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Models of risky choice: A state-trace and signed difference analysis

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Abstract

Models of risky choice fall into two broad classes; fixed utility models that satisfy the condition of *simple scalability* and everything else. While it is known that choice behaviour can be observed that is inconsistent with all models, this has largely been based on the construction of special cases. We use state-trace analysis and signed difference analysis to test a set of models on a set of ecologically representative risky choices. An advantage of this approach is that there is no requirement to posit a particular form for the error function that links the difference in the utilities of two gambles, A and B , with the probability of choosing A over B . We presented groups of participants with 30 *variable* gambles (A), each paired with one of four *fixed* gambles (B). We use state-trace analysis to test the prediction of all fixed utility models that the probability of choosing each A has the same order for all B . The results show that this prediction is not confirmed and a more complex model is required. We then use signed difference analysis to test two more complex models – the random subjective expected utility model based on Decision Field Theory and a fixed utility mixture model. We derive a key prediction from the random subjective expected utility model and show that it is confirmed by the data. In contrast, the data are shown to be inconsistent with the fixed utility mixture model.

Key words: risky choice, state-trace analysis, signed difference analysis, models of decision making, fixed utility models, mixture models, Decision Field Theory, independence.

1. Introduction

Risky choice consists of selecting the preferred gamble from a set of K different gambles. A gamble of size n is a set of n ordered pairs, (p_i, x_i) , where $p_i > 0$ is the probability of payoff, x_i , and $\sum_i^n p_i = 1$. If $n=1$ then $p_1 = 1$ and the gamble is called a *sure thing* or *certain outcome*, otherwise it is *risky*. Our focus in this paper on choices from sets of $K = 2$ risky gambles each of size 2.

A fundamental feature of risky choice is that it is stochastic (Reiskamp, 2008). When asked to choose between two or more gambles, responses vary both between different people and within the same person at different times. Yet this stochasticity is regular – while the probability of choosing the risky gamble, $\{(0.5, \$1), (0.5, \$0)\}$, against the sure thing, $\{(1, \$0.40)\}$, is close to 0.5, this probability increases monotonically as the risky payoff ($\$1$) increases (Mosteller & Noguee, 1951). In order to capture this stochastic element, many models of risky choice take the following general form:

$$P(A, B) = f \circ g(A, B). \quad (1)$$

Here, A and B are two gambles in a binary choice and $P(A, B)$ is the probability of choosing A over B . Let $S_n = [0, 1]^n \times \mathbb{R}^n$ be the space of all possible gambles of size n . Then the function, $g : S_n \times S_m \rightarrow \mathbb{R}$, maps a pair of gambles, A and B , of size n and m , respectively, to a real number representing the latent *preference* of A over B . The function, $f : \mathbb{R} \rightarrow [0, 1]$, is *strictly increasing* in its argument and maps the latent preference of A over B to the probability of choosing A over B . In this formulation, g can be thought of as instantiating the *deterministic* component of a model of risky choice, while f instantiates its *stochastic* component.

Not all models of risky choice can be represented by Equation (1), although many can be. One particular class of models consistent with Equation (1) is the class of *fixed utility* models (Rieskamp, Busemeyer, & Mellers, 2006). In this case,

$$g(A, B) = u(A) - u(B), \quad (2)$$

where, $u: S_n \rightarrow \mathbb{R}$ is a function that maps a gamble of size n to a real number representing its latent *attractiveness* or *utility*. Many well-known models of risky choice are fixed utility models, such as subjective expected utility theory (Edwards, 1954), prospect theory (Kahneman & Tversky, 1979), cumulative prospect theory (Tversky & Kahneman, 1992), and the transfer of attention exchange model (Birnbaum, 2005).

Fixed utility models satisfy a property called *simple scalability*, so-called because the effect of each gamble on choice probability depends only on a fixed value – its utility – that corresponds to a point on a common scale (Tversky & Russo, 1969). Simple scalability follows when $g(A, B)$ has the general form $g(u(A), u(B))$ and is strictly increasing in its first argument and strictly decreasing in its second argument. This requirement is satisfied by Equation (2). Tversky and Russo showed that simple scalability has three testable consequences:

1. *Strong stochastic transitivity*¹:

$$\text{if } P(A, B) \geq \frac{1}{2}, P(B, C) \geq \frac{1}{2} \text{ then } P(A, C) \geq \max(P(A, B), P(B, C)),$$

2. *Substitutability*:

¹ Both strong stochastic transitivity and substitutability depend on the additional assumption that $P(A, A) = \frac{1}{2}$.

$$P(A, C) \geq P(B, C) \text{ if and only if } P(A, B) \geq \frac{1}{2}.$$

3. Independence:

$$P(A, C) \geq P(B, C) \Leftrightarrow P(A, D) \geq P(B, D)$$

Independence follows directly from Equations (1) and (2). Accordingly,

$P(A, C) \geq P(B, C)$ if and only if $u(A) - u(C) \geq u(B) - u(C)$. If $P(A, C) \geq P(B, C)$ then $u(A) \geq u(B)$ and $u(A) - u(D) \geq u(B) - u(D)$ for any D . Thus, $P(A, D) \geq P(B, D)$ for any D . Paraphrasing Tversky and Russo (1969), if two gambles (A and B) are ordered according to their choice probabilities relative to some fixed standard (C) then this ordering is independent of the particular standard. We return to this implication later.

Tversky and Russo showed that the above three properties, along with simple scalability itself, are essentially equivalent and “capture the same principle that pairwise choice probabilities can be expressed as a monotone function of some underlying scale values in such a way that if two alternatives are equivalent in one context, they are substitutable for each other in any context” (p. 4). While violations of each of these properties have been consistently reported in previous studies (for a summary, see Reiskamp, Busemeyer and Mellers, 2006), this has been based on specially constructed special cases. Our aim in the present study is to use state-trace analysis to identify whether violations of independence arise in choices between a set of ecologically representative gambles of roughly equal expected value.

As noted, violations of independence have been observed in selected cases. An early example is the “Myers effect” (Busemeyer & Townsend, 1993). Myers and Sadler (1960) presented three groups of people with a series of binary choices between a sure thing, with payoffs of either +1 or -1, and a gamble consisting of an equal chance of receiving a whole

number payoff between $-X$ and $+X$, exclusive of $-1, 0, +1$. The value of X and hence the variance of the gambles varied between groups. When the sure thing was $+1$, the overall probability of choosing the gamble was greatest for the largest value of X , followed by the intermediate value, and then the lowest value. Independence requires that the same order should be found when the sure thing is -1 . However, although the range of probabilities of choosing the gamble was much reduced, the opposite order was found. That is, the probability of choosing the gamble was greatest for the smallest value of X , followed by the intermediate value and then the largest value. A similar effect was also found by Busemeyer (1985).

Later, Mellers and Biagini (1994) investigated violations of independence in a set of binary choices between two sets of gambles, labelled type a and type b . Participants were presented with 25 a gambles each paired with one of nine b gambles. The a gambles were constructed from the factorial combination of five probabilities, 0.1, 0.3, 0.5, 0.7 and 0.9, with five payoffs, \$13, \$17, \$51, \$81, and \$91. The b gambles were constructed from the factorial combination of three probabilities, 0.1, 0.5, and 0.9, with three payoffs, \$13, \$51, and \$91. For each pair of gambles, participants rated their preference for the a gamble over the b gamble on a scale from -100 (b gamble maximally preferred) to $+100$ (a gamble maximally preferred). If independence is satisfied then the order of mean preference ratings for a gambles should be the same for each b gamble². Mellers and Biagini (1994) presented results for three of the nine b gambles.

² This requires a generalization of Equation (1) in which function f maps preferences onto the rating scale that was used.

A convenient way of examining whether the order of preferences of different a gambles is the same for two different b gambles is to construct a *state-trace plot*. Let \bar{p}_{ij} be the mean preference for gamble a_i over gamble b_j , let N be the number of a gambles, and let $[N] = \{1, 2, \dots, N\}$. Then the set of points, $P = \{(\bar{p}_{ij}, \bar{p}_{ik}) : i \in [N]\}$, is the *state-trace* for b gambles, b_j and b_k . We say that a state-trace P is *monotonically increasing* (or simply *monotonic*) if, for each pair of points $(x, y), (\tilde{x}, \tilde{y}) \in P$, if $x > \tilde{x}$ then $y \geq \tilde{y}$ and if $y > \tilde{y}$ then $x \geq \tilde{x}$. Clearly, if the order of preferences for a gambles is the same for two b gambles then the corresponding state-trace must be monotonic (Dunn & Kalish, 2018; Dunn & Anderson, 2018).

Insert Figure 1 here

Figure 1 shows the results found by Mellers and Biagini in the form of three state-trace plots for each pairwise combination of the three b gambles, labelled b_1 , b_2 , and b_3 . Also shown is with the best-fitting monotonically increasing curve (explained in the next section). Based on visual inspection of these plots, each state-trace appears broadly monotonic and, while there are apparent violations, none appear to be large. Although Mellers and Biagini applied a statistical test, this was not specifically designed to detect departures from monotonicity. In addition, the structures of the a and b gambles were selected by Mellers and Biagini in order to maximize the chance of observing similarity effects (described below) which they proposed would lead to violations of both independence and strong stochastic transitivity. These considerations lead to two questions addressed by in the first part of the present study. First, are apparent violations of independence statistically reliable? Second, if they are reliable, can they be observed in relatively unselected gambles? We address these questions using the logic and recently developed statistical machinery of state-trace analysis.

State-trace analysis

In this section, we briefly review the logic of state-trace analysis (STA) and show how it can be used to test for independence among binary gambles. STA is a general method to determine properties of the latent structure that mediates the effects of one or more independent variables on two or more dependent variables (Bamber, 1979; Dunn & Kalish, 2018). If the effects of one or more independent variables are mediated by a single latent variable and if each dependent variable is a (possibly distinct) increasing function of that variable then a state-trace corresponding to each pair of dependent variables will be monotonically increasing. We apply this logic to the fixed utility model.

According to the fixed utility model, by Equations (1) and (2),

$$\begin{aligned}P(A, B) &= f \circ g(A, B) \\g(A, B) &= u(A) - u(B)\end{aligned}$$

Therefore,

$$\begin{aligned}P(A, B) &= f(u(A) - u(B)) \\&= f_B(u(A))\end{aligned}\tag{3}$$

where $f_B: \mathbb{R} \rightarrow [0, 1]$ is a strictly increasing function, distinct for each gamble B . Equation (3) re-states the independence property of fixed utility models, derived in the context of STA. Since it is a testable model under STA, we are able to apply the statistical machinery recently developed by Kalish, Dunn, Burdakov and Sysoev (2016) to identify violations of monotonicity and hence independence.

Kalish et al. (2016) developed a statistical test for the equality of two latent orders based on monotonic regression. In order to demonstrate its application in the present context, we analysed choice data from Tversky and Russo (1969). Although the stimuli used in this study

were not gambles, the structure of this study was comparable to that reported by Mellers and Biagini (1994). The stimuli used by Tversky and Russo were geometric figures varying in size and shape. Participants were asked to judge which of two pairs of figures had the greater area. One group judged pairs of rectangles and a second group judged pairs of lenses. Each type of figure came in two shapes; an elongated shape in which the ratio of height to width was 5:1 and a square (or round) shape in which the ratio was 3:2. Each pair of figures presented for judgment consisted of a *standard* stimulus and a *variable* stimulus. For each type of figure (rectangle or lens), there were two standard stimuli; one elongated and one square. These correspond to the *B* stimuli in Equation (3). There were eight variable stimuli determined by the factorial combination of two shapes (elongated and square) and four sizes; two smaller and two larger than the standard. These correspond to the *A* stimuli in Equation (3).

Insert Figure 2 here

Each pair of variable and standard stimuli was presented three times in the form of rectangles to 78 participants and in the form of lenses to 83 participants. The proportion of times that the variable stimulus was chosen as the larger, summed over replications and participants, is shown in Figure 2 in the form of two state-trace plots, one for rectangles and one for lenses. In each plot, filled circles correspond to square variable stimuli and filled triangles correspond to elongated variable stimuli. The *x*-axis corresponds to the observed proportion of times that the variable stimulus was chosen over the square standard and the *y*-axis corresponds to the proportion of times that the variable stimulus was chosen over the elongated standard.

As found by Mellers and Biagini for gambles, the overall impression is that the data, for both rectangles and lenses, reveal a monotonically increasing trend. While there are apparent

violations of monotonicity, they need to be supported by appropriate statistical tests. To this end, we submitted these data to the statistical procedure described by Kalish et al. The first step in this procedure is to calculate the fit of the best-fitting monotonic model. The predictions of this model for both rectangle and lens groups are shown in Figure 2 by white circles connected by straight lines. The fits of the model are given by the G^2 values in each panel. The second step is to compare the observed fit against the bootstrap distribution of expected fits under the null hypothesis of a common latent order of conditions on each of the two dependent variables. For both groups, the empirical p -value was found to be less than 0.001. While this is consistent with the conclusions reached by Tversky and Russo, the sizes of observed violations of monotonicity were not large, especially for the lens group, and the statistical significance of the result needs to be tempered by the fact that because each pair of stimuli was presented three times to each participant the observations are not all independent. As a result, the Type I error rate is likely to have been inflated.

The studies by Tversky and Russo (1969) and Mellers and Biagini (1994) looked for violations of independence in the context of *similarity effects*. Broadly speaking, a similarity effect occurs when discrimination between similar stimuli is enhanced relative to dissimilar stimuli. In these two studies, similarity was manipulated by constraining similar stimuli to vary on one stimulus property while dissimilar stimuli varied on two. An analogous example illustrates this idea. Imagine you are asked to decide whether Vancouver (Canada) or Portland (Oregon) is each to the north or south of Seattle (Washington). If the cities are familiar, this would be a simple enough task because their locations differ primarily in latitude. Similarly, many people would find it relatively easy to decide whether Paris (France) or Barcelona (Spain) is each to the north or south of Marseilles (France) as these locations also differ primarily in latitude. However, people may find it more challenging to decide if Paris or Barcelona is each to the north or south of Seattle or, equivalently, if Vancouver or

Portland is each to the north or south of Marseilles as these comparisons involve differences in both latitude and longitude³.

Insert Figure 3 here

Both Tversky and Russo (1969) and Mellers and Biagini (1994) found robust similarity effects in their respective experiments. However, these effects do not in themselves guarantee violations of independence. This can be seen in Figure 3 which plots data from Tversky and Russo (1969). The upper two panels show the observed similarity effect separately for rectangles and lenses. Here, the proportion of participants who selected the variable stimulus over the standard (averaged over elongated and square standards) is plotted against the relative area of the variable stimulus (where 1.0 is the area of the standard). Similar stimuli have the same shape as the standard (i.e., both elongated or both square) and differ only in area. Dissimilar stimuli have a different shape to the standard (i.e., one elongated and the other square) as well as differing in area. The similarity effect corresponds to a steeper function for similar stimuli compared to dissimilar stimuli. The lower two panels plot predicted proportions derived from the best-fitting monotonic model applied to all four dependent variables (elongated or square rectangles or lenses). By definition, the predicted values cannot violate independence. However, it is apparent that they show essentially the same similarity effect. In other words, the similarity effect does not necessarily lead to violation of independence.

If similarity effects do not lead to violations of independence, the violations observed by Tversky and Russo and by Mellers and Biagini may have more to do with the particular

³ Paris is further north of Seattle and Barcelona is further south. Both Vancouver and Portland are to the north of Marseilles.

stimuli used. Because both studies used stimuli generated from the orthogonal combination of levels of two different attributes, this may have led to special circumstances in which violations of independence could occur. For this reason, we conducted an experiment that used STA to identify violations of independence (if any) in an ecologically representative set of gambles.

2. Experiment

Previous studies suggest that violations of independence can be found for risky choice. However, these occur in the context of a strong monotonically increasing trend (consistent with independence) and lack statistical support customized to the hypothesis under test. Furthermore, even if some observed violations are statistically reliable (as in the Myers effect), they have been shown to occur only for a particular subset of gambles – specifically a comparison of specially constructed gambles against a sure gain or a sure loss of equal expected value. For these reasons, we investigated whether statistically reliable violations of independence can be found for a set of gambles involving neither sure things nor losses that had the following characteristics: (1) non-negative payoffs, (2) a wide range of probabilities of winning, and (3) broadly similar expected values (within the range 1.0 to 2.2). These characteristics guaranteed that risk and reward were negatively correlated across the gambles, mimicking ecological choice in the environment (Pleskac & Hertwig, 2014). In addition, we examined both one-outcome and two-outcome gambles.

The fixed utility model can be used to characterize choice behavior at several different levels. In the present study, we focus on the population level in which case the model may be instantiated as a random utility model in which the utility of gamble, A or B , for individual I is a random draw from an unknown distribution. Ideally, each pair of gambles would be presented to a separate participant but, for practical reasons, each participant

received 120 binary choices composed of 30 variable gambles each paired with one of four fixed gambles. If independence is satisfied at the aggregate or population level then the preference order of the 30 variable gambles should be the same for each fixed gamble. We tested this implication using state-trace analysis.

Method

Participants. A total of 219 participants were recruited online from Southwest University and South China Normal University in China, and were each given a small monetary compensation (10 RMB, approximately USD1.60) for taking part in the study. They were randomly allocated to two groups; 111 participants (78 female; $M_{\text{age}} = 20.8 \pm 3.9$ years) received pairs of *one-outcome* gambles and 108 participants (87 female; $M_{\text{age}} = 20.5 \pm 1.8$ years) received pairs of *two-outcome* gambles. Participants were excluded if they failed to answer a question that tested their understanding of the instructions. Fifteen participants were excluded from the one-outcome group (final $N = 96$, 68 female, $M_{\text{age}} = 20.9 \pm 4.1$ years) and 14 were excluded from the two-outcome gamble group (final $N = 94$, 77 female, $M_{\text{age}} = 20.6 \pm 1.8$ years). The study was approved by the Institutional Review Board of the Institute of Psychology, Chinese Academy of Sciences.

Insert Figure 4 here

Stimuli. The stimuli consisted of pairs of gambles each of size 2. That is, each gamble was of the form, $G = \{(p_1, x_1), (p_2, x_2)\}$, where $p_2 = 1 - p_1$, $x_1, x_2 \geq 0$ and $\min(x_1, x_2) \geq 0$. Thus, payoffs never involved losses. Participants received either one-outcome or two-outcome gambles. For one-outcome gambles, one of the payoffs was zero, that is, $\min(x_1, x_2) = 0$. For two-outcome gambles, both payoffs were greater than zero, that is, $\min(x_1, x_2) > 0$. There were 30 variable gambles each of which was paired with one of four

fixed gambles. Both sets of gambles are listed in Appendix A. Figure 4 plots the probabilities and payoffs of each gamble. As can be seen, they were selected so that probabilities and payoffs were negatively related and that the fixed gambles were distributed more or less evenly among the variable gambles. For variable one-outcome gambles, the mean expected value was 16.6 (range, 5.9-29.4) while for fixed one-outcome gambles, the mean expected value was 15.4 (range, 8.3-23.5). Two of these gambles offer a low probability (0.06 or 0.11) of winning a large amount of money (123 or 138), while the other two offer a moderate probability (0.49 or 0.51) of winning a moderate amount of money (32 or 48). For variable two-outcome gambles, the mean expected value was 24.9 (range, 15.8-38.7) while for fixed two-outcome gambles, the mean expected value was 18.7 (range, 17.8- 20.3). Two of these offer a low probability (0.07 or 0.11) of winning a large amount of money (135 or 83) and a high probability (0.93 or 0.89) of winning a modest amount of money (9 or 11), while the other two offer a moderate probability (0.38 or 0.46) of winning a moderate amount of money (21 or 23) and a moderate probability (0.62 or 0.54) of winning a modest amount of money (16 or 18).

Procedure. All participants completed the experiment online. All 120 pairs of gambles were presented in a random order. Participants were asked to select which gamble they preferred to play. At the end of the experiment, they were asked to provide demographic information (i.e., age, gender).

Results

The first step in applying STA to the present data is to make sure that each dependent variable varies significantly between conditions. If this is not the case then the relationship between dependent variables is trivially monotonic (Dunn & Kalish, 2018). In the current study, this means that choice frequencies should vary between variable gambles for each

fixed gamble. We tested this using a chi-square test for equal proportions which was highly significant for each fixed gamble; for one-outcome gambles, each $\chi^2(29) > 172$, and for two-outcome gambles, each $\chi^2(29) > 107$.

Insert Figure 5 here

Figure 5 shows the state-trace plots for each pairwise combination of fixed one-outcome gambles. Each point in each plot corresponds to one of the variable gambles. The axes correspond to the proportion of participants who chose the variable gamble over the indicated fixed gamble. The line in each plot connects the predicted points from the best-fitting monotonic model. The fit of the model is given by the G^2 value in each panel. The statistical significance of each fit was estimated using the bootstrap resampling procedure outlined by Kalish et al. As indicated by the asterisks in the Figure, the monotonic model can be rejected for four of the six plots. These results should be interpreted with some caution because each plot is not independent of the others. However, it is possible to conduct an overall test of significance using the Kalish et al procedure. This tests whether the latent order of variable gambles is the same for all of the fixed gambles⁴. The G^2 value for this model was 150.2 and was highly significant.

Insert Figure 6 here

Figure 6 shows the state-trace plots for each pairwise combination of fixed two-outcome gambles. The monotonic model could be rejected for four of the six plots. The G^2 value for the overall monotonic model was 277.2 and also highly significant.

⁴ That is, whether the observed state-trace is monotonically increasing in four-dimensional space.

Discussion

The results show that, despite an overall monotonic trend, violations of independence are common. For both one-outcome and two-outcome gambles, each pair of fixed gambles generated a highly significant violation of monotonicity except when the fixed gambles themselves were similar. In our experiment, fixed gambles 1 and 3 and fixed gambles 2 and 4 for both one-outcome and two-outcome gambles, were similar in terms of probabilities and payoffs (see Appendix A), and, for each of these pairs of gambles, no significant violations of monotonicity were observed. This makes intuitive sense as similar fixed gambles should have similar choice probabilities. That is, as $C \rightarrow B$, it follows that, $P(A,C) \rightarrow P(A,B)$, for all A .

It should be noted that the abovementioned similarity effect is not the same as that previously reported by Tversky and Russo (1969) and Mellers and Biagini (1994). The effect found in these studies is an example of a ‘double dissociation’ (Dunn & Kalish, 2018). As shown in Figure 3 in relation to the results found by Tversky and Russo, there is a large effect of changes in area (the independent variable) of the variable stimulus when it is similar to the standard. When it is dissimilar, the effect is reduced. This constitutes a ‘single dissociation’. If the two kinds of standard stimulus (elongated and square) are separately identified then the double dissociation results. That is, the effect of a change in the area of an elongated (square) variable stimulus is greater when compared to an elongated (square) standard stimulus than when compared to a square (elongated) stimulus. However, a double dissociation does not necessarily lead to a departure from a monotonic state-trace plot (Dunn & Kalish, 2018; Newell & Dunn, 2008). Everything depends on the location of the points in the state-trace plot. Accordingly, the data may fall on a monotonically increasing curve and yet still demonstrate a double dissociation. Such arrangements of points are apparent in Figures 1 and

2. In the latter Figure, it is most clearly demonstrated for lens stimuli. In contrast, the similarity effect that we have observed is based on the similarity of the fixed (or standard) stimuli. When these are similar (in both probability of size of payoffs) then independence is generally satisfied. Put another way, while violations of independence are likely to be present, they are numerically smaller than those observed between dissimilar fixed gambles.

The present results show that fixed utility models are inconsistent with observed patterns of risky choice in aggregated data. Geometrically, this means that the observed data do not fall on a one-dimensional curve in four-dimensional state-trace plot, as required by the fixed utility model. However, the data may conform to a higher-dimensional structure consistent with one or more alternative models. We explore this possibility with respect to two theoretical accounts; (1) that the observed patterns of risky choice are consistent with a *context-dependent* model of choice, or (2) that these patterns of choice arise from a *mixture* of individuals or subgroups whose choices each conform to a fixed utility model but with different utility values. In the following section we test the implications of both these accounts.

In a context-dependent model of choice, the utility of each gamble in a set is not fixed but varies depending on the content of the set. A prominent example comes from Decision Field Theory (Busemeyer & Townsend, 1993) in which the simplest model, which Busemeyer and Townsend call *random subjective expected utility (RSEU) model*, proposes that the probability of choosing gamble A over gamble B is given by

$$\begin{aligned}
 P(A, B) &= f \circ g(A, B) \\
 g(A, B) &= \frac{u(A) - u(B)}{\sqrt{\sigma^2(A) + \sigma^2(B)}}, \tag{4}
 \end{aligned}$$

where, f , $u(A)$ and $u(B)$ are defined as before and $\sigma^2(A)$ and $\sigma^2(B)$ are additional model parameters corresponding to the subjective variances of the utilities of gambles A and B , respectively⁵. Busemeyer and Townsend showed that the model given by Equation (4) is consistent with violations of independence. We show below that it is consistent with additional structure in the data.

The *fixed utility mixture model* proposes that aggregated choice probability consists of a mixture of the choice probabilities of two or more subgroups each of which conforms to the fixed utility model. Formally,

$$P(A, B) = \sum_{k=1}^N p_k \cdot f(u_k(A) - u_k(B)), \text{ subject to } \sum_{k=1}^N p_k = 1, \quad (5)$$

where N is the number of subgroups (each of which may contain a single individual), p_k is the proportion of observations in subgroup k , and $u_k(\cdot)$ is the corresponding utility function for this subgroup. A model of this kind is analogous to that proposed by Regenwetter, Dana and Davis-Stober (2011) who argued that violations of transitivity of preference judgments at a population or aggregate level arise as mixtures of transitive preference judgments at the individual or subgroup level. Similarly, it is readily demonstrated that choice probabilities that satisfy independence in each subgroup can lead to violations of independence when aggregated. An example is shown in Table 1. Consider two variable gambles, A_1 and A_2 , and two fixed gambles, B_1 and B_2 . Suppose there are two equal-sized subgroups or individuals. In subgroup 1, the utilities of the gambles are ordered, $u_1(A_1) < u_1(B_1) < u_1(A_2) < u_1(B_2)$.

⁵ The model may also include a parameter for the subjective covariance of the utilities of the gambles. For present purposes, we assume that this is equal to zero.

Independence is satisfied because $P(A_1, B_1) < P(A_2, B_1)$ and $P(A_1, B_2) < P(A_2, B_2)$. In subgroup 2, the utilities are ordered, $u_2(A_2) < u_2(B_2) < u_2(A_1) < u_2(B_1)$ and independence is satisfied because $P(A_1, B_1) > P(A_2, B_1)$ and $P(A_1, B_2) > P(A_2, B_2)$. However, as the illustrative figures in Table 1 show, independence is not satisfied in the aggregate.

Insert Table 1 here

Both the RSEU model and the mixture model are consistent with violations of independence. Because both contain more than one free parameter, in order to test them against the data, it is necessary to go beyond state-trace analysis. To do this, we apply *signed difference analysis*.

3. Signed difference analysis

Signed difference analysis (SDA) was first proposed by Dunn and James (2003) and recently extended by Dunn and Anderson (2018). It can be viewed as a generalization of state-trace analysis to models having more than one free parameter or latent variable. SDA is based on the observation that a model can be tested if it partitions the set of *sign vectors*⁶ into a *permitted* set and a *forbidden* set that are invariant under all possible monotonic transformations of the dependent variables (see Appendix C for examples). A *signed difference vector* is a sign vector consisting of the signs of the differences on each dependent variable between two different conditions. If the model is consistent with the data then each observed signed difference vector should be an element of the permitted set (subject to measurement error). In the present case, there are four dependent variables – the observed proportion of times that a variable gamble is selected over each of the four fixed gambles –

⁶ A sign vector is a vector in which each component is an element of the set, $\{+, -, 0\}$.

and 30 experimental conditions corresponding to the set variable gambles. Accordingly, there are $3^4 = 81$ distinct sign vectors and 435 possible signed difference vectors corresponding to each pairwise combination of variable gambles.

State-trace analysis is a special case of SDA in which the model has one latent variable. Accordingly, such a model also partitions the set of sign vectors into permitted and forbidden sets. With two dependent variables (corresponding to two fixed gambles), there are $3^2 = 9$ possible sign vectors. The fixed utility model requires that the preference order of two variable gambles be the same for any fixed gamble. Consider two variable gambles, A_1 and A_2 , each paired with two fixed gambles, B_1 and B_2 . Accordingly, if $P(A_1, B_1) \geq P(A_2, B_1)$ then $P(A_1, B_2) \geq P(A_2, B_2)$. Let $\mathbf{y}_1 = [P(A_1, B_1), P(A_1, B_2)]$ and $\mathbf{y}_2 = [P(A_2, B_1), P(A_2, B_2)]$ be the vectors of observed choice probabilities of, respectively, the variable gambles A_1 and A_2 over each of the fixed gambles B_1 and B_2 . Then $\text{sign}(\mathbf{y}_1 - \mathbf{y}_2)$ is the signed difference vector between conditions defined by A_1 and A_2 and, according to the fixed utility model, is an element of the set,

$$\{(0,0), (0,+), (0,-), (+,0), (-,0), (+,+), (-,-)\},$$

called the permitted set of the model. The remaining set of sign vectors, $\{(+,-), (-,+)\}$, is the forbidden set, so called because the fixed utility model (in this case) is refuted if a signed difference vector is found (subject to measurement error) that is an element of this set.

Any model that uniquely partitions a set of sign vectors into a permitted set and a forbidden set is said to be *orthomestic* (Dunn & James, 2003) and is associated with a unique *oriented matroid* (Dunn & Anderson, 2018). For present purposes, an oriented matroid may be viewed as a mathematical object that characterizes properties of a vector subspace that are

invariant under arbitrary monotonic transformations of the dimensions of the embedding space. An oriented matroid may be defined in several different ways in terms of different sets of sign vectors which, in turn, can be derived from a particular sign vector called the *chirotope* (Björner, Las Vergnas, Sturmfels, White & Ziegler, 1999). The chirotope of a vector space is defined as follows.

Let $V \subset \mathbb{R}^n$ be an m -dimensional vector subspace, $m \leq n$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a basis of V and let $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_m]$ be the corresponding $n \times m$ matrix. Then V is called the *column space* of \mathbf{V} . Let $I = \{i_1, i_2, \dots\}$ be the set of m -subsets of $[n]$ ordered lexicographically. An m -subset of $[n]$ is any selection of m distinct elements of $[n]$. For example, if $m = 2$ and $n = 4$ then $I = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. The components of each element of I are ordered from smallest to largest and the elements themselves are ordered lexicographically, increasing on the first component then the second and so on. Let i_k be the k th element of I and let \mathbf{V}_{i_k} be the $m \times m$ sub-matrix (i.e., minor) of \mathbf{V} with rows indexed by i_k . Let $\chi: \mathbb{R}^{n \times m} \rightarrow \{+, -, 0\}^{|I|}$ be the function,

$$\chi(\mathbf{V}) = \text{sign}\left(\det(\mathbf{V}_{i_1}), \det(\mathbf{V}_{i_2}), \dots\right). \quad (6)$$

Then $\{\chi(\mathbf{V}), -\chi(\mathbf{V})\}$ is called the chirotope⁷ of V .

Dunn and James (2003) showed that a model is orthomestic if the chirotope of the column space of its *integral Jacobian matrix* is unique. The integral Jacobian matrix of a

⁷ The term chirotope is part of oriented matroid theory. Dunn and James (2003) referred to it as the *signed Grassmann coordinates* of a matrix.

model is defined as follows. Let G be a convex subset of \mathbb{R}^m and let $g : G \rightarrow \mathbb{R}^n$, $m \leq n$, be a multivariate function. Let $\mathbf{x}, \tilde{\mathbf{x}} \in G$ and let $Dg(\mathbf{x})$ be the *Jacobian matrix* of g at \mathbf{x} . The Jacobian matrix at \mathbf{x} is the $n \times m$ matrix of partial derivatives of each component of g with respect to each component of \mathbf{x} . Let $\lambda(v) = \mathbf{x} + (\tilde{\mathbf{x}} - \mathbf{x})v$ be a straight path from \mathbf{x} to $\tilde{\mathbf{x}}$ in G . Then

$$\begin{aligned} g(\tilde{\mathbf{x}}) - g(\mathbf{x}) &= \int_0^1 Dg(\lambda(v)) dv (\tilde{\mathbf{x}} - \mathbf{x}) \\ &= \mathcal{D}g(\mathbf{x}, \tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \mathbf{x}) \end{aligned}$$

The matrix, $\mathcal{D}g(\mathbf{x}, \tilde{\mathbf{x}})$, is the integral Jacobian matrix from \mathbf{x} to $\tilde{\mathbf{x}}$.

Signed difference analysis of random subjective expected utility theory

The RSEU model given by Equation (4) has two free parameters, $u(A)$ and $\sigma^2(A)$, respectively the utility and subjective variance of the utility of gamble, A . As presented by Busemeyer and Townsend (1993), each of these is in turn a function of the probabilities and payoffs of the respective gambles. Let $A = \{(p_1, x_1), \dots, (p_n, x_n)\}$ be a gamble of size n . Then there are functions, $u(\cdot)$ and $\sigma^2(\cdot)$ such that,

$$\begin{aligned} u(A) &= u(p_1, \dots, p_n, x_1, \dots, x_n) \\ \sigma^2(A) &= \sigma^2(p_1, \dots, p_n, x_1, \dots, x_n) \end{aligned}$$

However, in line with our current approach, we make no assumptions concerning the nature of these functions. Rather, we treat the model parameters, $u(A)$ and $\sigma^2(A)$, as independent latent variables. We note that any prediction based on this assumption will also apply to any constrained version of the model. Accordingly, we parametrize the RSEU model as follows,

$$\begin{aligned}
P(A, B_i) &= f \circ g_i(a, s) \\
g_i(a, s) &= \frac{a - b_i}{\sqrt{s + t_i}}
\end{aligned} \tag{7}$$

Where, $f = (f_1, f_2, f_3, f_4)$ is a multivariate function with monotonically increasing components, $a \in \mathbb{R}$ is the utility of variable gamble A , $s \in \mathbb{R}^+$ is the subjective variance of the utility of A , $b_i \in \mathbb{R}$ is the utility of fixed gamble B_i , and $t_i \in \mathbb{R}^+$ is the subjective variance of the utility of B_i , for $i \in [4]$.

Analysis of the integral Jacobian matrix of the RSEU model given in Equation (7) shows that it is orthomestic for almost all $a \in \mathbb{R}$ and $s \in \mathbb{R}^+$ (for proof, see Appendix B). We also confirmed this result through numerical simulation. Furthermore, the analysis shows that the chirotope of the column space of the integral Jacobian matrix of the model depends only on the utilities of the fixed gambles corresponding to b_i in Equation (7). Let $g = (g_1, g_2, g_3, g_4)$ be the multivariate function with each g_i is defined by Equation (7). Let $\mathbf{x} = [a, s]$ and $\tilde{\mathbf{x}} = [\tilde{a}, \tilde{s}]$ be two sufficiently distinct parameter vectors and let $\mathcal{D}g(\mathbf{x}, \tilde{\mathbf{x}})$ be the integral Jacobian matrix of g from \mathbf{x} to $\tilde{\mathbf{x}}$. Let $I = ((1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4))$ be the set of 2-subsets of $[4]$ ordered lexicographically and let $i_k = (q_k, r_k)$ be the k th element of I . Then, according to the RSEU model, the chirotope of the column space of $\mathcal{D}g(\mathbf{x}, \tilde{\mathbf{x}})$ is the set $\{\chi(\mathcal{D}g), -\chi(\mathcal{D}g)\}$ where

$$\chi(\mathcal{D}g) = \text{sign}\left(\left(b_{r_1} - b_{q_1}\right), \left(b_{r_2} - b_{q_2}\right), \dots\right).$$

To illustrate, if $u(B_3) < u(B_1) < u(B_2) < u(B_4)$ then $\chi(\mathcal{D}g) = (+, -, +, -, +, +)$.

Insert Table 2 here

In the experiment described earlier, participants were never presented with pairs of fixed gambles. However, an order on the utilities these gambles can be obtained from the number of times that a variable gamble is selected over each fixed gamble. That is, if $u(B_i) < u(B_j)$ and $\sigma^2(B_i) \approx \sigma^2(B_j)$ then $P(A, B_i) > P(A, B_j)$, for any variable gamble, A . Accordingly, for each pair of fixed gambles, B_i, B_j , we calculated the proportion of times that $P(A, B_i) > P(A, B_j)$ across the set of 30 variable gambles, A . Table 2 presents the results separately for one-outcome and two-outcome gambles. The results for one-outcome gambles are reasonably straightforward and are consistent with the order (3,1,2,4). That is, $u(B_3) < u(B_1) < u(B_2) < u(B_4)$. The results for the two-outcome gambles are less clear cut but are most consistent with the order (1,3,4,2). That is, $u(B_1) < u(B_3) < u(B_4) < u(B_2)$.

Insert Table 3 here

A given 4×2 matrix may be associated with any of 24 distinct chirotopes that contain only non-zero components⁸. This is because not all $2^5 = 32$ combinations of + and - are possible⁹. Each chirotope defines an oriented matroid associated with an equivalence class of two-dimensional vector subspaces of \mathbb{R}^4 . Of the 24 chirotopes, 12 are consistent with a total order (and its opposite) on the utilities of the four fixed gambles. The oriented matroids associated with these chirotopes are called *acyclic* (Björner et al., 1999). The remainder are called *cyclic* and involve one or more intransitivities. For example, the chirotope, $\{\chi, -\chi\}$,

⁸ In the present case, a zero component indicates the unlikely possibility that two fixed gamble utilities are exactly equal.

⁹ While the chirotope in question has six components, only five are free to vary under the set of constraints called the Grassmann-Plücker relations.

$\chi = (+, -, -, -, +, -)$, requires that $u(B_1) < u(B_2)$ and $u(B_2) < u(B_4)$, but $u(B_1) > u(B_4)$.

Each of the 12 acyclic chirotopes can be paired with one of the 12 cyclic chirotopes that are, in a particular sense, orthogonal to each other¹⁰. Table 3 lists each pair of chirotopes corresponding to acyclic and cyclic oriented matroids, respectively, and the order of the utilities of the four fixed gambles associated with each acyclic oriented matroid.

Insert Figure 7 here

Using a modification of the procedure used by Kalish, Dunn, Burdakov and Sysoev (2016) we fit the model corresponding to each chirotope to data from both the one-outcome and two-outcome sets of gambles (for a similar application of this algorithm, see Stephens, Dunn and Hayes, 2018). In brief, each chirotope defines a distinct oriented matroid which, in turn, partitions the set of sign vectors into permitted and forbidden subsets as described by Dunn and Anderson (2018). As a further illustration, Appendix C lists the sets of permitted and forbidden sign vectors associated with three different chirotopes. To fit a given chirotope (so to speak), optimal predicted values are sought that are as close to the observed data as possible (in a weighted least squares sense) under the restriction that each predicted signed difference vector is an element of the permitted set of the associated oriented matroid¹¹. Given that we have identified a preference order for the four fixed one-outcome and the four fixed two-outcome gambles, the RSEU model predicts that the chirotopes consistent with these orders will fit best. The results are shown in Figure 7 which graphs the observed fit

¹⁰ Specifically, the subspaces associated with the pair of chirotopes are orthogonal.

¹¹ Because of the large number of conditions (30 variable gambles), in order to avoid combinatorial explosion, it was necessary to use an approximate version of the model fitting procedure (as also used by Stephens et al., 2018).

value (G^2) associated with each chirotope to both one-outcome and two-outcome gambles. Bars marked by an asterisk indicate that the observed fit value is significantly greater than zero ($p < 0.05$), based on the bootstrap test developed by Kalish et al. (2016). This reveals that the fits of all chirotopes associated with a cyclic oriented matroid are significantly greater than zero. For one-outcome gambles, the predicted order was (3,1,2,4). Numerically, the best-fitting acyclic chirotope was number 4, consistent with the order (3,1,2,4) or its opposite (see Table 3). Slightly greater fit values were found for chirotope numbers 3, 8 and 9 which correspond to the orders (3,1,4,2), (1,3,4,2), and (1,3,2,4) (and their opposites), respectively. For two-outcome gambles, the predicted order was (1,3,4,2). Numerically, the best-fitting chirotope was number 3 (3,1,4,2), closely followed by number 8 (1,3,4,2).

Discussion

Our earlier application of state-trace analysis to the aggregated data from one-outcome and two-outcome gambles showed that these data are not consistent with the predictions of the fixed utility model. However, application of signed difference analysis revealed that these data are nevertheless consistent with several two-dimensional state-traces, each characterized by its corresponding chirotope. The properties of this set of two-dimensional structures constitutes a regularity with which any viable model of risky choice needs to be consistent.

Second, the two-dimensional properties of the data were found to be broadly consistent with the predictions of the RSEU model based on Decision Field Theory. While this is a remarkable and unexpected result, it requires two caveats. First, the predictions derived from the RSEU model are based on an approximation that does not always hold when there are small differences between the utilities of fixed gambles. For the present set of stimuli, there are indications that the utilities of fixed gambles 1 and 3 and of gambles 2 and 4 respectively differ by relatively small amounts (for both one-outcome and two-outcome sets). To the

extent that these differences are sufficiently small, violations of the predicted order may occur. The second caveat is that the predictions may not be unique to the RSEU model. It is possible that other models may make the same prediction. It is to this issue that we now turn.

Signed difference analysis of the fixed utility mixture model

In this section, we apply a form of SDA to the fixed utility mixture model given in Equation (5). Our approach is to determine whether and under what conditions this model is consistent with the two-dimensional regularity that emerged from the signed difference analysis of the RSEU model. We were not able to repeat the analysis of the mixture model in terms of the properties of the integral Jacobian matrix because this component of the model cannot be separated from the (unknown) monotonic function. As a consequence, we rely on simulations.

We investigated the predictions of the mixture model in the following way. For each of $N > 1$ subgroups, we generated 30 variable gamble utilities and compared them to each of four fixed gambles. Let a_{ik} be the utility of variable gamble i for subgroup k and let b_{jk} be the utility of fixed gamble j for subgroup k . Each of these utilities was modelled as an independent draw from a standard normal distribution. We also investigated other plausible distributions (e.g., logistic) and found qualitatively similar results. The number of subgroups, N , was varied over a large range. Again, essentially the same results were found for all values of $N > 1$ although as N increases, the differences between conditions become smaller and G^2 values approach zero. Lastly, the mixture probabilities were all equal. That is, $p_k = 1/N$ for each subgroup k .

For each value of N , each combination of fixed and variable gamble utilities was converted into choice frequencies (based on 100 notional participants) using Equation (5) and

the normal cdf substituting for f . These simulated data were then fit by models corresponding to each of the 24 chirotopes discussed earlier. We were primarily interested in comparing two different scenarios in which the values of the variable gamble utilities were uncorrelated between subgroups. In the first scenario, fixed gamble utilities were also uncorrelated between subgroups while in the second scenario, they were constrained to be in the same order in each subgroup. In practice, this is equivalent to a re-ordering of the fixed gamble utility values generated from the first scenario. For ease of comparison to the results shown in Figure 7, the order of fixed gamble utilities was chosen to be the same as that found for one-outcome gambles. That is, (3,1,2,4).

Insert Figure 8 here

Figure 8 summarizes the key result. It shows the average fit values (G^2) for each acyclic chirotope under the two scenarios for $N = 10$ subgroups. The fits of the 12 cyclic chirotopes are not shown as they are uniformly poor and mask the differences between the fits of the 12 acyclic chirotopes. The choice of $N = 10$ is also arbitrary as similar results were found for any $N > 1$. The black bars show the fits of each chirotope under the first scenario in which the values of the fixed gamble utilities varied randomly between subgroups. It shows that each chirotope fits the data more or less equally. The gray bars show the fits of each chirotope under the second scenario in which the values of the fixed gamble utilities were constrained to have a common order (3,1,2,4) between subgroups. In contrast to the first scenario, it is clear that there are substantial differences in fit between the 12 acyclic chirotopes. The best fit is obtained for chirotope number 4 which corresponds exactly to the fixed gamble utility order (3,1,2,4) (and its opposite, see Table 3). There is also a graded increase in fit values for chirotopes containing increasing numbers of pairwise inversions. Thus, chirotope numbers 3, 6, 8, and 9, each of which is one pairwise inversion from the

target order, (3,1,2,4), all have relatively low fit values. This pattern is qualitatively similar to that observed in the experiment and shown in Figure 7.

Discussion

Signed difference analysis of aggregated data from both one-outcome and two-outcome gambles revealed an underlying two-dimensional structure consistent with the predictions of the RSEU model (Busemeyer & Townsend, 1993). We also investigated whether this pattern was consistent with a fixed utility mixture model in which violations of independence at the aggregate level, as revealed by state-trace analysis, arise as a consequence of combining choice probabilities across different subgroups each of which conforms to the fixed utility model with differing utility values. While summing over different subgroups easily leads to violations of independence at the aggregate level, our analysis of simulated data revealed that the critical two-dimensional structure emerged only when there was a common order of fixed gamble utilities across subgroups. Yet, while consistent with the observed data, this poses a problem for the mixture model. The problem is that if there is a common order of the fixed gamble utilities across subgroups then there is no reason that there should not also be a common order of the variable gamble utilities. That is, if subgroups agree on the utility order of one set of gambles, they should agree on the utility order of all gambles. But if this were the case then the mixture model is no longer a mixture and predicts the one-dimensional state-trace that has been ruled out by the earlier state-trace analysis.

Our simulation juxtaposed two extreme scenarios – uncorrelated fixed gamble utilities in the first scenario and perfect ordinal correlations in the second scenario. We have not explored all possible mixtures of orders and it is possible that a combination exists that is consistent with the observed data. However, this would offer an unconvincing *post hoc*

explanation that would depend critically upon a highly fortuitous combination of factors. In contrast, the RSEU model predicts almost exactly the pattern of data that is observed.

4. General discussion

In this paper we have tested three different models of risky choice using the related methodologies of state-trace analysis and signed difference analysis. In the first part, we used state-trace analysis to test for independence, a key property of fixed utility models. The singular advantage of this approach is that it enabled us to examine a set of ecologically relevant gambles and to test model predictions without having to specify the form of a specific monotonically increasing link function. This overcomes the problem of fitting models in which this function may be misspecified. For example, Reiskamp (2008) fit models of risky choice to data after assuming a stochastic link function, corresponding to a particular form of f in Equation (1), and found mixed results. While a fixed utility model (Cumulative Prospect Theory) fit the data reasonably well and outperformed a stochastic variant of the Priority Heuristic model (Brandstätter, Gigerenzer, & Hertwig, 2006), it was itself outperformed by Decision Field Theory (Busemeyer & Townsend, 1993). While these results are important they are limited by the fact that, in each case, the fit of the model depends on the appropriateness of the assumption of the stochastic link function.

Our application of state-trace analysis also showed that violations of independence do not depend on the presence of a similarity effect, as proposed by Mellers and Biagini (1994). Instead, they appear to be due to context effects arising from the component gambles themselves. This is supported by the finding that violations of independence (i.e., a non-monotonic state-trace) were not observed between pairs of similar fixed gambles (between gambles 1 and 3, and between 2 and 4).

In the second part, the question of how best to account for the observed data was explored using signed difference analysis. Two models were compared; the random subjective expected utility model derived from Decision Field Theory and a fixed utility mixture model. We interpreted the RSEU model as a context-dependent model of the data at an aggregated or population level. In contrast, according to the fixed utility mixture model, violations of independence at the population level arise as a mixture of heterogeneous subgroups each of whose pattern of choices is consistent with the fixed utility model. Consistent with previous demonstrations (Prince, Brown & Heathcote, 2012), we showed that such aggregation (or averaging) of underlying one-dimensional state-traces can lead to the observation of higher-dimensional state-traces.

We investigated the structure of the higher-dimensional state-trace using signed difference analysis. Based on the theory of SDA, we analyzed the integral Jacobian matrix of the RSEU model and derived the prediction that it is almost always consistent with a two-dimensional structure (in four-dimensional dependent variable space) characterized by an oriented matroid whose chirotope is determined by the order of the utilities of the fixed gambles. This is an elegant and surprising prediction which, perhaps even more remarkably, was found to be broadly consistent with the observed data both for one-outcome and two-outcome gambles. Through simulation, we demonstrated that this structure is also consistent with the mixture model but only under the assumption that the order of the fixed gamble utilities is the same for each component subgroup. While this is possible, it is not a natural prediction of this model and implies that the order of the variable gamble utilities should also be the same for each subgroup. This, in turn, implies contra the previous state-trace analysis that independence is not violated at the aggregate level.

Our application of STA and SDA is novel and while the results are suggestive, it is clear that further investigations are warranted. An important outstanding question concerns the

relationship between models that account for data at either population, subgroup, or individual levels. While we have found evidence consistent with the RSEU model at the population level, we cannot assume that this model also accounts for data at other levels of aggregation. Having said that, it is often difficult to obtain multiple independent responses to the same choice pairs at the individual level (Roelofsma & Read, 2000). Furthermore, determining the consistency of behaviour at the individual level invites the question of whether this may change over time or as a result of experience. For example, risky choice has been shown to be more consistent with expected utility or other fixed utility models if choices are offered on multiple occasions, either in fact (Baucells & Villasís, 2010; Hey, 2001) or by instruction (Keren, 1991; Rao, Dunn, Zhou & Li, 2015). The methods we have used in the present study could be used to investigate the viability of different classes of model under these conditions.

Our application of STA and SDA provides both a description of the data and a procedure for deriving predictions from models. As a description of the (aggregated) data, STA revealed that its structure was more complex than the unidimensional prediction of fixed utility models. Furthermore, SDA revealed that these data were well-described by a relatively simple (i.e., orthomestic) two-dimensional structure. This may provide a template for future investigations of risky choice or other kinds of judgment – given an n -dimensional dependent variable space, we can ask what is the smallest dimensionality that is consistent with the data. In addition, through the analysis of chirotopes corresponding to different data structures, it is possible to determine a regularity or set of regularities against which different models may be tested. The theory of SDA can then be used to derive predictions from models either analytically (if amenable) or through simulation. This, we suggest, offers a powerful tool for investigating the structure of models under monotonicity.

5. Appendix A - Probabilities (p_1) and payoffs (x_1, x_2) for one-outcome and two-outcome gambles.

<i>Type</i>	<i>No.</i>	<i>Gamble</i>				
		<i>One-outcome</i>		<i>Two-outcome</i>		
		p_1	x_1	p_1	x_1	x_2
<i>Variable</i>	1	0.04	161	0.05	165	8
	2	0.27	102	0.11	138	7
	3	0.32	64	0.23	102	12
	4	0.47	50	0.27	98	13
	5	0.36	47	0.32	89	15
	6	0.69	22	0.41	61	16
	7	0.56	27	0.49	51	13
	8	0.45	17	0.56	42	11
	9	0.83	9	0.61	36	12
	10	0.05	156	0.71	13	27
	11	0.25	98	0.09	127	11
	12	0.31	61	0.18	107	12
	13	0.46	52	0.26	82	13
	14	0.37	56	0.35	71	14
	15	0.67	23	0.44	57	13
	16	0.58	27	0.54	45	12
	17	0.73	18	0.67	31	13
	18	0.83	11	0.76	27	14
	19	0.17	102	0.86	15	28
	20	0.15	121	0.92	12	60
	21	0.72	21	0.07	140	8
	22	0.03	198	0.13	109	10
	23	0.28	105	0.25	56	11
	24	0.34	67	0.34	43	12
	25	0.48	56	0.45	32	13
	26	0.33	66	0.57	28	14
	27	0.71	23	0.65	21	20
	28	0.53	29	0.75	17	28
	29	0.63	17	0.85	15	40
	30	0.82	10	0.97	16	70
<i>Fixed</i>	1	0.11	123	0.11	83	11
	2	0.51	32	0.38	21	16
	3	0.06	138	0.07	135	9
	4	0.49	48	0.46	23	18

6. Appendix B – Derivation of the chirotope of the random subjective expected utility (RSEU) model.

We parameterize the RSEU model as follows:

$$\begin{aligned} P(A, B_i) &= f \circ g_i(a, s) \\ g_i(a, s) &= \frac{a - b_i}{\sqrt{s + t_i}} \end{aligned} \quad (\text{A.1})$$

Where, $f = (f_1, f_2, f_3, f_4)$ is a function with monotonically increasing components, $a \in \mathbb{R}$ is the utility of variable gamble A , $s \in \mathbb{R}^+$ is the subjective variance of the utility of A , $b_i \in \mathbb{R}$ is the utility of fixed gamble B_i , and $t_i \in \mathbb{R}^+$ is the subjective variance of the utility of B_i , with $i \in [4]$.

Let $g = (g_1, g_2, g_3, g_4)$ and let $g_{ij} = (g_i, g_j)$. Let $P \subseteq \mathbb{R} \times \mathbb{R}^+$ be the convex domain of g_{ij} and let $p = (a, s)$ be an element of P . The Jacobian matrix of g_{ij} at p is

$$Dg_{ij}(p) = \begin{bmatrix} \frac{1}{(s+t_i)^{\frac{1}{2}}} & \frac{b_i - a}{2(s+t_i)^{\frac{3}{2}}} \\ \frac{1}{(s+t_j)^{\frac{1}{2}}} & \frac{b_j - a}{2(s+t_j)^{\frac{3}{2}}} \end{bmatrix}.$$

Let $p_0 = (a_0, s_0)$ and $p_1 = (a_1, s_1)$ be elements of P such that $a_0 \neq a_1$ and $s_0 \neq s_1$. Because P is convex, the integrated Jacobian matrix of g_{ij} from p_0 to p_1 is defined as

$$\mathcal{D}g_{ij}(p_0, p_1) = \begin{bmatrix} \int_0^1 \frac{1}{(s_0 + v(s_1 - s_0) + t_i)^{\frac{1}{2}}} dv & \int_0^1 \frac{b_i - (a_0 + v(a_1 - a_0))}{2(s_0 + v(s_1 - s_0) + t_i)^{\frac{3}{2}}} dv \\ \int_0^1 \frac{1}{(s_0 + v(s_1 - s_0) + t_j)^{\frac{1}{2}}} dv & \int_0^1 \frac{b_j - (a_0 + v(a_1 - a_0))}{2(s_0 + v(s_1 - s_0) + t_j)^{\frac{3}{2}}} dv \end{bmatrix}.$$

Consider the two terms in the top row of the above matrix (the other terms follow by substitution). The first of these is

$$\int_0^1 \frac{1}{(s_0 + v(s_1 - s_0) + t_i)^{\frac{1}{2}}} dv = \left. \frac{-2(s_0 + v(s_1 - s_0) + t_i)^{\frac{1}{2}}}{(s_0 - s_1)} \right|_0^1$$

$$= \frac{2(z_{0i} - z_{1i})}{(s_0 - s_1)},$$

where $z_{0i} = (s_0 + t_i)^{\frac{1}{2}}$ and $z_{1i} = (s_1 + t_i)^{\frac{1}{2}}$.

The second term is

$$\int_0^1 \frac{b_i - (a_0 + v(a_1 - a_0))}{2(s_0 + v(s_1 - s_0) + t_i)^{\frac{3}{2}}} dv$$

$$= \left. \frac{-(s_0 + v(s_1 - s_0) + t_i)^{\frac{1}{2}} (3a_0(s_1 + t_i) - 3a_1(s_0 + t_i) + 3b_i(s_0 - s_1) - (a_0 - a_1)(s_0 + v(s_1 - s_0) + t_i))}{3(s_0 - s_1)^2} \right|_0^1$$

$$= \frac{3(b_i(s_0 - s_1) - a_1 z_{0i}^2 + a_0 z_{1i}^2)(z_{0i} - z_{1i}) - (a_0 - a_1)(z_{0i}^3 - z_{1i}^3)}{3(s_0 - s_1)^2}.$$

Consequently,

$$\det(\mathcal{D}g_{ij}(p_0, p_1)) = \frac{2(z_{0i} - z_{1i})}{(s_0 - s_1)} \cdot \frac{3(b_j(s_0 - s_1) - a_1 z_{0j}^2 + a_0 z_{1j}^2)(z_{0j} - z_{1j}) - (a_0 - a_1)(z_{0j}^3 - z_{1j}^3)}{3(s_0 - s_1)^2}$$

$$- \frac{2(z_{0j} - z_{1j})}{(s_0 - s_1)} \cdot \frac{3(b_i(s_0 - s_1) - a_1 z_{0i}^2 + a_0 z_{1i}^2)(z_{0i} - z_{1i}) - (a_0 - a_1)(z_{0i}^3 - z_{1i}^3)}{3(s_0 - s_1)^2}$$

Expanding and rearranging terms, we get

$$\det(\mathcal{D}g_{ij}(p_0, p_1)) = U(b_j - b_i) + V_1 - V_2 - V_3,$$

where,

$$\begin{aligned}
U &= \frac{2(z_{0i} - z_{1i})(z_{0j} - z_{1j})}{(s_0 - s_1)^2} \\
V_1 &= \frac{2(a_0 - a_1)}{(s_0 - s_1)^3} (z_{0i} - z_{1i})(z_{0j} - z_{1j})(z_{0j}^2 - z_{0i}^2) \\
V_2 &= \frac{2(a_0 - a_1)}{3(s_0 - s_1)^3} (z_{0i} - z_{1i})(z_{0j} - z_{1j})(z_{0j}^2 - z_{0i}^2) \\
V_3 &= \frac{2(a_0 - a_1)}{3(s_0 - s_1)^3} (z_{0i} - z_{1i})(z_{0i}z_{1j} - z_{0j}z_{1i})(z_{0i} + z_{1i}).
\end{aligned}$$

It is clear that $U > 0$. Less obviously, if $t_i = t_j$ then $V_1 = V_2 = V_3 = 0$ and hence

$\text{sign}(\det(\mathcal{D}g_{ij}(p_0, p_1))) = \text{sign}(b_j - b_i)$. Even less obviously, if $t_i \neq t_j$ then $V_1 \approx V_2 + V_3$ and

hence, for almost all $p_0, p_1 \in P$, $\text{sign}(\det(\mathcal{D}g_{ij}(p_0, p_1))) = \text{sign}(b_j - b_i)$.

The approximation, $V_1 \approx V_2 + V_3$, is shown as follows. Because $V_2 = \frac{1}{3}V_1$, it is sufficient to show that $V_3 \approx 2V_2$. Noting that, $z_{0j}^2 - z_{0i}^2 = t_j - t_i$, we want to show that

$$\frac{(z_{0i}z_{1j} - z_{0j}z_{1i})(z_{0i} + z_{1i})}{2(z_{0j} - z_{1j})} \approx y_j - y_i.$$

We use the identity,

$$\frac{1}{2}(b+d)(a-c) - \frac{1}{2}(b-d)(a+c) = ad - bc,$$

to give,

$$\begin{aligned}
\frac{(z_{0i}z_{1j} - z_{0j}z_{1i})(z_{0i} + z_{1i})}{2(z_{0j} - z_{1j})} &= \frac{(z_{0j} + z_{1j})(z_{0i} - z_{1i})(z_{0i} + z_{1i})}{4(z_{0j} - z_{1j})} - \frac{(z_{0j} - z_{1j})(z_{0i} + z_{1i})(z_{0i} + z_{1i})}{4(z_{0j} - z_{1j})} \\
&= \frac{(z_{0j} + z_{1j})(z_{0i}^2 - z_{1i}^2)}{4(z_{0j} - z_{1j})} - \frac{(z_{0i} + z_{1i})^2}{4} \\
&= \frac{(z_{0j} + z_{1j})^2(z_{0i}^2 - z_{1i}^2)}{4(z_{0j}^2 - z_{1j}^2)} - \frac{(z_{0i} + z_{1i})^2}{4}.
\end{aligned}$$

Noting that $z_{0i}^2 - z_{1i}^2 = z_{0j}^2 - z_{1j}^2 = s_0 - s_1 \neq 0$,

$$\begin{aligned} \frac{(z_{0i}z_{1j} - z_{0j}z_{1i})(z_{0i} + z_{1i})}{2(z_{0j} - z_{1j})} &= \frac{1}{4}(z_{0j} + z_{1j})^2 - \frac{1}{4}(z_{0i} + z_{1i})^2 \\ &= \frac{1}{4}(z_{0j}^2 - z_{0i}^2 + z_{1j}^2 - z_{1i}^2 + 2(z_{0j}z_{1j} - z_{0i}z_{1i})) \\ &= \frac{1}{2}(t_j - t_i + z_{0j}z_{1j} - z_{0i}z_{1i}). \end{aligned}$$

And, because $z_{0j}z_{1j} - z_{0i}z_{1i} \approx t_j - t_i$, it follows that

$$\frac{(z_{0i}z_{1j} - z_{0j}z_{1i})(z_{0i} + z_{1i})}{2(z_{0j} - z_{1j})} \approx t_j - t_i.$$

Therefore, $V_1 \approx V_2 + V_3$, and $\text{sign}(\det(\mathcal{D}g_{ij}(p_0, p_1))) = \text{sign}(b_j - b_i)$ holds for $|b_j - b_i|$ sufficiently greater than zero. To give some indication of the limits of this approximation, we conducted the following simulation. Values of a_0, a_1, b_1 and b_2 were drawn from a uniform distribution on the interval $(-1, 1)$. Values of s_0, s_1, t_1 and t_2 were drawn from a uniform distribution on the interval $(0, 2)$. A sample size of 100,000 was used. The results are shown in Figure A-1. Changes in sign between the determinant, $U(b_j - b_i) + V_1 - V_2 - V_3$ and $U(b_j - b_i)$ are indicated by the black points and accounted for 0.25% of observed values. They almost entirely occur when $|b_j - b_i|$ is within 5% of zero.

Insert Figure A-1 here

7. Appendix C – Permitted and forbidden sets of sign vectors associated with each of three example chirotopes (for more information, see Table 3).

<i>Set</i>	<i>Chirotope</i>		
	$\pm(+, -, +, -, +, +)$	$\pm(+, +, +, -, -, +)$	$\pm(+, -, -, -, +, -)$
Permitted	$(0, 0, 0, 0)$ $\pm(0, 0, 0, +)$ $\pm(0, 0, +, -)$ $\pm(0, 0, +, 0)$ $\pm(0, 0, +, +)$ $\pm(0, +, -, 0)$ $\pm(0, +, -, +)$ $\pm(0, +, 0, -)$ $\pm(0, +, 0, 0)$ $\pm(0, +, 0, +)$ $\pm(0, +, +, -)$ $\pm(0, +, +, 0)$ $\pm(0, +, +, +)$ $\pm(+, -, 0, -)$ $\pm(+, -, 0, 0)$ $\pm(+, -, +, -)$ $\pm(+, -, +, 0)$ $\pm(+, 0, -, 0)$ $\pm(+, 0, -, +)$ $\pm(+, 0, 0, -)$ $\pm(+, 0, 0, 0)$ $\pm(+, 0, 0, +)$ $\pm(+, 0, +, -)$ $\pm(+, 0, +, 0)$ $\pm(+, 0, +, +)$ $\pm(+, +, -, 0)$ $\pm(+, +, -, +)$ $\pm(+, +, 0, -)$ $\pm(+, +, 0, 0)$ $\pm(+, +, 0, +)$ $\pm(+, +, +, -)$ $\pm(+, +, +, 0)$ $\pm(+, +, +, +)$	$(0, 0, 0, 0)$ $\pm(0, 0, 0, +)$ $\pm(0, 0, +, -)$ $\pm(0, 0, +, 0)$ $\pm(0, 0, +, +)$ $\pm(0, +, -, -)$ $\pm(0, +, -, 0)$ $\pm(0, +, -, +)$ $\pm(0, +, 0, -)$ $\pm(0, +, 0, 0)$ $\pm(0, +, 0, +)$ $\pm(0, +, +, 0)$ $\pm(0, +, +, +)$ $\pm(+, -, -, -)$ $\pm(+, -, -, 0)$ $\pm(+, -, 0, -)$ $\pm(+, -, 0, 0)$ $\pm(+, -, +, -)$ $\pm(+, -, +, 0)$ $\pm(+, 0, -, +)$ $\pm(+, 0, +, -)$ $\pm(+, 0, +, 0)$ $\pm(+, 0, -, -)$ $\pm(+, 0, -, 0)$ $\pm(+, 0, 0, -)$ $\pm(+, 0, 0, 0)$ $\pm(+, 0, 0, +)$ $\pm(+, 0, +, -)$ $\pm(+, 0, +, 0)$ $\pm(+, 0, +, +)$ $\pm(+, 0, +, 0)$ $\pm(+, 0, +, +)$ $\pm(+, +, 0, -)$ $\pm(+, +, 0, 0)$ $\pm(+, +, 0, +)$ $\pm(+, +, +, 0)$ $\pm(+, +, +, +)$	$(0, 0, 0, 0)$ $\pm(0, 0, 0, +)$ $\pm(0, 0, +, -)$ $\pm(0, 0, +, 0)$ $\pm(0, 0, +, +)$ $\pm(0, +, -, -)$ $\pm(0, +, -, 0)$ $\pm(0, +, -, +)$ $\pm(0, +, 0, -)$ $\pm(0, +, 0, 0)$ $\pm(0, +, 0, +)$ $\pm(0, +, +, -)$ $\pm(0, +, +, 0)$ $\pm(+, -, 0, -)$ $\pm(+, -, 0, 0)$ $\pm(+, -, 0, +)$ $\pm(+, -, +, -)$ $\pm(+, -, +, 0)$ $\pm(+, 0, -, -)$ $\pm(+, 0, -, 0)$ $\pm(+, 0, 0, -)$ $\pm(+, 0, 0, 0)$ $\pm(+, 0, 0, +)$ $\pm(+, 0, +, -)$ $\pm(+, 0, +, 0)$ $\pm(+, 0, +, +)$ $\pm(+, +, -, -)$ $\pm(+, +, -, 0)$ $\pm(+, +, 0, -)$ $\pm(+, +, 0, 0)$ $\pm(+, +, +, -)$ $\pm(+, +, +, 0)$
Forbidden	$\pm(0, +, -, -)$ $\pm(+, -, -, -)$ $\pm(+, -, -, 0)$ $\pm(+, -, -, +)$ $\pm(+, -, 0, +)$ $\pm(+, -, +, +)$ $\pm(+, 0, -, -)$ $\pm(+, +, -, -)$	$\pm(0, +, +, -)$ $\pm(+, -, -, +)$ $\pm(+, 0, -, +)$ $\pm(+, +, -, -)$ $\pm(+, +, -, 0)$ $\pm(+, +, -, +)$ $\pm(+, +, 0, -)$ $\pm(+, +, +, -)$	$\pm(0, +, +, +)$ $\pm(+, -, -, -)$ $\pm(+, -, -, 0)$ $\pm(+, -, -, +)$ $\pm(+, 0, -, +)$ $\pm(+, +, -, +)$ $\pm(+, +, 0, +)$ $\pm(+, +, +, +)$

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Table 1: Violation of independence due to aggregation.

<i>Option</i>		<i>Probability of choosing variable option</i>		
<i>Variable</i>	<i>Fixed</i>	<i>Subgroup 1</i>	<i>Subgroup 2</i>	<i>Aggregate</i>
A_1	B_1	0.2	0.7	0.45
A_2	B_1	0.6	0.4	0.5
A_1	B_2	0.4	0.6	0.5
A_2	B_2	0.7	0.2	0.45

Table 2: Proportion of times that the probability of choosing a variable gamble over the fixed gamble indicated by row is greater than the probability of choosing the same variable gamble over the fixed gamble indicated by column.

	<i>One-outcome gambles</i>					<i>Two-outcome gambles</i>			
	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>		<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>
<i>1</i>	-	0.10	0.93	0.10	<i>1</i>	-	0.70	0.60	0.59
<i>2</i>		-	1.00	0.17	<i>2</i>		-	0.27	0.10
<i>3</i>			-	0.03	<i>3</i>			-	0.57
<i>4</i>				-	<i>4</i>				-

Table 3: List of 12 pairs of acyclic and cyclic chirotopes and associated preference orders.

<i>Number</i>	<i>Acyclic Chirotope</i>	<i>Preference Orders</i>	<i>Cyclic Chirotope</i>
1	$\pm(+, -, -, -, -)$	(2,1,3,4), (4,3,1,2)	$\pm(+, -, +, +, -, -)$
2	$\pm(+, -, -, -, +)$	(2,1,4,3), (3,4,1,2)	$\pm(+, +, -, -, +, +)$
3	$\pm(+, -, +, -, +)$	(2,4,1,3), (3,1,4,2)	$\pm(+, +, -, +, +, +)$
4	$\pm(+, -, +, -, +, +)$	(3,1,2,4), (4,2,1,3)	$\pm(+, -, -, +, +, +)$
5	$\pm(+, +, -, -, -)$	(2,3,1,4), (4,1,3,2)	$\pm(+, -, +, +, +, -)$
6	$\pm(+, +, -, +, -)$	(3,2,1,4), (4,1,2,3)	$\pm(+, -, -, +, +, -)$
7	$\pm(+, +, +, -, -)$	(1,4,3,2), (2,3,4,1)	$\pm(+, -, +, -, +, -)$
8	$\pm(+, +, +, -, +)$	(1,3,4,2), (2,4,3,1)	$\pm(+, +, -, +, -, +)$
9	$\pm(+, +, +, -, +, +)$	(1,3,2,4), (4,2,3,1)	$\pm(+, -, -, +, -, +)$
10	$\pm(+, +, +, +, -, -)$	(1,4,2,3), (3,2,4,1)	$\pm(+, -, -, -, +, -)$
11	$\pm(+, +, +, +, -, -)$	(1,2,4,3), (3,4,2,1)	$\pm(+, +, -, -, +, -)$
12	$\pm(+, +, +, +, +, +)$	(1,2,3,4), (4,3,2,1)	$\pm(+, -, +, +, -, +)$

10. Figure Captions

1. State-trace plots of mean preference ratings of gambles from Mellers and Biagini (1994). Each point represents one of the 25 a gambles. The axes correspond to pairs of b gambles; $b_1 = \{(0.1, \$51), (0.9, \$0)\}$, $b_2 = \{(0.5, \$51), (0.5, \$0)\}$, $b_3 = \{(0.9, \$51), (0.1, \$0)\}$. The line connects the points predicted by the best-fitting monotonic model.
2. State-trace plot of results found by Tversky and Russo (1969). The axes correspond to the proportion of times that the variable stimulus, either elongated or square, was judged larger than the standard stimulus, either elongated or square. The line connects the points predicted by the best-fitting monotonic model.
3. Data from Tversky and Russo (1969). The top row plots the observed proportion of times the variable stimulus is selected over the standard stimulus for rectangles and lenses. The similarity effect corresponds to the steeper curve associated with variable stimuli similar to the standard compared to variable stimuli that are dissimilar to the standard. The bottom row plots the predicted proportions derived from the best-fitting monotonic model.
4. Plot of the probabilities and payoffs of gambles used in the present study. (a) One-outcome gambles; (b) Two-outcome gambles.
5. State-trace plots for each pairwise combination of one-outcome gambles from the present study. Each point represents one of the 30 variable gambles. The axes correspond to the proportion of participants that selected the variable gamble over the indicated fixed gamble (range: 0 to 1). The line connects the points predicted by the best-fitting monotonic model. The G^2 value is the fit of the monotonic model. One

asterisk indicates that $p < 0.05$. Two asterisks indicate that $p < 0.01$. Otherwise, $p > 0.05$.

6. State-trace plots for each pairwise combination of two-outcome gambles from the present study. Each point represents one of the 30 variable gambles. The axes correspond to the proportion of participants that selected the variable gamble over the indicated fixed gamble (range: 0 to 1). The line connects the points predicted by the best-fitting monotonic model. The G^2 value is the fit of the monotonic model. One asterisk indicates that $p < 0.05$. Two asterisks indicate that $p < 0.01$. Otherwise, $p > 0.05$.
7. Fits (G^2) of models determined by each pair of acyclic and cyclic chirotopes. (a) One-outcome data; (b) Two-outcome data. Asterisks indicate a significant difference from zero ($p < 0.05$).
8. Fits (G^2) of models determined by each acyclic chirotope to simulated data based on the fixed utility mixture model with 10 subgroups.

A-1 Plot of approximate against exact values for the determinant of the integrated Jacobian matrix of the random subjective expected utility model. Gray points indicate consistent signs, black points indicate inconsistent signs. Based on 100,000 samples, 0.25% were inconsistent. (a) All values; (b) Values close to zero (note change of scale).

Figure 1

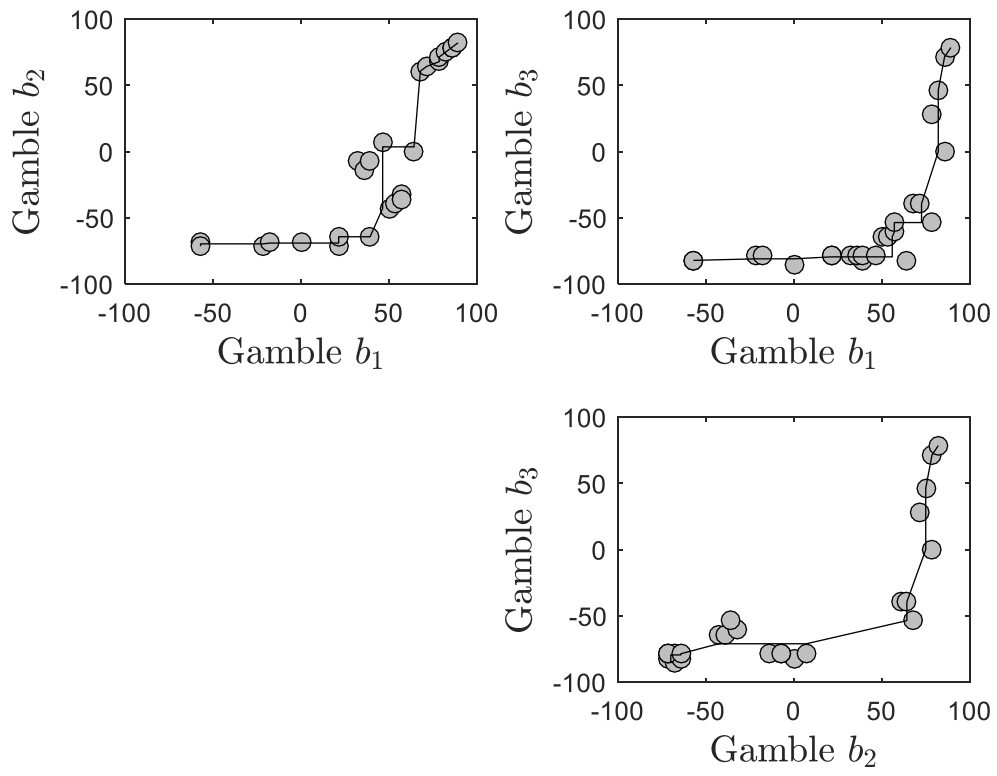


Figure 2

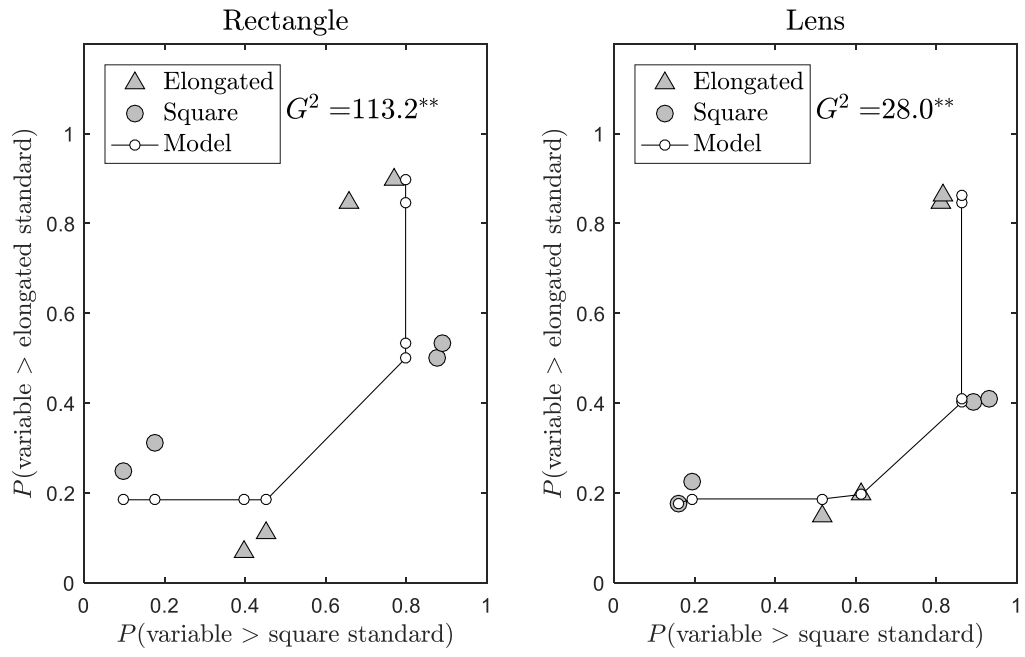


Figure 3

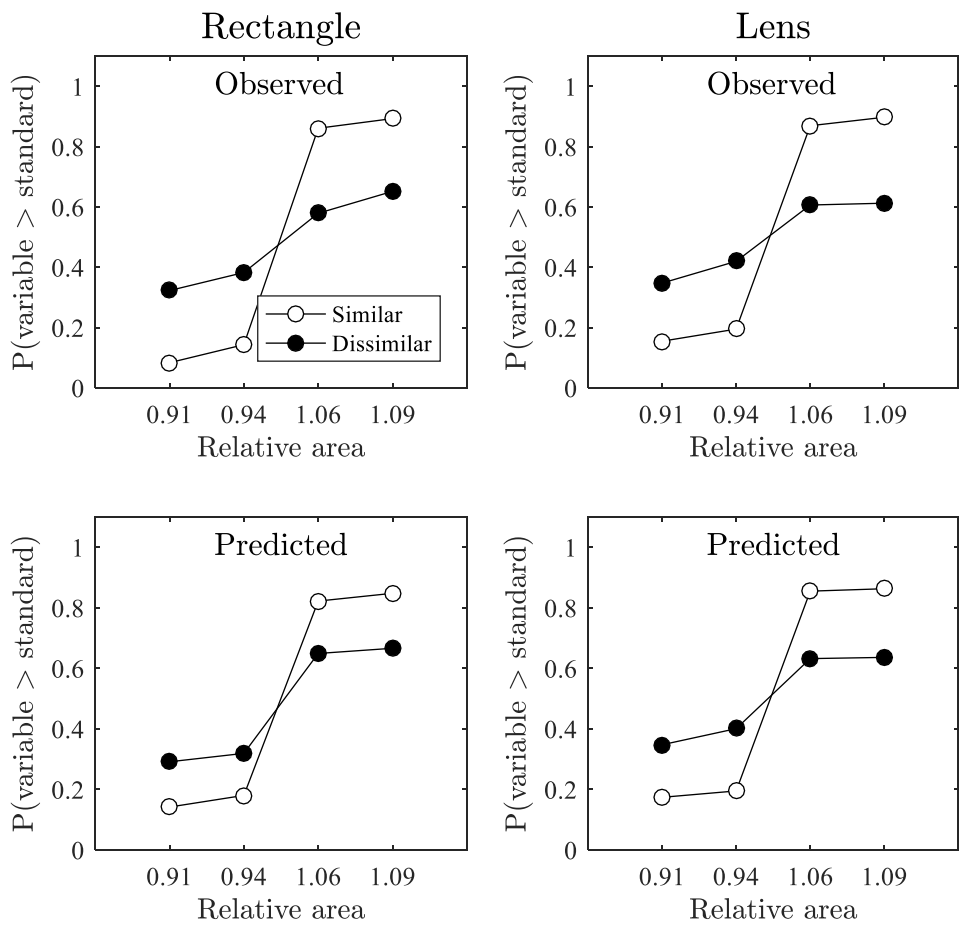


Figure 4

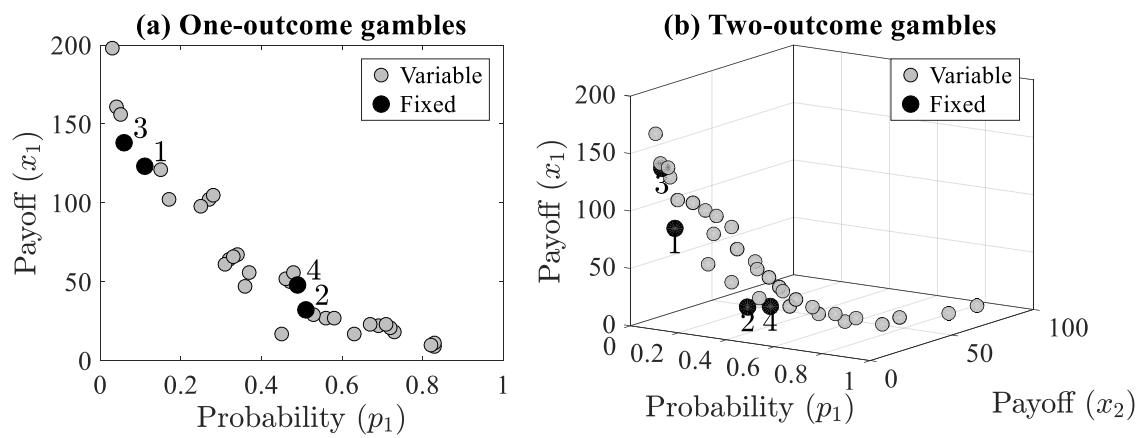


Figure 5

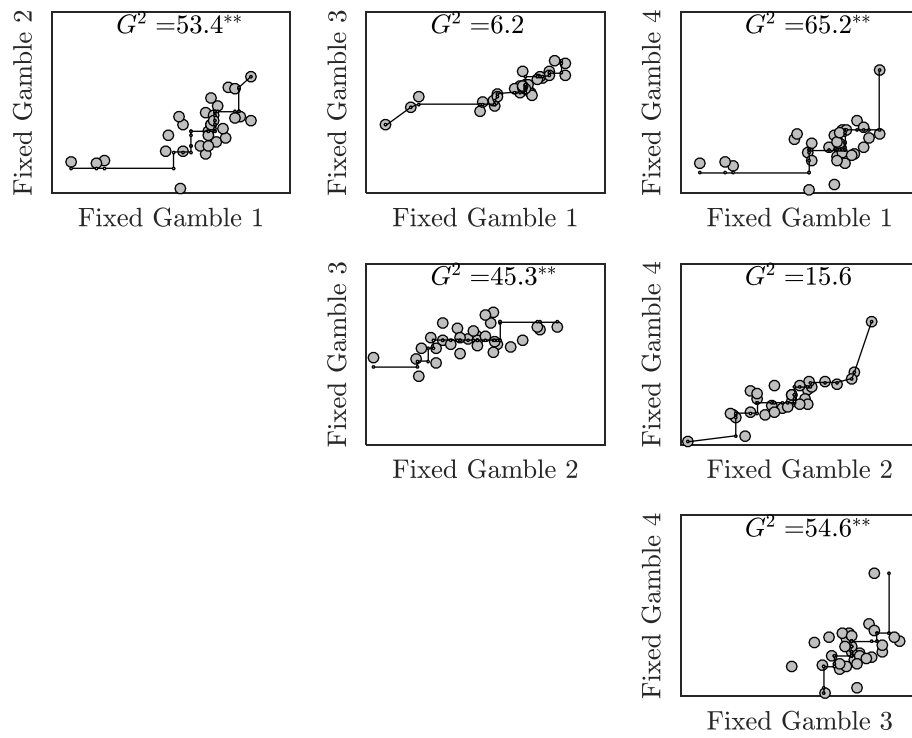


Figure 6

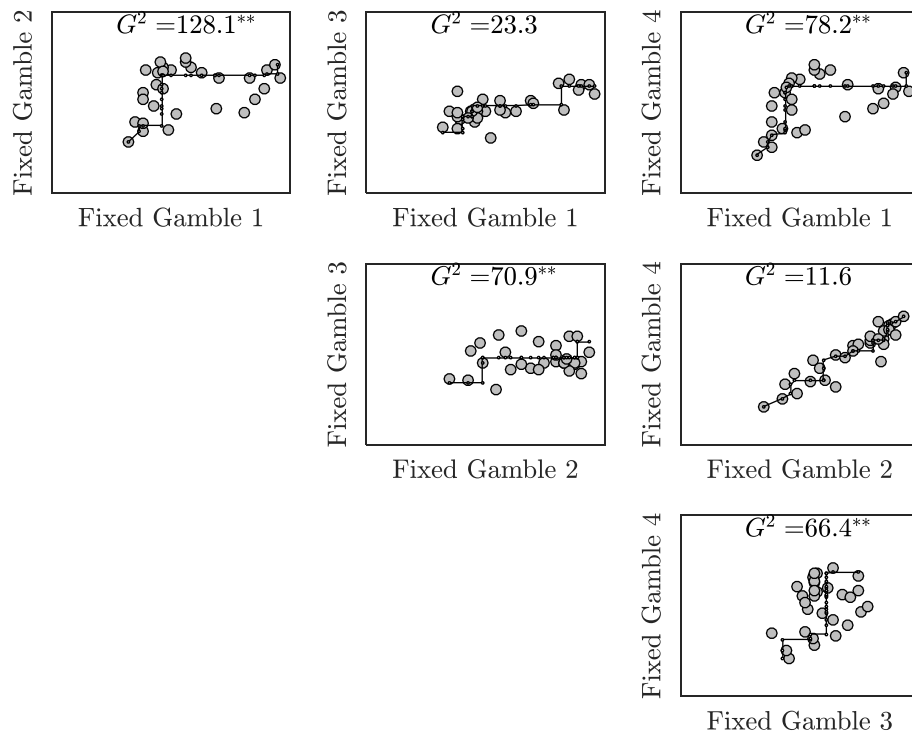


Figure 7

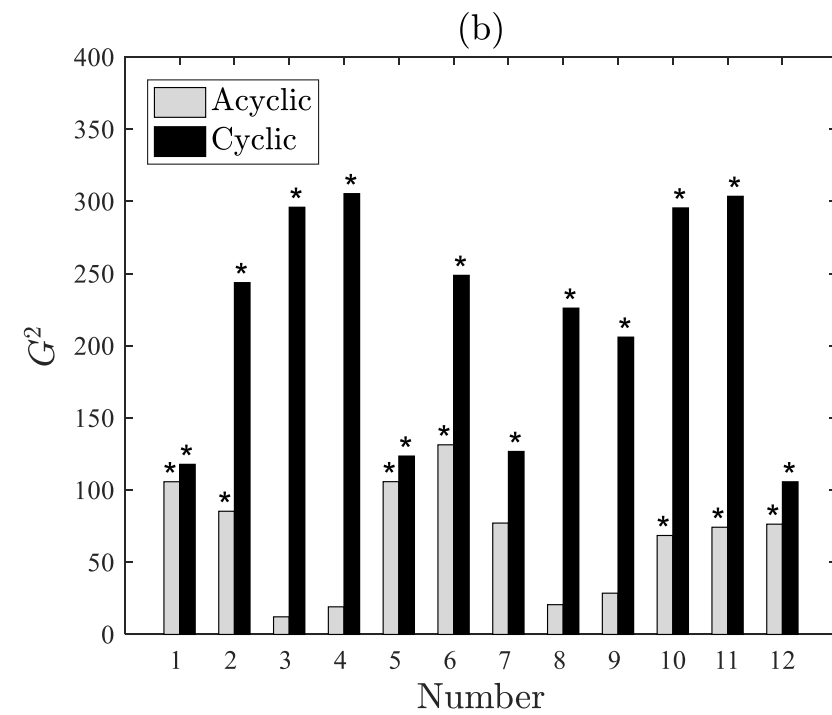
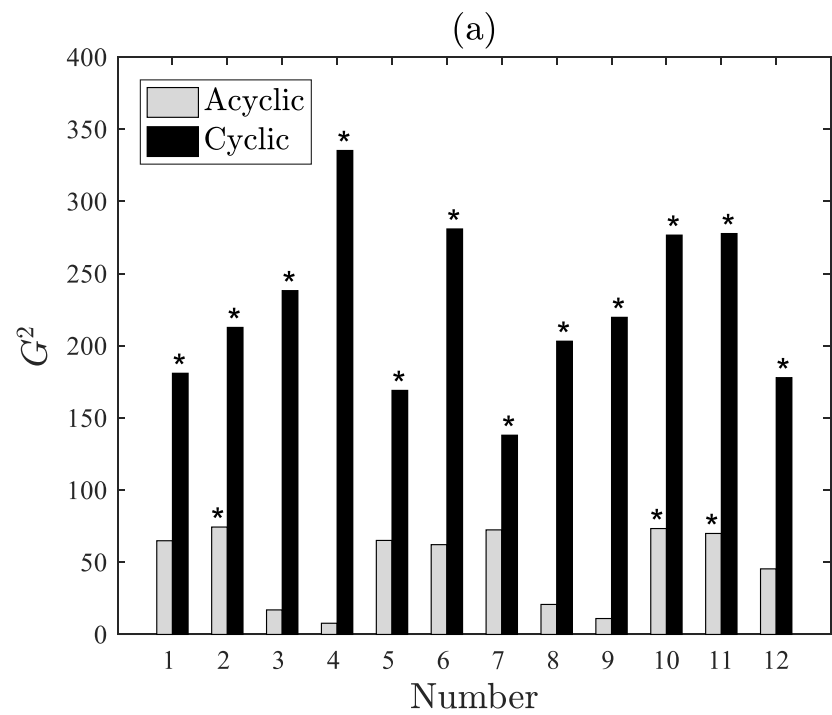


Figure 8

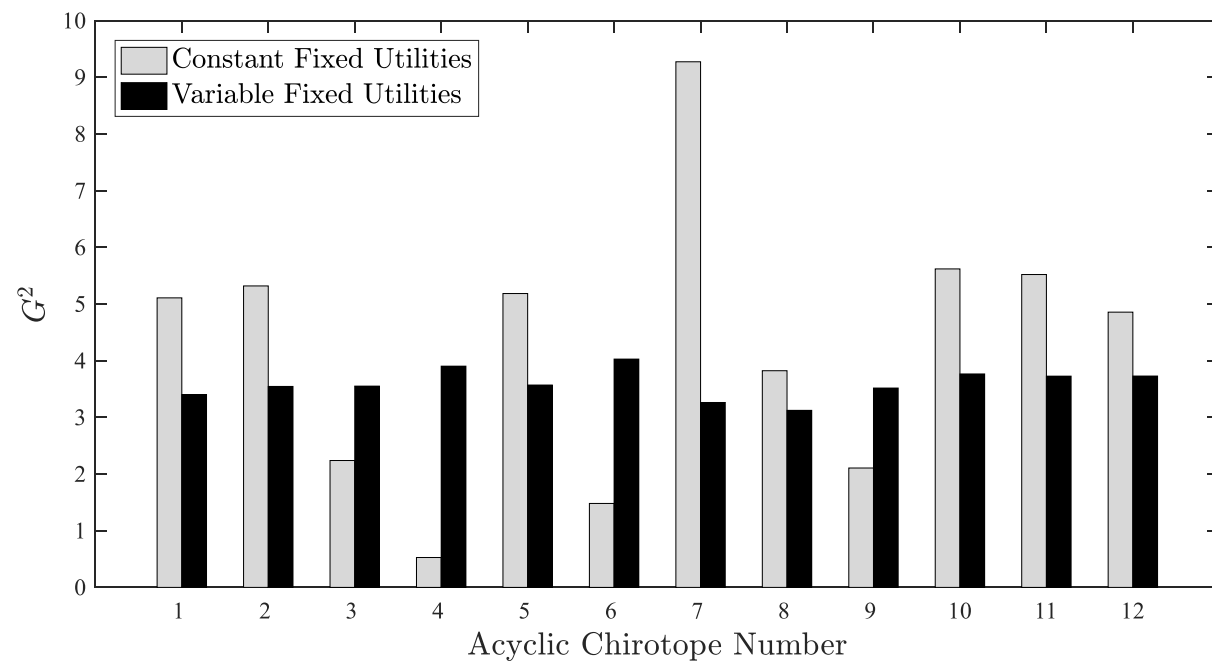
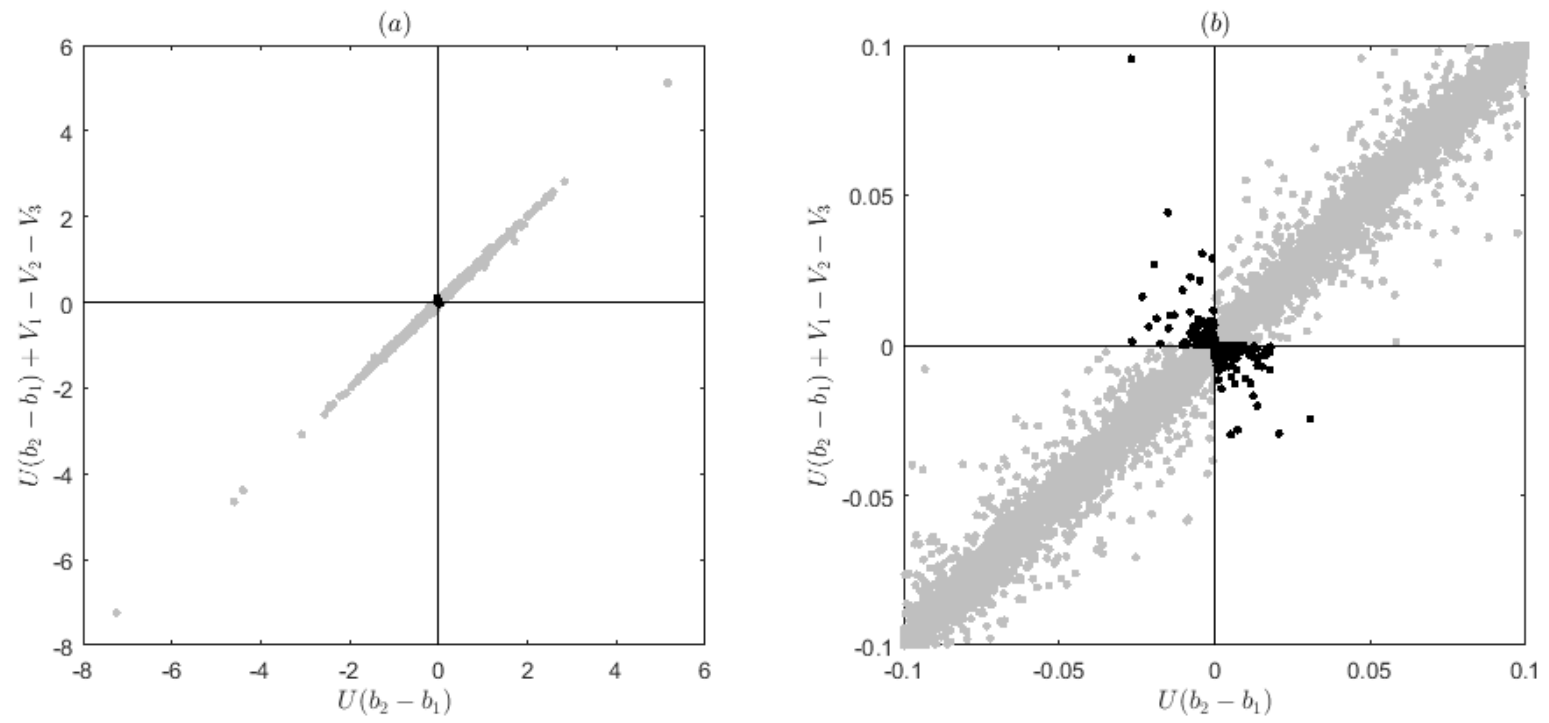


Figure A-1



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