On the other hand, others argue that procedural knowledge should be founded on conceptual knowledge (National Council of Teachers of Mathematics, 2014a). Evidence suggests that a conceptual to procedural approach assists students in developing and retaining procedures (Fuson, Kalchman, & Bransford, John, 2005; National Council of Teachers of Mathematics, 2014a). It is this perspective, conceptual-to-procedural knowledge, that drives much of the current reform in curriculum documentation (Rittle-Johnson et al., 2015). A third perspective is offered by Rittle-Johnson et al., (2015), who argue that the relationship between conceptual and procedural knowledge is bidirectional; building one form of knowledge builds the other. Rittle-Johnson et al. state (2015):

> *Mathematical competence rests on developing both conceptual and procedural knowledge… It is not a one-way street from conceptual knowledge to procedural knowledge; the belief that procedural knowledge does not support conceptual knowledge is a myth* (p. 594).

Implications for this study

The focus of this study was to build students' computational fluency in multi-digit multiplication. The work with students was on students' own strategies for solving multi-digit multiplication problems. Based on curriculum documentation (Board of Studies NSW, 2002), it was anticipated that this would be most students' first experience with multi-digit multiplication problems. The focus of instruction was on building students' fluency through number sense, through connecting students' procedural and conceptual knowledge. This study adopted the bidirectional relationship between procedural and conceptual knowledge offered by Rittle-Johnson et al. (2015). Conceptual understanding and procedural competence were used to support each other. Instruction was focused on building students' conceptual understanding of the structure of multiplication, and on discovering how multiplicative properties can be harnessed to create and use procedures to solve problems. In the same way, students' invented procedures were used to explore the multiplicative structure and identify the properties that lie behind the computational properties used.

Understanding

Up to this point, the review of literature has highlighted the fact that computational fluency, based on number sense, is built through conceptual and procedural understanding. It is pertinent to consider the bigger question: what is the nature of mathematical understanding? Although research and curriculum documentation promote mathematical understanding, interpretations of what constitutes understanding vary. The following section presents key work on the topic. It highlights that understanding comprises both conceptual and procedural knowledge and is developed through the formation of connections between this knowledge. Finally, the section presents the role representations play in the process of developing understanding.

Mathematical Understanding

A common goal of curriculum documentation is to develop students' mathematical understanding (ACARA, 2017b; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). This goal is illustrated in the current Australian Curriculum: Mathematics (ACARA, 2017b), which aims to ensure all students develop an increasingly sophisticated understanding of mathematical concepts. A multitude of literature exists on mathematical understanding (Cai & Ding, 2017; Davis, 1992; Hiebert & Carpenter, 1992; Nickerson, 1985; Pirie & Kieren, 1994; Sierpinska, 1994; Skemp, 1976), yet definitions of and processes for obtaining understanding vary significantly, highlighting the complexity of the topic (Cai & Ding, 2017). Recognising *understanding* as both a noun and verb, research has been directed by two different perspectives: the process by which students develop understanding, and understanding as a product of student learning. These two perspectives are highlighted by Sierpinska (1994) who described understanding using the phrases *acts of understanding,* and *processes of understanding*. First, the act of understanding is described as the function of mentally relating the object of understanding with the basis for that understanding. To illustrate, Sierpinska (1994) writes: "…my object of understanding can be a mathematical word problem, and in the act of understanding I may recognize the problem as following a certain well-known pattern. This pattern would be the basis of my understanding" (p. 29). The process of understanding is forming connections between the various acts of understanding through the use of mathematical reasoning.

Pirie & Kieren (1994) followed the perspective of understanding as a process, stating that "understanding is characterised as occurring in action and not as a product resulting from such actions" (Pirie & Kieren, 1994, p. 127). Throughout their research, Pirie & Kieren (1994) identified eight levels of understanding. They represented these eight levels as nested circles, each level dependent on the preceding levels and influencing growth into proceeding levels. Students' movement through the layers is described as non-linear and multi-directional, acknowledging students will *invocatively* return to inner levels when faced with a problem that is not immediately solvable, so as to extend their current inadequate and incomplete understanding (Pirie & Martin, 2000). They term this back and forward movement 'folding back' (Pirie & Kieren, 1991):

> *A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific local or intuitive understandings. This returned to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing one's current understanding on an earlier understanding, and the idea that understanding is somehow 'thicker' when inner levels are revisited. This folding back allows for the reconstruction and*

elaboration of inner level understanding to support and lead to new outer level understanding (p. 172).

More research has focused on understanding as a product of learning, producing multiple definitions for the term understanding. A common thread running through definitions is the importance of one creating connections between new learning and existing knowledge (Hiebert & Carpenter, 1992). Skemp (1976) described understanding as “building up a conceptual structure” (p. 14). Similarly, Davis (1992) stated that students understand a concept when new knowledge is connected to existing knowledge: “one gets the feeling of understanding when a new idea can be fitted into a larger framework of previously assembled ideas” (p. 228). The notion of “previously assembled ideas” presented in this quote is significant. It is necessary that existing knowledge is already connected, that is, it is already understood. For new understandings to develop, new knowledge must be placed within a framework of knowledge already understood.

The depth of students’ mathematical understanding is proportional to the quantity, strength and organisation of the connections that are formed (Barmby et al., 2009; Hiebert & Carpenter, 1992; Nickerson, 1985). Hiebert & Carpenter (1992) assert that an idea, fact or procedure is thoroughly understood when it is linked to existing networks with numerous, strong links:

A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections (p. 67).

Accompanying their definition, Hiebert et al. (1992) group forms of connections into two categories. The first group is centred on similarities and difference. Through mathematical activity, students make connections as they observe similarity and difference between mathematical representations. Hiebert et al. (1992) use base-ten blocks to illustrate—as students observe relationships between the size and composition of the blocks, they recognise the value of each block and form connections to important concepts behind the structure of place value. They note that these connections are not instantaneous but rather are built across multiple experiences.

The second category identified was based on notions of inclusion. Such connections are constructed at a relatively high level of abstraction as students generalise across mathematical cases. Again, using the example from Hiebert et al. (1992), students' early solutions to additive problems are based on counting strategies directly linked to the context of the problem. Through multiple experiences, students build up an internal network of relationships between smaller parts that build to make the whole. As all addition and subtraction is derived from part-whole relationships, the internal structure formed by students becomes a generalised way of thinking about all additive situations.

The Role of Representations

Students' abilities to work with mathematical representations flexibly and their capacities to interpret and connect representations is key to the process of building mathematical understanding (Goldin & Shteingold, 2001; Gravemeijer, 1999; National Council of Teachers of Mathematics, 2000). The National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000) define representation in the following manner:

> *Representations are necessary to students' understanding of mathematical concepts and relationships. Representations allow students to communicate mathematical approaches, arguments, and understanding to themselves and to others. They allow students to recognize connections among related concepts and apply mathematics to realistic problems.*

From this definition, two important components of representations can be seen. First, representations are an essential means to make sense of, and build connections between, mathematical ideas (Fosnot & Dolk, 2001). Representations in mathematics can be seen as a “structure-preserving map” (Cuoco, 2001, p. x) of and between mathematical phenomena that “mathematicians use as they organise their activity, solve problems, or explore relationships” (Fosnot & Dolk, 2001, p. 73). Secondly, representations are a powerful tool that allows for an individual’s understanding of mathematical concepts to be communicated and discussed (Gravemeijer, 1999; Siemon et al., 2011; Van den Heuvel-Panhuizen, 2003).

The literature distinguishes between different forms of mathematical representations. Goldin et al. (2001) use the terms *internal representations* and *external representations*. External representations are physical tools used to embody mathematics that is to be taught. In this sense, an external representation in mathematics makes concrete that which is abstract, thus making more abstract concepts accessible to students (Gravemeijer, 2004). Formal, external representations allow for connections and relationships between mathematical concepts to be more easily observed. As connections and relationships are observed, external representations become tools that help develop generalisation and abstraction (Zazkis & Liljedahl, 2004). External representations are also recognised by Lamon (2001) in her study on fractions and rational numbers, where she refers to such representations as *presentational models.* These models are used for instructional purposes. Lamon (2001) acknowledges that although these models embody the mathematics which the teacher already knows and understands, it is not necessarily true that the same model will hold the same meaning for students. An external representation is useful, or limited in its usefulness, based on how individuals understand them (Goldin & Shteingold, 2001).

An individual’s interpretation of external representations is dependent on the internal representations of the individual (Goldin & Shteingold, 2001). Internal representations refer to the mental constructions of a learner. Lamon (2001) refers to these informal models as representational models and explains that they are produced by the student in learning and are therefore owned by the student and reflect their understanding of a concept. Lamon (2001) writes:

> *When a student truly understands something, in the sense of connecting or reconciling it with other information and experiences, the student may very well represent the material in some unique way that shows his or her comfort with the concepts and processes* (pp. 156-157).

This idea corresponds with Harries and Suggate (2006) who explain that "the way we represent mathematical concepts strongly affects the way in which we understand and develop such concepts and process numbers using our private mental methods" (p. 53).

An individual's internal representations do not exist in isolation. Rather, they are highly structured with complex relationships between the different forms of mental constructs (Goldin & Shteingold, 2001). The process of structuring and connecting internal representations is the process of building understanding (Hiebert & Carpenter, 1992). Limitations in students' mathematical ability can often be attributed to incomplete or misconstrued internal representations (Goldin & Shteingold, 2001). Sfard (1991) describes these informal representations as abstract, intangible, invisible constructs that are only visible in the learner's mind's eye:

> *Being capable of somehow "seeing" these invisible objects appears to be an essential component of mathematical ability; lack of this capacity may be one of the major reasons because of which mathematics appears practically impermeable to so many "well-formed minds"* (p. 3)

This conclusion correlates with the description of internal models given by Goldin and Kaput (1996). They explain that internal representations are configurations of an individual and, as such, are not observable; nor can they be fully understood from the outside. This notion of internal constructions and understandings is also reflected more broadly by Cobb et al. (1992) when considering general mathematical understanding constructed by individuals in a mathematics community. They assert that meaning and understanding for individual students will grow out of mathematical activity. As these meanings and understandings are subjective in nature, they are only truly knowable by the individual who constructed them. It is through further interaction and discourse that these meanings and understandings can be *taken-as-shared* by the mathematical community and collective understanding can be established.

Implications for this study

It is pertinent to consider how external and internal representations operate within the context of this study. Goldin et al. (2001) present a goal for school mathematics linked to representation as "the development of efficient (internal) systems of representations in students that corresponds coherently to, and interact well with, the (external) conventionally established systems of mathematics" (p. 3). One of the aims of this study is to explore students' internal representations and their connections to conventionally established external representations. It is the assertion of this study that through mathematical experiences, students will form their own mathematical realities. Representation bridges the gap between the informal understanding connected to the 'real' and imagined reality on the one side, and the understanding of formal systems on the other (Van den Heuvel-Panhuizen, 2003). Students need to be encouraged to externalise their internal representations through their own invented mathematical models as a way of communicating their understanding and connecting to more formalised mathematics. Treffers (1991) states:

> *It is especially by means of strong models that children are given the opportunity to bridge the gap between informal, context-bound work and the formal, standardized manner of operation through the constructive contribution of the children themselves* (p. 33).

Formal mathematics necessarily grows out of student experience. Students need to be involved in mathematical activity that allows them to form and structure their internal representations so they can organise their thinking and understanding. Gravemeijer (2003) explains that as students act with their own internal representations, they will "(re)invent the more formal mathematics that constitutes the goal of instruction" (p. 53). As students represent their thinking in experientially real mathematical problems, their representations will go beyond a tool to solve just one problem; they will become more generalised tools and strategies. This is supported by Fosnot et al. (2001) who discuss models as representations of mathematical relationships that have been reflected upon, creating generalised ideas, strategies and representations.

Therefore, the manner in which external representations are introduced to students is crucial. Stephan et al. (2001) stress that representations, such as models and tools, should be introduced to students as a pathway towards a solution to a problem. However, students are given the models too often before they even realise that there is a problem. As such, students struggle to make meaning of the representations and their creativity is stifled. Representations need to be sequenced as a part of the planned learning process to ensure they continually link to and build on students' internal representations and prior experiences. In studying partitioning strategies, Lamon (1996) showed that students' understanding was constructed from external information and the internal cognitive models the students had constructed from their past experience. Through this process the "children's cognitive models developed from primitive to more refined states" (Lamon, 1996, p. 188). Gravemeijer (2003) writes about this process as a model (or representation) of informal mathematical activity and generalising this informal model to a model for formal mathematical reasoning. Students need to be presented with problems that will allow them to generate informal models which reinvent the formal models. Through experience with similar problems, the mathematical community creates a more generalised, taken-as-shared model that can then be used by the teacher for instruction. Students are able to move flexibly between the informal and formal models as they move to generalise a concept.

Conclusion

This chapter has presented an overview of the literature concerning computational fluency and mathematical understanding. Computational fluency based on numbers sense has been shown to be a product of mathematical understanding as both depend on conceptual and procedural knowledge. Mathematical understanding is developed through connecting one's internal representations of a concept. Internal representations are formed and connected through student activity with external mathematical representations. The role of internal and external representations in the development of students understanding of multi-digit multiplication presents implications for this research project.

CHAPTER 3

LITERATURE REVIEW: MULTIPLICATION

Introduction

Through this chapter, the literature concerning different aspects of multiplication is reviewed. First, the literature regarding multiplicative thinking is presented, with a particular focus on how students transition from additive to multiplicative thinking. The categorisation of multiplicative problems is explored and linked to the array as a model of multiplication. The final section of the chapter discusses the key understandings in multiplication and how these understandings relate to students' invented strategies for multi-digit multiplication.

Multiplicative Thinking

Multiplicative thinking is critical to students' mathematical development (Barmby, Harries, Higgins, & Suggate, 2009; Clark & Kamii, 1996; Park & Nunes, 2001; Siemon, Breed, Dole, Izard, & Virgona, 2006; Siemon et al., 2011; Sowder et al., 1998; van Dooren, de Bock, & Verschaffel, 2010) as it underlies important mathematical concepts (Hurst & Hurrell, 2016). Multiplicative reasoning is the ability to *think* and *reason* using a deep conceptual understanding of the multiplicative structure. According to Siemon et al. (2006), it is characterised by three elements:

> *(i) a capacity to work flexibly and efficiently with an extended range of numbers…, (ii) an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion, and (iii) the means to communicate this effectively in a variety of ways* (p. 113).

The move from additive thinking to multiplicative reasoning is complex and is acquired over many years (Clark & Kamii, 1996; Siemon, Izard, Breed, & Virgona, 2006; Simon & Blume, 1994; Vergnaud, 1983). Initially, students rely on additive thinking strategies to solve multiplicative problems (Mulligan & Mitchelmore, 1996, 1997). Additive thinking is based on *part-whole structures* (Sophian, 2007; Young-Loveridge & Mills, 2009), that is, individual numbers are added together to give a whole. At this stage, a learner's understanding of multiplication does not always appreciate the need for repeated equal groupings of objects (Sophian, 2007; Steffe, 1994); the learner "[focuses] on the individual unit items of numerical composites—they focus on, for example, three ones rather than one three" (Steffe, 1994, p. 129).

As students build an appreciation of the replication of equal groupings and recognise skip counting as a form of repeated addition (Wright, Stanger, Stafford, & Martland, 2015), the cognitive conversion of *three ones* to *one three* presents itself as a more mathematically sound method of solving multiplication problems (Steffe, 1994). At this point, children's thinking is still quite *linear,* or additive (Young-Loveridge & Mills, 2009), but they have an appreciation that equal counts to solve multiplication is not only viable—it is more efficient. In their study looking at Year 2 and 3 students' intuitive models for multiplication and division, Mulligan and Mitchelmore (1997) classified skip counting and repeated addition as one strategy, and noted the hierarchical development of each approach. Steffe (1994) provides more distinction between these approaches. The move from that which Steffe (1994) refers to as *numerical composite units* to *abstract composite unit*s involves students reorganising their thinking from purely skip counting orally to the more abstracted appreciation of the iterable unit, or the ability to repeatedly add equal groups of objects together, whether all objects are perceived or not. The repeated count linked to composite units is not "primary, natural and basic" (Steffe, 1994, p. 135) for all students and therefore carefully choreographed experiences need to be provided. The danger of not separating the two strategies may result in the absence of these learning experiences.

A learner must be challenged to move beyond an understanding of multiplication as an additive process of skip counting or repeated addition (Siemon et al., 2006). To teach multiplication purely as repeated addition is to limit a child's thinking to a linear model. Teaching in this manner fails to recognise multiplicative situations such as one-to-many correspondence and equal shares (Siemon, 2013). Because of this, many researchers argue that a repeated addition model of multiplication is incomplete (Bell, Greer, Grimison, & Mangan, 1989; Clark & Kamii, 1996; Nesher, 1988; Park & Nunes, 2001). Jacobson (2009) states "defining multiplication as repeated addition disrupts the mathematical integrity of instruction and impedes the coherence of subsequent learning" (p. 68). Rather, deep understanding of multiplication comes from the ability of children to distinguish between two different relationships: *how many* (the number of groups) and *how much* (the number in each group) (Siemon, 2013). While students' multiplicative thinking skills grow from additive strategies, they require a higher level of abstraction, as the child is expected to work simultaneously with these different relationships (Piaget, 1983). This is supported by Steffe's (1994) investigation into the counting scheme for multiplication, which concluded that when thinking multiplicatively, children can think simultaneously about single units and multiple units.

It is not surprising that while students may be able to work procedurally with multiplication and division, the ability to think multiplicatively is often an area of weakness (Clark & Kamii, 1996; Siemon, Breed, et al., 2006). Based on a longitudinal study of middle years students, Siemon (2013) claims that students' ability to think multiplicatively, or lack thereof, is the single most significant factor responsible for the eight-year gap in middle years mathematics for Australian students. Multiplication lays the foundation of one's ability to reason multiplicatively and is therefore an important stepping stone to other mathematical skills, including concepts such as fractions, decimals, proportional reasoning and ratios (Siemon et al., 2011). Fluency with multiplication, based on deep understanding, opens the door to many other areas of mathematics.

Types of Multiplication Problems

There has been substantial work classifying types of addition and subtraction problems but limited work in the case of multiplication and division (Greer, 1992; Larsson, 2016; Vergnaud, 1983). An influential study conducted by Vergnaud (1983) introduced the developmental theory of a *multiplicative conceptual field*. The intent of the theory is to describe and analyse the progressive complexity of the multiplicative domain and the different concepts and situations contained by the field (Vergnaud, 2009). Vergnaud (1983) identified three multiplicative situations or problem types. The first, isomorphism of measures, comprises problems that involve proportion between two measures. Problems within this category include questions based on equal groups, partitive and quotative division, ratio, constant rate and direct proportion (Tillema, 2013). The second category, product of measures, encompasses Cartesian product situations between two independent measures into a third measure. The final category is multiple proportions, where one measure is proportional to two independent measures.

Four categories of multiplicative problems with whole numbers were proposed by Greer (1992): equal groups (including proportional sharing and rate), multiplicative comparison, rectangular array and area, and Cartesian product. Greer described a further six categories for work with rational numbers. In discussing Greer's four categories of problems with whole numbers, Larsson (2016) suggests merit in separating rectangular array and area, reasoning that the array is comprised of discreet objects whereas an area model is made up of continuous quantities.

Greer's (1992) categories of equal groups, multiplicative comparison and rectangular array and area align with Vergnaud's (1983) isomorphism of measures, and Cartesian product corresponds with Vergnaud's (1983) product of measures. This illustrates that classification is somewhat subjective as "categories can be extended, collapsed, or refined depending on the purpose of the investigation" (Mulligan & Mitchelmore, 1997, p. 310). Mulligan et al. (1997) state that what is of primary importance is the problem type determined by the research prior to its presentation.

Implications for this study

The New South Wales Syllabus (Board of Studies NSW, 2002), in use at the time of this study, focuses on concepts related to isomorphism of measure problems, specifically equal groups and rectangular arrays/area. It may be argued that this is a narrow view of multiplication, however exploring the specific content of the syllabus was not in the scope of this research. Based on the syllabus documentation, two categories of problems were used in my research: equal groups and rectangular arrays/area, with a particular focus on rectangular arrays and area. I chose not to categorise arrays and areas as distinct groups, as suggested by Larsson (2016). Rather I elected to make the focus of study the progression from the array model to the area model. The proceeding section of this chapter discusses the literature on the array and area as a model for multiplication.

Arrays and Area as Models for Multiplication

The array is a powerful model to illustrate multiplication (Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Davis, 2008; Fosnot & Dolk, 2001) that allows access to the important theoretical constructs of multiplication (Barmby et al., 2009). The array has been described as "the most flexible and robust interpretation of multiplication" (Davis, 2008, p. 8). This two-dimensional multiplicative representation highlights equal groupings and how the composite units build on each other to produce a whole.

Three levels in the development of understanding the spatial structure of the array were identified by Battista et al. (1998). Students first structure the array as one-dimensional paths, not recognising equal groupings or coordinated columns. Battista et al. (1998) use the analogy of traveling along a road or in a tunnel: "they follow these paths as if they are traveling along a road and have no awareness of their surroundings, as if in a tunnel" (p. 528). At the second level, students are able to recognise the rows and columns, moving to a more two-dimensional perspective. It is not until students reach the third level of understanding that they fully appreciate the coordination of rows and columns and can then interact with the rows and columns simultaneously.

The model of the array is used in two distinct ways within the literature. First, it is used as a tool to develop an understanding of the multiplication operation through the process of manipulating the array to carry out calculations (Barmby et al., 2009; Izsak, 2004; Young-Loveridge & Mills, 2009). Secondly, it is used as a tool for mathematical thinking and reasoning. In this context, models are not given to students, but are constructed by them as they “generalise ideas, strategies and representations across contexts” (Fosnot & Dolk, 2001, p. 73). As such, models become a lens by which to view mathematical structure and are mental maps of students’ reasoning (Fosnot & Dolk, 2001). Using the array in this manner is supported by Gravemeijer (1999) who reasons that the role of a model is not as an alternate form of an algorithm, but rather as a tool for thinking and reasoning in mathematics.

The construct of the array is a recommended model for multiplication (ACARA, 2017b; Board of Studies NSW, 2002; NSW Education Standards Authority, 2012) but has been the focus of limited research with multi-digit multiplication (Barmby et al., 2009; Izsak, 2004; Young-Loveridge & Mills, 2009). Young-Loveridge & Mills (2009) used what they termed a *dotty array*—an array made up of small dots pre-partitioned into smaller 10 x 10 arrays. It was designed this way to build students’ appreciation for the differences in the size of partial products, the impact of place value and awareness for the distributive property of multiplication. The study showed that “arrays can be useful for enhancing students’ understanding of multi-digit multiplication, providing there is a good match with their learning needs” (p. 7). Those who were successful in solving the problems demonstrated strong multiplicative thinking skills, progressing to the more abstract grid method (Fuson, 2003) for multiplication, suggesting that their learning needs were beyond that of the presented dotty array.

Similar results were obtained in a study conducted by Izsak (2004) involving Year 4 students. This research focused on three constructs of the array. First, students were presented with a grid array with all items perceived, then moved to a more abstracted form of the array with parts partitioned into hundreds, tens and ones. Finally, they were presented with an open array, or area model, where students chose how to partition numbers. Strategies used by students demonstrated an implicit understanding of the distributive property as they were able to represent factors and products geometrically. Izsak (2004) did note that some students connected the dimensions of the array to numerical labels more readily than others.

Barmby et al. (2009) adopted the use of the array in a study with Year 4 and Year 6 students. Again, the array model initially presented to the students was with all units perceived and pre-partitioned, this time into groups of 5 x 5. The study revealed that the array representation did indeed support calculations strategies and was perceived positively by the students. However, Barmby et al. (2009) did identify two concerns about the array. First, it was observed that students who did not think multiplicatively about multiplication demonstrated a lack of understanding of the array. The researchers also found that the model used with all parts of the array shown resulted in students relying on over-use of laborious counting strategies.

Overuse of counting strategies highlights the need to move to a more abstract model of the array (Sullivan, Clarke, Cheeseman, & Mulligan, 2001). In their study on the spatial structuring of two-dimensional arrays, Battista et al. (1998) demonstrated that abstraction of the array, with limited perceivable items, enables students to be more multiplicative in their thinking and encourages more sophisticated strategies beyond counting to solve calculations. It is at this point that students "see the whole plan of the scheme from the start and to utilize it in the absence of the original perceived material" (p. 529), making the cognitive transition that multiplication is more than just repeated groups. A connection to the area model (Jacobson, 2009) or open array (Fosnot & Dolk, 2001) is established. Jacobson (2009) stated: "much will be gained if students associate multiplication more closely with the area model… a powerful and suggestive conceptual model" (p. 71). The use of the area model most clearly illustrates the distributive property of multiplication and makes sense of multi-digit calculations (Jacobson, 2009).

Implications for this study

The literature provides clear insight into students' understanding of the array, particularly in single-digit multiplication where all parts of the array are visible. There is less research available for how the array supports multi-digit multiplication. The studies presented by Barmby et al. (2009) and Young-Loveridge & Mills (2009) examined how students interacted with a specific form of the array and how the array developed their understanding of multiplication. The study presented by Izsak (2004) used multiple arrays that focused students' attentions on the distributive property of multiplication, based on the place value properties of the factors being multiplied. There is scope to build on this research, particularly examining students' interactions with less-structured forms of the array.

Students' interactions with models develop and change through the course of an instructional sequence. In discussing the role of emergent models, Gravemeijer (1999) explains this process of change. Initially, students model their informal mathematical activity set within the context of a problem. At this stage, the model is situation specific; it is a *model of* a contextualised situation. As their understanding of mathematical concepts grow, students' use of models becomes more abstract. The model moves to become a *model for* more generalised mathematical reasoning. This flexibility in a model to move from a *model of* to a *model for* also works in reverse; students are able to revert to a lower level of the model to support the learning process (Van den Heuvel-Panhuizen, 2003).

It is important to recognise that models have different manifestations (Van den Heuvel-Panhuizen, 2003) and that different forms of a model serve different functions (Saxe, 2002, 2004; Teppo & van den Heuvel-Panhuizen, 2014) For example, Teppo et al. (2014) reported on the functions that various forms of the number line served in the teaching and learning of mathematics. In this present study, students used multiple forms of the array. Students' use of and movement between arrays with all parts perceivable through to an open array, or area model, was part of the transition from a *model of* to a *model for* and formed an important question for investigation:

How does the array develop from a model of a contextual situation to a model for more generalised mathematical reasoning in multi-digit multiplication?

Key Developmental Understandings in Multiplication

The construct of a Key Developmental Understanding (KDU) was introduced by Simon (2006). He explains that a KDU is a conceptual advance for students that develops across multiple experiences rather than through direct instruction. In multiplication, the commutative, distributive and associative properties have been identified as important understandings that need to be developed (Ambrose et al., 2003; Larsson, 2016). It is therefore important to develop an understanding of their relevance for the purpose of this study.

Commutative Property

The commutative property of multiplication (i.e. *ab* = *ba*) lays the foundation for algebraic thinking (Bruner, 1962; Warren, 2002). According to Bruner (1962), the power of the commutative property lies “in the power of the idea to create a way of thinking about number that is lithe and beautiful and immensely generative” (p. 121). However, commutativity in multiplication is not necessarily intuitive for students (Vergnaud, 1983) and, as such, can prove to be a difficult concept for students (Squire, Davies, & Bryant, 2004). Many students do not readily accept the commutative property (Park & Nunes, 2001) and without multiple experiences and careful instruction, multiplicative commutativity is not easily understood (Petitto & Ginsburg, 1982). In a study on students’ invented strategies for multiplication and division, Ambrose et al. (2003) showed that students did not tend to make use of the commutative property despite making the calculation easier. They reasoned that students’ strategies were closely aligned with the context of the problem and, in most instances, remaining connected with the context did not support the use of the commutative property.

Distributive Property

The distributive property forms a critical understanding required to work with larger numbers in multiplication (Anghileri, 2008), as more complex multiplication problems can be solved by adding together simpler, known products (Squire et al., 2004). The distributive property allows the *chunking* of numbers into groups. For example, the equation 26 x 13 may be solved as (20 x 10) + (20 x 3) + (6 x 10) + (6 x 3). It may also be solved as 26 x 13 = (25 x 10) + (1 x 10) + (3 x 25) + (1 x 3). The distributive property is not only an important arithmetic concept, it is foundational to later learning in algebra (Ding & Li, 2010).

The findings of research concerning students' understanding and use of the distributive property are varied. Based on a study conducted with children aged nine to eleven, MacCuish (1986) found evidence that the distributivity property was a difficult concept for students to grasp. When posed with numerical multiplication problems based on distributivity, students were only able to correctly answer about 20% of questions. In similar findings, Squire et al. (2004) observed students had a poor understanding of distributivity. They noted that students employed additive strategies in the multiplicative context and suggested that instruction around the distributive property should be focused on equal groups rather than area contexts.

Conversely, other studies have shown the distributive property is an intuitive concept for students (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004), with Benson, Wall and Malm (2013) stating "…the distributive property is naturally logical to most students if first we allow them to think through problems…from multiple perspectives—numerically and geometrically" (p. 505). Ambrose et al. (2003) found students in Years 3 to 5 made implicit use of the distributive property in their created strategies to solve multiplication and division problems. This was reinforced by Izsak (2004), who noted students' implicit use of the distributive property, particularly in geometric contexts. In their early use, students may not be able to name the distributive property, but they do recognise the viability of grouping and partitioning numbers to solve complex multiplication problems (Barmby et al., 2009; Schifter, Monk, Russel, & Bastable, 2008). Barmby et al. (2009)

concludes that focused class discussion around students' strategies and distributivity is important in making the connections explicit.

Associative Property

Although the associative property is identified in documentation as a fundamental understanding required by students (Board of Studies NSW, 2002; NSW Education Standards Authority, 2012), there is very limited research based on this property (Ding, Li, & Capraro, 2013). One reason for this may be that primary school students rarely encounter more than one factor in multiplication problems (Schifter et al., 2008). The associative property means that the result of a multiplication calculation does not depend on the way in which the objects are grouped i.e. $(ab)c = a(bc)$. In one notable study, Ding et al. (2013) evaluated pre-service teachers' understanding of the associative property of multiplication. Their findings showed that the teacher education students possessed general misunderstandings around the associative property, primarily confusing it with the commutative property. They concluded that teachers' poor understanding of the property would in turn limit students' conceptual understanding, and thus opportunities to explore the associative property in the classroom would be overlooked. Due to teachers' poor understanding about the associative property, the researchers stated "teachers may at most teach procedures for computation but lack the ability to recognise and discuss the underlying principles and help students develop mathematical reasoning abilities" (Ding et al., 2013, p. 49) and that as such "[pre-service teachers] did not possess a clear understanding of the basic meaning of multiplication itself" (Ding et al., 2013, p. 49).

Fosnot et al. (2001) demonstrated the power of a conceptual understanding of the associative property of multiplication. They showed students' use of the associative property through halving and doubling strategies. Similarly, Schifter et al. (2008) found that students were able to recognise and justify the associative property for multiplication, explaining that the factor of one number would also be factors of its multiples. This understanding allowed the students to use factors to employ strategies such as halving and doubling.

Students' Strategies for Multi-Digit Multiplication

A number of studies have examined students' invented strategies for multi-digit multiplication (Ambrose et al., 2003; Baek, 2005; Barmby et al., 2009; Harries & Suggate, 2006; Larsson, 2016; Young-Loveridge & Mills, 2008). The section following presents the strategies observed in these studies and classifies the strategies into two groups: additive strategies and multiplicative strategies. It draws connections back to the KDUs already identified in this chapter.

Additive Strategies

Students' difficulty with multiplicative reasoning means that many rely on additive strategies in multiplicative contexts. These strategies included repeated addition, repeated doubling and forms of re-grouping (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004; Young-Loveridge & Mills, 2009). The most elementary additive strategy observed was counting, which was noted in two studies. The first was conducted by Ambrose et al. (2003). They noted the inefficiency of different forms of counting strategies used by students. Students were not provided with a representation and many students chose to model the problem using concrete materials then proceeded to use simple counting strategies to find the total. The second study was conducted by Barmby et al. (2009) with Year 4 and Year 6 students. The students were presented with an array representation which had been pre-partitioned into 5 x 5 groups. Skip counting in groups of five was a predominant strategy for many of the Year 4 students, yet not with the Year 6 students. In both studies, counting was enabled through the use of concrete materials or a visual representation.

Overreliance on Additive Strategies

While additive strategies may result in correct answers, there are also dangers associated with their use. The use of skip counting is inefficient with multi-digit numbers and there is a likelihood that students will lose track of their count (Ambrose et al., 2003; Barmby et al., 2009). Additionally, an overgeneralisation of additive strategies can impede students' understanding of the multiplicative structure (Larsson, 2016; Simon & Blume, 1994). While students accept that factors can be split into smaller parts, they do not immediately recognise all the partial products formed (Young-Loveridge & Mills, 2009). For example, when calculating 24 x 32, the multiplier and multiplicand can both be split into tens and ones to form four partial products that are then added together to form the total, as shown by 24 x 32 = (20 x 30) + (20 x 2) + (4 x 30) + (4 x 2). However, a common error is for students to partition each number into tens and ones, multiply the tens together, and then multiply the ones: 24 x 32 = (20 x 30) + (4 x 2) (Murray, Olivier, & Human, 1994; Young-Loveridge & Mills, 2009). While this holds true for addition, it does not appreciate the binary nature of multiplication (Barmby et al., 2009; Larsson, 2016).

A further influence of additive thinking can be seen in students' use of regrouping and rearranging strategies. Barmby et al. (2009) observed students rearranging arrays to aid computation using an array pre-partitioned into smaller 5 x 5 arrays. They noted that most often these rearranging strategies were used to created full groups of 25 that could easily be skip counted. However, in so doing, students lost the rectangular structure of the array which is inherent to multiplication (for example, see Figure 3.1). A very similar error was noted by Young-Loveridge & Mills (2008), with students completing groups of 100 (see Figure 3.2).

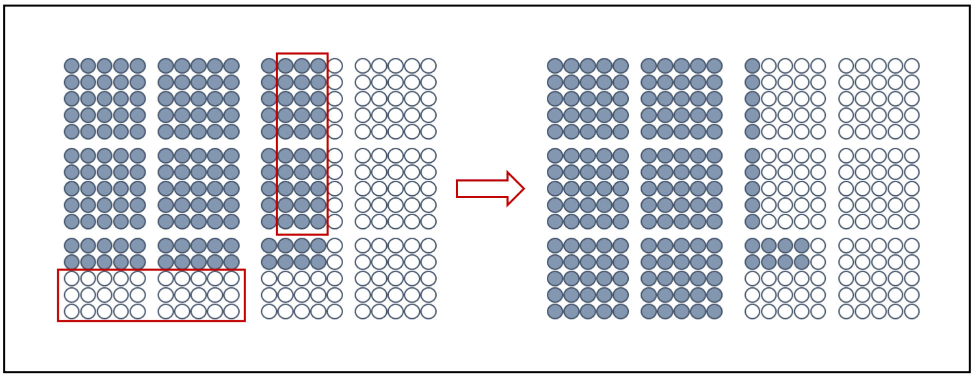


Figure 3.1 —Strategy used by students in Barmby et al. (2009)

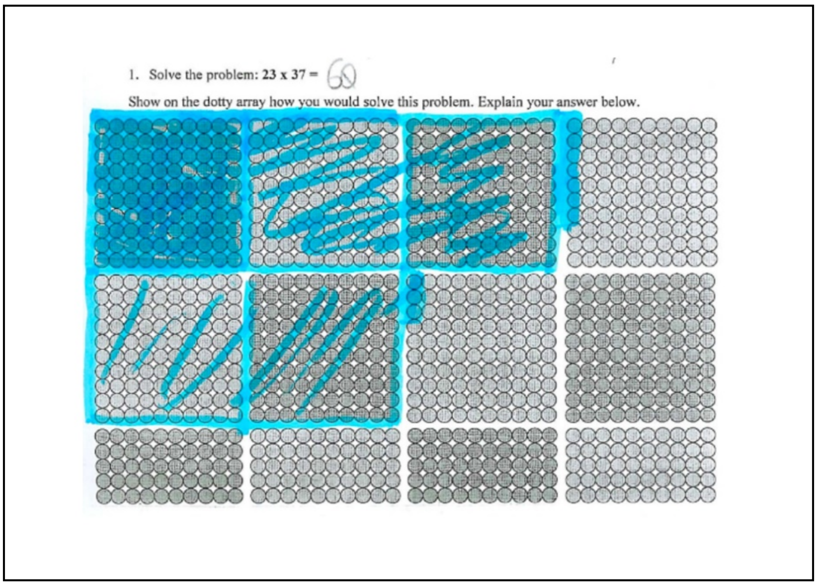


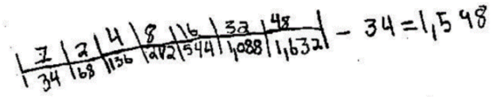
Figure 3.2—Figure from Young-Loveridge & Mills (2009)

Linking Additive and Multiplicative Strategies

Despite the inefficiencies of additive strategies and the misconceptions that can be developed, there is scope to extend these strategies to more efficient ways of computing using multiplicative reasoning (Ambrose et al., 2003). Treffers & Buys (2008) note that students' strategies develop in sophistication across three levels: counting, structuring and flexible/formal. How these three levels reveal a shift from additive to multiplicative approaches to computation will be explored later.

Repeated doubling was used across a number of studies to solve multi-digit multiplication problems (Ambrose et al., 2003; Baek, 2005; Murray et al., 1994). In their study, Ambrose et al.

(2003) noted that doubling is intuitive for students and formed a natural step in grouping to make calculation easier. Through doubling “the child creates a new iterable unit by clumping two of the multiplicands together” (p. 55) and this paves the way for more complex strategies based on the multiplicative properties of distributivity and associativity. For example, the multiplier may be repeatedly doubled, by first being partitioned into tens and ones, thus drawing on the distributive property. Similarly, in an example given by Ambrose et al. (2003), 47 x 34 can be solved by partitioning the multiplier into groups of 32, 16 and 1 as illustrated in Figure 3.3. This strategy also employs the associative property (Ambrose et al., 2003), illustrated in Figure 3.3. Discussion is important in highlighting mathematical structure and making students aware of the multiplicative properties that they are employing (Barmby et al., 2009). Ambrose et al. (2003) concluded that students’ “creation of new units from additive processes may provide a transition to the multiplicative reasoning that underlies proportional reasoning among other things” (p. 59).



$$\begin{aligned}47 \times 34 &= (32 + 16 - 1) \times 34 \\ &= [(2 \times 2 \times 2 \times 2 \times 2) + (2 \times 2 \times 2 \times 2) - 1] \times 34 \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 34 + 2 \times 2 \times 2 \times 2 \times 34 - 1 \times 34 \\ &= 2(2(2(2(2 \times 34) + 2(2(2(2 \times 34) - 34\end{aligned}$$

Figure 3.3—Figure of a student's strategy from Ambrose et al. (2003)

Multiplicative Strategies

Numerous studies observed students making implicit use of the distributive property in their invented strategies for multi-digit multiplication problems (Ambrose et al., 2003; Baek, 2005; Barmby et al., 2009; Murray et al., 1994; Young-Loveridge & Mills, 2009). Across these studies, students’ strategies varied from partitioning only one number to partitioning both numbers. In their study, Murray et al. (1994) noted that some students would partition numbers at different points during the calculation. In their evaluation of strategies used, Murray et al. (1994) concluded that "the most useful theorem in action seems to be the distributive property, which pervades all strategies" (p. 402).

In the research conducted by Ambrose et al. (2003), students were presented with word problems. It became evident that students' choice of strategies was closely linked to the context of the word problems. As such, the students rarely split the multiplicand as this did not make sense within the problem. For example, when students were asked to calculate the total number of Thanksgiving Day cards if there were 42 boxes with 24 cards in each box, they tended to decompose the 42 into smaller parts. One student created four groups of 10 boxes, recognising that there would be 240 cards in each of these collections, and then proceeded to add the two additional boxes. Splitting the number of boxes was typical of other students. Only one student, Sean, used the distributive property to decompose both numbers and the researchers reasoned that "when Sean decomposed both numbers, it seems that he lifted the quantities out of the particular context of the problem and worked with them independently" (Ambrose et al., 2003, p. 58).

Representation can also influence students' choice of strategy. As previously mentioned, Barmby et al. (2009) presented students with a large array pre-partitioned into smaller 5 x 5 arrays. Students' distributive strategies evolved from the pre-partitioned structure allowing them to quickly detect groups of 25. Similarly, Izsak (2004) and Young-Loveridge & Mills (2008) used array structures based on hundreds, tens and ones which were then utilised by students to calculate using place value.

The associative property was only mentioned by Ambrose et al. (2003) and Baek (2005), showing that students repeated doubling strategies drew on this multiplicative property. Murray et al. (1994) also observed students using repeated doubling but did not link this strategy back to employing the associative property of multiplication.

Implications for research

Students' strategies for multi-digit multiplication link to the underlying properties. The research demonstrates that, while students may not be able to name them, they often possess an intuitive understanding of multiplicative properties (Ambrose et al., 2003; Barmby et al., 2009; Benson et al., 2013; Izsak, 2004; Schifter et al., 2008). Researchers have made these claims based on the strategies that students chose to use to solve problems. This highlights the importance of connecting students' strategies with the multiplicative properties from which they are derived to gain insight into students' understandings of multiplication.

There is an important emphasis in the current literature on the distributive property of multiplication, particularly based on place value (Barmby et al., 2009; Izsak, 2004; Young-Loveridge & Mills, 2008, 2009). This is an efficient strategy for multi-digit problems that lays the foundation for the formal multiplication algorithm. Opportunity exists to explore the associative property and related strategies. It is interesting to note that the errors and misconceptions reported in the literature (and discussed earlier in this chapter) all relate to students use of additive thinking, indicating these students did not appreciate the structure of multiplication. Through this research, it was considered of primary importance to focus on the multiplicative structure through the properties of commutativity, associativity and distributivity. This forms two important question for analysis:

> *What key developmental understandings (KDUs) in multi-digit multiplication are reflected in a hypothesised learning trajectory?*
>
> *What strategies do students reinvent through the implementation of the learning trajectory?*

Conclusion

This chapter has reviewed relevant literature concerning key aspects of multiplication including: multiplicative thinking; categorisation of multiplicative problems; the array as a model of multiplication; key developmental understandings; and students' invented strategies for multi-digit multiplication. Through the chapter, the implications for this research project have been highlighted. These implications were then used to design the research questions of the study which are explored in the Methodology & Methods chapter which follows.

CHAPTER 4

METHODOLOGY AND METHODS

Introduction

This chapter describes the research strategy used to answer the research questions on how students develop understanding in multi-digit multiplication. The chapter is divided into two main sections: methodology and methods. The methodology section presents the research questions, the theoretical framework and specific methodology, Design Research, used to guide the research. The second section entitled Methods, details the specific methodological practices I used, including the participants involved in the study, the data gathered and the process of data analysis.

Research Questions

This study explored how a proficient understanding of multi-digit multiplication is developed from the perspective of the learner. In particular, it examined how learners move from current levels of reasoning to more sophisticated, complex mathematical thinking and reasoning.

The aim of this study was to explore how students construct understanding in multi-digit multiplication and how this process of understanding could be reflected through a learning trajectory. This aim was guided by the following specific questions:

1. How does the array develop from a model of a contextual situation to a model for more generalised mathematical reasoning in multi-digit multiplication?
2. What key developmental understandings (KDUs) in multi-digit multiplication are reflected in a hypothesised learning trajectory?
3. What strategies do students reinvent through the implementation of the learning trajectory?

4. What social mathematical practices emerge in the classroom through the implementation of the hypothesised learning trajectory?
5. How can students' learning be supported through the implementation of such a learning trajectory?

In summary, the over-arching aim explores the construction of understanding in multi-digit multiplication. The focus on construction of understanding is supported by questions one, two, three and four; namely, the *models* students use to organise their thinking, the *strategies* and *key developmental understandings* (Simon, 2006) used to solve problems and the emergence of *mathematical practices* (Cobb & Yackel, 1995) in the classroom. Question five is concerned with *instruction* for understanding. It is focused on how students learning can be actively supported by the teacher through the implementation of the learning trajectory. The words italicised indicate commonly used terms in this thesis.

Theoretical Framework

The theoretical framework provides a "conceptual framework explains the path of a research and grounds it firmly in theoretical constructs" (Adom, Hussein, & Agyem, 2018, p. 438). The theoretical framework of this research is constructed on two levels, the grand theoretical frame and the intermediate frame as presented by Kieran, Doorman, & Ohtani, (2015). The different levels represent a different purpose and influence the theory had on the design and development of the research. In this project, the grand theoretical frame of constructivism provided an overarching theory to understand the learning of students, which in turn guided the analysis of data. The intermediate theoretical frame, Realistic Mathematics Education (RME) provided a theory of design to focus the construction of the learning trajectory, guided by the three design heuristics of guided reinvention, experientially real contexts for learning, and emergent modelling. This theoretical framework is represented in Figure 4.1. Constructivism and RME, and the way that they influenced the design and development of this research, are each described in the proceeding section.

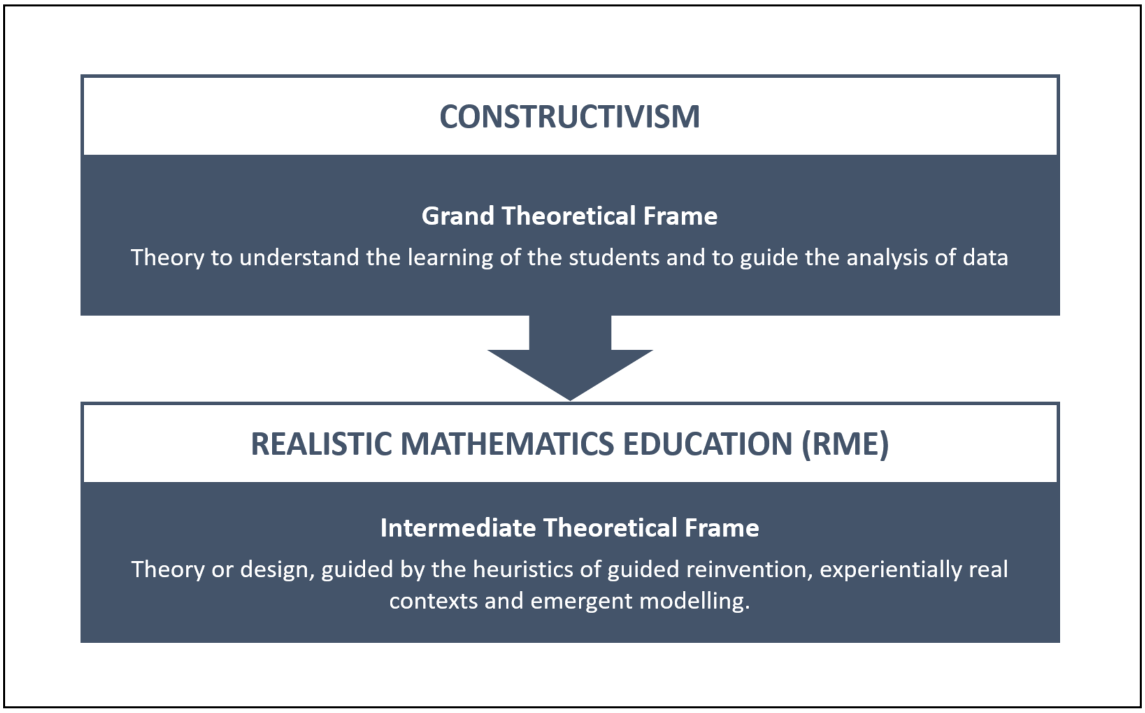


Figure 4.1—Theoretical framework

Constructivism

Constructivist theory describes the reality of knowledge and how it is acquired. It states that knowledge is not something to be transferred, but rather knowledge is constructed by people engaged in meaning-making (Fosnot & Perry, 2005). Knowledge is constructed as people act on their world, interpreting, organising and inferring new information which is mediated through prior experience and understandings (Fosnot & Perry, 2005; Simon, 1995). Central to this view of learning and knowledge is the individual. As such, a constructivist perspective holds the belief that it is not possible to obtain objective reality and knowledge (Simon, 1995). Our understanding of reality and our knowledge will always be influenced by our own experiences and perceptions. In the classroom, this means students must be active participants and, through activity, build their own mathematical knowledge. The role of the teacher is not to transmit knowledge, but support students in the process of learning.

There are varied interpretations of constructivism, with the major difference seen in the priority given to either the social aspect of learning or the cognitive constructions of the individual (Simon, 1995). The early rise of constructivist views on learning adopted a cognitive-psychological perspective, particularly through the work of Piaget, and have evolved into such theories as Radical Constructivism. The work of Vygotsky, some decades later than Piaget, introduced the importance of the social aspects of learning (Kieran et al., 2015) giving rise to such socio-cultural theories such as socio-constructivism and symbolic interactionism (Blumer, 1969).

Both social and cognitive aspects of learning are coordinated in this thesis (as in Cobb & Bowers, 1996; Cobb & Yackel, 1995; Gravemeijer, Bowers, & Stephan, 2003). Rather than seeing one having dominance over the other, this research realises a cooperative relationship between the constructions of the individual and the social culture of learning in the classroom. Cobb et al. (1995) refer to this relationship as the emergent perspective. They argue that individuals construct new knowledge and understandings through mathematical activity while participating in the social role of learning in the classroom. In turn, students' interactions and contributions influence the evolving learning culture in the classroom. As such, the social and cognitive aspects of learning cannot exist in isolation. Concern is not about which perspective is more dominant, rather how the two aspects work together to support the development of students' mathematical learning.

Realistic Mathematics Education

The origin of RME goes back to the early 1970s from the work of Dutch mathematician, Hans Freudenthal. Most commonly used as a theory for instructional design, RME also provides a theory for how mathematics is learnt. Freudenthal did not see mathematics as a discipline of structures and procedures to be transmitted (Fosnot & Jacob, 2010), rather, he interpreted mathematics as a human activity (Freudenthal, 1971, 1973) which uses mathematical pattern and structure to organise the world (Gravemeijer, 1994). Freudenthal named this process of organisation *mathematising*. He believed that learning mathematics should be through the experience of mathematising one's reality (Gravemeijer & Cobb, 2013).

The process of mathematisation is two-dimensional (Van den Heuvel-Panhuizen, 2003). Horizontal mathematising is the process of organising and structuring our world and problematic situations. Vertical mathematising, on the other hand, is the process of mathematising our mathematical activity. It is in this stage that we reflect on our mathematical activity and thinking and develop reasoning, solutions and strategies that are increasingly more sophisticated. As growth occurs on both the horizontal and vertical axis, deeper understanding and greater mathematical proficiency are developed (Fosnot & Jacob, 2010).

Three heuristics guide the implementation of RME as a theory for domain-specific instructional design: guided reinvention, experientially real contexts for learning and emergent modelling. These heuristics work in unison to develop students' mathematical understanding and practices and formed the theory for design in this research.

Guided reinvention

Guided reinvention is the backbone of RME. It is built on the view that 'mathematics can and should be learned on one's own authority, through one's own mental activities' (Freudenthal, 1973 in Gravemeijer 1994, p.172). Freudenthal believed that this process was not only possible but necessary for the development of understanding. Students should not be passive recipients in learning, rather they should be afforded the opportunity to reinvent mathematical methods and strategies for themselves. This involves students in the process of mathematising problem situations as well as their own mathematical activity (Gravemeijer, 2004).

For the teacher, learning must be planned so that the inventions of the students are harnessed. The designer needs to make hypotheses about anticipated learning routes of students, so they can then plan activities and tools to allow students every opportunity to invent the intended mathematics for themselves. As students engage in activities that cause reflection on their informal understandings, they work to develop more formal and mathematically sophisticated understanding and reasoning.

Experiential real contexts for learning

The second design heuristic of RME is that of experientially real contexts for learning. What students know from the world can, and should, be used as a context in which to situate the teaching of mathematical concepts and ideas:

> *...rather than looking around for material to concretise a given concept, the didactical phenomenology suggests looking for phenomena that might create opportunities for the learner to constitute the mental object that is being mathematised by that very concept.* (Gravemeijer, 2004, p. 116)

Real situations are not limited to everyday contexts but draw on broader situations such as fantasy worlds and within mathematics itself (Van den Heuvel-Panhuizen, 2003). Context gives meaning and as students move from the informal context to the more formalised mathematics, they develop understanding and form generalisations. To find relevant contexts, the designer must conduct a didactical phenomenological analysis. This requires the designer to identify the mathematical concept to be explored and place the concept in the context of an applied problem, or phenomena (Gravemeijer, 2004). The phenomena must allow for mathematising, that is, the construction of understandings and strategies that are being organised through the situation itself.

Emergent modelling

The third and final design heuristic of RME is emergent modelling. Rather than attempting to concretise abstract mathematical concepts, emergent modelling encourages learners to generate their own informal models through the reinvention process (Gravemeijer, 2004). The model emerges through students' informal mathematical activity within a contextual situation and helps to organise students' mathematical thinking and highlight mathematical relationships that are being constructed through the problem-solving process. As students interact with this informal model, it is progressively removed from the context and the model emerges as a tool for more formalised mathematical thinking. This transition is described by Gravemeijer (2004) as a *model of* situated activity to a more abstracted *model for* mathematical reasoning.

Instructional sequences need to provide students with the opportunity to model their informal mathematical thinking, requiring the designer to anticipate the sort of mathematical activity and practices that will emerge through tasks (Gravemeijer et al., 2003a). The designer needs to anticipate the ways in which students might represent their thinking and reasoning, so that the model can be developed as a meaningful representation for the class community. From here, tasks and mathematical activity can be planned with the model as a powerful tool for reasoning that sits across the whole instructional sequence, and that can also be transferred to similar mathematical situations in the future. Designing learning in this manner allows the evolutionary process of students' *model of* informal mathematical activity to a *model for* increasingly sophisticated mathematical thinking.

These three heuristics do not stand in isolation from each other, rather they work in unison to support students' development of increasingly sophisticated mathematical understanding and practices (Gravemeijer et al., 2003a).

Methodological Approach

In considering the methodological design for the project, the research questions were more closely reflected upon. First, as the research questions focused on construction of, and instruction for, understanding, the research plan needed to enable the parallel inquiry into the process of learning and then the way in which this learning could be supported and organised. Secondly, the opportunity to actively observe, explore and develop students' mathematical modelling, reasoning and use of strategies in the specific domain of multi-digit multiplication was also paramount to successfully answer the research questions. With attention given to these factors, it was decided that the research was best positioned in a classroom setting through qualitative methods as qualitative methods fuse theory and practice (Neuman, 2006). In this research, existing theory was used to inform the practice adopted in the classroom experiment and then this worked to produce new theory to inform future practice. As the researcher, I took an active role in the methodological approach, qualitative methods were deemed appropriate (Neuman, 2006).

The specific methodological approach of Design Research was employed, as described by Cobb et al. (2008; 2013). Design research is exploratory in nature and “aims at creating innovative learning ecologies in order to develop local instruction theories on the one hand, and to study the forms of learning that those learning ecologies are intended to support on the other hand” (Gravemeijer & Cobb, 2013, p. 73). This form of Design Research is seen as a validation study; the purpose is not to validate an existing theory or process, but rather “to develop or validate theories about such processes and how these can be designed” (Plomp, 2013, p. 16). The focus of this study was to explain how students construct understanding in multi-digit multiplication and how the pedagogical practices employed supported such development.

Design Research

This section explores the tenets of Design Research in closer detail. Design Research employs three research phases: the preparatory thought experiments, the teaching experiment and a retrospective analysis. Figure 4.2 represents the structure of the project.

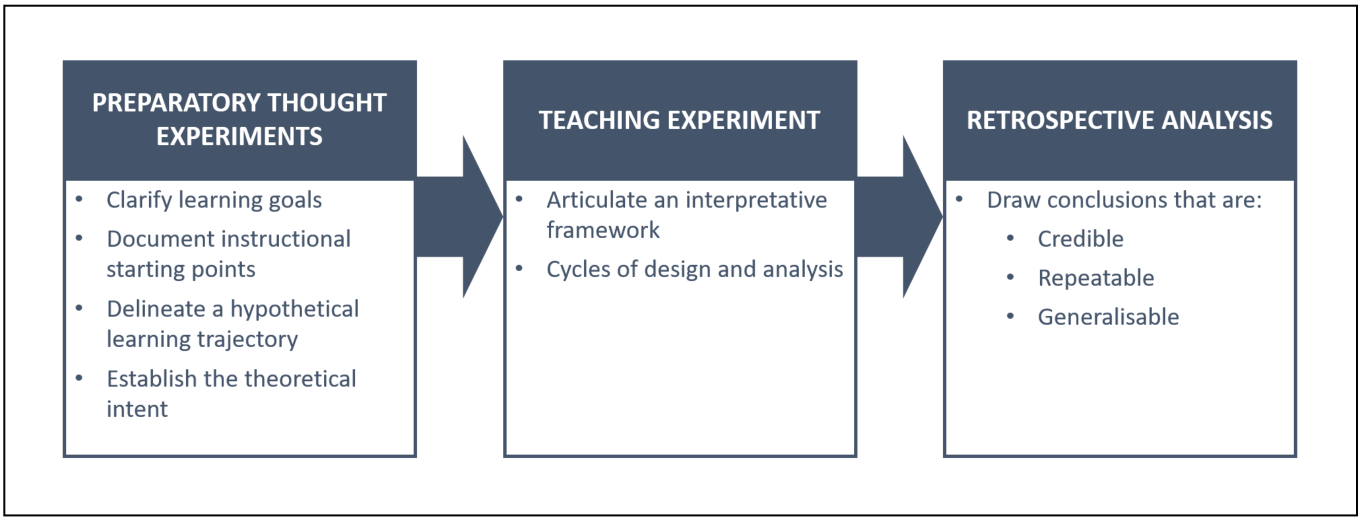


Figure 4.2—The process of Design Research

Preparatory Thought Experiments

The preparatory phase formed the foundation of the project. A detailed analysis of relevant literature was the basis for anticipatory thought experiments (Gravemeijer, 2004). The outcome of these thought experiments assisted with a preliminary answer to the overarching aim on the construction of understanding; that is, looking at how the research in the domain of multi-digit multiplication helps delineate a learning trajectory. The thought experiments also gave insight into questions one, two, three and four on students' constructional understanding. This was achieved by clarifying the learning goals, documenting instructional starting points and then, from this, delineating a learning trajectory. This process was both influenced by and helped to define the theoretical intent of the research. The structure of the thought experiments is illustrated in Figure 4.3. The following section looks more closely at these specific elements.

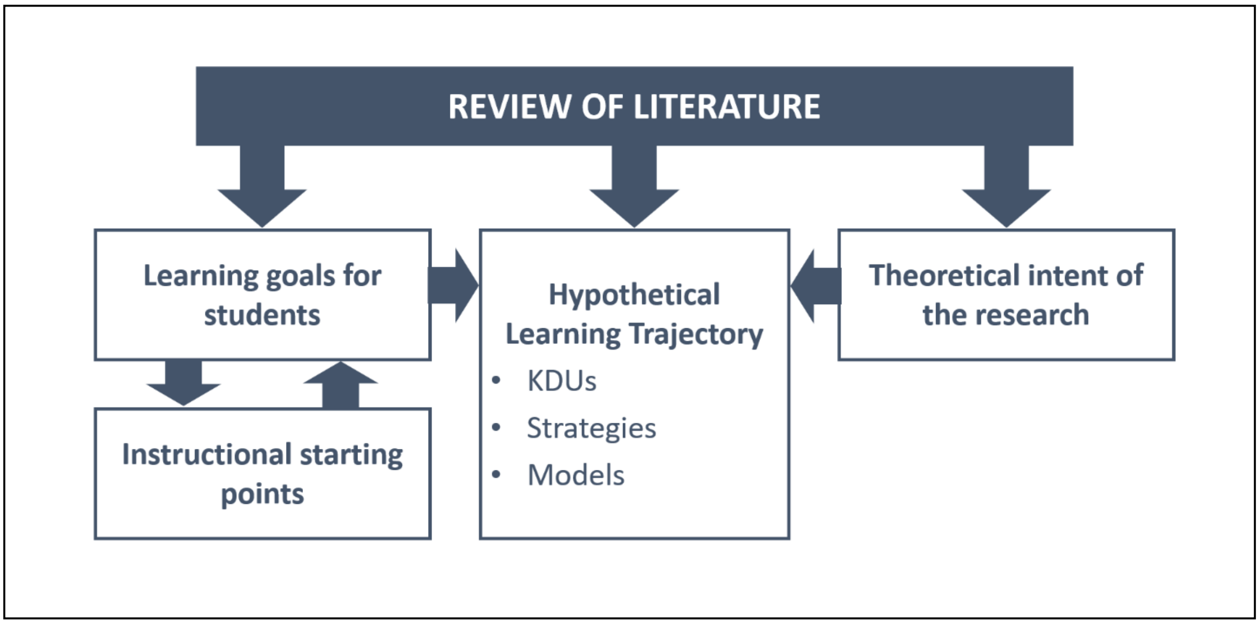


Figure 4.3—The process of the preparatory thought experiments

Clarify learning goals for students

This study aimed to influence the rethinking of achievable learning goals for students in multi-digit multiplication and how these goals can be developed by validating what is possible for students' mathematical learning (Cobb & Gravemeijer, 2008). As such, it was important that the project did not just accept the institutionalised goals for this domain as set out in different curriculum documentation. Rather, a synthesis of relevant literature, identifying Key Developmental Understandings (KDUs) (Simon, 2006) and documenting the development of students' knowledge and skills, was conducted. This synthesis enabled the establishment of learning goals that were used to guide the study. These learning goals are described in Chapter 5.

Document instructional starting points

The previously described review and synthesis of literature also allowed for the documentation of instructional starting points. The current levels of reasoning of students in this study were hypothesised, based on their instructional history according to curriculum requirements, and the findings in relevant literature. The documentation of these starting points was also supported by a one-to-one, pre-assessment interview that was delivered (see Appendix 1). The pre-assessment tool was established to understand and document the students' current levels of reasoning rather than accepting the expected proficiency of students based purely on curriculum documentation. An interview procedure was used in response to the limitations recognised in the traditional 'pen and paper' test to assess students' mathematical knowledge (Clements & Ellerton, 1995). Clements & Ellerton (1995) compared the data gained on students from a written test (comprising of multiple choice and short answer questions) to that of a one-to-one interview. They observed that, while students may have been unable to perform well in written papers, an interview revealed many of these same students had a strong conceptual knowledge.

Interviews are being used more widely in the mathematics classroom (Clarke et al., 2001; Clarke & Mitchell, 2005). Clarke et al. (2005) notes that "the late 1990s, in Australia and New Zealand, saw… the development and use of research-based one-to-one, task-based interviews on a large scale, as a professional tool for teachers of mathematics" (p. 2). Although time consuming, such task-based interviews have been shown to allow for true insight into each student's thinking, understandings and approaches to solving mathematical problems, rather than purely assessing the student's ability to recall taught procedures (Clarke et al., 2001).

An interview was specifically constructed for the purposes of this study based on the key issues raised in the literature reviewed. From here the initial learning challenges were designed. A shortened version of the interview was delivered to students at the end of the teaching episodes to establish whether the trajectory implemented had resulted in growth of students' understanding.

Delineate a learning trajectory

An essential element of the project's design was the delineation of a learning trajectory. A learning trajectory was used as the vehicle for planning learning in a mathematical domain. In concordance with the grand theoretical frame of constructivism explained earlier, the learning trajectory was concerned with articulating the cognitive-psychological aspect of learning and the classroom culture and practices to support that learning. In this first stage of the research, the preparatory thought experiments, the learning trajectory is only hypothetical. The teaching experiment in the second phase of the research design allows for its testing, modification and refinement before the final trajectory can be proposed.

Chapter 5 describes the process of design and makes explicit the conjectures that were made about how students' reasoning in multi-digit multiplication might develop. It also describes how the design heuristics of RME were used to inform the design and influence instructional tasks.

Theoretical intent

The primary aim of Design Research is to establish an empirically grounded theory that can guide teachers as they support students' learning in other settings (Cobb, Jackson, & Dunlap, 2016). Additionally, the theoretical context of the experiment is extendable to other areas of mathematical learning in the classroom (Gravemeijer & Cobb, 2013). As such, the theoretical intent of this experiment was twofold. First, it was to provide an empirically grounded instructional theory for multi-digit multiplication, guided by an RME design framework. This was achieved through the construction and refinement of a learning trajectory to be used as a point of reference by researchers and practitioners to guide future design and teaching (Gravemeijer & Cobb, 2013). Of particular focus was the role of semiotic processes (Gravemeijer & Cobb, 2013) in building students' understanding and fluency in the domain; specifically, modelling and informal recordings. This was the focus of investigation for addressing Research Questions 1, 2 and 3.

Secondly, the theoretical intent was to articulate the collective classroom mathematical practices that emerged through the teaching experiment that supported the development of understanding, as identified in Research Question 4. The RME framework adopted influenced the type of tasks and the expectations for the way that students interacted with the mathematics and with each other. As such, it was regarded as important to explore the impact that mathematical tasks would have on the development of learning through such a trajectory.

The Teaching Experiment

The second phase of the research was the experimental stage. In this phase, a connection between the practice of research and the practice of teaching was established, with a teaching experiment used as the catalyst through which the teacher-researcher examined the learning trajectory. The purpose of experimentation in Design Research is not primarily to demonstrate whether the hypothesised learning trajectory works, rather, experimentation becomes the catalyst to understand clearly students' mathematical construction to best define the learning trajectory, and any possible misconceptions (Cobb & Gravemeijer, 2008). Understanding students' mathematical constructions comes through observing and exploring their symbolisation and actions as well as investigating their mistakes and misconceptions, or what can be called constraints (Steffe & Thompson, 2000). Steffe et al. (2000) explain that the primary reason for conducting a teaching experiment is for the researcher to experience these constraints directly. As such, the teaching episodes were designed so students were afforded the opportunity to construct their own mathematical realities based on exploration of concepts through collaborative problem solving, reasoning and dialogue with peers. The specific classroom culture established and its alignment with RME as a theory for design is discussed in Chapter 5.

This phase was a cyclic process of designing and then testing and refining the instructional sequence in the classroom setting (Gravemeijer, 2003). Through this process, modifications were reliably made to the learning trajectory.

This cyclical phase contributed to meeting the overall aim of the project as well as the specific research questions. The testing and refining of the preliminary insights into the construction of students' understanding, and the development of KDUs and strategies and models to support reasoning were achieved in this phase. The answering of the fifth research question, instruction for understanding, was also achieved in this stage through the designing, testing and refining of an instructional sequence for multi-digit multiplication using specific tasks and adopting particular classroom norms that supported the learning.

The teaching experiment phase was enacted through three key stages: the construction and implementation of an interpretive framework, teaching episodes that were developed through cycles of design and analysis, and the generation of a learning trajectory for multi-digit multiplication.

Interpretative framework

This phase involved the collection and ongoing analysis of data. An interpretive framework was used to justify the interpretations and assumptions made in the review of data. To address the research questions, it was important that the interpretative framework enabled the analysis of data on two levels. First, to meet the overarching aim and Research Questions 1, 2, 3 and 4, it was necessary that the data analysis focused on the *construction* of students understanding. Secondly, to answer Research Question 5, the interpretative framework needed to address how students' learning can be best supported in the social setting of the classroom, or *instruction* for understanding.

In alignment with the grand theoretical frame of constructivism, the emergent perspective framework as developed by Cobb et al. (1995) was employed as shown in Figure 4.4. As such, the framework did not stand apart from the experimentation process but was grounded in it. As stated earlier in the chapter, according to the emergent perspective, neither the psychological nor social aspect of learning takes precedence over the other in students' construction of knowledge. Rather, they have a reflexive relationship where one builds the other and neither stands in isolation from the other. The emergent perspective framework was introduced by Cobb and Yackel (1995) in an effort to coordinate the differing psychological and social theoretical viewpoints on learning.

In line with the work of Cobb, Stephan, McClain & Gravemeijer (2001), the interpretative framework was used as a conceptual tool to organise the analysis of the individual and collective mathematical events in the classroom. It brought to the fore three ways of acting and reasoning that are part of the social culture of the classroom and the corresponding psychological aspects which organised the diversity of students' participation. Events of the classroom and participatory regularities that were observed were interpreted according to these categories and, from this, conjectures made about the route of learning. These regularities informed modifications to the trajectory including the documentation of classroom practices, the types of thinking used by students and how learning was best supported. The following section provides a description of the elements of the framework. A description of how the framework was specifically used to organise and analysis the classroom happenings is documented in the methods section later in this chapter.

SOCIAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 4.4—The emergent perspective interpretative framework

Classroom Social Norms

The social norms of the classroom are the expectations and responsibilities for acceptable participation in the activity of learning (Cobb & Yackel, 1995). While the teacher might initiate and employ strategies to maintain social norms, all in the class must adhere to these norms and participate accordingly to make them principles of practice. Cobb et al. (1995) explain that the social norms include the way students explain and justify solution methods, listen to and make sense of other students' methods and solutions, indicate agreement or disagreement and ask clarifying questions when conflicting interpretations arise. As they make contributions to the social norms, students reorganise their own beliefs about their role, the role of others and the general nature of mathematical activity, which forms the correlating psychological perspective.

Sociomathematical Norms

The social norms of the classroom are relevant to learning in all subject areas, whereas the sociomathematical norms are specific to the discipline of mathematics. Such norms include what counts as an acceptable explanation, a different solution, an efficient solution, and sophisticated reasoning (Yackel & Cobb, 1996). As students participate in these sociomathematical norms and make mathematical judgements about explanations and reasoning, they are reorganising their beliefs and attitudes about their mathematical dispositions which forms the psychological perspective (Cobb, Stephan, McClain, et al., 2001).

Classroom Mathematical Practices

The mathematical practices relate specifically to the implementation of the learning trajectory and describe the domain-specific development of students. For this reason, they are of particular focus in the process of data analysis. Cobb et al. (2008) argue that a learning trajectory cannot anticipate the mathematical learning of all students and that learning is not a linear process—it is messy—and so anticipating specific learning goals for each individual student is problematic. Yet, as educators, we need to plan for learning to progress in a particular direction. Cobb et al. (2008) present the argument that learning should focus on the collective learning of the class as whole and present the construct of a learning trajectory focused on a sequence of mathematical practices, rather than tasks. A trajectory of this nature documents the socially constructed practices that emerge in the classroom and a means by which these practices can be supported. Creating a trajectory of mathematical practices involves an analysis of the social aspect of practices that emerge and correlating psychological perspective of individual students' constructions and reasoning.

Cycles of design and analysis

The cycles of design and analysis were fundamental to the experimental phase. Their purpose was threefold:

1. Establishing mathematical practices;
2. Identifying teaching episodes; and
3. Testing and refining the hypothesised learning trajectory

The cycles of design and analysis occurred on two levels: the micro level, within a single experiment, and the macro level, in the iteration of a second experiment. This is illustrated in Figure 4.5.

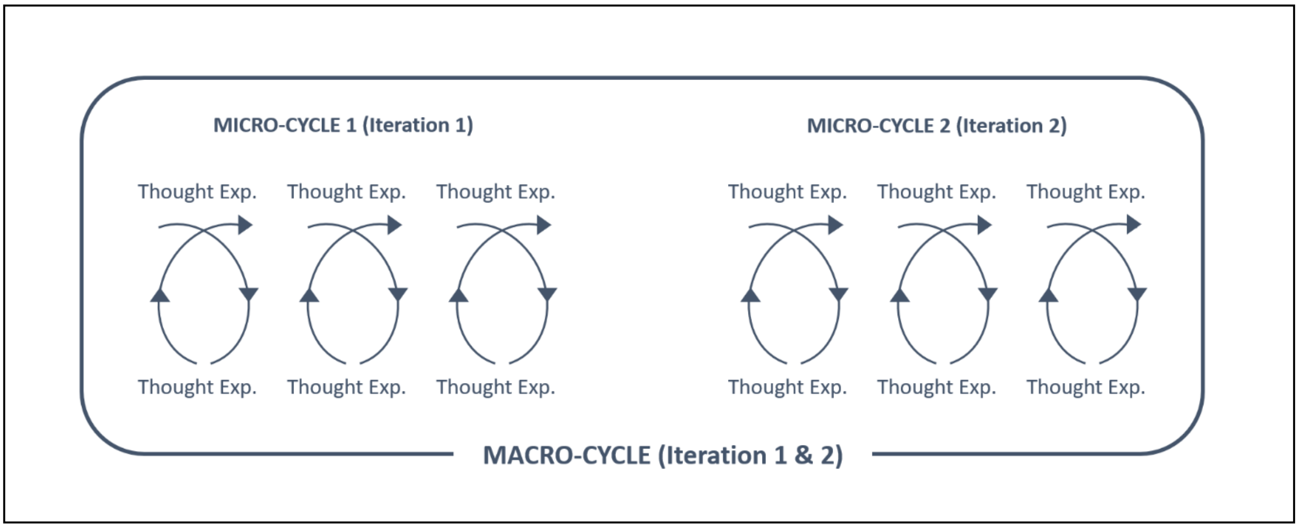


Figure 4.5—Cycles of design and analysis across multiple iterations of the teaching experiment

Cobb et al. (2013) explain that at the heart of design experimentation lies the cyclical process of design and testing. This process initially involved the identification of task types that support learning and that fitted within the established theoretical context. The researcher then worked in the classroom and, after each lesson, analysed the student activity based on the previously established interpretive framework. Learning and activity were seen as important when it was observed as regularity across multiple cases (Cobb et al., 2008). The interpretations also explored how shifts in students' reasoning were supported. After this, subsequent teaching episodes were designed, comprising of tasks and approaches to best support learning, and then modifications were made as necessary to the learning trajectory. This process continued and was conducted after every teaching episode throughout the teaching experiment.

The macro level of cyclical design and analysis was a much broader view of analysis, allowing for the first experiment to be placed within the larger context of multiple experiments. This was achieved through the iteration of a second experiment, and the results generated through each experiment compared and analysed. The aim of multiple experiments is to add rigour to the testing of the learning trajectory and, primarily, to generate a learning trajectory for multi-digit multiplication (Cobb et al., 2008).

A detailed explanation of the process of analysis and the specific methods employed is described later in the chapter.

Generation of a learning trajectory

The primary purpose of the preparatory thought experiment phase of this project was to hypothesise a domain-specific learning trajectory for multiplication. The cycles of design and analysis allowed for the modification and refinement of this trajectory, which, in turn, led to the development of the final domain-specific learning trajectory. Cobb et al. (2008) explain that it is called a domain-specific trajectory "to emphasise that [the] scope is restricted to significant learning goals in a particular domain" (p. 77). Specifically, this research established the learning goal as being fluent in multi-digit multiplication. Through the process of testing and refining instructional sequences across multiple classrooms, the trajectory was tested and modified and the processes by which this learning can be supported, was also investigated.

Retrospective Analysis

The focus of the experimental phase was on the analysing of learning and the means by which it is supported. In this phase, a shift is made to situate the learning process into the "broader theoretical context as a paradigmatic case of a more encompassing phenomenon" (Cobb & Gravemeijer, 2008, p. 83). It was in this phase that credible, repeatable and generalisable conclusions and implications were made.

Credibility

All conclusions and inferences made were based on a systematic analysis of the data in relation to the interpretive framework. Structured coding systems were used, and ongoing documentation linked these conclusions and inferences directly to the data. To establish further credibility, the critique of the data analysis was sought from an independent researcher.

Repeatability

As Simon (1995) helpfully points out "the only thing that is predictable in teaching is that classroom activities will not go as predicted" (p. 133). It goes without saying that the tasks and instructional sequence played out differently in the two classrooms! As the analysis of data allowed for conclusions and implications to be drawn, emphasis was placed on the aspects of the learning that could be repeated, rather than the unique situations created through the experiment. In keeping with the research questions, two key elements were focused on in the retrospective analysis of data:

- Construction of understanding—The development of specific forms of reasoning in multi-digit multiplication and how shifts and reorganisations of these specific forms of reasoning were developed along the learning trajectory; and
- Instruction for understanding—Aspects of the learning environment that supported these successive forms of reasoning

Generalisability

Through the process of establishing the connection between conclusions and the data sets, a more formalised learning trajectory was established. As of the work of Steffe et al. (2000), this learning trajectory served as an explanatory framework. This allowed the learning theory to be used as an exemplar for use or further research within the domain of multi-digit multiplication (Cobb et al., 2008).

Methods

The previous part of this chapter looked at the specific methodology employed for the research. This final section of the chapter explores the actual methods employed for the teaching experiment and the data collection and analysis. This encapsulates the methods of the Teaching Experiment and Retrospective phases of the methodology. The specific methods of the Preparatory Thought Experiment phase are documented in Chapter 5. This phase involved the development of the proposed learning trajectory and the design of the teaching experiment.

Context

The participants were recruited in the thought experiment phase of the research so as pre-assessment could be conducted in preparation for the commencement of the experiment. Two independent schools in New South Wales accepted the invitation to participate in the study. There was not coercion to participate. Signed consent was obtained from the schools in the form of a letter signed by the Principal. (See Appendix 2) The schools were from different parts of the Sydney metropolitan area. This participant demographic was selected due to the researcher's role as a Mathematics Consultant for *The Association of Independent Schools of NSW*, affording easy access to NSW independent schools. Year 5 students from these schools received written documentation clearly outlining the study’s purpose and intended outcomes. Signed consent was gathered from the students and from their parents. Participation was voluntary; schools, students and/or their parents could elect not to participate. Any students who elected not to participate were supplied with similar learning activities and appropriate supervision as negotiated with the school prior to the commencement of the study.

The project was conducted in Year 5 classes; 23 students in the first and 22 in the second class, creating a sample size of 55 students. The first school was a co-educational school located on the Northern Beaches of Sydney, a relatively high socio-economic area. The students from the school performed well in national testing (ACARA, 2010). In the researcher's role as a consultant in the sector at the time of the research, there had been limited contact with the school previously. The school had single streamed classes for each year group. There were 24 students in the experimental class, with one student electing not to participate. Of the 23 who participated in the study, 13 were boys and 10 were girls.

The second school was an all-girls school in the inner-western suburbs of Sydney. Again, this school was in a relatively high socio-economic area of Sydney with the students performing well in national testing (ACARA, 2010). As a consultant, the researcher had had more contact with this school than the first school in the study. Most of the contact was limited to working with teaching staff and very little time had been spent with the students who participated in the study. This school was substantially larger than the previous school, with three Year Five classes. The students were streamed into four ability groups; top, higher core, lower core, bottom. The research was conducted with the students in the higher core class, as this was the class offered for the study by the school. This class had 24 students and 22 of these students participated in the teaching experiment. Both groups of students were pre-assessed using the one-to-one interview developed by the researcher. The interview development and the students' results are discussed in more detail in Chapter 5.

In both teaching experiments, the researcher adopted the role of the teacher. The students were able to meet the researcher through the process of the pre-assessment one-on-one interviews, which also enabled the researcher to establish a rapport with the students. Adopting the role of the researcher-teacher meant that I could control the learning environment and teaching episodes across both experiments. It also enabled the researcher to experience first-hand the events of the classroom, thus enriching the ongoing cycles of data analysis and experimentation. The students' regular teacher was also present in the classroom and helped facilitate student activity. Prior to the commencement of the teaching experiment, the researcher met with the class teacher to discuss the hypothesised learning trajectory and the learning culture that was expected during the teaching episodes. An important part of this meeting was also to discuss the interpretative framework based on the emergent perspective.

Data

To analyse the data collected, the Constant Comparative Method (Glaser & Strauss, 1968) was used and adapted to the needs of Design Research by Cobb & Whitenack (1996). The Constant Comparative Method is an interpretivist approach to qualitative data analysis that uses a combined process of coding and analysis to create a coherent and trustworthy theory from the data. Using this method, collected data is used to document incidents of participants' activity and, when these incidents are compared, common categories and themes emerge. As new data is produced, it is compared with the currently conjectured categories and themes. Cases that seem to contradict the established categories are considered especially interesting and are used to refine the categories. The ongoing process of constant comparison leads to the ongoing refinement of the theoretical categories developed from the data. Glaser and Strauss (1968) describe the method using a four-stage process:

1. Comparing incidents applicable to each category
2. Integrating categories and their properties
3. Delimiting the theory
4. Writing theory

Cobb & Whitenack (1996) altered this process to suit the needs of Design Research, specifically, identifying mathematical practices across a sequence of lessons. They observed that the most significant difference with their approach was seen in the creation of categories. The Constant Comparative method require categories to emerge from a given set of data. Cobb et al. (1996) acknowledged that they approached the data with the preconceived category of a *mathematical practice* and this category was not limited to one set of data but could be used across multiple sets. Cobb et al. (1996) describe a two-phase process, in line with the experimental phase and the retrospective analysis of Design Research.

Phase 1—Analysis in the Teaching Experiment

Data were collected and analysed chronologically through the teaching experiment phase. Based on this ongoing analysis, conjectures were made about ways of reasoning and communicating that are normative in the classroom culture and selected students' reasoning was also recorded. Documentation was kept, outlining conjectures of these developing mathematical practices along with all supporting evidence.

Phase 2—The Retrospective Analysis

The documentation of Phase 1 became the focus of analysis in Phase 2 to develop a grounded theory on the development of mathematical learning in the classroom. The conjectures about the possible mathematical practices were tested across the complete data set. Cases of data that appeared to contradict conjectures were of particular interest and helped refine the categories of practices that were identified.

In the following section, how these two phases were enacted in this project are outlined. A description of the data collected is provided and how it was analysed.

Phase 1—The Teaching Experiment

Data Gathered

In deciding the type of data that needed to be produced and collected, the theoretical intent of the research was revisited. The experiment was to provide an empirically grounded instructional theory for the domain of multi-digit multiplication. To achieve this, the data collected needed to elicit evidence of students' reasoning, shifts in their reasoning, and the means by which these shifts were supported and organised. Based on this, four key forms of data were collected: pre- and post-assessment interview results, student work samples, classroom video recordings and field notes.

Pre- and post-assessment interview results

Each student participated in a pre- and post-assessment one-to-one interview. The interview was designed so that KDUs were assessed, and that the questions moved from easier to more complex questions. The interview concluded when students made an error.

The data from the pre-assessment was used to help inform the learning goals and the instructional starting points for the sequence. At the conclusion of the study, the same interview was used. Students were asked questions based on their performance in the first interview. The questions started from when they experienced difficulty in the pre-assessment interview. The scores from the pre- and post-assessment were recorded to look at students' growth as a result of the teaching experiment. The interview design and pre-assessment results are discussed in Chapter 5.

Work samples

Student work samples were collected from each teaching episode. Students were provided with blank sheets of A3 paper to develop solutions to the problems that were presented. The expectation was that one work sample would be produced for each pair or group of three. Students were encouraged to not only provide an answer to the problem, but to clearly show how they arrived at their solution. This meant that within the groups of two or three, consensus was needed on how best to represent their workings so that there was enough detail for someone else to make sense of their strategy using only the physical recordings.

Video

Video recordings were made of different aspects of the teaching lesson. As the researcher or the classroom teacher interacted with students, short recordings were made as the students explained their reasoning. Whole class discussions were also video-taped. This video footage was later transcribed as needed in the Retrospective Analysis phase.

Field notes

Before each lesson, I met with the class teacher to discuss the planned activities for the day. In this time, I highlighted the observational data that should be collected as anecdotal notes. The object of observational notes was focused on the interpretative framework which had previously been discussed with the teacher, and as discussed earlier in the chapter. As the lesson was conducted, the class teacher and I interacted with students and took any necessary anecdotal notes. At the conclusion of the lesson we met again to discuss the notes taken and any further observations. We also used this meeting time to discuss the ways in which the students' inventions could be capitalised upon in the forthcoming lesson.

Cycles of Design and Analysis

The purpose of the teaching experiment was to improve the conjectured learning trajectory designed in Phase 1 and to document the means by which learning could be supported in the classroom. To achieve this goal, ongoing cycles of design and analysis were employed (see Figure 4.5). These cycles employed a reflexive relationship between the ongoing thought experiments based on the envisioned learning trajectory and the subsequent testing and revising of conjectures, based on the work of Cobb & Gravemeijer (2008) and Simon (1995). After each lesson, interpretations of classroom events were made based on the data gathered which allowed modifications to be made to the trajectory, new conjectures to be made and subsequent tasks to be planned. This section provides a description of three stages that were employed to guide the cycles of design and analysis (Figure 4.6).

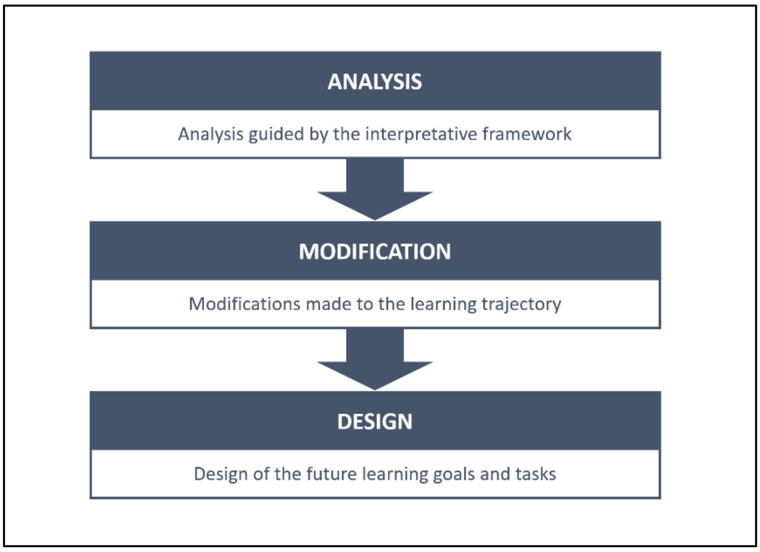


Figure 4.6—Three stages used to guide the cycles of design and analysis

Analysis

The first step in the process of analysis was the most intensive part of the ongoing thought experiments. After each lesson, the new data was compiled with the existing data and reviewed. In keeping with the interpretative framework and the research questions, the dataset was analysed from three perspectives. The first two perspectives aligned with Research Questions 1, 2, 3 and 4, and also the interpretative framework: the social perspective and psychological perspective. Analysis from the social perspective was concerned with identifying the evolving classroom mathematical practices and provided conclusions for Research Question 4. The focus of the psychological perspective was to document the mathematical conceptions of the individual students, which helped to answer Research Questions 1, 2 and 3. The third perspective used to analyse data was to answer Research Question 5, that is, how the learning of the students could be best supported in the classroom. The method of analysis from each of these perspectives is explained in the proceeding section.

The Psychological Perspective—Mathematical Constructions

A structured coding system was used to categorise the data after each lesson. First, the strategies students used were recorded using a coding system (Table 4.1). Next the key developmental understandings (KDUs) evident in students' solution were listed against those codes. Creating a coding system for the use of the model, in this case the array, became problematic and so a brief description of how the array was used was recorded. The final element that was recorded were any mistakes or misconceptions. Again, a brief description was provided for errors and misconceptions rather than a specific code.

Code	Strategy Description
C1	Count in 1s
SC#	Skip count where # will represent the size of the count, e.g. SC5 means the students skip counted in 5s.
R+	Repeated addition
MD	Multiplicative doubling, e.g. 24 x 8 = 24 x 2 x 2 x 2
P-10	Partition number into groups of 10's, e.g. 24 x 8 = (10 x 8) + (10 x 8) + (4 x 8)
P2-10	Partition both numbers - 10's, e.g. 24 x 12 = (10 x 10) + (10 x 10) + (4 x 10) + (2 x 10) + (2 x 10) + (2 x 4)
P1-PV	Partition one number - PV, e.g. 24 x 12 = (20 x 12) + (4 x 12)
P2-PV	Partition both numbers – PV, e.g. 24 x 12 = (20 x 10) + (20 x 2) + (4 x 10) + (4 x 2)
CP	Multiplicative compensation, e.g. 24 x 8 = (25 x 8) – (1 x 8)

Table 4.1—Coding used to classify students' strategies

After coding the data, it was analysed on two levels. The first focused on the emerging uses of the array as a *model of* a contextual situation to a *model for* mathematical reasoning, as aligned with Research Question 1. The second was to explore the pathways of students' reinvention of strategies and the underlying KDUs, as in Research Questions 2 and 3. The specifics of how each was used in the analysis of data are explored in the following section.

Code	Key Developmental Understandings (KDUs)
+ve	Additive thinking
xve	Multiplicative thinking
dist	Distributive property
ass	Associative property
pv	Place value
com	Commutativity

Table 4.2—Coding used to identify KDUs associated with students' strategies

a) The array as a model of to a model for

Models as a representational tool in mathematics play the dual role of supporting student learning as well as communicating student reasoning. As such, analysing the model and its use was an important part of the ongoing thought experiments to plan future learning activities and influence modifications to the learning trajectory. It is important that models emerge naturally from students' activity and that they allow students to develop more sophisticated mathematical understanding and reasoning (Gravemeijer, 1994). The model used in this research was the array. Students' use of the array was tracked across the teaching experiment and descriptions of how the students used the array was grouped in two ways: according to student and according to the solution method.

Grouping the use of the array according to the individual students allowed consideration as to whether the model emerged naturally through their activity. The model needed to correspond with students' thought processes. It was also important to observe whether the students were able to use the array in different ways and in new problematic situations. The most significant benefit of grouping in this way was to observe whether the model held power for the students which would be evident through moving from a *model of* the contextualised situation to a *model for* more generalised mathematical reasoning.

b) Students' reinvention of strategies and underlying KDUs

It was important to track the pathway of reinvention of strategies used in the classroom. Students' work samples, the video recordings and anecdotal notes were used to inform the assumptions made and the solutions to problem solving tasks, observational notes and any errors were accumulated in a table. This table served to track the learning path of individual students as well as provide a picture of the class as a whole.

The variety and spread of the solutions used by the students was analysed and used to directly inform the learning trajectory. The variety of solution methods illustrated the strategies and reasoning that were to be expected by students in the reinvention process. The solution level was evaluated based on whether strategies were efficient, additive or multiplicative in nature and whether students were moving towards more generalised forms of reasoning. The solution level revealed the learning path to be followed.

The dispersal of strategies also informed how learning could be best supported. While reinvention requires students to be self-reliant in their solution development, the role of the teacher is to coordinate activity so that students thinking is challenged to promote higher levels of reasoning. This was primarily done through class discussions where students shared their strategies and reasoning. In this way, the students' inventions challenged others' thinking, it was not necessarily the teacher who challenged. Gravemeijer (1994) explains why the dispersal of strategies is critical for discussions:

> *A productive classroom discussion is only possible when there is a difference both in the solutions of various students and in the solution level. If the students all use the same procedure, then there can be no discussion. If all solutions are on the same level, then the teacher has no other choice than to exert a guiding influence* (p. 187).

Analysing the dispersal of strategies allowed the researcher to anticipate the solutions that were expected. Based on this activity and discussion could be planned so that the students' inventions could be harnessed to guide and build students' thinking accordingly.

The data on individuals were analysed to determine whether students were following the path of reinvention. Attention was given to flexible and appropriate use of strategies. It was also important to consider whether the students were simply following other students or the teacher's direction. This was primarily informed through observation, although a variety of strategies evident through students working was also informative. Another key indicator was whether students adopted a bottom-up approach to problem solving. Faced with new and unfamiliar questions, it is to be expected that students will shuffle back and forth between the problem and their existing knowledge (Gravemeijer, 1994). This results in a forwards and backwards movement on the learning trajectory.

The Social Perspective—Mathematical Practices

To determine the emergence of mathematical practices, the coded data, the student work samples and video footage were analysed for any regularities and patterns in the way students acted, interacted or reasoned mathematically. These events were considered significant when they were observed over multiple cases and influenced the collective knowledge of the class. They were defined as taken-as-shared practices (not yet an established mathematical practice) and were systematically documented along with the supporting evidence that formed the basis for the assumptions. This resulted in a chain of conjectures and inferences about the potential mathematical practices that were emerging.

Supporting Students' Learning

To assist in documenting how learning might be best supported, an analysis of the context and associated questions was conducted. The aim at this point was not to show that the context worked, but to look for aspects of the context that provided students with footholds in their development of understanding in the domain. To analyse the context, the learning goals for each lesson were revisited along with the collected data.

It was important that the context used was suitable. This meant that students needed to engage both in vertical and horizontal mathematising. It was also important to ensure that the context did not distract students from the mathematics being studied. For each lesson, the learning goal was revisited and then students work was studied to see if the appropriate mathematical objectives were being reached. The specific elements of the context that provided powerful exploration of mathematical content was documented.

The applicability of the context was also explored. The context needed to facilitate a growth in knowledge across the class in the domain of multi-digit multiplication. While solutions were initially localised, over time they needed to hold communal value allowing mathematical knowledge to develop (Gravemeijer, 1994). The areas of the context that enable application to mathematical development of the class were documented.

Modification and Design

After the data were analysed based on the interpretative framework and in light of the research questions, modifications were made to the learning trajectory. The modifications were not made solely on the events of the previous teaching episode but looked across all the episode that had been conducted up to that point. Any modifications made were not seen as final, but rather offered new assumptions that need to be tested and refined. Based on the modifications and analysis conducted, learning goals and tasks for the following teaching episode were planned.

Phase 2—Retrospective Analysis

The second round of data analysis was the retrospective analysis. In this phase the documentation created in the previous analysis became the focus of analysis to develop a grounded theory on the development of mathematical learning in the classroom. The inferences and conjectures made in Phase 1 were taken back through the data to create a unified sequence of events so that the taken-as-shared practices and the mathematical constructions of the individuals could be viewed from a more holistic perspective. This systematic backtracking through the data to refine and refute of conjectures ensures the trustworthiness of the final claims and assertions made (Cobb & Gravemeijer, 2008). In addition to this, the retrospective analysis served to answer the overarching research aim along with the five research questions. The following section discusses the process of the retrospective analysis.

Mathematical Practices

Initially, the data preceding the emergence of each mathematical practice were re-analysed to identify students' common activity and reasoning (Stephan & Rasmussen, 2002). This was done for three reasons. First, it helped confirm the establishment of each mathematical practice and, indeed, when it occurred. This served to answer Research Question 4. Secondly, a second analysis of data enabled the link between the students' activity and reasoning, and the establishment of the practice to made explicit. This was critical in answering Research Question 5, specifically how the development of the mathematical practices was supported in the implementation of the learning trajectory. Retracing the establishment of each practice, revealed those aspects of the classroom environment that were necessary in the negotiation and evolution of mathematical practices. Finally, the re-analysis allowed the mathematical practices to be categorised according to the three types of mathematical practices as identified by Cobb et al. (2001): taken-as-shared purposes, taken-as-shared ways of reasoning with tools and symbols, and taken-as-shared forms of argumentation.

The documentation based on the field notes, work samples and video compiled in the first round of analysis was again investigated. First, the collective activity of the students and their normative behaviours were categorised according to the interpretative framework, specifically as to whether the students were adhering to and exhibiting social or sociomathematical norms. Through the course of analysis, an additional set of norms emerged. Initially, they were categorised under the heading *other* and then later labelled *mathematical norms* (these are discussed in detail in Chapter 7). Each practice was analysed individually and then commonalities were sought across all practices. These commonalities were identified as significant and, as such, classified as necessary in the negotiation and evolution of the mathematical practices. After this, commonalities were also sort across the practices. Each practice was categorised according to the three types of practices identified by Cobb et al. (2001) allowing the practices to be merged or aligned if commonalities were observed. Commonalities were also looked for across the three categories. Where alignment was observed the taken-as-shared practices were then seen as a subset contributing to the formation of a larger, encompassing mathematical practice.

Mathematical Constructions

The students' individual constructions were also analysed for a second time as part of the retrospective analysis. Analysis in Round 1 was concerned with tracking students use of the array and their pathways of reinvention with strategies. Analysis shifted in this second round to look more closely at generalisations that could be made about students' mathematical constructions. Analysis focused on two aspects of students' constructions: the array and students' reinvented strategies.

The array as a model of to a model for

The purpose of this round of analysis was to map the process of the array moving from a *model of* a contextualised situation, to a *model for* more generalised mathematical reasoning and, as such, provide an answer to Research Question 1. Saxe's (2004) form-function framework was used to explore the different forms of the array used by students and what function each form of the array served in students' working. To do this, I first defined the *form* of the array as specific visual features, and its *function* as the way the students chose to interact with the array in their work. Having established these definitions, I re-analysed the documentation created in the Round 1 analysis to note the visual form of the array used by students with a brief description of the function the array served in this form. Three forms of the array were observed across the instructional sequence: arrays with all parts visible, pre-partitioned array and an open array. The dataset was then grouped according to the three forms of the array observed so commonalities could be identified, and how students' form-function use shifted over the course of the instructional sequence could be noted. The dataset was then re-grouped, this time based on the array's function. Grouping in this way served to confirm the commonalities that were identified, students' evolving use of the array and to highlight any anomalies.

The final step in the analysis was to explore the form and function of the array based on students' diverse conceptions and strategies. To do this, data were grouped based on the form and function of the array that the students chose to use as they developed solutions to the problems.

a) Students' reinvention of strategies & underlying KDUs

The purpose of this round of analysis was to better understand students' pathways of reinvention and to answer Research Question 3. To achieve this, students' strategies were classified according to Treffers & Buys' (2008) calculation levels. Treffers & Buys (2008) note three levels of calculation strategies for whole number computation: calculation by counting, calculation by structuring and flexible, formal calculation. After the strategies used were classified, the identified

KDUs were aligned with these strategies. Of particular interest were students' movements from additive to multiplicative thinking and when students employed strategies based on the distributive property. This movement was used to identify important shifts in students thinking in the reinvention process. Based on this analysis, new sub-levels were added to Treffers & Buys (2008) calculation levels.

As with the second analysis of the array data, students' misconceptions and errors were also mapped alongside this classified data. The strategies and the underlying KDUs were identified when errors were made, or misconceptions made apparent. The subsequent strategies that students used after faced with an error and/or misconception, were then mapped.

The final part of the analysis was to look at students use of the array alongside their strategy use. The form and function of the array was aligned to the strategies and calculation levels. Doing this enabled an exploration into how the array supported students in the process of reinvention. It also allowed an exploration into students understanding of the multiplicative structure based on the strategies that they used.

The Final Learning Trajectory

The final stage of the retrospective analysis was to construct the final learning trajectory. The documentation of the trajectory was reorganised based on the analysis of results. The final trajectory was structured around five elements: a description of the instructional sequence, the mathematical practices that emerged, the focus KDUs and strategies for each teaching episode, the expected form and function of the array and finally topics for mathematical discourse. Finally, in answer to question five, the teaching strategies and classroom culture to support successful future implementation were recorded. The retrospective analysis also included a modification to the interpretative framework used in the analysis of results.

Conclusion

This chapter has presented the methodology used for the current research project. It discussed the theoretical two-level theoretical framework of constructivism and RME and explored how the framework complemented the research methodology of Design Research. The specific methods of the project were then discussed, including the participants used, the data gathered, and methods of analysis employed. The next chapter discusses the first phase of the Design Research methods, the delineation of the hypothetical learning trajectory.

CHAPTER 5

THE LEARNING TRAJECTORY

Introduction

Central to this project was the development and refinement of a learning trajectory for multi-digit multiplication. The chapter is divided into two main sections documenting the work of the preparatory thought experiments used to develop the initial hypothesised trajectory. The first section gives a brief history on learning trajectories and describes elements common to trajectories. The section articulates the form of learning trajectory used in this research project. The second section of this chapter focuses on the specific work of the preparatory thought experiments. The design features of Realistic Mathematics Education (RME) are explained. Following this, the hypothetical learning trajectory that was implemented in the teaching experiments is described.

Framing the Learning Trajectory

The theoretical foundation of this project (as described in Chapter 4) was built on two levels: a grand-theory level, and an intermediate level (Kieran et al., 2015). The grand-level theory of constructivism was used as a theory of learning. Constructivism was interpreted in accordance with the emergent perspective offered by Cobb & Yackel (1995). The intermediate level used RME as a theory for design. The theory of design was critical in the development of the trajectory.

In this section, the theoretical framework of the project is placed into the planning and design of the learning trajectory. I begin by looking at learning trajectories more generally and explore why a trajectory fits with the constructivist view of learning. While learning trajectories are common in developmental research, they are interpreted differently. An interpretation for the learning trajectory used in this project is established. The theoretical and practical implications for the trajectory used are also discussed. The three design heuristics of RME that were used to plan the students' learning are then examined.

Learning Trajectories

Historically, curriculum design has focused on the linear transmission of content that progressively becomes more abstract (Fosnot & Jacob, 2010). While developing mathematical knowledge may appear to be a linear progression, one skill building upon another, the cognitive development of the child is neglected (Fosnot & Dolk, 2001). Learning is not linear; learning is messy (Duckworth, 1987; Fosnot & Perry, 2005). From a constructivist viewpoint, students draw on their individual experiences, knowledge and social interactions to construct a learning pathway that is unique to them (Fosnot & Perry, 2005).

Many recent studies on teaching and learning in mathematics have acknowledged this view on learning through the adoption of a learning trajectory. The unique strength of learning trajectories lies in the interconnectedness of the psychological developmental of the child and instructional sequences; the focus of teaching shifts from the expert deconstructing the content in a manner that makes sense to them, to acknowledging and using students' constructions and thinking in planning and presenting subsequent instruction.

The notion of learning trajectories was first introduced by Simon (1995), who used the phrase *hypothetical learning trajectory* (HLT). Simon acknowledged that the theory of constructivism provided a useful framework to think about learning and learners but did not specify a model for teaching. He presented the model of a HLT for "what teaching might be like if it were built on a constructivist view of knowledge development" (Simon, 1995, p. 115). A HLT predicts a path by which learning might proceed, but it is hypothetical because "the actual learning trajectory is not knowable in advance" (1995, p. 135). To illustrate, Simon (1995) used the metaphor of a sailing voyage:

> *You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of destinations. You modify the length and nature of your visits as a result of the interactions with the people along the way. You add destinations that prior to your trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your 'hypothetical trajectory'* (pp. 136-137)

A learning trajectory is "a vehicle for planning learning of a particular concept" (Simon & Tzur, 2004, p. 93) which shifts the focus of teaching from transmitting content to an emphasis on the cognitive constructions of the individual students (Gravemeijer, 2004). Trajectories provide teachers with a framework of reference to be used as a source of inspiration. Teachers plan lessons that enable students to reinvent conventional mathematics and so refine and expand their current ways of knowing (Gravemeijer, 2004). Students direct their own pathway through the trajectory through strategies employed and mathematical reasoning.

The concept of *learning trajectories* has evolved since Simon first introduced the term (Siemon et al., 2017). Simon's first HLT (1995) was based on a single instructional episode. Now others have applied the notion of learning trajectories to sequences over a longer period of time, including: instructional sequences spanning multiple lessons (Gravemeijer, 2004; Gravemeijer et al., 2003a), a specific aspect of curriculum content taught over multiple years (Clements & Sarama, 2014) or a progression of interconnected content taught over an extended period of time (Confrey, Maloney, & Corley, 2014; Siemon, Breed, et al., 2006).

Despite the variation in use, common interpretations agree on three common elements. First, learning trajectories state and target specific learning goals for students in a mathematical domain (Cobb & Gravemeijer, 2008). Secondly, the trajectory forms a developmental progression or "learning path through which children move through levels of thinking" (Clements & Sarama, 2014, p. 17). This framework is then used as a reference point for planning learning, typically incorporating the anticipated and desired construction and thinking of the students. Finally, the trajectory provides an instructional sequence that consists of mathematical tasks (Simon & Tzur, 2004) or mathematical practices (Cobb, Stephan, Mcclain, & Gravemeijer, 2001) that are used to promote student learning. Each of these elements are identified in the following definition of learning trajectories offered by Clements & Samara (2009):

> *…descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain* (p. 83).

In this thesis a learning trajectory for multi-digit multiplication is described. Learning trajectory here is used to mean an instructional sequence over the course of approximately ten teaching episodes (as in Gravemeijer et al., 2003; Gravemeijer & Cobb, 2013; Prediger, Gravemeijer, & Confrey, 2015). The intent of the trajectory is both theoretical and practical (Cobb, 2003). The trajectory describes a potential taken-as-shared learning route, and a means by which students' learning might be supported (Stephan, 2003b). A domain-specific instructional theory for multi-digit multiplication is provided, focused on substantial mathematical ideas and a demonstrated means of supporting students' learning. It is expected that the theory will prove to be "useful in providing guidance to others as they attempt to support learning processes" (Cobb & Gravemeijer, 2008, p. 72).

The remainder of this chapter focuses on describing the construction of the initial trajectory before its implementation and its refinement through a classroom teaching experiment. First the theory for design (RME) is discussed. Three design heuristics are introduced and the implications for the trajectory are considered.

Realistic Mathematics Education

The theory of RME guided the design and development of the instructional sequence. RME emerged through the work of the German-born Dutch mathematician, Hans Freudenthal (1905-1990), who saw mathematics as a human activity. He argued that mathematics is not a closed body of knowledge to be transmitted, rather, it is an exercise in which learners are active participants (Van den Heuvel-Panhuizen, 2003). Mathematics is learnt through one's own mental processes, whereby one 'reinvents' conventional mathematics (Gravemeijer, 2004). In the context of the classroom, students engage in tasks that require them to develop tools and strategies as they solve experientially real problems. Students form and organise new knowledge and develop their own mathematical insights (Van den Heuvel-Panhuizen & Drijvers, 2014), a process Freudenthal called *mathematisation*.

The aim of RME is to support students' *progressive mathematisation*, or level-raising (Gravemeijer et al., 2003a). To achieve this, three design heuristics are used to construct learning experiences: experientially real contexts for learning; guided reinvention; and emergent modelling. The heuristics work in unison to develop students' mathematical understanding and practices. The following section provides a description of each heuristic.

Heuristic 1—Experientially Real Contexts for Learning

A central feature of design in RME is the use of realistic contexts as a starting point for learning new concepts. By adopting an appropriate context, students are encouraged to engage meaningfully in tasks and to mathematise situations (Gravemeijer et al., 2003a). Realistic, in this instance, refers to situations that can be easily imagined by students, whether real, fantasy or even taken from the abstract world of mathematics (Van den Heuvel-Panhuizen, 2003), as long as the situation is real in the minds of the students. To find such a context, the designer must carry out a didactical phenomenological analysis to identify scenarios which are ordered by the mathematical concept being explored. A didactical phenomenological analysis takes what students know from their world as a context to situate the teaching of mathematical concepts and ideas: "rather than looking around for material to concretize a given concept, the didactical phenomenology suggests looking for phenomena that might create opportunities for the learner to constitute the mental object that is being mathematised by that very concept" (Gravemeijer, 2004, p. 116). This then becomes the starting point for instruction, with the mathematics progressively abstracted throughout the course of the sequence.

Heuristic 2—Guided Reinvention

Guided reinvention encourages students to *reinvent* important mathematical concepts and strategies for themselves. The designer's role is to plan "a route…that allows the students to invent the intended mathematics for themselves" (Gravemeijer, 2004, p. 114). The designer must imagine the pathway that students will take, based on the history of mathematics and relevant research into the mathematical domain (Gravemeijer et al., 2003a). Tasks must be carefully planned and sequenced to allow the students to experience a process similar to the process when the mathematics was first developed (Stephan, Underwood-Gregg, & Yackel, 2014).

Heuristic 3—Emergent Modelling

The final heuristic is that of emergent modelling. A 'model' is a broad term that encompasses varied representations of mathematical concepts and structures (Van den Heuvel-Panhuizen, 2003). In RME, the model is not a ready-made representation trying to make concrete mathematical concepts. Rather, the model is developed out of the context and, ideally, is created by the students themselves (Gravemeijer, 2004). Its purpose is to help students in the process of progressive mathematisation. The model, Van den Heuvel-Panhuizen explains (2003), acts like a bridge. On one side are the informal understandings bound within the context of the problem, and on the other side are the formalised mathematical concepts. It is students' interactions with the model that allow them to cross this bridge.

The nature of the model changes through students' activity. It moves from a *model of* a situation to a *model for* mathematical reasoning (Gravemeijer, 2004; Van den Heuvel-Panhuizen, 2003). Initially, the model is closely connected to the context of the problem: it is a *model of* a particular situation and students use it to make sense of the problem at hand. As the students work with the model over multiple experiences within a sequence or across several sequences, students build an appreciation for the mathematical concept or structure that the model embodies. Their understanding of the model becomes more generalised and it becomes a *model for* mathematical reasoning. The model is reified. As Gravemeijer (2004) explains,

> *the model first comes to the fore as a model of the students' situated informal strategies. Then, over time, the model gradually takes on a life of its own. The model becomes an entity in its own right and starts to serve as a model for more formal, yet personally meaningful, mathematical reasoning* (p. 117).

This section has outlined the three heuristics that guide the work of design in RME. The following section moves to describe the preparatory thought experiments conducted as part of the research process.

PREPARATORY THOUGHT EXPERIMENTS

In developmental research, the first phase of the research process is the preparatory thought experiments. These thought experiments use the available literature to hypothesise the path that students might take in their learning journey. In this section, the preparatory thought experiments for this project are documented.

Emergent Modelling

The design process commenced by selecting a representation to stimulate students' progressive mathematisation. Decisions concerning which model to use were based on the review of literature in Chapter 3. A majority of studies drew on the array structure as a tool to build students' understanding in both early multiplication and multi-digit multiplication (Barmby et al., 2009; Battista et al., 1998; Davis, 2008; Izsak, 2004; Young-Loveridge & Mills, 2008, 2009). This model was consistent with the New South Wales Syllabus documentation which was implemented in schools at the time of the research (Board of Studies NSW, 2002).

The review of literature noted that the array was presented differently to students in different studies. Barmby et al. (2009) grouped the array in five rows of 5, creating squares of 25 dots, whereas Young Loveridge et al. (2008, 2009) partitioned the array in 10s to create groups of 100. In both these studies, the grouping of the array influenced the strategies used by the students. To encourage flexible partitioning, the array would need to be presented in different forms and integrated seamlessly into the context presented to students.

It was important that students' use of the model moved from being a *model of* a specific context to a *model for* more generalised mathematical reasoning. It was anticipated that this would occur on two levels:

1. Progressive abstraction of the array structure
2. Shifts in students' use of the array as a tool

Progressive abstraction of the array structure

The progressive abstraction of the array was a design feature of the sequence (see Figure 5.1). It was decided that the array structure would be introduced to students with all parts of the array visible. In this form, the array was pre-partitioned into smaller sections. This enabled students to use counting strategies if needed and allowed them to show how the items in the array could be partitioned differently or rearranged, at the discretion of the individual student. From here, parts of the array were hidden, introducing a more abstract form of the array. The final stage of abstraction moved to an 'area' model of multiplication (Siemon et al., 2011), or 'open array' (Fosnot & Dolk, 2001), which is the term used in this thesis.

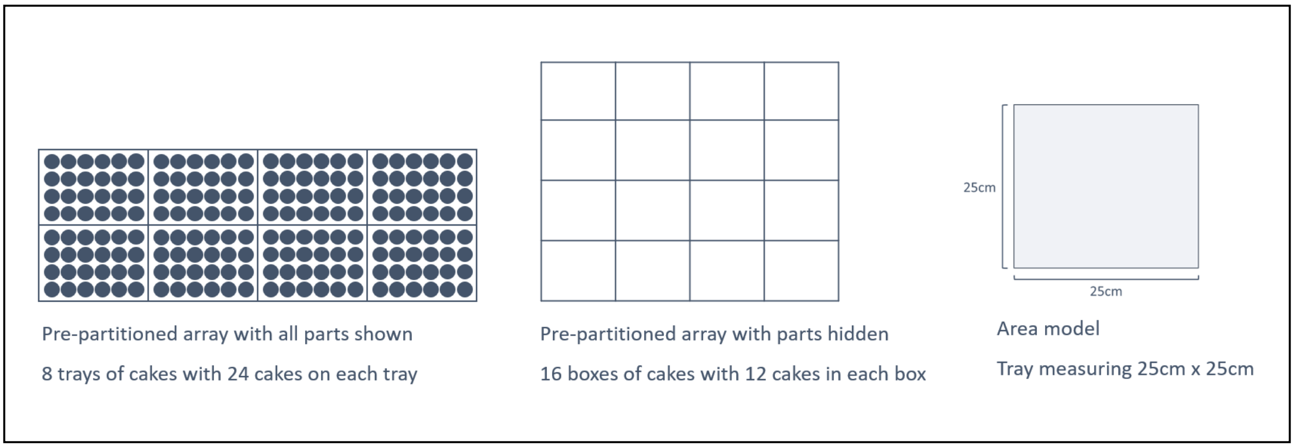


Figure 5.1—Progressive abstraction of the array

Shifts in students' use of the array as a tool

Models, and students' interactions with them, are closely tied to the context of the situation (Fosnot & Dolk, 2001). Therefore, consideration was given to the context and the type of activity the context would produce. The instructional sequence sought to promote different sorts of activity with the array. This was accomplished in two ways. First, the context of each problem encouraged particular sorts of activity. Initial problems focused on practical situations where the array could be easily partitioned or manipulated. The questions then progressed to contexts where it did not make sense for the array to be physically manipulated. This encouraged students to work independently from the context and use the array as a tool for reasoning. By the end of the sequence, problems were presented to the students in numerical form, and students could choose to use the array if they found it helpful to solve the problem.

The second way that the instructional sequence promoted different sorts of activity was through the development of each individual problem. When a problem was posed, students would construct a solution using the context of the problem. Students would share their strategies in a class discussion and consider similarities and differences between the different solutions presented.

Students would be encouraged through the course of discussion to consider more formal mathematical concepts. Again, they were starting to work independently of the context, and recognise and employ the array as a tool for mathematical reasoning.

In keeping with the design heuristic of emergent modelling, it was important that the model emerged naturally from a meaningful context. The next section discusses the context of the project and how it would be developed.

An Experiential Real Context

A didactical phenomenological analysis was enacted in order to create a context for the sequence. It was important that the array arose naturally from the context, and that the context allowed the array to be manipulated as students made sense of their strategies and the methods used by others. The context of a bakery was selected, for two reasons: it was a context easily imagined by students, and the array model could be developed through the imagery of cupcakes in baking trays and boxes. An ongoing narrative could be used throughout the sequence as the students encountered new problems to be solved.

Guided Reinvention

The heuristic of guided reinvention was used to guide the planning of the HLT. It is divided into two parts: the route of reinvention, and the role of the teacher.

The route of reinvention

The overarching objective of the project was to map an instructional theory that developed students' computational fluency in multi-digit multiplication. This meant the learning sequence needed to focus on developing students' conceptual and procedural knowledge. The aim was to build students' flexibility with numbers, rather than necessarily having students revert to preferred strategies or influences from prior teaching. A summary of students' expected activity in the reinvention process was constructed based on the review of literature in Chapter 3 . Students' reinvention activity was anticipated to unfold as follows:

1. Students would explore doubling and simple partitioning of numbers, creating more *friendly* computations. This activity would be heavily supported by the array as a tool.
2. Students would use place value as an efficient and effective way to partition numbers, using the array to support their calculations.
3. Students would recognise that some calculations could be solved using efficient mental computation and/or calculations using the associative property and compensation when using the distributive property.
4. Students would perform calculations using their understanding of the multiplicative structure without reliance on the array as a tool, but rather using the array as model for mathematical reasoning.

There were three key aspects to this hypothesised reinvention route: the array; the strategies used and the key developmental understandings (KDUs) (Simon, 2006). The development of the array was previously discussed in this chapter. The strategies and KDUs and implications for the design of the learning sequence are discussed below.

Strategies

While there is significant research on addition with multi-digit numbers, the literature concerning the teaching of multi-digit multiplication is limited (Ambrose et al., 2003). A number of studies reported strategies invented by students (Ambrose et al., 2003; Baek, 2005; Barmby et al., 2009; Fosnot & Dolk, 2001; Izsak, 2004; Murray et al., 1994; Young-Loveridge & Mills, 2008, 2009) and in most cases, the strategies they used were presented in some form of a hierarchical order. Two sources, Fosnot & Dolk (2001) and Ambrose, Baek, & Carpenter (2003), focused on the development of understandings and skills over time. The available literature provided insight into the students' possible reinvention processes. It was anticipated that the most primitive strategies used would be based on counting. These strategies would be followed by repeated addition strategies and partitioning strategies. It was expected that students would mostly partition numbers based on place value. It was predicted that multiplicative doubling strategies would develop from here: for example, repeatedly multiplying by two or halving one number and doubling the other to make the calculation simpler.

The progression of strategies was used as a tool to think about the complexity of strategies and the sophistication of students' mathematical reasoning. Earlier in this chapter, I acknowledged that learning is messy, not linear. It was anticipated that students would move back and forth between stages as their understanding and skills developed. The progression was subject to modification based on the results of the teaching experiment.

Key Developmental Understandings (KDUs)

The strategies used by students reflect an understanding of, or intuition for, essential KDUs. KDUs are "the central, organising ideas of mathematics—principles that define mathematical order" (Fosnot & Dolk, 2001, p. 10) and are critical to the development of understanding of mathematical concepts (Simon, 2006). Simon (2006) argues that learning goals for students should focus on KDUs if teachers are to teach for understanding.

Three KDUs for multi-digit multiplication were identified from the literature:

- Commutative property—changing the order in which terms are given does not affect the result, i.e. $a \times b = b \times a$;
- Distributive property—multiply a sum by multiplying each addend separately and then add the products, i.e. $ab \times c = (a \times c) + (b \times c)$; and
- Associative property—the order in which operations take place does not affect the result, i.e. $(a \times b) \times c = a \times (b \times c)$

In planning the route of reinvention, the KDUs were linked to strategies. It was expected that students using these strategies would have some level of understanding of these key concepts.

It was considered likely that students would have explored the commutative property of multiplication from the early years of school; the distributive and associative properties less so. The decision was made to focus on the associative and distributive properties.

Additive and multiplicative thinking were seen as overarching ideas and ways to group the strategies used by the students. Multiplicative thinking is much broader than just being able to perform multiplication calculations, and it develops over a long period of time (Siemon et al., 2011). It was hoped that the sequence would develop students' abilities to think multiplicatively.

The role of the teacher

Up to this point, the route of reinvention has been described in terms of students' cognitive constructions. An important aspect of the guided reinvention design heuristic, as indicated in the name, is that the reinvention pathway is guided. This was realised in two ways: the careful design of the learning sequence (described later in the chapter) and the role the teacher played in orchestrating the social culture of the classroom (Stephan et al., 2014).

Five practices of the teacher in the guided reinvention classroom are described by Stephan et al. (2014): initiating and maintaining social norms; supporting the development of sociomathematical norms; capitalising on students' imagery to create inscriptions and notations; developing small groups as communities of learners; facilitating genuine mathematical discourse. These five practices were interpreted and implemented through three key practices of the researcher/teacher in this project:

1. Establishing and maintaining classroom social and sociomathematical norms;
2. Capitalising on students' inventions; and
3. Developing a knowledge-building culture in the classroom;

Practice 1—Establishing and maintaining classroom social and sociomathematical norms

In classrooms, there are expectations and responsibilities that determine interactions in the activity of learning. These expectations are the *social norms* of a classroom (Cobb & Yackel, 1995). While the teacher is responsible for initiating and maintaining the social norms, all in the class must participate to make them principles of practice. In the guided reinvention classroom, students need to:

- explain and justify solution methods;
- listen to and make sense of other students' methods and solutions;
- indicate their agreement or disagreement; and
- ask clarifying questions as the class works towards consensus (Cobb & Yackel, 1995; Sowder, Cobb, Yackel, Wood, & Merkel, 1988; Stephan et al., 2014).

These norms play an important role in mathematics, but they are also relevant for meaningful participation in any school subject. As such, social participation in mathematics employs additional, specific norms known as *sociomathematical norms* (Cobb & Yackel, 1995). Stephan et al. (2014) explain that the sociomathematical norms of a classroom articulate acceptable, different, efficient and sophisticated mathematical solutions, focusing on conceptual over procedural reasoning. They state (2014),

> *…in guided reinvention classrooms, students' explanations are acceptable if they meet the criterion that they describe the students' actions on mathematical objects that are experientially real to them. Descriptions of only procedural steps are not counted as acceptable. Descriptions of procedures for finding an answer must be accompanied by the reasons for the calculations as well as what these calculations and their results mean in terms of the problem* (p. 43).

These social and sociomathematical norms were part of the interpretative framework which was used to make sense of the happenings in the classroom. In my role as a visiting teacher/researcher to the schools, I spent time with the class prior to the teaching experiment in order to start negotiating the desired norms. I played an active role in maintaining the norms through the teaching episodes. Class discussions emphasised conceptual reasoning over procedural explanations and the students were encouraged to ask questions and indicate their own thinking. As the teacher/researcher, I stimulated class discussions using two key strategies:

- Encourage listening and sense-making—When students presented solutions, they were encouraged to speak to the class and not to the teacher. After students presented their solutions, two or three students were selected at random to put the solution method into their own words. Students were allowed to pass to another student if they did not feel confident to do so.
- Questioning, agreement, disagreement and consensus—After students presented their solutions and fellow students put the method into their own words, students discussed the solution with a classmate sitting next to them. Students were then allowed to ask questions or make comments to those presenting.

Practice 2—Capitalising on students' inventions

In many mathematics classrooms, pre-established procedures and materials are introduced and explained, and students are expected to use them to answer questions and solve problems. In such a scenario, students have little to no ownership of the procedures or materials, and there is a danger that the procedures hold little meaning. In RME, the teacher's role is not to implement pre-established procedures or materials, rather, the teacher's role is to capitalise on the inventions of the students. The use of experientially real contexts and the development of the model creates scope for powerful imagery. Tools for solving problems, such as strategies, models and notations, are developed through imagery and the students' mathematical activity. Further imagery can be used to develop new or more sophisticated use of tools and strategies (Gravemeijer, 2004).

Involving students in the process of reinvention requires the teacher to harness the inventions of the students as a means of teaching important mathematical concepts and procedures.

In this research, I use the array through the course of the sequence to create rich imagery and capitalise on the opportunities of students' creations. The manner in which the students' creations and imagery could be harnessed were part of the reflective process between myself and the class teacher at the end of each lesson.

Practice 3—Establishing a knowledge-building culture in the classroom

A classroom culture that is focused on collectively building understanding and knowledge requires meaningful interaction between students, whether in participating in class discussions or in small group work. It was decided that class discussions would be used at the end of each lesson to look at the strategies used by students, and to make connections between these strategies and prior learning in the sequence. Discussions would be focused on the students' solutions but would also be used as an instructional opportunity. As the students shared solutions, the researcher/teacher could model correct mathematical practices, such as vocabulary and recording methods. To plan for this, the potential themes that could arise in discussions would need to be considered in the planning of the sequence.

Just as there are classroom social norms, similar norms guide the work of small groups in the classroom. In this research, students worked on tasks with a partner. As such, it was important to identify how students should interact in groups and to establish such interactions amongst the students. Stephan et al. (2014) give five norms to guide small group work:

1. Solutions should be personally meaningful;
2. Each person should explain their thinking and reasoning;
3. Group members should listen to, and make sense of, others' explanations;
4. Groups should work together with a productive disposition; and

5. Groups should work cooperatively on tasks to completion and reach consensus through questioning.

The heuristics of RME provided the groundwork for designing the learning trajectory. The next part of the chapter describes the development of the learning trajectory, stating the learning goals for students and then presenting the learning progression where the tasks of the trajectory are also described.

Learning Goals

The primary concern in devising learning goals for the trajectory was to understand the consequences of earlier instruction and how they may impact upon students' current levels of reasoning (Cobb et al., 2013). To achieve, this an inquiry into curriculum documentation for New South Wales (NSW) schools was conducted, as this was the document in use in the schools involved in the study. To support the findings from this review, a one-to-one, pre-assessment interview was conducted between the researcher and the individual students involved in the study.

Curriculum documentation

The curriculum document in use at the time of the research was the NSW K-6 Mathematics Syllabus (Board of Studies, 2002), a document that was mandated for use in all schools in NSW. In the primary years, multiplication is taught within the *Multiplication and Division* sub-strand, which is contained within the *Number* strand. The concept of multiplication is first taught in the first year of schooling (Kindergarten) and multiplication continues to be taught as a distinct area of study through to the final year of primary school (Year 6). Students' work with multiplication in the primary years is almost entirely limited to whole numbers, though there is reference to multiplying whole numbers by a fraction for Year 6 students. Beyond the primary years, all the operations are extended into work with integers, fractions, decimals and percentages.

The exploration of the multiplicative structure is the focus of study in the first three years of schooling. This is achieved through addressing three important understandings. The first is that of equal grouping, with students working across the number strand using concrete materials to form equal-sized groups, skip count and work with repeated addition. Second, the key developmental understanding of the commutative property is referenced in Year Two where students are introduced to “modelling the commutative property of multiplication” (Board of Studies NSW, 2002, p. 53). Finally, the array structure is introduced, with a focus on modelling the array with concrete materials and building an understanding of rows and columns.

In the middle years of primary school, Years 3 and 4, the syllabus emphasises the use of mental strategies and informal recording methods (Board of Studies 2002). In these years, the focus is on building understanding and recall of single-digit multiplication and how these facts can be used to derive unknown facts. Derived strategies include partitioning numbers using known facts, place value, factorising and doubling. These strategies are grounded in an understanding of the associative and distributive properties, though neither of these multiplicative properties are referenced in the syllabus until the secondary years of schooling.

The formal algorithm for multiplication is not introduced until Year 5 in NSW schools. The teachers of the classes involved in the study had not taught the multiplication algorithm prior to the study, although, conversations with classroom teachers indicated that some students had been introduced to the algorithm by tutors or parents.

The NSW Syllabus focused instruction on just two forms of multiplication: equal groups (including the array) and rate. There was no mention of proportion or Cartesian product in the primary years syllabus, both of which draw on different key developmental understandings. Consequently, it was decided that the instructional sequence of the project would focus on equal groups and rate forms of multiplication.

Pre-assessment interview

A pre-assessment interview was conducted between the researcher and students involved in the study (Appendix 1). It focused on assessing students' fluency with single-digit multiplication, understanding of the array, understanding of the distributive property and strategies students may already have for multi-digit multiplication. The results of the interview for both iterations shown in Table 5.1.

	Question number and the number of students who answered correctly								
	1a	**1b**	**1c**	**1d**	**1e**	**2**	**3**	**4**	**5**
Iteration 1 *21 students interviewed*	21	20	21	19	19	15	15	21	4
Iteration 2 *20 students interviewed*	20	20	20	19	19	17	14	19	6

Table 5.3—Pre-assessment interview results

Both groups of students demonstrated a good understanding of the array structure and single-digit multiplication. If students were unable to recall a multiplication fact, they predominantly used efficient derived strategies to come to a solution. An example of a derived strategy that was used by multiple students was 6 x 9 = (10 x 6) - 6. This strategy demonstrated an understanding of the commutative property of single-digit multiplication and use of compensation with the distributive property. Of the forty-three students interviewed, only eight reverted to skip counting strategies to help them solve unknown problems.

The students in the first iteration of the experiment did not use the distributive property as much as those in the second iteration. In the second iteration, nearly all students used the distributive property to solve 19 x 7, by partitioning 19 into its place value parts. The question 19 x 7 also raised some concern about a few students' grasp of place value concepts. Four students across both iterations partitioned 19 incorrectly and attempted to solve the problem as (1 x 7) + (9 x 7). Another student used (100 x 7) + (9 x 7) as their solution strategy.

Very few students were able to solve the final question which asked them to calculate how many cupcakes there would be if 35 trays with 24 cakes in each tray were baked. Nearly all the students were aware that the two numbers needed to be multiplied but were unsure how to do this. Ten students out of twenty-two in the second group attempted to partition the numbers into place value parts but failed to find the correct products of each part. Only ten students out of the forty-three total used the formal algorithm to solve multi-digit problems.

From these results, students appeared to have a good grasp of the multiplicative structure, as they had skills to solve single-digit questions including strategies that drew on an understanding of the distributive property.

Specific learning goals

Based on the curriculum review and the results from the pre-assessment interview, one overarching learning goal was established for the sequence: to develop students' *understanding of* and *fluency in* multi-digit multiplication.

This overarching goal was supported by three more focused goals:

1. Students will gain an understanding of the distributive and associative properties and apply this understanding to help them solve mathematical problems involving multi-digit multiplication;
2. Students' use of the array will move from being a model of a context-bound situation through to a model for more sophisticated mathematical reasoning; and
3. Students will build a repertoire of strategies to solve multi-digit multiplication problems and apply these strategies appropriately, based on the numbers used in the questions

These learning goals were reflective of the project's research questions, as outlined in the fourth chapter.

The Hypothetical Learning Trajectory

This section describes the instructional sequence that was constructed as part of the preparatory thought experiments. The sequence is presented in two forms: a summary of the sequence (Table 2) and an elaboration of the sequence.

The summary table is divided into five key sections: the teaching episode, context and imagery, the mathematical practices, the development of the array as a model and finally the mathematical discourse topics. These sections were identified by Gravemeijer, Bowers and Stephan (2003a) as important in their development of a learning trajectory for early measurement and flexible arithmetic. These sections were also identified as significant in the guided reinvention process.

The imagery in the sequence was to be built through the bakery narrative, presented at the start of each teaching episode. The model of the array was intended to emerge through the narrative and to progressively abstract the array as the sequence progressed. It was anticipated that some common mathematical practices would emerge through the students' problem solving, which would raise topics and themes for class discussions and discourse between students.

The sequence was divided into five teaching episodes. Each episode explored the development of a mathematical concept or idea through multiple tasks and, as such, was conducted across two or three mathematics lessons of one hour. Lessons were conducted on consecutive days.

Each teaching episode followed a similar structure.

1. Each episode commenced with a task that required students to shift their mathematical thinking and reasoning (Stephan & Akyuz, 2012). These tasks were presented as part of the cupcake bakery narrative.
2. Groups of two to three students worked together to solve the problem and construct an explanation of their solution strategy on a sheet of A3 paper, referred to as a solution poster. This poster was to show clearly the solution method used, any invented strategies used and how they worked, and the answer obtained. It was expected that

someone reading the poster would be able to make sense of the solution method without needing a verbal explanation from the authors. Each poster was displayed in the classroom allowing students to view each other's work and make sense of the different strategies used in the classroom.

3. A class discussion was conducted where students presented their strategies to the class. Students were selected to present based on the strategies that they had used. The teacher/researcher then mediated a discussion with the students on particular themes that emerged through the problem-solving process.
4. *Number strings* were used to look more closely at some of the different strategies employed by students in the classroom. Number strings are a sequence, or string, of calculations that are carefully crafted to assist students in constructing key understandings and building strategies in a mathematical domain (Fosnot & Dolk, 2001).
5. Similar problems were now posed to the students, which used the same context but varied the numbers.
6. A final class discussion was conducted to discuss themes that emerged through the episode and to review and reflect on new learning.

It was anticipated that the proposed instructional sequence would unfold as shown in Table 5.2 on the following page.

Instructional Episode	Context &Imagery	Mathematical Practices	Development of the Model	Mathematical Discourse
Episode 1: 2- x 1-digit multiplication Calculate the total number of cakes	A cupcake baking tray with 12 cakes in a 6x4 array 8 cupcake trays in a 4x2 array	The array can be partitioned into smaller, more 'friendly' parts - distributive property	8 baking trays set out in an array with 12 cupcakes in each tray as an array Array partitioned by the trays All parts of the array can be seen	Partitioning of the array Links between the strategies that were used Efficiency of strategies
Episode 2: 2- x 2-digit multiplication Calculate the total number of cakes packed in boxes	An order of 16 boxes with 12 cakes in each box	The array can be manipulated e.g. halve and double – associative property	16 cake boxes arranged in a 4x4 array with 12 cakes in each box Abstracting of the array – region model (cakes not visible)	Why is the value of 12x16 the same as 24x8? Double and halving and when this is an efficient strategy
Episode 3: 2- x 2-digit multiplication Calculate the number of squares on a cooling rack	A rectangular cooling rack with 28 rows of 18 squares	Partitioning based on place value Compensation strategies can be used	Cakes on the cooling rack hiding most of the squares Region model of the array	Efficiency of strategies
Episode 4: - 2- x 2-digit multiplication Calculate the area of two trays	Comparing the area of two trays to decide which was the most efficient use of space	All parts of the array must be added together when it is partitioned into smaller parts Partitioning based on place value is efficient and effective	Array completely abstracted - open array introduced Students partition array flexibly	Partitioning based on place value properties
Episode 5: 3- x2- digit multiplication Calculate the money raised through cake orders	Trays and boxes of cakes	Students apply understandings of 2-x2-digit multiplication to solve 3-x2-digit	Questioning not directly linked with the array - a model for mathematical reasoning	Strategies used for 2-x2-digit can be used for larger numbers

Table 5.2—The hypothetical learning trajectory

Episode 1

Episode 1 was planned to begin with the following narrative:

Charlie the baker has his own cupcake shop. Each day he makes and sells eight different flavours of cakes. The cakes are baked in a tray that has four rows with six cakes in each row. He bakes one tray of each flavour. How many cakes does he bake each day?

Students were shown a picture of a baking tray and an image of the cakes that are baked (see Figure 5.2).

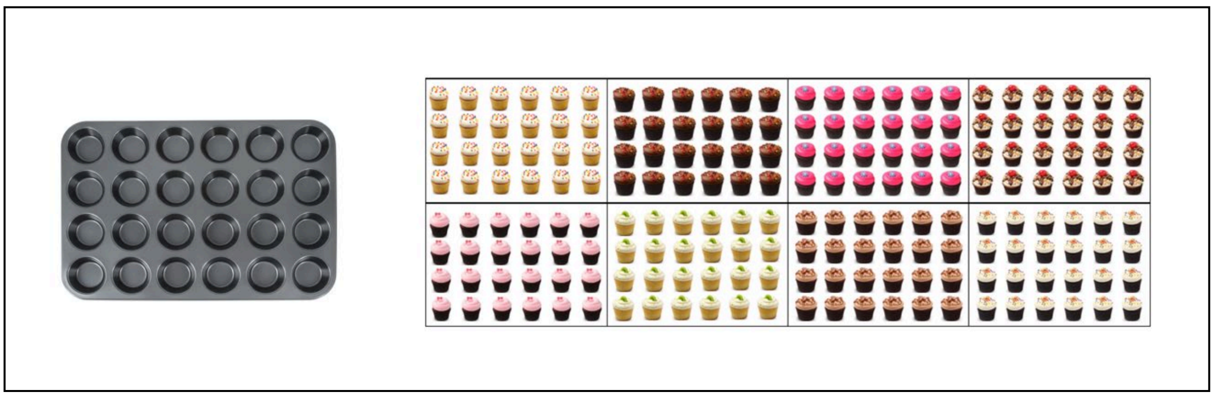


Figure 5.2—Episode 1 images

The intention of this first episode was to present students with a 2- by 1-digit multiplication problem to build on their understanding of single-digit multiplication. It was anticipated that the students would draw on a variety of strategies, with some students reverting back to additive strategies such as skip counting and repeated addition. It was also predicted that students would demonstrate an emerging understanding and use of the distributive property of multiplication, breaking the array apart into smaller, known parts, facilitated by the presentation of the array as eight separate trays.

Another important aspect of this phase of the sequence was the introduction of the array as a model of the multiplicative structure. It was conjectured that the array would be a critical tool used by students in solving the problem and reasoning mathematically. The array would be introduced through the narrative of the bakery with cupcakes arranged in equal rows and columns. All parts of the array would be visible, allowing counting strategies through to partitioning strategies. The visual nature of the array would help students develop the understanding that the array could be partitioned into smaller, more *friendly* parts and that adding these parts together would give the total number in the collection.

Episode 2

The narrative for Episode 2 built on Episode 1:

Charlie the baker received a large order of cupcakes. A local school has asked for 16 boxes of cakes for their annual fete. Each of the boxes holds 12 cakes. He has made all the cakes and the boxes are sitting in the shop ready to be delivered. How many cakes are there in the order?

Students were to be shown a physical cupcake box that held 12 cakes and a picture of 16 boxes in a 4 x 4 array (see Figure 5.3).

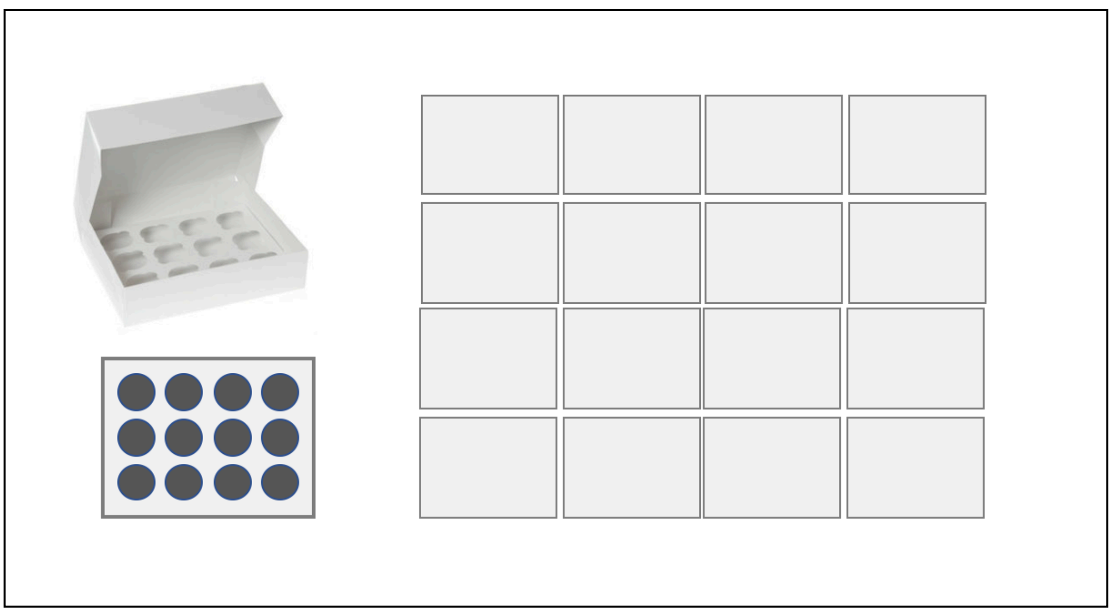


Figure 5.3—Episode 2 images

This section of the sequence introduces 2- x 2-digit multiplication, with the array again presented through the introductory story. The array is somewhat abstracted as the lids of the boxes are closed, obscuring the individual items in the array. This moves to use of a partially abstracted array. The question was designed in this manner to encourage strategies beyond skip counting, specifically partitioning strategies built on an understanding of the distributive property of multiplication.

It was believed that the discourse for this episode would revolve around the fact that this question has the same answer as the question in the first episode. Part of the problem solving for this episode is to understand the relationship between 16 x 12 and 24 x 8. Ideally, students would see that the boxes can be rearranged into an array that is 8 x 2, thus arranging the cakes into a 24 x 8 array (see Figure 5.4). This process of halving and doubling demonstrates that the array can be partitioned and manipulated, and thus, introducing the associative property of multiplication.

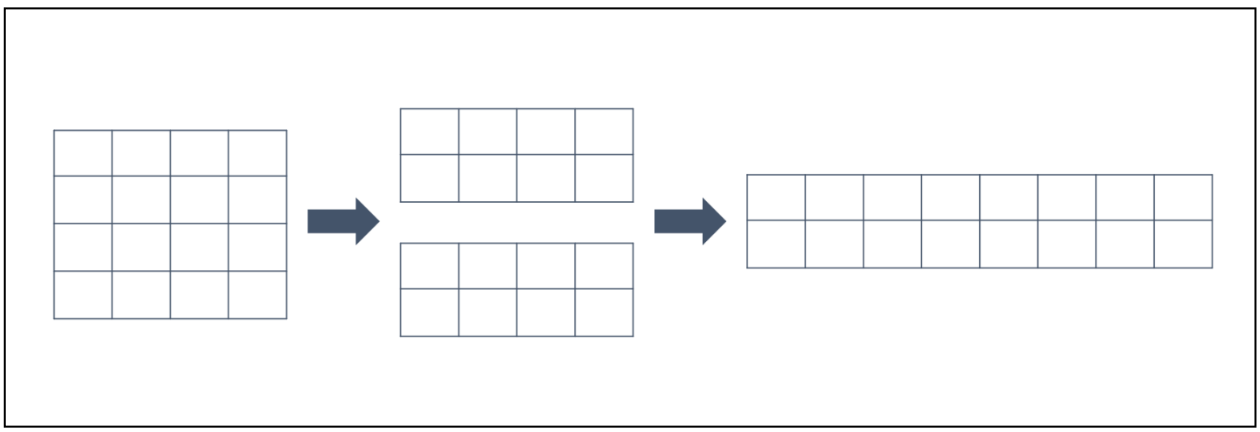


Figure 5.4—Rearrangement of the cake boxes

Episode 3

The narrative to initiate activity at the start of Episode 3 again built on from the previous episode:

When the cupcakes were cooked, Charlie would lay them out on a wire rack to cool. Each tray of cupcakes would fit onto one wire rack, in an array of 4 by 6. Charlie would take the cakes one-by-one and *ice them. One day, as he was doing this, he wondered how many little squares were on the wire rack. Can you answer Charlie's question: how many squares are on the wire rack?* (See Figure 5.5.)

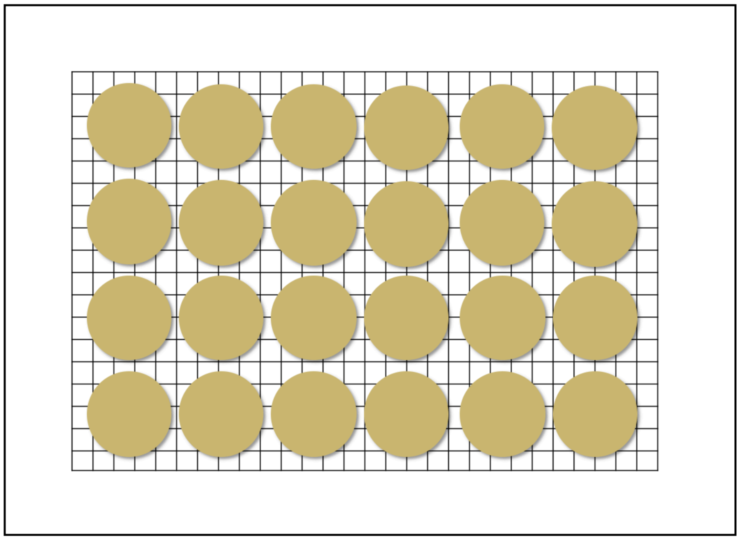


Figure 5.5—Episode 3 image

The main purpose of this task was to allow students the opportunity to explore different partitioning strategies. Before calculating the total, the class would work together to reach consensus that there were 18 squares in each column and 28 squares in each row. As the array is mostly covered, it is not possible to count the number of squares.

It was anticipated that students would partition the array predominantly based on place value. There would also be opportunity for students to use compensation strategies. For example, students could add two columns, multiply 30 x 18 and then subtract 2 x 18, or add two rows, multiply 28 x 20 and then subtract 28 x 2.

Episode 4

The planned introductory narrative for this section of the sequence was as follows:

Charlie the baker receives an order of boxes that hold 12 cupcakes. The boxes are a different design from the previous boxes he used. These boxes have the cakes in a 'skewed array'. He wonders why this might be.

This episode was planned to begin with discussion on the different arrangements of cakes in the boxes (see Figure 5.6), to introduce the main investigation for the episode: does the skewed array take up less space than the array arranged in clear rows and columns? The dimensions of the two boxes would then be given to the students.

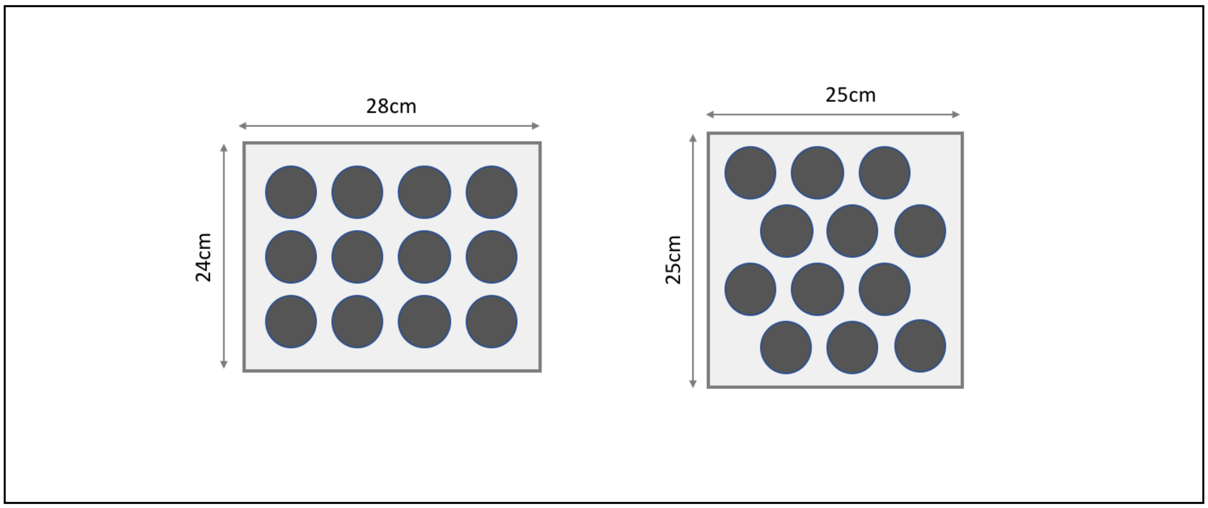


Figure 5.6—Episode 4 image

In this context, the array is fully abstracted to an open array. The students are being asked to calculate the area of two rectangles. This task was designed to address two key misconceptions, or errors, common among students that were raised throughout the literature and discussed in Chapter 3: errors associated with place value and errors associated with a partial distribution of the array. The task was designed so that students would grapple with distribution and place value concepts as they worked.

Episode 5

The final episode was to be presented with the following narrative:

To fill all the orders over the course of one month, Charlie had to bake 64 trays of cakes. There were 24 cakes in a tray. He packed the cakes into boxes of 12. Each box was sold for $28. How many cakes did Charlie bake and how much money did he get from all his cake sales?

The final episode was designed for students to work with the array as a tool for mathematical reasoning. The array was not part of the narrative, but students could use it to help answer the questions if needed.

The questions could be answered by partitioning, but they also provided the opportunity to explore other possibilities, like halving and doubling, and perhaps coming to the realisation that 64 x 24 = 132 x 12. While all questions to this point had been focused on the number of items in an array or area, this question introduced a rate problem as students were asked to calculate the cost of 132 boxes sold for $28 a box. It was anticipated that students would draw on strategies and skills that they had developed over the course of the sequence.

This section has outlined the proposed sequence for use in the teaching experiment. Modifications would be made to the design of the sequence during the course of the teaching experiments. Reflection at the end of each lesson would inform any changes that were required. A key consideration would be whether the sequence allowed students to reinvent the mathematics for themselves.

Conclusion

This chapter has looked at the design of the learning trajectory in the preparatory thought experiments. These thought experiments were the first phase of the research design. The interpretation of the learning trajectory used in the research was discussed as well as the way the specific design heuristics of RME informed and influenced the design of the learning experiences and the classroom culture. The focus of the chapter then shifted to the actual process of design. The goals for students' learning were identified and finally, the tasks and anticipated progression of students' reasoning were described.

The next chapter explores the second phase of the research design: the teaching experiment. In this phase, the learning trajectory was implemented in the classroom. The chapter explores the results of each teaching episode and cyclical process of design and experimentation.

CHAPTER 6

THE TEACHING EXPERIMENT

Introduction

This chapter reports the results of the teaching experiment phase of the research project, where the hypothetical learning trajectory on multi-digit multiplication was implemented across two classes. It identifies four socially constructed mathematical practices that emerged over the course of the instructional sequence and five mathematical norms that played an important role in the negotiation of the mathematical practices. The chapter documents the cognitive shifts made by individual students, termed *mathematical constructions*. These constructions are made visible through students' participation in, and contributions to, the mathematical practices.

Structure of results

The teaching experiment phase of the project was implemented as a two-week instructional sequence in two different classes. The instructional sequence comprised of four teaching episodes. Each was characterised by a focus on a distinct concept and spanned two or three one-hour lessons. Four mathematical practices were identified over the course of the instructional sequence. Mathematical practices were defined as taken-as-shared ways of reasoning and arguing mathematical (Stephan & Rasmussen, 2002). The four mathematical practices and their associated teaching episodes are used to structure this chapter.

In this chapter an account of each teaching episode is provided. Each account commences with a short narrative excerpt from the teaching episode, then discusses the mathematical practice that emerged and the associated constructions of the students. The emergence and acceptance of the mathematical practices were linked to what can be termed *conceptual events* rather than specific actions. Events were considered *conceptual* when a shift in the collective reasoning of the class was observed. For example, in the class discussion in Episode 1, students recognised a connection between strategies based on the distributive property of multiplication. Recognising this connection allowed the class to form a deeper understanding of the structure of multiplication and improved efficiency and sophistication of strategies. Conceptual events were considered significant when they occurred in both iterations of the experiment.

The mathematical practices that emerged in both iterations of the teaching experiments were similar. For each mathematical practice, the sequence of events of one class are used to illustrate the conceptual events in both classes. *Class 1* refers to the group of students in the first teaching experiment and *Class 2* refers to the second.

The structure of the instructional sequence is illustrated in Figure 6.1.

Teaching Episode 1 3 lessons	**Teaching Episode 2** 3 lessons	**Teaching Episode 3** 2 lessons	**Teaching Episode 4** 2 lessons
Mathematical Practice 1 **THE ARRAY AS A TOOL FOR SENSE-MAKING** Partitioning the array	***Mathematical Practice 2*** **THE ARRAY AS A TOOL FOR SENSE-MAKING** Manipulating the array	***Mathematical Practice 3*** **WAYS OF WORKING MATHEMATICALLY** Thinking multiplicatively	***Mathematical Practice 4*** **WAYS OF WORKING MATHEMATICALLY** Looking for 'friendly' numbers

Figure 6.1—The mathematical practices that emerged through the instructional sequence

Mathematical Practice 1 and 2 (in Figure 6.1) were classified as centred around the way in which students used the array to make sense of computations and to reason mathematically. As such, these practices have been grouped under the heading *The array as a tool for sense-making*. The specific title given to each practice clarified how the array was used as a mathematical tool. Mathematical Practices 3 and 4 were classified as *Ways of working mathematically*. As with the practices relating to the array, specific titles were given.

Each section of this chapter concludes with a description of a single student's constructions and reasoning. These descriptions are not case studies, as individual students were not tracked closely through the course of the experiment. Rather, they serve as illustrative examples of the cognitive constructions observed in the classes, and how the students' constructions influenced the negotiation of mathematical practices.

Mathematical Practice 1 - The array as a tool for sense-making: Partitioning based on place value

Teaching Episode 1—Charlie the Baker

The first teaching episode comprised three lessons, each of one hour in duration over three days (see Figure 6.2).

TEACHING EPISODE 1		
Lesson 1	**Lesson 2**	**Lesson 3**
Solving the problem - How many cakes were baked? Gallery walk	Class discussion – Similarities and differences in the strategies used Students comparing strategies used	Number strings based on the different strategies used Additional problems to solve
Mathematical Practice 1 **- THE ARRAY AS A TOOL FOR SENSE-MAKING:** Partitioning the array		

Figure 6.2—The first teaching episode

Episode 1: Lesson 1

Lesson 1 introduced the students to the context of a bakery that sold cupcakes. Students were presented with the following narrative and the associated pictures (Figure 6.3):

Charlie the baker has his own cupcake shop. Each day he makes and sells eight different flavours of cakes. The cakes are baked in a tray that has four rows with six cakes in each row. He bakes one tray of each flavour. How many cakes does he bake each day?

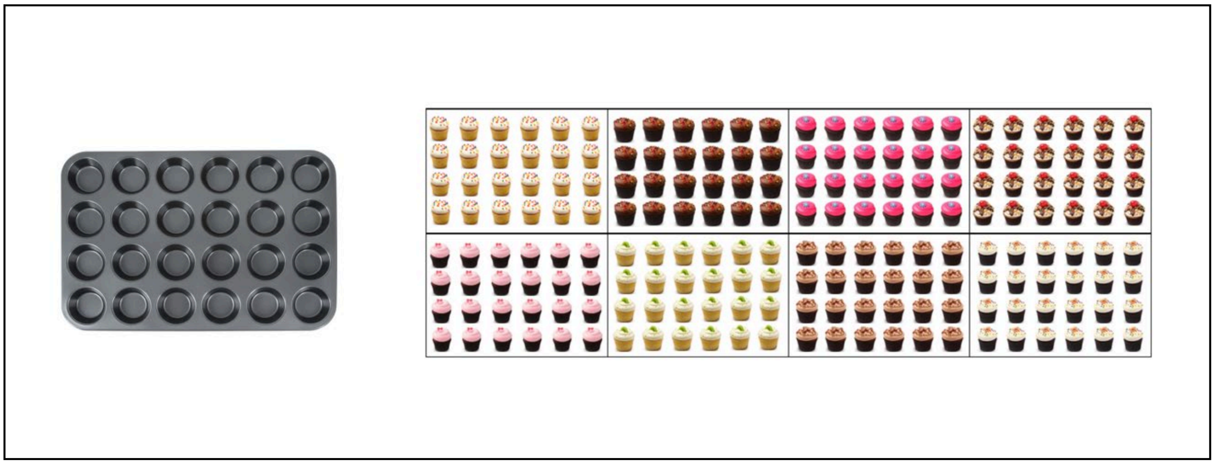


Figure 6.3—Pictures used in the narrative for Episode 1

The students worked in pairs to construct a solution to the problem. In accordance with the established classroom social and sociomathematical norms, students needed to give an answer to the question and, more importantly, an explanation of how they obtained their answer. Each pair was provided with a blank sheet of A3 paper and asked to create a poster of their solution. The poster needed to 'stand on its own': to contain sufficient information about the strategy used, so that others could make sense of the working without requiring further explanation. To assist in this process, students had access to a collection of resources including counters, grid paper, Multi-Attribute Blocks (MAB), interlocking cubes and a picture of the cakes as shown in Figure 6.3. Students were not required to use these resources but could choose to if they deemed them helpful.

All but two pairs of students in Class 1, and all of Class 2, chose to use the picture of the cupcakes (Figure 6.3) on their poster. Across both classes, students either partitioned the array by cutting or drawing lines, or they grouped parts of the array using circles to explain and justify their strategy. The strategies used by the students were categorised into five groups.

Group 1—Skip counting

One student, Jake (Class 1), used skip counting to solve the problem. Jake performed below his peers in mathematics and he struggled to understand others' reasoning. His request to work on his own was allowed. Jake noted that each tray of cakes contained four rows of 5 cakes and one remaining column of 4 cakes. He circled the four rows of 5 and the one column of 4 on the first tray. He then circled the column of 4 on each remaining tray to segregate these cakes from his initial count. Jake proceeded to skip count the groups of 5 on each tray. Finally, he counted the columns of 4s in ones, although his recording indicated that he had added 4 onto the total eight times (as shown in Figure 6.4). Jake's work is discussed in more detail later in this section.

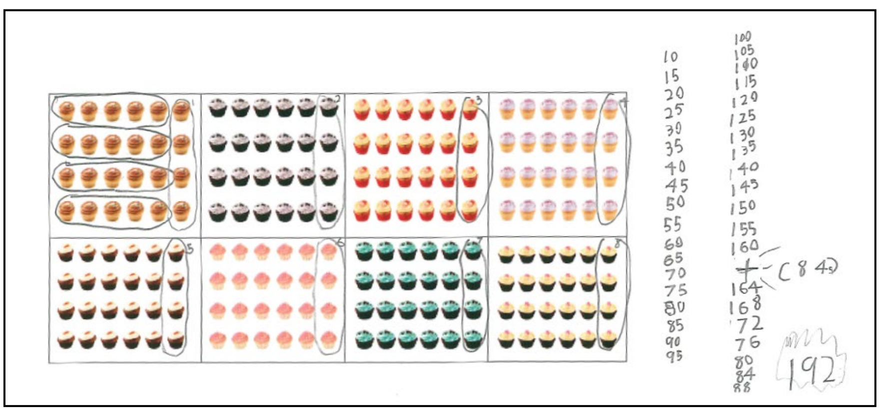


Figure 6.4—Jake's work sample from Teaching Episode 1

Group 2—Repeated addition

Across both classes, four pairs of students repeatedly added 24 to find the total. Three of these pairs used a formal addition algorithm. The fourth pair (Mitchell and Jack of Class 1) verbally indicated that they added 20 eight times and then added 4 eight times (see Figure 6.5).

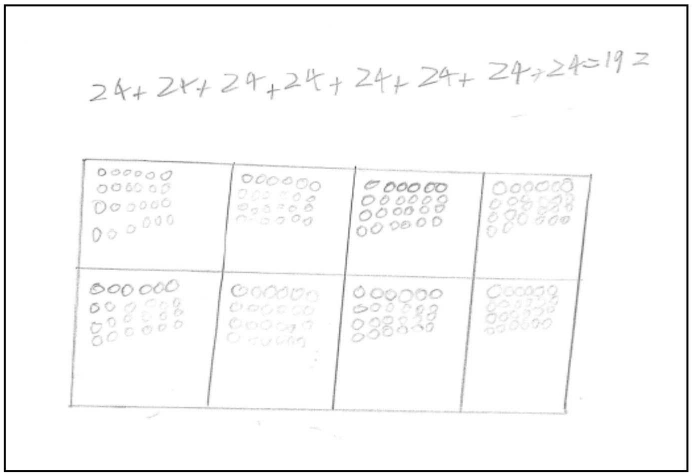


Figure 6.5—Mitchell and Jack's work sample from Teaching Episode 1

Group 3—Repeated doubling

Four pairs of students in Class 1 and one pair from Class 2 used repeated doubling. Two distinct forms of doubling were noted. Ashley and James (Class 1) named their approach the *ice-cream strategy* based on its resemblance to an ice-cream cone (see Figure 6.6). They repeatedly doubled 24, then 48 and finally 96. The other groups of students doubled 24 three times and indicated how this worked using the array (see, for example, Figure 6.7).

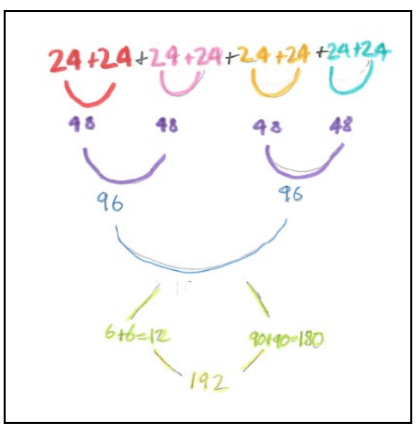


Figure 6.6—Ashley and James's work sample from Teaching Episode 1

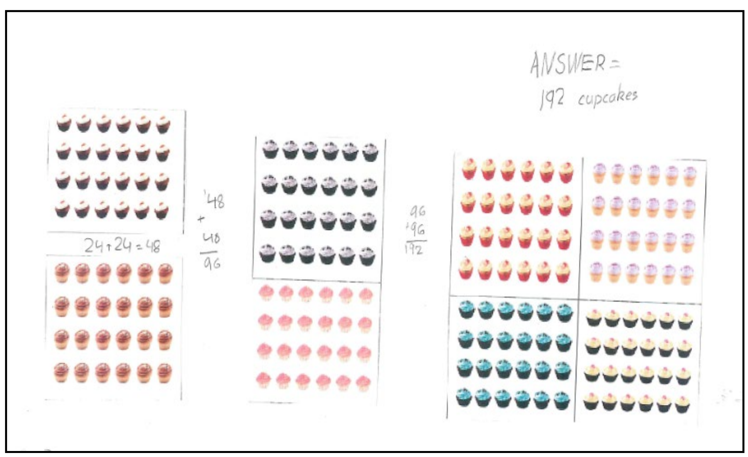


Figure 6.7—Jade, Joy and Olivia's work sample from Teaching Episode 1

Group 4—Using place value

The strategy of distributing the array based on place value was used in two different ways. The first method is illustrated by Zoe and Lucille (Class 1) who subdivided the large array and worked with the smaller eight trays of cakes (see Figure 6.8). They divided these trays into 20 and 4, then multiplied 20 and 4 by 8. The second method, illustrated by Layla and Maddie (Class 2), partitioned the large array into one group of 20 by 8 and one group of 4 by 8 (see Figure 6.9).

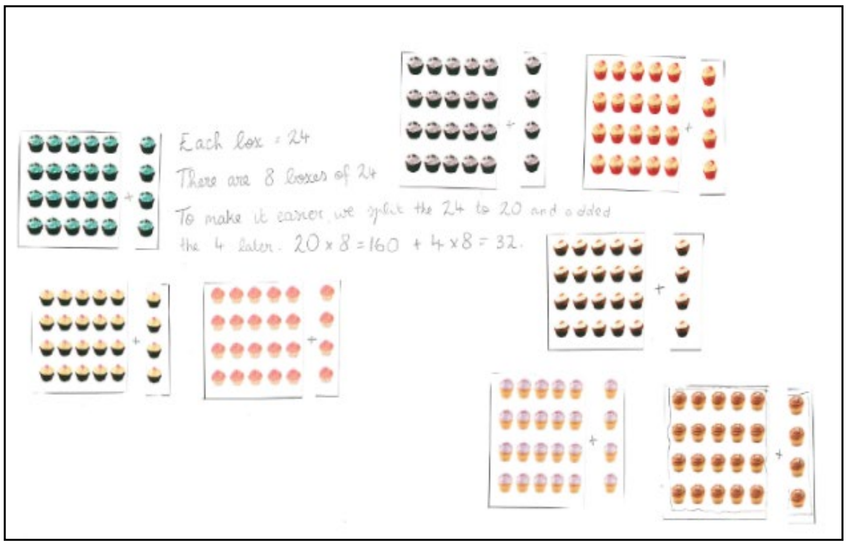


Figure 6.8—Zoe and Lucille's work sample from Teaching Episode 1

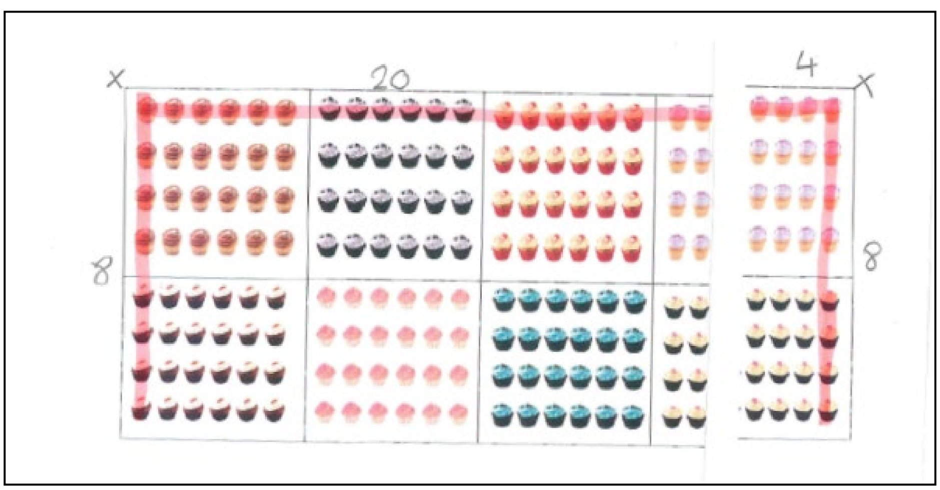


Figure 6.9—Layla and Maddie's work sample from Teaching Episode 1

__Group 5—Multiplicative Compensation__

Ryan and Dylan (Class 1) used a strategy termed *multiplicative compensation*. Their strategy involved multiplying 25 by 8, then subtracting 8. They were able to illustrate how the strategy worked using the array (see Figure 6.10).

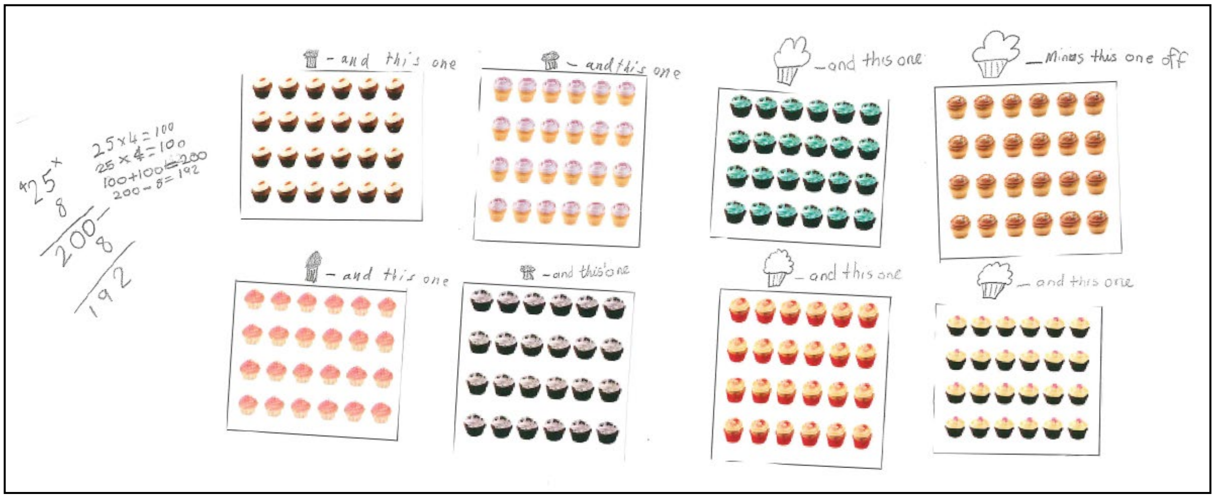


Figure 6.10—Ryan and Dylan's work sample from Teaching Episode 1

Lesson 1 concluded with a *Gallery Walk*: students were invited to proceed individually around the classroom, examining and making sense of other students' work, asking questions and leaving comments via sticky notes attached to the posters. The posters were then transferred to the classroom wall for the remainder of the school day, allowing students more time to consider the working of their classmates.

The classroom teacher and I then conferred and agreed that Lesson 2 would commence with a whole-class discussion on the different partitioning and doubling strategies that had been utilised in Lesson 1.

Episode 1: Lesson 2

The second lesson commenced with a whole class discussion. Selected students presented their work and the class was asked to consider similarities and differences between their strategies. Following this, the students were divided into small groups, and examined the presented strategies more closely with a view to refining their own strategies. As the teacher, I was deliberate not to promote particular strategies as being 'better' than others. I encouraged students to explore connections between solutions and to consider the efficiency and sophistication of strategies, in keeping with the classroom sociomathematical norms.

Episode 1: Lesson 3

The final lesson of the first teaching episode commenced with a presentation of two number strings to the students. A number string is a 'sequence of calculations that are carefully crafted to assist students in constructing key understandings and building strategies in a mathematical domain' (Fosnot & Dolk, 2001). The first number string was focused on using place value partitioning to solve multi-digit multiplications and the second focused on doubling as a strategy (Appendix 3). Students were subsequently presented with a further collection of multiplication problems, set in the original bakery context, which are documented later in this section. I provided students with corresponding arrays for each problem. Students were given the choice of whether to use these new arrays.

The Emergence of Mathematical Practice 1

The following paragraphs describe the emergence of the first mathematical practice: the array as a tool for sense-making using partitions based on place value. First, it is established that the mathematical practice was not already taken-as-shared in the classes. Three classroom events which were considered significant for this practice becoming taken-as-shared are then described. Excerpts from class discussions are used to illustrate how the mathematical practice of partitioning based on place value emerged in the classes and became taken-as-shared by the students. Finally, the journey of one student is described, demonstrating how his personal shifts in understanding and reasoning contributed to the development of the mathematical practice.

Not yet taken-as-shared

To state that a mathematical practice *emerged* in a teaching episode infers that the practice was not already taken-as-shared in the classroom. Prior to the teaching episode, understandings and usage of the array differed from one student to another. These differences in usage were classified into two general categories that were observed in both classes.

Students in the first category saw the array as a whole and used the structure of coordinated rows and columns as a method of partitioning the array. Drawing on the distributive property of multiplication, these students tended to sub-divide complete rows and/or columns to create smaller, more manageable parts.

Students in the second category focused on the individual parts of the array. Rather than using the opportunities afforded by the array's structure, these students identified and worked with countable parts or parts that could be easily added together. It can be assumed that students in this category did not fully appreciate the structure of the array, based on two key observations. First, a difference was noted in the way that these students divided the array. Rather than separating a complete row or column, they tended to remove a section of a row or column, as illustrated in Figure 6.11. The second observation comes from the notations these students made on the array itself. Students in the first category, who worked with the array as a whole, used straight lines or cuts to indicate their partitioning of the array. This second category of students, who saw the array as individual items, generally (although not exclusively) used circles to indicate small groups in the array. While students in the first category used straight line partitions to make smaller parts that were then multiplied, conversation with students in the second category revealed that these circles were their way of keeping track of their count. This is illustrated in Jake's earlier work sample (Figure 6.4).

Figure 6.11—Not separating a full row or column

While each class appeared to have a taken-as-shared view of the array as a tool for reasoning, the varying use of the array demonstrated that neither classroom had a shared view of the how the array and place value properties might be best utilised. The strategies used by students suggested that they understood how to utilise place value to aid computation, but an appreciation for how place value parts could be connected to the structure of the array was not evident in their working. This was somewhat surprising given the pre-assessment interview indicated that students were familiar with the array. For this reason, the whole-class discussions at the start of the second lesson focused on comparing the alternative uses of the array.

The following section of the chapter describes the *conceptual events* that were linked to the emergence and acceptance of the first mathematical practice. As previously described, events were considered *conceptual* when a shift in the collective reasoning of the class was observed in both iterations of the teaching experiment.

Conceptual Event 1—Using complete rows and columns to partition the array

As previously stated, the class discussion at the start of the second lesson was centred around alternative uses of the array. The purpose of the discussion was to examine similarities and differences between student strategies, and for individuals to use these observations as a catalyst for modifying and refining their own strategies. The discussion commenced with four students sharing their partitioning strategies with the class: Zoe and Lucille (Figure 6.8), Jake (Figure 6.4) and finally Jasper (Figure 6.12), who had also worked on his own. First, Zoe and Lucille explained their strategy of partitioning the smaller trays into eight groups of 20 and 4. They showed that the structure of the array allowed for a straight cut to create a group of 20 and a group of 4.

Zoe: *We cut it down here (pointing to an array on their poster) to make a group of 20 and a group of 4.*

Teacher: *Why did you use 20 and 4?*

Zoe: *This one is 5 times 4 and this one is 1 times 4. You can do it just by cutting down here (showing the cut made on one of the trays of cakes).*

Lucille: *Yeah, we just cut off one row…also we thought that 20 and 4 would be easy numbers…they would be easy for us to use.*

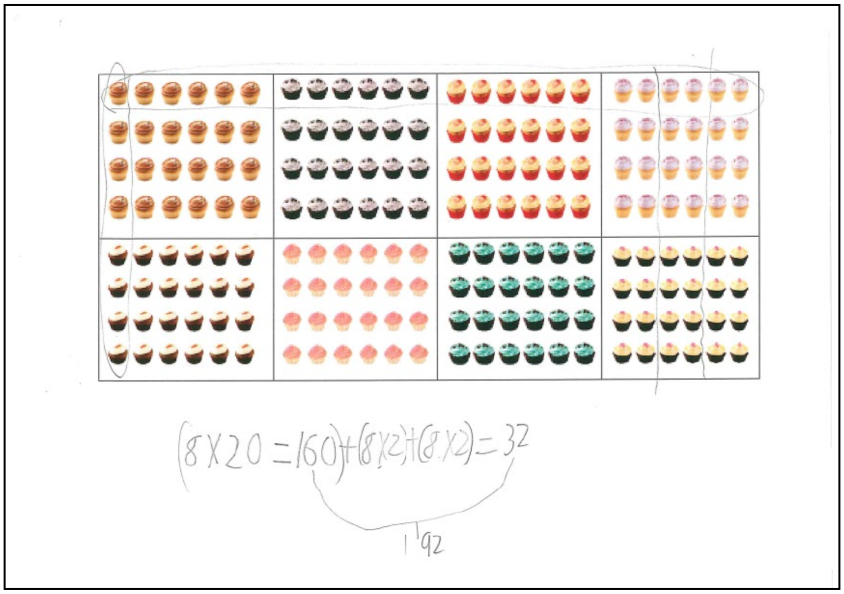


Figure 6.12—Jasper's work sample from Teaching Episode 1

The students were asked to consider if it was mathematically sound to 'just cut off one row', or whether splitting the array like this would change the end result. This was used as an opportunity to reiterate the sociomathematical norms introduced into the classroom. As the teacher, I indicated that consensus from the class was required before moving on. All members of the class needed to indicate their agreement or disagreement and, if necessary, ask clarifying questions. Students reasoned that all the cakes were still being considered in the total and so it was mathematically reasonable to partition the array in such a fashion, or as one student put it: 'It's not like you are losing any of the cakes or anything'. A number of students indicated they had used a similar strategy. Consensus across the class indicated that partitioning the array in such a fashion was a mathematically sound way of working with the array.

A further two strategies were shared with the class. Jake explained his strategy of skip counting fives and then adding on the remaining fours. Jasper explained how he used the larger array to partition 20 x 8 and two groups of 2 x 8, as these were multiplication calculations that he could perform mentally.

At this point the students were asked to compare the three strategies presented and to describe how they saw them as similar and different.

The first observation made by the class was the different ways the cakes were grouped: Zoe and Lucille had cut the array, Jasper had drawn lines and Jake had used circles. When the class was asked if they saw a similarity between these three methods, one student commented that Jasper, Zoe and Lucille had all partitioned multiple rows with a straight line or cut, whereas Jake had circled cakes in a single row. Jake agreed and commented that he could have used rows as well.

Teacher: *Do you agree with those comments Jake?*

Jake: *Yes, I think so. Well, I think that I just counted 5 in the first row and then I counted 5 in the next row and then I just kept going and I saw that the groups of 5 just kept going and so I stopped circling them.*

Teacher: *Could you have split your array in a similar way to Zoe, Lucille and Jasper?*

Jake: *Yes…I think …I could have just cut down there and… just cut it like Zoe and Lucille's. I think our ways are sort of the same …more than Jasper's anyway.*

Teacher: *Why is your way more like Zoe and Lucille's?*

Jake: *Well, where I circled is sort of like... it's just like where they cut. They are just the same really.*

Teacher: *Do people agree with Jake? Do you think his strategy is like Zoe and Lucille's?... Frederique?*

Frederique: *Well, I think it is sort of. You can see that they both used the 4s at the end…*

Lucy: *Jake could use 20 too because he has 20 with his 5s.*

Jake's contribution was an important occurrence in establishing the first mathematical practice as taken-as-shared by every member of the class. It was evident that Jake, and those who had used similar reasoning to Jake, now had a greater appreciation of the array as a tool for sense-making as he recognised its structure of coordinated rows and columns. The class realised that Jake could create groups of multiple rows, rather than focusing on just one row at a time, and saw that his multiple rows of 5s were similar to Zoe and Lucille's method of splitting the smaller arrays into four rows of 5 with one column of 4 remaining. The similarity between Jake's strategy and Zoe and Lucille's strategy was accepted and reinforced by other class members who indicated how the partitioning used by Zoe and Lucille was evident in Jake's skip counting.

At this point, I directed the class back to Zoe and Lucille's work and to Jasper's work to look more closely at the similarities and differences. When asked what the main difference was between the two strategies, the class agreed that Jasper had used the large array while Zoe and Lucille had used the smaller arrays and concluded that Jasper's way would be more efficient. When asked how the strategies were similar, they observed that both used 20 x 8 and 4 x 8, although Jasper had partitioned his 4 x 8 into two groups of 2 x 8. I asked the class how this was possible using different arrays.

Teacher: *How could they both use the same calculations?*

Luke: *If you look at Jasper's there is 24 across the top and there is 8* [motioning down the column]*. You can just make 20 by putting in the line and then there is 4 on this side. And then in that one* [pointing at Zoe and Lucille's] *they have 20 and then they have 4 and…if you count…there are 8 of them.*

The class agreed with Luke's explanation. They also agreed that using the larger array, as Jasper did, would be a lot quicker than using all the smaller arrays. Notably, the students continually referenced the array in their reasoning. This indicated that, collectively, the students accepted the array as a useful tool to explain their thinking and to make sense of the thinking of others. Further to this, the array provided a means for students to reason conceptually, not just procedurally, about multiplication. While there was some symbolic recording on students' work, this was not referenced during the discussions. Students' argumentation, even their gesturing, centred on the array.

Conceptual Event 2—The Use of Place value Across Strategies

The class discussion continued on to explore two student strategies focused on doubling. Ashley and James shared their *ice-cream strategy* (Figure 6.6) followed by Tom and Frederique who doubled three times (Figure 6.13). The class also compared how these strategies were similar but different. As part of this comparison, James instigated a discussion which was particularly relevant to the first mathematical practice.

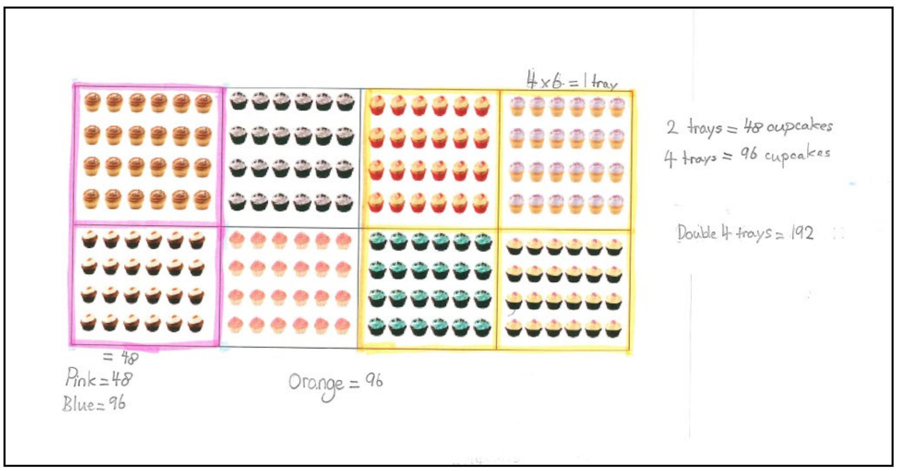


Figure 6.13—Tom and Frederique's work sample from Teaching Episode 1

James: *I think our way is a bit like Zoe and Lucille's and Jasper's.*

Teacher: *What do you mean by that? Can you explain what you mean to the class?*

James: *Well, when I doubled the 24, I thought about 20 and then I did the 4. It's sort of the same.*

Teacher: *What do others think of that? Do you understand what James is saying? Do you think that they are similar?*

Lucille: *I guess it is sort of the same but it sort of isn't as well. You didn't use the array like we did.*

James: *But I think...yes...we could. We could just put them at the top and make 20 and 4 like you did. It would look like yours really, just with the arrays at the top.*

This interaction was significant for two reasons. First, although James and Ashley had not used the array on their poster, James chose to use the array to explain how he added the numbers together and how this linked to other strategies in the class. Second, the interaction highlighted connections between the strategies observed by the students, specifically the use of place value. While James and Ashley had not used an explicitly distributive strategy, James had appreciated the significance of place value in calculating the total.

The use of place value to aid computation became the focus of the class conversation. Students were asked to consider if and how they used place value in their calculations. At this point in the discussion, I used symbolic recording to illustrate the place value connection between Ashley and James' 'ice-cream' strategy and Zoe and Lucille's partitioning strategy, as illustrated in the first part of Figure 6.8. After receiving agreement from Ashley and James and then from Zoe and Lucille that this was in fact representative of their working, I asked the class if they appreciated how both strategies drew on place value. I then added Jasper's strategy in symbolic form, as shown in the second part of Figure 6.12. The students understood the way place value was used and there was consensus that multiplying whole 10s was easily performed mentally.

Conceptual Event 3—A Shift in Students' Strategies

In the third lesson the students were asked to solve additional problems. The questions used the original bakery context but changed the numbers slightly. This allowed students the opportunity to explore new computational methods or refine their existing strategies. For each question, an associated array was provided if the students wanted. The questions were as follows:

1. What if the trays held a different number of cakes?
    a. How many cakes would Charlie bake if he made 8 trays with 16 on each tray?
    b. How many cakes would Charlie bake if he made 8 trays with 32 on each tray?
2. What if he baked a different number of trays?
    a. How many cakes would Charlie bake if he baked 6 trays with 25 on each tray?
    b. How many cakes would Charlie bake if he baked 9 trays with 18 on each tray?

In answering these questions students predominantly used place value parts to aid computation, with a few students using doubling strategies. The array was used by all students to explain conceptually how they solved the problems. A shift in the usage of the array and place value was observed on two fronts. First, the students' strategies indicated an understanding of how the structure of the array could be used to make conceptual sense of strategies and to reason mathematically. Students were also able to link the structure of the array to the place value structure of numbers. They appreciated how the array could be partitioned based on the rows and columns of the array. To illustrate these observations, consider the work of two students, Robyn and Bella (Figure 6.14).

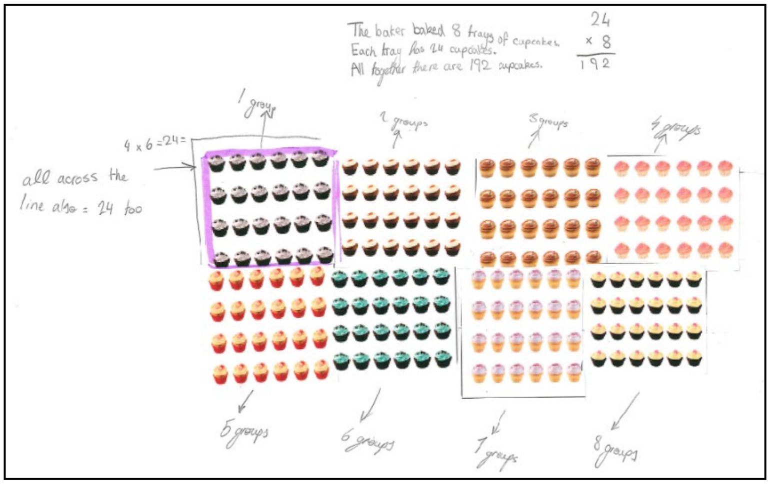


Figure 6.14—Robyn and Bella's work sample from Teaching Episode 1

Robyn and Bella solved the original problem in Lesson 1 using a formal algorithm. They indicated the eight groups of 24 on the array but could only give a procedural explanation for how the algorithm worked. They were unable to explain how the calculations of 20 x 8 and 4 x 8 related to their use of the array. When asked to show 20 in either the smaller arrays or the larger array, Robyn began to count the individual cakes on one tray, not appreciating the structure of the array. Robyn and Bella's subsequent strategies in Lessons 2 and 3 indicated a conceptual shift in their explanations. They used similar recordings to those I had used with the class, along with the formal algorithm. Their solutions reflected an understanding of how the calculations in the algorithm related to place value and to the structure of the array (Figure 6.15).

Both classes worked similarly. An important observation on this episode was the amount of time students devoted to each question. Students spent time developing and refining their own mathematical constructions and understandings as they worked in their small groups. An important element in the development of Mathematical Practice 1 was the time allowed by the teacher to explore a few carefully designed problems, rather than completing many problems in a shorter time frame.

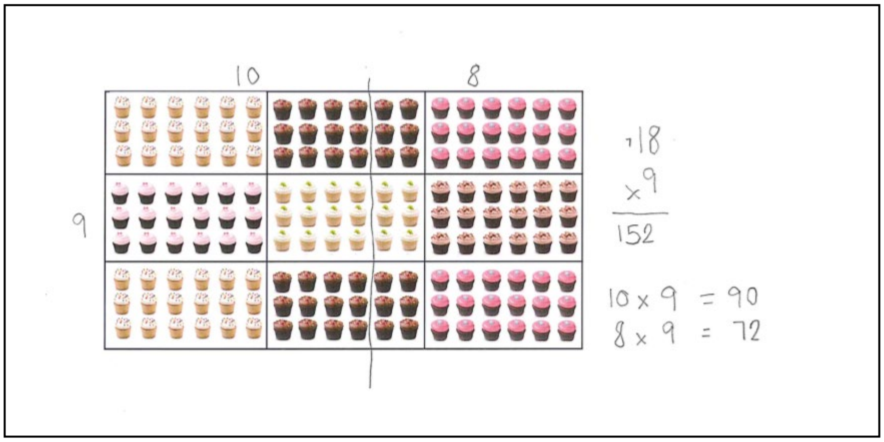


Figure 6.15—Robyn and Bella's second work sample from Teaching Episode 1

The Story of Jake

This section of the chapter looks more closely at the mathematical constructions of Jake from Class 1 (who was introduced earlier in the chapter). Shifts in Jake's mathematical thinking were observed through the development of his use of particular strategies, and his interactions with the array as a tool for mathematical reasoning.

To solve the initial cupcake problem, Jake used a basic skip counting strategy (Figure 6.4). Knowing he had obtained the correct answer, Jake was very keen to share his work during the class discussion. It is reasonable to say that this was a significant confidence boost for Jake, and there was a positive shift in his perceptions about himself as a mathematician and the role that he played in the collective learning of the class. Jake's contribution to and participation in the class discussion, as previously documented, and his subsequent activity, were significant on two levels. First, Jake's constructions and his participation in the class discussion contributed to the emergence and acceptance of the first mathematical practice for the class—*The array as a tool for sense making: Partitioning the array.* Secondly, changes in Jake's mathematical working were observed as he accepted this practice as his own.

Following the class discussion, Jake chose to work with two other students. The class was asked to continue comparing the presented work samples and to make sense of the different strategies that had been used. Jake reviewed his strategy and recognised that he could use the large array to count in 5s or 10s. He saw that it was possible to make two groups of 10 across each row. Another group member showed Jake that he could make groups of 10 across the top row and there would be a row of 4 remaining. Jake circled the 10s in the second row and saw that the pattern continued down the array. He partitioned the arrays into two groups of rows of 10 and one group of rows of 4. He proceeded to skip count the 10s and then add on the 4s. He was using a primitive form of the distributive property based on place value parts. Although holding onto skip counting strategies, Jake now recognised the array as coordinated rows and columns and understood how this could be used to partition rows and columns based on place value parts.

In the third lesson of Episode 1, Jake focused on answering the first of the two additional questions that were posed to the class:

1. What if the trays held a different number of cakes?
    a. How many cakes would Charlie bake if he made 8 trays with 16 on each tray?
    b. How many cakes would Charlie bake if he made 8 trays with 32 on each tray?

To solve the questions, Jake worked with the rows of the array (Figure 6.16). The strategies he used reflected an eagerness to experiment with concepts raised in the discussion. He had made some significant cognitive shifts. For the first problem (1a), he counted and circled 10 cakes across the top row and then continued to count 10 cakes along the second and third rows. Recognising that these groups of 10 aligned, he used a line to partition the 10 from the remaining 6, and skip counted the rows of 10. Jake then partitioned the eight rows of 6 into rows of 5 and 1 so that he could skip count the 5s and count on the 1s. He used a similar strategy for the second problem (1b), partitioning three groups of 10 in a row and a final group of 2. He skip counted the first group of 10 to get 80 and, at this point, I asked him to predict how many cakes would be in the second group of 10. He stated that he thought there would also be 80 as there were the same number of rows. He counted in 10s to check and immediately recorded 80 at the bottom of the second and third groups.

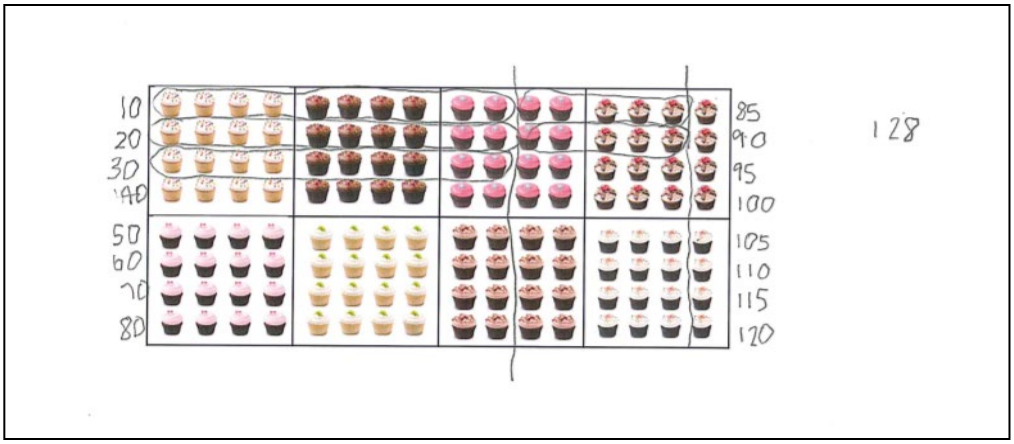


Figure 6.16—Jake's second work sample from Teaching Episode 1

Conclusion—Mathematical Practice 1

The previous section documented the emergence and development of the mathematical practice of partitioning based on place value becoming a taken-as-shared way of reasoning with an array over a series of three teaching episodes.

Three key conceptual events are used to illustrate the negotiation of this practice. First, students negotiated then recognised how the place value properties of a number could be used to distribute complete rows and columns in the array. The second event was students' recognition of similarity between strategies that drew on place value. The final conceptual event indicating that the mathematical practice was taken-as-shared in the class was revealed through shifts in students' strategy use, demonstrating an increased awareness of how the distributive property could be used to aid computation. This shift in strategy use was illustrated through Jake's work.

Mathematical Practice 2 - The array as a tool for sense-making: Using factors to manipulate the array

Teaching Episode 2 – The Cupcake Order

The second teaching episode comprised three lessons (see Figure 6.17).

TEACHING EPISODE 2		
Lesson 1	**Lesson 2**	**Lesson 3**
Solving the problem - How many cakes were in the order? Gallery walk	Class discussion – Looking at the strategies used Exploring why 16 x 12 = 24 x 8 Class discussion – explaining why 16 x 12 = 24 x 8	Number strings based on halving and doubling Additional problems to solve
Mathematical Practice 2 - **THE ARRAY AS A TOOL FOR SENSE-MAKING:** Manipulating the array		

Figure 6.17—The second teaching episode

Episode 2: Lesson 1

Students were separated into pairs and presented with a narrative continuing the story of Charlie the baker:

Charlie the baker received a large order of cupcakes. A local school has asked for 16 boxes of cakes for their annual fete. Each of the boxes holds 12 cakes. Charlie has made all the cakes and the boxes are sitting in the shop ready to be delivered. How many cakes are there in the order?

The class was shown a picture of filled cake boxes sitting on a bench, ready to be delivered to the school fete (see Figure 6.18). The array was somewhat abstracted in the picture, as the individual cakes were not visible. However, a further diagram revealed that the cakes in the closed boxes were configured in three rows of 4. This was the first 2- x 2- digit multiplication question students encountered as part of the instructional sequence.

In Class 1, two distinct student strategies were used: those that employed partitioning, and those that used doubling. All Class 2 students used partitioning to solve the problem and so it was assumed that they drew on their learning from the first teaching episode.

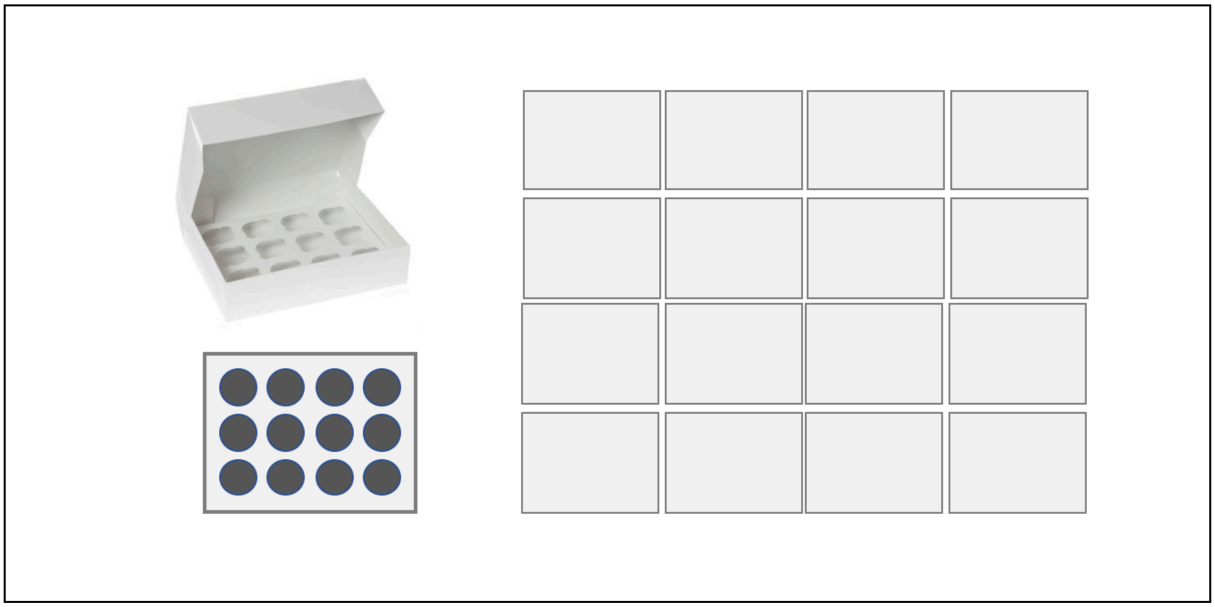


Figure 6.18—Pictures used in the narrative for Teaching Episode 2

Partitioning Strategies

The partitioning strategies observed employed distribution based on place value. The array formation in the scenario presented to students was not visibly conducive to the use of place value partitioning. Despite this, a number of students used place value to support their computation. One example is shown in Figure 6.19. The students sketched some of the cakes into the array representing the closed boxes, and then partitioned the array into two parts. Mitchell and Harley partitioned the array into two segments of 12 rows, one segment containing 10 columns and the other 6. They recorded the result of 10 x 12, then recognised that they did not know how to calculate the result of 6 x 12, so divided this second segment into two smaller parts.

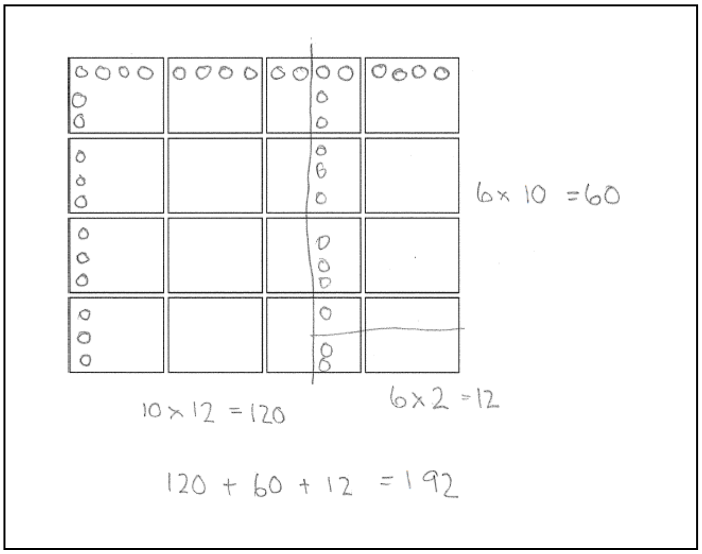


Figure 6.19—Jack and Harley's work sample

Doubling Strategies

Two pairs of students from Class 1 used doubling strategies. Ashley and James reused their *ice-cream strategy*, doubling 12 sixteen times. A second group, Jade and Joy, used multiplicative doubling, multiplying 12 by 2 four times (illustrated in Figure 6.20).

As in Episode 1 students were expected to adhere to the classroom's social and sociomathematical norms by conceptually explaining their solution strategy. All students in both classes chose to use the array to explain their solution, indicating that the array was becoming a powerful tool for sense-making and mathematical reasoning. The lesson concluded with a Gallery Walk, where students examined and made sense of the reasoning used by the other students.

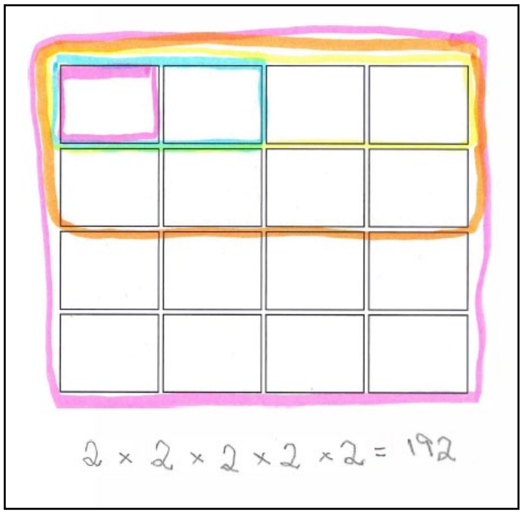


Figure 6.20—Jade and Joy's work sample from Teaching Episode 2

Episode 2: Lesson 2

At the start of the second lesson, a brief class discussion was conducted to review some of the different strategies used to solve the problem, and to link these strategies back to the mathematical learning of Episode 1. After the discussion students were asked to consider why the total number of cakes in the order was the same as the total number of cakes baked from Episode 1. In other words, they were asked to explain why 16 x 12 had the same value as 24 x 8.

The remainder of the second lesson was allocated to investigating the equivalency between the two calculations more deeply. The picture from Episode 1 (Figure 6.3), and Figure 6.18 were made available for the students to use in their working. A new strategy emerged during this exploration. Some groups recognised that the 16 x 12 array could be split and rearranged to make a 24 x 8 array as illustrated in Ryan and Dylan's work sample (Figure 6.21). The idea of halving one side of the array and adding it to the other side, thus doubling this side's value, became the focus of a class discussion at the conclusion of the lesson. This strategy is discussed as part of the analysis in the following chapter, Chapter 7.

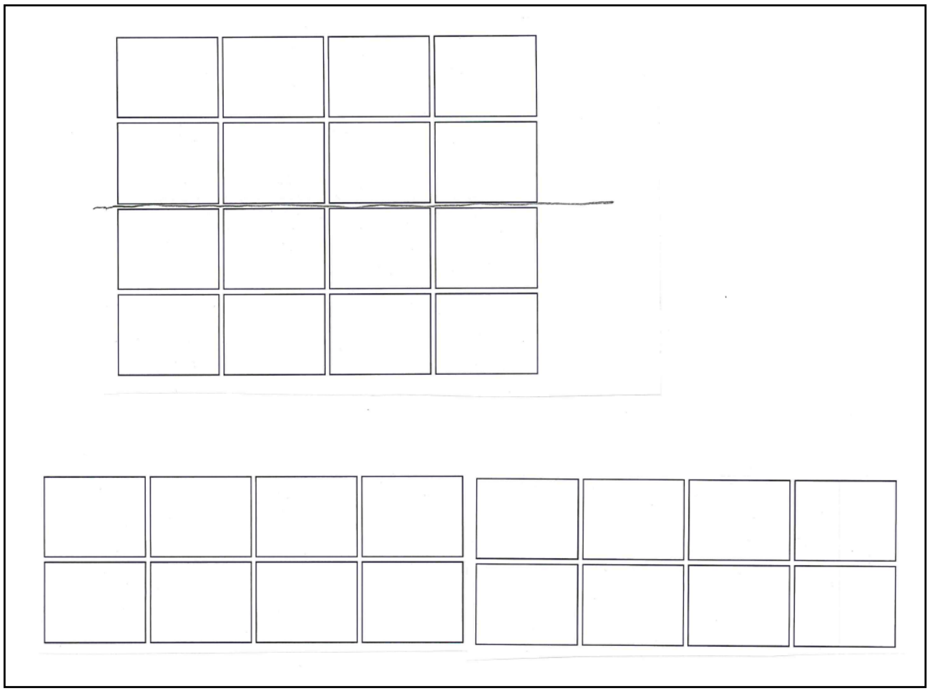


Figure 6.21—Ryan and Dylan's work sample from Teaching Episode 2

Episode 2: Lesson 3

Lesson 3 started with a whole-class discussion in which the class was introduced to a number string (Fosnot & Dolk, 2001) (Appendix 3). As previously mentioned, number strings are a sequence, or string, of calculations that are carefully crafted to assist students in constructing key understandings and building strategies in a mathematical domain. In this instance, the focus was on halving and doubling which can generalise into an understanding of the associative property. The various ways in which groups had split and rearranged the array were explored. After the discussion students were asked to decide within their groups when halving and doubling might be a useful strategy. They were also asked to consider if the strategy of splitting and rearranging could be used for other numbers. To assist with answering both questions, students were given a series of calculations to stimulate their investigation (Figure 6.22).

5 x 16	50 x 21	25 x 16	8 x 14	13 x 17
5 x 13	13 x 50	25 x 14	16 x 14	14 x 18
22 x 5	50 x 12	20 x 25	14 x 32	17 x 14

Figure 6.22—Multiplication number strings used in Teaching Episode 2, Lesson 2

The Emergence of Mathematical Practice 2

Over the course of Episode 2 the second mathematical practice emerged and was accepted across both classes. The following section explores three conceptual events that were significant in the development of the mathematical practice—*The array as a tool for sense-making: Manipulating the array*. Class 1 is used to illustrate the work of both classes. The section concludes by tracking the contributions and participation of one particular student, Archie, who recognised the advantages and limitations of using strategies based on the associative property.

Conceptual Event 1—Noticing a relationship between the numbers

Following the Episode 2, Lesson 1, the pairs investigated why the value for 16 x 12 was equal to 24 x 8. During the course of this investigation, a number of student pairs noted a certain relationship between the numbers involved. An excerpt from the recorded interactions among one pair of students illustrates the 'light-bulb' moment when this relationship was realised:

Ryan: 12 is half of 24.

Dylan: You could join two of the boxes together to make 24 then… wait, that's 8 groups…yeah…that's 8 groups because 8 twos are 16.

Ryan: So then, 8 is also half of 16 so they are both half.

Dylan: Yeah, I think that's right.

The two boys explored this relationship further and eventually recognised that they could rearrange the array to make a 24 x 8 array of cakes as in the first question in Episode 1 (see Figure 6.21). This observation was the focus of discussion in the ensuing class conference.

Tom and Frederique also shared their reasoning, which mirrored Dylan and Ryan's thinking.

Frederique: Well, we noticed that you could turn the array into 24 by 8 by cutting the boxes in half and then moving one half of them down here (gesturing on the array). That made 24 down here and 8 across the top…yeah.

Teacher: So, Tom, you halved the length of one side and doubled the length of the other? Is it OK to do that?

Tom: Well, yeah, it's fine.

Teacher: Why? Why do you say its fine?

Tom: Well, like…all the cakes are still there. It just shows why they are the same.

Teacher: What do people think about that? Is it OK to move the array around? Do you agree they're the same?

Dylan: Yeah, I agree. I did it a bit the same…and Ryan…

Dylan and Ryan briefly shared their working with the class and presented their observation that 24 could be made by doubling 12, and 8 could be created by halving 16. After further discussion and clarification, the class agreed that the array could be manipulated in this way and that the two arrays contained the same number of cakes.

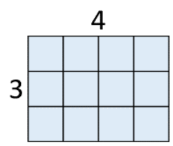
At the end of the lesson, the class teacher and I agreed that the idea of manipulating the array had become taken-as-shared. We also agreed that the connection between halving and doubling needed further exploration before it was taken-as-shared in the classroom. We needed a further exercise in order for the students to extend their understanding of the factors involved in manipulating the array. It was for this reason that we devised a number string to be introduced at the start of Lesson 3.

Conceptual Event 2—Using factors to manipulate the array

The third lesson started with the presentation of a number string (Figure 6.22). Working within their groups, students were then asked a number of associated multiplication problems. As part of their answer they were required to represent the question as an array constructed using grid paper. After they had done so, the class was asked to use their answers to the associated multiplication problems to help answer a new set of questions.

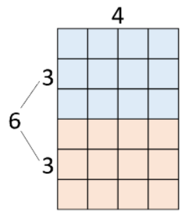
The following progression of activity records the numbers string and associated arrays used in Class 1 and Class 2 (Figure 6.23). The first problem required students to solve 3 x 4 and then represent 3 x 4 as an array. The question was easy for students to answer. It was noted that students presented the array in two different orientations—3 x 4 and 4 x 3. There was agreement that both arrays were the same size based on the commutative law. For consistency, it was also decided that the first multiplicand would represent the number of rows and the second would represent the number in each row.

3 x 4



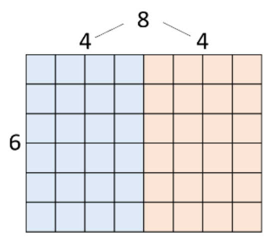
3 x 4 = 12

6 x 4



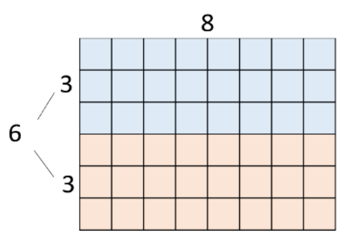
6 x 4 = 2 x (3 x 4) = 24

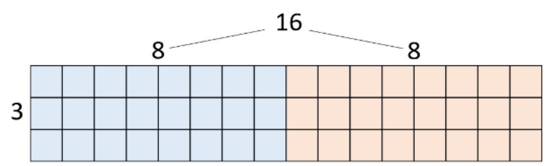
6 x 8



6 x 8 = 2 x (6 x 4) = 48

3 x 16





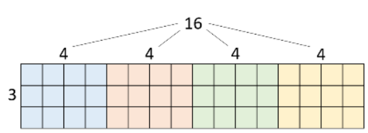
Halve the 8

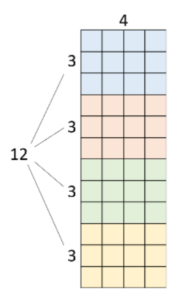
6 x (2 x 4) = 48

Double the 6

(6 x 2) x 4 = 48

12 x 4





Divide the 16 into 4 to create

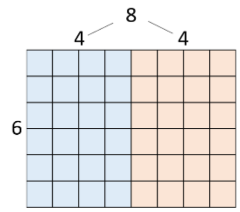
3 x 4 x 4.

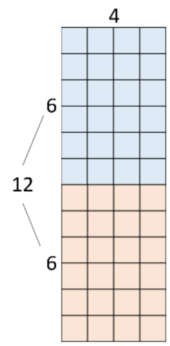
The 16 is here - 3 x (4 x 4)

We did it this way to create 12 x 4. *The*

12 is here (3 x 4) x 4.

3 x 16 = 12 x 4





Divide the 8 into 2 to create 6 x 2 x 4.

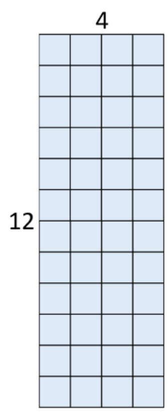
The 8 is here - 6 x (2 x 4)

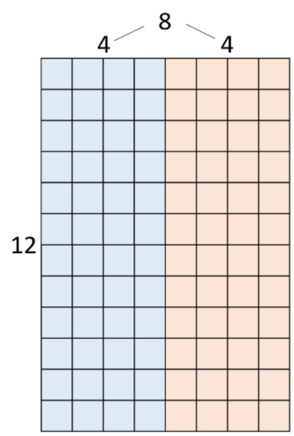
We did it this way to create 12 x 4. *The 12 is here* (6 x 2) x 4.

6 x 8 = 12 x 4

---

12 x 8





12 x 4 = 48

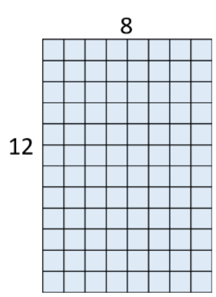
(12 x 4) x 2

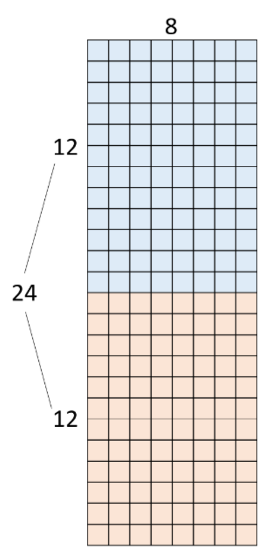
12 x (4 x 2)

12 x 8 = 96

---

24 x 8





12 x 8 = 96

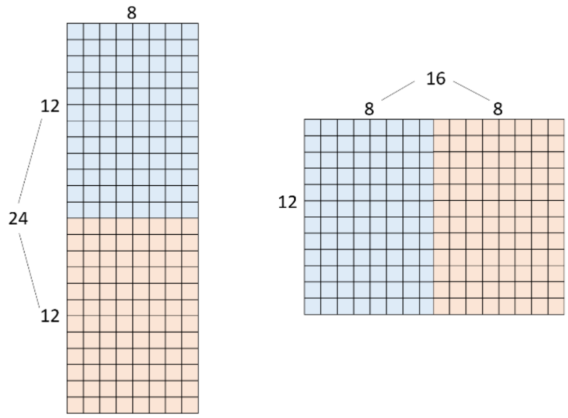
2 x (12 x 8)

(2 x 12) x 8

24 x 8 = 192

---

12 x 16



24 x 8 = 192

(12 x 2) x 8

12 x (2 x 8)

12 x 16 = 192

Figure 6.23—Number string used in Episode 2

The second question, 6 x 4, built on the first. The students recognised they could double 3 x 4 to create 6 x 4. Using the array, they just needed to double the number of rows. Similar reasoning was applied to the third question (6 x 8). The fourth question (3 x 16) started combining ideas of halving and doubling. Students saw that the number of rows could be halved and then added onto the number in each row, thus doubling this dimension. The next question (12 x 4) reinforced this manipulation of the array. Some students used 6 x 8 as a supporting computation, by halving the number in each row and then doubling the number of rows. A second group of students used 3 x 16 as a foundational computation. They divided 16 into quarters and then multiplied the number of rows by 4. This strategy prompted a discussion with the class on the use of factors.

Flora: *I divided the 16 by 4 and multiplied the 3 by 4.*

Teacher: *So, you divided the number in each row by 4. And then multiplied the number of rows by 4.*

Flora: *Yes.*

Teacher: *Could you have done it with 3? I mean, that would make a different computation, but could you do it by 3?*

Flora: *Well…no…I don't think so. No, you can't divide 16 by 3. Well, you can but you get a remainder and so that won't work.*

Teacher: *Do others agree?...Harvey?*

Harvey: *You can divide by a number that will go in evenly, they have to be…what do you call those numbers again, the ones that divide into a number?*

Teacher: *Factors?*

Harvey: *Yes, that's it—factors. You can divide by a number if it is a factor. So…yeah, couldn't do 3 for 16 because it isn't a factor, but you could do 2 or 4, oh they already did that… or you could do…8.*

Teacher: *So, if the number was 12, I could divide by a few numbers? What does someone else think?...Tom?*

Tom: *For 12, you could divide by 2, 3 or 4…or 6.*

Following this interaction, I used symbolical recording to show what was happening when the arrays for 6 x 8 and 3 x 16 were manipulated to form 12 x 4. The recording and my accompanying commentary are shown in Figure 6.23. Through the course of discussion, the class agreed that this halving and doubling strategy applied the associative property of multiplication. The students were happy with the way I had recorded the manipulation of the arrays. They were then asked to think about how they might record their working from this point forward.

The subsequent questions in the string led to the numbers used in the questions from Episode 1 and 2. The students were asked to solve 12 x 8 and then 24 x 8. The students recognised that the 4 could be doubled to make 8 in the first question, and then in the second question, the 12 could be doubled to form 24. Many students represented this symbolically as 12 x 2 x 8. The class agreed that this representation accurately recorded what was being done mathematically. The final question in the string was 12 x 16 which students immediately recognised was an example of halving and doubling 24 x 8.

Following the number string, the students were given a set of multiplication questions (Figure 6.22). Students were to explore how the associative and distributive properties could be used to calculate the total for each question, and to determine when the associative property might be a useful strategy. This sequence of questioning proved to be an influential activity in developing their understanding of the associative property which, in turn, reinforced the emergence of the second mathematical practice—*The array as a tool for sense-making: Manipulating the array*. Three generalisations were formed by the class:

- Halving and doubling would only work neatly when at least one number was even;
- Repeated doubling was a useful strategy for numbers that were powers of 2; and
- The distributive property was a more efficient strategy to answer most questions.

In the initial analysis of the lesson, these generalisations formed an important finding, which increased in its significance as the lesson progressed. The students started making reference to numbers that were easily factorisable, for which I introduced the term *friendly number*. Students also noted that some numbers could be easily converted into easier numbers. For example, 5 could be converted to 10 by multiplying by 2, or 25 could be converted into 100 by multiplying by 4. In their problem solving and reasoning, students started to consider the multiplicand and multiplier and the strategies that would work best with the numbers presented in the different problems. The students' generalisation not only supported the development of the second mathematical practice, but it was important in the genesis of the fourth mathematical practice: the use of friendly numbers, described in more detail later in this chapter.

The Story of Archie

One student from Class 1, Archie, made a significant contribution to the establishment of the second mathematical practice. His work on the final series of multiplication questions is significant enough to warrant deeper exploration. It focusses on his activity in the final series of multiplication questions (Figure 6.22), particularly his personal assessment of when the associative property was a useful strategy to use.

Two important shifts were noted in Archie's constructions as he worked through this sequence of questions. The first shift was the connection of place value to the use of the associative property to multiply numbers. Archie began with the sequence of multiplication questions relating to 5 (first column of Figure 6.22). He observed that the calculation could be simplified by doubling the 5 to make 10 and then halving the remaining number. Archie then extended this strategy to the questions relating to 50 (second column of Figure 6.22). He recognised that halving 13 to solve 5 x 13 and 13 x 50 posed a challenge, but he still devised a way to use the strategy: He could use his understanding of place value and multiply 6.5 by 10 and by 100. In the concluding class discussion, Archie shared this discovery with the class. Students were interested by Archie's explanation of the connection between using an understanding of place value and the use of the associative property.

The second shift noted in Archie's constructions related to *friendly numbers*. As he worked with his partner Tom, the boys started to underline numbers in the questions, as illustrated in Figure 6.24. Archie explained that these were the helpful numbers that were used to initiate use of the associative property. I asked him why he had not underlined any numbers in the final group of questions. He answered that it was unnecessary, as he could just use the distributive property with these numbers. In the ensuing class discussion, Archie's comments played an important role in the class's appreciation that some numbers lent themselves to use of the associative property.

5 x 16	50 x 21	25 x 16	8 x 14	13 x 7
3 x 13	13 x 50	25 x 14	16 x 14	14 x 18
22 x 5	50 x 12	20 x 25	14 x 32	17 x 14

Figure 6.24—Archie's work sample from Teaching Episode 2, Lesson 3

Conclusion—Mathematical Practice 2

This section has presented a summary of the development of the second mathematical practice. It described how manipulating the array based on factors became taken-as-shared in both classes. Two significant conceptual events were identified in the negotiation of this practice. The first event was the students coming to a shared understanding that the associative property could be employed when particular numbers shared common factors. This understanding led to the second conceptual event: the utilisation of common factors to divide and then rearrange the array to aid computation.

The story of one particular student, Archie, was used to demonstrate the cognitive shifts made by all the students as they participated in and contributed to the development of the social mathematical practice.

A Change to the Hypothetical Learning Trajectory

At the conclusion of this episode with Class 1, I decided not to proceed with the third episode of the hypothetical learning trajectory, as originally planned and documented in Chapter 5. As such, Episode 3 was not taught in Class 2 either. This decision was made for two reasons. First, time was a major consideration. I only had access to the class for 10 lessons over a two-week period. The first two episodes had taken six lessons, and I felt I would need the remaining four lessons for the final two teaching episodes and to draw reliable conclusions for the project.

Secondly, the students were demonstrating sound skills in partitioning and manipulating the array. As such, the third episode which was intended to explore partitioning strategies was of low priority. I concluded that the students were ready to progress to the fourth episode, which was focused on comparing the areas of box designs and abstracted the array to an open array.

To maintain the consistency of numbering, Episode 4 as documented in the HLT is referred to as Episode 3 in this chapter and in Chapter 7. Similarly, Episode 5 as documented in the HLT is referred to as Episode 4.

Mathematical Practice 3—The array as a tool for sense-making: Thinking multiplicatively

Teaching Episode 3—Comparing Box Designs

The third teaching episode was made up of two lessons as shown in Figure 6.25.

TEACHING EPISODE 3	
Lesson 1	**Lesson 2**
Solving the problem – Which has the larger area: 25 x 25 or 28 X 24? Students' hypothesised which they thought was larger and then looked at whether their reasoning was correct.	Class discussion – Additive and multiplicative situations has key structural differences. Exploring the strategies of other students.
Mathematical Practice 3 – **WAYS OF WORKING MATHEMATICALLY:** Thinking multiplicatively	

Figure 6.25—Teaching Episode 3

Episode 3: Lesson 1

The third teaching episode continued the narrative of Charlie the baker:

Charlie the baker receives an order of boxes that hold 12 cupcakes. The boxes are a different design from the previous boxes he used. These boxes have the cakes in a 'skewed array'. He wonders why this might be?

In both iterations of the experiment, students were shown the trays from inside the different cupcake boxes, and they discussed why one array was skewed and the other was not (see Figure 6.26). Both classes offered similar suggestions for the skewed array.

One popular suggestion for the skewed array was that it looked more decorative. A second suggestion was that the cakes would not touch in the skewed array, but they might in the regular array. To check if this would be true, students measured between the cupcake holes and noted that both trays had the holes spaced at the same distance. A further suggestion was that the skewed array took up less space than the regular array. Therefore, its box would not use as much material and consequently be cheaper to make. This hypothesis was the focus of the teaching episode.

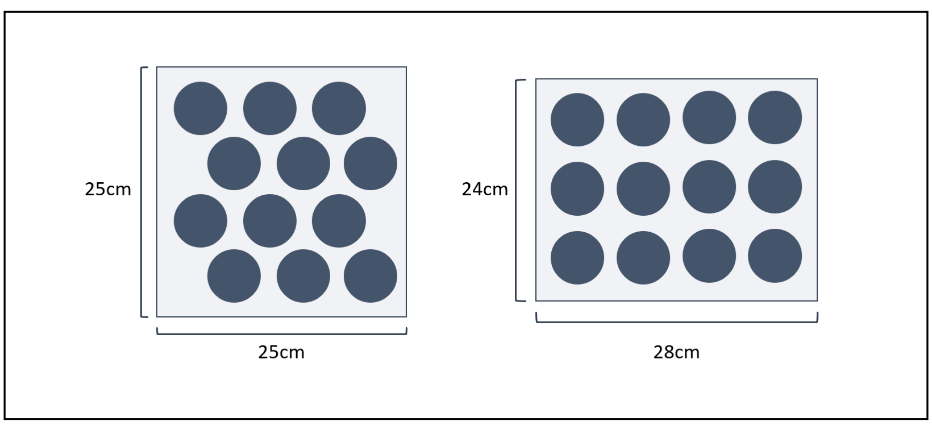


Figure 6.26—Comparing the area of two trays in Teaching Episode 3

The Emergence of Mathematical Practice 3

Not Yet Taken-as-Shared

The initial plan for the class activity was to have students use their own strategies to work out which tray had the larger area. In Class 1, the students were asked to predict if it was true that a tray with dimensions 28cm x 24cm was larger than a tray with dimensions 25cm x 25cm. The students' responses were interesting and unexpected.

Mia: *28 x 24 is like 26 x 26.*

Teacher: *Why is that? What are you thinking?*

Mia: *Well, if you take 2 off the 28 that makes 26, and then you just put that 2 with the 24, then it is 26 too. So, it is 26 and 26 and you just times them together and it's obvious…26 and 26 will make more than 25 and 25.*

Ashley: *Well, both of them are just 20 multiplied by 20. Then… well, with that one* [indicating the 25cm x 25cm array] *you do 5 times 5 and then with that one (indicating the other array) you do 4 times 8 and I think…um…wait, it's, yeah, it's 32 and the other one is 25, and 32 is bigger than 25.*

Up to this point in the sequence, the students' responses had demonstrated sound mathematical understanding. However, both responses given above were based on incorrect mathematical reasoning. The other members of the class were asked what they thought of this reasoning. While there was some scepticism, all students agreed with one or the other of the reasons presented. The focus of the episode then shifted from having students work out the total areas of each tray, to having students show how their explanations worked mathematically and whether they were correct.

In response to the outcome of this episode in Class 1, two alternative approaches to the lesson were planned for Class 2. If students made mathematically sound predictions in the whole-of-class opening discussion, the students would be asked to calculate the total area of the trays. However, if Class 2 made predictions similar to those of Class 1, the lesson would follow the same path as in Class 1: the students would be asked to show that the mathematical reasoning behind their predictions was correct.

When the students of Class 2 were asked to predict which area was larger, the following responses were given:

Ellena: *Well, to work out the bigger tray you just do 20 x 20, and that's 400, and then you do 8 x 4 and that is 32. So that one is…it's 432. The other one will be…it will just be…425, yeah 425… because 5 x 5 is 25 so it will be smaller than the other one…*

Meera: *If you just take 2 from the 28 and then you just put it on the 24 you will get 26 and 26 and, yeah, that's bigger, it's bigger than the other tray.*

It was surprising that the reasoning errors of Class 1 mirrored the errors in Class 2. Some difference in reasoning between the two classes was anticipated. The students in Class 2 were asked to show if their reasoning was correct.

The same two misconceptions were observed in both classes. First, students were not recognising all the partial products in a multi-digit multiplication. The second misconception centred on compensation: it is incorrect to use additive compensation strategies in a multiplicative context. It was clear that the students in both classes still lacked an understanding for what it meant to work and reason multiplicatively.

In both classes, the students were asked to choose to explore more deeply one of the predictions they had presented. Using a calculator, they quickly realised that the 28cm x 24cm area was larger, but also that neither strategy produced the correct answers for the total area of each box insert. The class investigation then shifted to a consideration of how the reasoning behind their strategies had resulted in incorrect answers, and how their reasoning could be modified to calculate the correct totals. This process followed similar lines in both classes, and the following pages detail that process as it occurred in both classes. Illustrative examples of students' solution methods, and a description of the resulting class discussion and activity, have been selected from Class 2 in this instance.

The two investigations and resulting discussion and activity are recorded as significant conceptual events in the emergence and acceptance of the third mathematical practice.

Conceptual Event 1—Recognising and adding all partial products

Five student pairs in Class 2 chose to investigate the validity of the comment that 28 x 24 was bigger than 25 x 25 because 8 x 4 has a greater total than 5 x 5. To begin, the students were asked to solve two questions:

- Which is bigger—28 x 16 or 26 x 19?
- Which is bigger—26 x 18 or 29 x 16?

The students used similar reasoning to answer both questions: 26 x 19 is bigger than 28 x 16 because 6 x 9 is bigger than 8 x 6; 29 x 16 is bigger than 26 x 18 because 9 x 6 is bigger than 6 x 8.

Students checked their predictions using a calculator and realised that, while 26 x 19 was indeed bigger than 28 x 16, 29 x 16 was *smaller* than 26 x 18. They also noted that their calculations were incorrect: the answer for 26 x 19 was not the result of 20 x 10 and 6 x 9 as assumed. At this point, there was obvious confusion for the students. I responded by asking students to return to the question from the beginning of the lesson: which is bigger—28 x 24 or 25 x 25?

Students instinctively reverted to an array as a means of solving the calculations and to make sense of their errors. This demonstrated that utilisation of the array as a tool for sense-making was indeed taken-as-shared. The students partitioned the array into place value parts and found that, when the array for a 2- x 2-digit multiplication was distributed based on place value parts, four partial products were formed. In their initial calculations the students had only acknowledged two of these parts. This learning was the focus of a whole-class conference.

At the start of this conference, I reminded the students of Ellena's initial reasoning used for the first prediction. I used the following symbolic recording:

Prediction 1

28 x 24 → (20 x 20) + (8 x 4) → 432

25 x 25 → (20 x 20) + (5 x 5) → 425

Ellena and Taylor presented their findings (Figure 6.27) from the investigation to the class, and stated that the problem with Ellena's initial reasoning was that she didn't use the array.

Taylor: *I think that if Ellena used the array, she would have got it right.*

Ellena: *Yeah, I agree. I think I would have got it right. I just needed the array to see all the parts.*

Teacher: *What do you mean by that Ellena—that you needed the array to see all the parts?*

Ellena: *Well, I forgot these two (pointing to the work sample) and so I didn't add them. Well actually, I didn't really know they were there.*

Teacher: *Does everyone understand what the girls are saying? Do you understand why they think Ellena made the mistake in her calculations?*

Layla: *Yes, when you divide it down that way (pointing down the array) and then along that way (pointing across the array) you get four parts in the array and you need to add them all together and Ellena only added two of them.*

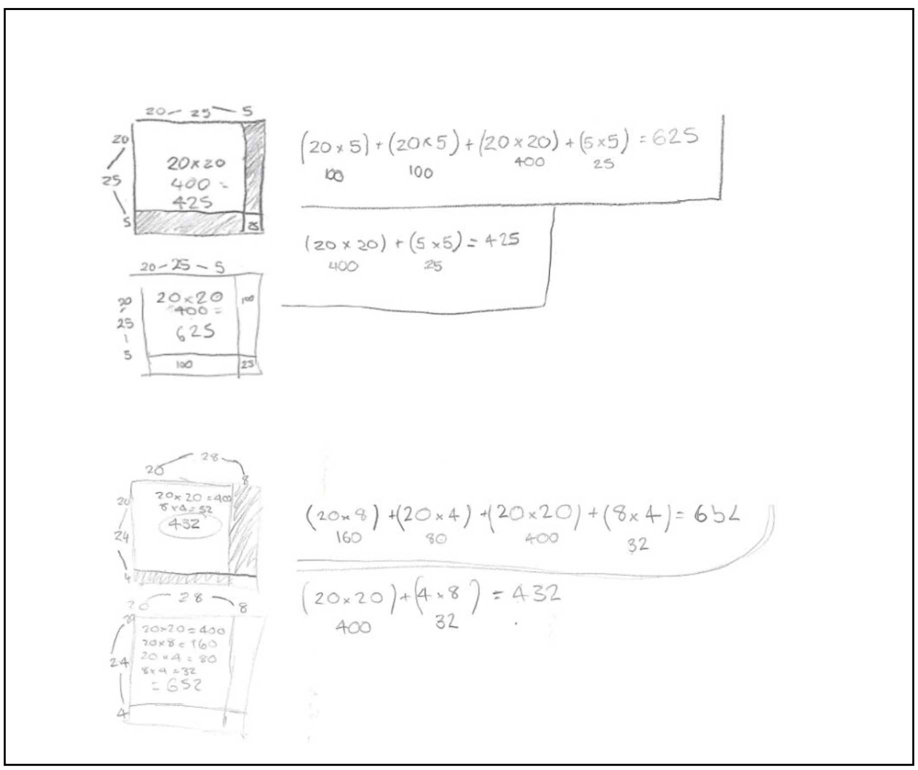


Figure 6.27—Ellena and Taylor's work sample from Teaching Episode 3

At this point, the students were invited to use Ellena and Taylor's modified strategy to solve 26 x 18 or 29 x 16. As they worked, the students were asked to consider how the numbers corresponded to the partial products created by distributing the array.

The first observation made by the class was that the number of parts in the distributed array corresponded to the number of digits in the problem. Students then realised that each part of both numbers was multiplied together. One student, Darcy, explained her observation symbolically as shown in Figure 6.28. Darcy showed that each place value part in the first number needed to be multiplied by each place value part of the second number. In her explanation, she used a number line to illustrate how multiplying two 2-digit numbers differed from adding the numbers together.

Darcy: *In addition, you just need to add one number to another number. You can add the tens and then you can add the ones, but you don't add the tens in the first number to the tens and the ones in the other number. You just add it once.*

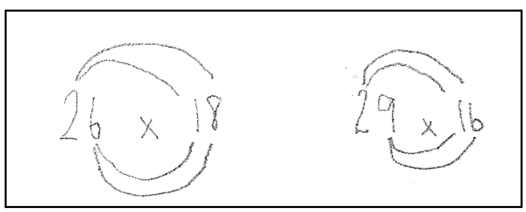


Figure 6.28—Darcy's work sample from Teaching Episode 3

Three specific occurrences in the sequence of events were significant in the negotiation of the third mathematical practice (The array as a tool for sense-making: Thinking multiplicatively). The first event was students' recognition of all the partial products formed in a multi-digit multiplication. The various steps in this sequence of activity revealed to students that the two-dimensional structure of multiplication created multiple sections when the array was distributed. As a result of recognising these sections, the students appreciated the importance of multiplying and adding together all the partial products formed. Further to this, they recognised that the partial products in this instance were formed by the place value parts of the numbers.

The second significant moment occurred when multi-digit addition and multi-digit multiplication were compared. As the teacher, I deliberately introduced this comparison in Class 1, and it played a significant role in the development of students' abilities to work multiplicatively. Recognising that multiplication and addition differ *structurally* assisted the students in identifying how and why the *processes* of addition and multiplication differ. The structural distinction was further supported by the use of different models: the number line for addition and the array for multiplication. The structural difference between addition and multiplication and the the uniqueness of working multiplicatively was elegantly summarised by Marcus, a student in Class 1:

Marcus: *Each number gets multiplied twice. It's just like you first multiply it by the vertical numbers and then you multiply it by the horizontal numbers.*

The final significant observation in the negotiation of the third mathematical practice was the changing nature of students' use of the array, together with their notations and symbolisation. In their exploration of finding all partial products, the students abstracted the array to an area model or open array, in keeping with the model used in the context. It was observed that students had a greater reliance on arrays constructed from grid paper at the start of the lesson, as they made sense of their error. As they returned to the original question about the area of the two trays, students' use of the array became more abstracted, and as the students progressed to a more abstracted model of the array, their use of symbolisation increased. Ellena and Taylor's work sample, displayed in Figure 6.27, is typical of this progression. On the basis of the observation of the increasing abstraction of the array, it is reasonable to conclude that students' focus was shifting as they progressed through the instructional sequence; rather than a tool for aiding calculations, the array had become a tool for making sense of findings that did not initially align with their current understandings.

Conceptual Event 2—Recognising the difference between additive and multiplicative compensation

The first group of students investigated the distributive property of multiplication, specifically all the partial products formed by distributing the array. The second group of students looked more closely at the associative property of multiplication. These students investigated whether 28 x 24 had the same value as 26 x 26. The students had reasoned that two could be taken from the 28 and added onto the 24 to create 26 x 26.

As with the group described in Conceptual Event 1, students began their investigation by using a calculator to check if the two multiplications had the same value. Students realised the two equations were not equal and were asked to determine *why* they were not equal. As with the previous questions, all students used an array to make sense of what was happening.

This problem proved to be complex. Two key strategies were observed. The first strategy, used by just one pair of students, focused on comparing the common areas of the arrays. Mitchell and Tom constructed two arrays, the first measuring 28 x 24 and the second 26 x 26. They proceeded to overlay the two arrays and measured the size of the overhanging sections. Mitchell and Tom reasoned that 28 x 24 was not the same as 26 x 26 as the size of the overhangs were not equal (see Figure 6.29). Using the same strategy, they showed that the area of the 28 x 24 array was bigger than the area of the 25 x 25 array.

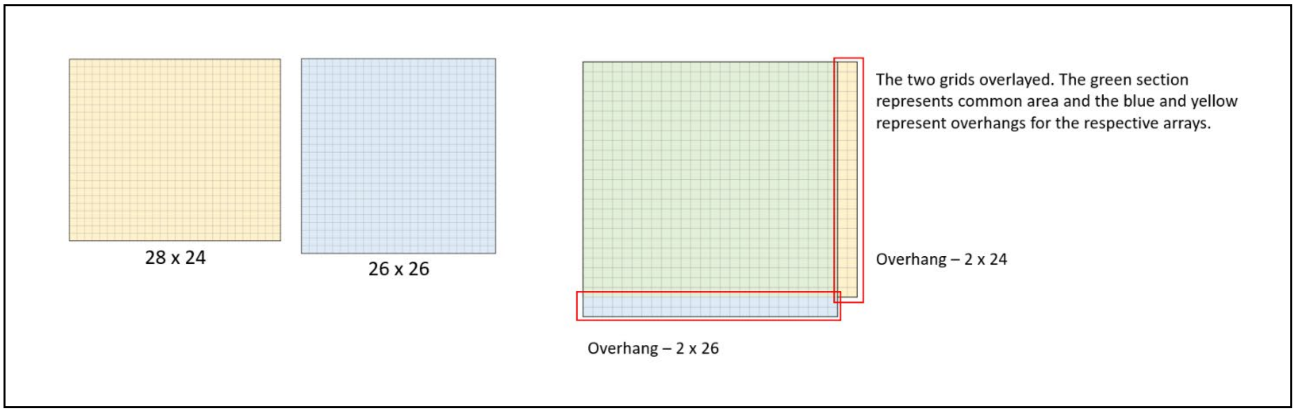


Figure 6.29—Strategy 1 for comparing 28 x 24 and 26 x 26

A second strategy was used by two other student pairs. They constructed an array measuring 28 x 24, then physically cut two columns off the 28 and added these to make two new rows below the existing 24 rows (Figure 6.29). These students observed that a gap of four squares had been revealed, and so concluded that the product of 28 x 24 must be 4 less than the product of 26 x 26. This strategy demonstrated the second mathematical practice: that is, the array can be manipulated.

In the ensuing class discussion, both strategies were shared with the whole class. Of particular significance to the emergence of the third mathematical practice was the conversation relating to the second strategy. Amelie shared how she had used the array to show why 28 x 24 did not equal 26 x 26 (as in Figure 6.30).

Amelie: *I made an array that was 28 across and 24 down and then I cut off two rows from the 28 and I stuck it onto the bottom of the 24. But what I noticed was that the rows are different. These ones are 24 long and the rows this way are 28 long and so they don't match up…these ones are shorter and so it won't work.*

Teacher: *So, you can't just take two off the 28 and put it on the 24 then?*

Amelie: *No, it won't work because the rows are different. You can see it here on my array.*

Teacher: *Does everyone understand what Amelie did? Do you get what she is saying?*

Maddie: *Yes, but I am really surprised though!*

Teacher: *Why are you surprised?*

Maddie: *I thought you could. I mean you can do it in addition, so I just thought that you could do it in multiplication, but now I can see that you can't. I can see it. I am just really surprised.*

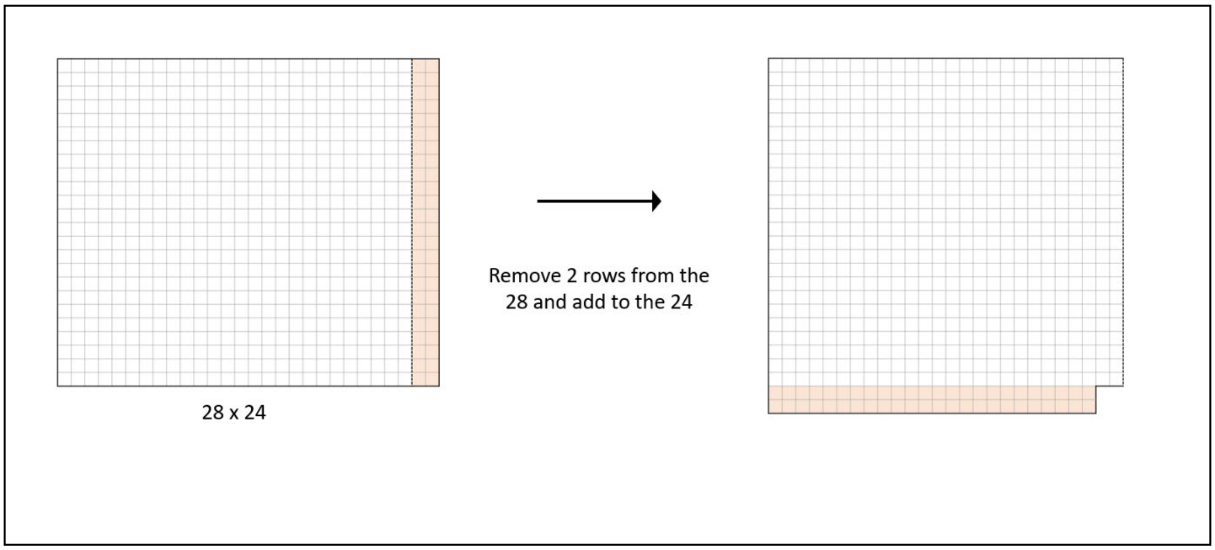


Figure 6.30—Strategy 2 for comparing 28 x 4 and 26 x 26

Once again, the students were given the time to explore the strategy for themselves. In particular they were asked to consider why you can use such a strategy in addition and not for multiplication, as raised by Maddie. In her reasoning, Samar referred back to the previous teaching episode.

Samar: *Last time we halved and doubled. When you halved, the two parts were the same and so you could move the parts of the array. But when you take just two off, the* *parts are not the same and so they won't match. So, I think that you can divide but not subtract.*

Teacher: *What do you mean that you can 'divide'?*

Samar: *You can divide by 2 and then double or divide by 3 or divide by 4…and you can keep going. But you can't just subtract 2 or 3 or 4. You need the parts [of the array] to be the same.*

Two key observations were evident from this interaction with Samar. First, the mathematical practice of manipulating the array was taken-as-shared in the class. All students readily accepted that two rows could be removed and then added to another part of the array without changing the total area of the array. Secondly, it became clear that *how* the array could be manipulated to maintain a rectangular shape was *not* previously taken-as-shared. The investigations of the second teaching episode had centred on using factors. The students had been asked to explore strategies that would work. They had not been asked to explore strategies that would not work. That the array could be manipulated based on factors was taken-as-shared. That this was the only mathematically sound way the array could be manipulated to maintain its rectangular form was not taken-as-shared. This notion of working multiplicatively versus additively was new learning. The class had made an important distinction between additive and multiplicative thinking.

The students were given some time to explore Samar's reasoning and came to the consensus that there was a difference between multiplicative compensation and what was termed additive compensation.

The story of Amelie

The cognitive shifts made by students across both classes are illustrated by the ways in which one student, Amelie, participated in and contributed to the development of the third mathematical practice. The pre-assessment interviews revealed Amelie to have a good grasp of early multiplication concepts, and this finding had been reinforced by her thinking and reasoning up to this point in the experiment. Amelie confidently sided with the view that 28 x 24 would be equal in area to 26 x 26, and she chose to explore this concept with another student in the activity of the teaching episode. The class teacher was working Amelie and her partner.

Amelie created a 28 x 24 array from grid paper to help make sense of why the areas would be equal. For some time, she was unsure how to proceed, expressing that she did not know how to manipulate the array to take 2 from the 28. Prompted by her class teacher, Amelie cut off two columns from the 28 and taped them to the bottom of the array. However she noticed a 2 x 2 corner missing, which left her puzzled. Once again, her teacher prompted her, suggesting she explore some other arrays to understand what was happening.

Teacher: *What about trying some smaller arrays, like 12 x 8. Will 12 x 8 be the same as 10 x 10?*

Amelie: *I know that won't work! 10 x 10 is 100 and 12 x 8 is not 100.*

Teacher: *Well, what about some others like 15 x 9 and 12 x 12? How about you give them a go. It is the same reasoning.*

Amelie proceeded to explore a number of arrays. First, she noted that taking two columns and moving them to form two new rows would always result in a missing 2 x 2 block. She then experimented with some other sets of numbers and realised that, when attempting to form a square, a square corner with the dimensions of the number removed would be missing. Time limited Amelie's investigations.

This process helped Amelie realise that additive compensation could not be used in a multiplicative context. In the analysis of the lesson, I agreed with the class teacher that an important aspect of Amelie coming to this realisation was her exploration of several examples, including examples that she knew would not work, such as 12 x 8. We noted that Amelie was not investigating to see *if* the examples worked but was exploring to see *why* they did *not* work. This investigation also led Amelie to see a mathematical pattern and form some generalisations of her own. Amelie was able to share these findings with the class, with part of the transcript supplied earlier. As such, Amelie's constructions played an important role in the class accepting the mathematical practice as their own.

Conclusion—Mathematical Practice 3

In summary, this section has documented the development of the third mathematical practice of working multiplicatively becoming taken-as-shared in both classes. Two key conceptual events were highlighted in the negotiation of this practice. First, the students developed an understanding of all the partial products that were formed when an array was distributed based on place value. They also learnt that all these parts needed to be added together to reach the correct total. The second event was when the students recognised the difference between additive and multiplicative compensation. Both these events were driven by students comparing how additive and multiplicative strategies differed. The story of one student, Amelie, illustrated the cognitive shifts made by students, and how these shifts influenced the acceptance of the mathematical practice through an individual's participation and contribution to the class. The following section describes the emergence of the fourth mathematical practice.

Mathematical Practice 4—Ways of working mathematically: Looking for 'friendly' numbers

Teaching Episode 4—Pricing Orders

The final teaching episode comprised two lessons as illustrated in Figure 6.31.

TEACHING EPISODE 4	
Lesson 1	**Lesson 2**
Solving the problem – 64 trays of 24 cakes were packed into boxes of 12. Each box was sold for $28. How much money was made? Class discussion – Looking for friendly numbers	Applying strategies to new problems Class discussion – Extending the notion of using friendly numbers for calculations
Mathematical Practice 4 – **WAYS OF WORKING MATHEMATICALLY:** Looking for 'friendly' numbers	

Figure 6.31—Teaching Episode 4

Episode 4: Lesson 1

The final teaching episode commenced in the same way as the other episodes, with a narrative set in the context of the bakery:

To fill all the orders over the course of one month, Charlie had to bake an additional 64 trays of cakes. There were 24 cakes in a tray. He packed the cakes into boxes of 12. Each box was sold for $28. How many cakes did Charlie bake and how much money did he get from all his cake sales?

This problem presented students with a multi-step problem which could extend their existing strategies to enable them to solve a 3- x 2-digit multiplication question. During the course of this teaching episode, two key developments were observed. First, students' use of the array generally decreased. However, those who continued to employ the array made some representational errors. This had not been previously observed, and it was hypothesised that these representational errors were based on a shift in the type of multiplication context. Previous problems focused on contexts that lent themselves to array-based multiplication; that is, the contexts drew on cakes in equal rows and columns. But this new problem, which asked for the total cost of multiple boxes at a set price, introduced a rate-based context. This context could not be easily represented as an array and the array was not a helpful model for the question.

The second key observation was an increase in students' use of symbolisation, with many students solely using equations to demonstrate their working. Across both classes, the students used various instantiations of the distributive property classification. The strategies were further sub-divided into two groups.

Group 1—Partitioning both numbers based on place value

The majority of students from both classes used the place value properties of the numbers to form simpler calculation. In Figure 6.32, Ava and Hannah partitioned all of the numbers into their place value parts, with little consideration for the particular numbers used. This strategy was judged as a sophisticated and efficient strategy and noted as a means of distinguishing it from the second group of strategies.

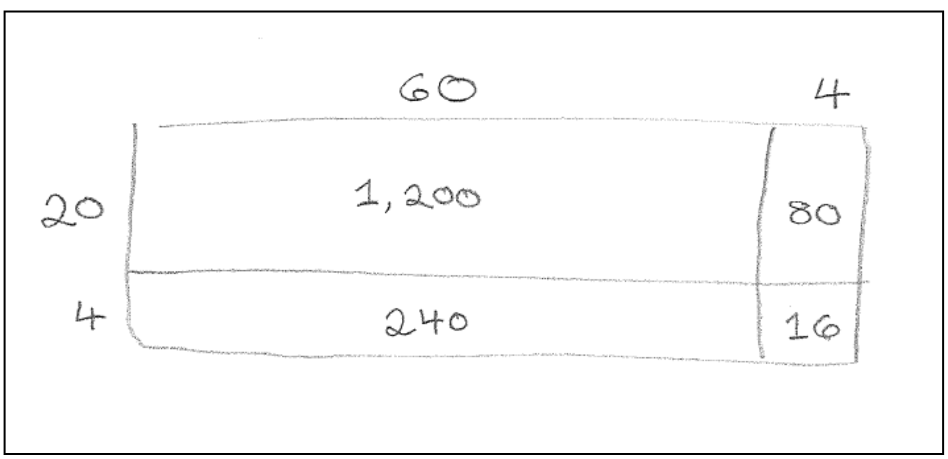


Figure 6.32—Ava and Hannah's work sample from Teaching Episode 4

Group 2—Partitioning numbers as needed

A second group of students focused on partitioning numbers to create multiplication computations that they could perform mentally, generally distributing numbers at different times through the course of calculation. While their partitions were predominantly based on place value, it was observed that the students carefully considered the numbers before partitioning. Some students only partitioned one number, while others created increasingly smaller partitions as needed.

Episode 4: Lesson 2

In the second lesson, the students were presented with a series of multiplication problems. The first group of problems, presented verbally at the start of the lesson, asked the students to respond with an emphasis on the strategy they would use to solve the problem, rather than necessarily giving an answer. Next, they were given a further series of multiplication problems on paper. In seeking to solve these problems, they were encouraged to work with another student and, in their working, carefully consider the numbers in the problems.

The Emergence of Mathematical Practice 4

This sequence of activity led to the negotiation of the fourth mathematical practice: looking for 'friendly' numbers. In the negotiation of this practice, one event was noted as significant: 'looking for efficiency'. The previous teaching episodes contributed in some manner to the development of this practice amongst the students, culminating in one significant event described in the following section. The sequence of occurrences in Class 2 is used to illustrate the emergence and acceptance of the mathematical practice across both classes. The story of one student, Louisa, then describes the way in which she contributed to and participated in the negotiation of the mathematical practice of looking for friendly numbers.

Conceptual Event 1—Looking for efficiency

The end of the first lesson concluded with a class discussion focusing on the differing ways that students had distributed and then multiplied the numbers. Ava and Hannah, in presenting their strategy for 64 x 24 based on the place value of the numbers (Figure 6.32), reasoned that breaking the numbers into place value parts created calculations that were easy to perform. Following Ava and Hannah, Sarah and Olivia presented their strategy for the same calculation (Figure 6.33). In this case, the students had left 64 as a whole number and split 24 into 20 and 4. They also reasoned that these were calculations that were easy for them to perform.

Sarah: *We timesed (sic) 64 by 20 which is really just like doing 64 times 2 and then adding a zero. And then we just timesed (sic) 64 by 2, and then doubled again to get 64 times 4.*

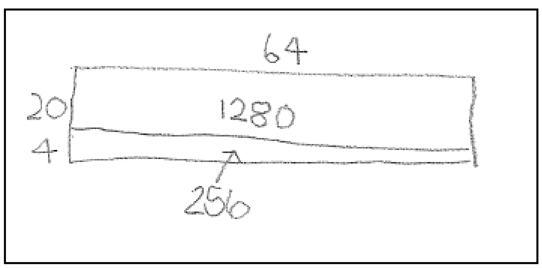


Figure 6.33—Sarah and Olivia's work sample from Teaching Episode 4

Further methods of distributing the numbers were presented by students. The discussion shifted to the sophistication and efficiency of these strategies, in keeping with the sociomathematical norms of the classroom. It was discussed that numbers could be partitioned in many ways but, most importantly, numbers should be partitioned in a way to perform calculations efficiently and accurately.

The students started to discuss the numbers in the solutions that had been presented. They noted that, in a strategy like Sarah and Olivia's (Figure 6.33), doubling was efficient and accurate. Indeed, according to Sarah and Olivia, it was possible for them to perform the entire calculation in their head. The students also observed that partitioning into place value parts, such as Ava and Hannah's strategy, created simple calculations. Hannah observed that it was easy to keep track using this strategy.

Hannah: *…Well if you just follow the pattern…you times all the numbers in the first number by all the numbers in the second number then you know what numbers you have already times together and you don't have to think too much about it.*

The students arrived at a consensus on two important generalisations through this discussion. First, distributing numbers into place value parts would always produce 'friendly numbers' to work with. There was common agreement that place value partitioning would be the most efficient strategy in most cases. It was at this point I raised the standard multiplication algorithm and explained that this was the basis for the way that it worked, although the algorithm was not explored at this point. The second generalisation students agreed on was the importance of first considering the numbers to be multiplied. Although students agreed that partitioning numbers into place value parts would be the most efficient strategy in most cases, there was also a consensus that, sometimes, the numbers would enable efficient and accurate use of strategies drawing on the associative property. This reflected Sarah and Olivia's strategy, in which they used place value and doubling to mentally calculate the total number of cakes efficiently.

The Story of Louisa

The second lesson began, as detailed earlier, with the students being presented verbally with a series of multiplication problems. For each problem, students were asked to respond with an emphasis on the strategy they would use to solve the problem, rather than necessarily giving an answer. The aim was to encourage students to carefully consider the numbers in the problem and to weigh up the different strategic options before responding with their chosen strategy.

The first problem given to the students was 24 x 8, a repeated problem from the first teaching episode. The students agreed that this could be efficiently calculated using doubling or place value partitioning and that such calculation could be performed mentally. Similar problems were then given to the students, and strategies to solve each were suggested by the students. The last multiplication in the sequence was 32 x 25. One student, Louisa, responded immediately.

> *Louisa:* *25 is a friendly number because you just multiply it by 4 to get 100, so you divide the 32 by 4 to get 8, so it is just the same as 8 x 100.*

This quick response was surprising. Through the course of the experiment, Louisa had demonstrated a procedural knowledge of multiplication. In earlier tasks she had relied on the algorithm to solve problems and found it difficult to reason conceptually, which was in keeping with the sociomathematical norms. Her answer to this question, an answer given so quickly and confidently, demonstrated significant growth in Louisa's own understanding of multi-digit multiplication. What was equally surprising was the response of her classmates. I was quietly excited with such reasoning, whereas there was little reaction shown by her classmates. It was not that they were unimpressed; it was simply that Louisa's response was an expected way of working. The mathematical practice of looking for and using friendly numbers to employ sophisticated and efficient solutions was indeed taken-as-shared.

Following these verbal questions, the students worked with a partner on some additional written questions. The shift in Louisa's way of reasoning continued to be evidenced. Instead of working procedurally, she was now clearly working and reasoning conceptually. As she continued to work on the additional problems, her ability to reason conceptually was reinforced. She discussed alternative methods for performing calculations and considered the efficiency of different strategies. She was not limited to just one procedure to perform a multi-digit multiplication calculation but evaluated different strategies and their effectiveness.

Louisa did revert to using the formal algorithm for a number of the questions in this last set of problems, but there was a notable difference in the way she used it. She was now able to explain how the algorithm worked, demonstrating her understanding of the distributive property.

Conclusion—Mathematical Practice 4

In summary, this section documented the development of the fourth mathematical practice. This mathematical practice described looking for 'friendly numbers'-aided computation and became a taken-as-shared way of working mathematically in both classes. It was described that this mathematical practice grew from the last three practices and, as such, one significant event in this fourth episode united the learning from the previous three episodes. This event was described as students looking for efficiency. The work of Louisa was used to illustrate the cognitive shifts made by students and how one student's learning contributed to the collective learning of the class.

Mathematical Norms

In this chapter, four mathematical practices that emerged and evolved in the workings of the two classrooms were described. The negotiation of these practices was facilitated by the social and sociomathematical norms instituted in each classroom (described in detail in Chapter 5). An additional set of five norms emerged as significant in the negotiation of the mathematical practices. I have termed these *'mathematical norms'*: looking for similarity and difference, making inferences, utilising representations, justifying mathematical thinking and forming generalisations. These five mathematical norms were introduced and maintained by the teacher/researcher, in the same way that the social and sociomathematical norms were introduced and maintained. The mathematical norms became central to students thinking and reasoning, and therefore, central to the development of the identified mathematical practices.

This section of the chapter documents how students employed each of these mathematical norms as they participated in classroom activity. Chapter 7 discusses the significance of these norms in the development of learning in the classroom and how they fit into the interpretative framework used to analyse the data collected. It is important to acknowledge that while each norm is discussed separately, all were interconnected in the way they played out in the students' learning. For example, seeing similarities led students to recognise patterns and relationships, and so form inferences. Based on these inferences, students were then able to make generalisations. To make generalisations, students used representations to investigate the multiplicative structure. The representations were then used to justify their thinking. As such, the mathematical norms were inextricably intertwined.

Looking for similarity and difference

An important focus for learning across all the teaching episodes was the observation of similarities and difference, particularly between the different strategies that were used to solve problems, and identify the connections that existed between them. Comparison in this context was not designed to promote some methods as ‘better’, even though greater sophistication and efficiency were evident in some strategies than others. Comparing strategies allowed students to meaningfully refine their own thinking and reinvent strategies based on the working of others.

An example of noticing similarity was seen in Episode 1. In the second lesson of Episode 1, selected work samples were compared in the class discussion. While all the work samples presented a different strategy, the students noticed strong similarities between solutions. Noticing similarity created greater cohesion and a sense of shared purpose between the students. Rather than just presenting a collection of different strategies in the class discussion, students were encouraged to look at how one strategy built on another. In doing so, students realised that, although their strategy may look different from others, conceptually and structurally there were important commonalities. Students could use fellow students’ reasoning to build on their existing understandings. New strategies were not presented as something that had to be learned. Instead, different strategies presented students with opportunities to reflect on their own thinking in a slightly different manner and allowed students to modify and refine their own strategies. The benefits of noticing similarity and difference in this context helped to establish this as a common way of working in each classroom.

The focus of Episode 3 was on difference. In this episode, the students noted the difference between working additively and working multiplicatively. In Class 1, I initiated the comparison of the two ways of working as part of the class discussion. In Class 2, it was initiated by the students. Noticing the difference between working additively and multiplicatively enabled students to notice their errors, correct their working and refine their own understanding. Additionally, the students recognised the difference between the additive and the multiplicative structure. An example of this was seen in the work of Darcy. Darcy compared the models of a number line and an array to help explain the difference between multiplicative and additive thinking. In her comparison, Darcy illustrated addition as one-dimensional using the number line, and multiplication as two-dimensional using the array. Sharing her observation helped other students in the process of sense-making.

Making Inferences

In each teaching episode, a considerable amount of time was allocated to investigating a single question. This allowed students time to construct new learning based on observed patterns and relationships, or, put more simply, to make inferences. Episodes 2 and 3 provide examples of students making important inferences.

In Episode 2, the students were asked to consider why the value of 24 x 8 was equivalent to 12 x 16. Based on their exploration, Dylan and Ryan noted that halving 24 gave 12 and doubling 8 gave 16, and introduced the concept of the associative property to the class. Further exploration using the array allowed Archie to realise conditions where such strategies would be beneficial. Connecting his knowledge of place value, Archie recognised that any number multiplied by 5 could be halved and then multiplied easily by 10. Similarly, any number multiplied by 50 could be halved then multiplied by 100. He was also able to infer that there were instances when using factors in this manner would not be as efficient as using strategies based on the distributive property of multiplication.

In Episode 3, Amelie used her existing understanding of the multiplicative structure to explore and then explain why additive compensation strategies could not be used in multiplicative contexts. In this case, Amelie recognised the relationship between the numbers being multiplied and that manipulating the array using factors maintained its rectangular, or multiplicative, structure.

Utilising Representations

During the course of the instructional sequence, the problem-solving process was left open to students. They could choose to use any concrete resources, representations or symbolic recording that they deemed helpful. In each teaching episode the students chose to use the array to support their working.

In Episode 1, the students used the array as a tool to make sense of their own strategies and those of other students. Through the array, students observed similarities between the different strategies that were used. These similarities may not have been as obvious through symbolic recording alone, as illustrated through the account of Jake given earlier in this chapter. In solving the first question presented in the instructional sequence, Jake used skip counting to calculate the total, which was indicative of Jake's understanding and skills in multiplication given his results in the pre-assessment interview. Jake represented his strategy on the array by circling groups, which also served as a mechanism of keeping track of the count (Figure 6.4). Jake compared his representation with those of students who used the distributive property and recognised that the coordinated rows and columns of the array could be used to extend his strategy to use the distributive strategy based on place value. The use of the representation was critical in making this shift, indeed, there was little or no connection seen between the different forms of symbolic recording used by the students.

In Episodes 2 and 3, students' use of representation was critical to developing their understanding of the associative property. In particular, it was the physical manipulation of the array that proved most powerful. Students recognised that halving and doubling the array maintained the rectangular structure, while additive compensation did not. This realisation launched a deeper exploration into the structure of multiplication.

It was not just that students used representation that proved to be significant in this study; it was how they used representations. It was students' exploration of the multiplicative structure that was of primary importance. It was not simply a tool to aid calculation. It is also important to note that students switched between different forms of the array dependent on the function that they needed it to serve. On several occasions, the form of the array that they used differed to the array presented in the question. This was true in Episode 3 as previously discussed. While the students were presented with an open array, or area model for multiplication, the majority of students across both classes created their own arrays using grid paper. This form of the array, with all parts represented with squares, supported their thinking in a way that the open array did not. Students utilised the representation in a manner that was most meaningful to them in the process of sense-making.

Justifying Mathematical Thinking

A key feature of the instructional sequence was the limited number of questions that students were asked to solve. Restricting the number of questions provided students with a significant amount of time to construct solutions for each problem. In keeping with the instituted social and sociomathematical norms, students were expected to justify their mathematical thinking as part of their worked solutions. Rather than the teacher telling students whether their answers were correct, students' justifications satisfied them as to whether they were correct. Justification was also an important part of the class discussions. When sharing their strategies with the class, students were also expected to justify their strategies. Most often, these justifications were the focus of discussion rather than the strategies themselves.

Forming Generalisations

The aim of the instructional sequence was to build students' computational fluency in multi-digit multiplication. Central to this was students' ability to form generalisations about multi-digit multiplication. The students commenced with their own reinvented strategies that drew from their current levels of understanding. As their understanding developed, so did their strategies. The refinement of students' understanding, and strategies, led to generalised knowledge that could be applied to new and different situations. As such, these generalisations were a key outcome of the learning trajectory.

In summary, this section has documented the five important mathematical norms. In the same way that social and sociomathematical norms are central to the culture of learning in the classroom, so were these mathematical norms. This is discussed in further detail in the following chapter where the mathematical norms are placed within the interpretative framework used in this research.

Conclusion

This chapter has reported the results of the teaching experiment phase of this research project, in which the hypothetical learning trajectory on multi-digit multiplication was implemented in two classrooms. The results were presented as a coordinated analysis of the social and individual aspects of learning. To achieve this, a narrative of each teaching episode was presented and then a description was provided of the significant events that led to the acceptance of each mathematical practice. In all, four mathematical practices were identified:

1. The Array as a Tool for Sense-Making: Partitioning based on place value;
2. The Array as a Tool for Sense-Making: Using factors to manipulate the array;
3. Ways of Working Mathematically: Thinking multiplicatively; and
4. Ways of Working Mathematically: Looking for ‘friendly’ numbers

Additionally, five mathematical norms were identified through the instructional sequence. These mathematical norms were instrumental in the establishment of the mathematical practices and the analysis of results. The five norms identified were:

1. Looking for similarity and difference;

2. Making inferences;
3. Using representations;
4. Justifying mathematical thinking; and
5. Forming generalisations

The coordinated analysis of social learning and that of individual students served to give a comprehensive picture of the implementation of the learning trajectory on multi-digit multiplication and enabled modifications and refinements to be reliably made to the learning trajectory.

CHAPTER 7

THE RETROSPECTIVE ANALYSIS

Introduction

This chapter presents the final phase of the Design Research process—the retrospective analysis. The study's findings are discussed and documents the final version of the learning trajectory for multi-digit multiplication. Through the analysis of results, three themes emerged as significant:

- the reification of the array;
- the interaction between strategies and key developmental understandings (KDUs); and
- the social and individual use of mathematical norms

The three themes form the structure of the chapter. The first section of this chapter discusses the reification process of the array and explicitly addresses the first research question. The second theme, the interaction between students' strategies and the key developmental understandings (KDUs) of multiplication, forms the second section of the chapter in response to the second and third research questions. The social and individual use of mathematical norms form the third section of the chapter. This section reiterates the mathematical practices that emerged through the instructional sequence, which were presented in Chapter 6 and, in so doing, answers the fourth research question. This part of the chapter also discusses five mathematical norms that were evident in students' working. The chapter concludes by presenting the final learning trajectory produced as a result of this research.

The Reification of the Array

This section of the chapter addresses the first research question:

How does the array develop from a model of a contextual situation to a model for more generalised mathematical reasoning in multi-digit multiplication?

The array is recognised as a powerful tool for modelling multiplication (Battista et al., 1998; B. Davis, 2008; Fosnot & Dolk, 2001). What is less evident in the literature is the evolution of the array from a *model of* a contextual situation to a *model for* mathematical reasoning (Gravemeijer, 1999). This study examined the power of the array as a *model of* a contextual situation through to a *model for* mathematical reasoning as enacted by the students through their mathematical activity over the course of an instructional sequence. Gravemeijer (1999) described this shift as a process of reification, where mathematical activity takes on object-like character as a result of student activity. In this instance, the array is reified through student action. The array moves from being a *model of* acting with arrays of cakes to a *model for* reasoning with the multiplicative structure. According to Gravemeijer (1999), there are two stages to the process of reification. First, students' activity is bound in the context of the problem, a stage known as the referential level. The second stage is the general level, where students' interpretations and solutions operate separately to the contextual imagery.

Saxe's (2002) form-function shift framework provided a useful lens for studying the students' progressive reification of the array. The framework helped explain how students' use of the array shifted over the course of the instructional sequence, to serve differing functions. In Saxe's terms, the array serves as a cultural artefact in the classroom, which can be used to accomplish different mathematical goals. Saxe (2002) explained that, in themselves, these cultural artefacts do not hold any mathematical meaning. It is only through student activity that artefacts, or forms, take on mathematical meaning. In the process of activity, particular forms will serve particular functions to achieve mathematical goals. The emergence and change of functions for different forms can be viewed on three levels (Saxe, 2002):

- Microgenesis: The way in which particular forms, and the functions they afford, are utilised by students through activity to achieve mathematical goals.
- Sociogenesis: The appropriation and spread of forms and functions in communities of learners.
- Ontogenesis: The shifting relations between forms and the functions that they serve in individual activity over an individual's development.

Saxe (2002) presented these three processes of development as interconnected (see Figure 7.1). Individual students use different forms to accomplish emerging mathematical goals in their microgenesis constructions. Students share their constructions with the class, facilitating the appropriation and spread of the microgenesis constructions that is the root of the sociogenesis of knowledge in collective practices. As Saxe (2002) explains, the "uptake by additional others may lead to an eventual institutionalization of such means as the received or reified convention" (p. 297). The ontogenesis development is represented as shifts in the micro- and sociogenesis developmental processes, that is, shifts in individual and collective use and organisation of forms in solving similar problems.

Saxe's form-function shift framework (Saxe, 2002) was used by Teppo & Van den Heuvel-Panhuizen (2014) for a similar purpose when studying the number line. They observed that different forms of the number line served different cognitive and didactic functions. Teppo & Van den Heuvel-Panhuizen (2014) noted that there was not one ubiquitous 'number line'. The form of the number line changed dependent on the intended function. Similarly, in my research, there was not one ubiquitous array; students chose to use different forms of the array to serve different functions. The process of reification and abstraction of the array was intrinsically tied to the function of each form of the array. As such, the form-function shift framework was used to analyse the array and to show how the array moved from a *model of* a contextual situation to a *model for* generalised mathematical reasoning.

For the purpose of this research, the *form* of the array refers to its specific visual features. The *function* of the array is defined on two levels: the pedagogical functions of the various array models and the student-generated functions developed through mathematical activity. Three forms of the array were used in the instructional sequence: a pre-partitioned array, an open array and an array with all parts visible. The development of, and interaction between, the micro-, socio- and ontogenesis processes for each form of the array are described in turn in the following sections.

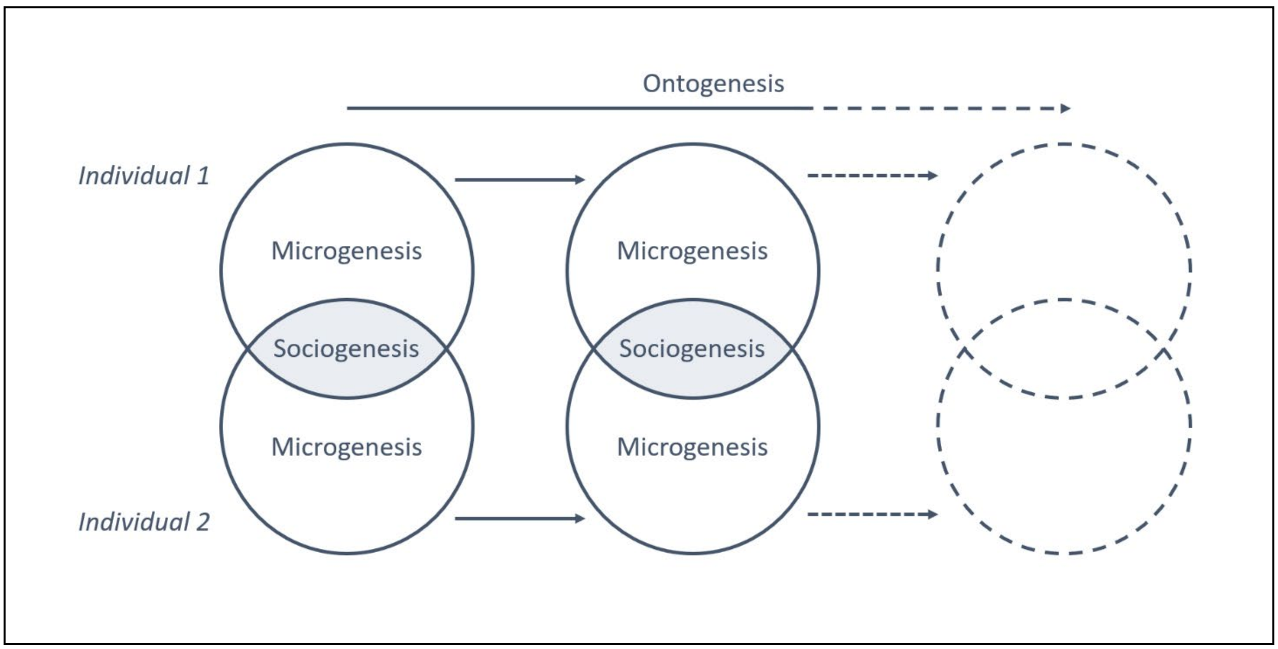


Figure 7.1—Saxe's form-function shift process

Pre-partitioned array

Pre-partitioned arrays were used in the first two teaching episodes of the instructional sequence (see Figure 7.2). Both arrays served one overarching pedagogical function: to highlight specific multiplicative properties. The first array focused student attention on the distributive property of multiplication. The second highlighted the associative property of multiplication through the strategy of doubling and halving. In the literature, this form of the array as a tool for highlighting key multiplicative properties is the most widely recognised. Partitions based on ten have been used to accentuate the distributive property and the power of strategies based on place value (Bobis, 2007; Izsak, 2004; Young-Loveridge & Mills, 2008, 2009). Barmby et al. (2009) used 5 x 5 partitions to allow students to draw on the power of five, enabling links to be made to place value but also demonstrating that splitting the array need not be restricted to place value parts only.

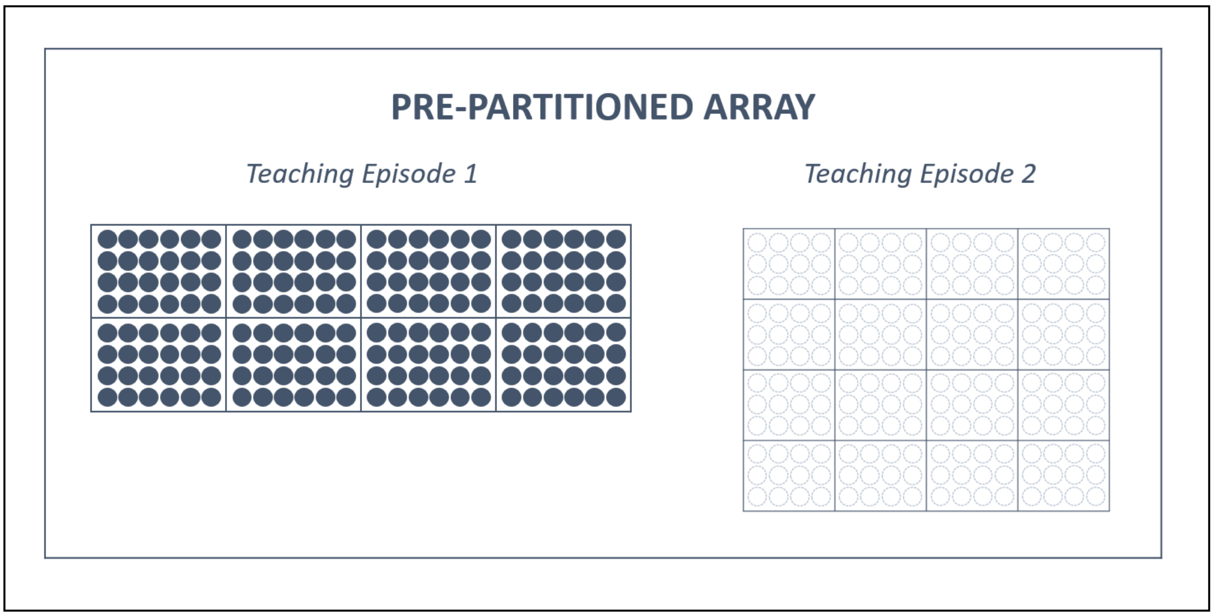


Figure 7.2—Pre-partitioned arrays used in the instructional sequence

The pedagogical function of the pre-partitioned array was realised through two student-generated functions: the use of place value to partition the array and the manipulation of the array based on factors. These functions evolved into socially accepted ways of working in the class and are identified as mathematical practices for the instructional sequence. The micro-, socio- and ontogenetic processes are discussed in the following section for each student-generated function with reference to the overarching pedagogical function.

Function 1—Partitioning based on place value

The first teaching episode presented students with a 24 x 8 array (see Figure 7.2). The array was carefully designed so that the structure revealed 24 x 8 in two ways: eight trays of 24 cakes were presented in the array and the individual cakes on trays were presented in a 24 x 8 arrangement. These two arrangements are illustrated in Figure 7.3. The design of the array did not focus attention on the use of place value to solve the calculation. Rather, the power of the distributive property was unveiled through the microgenetic constructions of students.

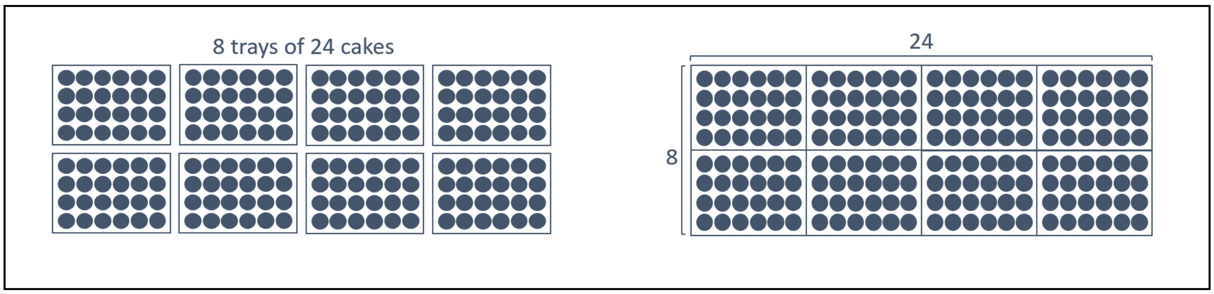


Figure 7.3—Pre-partitioned array used in Teaching Episode 1

The microgenesis process of development is concerned with how individuals organise cultural artefacts, or forms, to achieve mathematical goals. In this case, an overarching mathematical goal was inherent in the question posed to students: they needed to determine the total number of cakes baked in the context of the problem. A more specific goal emerged for students—simplifying the calculation into smaller, known parts in order to calculate the total. One way in which this was achieved was to partition the array into place value parts. The mathematical activity of two pairs of students, Zoe and Lucille and Layla and Maddie, illustrate the microgenetic process.

Zoe and Lucille used the pre-partitions to create eight separate trays, and, in doing so, created eight smaller calculations (see Figure 7.4). For each tray, a group of 20 and a group of 4 could be easily identified. They then multiplied the 20 by 8 and the 4 by 8. Layla and Maddie utilised the structure of the larger array, creating one large group of 20 x 8 and a smaller group of 4 x 8 (see Figure 7.5). In both cases, the function of the array was the same, despite the function being realised using different strategies.

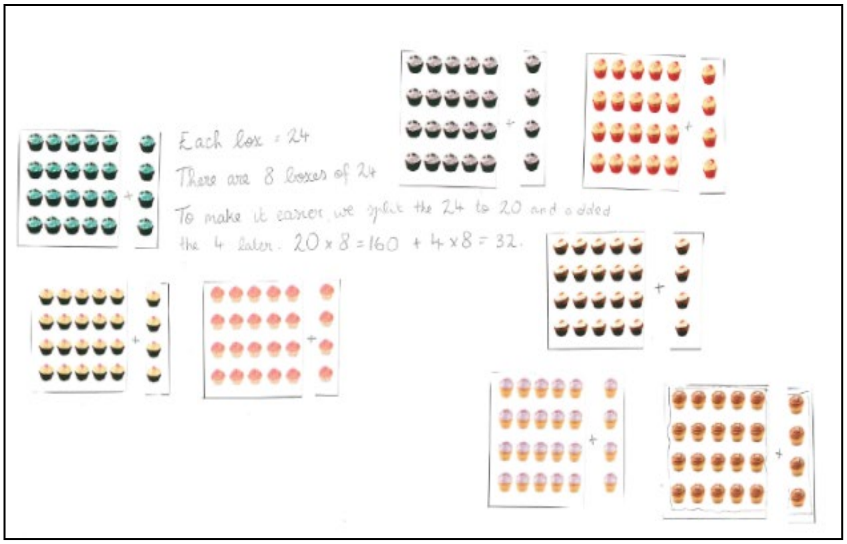


Figure 7.4—Zoe and Lucille's work sample from Teaching Episode 1

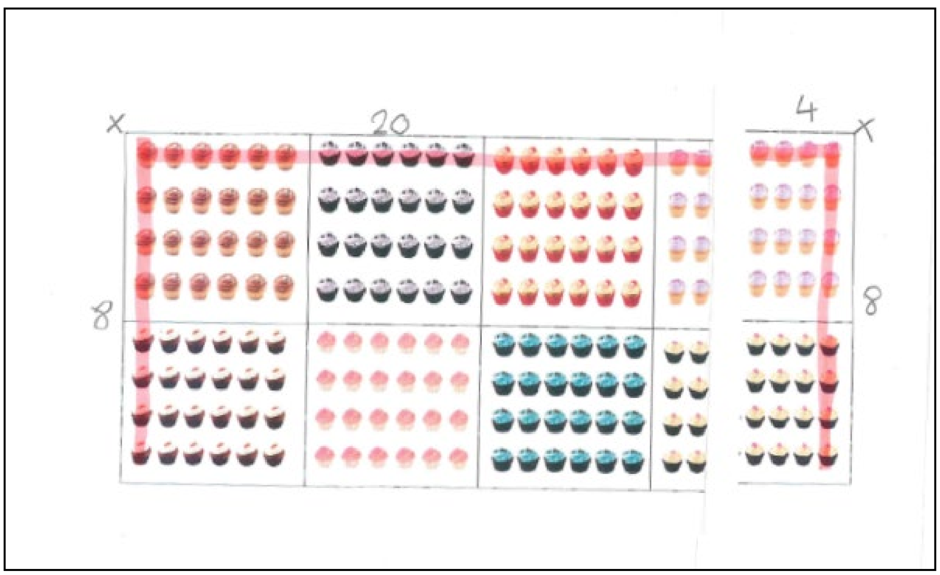


Figure 7.5—Layla and Maddie's work sample from Teaching Episode 1

The sociogenesis process took place when the strategy of partitioning the array was adopted across the class because of its efficiency and sophistication. The function of partitioning then became part of the microgenetic processes of individuals in the class. To illustrate, consider the work of one student, Jake. Through the course of class discussion, Jake recognised connections between his rudimentary strategy of skip counting and Layla and Maddie's strategy of distributing the larger array based on place value. Jake associated counting in 10s to partitioning the array in groups of 10 (see Figure 7.6). He folded the function of distributing the array into his own microgenetic processes, applying the appropriated function in keeping with his own levels of understanding.

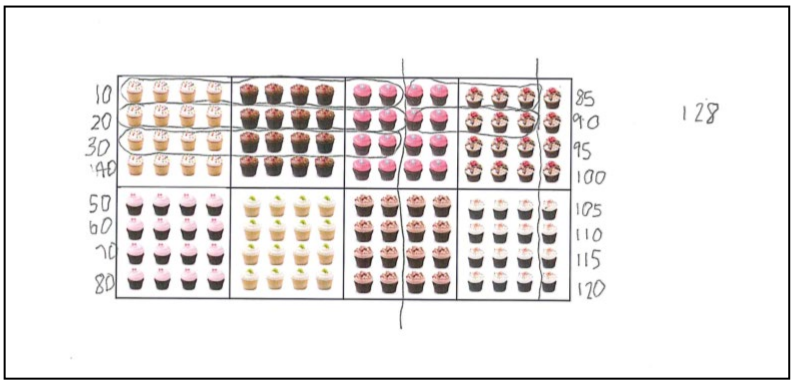


Figure 7.6—Jake's work sample from Teaching Episode 1, Lesson 2

In the process of ontogenesis, students develop new functions for forms. They may also appropriate new forms and use them for previously understood functions (Saxe, 2002). In this case, both the development of new functions and the appropriation of new forms were observed. Students transferred the function of partitioning the array based on place value to the open array and also to their numerical representations. Partitioning the open array into place value parts to identify all partial products formed became common practice in the classroom. In this context, the function of the open array became a more generalised way of supporting calculations. Partitioning based on place value was also transferred to numerical recording of calculations as students shifted from array-based strategies to more abstract representations.

Function 2—Manipulation of the array based on factors

An additional function that emerged for the pre-partitioned array was manipulation of the array based on factors. Through this function, the array moved from a static tool to a dynamic representation. This function developed through the second teaching episode using a pre-partitioned array designed to highlight the associative property of multiplication.

At the start of the instructional sequence, students used the array to aid calculations, without modifying its arrangement. In the second teaching episode, students were asked to calculate the total number of cakes in a collection of cakes presented in an array of 16 x 12. Here students' interactions with the array shifted; they used the partitions present in the array to alter its arrangement. This is illustrated through the microgenetic constructions of one pair of students, Ryan and Dylan.

Ryan and Dylan partitioned the array based on place value and noted that the result for this collection of cakes, 16 x 12, was the same as the total number of cakes in the first teaching episode, an array of 24 x 8. This led into investigating a second mathematical goal—why did 16 x 12 = 24 x 8? Recognising that 16 could be halved to make 8 and that 12 could be doubled to give 24, the pair divided the array in half and rearranged it to transform 16 x 12 into 24 x 8 (see Figure 7.7). Through their mathematical activity, they established a new function for the pre-partitioned array, that the array could be manipulated.

The appropriation and spread of manipulating the array was realised through the whole class discussion of the second teaching episode. The focus of the discussion was on halving and doubling and how this could be done with the array while preserving its rectangular structure. Two students, Tom and Frederique, shared their reasoning with the class:

Frederique:	*Well we noticed that you could turn the array into 24 by 8 by cutting the boxes in half and then moving one half of them down here (gesturing on the array). That made 24 down here and 8 across the top…yeah.*
Teacher:	*So, Tom, you halved the length of one side and doubled the length of the other? Is it OK to do that?*
Tom:	*Well, yeah, it's fine.*
Teacher:	*Why? Why do you say it's fine?*
Tom:	*Well, like…all the cakes are still there. It just shows why they are the same.*

Given time to explore this strategy further, there was class consensus that the strategy was correct as all the items in the original array were preserved, as was the structure of the array; indeed, all that was happening was a rearrangement of boxes. Following this discussion, students began to interact with the array as a dynamic tool rather than as a static tool.

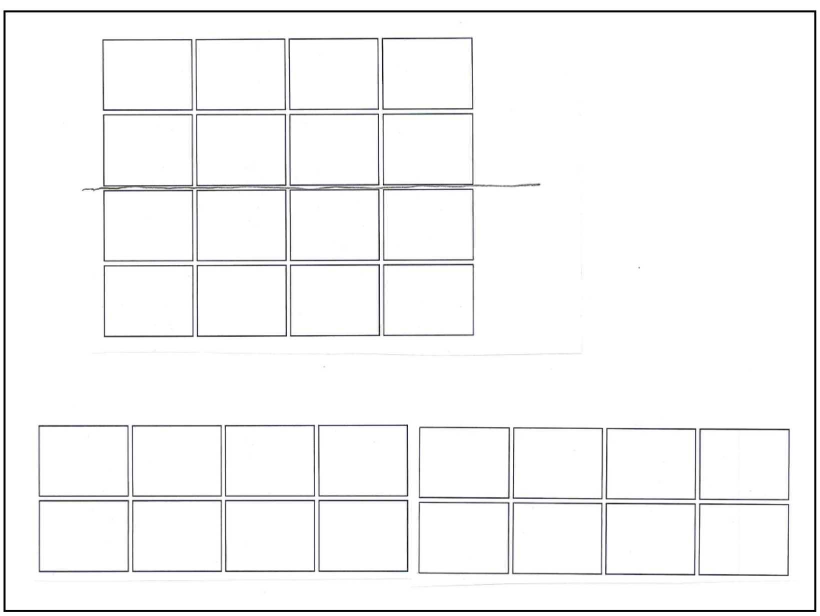


Figure 7.7—Ryan and Dylan's work sample from Teaching Episode 2

The process of ontogenesis was seen in the transfer of this function to two new forms of the array: the array with all parts visible and the open array. In both cases, the students used the function of manipulating the array in the process of sense-making and refining their understandings.

Open array

The curriculum in use at the time of the study (the NSW Syllabus) recommended the use of an open array, or area model, to represent strategies for multi-digit multiplication (Board of Studies NSW, 2002). The students were introduced to this form of the array in the third teaching episode when they were asked to compare the areas of two rectangular trays (see Figure 7.8). The pedagogical function of the array in this form was to support students' computation. The student-generated function for the open array was as a tool for mathematical reasoning, based around the organised development of the conceptual understanding of the multiplicative structure.

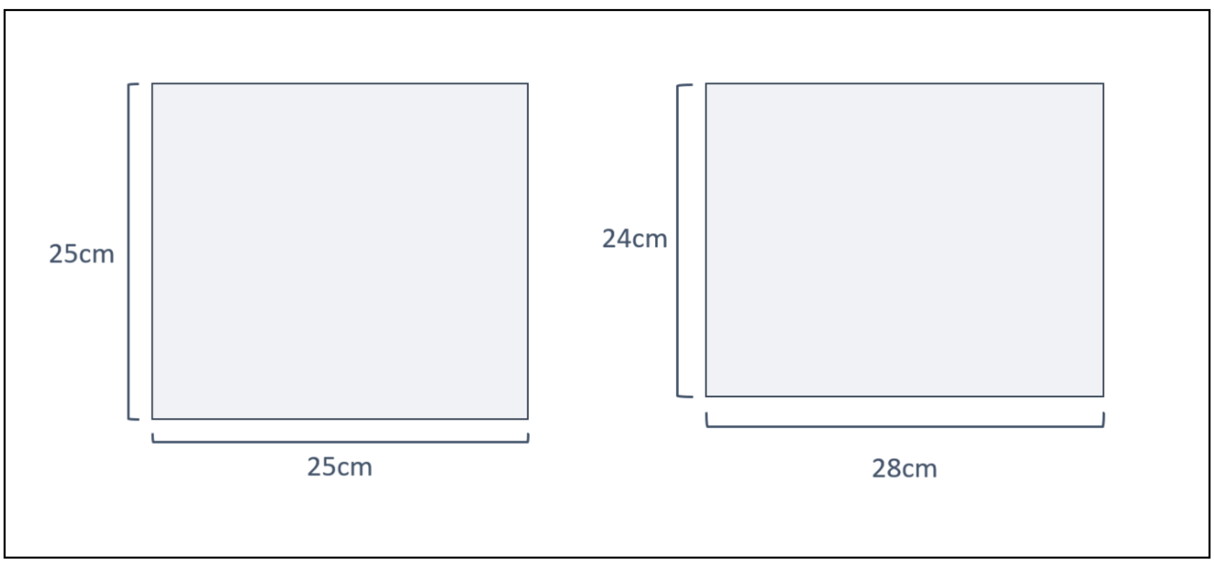


Figure 7.8—The open arrays used in Teaching Episode 3

Function—Reasoning mathematically

The students' microgenetic constructions using the open array were based on simple notations. Rather than partitioning the open array to represent numbers proportionally, students used the model to support their calculations. As students' use of numerical recording increased, their array sketches became more simplistic. This suggests that the open array served as a tool for more generalised mathematical reasoning as students progressed from the visual model to numerical strategies. A similar observation was also made by Bobis (2007) in a small-scale study where a student was introduced to the open array, and moved from reliance on the model to pure numerical recording over a two-week period.

Comments on the socio- and ontogenetic processes around the use of the array were more difficult to make. It was not until the final teaching episode that students effectively interacted with the function of the open array, and so further time would have been required to study the appropriation and development of the array in this form. The open array was first introduced in the third teaching episode, however, in the process of problem solving, the students required greater structural support. The open array did not prove robust enough for students' sense-making processes when the students were faced with their own misconceptions and so they reverted back to a grid array, or an array with all parts visible.

While some studies have shown that relating rectangular area to multiplication can prove to be problematic for students (Izsak, 2004; Simon & Blume, 1994; van Dooren et al., 2010), this was not the basis for students' difficulties with the open array in this study. Students' reasoning, particularly in the third episode, was additive in nature. Their errors indicated that these Year 5 students did not fully appreciate the binary nature of multiplication, even though instruction was introduced as a curriculum requirement from a much earlier age (Board of Studies NSW, 2002). Therefore, it should not be assumed that students automatically comprehend the coordination of rows and columns in arrays. To function effectively with the open array as a tool for mathematical reasoning, students need to operationalise the row-by-column structuring to see the whole scheme in the absence of the original visible parts (Battista et al., 1998).

Array with all parts visible

A third form of the array emerged through the instructional sequence: a rectangular arrangement of discrete objects (Siemon et al., 2011), or, as described in this research, an array with all parts visible (see Figure 7.9). This array arose naturally out of student activity, rather than being presented to the students. As this form of the array was not part of the planned instructional sequence, it did not serve any particular pedagogical functions. It was a student-generated function that emerged as a tool for sense-making as students faced difficulty.

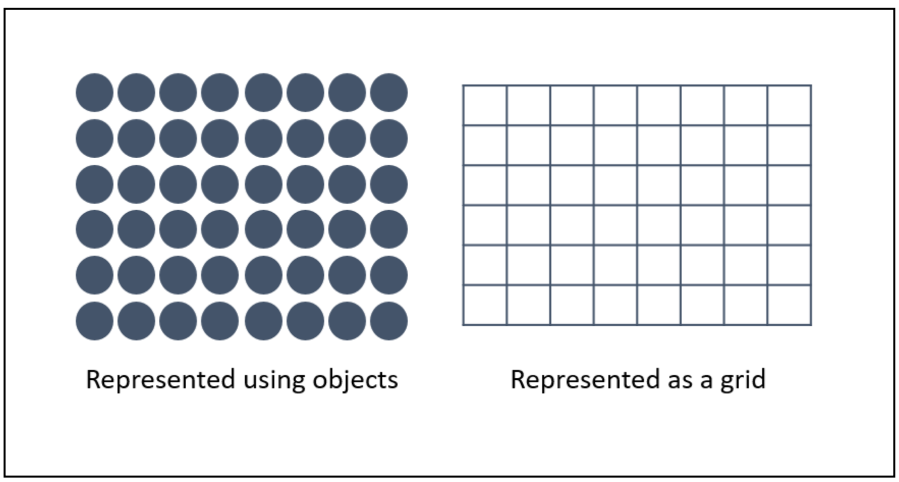


Figure 7.9—Array with all parts visible

Function—A tool for sense-making

Each teaching episode commenced with a contextualised problem, with the problem fashioning the mathematical goal for students. In each case, the form of the array was connected to the imagery generated by the problem. Students did not appear to have difficulty using the various forms of the array while working within the context of the problem. Problems arose, however, when students shifted from working within the context to working in the domain of more formal mathematics. In this domain, students were faced with their own incomplete or insufficient internal representations (Goldin & Shteingold, 2001) and they required a previously understood external representation of multiplication (an array with all parts visible) to assist in modifying and clarifying their understanding. This familiar form of the array highlights the binary nature of multiplication as it displays the number of groups, the size of each group and the total number in each group (Siemon et al., 2011). These qualities of the array assisted students in making sense of the multiplicative structure. This is illustrated by the microgenesis constructions of one student, Amelie, in the third teaching episode.

In the third teaching episode, the students were shown two rectangular areas (represented as open arrays) and asked to hypothesise which area they thought would be bigger: 25 x 25 or 28 x 24. Amelie reasoned that 28 x 24 would be bigger, arguing that 2 could be taken from 28 and added to 24 resulting in 26 x 26 (which is larger than 25 x 25). While 28 x 24 was indeed bigger, Amelie came to see that her reasoning was incorrect. A new goal emerged: why was 28 x 24 not equivalent to 26 x 26? To achieve the mathematical goal, Amelie regressed from an open array to an array with all parts visible which she constructed from grid paper. She physically represented the problem by cutting two rows off the 28 and adding them to the 24, and recognised that the rectangular structure of the array was lost as a smaller 2 x 2 array was missing from the corner. By repeating this activity with different numbers, Amelie was able to make sense of her misconceptions.

The array with all parts visible was used as a tool for sense-making across the class, demonstrating the process of sociogenesis. A process of ontogenesis was not apparent, as the function of the array did not vary. Nor did students transfer the function of sense-making to another form of the array.

The Reification of the Array

The process of reifying the array describes a shift in students' thinking and actions as the array becomes an entity in its own right; the array becomes a means to reason mathematically rather than a tool to symbolise mathematical activity within the context of a problem (Gravemeijer, 1999). The array moves from a *model of* a contextual situation to a *model for* more formalised mathematical reasoning. In the course of this instructional sequence on multi-digit multiplication, the reification of the array occurred through students' interaction with multiple forms of the array and the functions that emerged for each form (see Figure 7.10). Two factors were important to this process: the flexibility for students to fold back between multiple forms of the array, and the opportunity for students to explore the structure of multiplication.

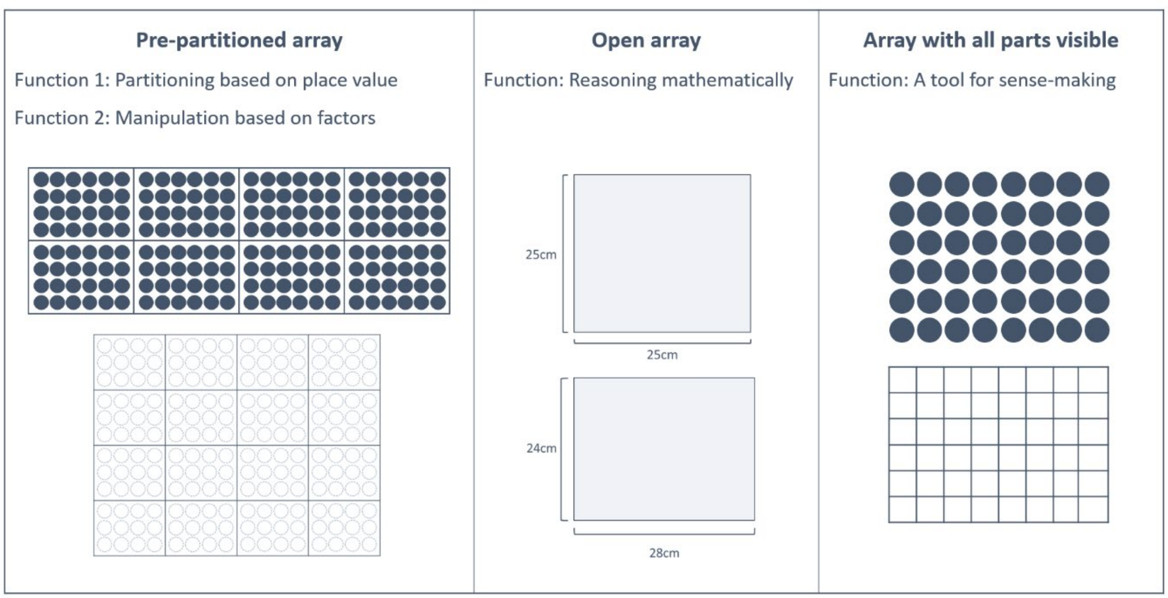


Figure 7.10—The multiple forms of the array used through the instructional sequence

Flexibility to fold back

Central to the process of reification was the flexibility for students to move between the forms of the array according to their cognitive needs. On several occasions when using the pre-partitioned or open array, students were faced with their own insufficient or incomplete internal representations (Goldin & Shteingold, 2001). In these instances, students would 'fold back' (Pirie & Kieren, 1991, 1994; Pirie & Martin, 2000) to the simpler form of the external representation: the array with all parts visible. They would use this form of the array to explore the multiplicative structure and to make sense of what was happening mathematically. Students were creating new connections and strengthening existing connections between their internal representations and, in so doing, building a deeper mathematical understanding of multiplication. Pirie & Kieren (1991) describe the process of folding back as 'thickening' understanding:

> *The metaphor of folding back is intended to carry with it notions of superimposing one's current understanding on an earlier understanding, and the idea that understanding is somehow thicker when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding* (p. 172).

In this study, students were introduced to new content through a contextual situation. The students worked within the context of the problem to generate solutions using the array. The form of the array arose from the context and focused students' attention on specific multiplicative properties. As students explained their strategies and observed similarities and differences between their work and that of others in the community, they stepped away from the context and into the domain of more formal mathematics. Stepping away from the context drew students' attention to their incomplete or insufficient understandings. At this point, students would fold back to a previously understood form of the array, the array with all parts visible. Folding back to this form of the array would assist them in correcting inconsistencies in their internal representations and to support the formation of new connections. When satisfied that their internal representations matched the external representations generated through the problem, the students would proceed to use the more abstracted external representations.

The flexibility for students to choose the form of the array they needed was significant. A study conducted by Izsak (2004) also had students work with multiple forms of the array, shown in Figure 7.11. The design of the arrays supported students' use of the distributive property and, eventually, the formal algorithm. As the instructional sequence progressed, new forms of the array were introduced, and old forms of the array were dropped. Izsak (2004) noted that a number of students struggled when the unit squares were removed, and students had difficulty connecting dimensions of rectangular arrays to numerical labels. Izsak (2004) concluded that multiple approaches, including multiple forms of the array, can afford opportunity to make connections between different representations and "make mathematical connections and flexible problem solving realizable in the classroom" (Izsak, 2004, p. 75).

The flexible movement between multiple forms of the arrays in my own study allowed for students' understanding to become *thicker*. Until faced with their own misconceptions or incomplete understandings, it is difficult for students to know what it is they need to learn. Cognitive conflict and challenge arose for students through more abstract forms of the array. New learning occurred through the process of folding back to previously understood forms of the model. As Hiebert & Carpenter (1992) state: "Growth can be characterized as changes in networks as well as additions to networks. The changes sometimes are manifested as temporary regressions as well as progressions" (p. 69)

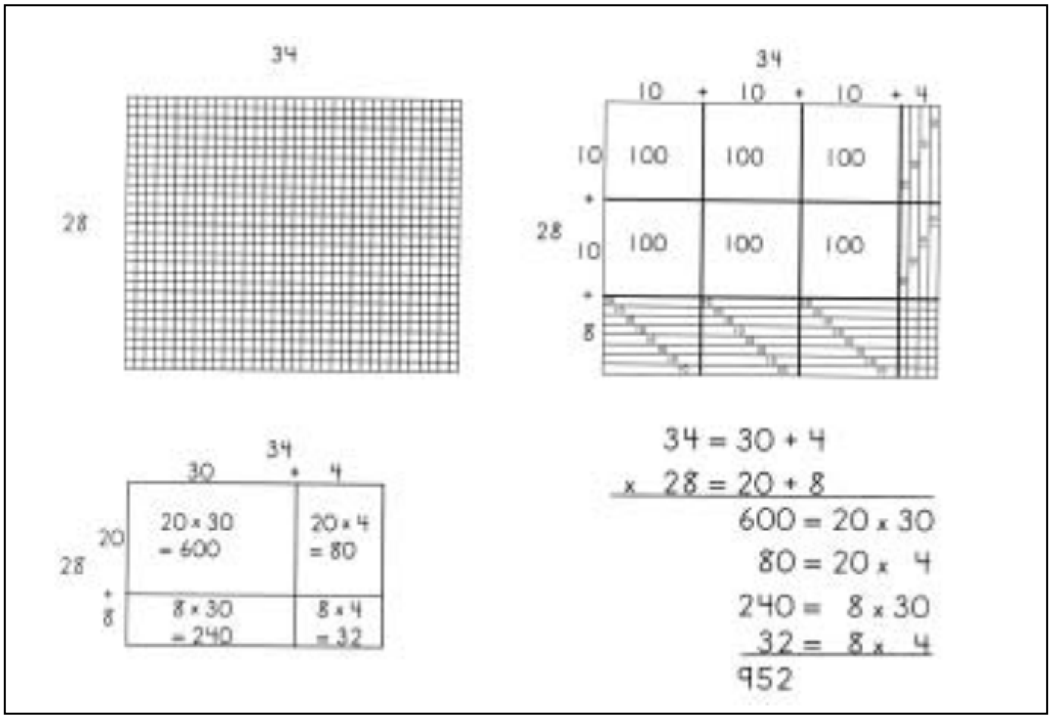


Figure 7.11—Multiple forms of the array used by Izsak (2004)

Exploring the structure of multiplication

This research explored the way the array moved from a *model of* a contextual situation to a *model for* more formal mathematical reasoning. As discussed at the start of this chapter, Gravemeijer (1999) explains the process of a model being reified occurs over two levels: referential and general. At the referential level, students' activity is bound with in the context of the problem. When interpretations and solutions operate independently of context-specific imagery, students are working at a general level. Gravemeijer (1999) acknowledges that this is not a neat, linear progression. Students move back and forth between the two levels as they encounter new problems and develop understandings.

Based on the evidence of this study, I argue that an interim level exists in this process which I have termed *structuring*. The previous section in this chapter described the process of students folding back. In the process of making sense of the multiplicative structure, students worked independently of the context with a previously understood form of the array. They were no longer working at the referential level, nor were they generalising. The array model had not yet become a tool for more formal mathematical reasoning as students were not engaged in reflection, explanation and justification (National Research Council, 2001). Students' activity was focused on sense-making through an exploration of the multiplicative structure, removed from the context of the problem.

Students needed to work independently of the context of the problem in order to make sense of the mathematical properties of the array. As powerful as a context can be in enabling students' access to mathematical ideas, it can also be a hindrance. In studying multi-digit multiplication, Ambrose et al. (2003) observed that students aligned their strategies with the story of the problem. As a result, the students did not freely make use of multiplicative properties or strategies if the properties or strategies did not make sense within the particulars of the problem. The students' most efficient strategies emerged when they worked independently of the context. Similarly, if student activity with the multiple forms of the array is limited to the context of a problem, and time is not spent developing an understanding of the array's mathematical properties, students may fail to recognise the array as a representation of multiplication. This is affirmed by Barmby et al. (2009) who stated that "in utilising these more abstract representations, we must recognise that when children are first introduced to these representations, they must become part of their understanding of the mathematical concept" (p. 236). This does not mean that contextual situations should not be used to introduce mathematical content. This study asserts the opposite: contexts provide students with the opportunity to reinvent formal mathematics for themselves. However, students must have the opportunity to work independently of the context in order to connect the array representation with the mathematical concept being explored. It is the array representation, not the context, that highlights important theoretical properties of multiplication. As Wittmann (2005) argues:

Contrary to real objects or models of real situations which are charged with various constraints mathematical objects allow for unlimited operations and for establishing theoretical knowledge which is more applicable than knowledge directly derived from mathematizing real situations (p. 19).

In the process of reification, student activity at the level of structuring is focused on two important elements. First, activity is supporting sense-making that will lead to reasoning. Secondly, students' activity with the representation must be removed from the potential limitations of a context.

The notion of structuring is inextricably linked to the properties of multiplication and, therefore, key developmental understandings. The next section of this chapter discusses the KDUs that were fundamental to the learning trajectory. It also explores the strategies that students employed based on their understanding of the identified KDUs.

Key Developmental Understandings (KDUs) and Student Strategies

This section of the chapter addresses the second and third research questions:

What key developmental understandings in multi-digit multiplication are reflected in the learning trajectory?

What strategies do students reinvent through the implementation of the learning trajectory?

The aim of the study was to examine students' development of understanding and computational fluency in multi-digit multiplication. The literature review presented in Chapter 2 reported that computational fluency is built on number sense and that number sense requires a balanced and connected knowledge of both procedures and concepts. This segment of the chapter presents students' development of procedural and conceptual knowledge. The variety of procedures invented by the students was documented in the results chapter. The following section pays particular attention to the hierarchical development of students' strategies. KDUs are linked to the strategies used by students, creating a connection between students' procedural knowledge and their conceptual knowledge. The section concludes with a discussion on the bidirectional relationship between procedural and conceptual knowledge as observed through the course of this research.

The Hierarchical Development of Students' Strategies

The reinvented strategies that emerged through the course of this study were comparable to strategies reported in similar studies (Ambrose et al., 2003; Barmby et al., 2009; Izsak, 2004; Murray et al., 1994; Young-Loveridge & Mills, 2009). To understand students' pathways of reinvention, strategies used were mapped according to Treffers & Buys' (2008) three calculation levels: calculation by counting, calculation by structuring, and flexible, formal calculation. It was possible to align the strategies used in this study with these three levels to illustrate a developmental progression in calculations. The classification of strategies is illustrated in Figure 7.12. Additionally, student strategies were linked with KDUs and the type of thinking employed by the student, whether additive or multiplicative. In this case, the strategies drew on students' understandings of the associative and distributive properties of multiplication and a progression from additive to multiplicative thinking could be observed.

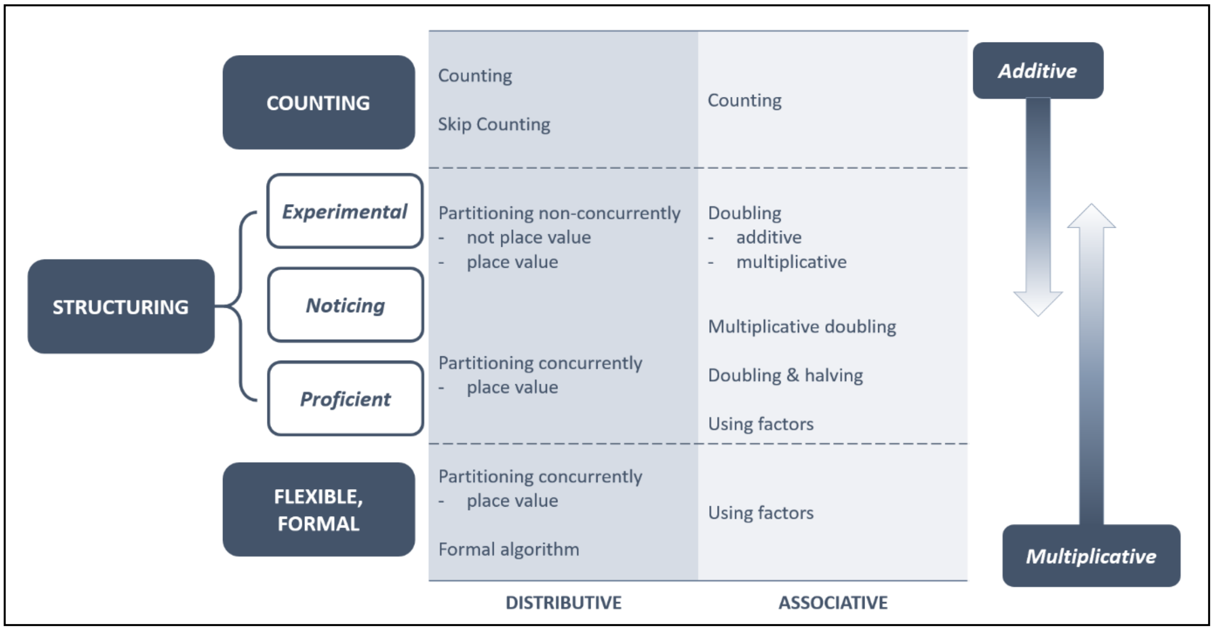


Figure 7.12—Classification of students' strategies

Before discussing the strategies in more detail, it is important to acknowledge two significant points. First, the array played a critical role in the development of students' understanding. The array is not the focus of analysis in this section, as it was discussed in detail in the preceding part of this chapter. Despite this, the array is not considered separate to strategies and understandings. The array is integral to students' development. As Gravemeijer explains (1999), mathematical models bridge informal and formal mathematics. In this case, the array model connected students' informal, reinvented strategies to a deeper understanding of the distributive and associative properties. This allowed students to engage in more formal mathematical reasoning.

Second, although the development of strategies is presented here in a linear fashion, students' movement between strategies was not necessarily linear. Regression was noted when students were faced with new and unfamiliar situations, and sideways movements were observed as students used their understanding of associativity and distributivity in their calculations. Shuffling back and forth between strategies is to be expected on the reinvention pathway (Gravemeijer, 1994) as students experimented with the multiplicative structure and moved towards flexible strategy use based on number sense.

The distributive property

Students made intuitive use of the distributive property, partitioning numbers into new, more readily usable units. A progression could be seen in students' strategies based on the distributive property, as they moved from basic counting strategies through to efficient partitioning based on place value. Although there was limited use of counting strategies, a clear connection was made between Jake's skip counting in the first teaching episode and the distributive property of multiplication. By recognising the connection between counting groups of 10 and partitioning based on 10, Jake was able to utilise the distributive property to solve subsequent problems. A similar connection between counting strategies and the distributive property was recognised by Ambrose et al. (2003), when modelling problems with base-ten materials, students linked partitioning in 10s and 100s with skip counting using tens and hundreds blocks.

At the level of structuring, students' partitioning of numbers progressed over two stages. First, it was noted that a number of students dealt with the numbers being multiplied separately. They would partition one number immediately to create a usable unit and partition the other number later when they faced difficulty and needed to simplify the calculations. This was referred to as non-concurrent partitioning. An example of this strategy was observed in the second teaching episode when students were asked to solve 16 x 12. Jack and Harley (students in the first class) partitioned 16 into 10 and 6, multiplied the 10 by 12, and then partitioned 12 into 10 and 2, multiplying each part by 6, as shown in Figure 7.13. From a discussion with the boys, it was clear that they thought about the calculation in stages. Other students were observed using similar non-concurrent partitioning strategies early in the instructional sequence. Later in the sequence students moved towards partitioning numbers concurrently based on their place value parts, a more efficient strategy. Initially calculations with this form of partitioning were supported by the array. The strategy became formalised as students progressed from the array to numerical recording.

Students' difficulty with the distributive property became apparent in the third teaching episode. Without the support of the array, a number of students did not recognise all the partial products formed when the numbers were partitioned into their place value parts. Similar difficulties were identified in other studies (Murray et al., 1994; Young-Loveridge & Mills, 2009). Students making such an error did not appreciate the binary nature of multiplication, as the evidence suggests they were still thinking additively, while working with multiplicative structure.

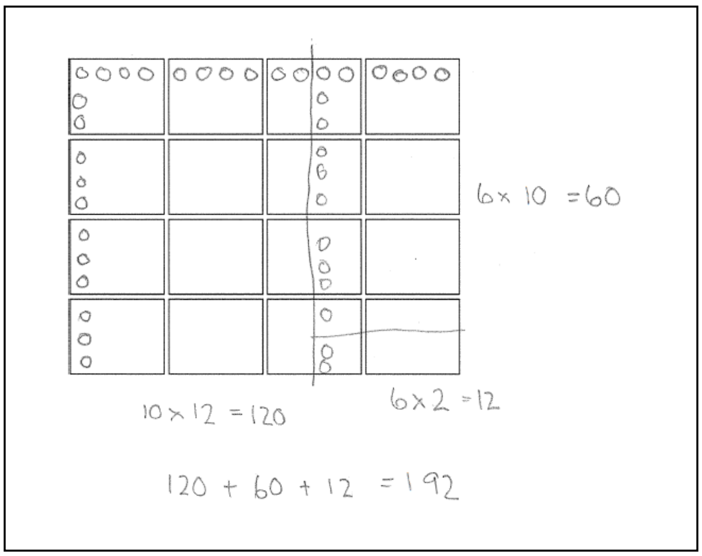


Figure 7.13—Jack and Harley's work sample

The associative property

Students made intuitive use of the associative property. In the first teaching episode, students used doubling strategies of varying complexities. While some students added 24 together eight times, others recognised that multiplying 24 by 8 was the same as doubling 24 three times. The notion of doubling came naturally to students, an observation also made by Ambrose et al. (2003). Students could then transfer doubling to other problems, leading into doubling and halving strategies and the use of other factors. Students observed that the use of the associative property based on factors was best suited to just some numbers.

There is very limited research on students' understanding and use of associative property for multiplication, particularly with multi-digit numbers. Yet, it is the associative property that explains the ubiquitous adding of a zero in the formal algorithm for multiplication with multi-digit numbers (Barnett & Ding, 2018). The students' misconceptions and errors highlight the importance of an explicit focus on the associative property of multiplication. The most significant error was seen in the third teaching episode where the students were asked to predict which area was bigger: 25 x 25 or 28 x 24. A large proportion of students reasoned that you could take two from the 28 and add this two to the 24 to make 28 x 24 = 26 x 26. The students did not appreciate that this additive reasoning was not mathematically sound in a multiplicative context (see Figure 7.14). Similarly, Barmby et al. (2009) noted students using what they termed 'rearranging strategies' to aid computation. In this case, the students' rearrangement of the array was used to simply aid counting and, in doing so, the rectangular structure of the array was lost. Barmby et al. (2009) concluded that wider application of such strategies would be problematic.

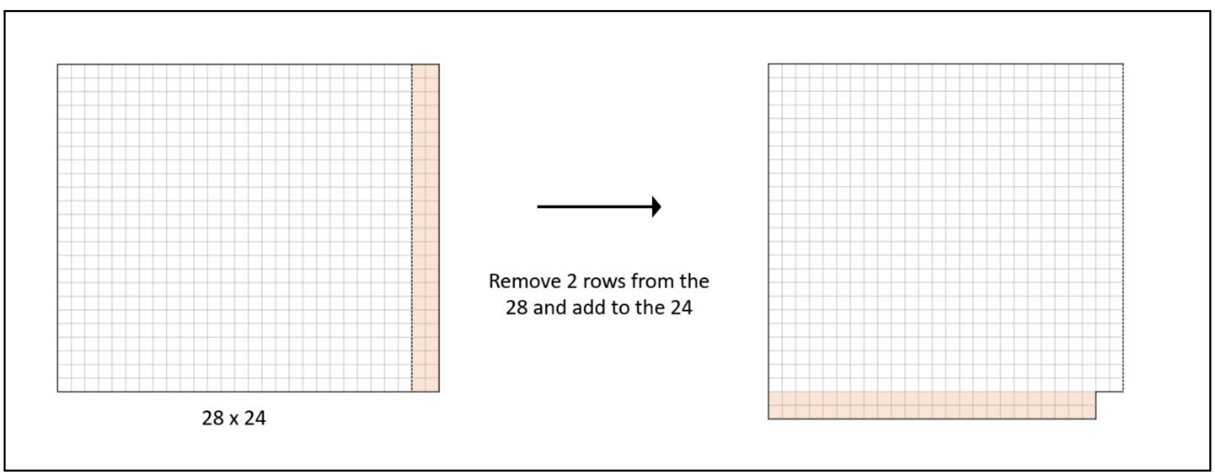


Figure 7.14—Students' error with the associative property

The importance of the associative properties is not limited to its application to calculations. Students require an explicit awareness of the role of arithmetic properties to justify their solutions and to form generalisations (Barnett & Ding, 2018). This stage of formalisation is critical in the development of understanding mathematical concepts (Pirie & Kieren, 1991). Hiebert & Carpenter (1992) argued that students' understanding can be limited if only some aspects of a concept are explored. A robust understanding is founded on an interconnected web of the relationships related to a concept.

The shift from additive to multiplicative thinking

Students' transition from additive to multiplicative thinking may take many years (Siemon, Breed, Dole, Izard, & Virgona, 2006; Siemon, 2013; Vergnaud, 1983), and this transition in not as smooth as suggested in curriculum documentation (Siemon, 2013). A progression from additive to multiplicative thinking was evident in the students' strategies, but a clear transition point was not apparent. In the analysis of results, I met with an independent researcher to validate my findings. As we examined students' strategies and their accompanying explanations, there were times when we could not reach consensus on whether students were thinking multiplicatively or additively. What we were able to conclude was this transition occurred at the level of calculation by *structuring*.

Treffers et al. (2008) describe structuring as derived calculations, where students use known facts to work out unknown facts, usually with the support of a visual or concrete model. During the course of the instructional sequence, most of the students' strategies were at the level of calculation by structuring. Central to their activity was the model of the array as a tool to support calculations and make sense of the multiplicative structure. Based on the evidence of this study, I identified three sub-levels of structuring: experimental, noticing and proficient.

Experimental

Experimental strategies were observed predominantly at the start of the instructional sequence in both classes. Students recognised that numbers could be partitioned or broken into smaller, more readily usable units. At this level, students' understanding of the properties of multiplication were rudimentary and application of strategies was somewhat flawed, even though they may have resulted in the correct answer, as evidenced in the work of Meera and Maddie (Figure 7.15). Rather than distributing 24 into 20 and 4 with one straight partition, they removed a group of 4 from the bottom row, and, in so doing, lost the rectangular array structure. While they were aware that it was possible to make smaller groups, they did not recognise how the structure of coordinated rows and columns should be harnessed to support distribution of the array. A parallel can be drawn with the previously mentioned ‘rearranging strategies’ that emerged in the study conducted by Barmby et al. (2009). Students recognised that the array could be rearranged to aid computation, but their methods of rearrangement lost the rectangular arrangement of the array. As seen from the fact that students were not applying strategies based on an understanding of multiplicative properties, students’ work at this stage was additive in nature.

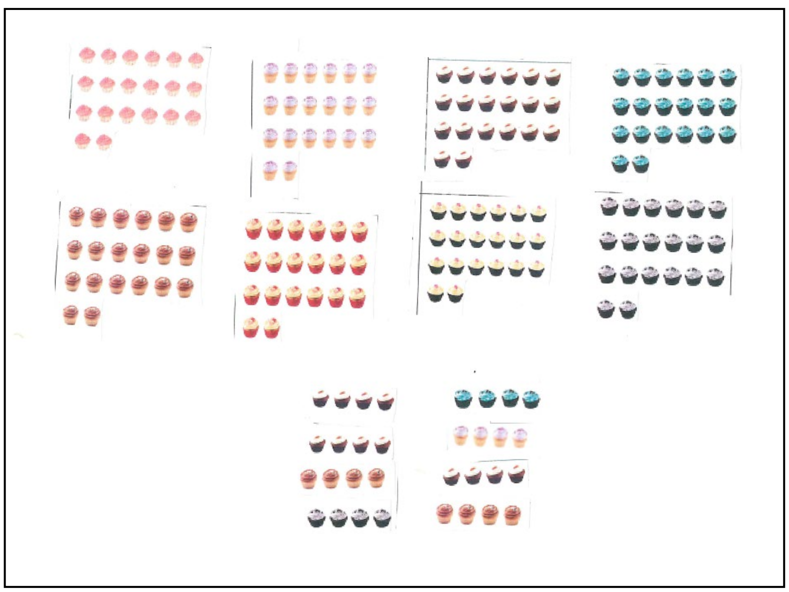


Figure 7.15—Meera and Maddie's work sample from Teaching Episode 1

Noticing

At this sub-level, students were discerning how the multiplicative structure and key properties formed the basis for calculations. They were able to explain why strategies worked (or did not work) based on a recognition, and growing understanding of, the associative and distributive properties of multiplication. For example, Ryan and Dylan used the associative property to aid calculation in the second teaching episode (see Figure 7.7). They recognised that halving one factor and doubling the other would not affect the product and demonstrated how this worked with the array. Students' strategies at this point were generally multiplicative in nature as they were able to use the structure of multiplication to justify their solutions.

For other strategies at this level, it was less clear as to whether students were thinking multiplicatively or additively. In Episode 1, Ashley and James used a repeated addition strategy, illustrated in Figure 7.16. They were able to demonstrate how their strategy worked using the multiplicative structure, but their method of calculation was still based on addition. Discussion allowed for parallels to be drawn between this strategy and doubling 24 three times.

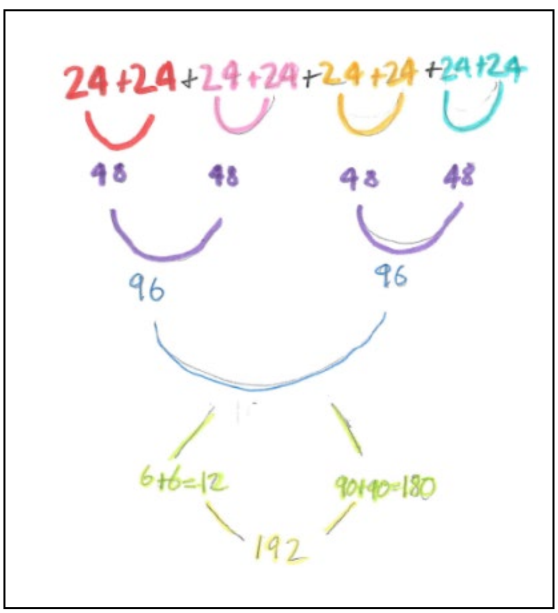


Figure 7.16—Ashley and James's work sample from Teaching Episode 1

Proficient

At this level, students were using derived strategies that drew on a sound understanding of associativity and distributivity. The array was still used for support as the students developed numerical recording methods. Strategies were not yet in the domain of flexible, formal calculations.

The importance of structuring

The importance of the structuring phase was seen through the development of procedural and conceptual knowledge, which did not develop independently. There was a bidirectional relationship between the two. Students explored the properties of multiplication through the application of procedures, while their conceptual knowledge influenced the procedures that they used. This finding concurs with similar discussions by several authors (Baroody, Feil, & Johnson, 2007; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Schneider, & Star, 2015; Rittle-Johnson, Siegler, & Alibali, 2001). As expressed by Rittle-Johnson et al. (2001), "conceptual and procedural knowledge appeared to develop in a gradual, hand-over-hand process" (p. 358).

Rittle-Johnson et al. (2001) observed that students with a greater conceptual knowledge were more able to represent problem situations correctly, which in turn improved their use of procedures. I made the same observation in my study: students' conceptual knowledge of multiplication enabled them to accurately represent problems using the array. Their conceptual knowledge supported them in their use of procedures and, in so doing, built greater procedural knowledge. This scenario was equally true when faced with their own procedural errors, as evidenced in the third teaching episode. When students recognised that their procedures were incorrect, they used their conceptual knowledge to represent the problem situation. This highlighted inconsistencies in the procedures used and the students were then able to remedy calculation errors and misconceptions, thus building greater procedural knowledge. Students' conceptual knowledge improved their procedural knowledge.

Students' conceptual knowledge also guided their strategy choice. At the start of the teaching episode, students often justified their strategy choice based on personal preference, using statements such as *this is my favourite strategy* or *I know how to use this strategy*. Growth in students' conceptual knowledge resulted in improved strategy choice. Students selected appropriate strategies based on features such as the numbers involved and the context of the problem, rather than personal preference. This usage of correct and appropriate procedures is indicative of strong procedural knowledge (Rittle-Johnson & Alibali, 1999).

Procedural knowledge also positively impacted conceptual knowledge. Students invented their own procedures to solve problems. They then justified why these procedures made sense conceptually, which furthered their conceptual knowledge. When students followed the steps of a procedure correctly yet obtained incorrect results (see Episode 3, for example), they became aware of inconsistencies in their conceptual knowledge that needed to be remedied. The fact that improved procedural knowledge can highlight misconceptions in students' conceptual knowledge is also recognised by Rittle-Johnson et al. (2001).

The bidirectional development of conceptual and procedural knowledge was evident throughout the sequence. It is important to acknowledge, however, that this instructional sequence is just a small snapshot of a much larger picture of learning in multiplication. It is not possible to comment on whether conceptual knowledge precedes procedural knowledge or vice versa. It is however possible to state that in the context of this sequence of learning, there was an iterative development between the two forms of knowledge. The evidence strongly suggests that focusing on one type of knowledge at the expense of the other would have inhibited the development of students' understanding.

Mathematical Norms

Chapter 6 presented a coordinated analysis of the social and individual learning in the two Year 5 classes. The four taken-as-shared mathematical practices that emerged through the instructional sequence were presented and the reasoning of individual students reasoning to illustrate the practices. The four mathematical practices are presented in Figure 7.17. This was in keeping with the emergent perspective, the theoretical basis of the research. The emergent perspective was also the foundation of the interpretative framework (Cobb & Yackel, 1995) used to analysis results (see Figure 7.18). I return to the emergent perspective interpretative framework are the end of this section.

Teaching Episode 1 3 lessons	**Teaching Episode 2** 3 lessons	**Teaching Episode 3** 2 lessons	**Teaching Episode 4** 2 lessons
Mathematical Practice 1 **THE ARRAY AS A TOOL FOR SENSE-MAKING** Partitioning the array	***Mathematical Practice 2*** **THE ARRAY AS A TOOL FOR SENSE-MAKING** Manipulating the array	***Mathematical Practice 3*** **WAYS OF WORKING MATHEMATICALLY** Thinking multiplicatively	***Mathematical Practice 4*** **WAYS OF WORKING MATHEMATICALLY** Looking for 'friendly' numbers

Figure 7.17—Mathematical practices that emerged through the instructional sequence

SOCIAL PERSPECTIVE	**PSYCHOLOGICAL PERSPECTIVE**
Social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 7.18—The emergent perspective interpretative framework

The notion of social and sociomathematical norms was introduced by Cobb and Yackel (1995) as a way of making sense of the happenings in the classroom. These norms, along with mathematical practices, describe the communal aspects of learning. Cobb et al. (1995) described social norms as general social behaviours, and ways of acting that apply to any subject matter, whereas sociomathematical norms are unique to mathematics. Since their introduction, these norms have become more than just a tool for analysing of learning. Yackel & Cobb (1996) have studied teachers' processes of implementing social and sociomathematical norms in their classrooms. Stephan, Underwood-Gregg & Yackel (2014) describe the initiation and development of these norms as an essential teaching practice in classrooms that support guided reinvention. Social and sociomathematical norms have become a way of describing the practice of teaching and a desired culture of learning in a classroom.

Social and sociomathematical norms played an important role in the development of concepts in multi-digit multiplication. An additional set of norms were identified through the course of the instructional sequence, which I have termed *mathematical norms*. These norms are unique to mathematics but cannot be classified as sociomathematical norms as they are not uniquely social. Mathematical norms describe expected ways the students should work mathematically. Five mathematical norms were identified: noticing similarity and difference, making inferences, using representations, justifying mathematical thinking and forming generalisations.

These mathematical norms were interconnected and interdependent. One mathematical norm did not stand on its own. For example, students would justify their mathematical thinking using the array. Through their justifications and representations, students would observe similarities and differences which in turn allowed them to make inferences and form generalisations. The mathematical norms were also critical to the development of students' understandings.

Understanding is both procedural and conceptual in nature and is built on connections formed between one's internal representations. In Hiebert & Carpenter's (1992) discussion on understanding, two types of connections were identified: connections based on similarity and difference, and connections based on generality. Students build connections based on similarity and difference as they work with different mathematical representations. Students may build connections between representations, within the same representational form or based on their actions with representations. As students notice similarities and differences, they join pieces of information together to create a web of relationships. The networks formed through noticing similarity and difference are not necessarily hierarchical. Often these networks are connecting pieces of information on the same level. Connections based on generalisation, however, are hierarchical. In this process, internal representations are connected to more general principles, which, in turn, are connected to higher-level constructs. Relationships are formed based on a concept, where the concept serves as an umbrella for more specific cases.

The mathematical norms of noticing similarity and difference and forming generalisations are at the heart of forming deep understanding and so, due to their interconnected nature, *all* mathematical norms play an integral role in learning. Norms serve to focus teaching. The focus of teaching shifts from simply the acquisition of knowledge (although this will undoubtedly happen) to the construction of internal networks, whose connections are formed through application of the mathematical norms. To illustrate, consider how similarities between arithmetic procedures form connections (Hiebert & Carpenter, 1992):

> *By thinking about similarities and differences between arithmetic procedures, students can construct relationships between them. In this case, the instructional goal is not necessarily to inform one procedure by the other but, rather, to help students build a coherent mental network in which all pieces are joined to others with multiple links.*

Constructing relationships within a representation form often increases the cohesion and structure of the network (p. 68).

The emergent perspective interpretative framework

Promoting mathematical norms as expected ways of working in the classroom transcends a domain-specific instructional sequence. Such norms are generalisable to all aspects of mathematics study. As the emergent perspective interpretative framework details the communal and individual learning of a class (Gravemeijer et al., 2003a), the mathematical norms fit within the framework. The proposed revised version of the interpretative framework is illustrated in Figure 7.19.

The mathematical norms were communal and individual. They influenced the mathematical discourse of the classroom and the teacher-to-student and student-to-student interactions. They also became ways that individuals worked mathematically as they were involved in the process of mathematisation. The mathematical norms were central to analysing the learning of the class.

SOCIAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Mathematical norms	Mathematical norms
Classroom mathematical practices	Mathematical conceptions

Figure 7.19—Emergent perspective framework including mathematical norms

The Final Learning Trajectory

In this section, the final version of the learning trajectory is presented. The revisions made to the trajectory are documented and rationales for these revisions are provided. The revisions incorporate the preceding discussions on the array, KDUs and student strategies, as well as the mathematical practices that were reported in Chapter 6.

The hypothetical version of the learning trajectory that was presented in Chapter 5 was based on the work of Gravemeijer (2004) and Gravemeijer et al. (Gravemeijer et al., 2003a). These authors described a learning trajectory for developing early concepts in linear measurement, presented across four categories: tools, imagery, mathematical practices and mathematical discourse. These categories were used to anticipate the important aspects of learning that would develop through the course of instruction in this research on multi-digit multiplication.

A different set of categories emerged as significant in this research, relating to the three themes presented in this chapter. Four categories were used to describe the final learning trajectory: mathematical practices, the array, KDUs and strategies, and mathematical discourse. The final trajectory is presented in Table 7.1.

Category 1—Mathematical Practices

Four mathematical practices emerged over the course of the instructional sequence and were reported in Chapter 6 and shown in Figure 7.17. These practices related to the ways students interacted with the array and different ways that students worked mathematically. The mathematical practices are deliberately presented as the first category in the learning trajectory, in order to present a communal goal for learning.

Category 2—Array: Form and Function

Students' interactions with multiple forms of the array have already been discussed in detail in this chapter. Each form of the array served important pedagogical and student-generated functions. The pedagogical functions highlighted important conceptual elements to the students. This was particularly true for the pre-partitioned array, which built and supported students' understanding of the distributive and associative properties through the careful design of the array structure. The design features of the array and the pedagogical affordances are made explicit in the final form of the trajectory. Successful implementation of the learning trajectory beyond the bounds of this research would be reliant on teachers recognising and harnessing the design features during the course of instruction.

The student-generated functions for particular forms of the array are also made explicit in the final trajectory. These functions demonstrated students' levels of understanding and helped them to move forward in their learning.

Category 3—KDUs and Strategies

In the first version of the trajectory, the mathematical practices and associated understandings were described together. The final version of the trajectory separates the two. While they are related, they are distinct enough to address each separately. The understandings in the final trajectory based on the properties of multiplication and particular aspects of the multiplicative structure. They are linked to specific strategies that students used to solve problems. The notion of *structure* was a repeated theme in the research findings. In the process of reifying the array, students started to work independently of the context as they explored the structure of the array. Additionally, students moved from additive to multiplicative thinking at the calculation level of structuring. In both cases, student activity was focused on making sense of the multiplicative structure and the associative and distributive properties of multiplication. Structuring served as a bridge between students' activity with the array and their invention and development of procedural fluency.

Category 4—Mathematical Discourse

The classroom mathematical discussions were central to the emergence of the mathematical practices and the development of students' understanding and computational fluency. The discussions were carefully planned as part of the ongoing cycles of design and analysis. A key feature of these discussions was that they were conceptual in nature. Cobb (2003) distinguishes between conceptual and calculational discourse: contributions to calculational discourse focus on how a result was obtained, while conceptual discourse also includes the reason for calculating in particular ways. Conceptual discourse serves to support the development of computational fluency built on number sense. The conceptual focus of discussions for each teaching episode is explicitly stated in the final trajectory.

Context	Mathematical Practices	Structure & KDUs	Mathematical Discourse
Episode 1: 2- x 1-digit multiplication Calculate the total number of cakes on 8 trays of 24	The array as a tool for sense-making: Partitioning the array	Distributive property: The array can be distributed into smaller, more 'friendly' parts to aid computation	Use of the distributive property to aid calculation Connection between partitioning the smaller arrays and the larger array
Episode 2: 2- x 2-digit multiplication Calculate the total number of cakes packed in boxes	The array as a tool for sense-making: Manipulating the array	Associative property: The array can be manipulated using factors and multiples e.g. halve and double	Why is 16 x 12 = 24 x 8? Use of the associative property to halve and double
Episode 3: 2- x 2-digit multiplication Calculate the area of the area of two trays	Ways of working mathematically: Thinking multiplicatively	Distributive property: All partial products need to be added together when it is partitioned into smaller parts Associative property: The array can be manipulated using factors and multiples	Difference between working additively and working multiplicatively Recognising that the two-dimensional structure of multiplication and the way it impacts the working of strategies
Episode 4: 3- x 2-digit multiplication Calculate the money raised through cake orders	Ways of working mathematically: Looking for friendly numbers	Flexible use of the distributive and associative properties	Identifying friendly numbers in calculations and using the numbers to decide on strategy use

Table 7.1—The final learning trajectory

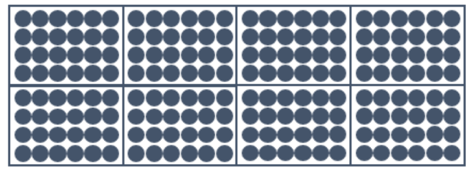
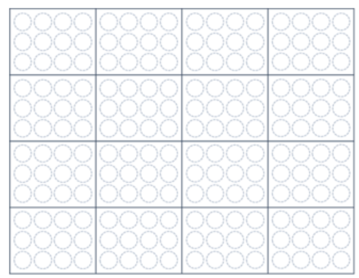
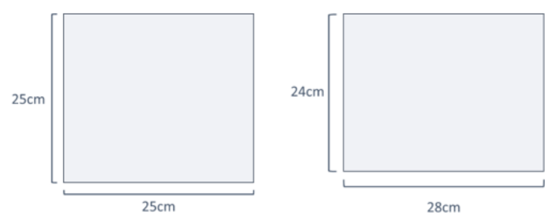
Context	Array - Form	Array - Form
Episode 1: 2- x 1-digit multiplication Calculate the total number of cakes on 8 trays of 24	Array form: Pre-partitioned array Tray with 12 cakes in a 6x4 array, 8 trays in a 4x2 array 	Pedagogical function: 24x8 can be seen in two ways • 8 trays of 24 cakes • 8 rows of cakes with 24 in each row
Episode 2: 2- x 2-digit multiplication Calculate the total number of cakes packed in boxes	Array form: Pre-partitioned array 16 boxes with 12 cakes in each box in a 4x3 array 	Pedagogical function: 16x12 can be seen in two ways • 16 boxes of 12 cakes • 12 rows of cakes with 16 in each row The array of boxes can be halved and doubled to form a 24x8 array
Episode 3: 2- x 2-digit multiplication Calculate the area of the area of two trays	Array form: Open array One tray measuring 25x25cm and another tray measuring 28x24cm 	Pedagogical function: Use of the open array to: • support and illustrate computational strategies • more formal mathematical reasoning

Table 7.1—The final learning trajectory cont.

Conclusion

This chapter documented the final learning trajectory for multi-digit multiplication. It also described how learning should be supported through the trajectory. Support for learning was associated with three themes: the reification of the array, the interaction between strategies and Key Developmental Understandings (KDUs) and the social and individual use of mathematical norms. The three themes aligned with the research questions that shaped the research. The following chapter, the conclusion, articulates answers to the research questions, acknowledges the limitations of the research and scope for further work in the domain.

CHAPTER 8

CONCLUSION

Introduction

Multiplicative thinking has been shown to be an area of weakness for many middle years students (Clark & Kamii, 1996; Siemon, Breed, et al., 2006). While there is substantial research on the development of multiplicative thinking and why students struggle, the field of multi-digit multiplication is under-researched (Ambrose et al., 2003). Fluency with multi-digit multiplication is an important skill for students which develops from their ability to think and work multiplicatively. The aim of this study was to explore the development of students' understanding in multi-digit multiplication and how this process could be reflected in a learning trajectory.

The intent of the research was both theoretical and practical. This thesis provides a domain-specific instructional theory for multi-digit multiplication in the form of a learning trajectory. The trajectory presents a viable, generalisable theory for instruction in multi-digit multiplication. It is an instructional *theory* that can be generalised, not an instructional *sequence* as no sequence would produce exactly the same results in multiple classrooms (Gravemeijer et al., 2003a). The theory for multi-digit multiplication presented in this thesis describes students' potential use of the array, the important understandings to be developed, the possible pathways of strategy reinvention and topics for mathematical discourse. The theory also includes ways that learning can be supported through mathematical norms. Practically, the learning trajectory provides a substantiated theory that can be used and adapted by teachers to design their own hypothetical learning trajectories, based on the needs of the students in their classroom (Gravemeijer et al., 2003a).

Design Research methods were used to develop and refine the learning trajectory. In the first phase of the research, preparatory thought experiments were used to develop a hypothetical version of the trajectory. The hypothetical learning trajectory drew on findings from previous research in the domain of multi-digit multiplication with Realistic Mathematics Education (RME) forming the basis of design for the learning trajectory. The trajectory was tested and modified over the course of two teaching experiments run in two separate classes, analysing the social and cognitive learning of students. Social learning was categorised as mathematical practices, that is, the taken-as-shared ways of reasoning and arguing mathematically that emerged through student activity. The analysis of the cognitive learning of individual students focused on paths of reinvention (Gravemeijer, 1994, 2004) as students developed and refined mathematical strategies. The changing and developing ways that students interacted with the array were also analysed.

The final stage of the research cycle was the retrospective analysis, which re-evaluated the analysis of results that occurred in the teaching experiments. The intent of this stage was to create a domain-specific instructional theory. Three significant themes emerged through the retrospective analysis which provided answers to the research questions. Through this chapter, I discuss each theme in turn, along with the associated research questions. The significance of the findings and their contribution to the field of mathematics education research are articulated. The chapter concludes by acknowledging the limitations of the research and presents future directions for research.

Theme 1: The Reification of the Array

The first theme of significance that emerged was the reification of the array. The literature reviewed in Chapter 3 discussed the capability of the array to model both single- and multi-digit multiplication (Battista et al., 1998; Davis, 2008; Fosnot & Dolk, 2001). Those studies that explored the array with multi-digit numbers focused on how students interacted with the model and the use of the array to build students' conceptual and procedural knowledge. My research took a different focus: I studied students' progressive abstraction of the array, that is, how students' interaction with the array developed and changed over the course of an instructional sequence. This focus was reflected in the first research question:

1. *What key developmental understandings in multi-digit multiplication are reflected in a hypothesised learning trajectory?*

The focus on the progressive abstraction of the array was closely linked to the theory of design for the research, Realistic Mathematics Education (RME). Chapter 4 presented the three design heuristics of RME, one of which was *emergent modelling*. In emergent modelling, the model develops through students' informal mathematical activity within a contextual situation. Through student activity, the informal form of the model is increasingly removed from the context. The model gradually transforms to a tool for more formalised mathematical thinking. This is described as a transition from a *model of* a contextual situation to a *model for* mathematical reasoning; a process of reification (Gravemeijer, 1999). The model moves from being bound within a context to having a life of its own in the realm of formal mathematics. The retrospective analysis, Chapter 7, presented three aspects of students' work that supported the process of reification: the form and function of the array, folding back to previously understood forms of the array, and the use of the array to make sense of the multiplicative structure.

Students engaged with multiple forms of the array during the instructional sequence. Two forms were introduced through the narrative of the sequence: a pre-partitioned array and an open array. The third form of the array was instituted by the students: an array with all parts visible. Each form served particular functions. In some instances, the functions of the array were pedagogical, as the array's structure was used to accentuate multiplicative properties. The pedagogical function of the array was most evident in the uses of the pre-partitioned array. In one teaching episode (Episode 1), the careful design of the array emphasised the distributive property; in the following episode, the array design highlighted the associative property. Students also generated functions for each form of the array through their process of sense-making and reasoning. This was particularly evident with the array with all parts visible, which functioned as a tool for making sense of the multiplicative structure.

Being able to move freely between different forms of the array based on function was important to the development of students' understanding. While one form of the array may have been used for pedagogical functions, it was evident that students may have required a different form in the process of constructing understanding. The flexibility to draw on multiple forms of the array allowed for both the pedagogical and student-generated functions to be realised. Understanding is dependent on connections forming between one's various internal representations of a mathematical idea or concept (Hiebert & Carpenter, 1992; Goldin & Shteingold, 2001). Internal representations and the connections between them are created through an individual's interaction with external mathematical representations (Goldin & Shteingold, 2001). The external representation of the array was used to highlight important aspects of the multiplicative structure through a process of careful design. If a student's internal representations were not aligned with the array form presented, students would fold back (Pirie & Kieren, 1991, 1994; Pirie & Martin, 2000) to a form of the array that did align. The flexibility for students to select the form of the external representation that best met their needs was critical to the success of the learning trajectory. The need to fold back was the reason for students' introduction of the array with all parts visible; a form of the array that was understood, and one that clearly displayed the multiplicative structure. It provided a stable foundation for learning.

The students' need to fold back was indicative of their need to make sense of the multiplicative structure. Throughout the course of problem solving, students were faced with their insufficient or incomplete understandings. Folding back allowed them to modify their understandings as needed. In the process of making sense of the multiplicative structure, students removed themselves from the context. While the context enables students to meaningfully engage in tasks and to mathematise situations (Gravemeijer et al., 2003a), it can also limit students' thinking and reasoning (Ambrose et al., 2003; Wittmann, 2005). In order to make sense of the mathematical properties of the array, students needed to work independently of the context and any potential limitations that the context may impose—they needed to work in the realm of more formal mathematics. As such, they were no longer working with the array as a *model of* a contextual situation, but not yet working with the array as a *model for* mathematical reasoning. Students were working at an interim level, which I termed *structuring,* in this thesis. The form of the array that proved most helpful to students in the process of structuring was the array with parts visible.

Theme 2: Interaction between KDUs and Strategies

The KDUs and strategies were addressed through two research questions:

2. *What key developmental understandings (KDUs) in multi-digit multiplication are reflected in a hypothesised learning trajectory?*

3. *What strategies do students reinvent through the implementation of the learning trajectory?*

Although Question 2 refers to KDUs and Question 3 refers to strategies, it was difficult to distinguish between the two in the analysis of results.

Students made intuitive use of the distributive and associative properties of multiplication in their process of reinvention. Chapter 3 showed that the majority of studies in multi-digit multiplication to date focused on the distributive property of multiplication, to the exclusion of the associative property. This is problematic as students' understandings are limited if only some aspects of a concept are explored (Hiebert & Carpenter, 1992 It was evident in this research that students did not fully understand the multiplicative structure, including the workings of the associative property. Therefore, the instructional sequence used in this research placed an explicit focus on that property. Students were confronted with their misconceptions and incomplete understandings through questions with an explicit focus on the associate property. As students worked through these questions, they corrected their understandings and, in so doing, gained a fuller understanding of the multiplicative structure. Their misunderstandings would not have been apparent to them or to their teachers if the sequence had not provided an explicit focus on the associative property. A robust understanding is built on an interconnected web of the relationships inherent in a concept (Hiebert & Carpenter, 1992).

The notion of *structuring* again proved to be significant in the development of conceptual and procedural knowledge. As students formed connections between the multiplicative structure and the strategies that they used, they were involved in the activity of *structuring.* Structuring activity occurred on three levels: experimental, noticing and proficient. It was observed that students' strategies would commonly shift from additive to multiplicative in nature at the structuring level of *noticing*. At this level, students were able to discern how the multiplicative structure and key properties formed the basis for calculations.

The importance of *structuring* was also seen through the development of procedural and conceptual knowledge. A bidirectional relationship was established between students' procedural knowledge and their conceptual knowledge. Students explored the properties of multiplication through the application of procedures, while their conceptual knowledge influenced the procedures that they used. Focusing on one type of knowledge at the expense of the other would have inhibited the development of students' understandings.

Theme 3: Mathematical Practices and Mathematical Norms

The emergent perspective was used to make sense of the happenings in the classroom, as described in Chapter 4. The teaching experiment results in Chapter 6 coordinated social and cognitive aspects of learning. Based on the social perspective, the taken-as-shared ways of reasoning and arguing mathematically were documented as mathematical practices. The cognitive perspective on learning was formed by individual students' reasoning and their participation in, and contribution to, the emergence of the mathematical practices. Presenting both the social and cognitive aspects gives a fuller description of the learning that took place in the classroom (Stephan, 2003).

Four mathematical practices emerged through the course of the instructional sequence:

1. The array as a tool for sense-making: Partitioning the array;
2. The array as a tool for sense-making: Manipulating the array;
3. Ways of working mathematically: Thinking multiplicatively; and
4. Ways of working mathematically: Using friendly numbers.

These practices addressed the fourth research question:

4. *What social mathematical practices emerge in the classroom through the implementation of the hypothesised learning trajectory?*

Five mathematical norms were also identified: noticing similarity and difference, making inferences, using representations, justifying mathematical thinking and forming generalisations. These norms build on the emergent perspective framework used in the analysis of results and the descriptions of social and sociomathematical norms (Cobb et al., 1995). Social norms describe the expected social learning behaviours for any subject, while sociomathematical norms are specific to the discipline of mathematics. In Chapter 7, I showed that the development and use of mathematical norms were interconnected and interrelated. As the mathematical norms were generalisable to all areas of mathematics, it was possible to place these norms into the emergent perspective framework, spanning the social and cognitive aspects of learning.

The discussion on mathematical norms established a relationship between these norms and the development of understanding. Earlier in this chapter, when discussing the reification of the array, it was stated that understanding is formed by connecting internal representations. Relationships formed between internal representations are based on two types of connections: similarity and difference and generality (Hiebert & Carpenter, 1992). It was shown in the discussion of Chapter 7 that the mathematical norms of noticing similarity and difference and forming generalisations are central to understanding. As mathematical norms are interconnected, *all* norms play an integral role in learning and serve to focus teaching. The intent of teaching shifts from the acquisition of knowledge to the construction of internal networks, whose connections are formed through application of the mathematical norms.

The Final Trajectory

The fifth, and final, research question asked:

5. *How can students' learning be supported through the implementation of such a learning trajectory?*

The final research question was not linked to one specific theme. Rather, it encapsulated all themes. As outlined in the documentation of the final learning trajectory, students' learning can be supported through:

- The varied and flexible use of the form and functions of the array;
- The bidirectional development KDUs and strategies;
- The development of mathematical practices and norms; and

- Conceptual mathematical discourse

The findings of the three main themes informed the documentation of the final learning trajectory (presented in Chapter 7). The trajectory also included the mathematical discourse that was expected, which was critical to the development of the mathematical practices and norms.

Limitations and Opportunities for Further Research

Several limitations of the research need to be acknowledged. The schools involved in the study were Independent schools from affluent areas of Sydney. These schools were chosen due to my role as an Education Consultant with the Association of Independent Schools of NSW. The schools were well resourced and achieved well, with the majority of students achieving in the top two bands on national testing (ACARA, 2010). Most students came from English speaking backgrounds and there was only a small minority with significant learning difficulties. While this presents a limitation, it also presents an opportunity for further research work. Testing the learning trajectory in less advantaged schools and with a more diverse group of students would add to the value of the instructional theory presented in this thesis.

The learning trajectory was implemented over a fairly short time frame. In a similar study, Gravemeijer et al., (2003a) worked with students over the course of ten weeks as they implemented a learning trajectory on early linear measurement. This time frame allowed ample time to analyse the development of mathematical practices and individual students' constructions. In contrast my own instructional sequence on multi-digit multiplication was implemented over a two-week period, as schools were not able to offer more time out of their schedule and I was working in a full-time role at the time of the study. Further research time would have provided the opportunity to explore students' development of more formal strategies for multi-digit multiplication. Again, an exploration of students' formal strategies presents an opportunity for further research. The learning trajectory could be extended beyond the multi-digit numbers presented in this sequence to include much larger numbers and decimals.

The research presented a coordinated analysis of the social and individual constructions of students. Chapter 6 presented the social constructions of students as mathematical practices and used illustrative examples of individual students' reasoning as they participated in, and contributed to, the development of the mathematical practices. In hindsight, two or three selected case studies of individual students would have added value to the research. Case studies would also have focused the collection of data in the class. While all students' written work was collected and analysed, the class teacher and I recorded only the conversations that we had with students. This meant that not all classroom discourse was recorded and so there were instances of students' reasoning and justifications that were not captured. If two or three students were selected as cases, their discourse could have been recorded and analysed. A case study analysis would have provided a fuller description of the developing and changing nature of individual students' participation in, and contribution to, the mathematical practices.

This research focused on the perspective of the learner which allowed for the analysis of the social and cognitive construction of understanding. The impact made by the teacher, and the instructional strategies used, were not analysed. There is little doubt that I influenced the learning of the students in my role as the teacher in both classroom teaching experiments. The role of the teacher presents an opportunity for research. The development of learning trajectories explores the cognitive development of students. It is intended that the learning trajectory constructed will be adapted and used by teachers for their own classrooms. Understanding the decision-making of teachers as they adapt learning trajectories, and the influence their own teaching styles have on the trajectory's implementation, presents a rich field of research.

The recurring theme of *structuring* presents scope for exploration. This research showed that students passed through a *structuring* phase in the process of reifying the array and in the development of strategies. The structuring phase cannot be generalised beyond this instructional theory without further research. Opportunity exists to examine whether a *structuring* phase is evident in the reification of other models, such as the number line. Further research would be needed to see if the three stages of *calculation by structuring* are present with other arithmetic operations. It was previously noted that, with further time, the instructional sequence could be extended to include multiplication with larger numbers and decimals. It would be interesting to see if students regressed to the three stages of *calculation by structuring* when faced with larger numbers and decimals.

The concept of mathematical norms was introduced in this thesis and a case was presented for how mathematical norms and understandings are connected. While literature exists on each norm individually, there is a lack of literature exploring these norms as a collective. Mathematical norms transcend the domain of multi-digit multiplication. They describe ways of working mathematically in mathematics generally. They are also applicable to all age groups. The relationships that exist between these norms and how they contribute to a culture of learning in varied classrooms presents a large field of continued research.

Conclusion

This research aimed to explore students' development of understanding in multi-digit multiplication and how this process could be reflected in a learning trajectory. Using Design Research methods and RME as a design theory, a trajectory was hypothesised, tested and refined. The trajectory developed through this research describes a domain-specific instructional sequence for multi-digit multiplication that can be used by teachers in designing learning for their students. The process of developing the trajectory and the findings of the research make significant contributions to the field.

Specifically, the contributions are recognised through the three themes identified in the retrospective analysis: the reification of the array, the interaction between KDUs and strategies and the identification of mathematical practices and mathematical norms.

It is hoped that the contributions of this research are realised in the classroom, enriching students' learning in multi-digit multiplication and mathematics more generally.

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Appendices

Appendix 1 – The Pre- and Post-Assessment Interview

MULTIPLICATION INTERVIEW	
Interview Instructions To be completed as a one-to-one interviewer. Italics are for instructions to the interviewer; normal type for the words you say. Student may use pen and paper to assist in calculations. ***Interview Resources*** Blue flash cards, yellow equation card, green card, photo of cupcakes.	***Interview Resources*** Blue flash cards Yellow equation card Green card Photo of cupcakes.
Multiplication Problems – Recall I am going to show you some questions. Please tell me the answers. *Show the student blue flash cards.* *After each answer ask…* How did you work that out? a) 3 x 6 b) 7 x 4 c) 5 x 5 d) 8 x 7 e) 6 x 9	
Distributive Property – Equation *Show child the yellow card*: 3 x □ = (3 x 5) + (3 x 1) Please tell me what number goes in the box. How did you work that out?	
Distributive Property – Problem *Show the child the green card 19 x 7.* What is 19 x 7? How did you work that out?	
Single-Digit Multiplication – Problem A baker bakes a tray of cupcakes. *Show picture of the cupcakes in array of 4 by 7.* How many muffins are there altogether? How did you work that out?	
Multi-Digit Multiplication – Problem The baker makes 35 trays of muffins. How many muffins did the baker make? How did you work that out? *If the student uses a formal algorithm ask…* Can you explain how the algorithm works? Is there another way that you could solve the problem?	

Appendix 2 – Ethics Approval

THE UNIVERSITY OF SYDNEY

RESEARCH INTEGRITY
Human Research Ethics Committee
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Email: ro.humanethics@sydney.edu.au

Address for all correspondence:
Level 6, Jane Foss Russell Building - G02
The University of Sydney
NSW 2006 AUSTRALIA

Ref: SA/JM

25 May 2012

A/Prof Janette Bobis
Faculty of Education and Social Work
The University of Sydney
Janette.bobis@sydney.edu.au

Dear A/Prof. Bobis,

Thank you for your correspondence dated May 14th 2012 addressing comments made to you by the Human Research Ethics Committee (HREC).

On May 23rd, 2012 the Chair of the HREC considered this information and approved your protocol entitled "A Journey to Understanding: Developing Computational Fluency of Year 5 Students in Multi Digit Multiplication ".

Details of the approval are as follows:

Protocol No.: 14726

Approval Date: 23rd May 2012

First Annual Report Due: 31st May 2013

Authorised Personnel: A/Prof. Janette Bobis
Dr Jenni Way
Miss Kristen Tripet

Documents Approved:

Document	Version Number	Date
Multiplication interview document	1	May 2012
Parent information statement	1	March 2012
Parental or guardian consent form	1	March 2012
School information sheet	1	March 2012

HREC approval is valid for four (4) years from the approval date stated in this letter and is granted pending the following conditions being met:

Condition/s of Approval

- Continuing compliance with the National Statement on Ethical Conduct in Research Involving Humans.

Manager Human Ethics
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CRICOS 00026A

- Provision of an annual report on this research to the Human Research Ethics Committee from the approval date and at the completion of the study. Failure to submit reports will result in withdrawal of ethics approval for the project.

- All serious and unexpected adverse events should be reported to the HREC within 72 hours.

- All unforeseen events that might affect continued ethical acceptability of the project should be reported to the HREC as soon as possible.

- Any changes to the protocol including changes to research personnel must be approved by the HREC by submitting a Modification Form before the research project can proceed.

Chief Investigator / Supervisor's responsibilities:

1. You must retain copies of all signed Consent Forms (if applicable) and provide these to the HREC on request.

2. It is your responsibility to provide a copy of this letter to any internal/external granting agencies if requested.

Please do not hesitate to contact Research Integrity (Human Ethics) should you require further information or clarification.

Yours sincerely

S. J. Assinder

Dr Stephen Assinder
Chair
Human Research Ethics Committee

cc: Kristen Tripet

This HREC is constituted and operates in accordance with the National Health and Medical Research Council's (NHMRC) National Statement on Ethical Conduct in Human Research (2007), NHMRC and Universities Australia Australian Code for the Responsible Conduct of Research (2007) and the CPMP/ICH Note for Guidance on Good Clinical Practice.

THE UNIVERSITY OF SYDNEY

Research Integrity
Human Research Ethics Committee

Tuesday, 27 August 2013

Dr Janette Bobis
Education and Social Work - Research; Faculty of Education & Social Work
Email: janette.bobis@sydney.edu.au

Dear Dr Janette Bobis,

Your request to modify the above project submitted on 2nd July 2013 was considered by the Executive of the Human Research Ethics Committee.

The Committee had no ethical objections to the modification/s and has approved the project to proceed.

Details of the approval are as follows:

Project No.:	**2012/379**
Project Title:	**A Journey to Understanding: Developing Computational Fluency of Year 5 Students in Multi Digit Multiplication**

Please do not hesitate to contact Research Integrity (Human Ethics) should you require further information or clarification.

Yours sincerely

Glen Davis
E

Professor Glen Davis
Chair
Human Research Ethics Committee

This HREC is constituted and operates in accordance with the National Health and Medical Research Council's (NHMRC) National Statement on Ethical Conduct in Human Research (2007), NHMRC and Universities Australia Australian Code for the Responsible Conduct of Research (2007) and the CPMP/ICH Note for Guidance on Good Clinical Practice.

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Appendix 3 – Number Strings

Teaching Episode 1, Lesson 3 - Doubling Number String

6 x 2	6 x 4	12 x 2	12 x 4	12 x 8	24 x 2	24 x 4	24 x 8

Teaching Episode 2, Lesson 3 - Doubling Number String

3 x 4	6 x 4	6 x 8	3 x 16	12 x 4	12 x 8	24 x 8