ON CLASSIFICATION OF TYPICAL REPRESENTATIONS FOR $\mathrm{GL}_3(F)$.

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Abstract. Let F be any non-Archimedean local field with residue field of cardinality q_F. In this article, we obtain a classification of typical representations for the Bernstein components associated to the inertial classes of the form $[\mathrm{GL}_n(F) \times F^\times, \sigma \otimes \chi]$ with $q_F > 2$, and for the principal series components with $q_F > 3$. With this we complete the classification of typical representations for $\mathrm{GL}_3(F)$, for $q_F > 2$.

1. Introduction

Let F be any non-Archimedean local field with residue field k_F of cardinality q_F. Let $\mathcal{A}_n$ be the set of isomorphism classes of irreducible smooth complex representations of $\mathrm{GL}_n(F)$. The theory of Bernstein decomposition gives a natural partition of the set $\mathcal{A}_n$

$$\mathcal{A}_n = \coprod_{s \in \mathcal{B}_n} \mathcal{A}_n(s).$$

Here, the set $\mathcal{A}_n(s)$ is defined in terms of parabolic induction. The parameter s is the inertial class containing the cuspidal support of an irreducible smooth representation of $\mathrm{GL}_n(F)$ (see Section 2). In the context of the local Langlands correspondence for $\mathrm{GL}_n(F)$, the parameter s determines the isomorphism class of the restriction to the inertia subgroup I_F of the Weil–Deligne representation associated by the local Langlands correspondence.

The reciprocity map of the local class field theory gives an isomorphism between the abelianization of I_F and $\mathfrak{o}_F^\times$, the group of units of the ring of integers of F. It is natural to ask for a relation between the representations of I_F, which can be extended to a Weil–Deligne representation, and the representations of the maximal compact subgroup $\mathrm{GL}_n(\mathfrak{o}_F)$. One natural way would be to understand the cuspidal support of a smooth irreducible representation from its restriction to $\mathrm{GL}_n(\mathfrak{o}_F)$. Indeed, in several arithmetic applications (see [BM02], [EG14]) it is desired to construct irreducible smooth representations τ_s of the maximal compact subgroup $\mathrm{GL}_n(\mathfrak{o}_F)$ such that, for any irreducible smooth representation π of $\mathrm{GL}_n(F)$,

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathfrak{o}_F)}(\tau_s, \pi) \neq 0 \Rightarrow \pi \in \mathcal{A}_n(s).$$

Such a representation τ_s is called a *typical representation* for s.

The existence of typical representations, for any s, follows from the theory of types developed by Bushnell and Kutzko. For all $s \in \mathcal{B}_n$, Bushnell and Kutzko explicitly constructed pairs of the form (J_s, λ_s), where J_s is a compact open subgroup of $\mathrm{GL}_n(F)$ and λ_s is an irreducible smooth representation of J_s such that, for any irreducible smooth representation π of $\mathrm{GL}_n(F)$, we have

$$\mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0 \Leftrightarrow \pi \in \mathcal{A}_n(s).$$

We may assume that $J_s \subseteq \mathrm{GL}_n(\mathfrak{o}_F)$. It follows from Frobenius reciprocity that any irreducible sub representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathfrak{o}_F)} \lambda_s \tag{1}$$

is a typical representation for s. In general, the representation (1) is not irreducible; therefore, we cannot expect to have a uniqueness result on typical representations for a general s. Now, it is natural to ask whether there exist any other typical representations which do not occur as subrepresentations of (1). Hence, the question we are interested in is the classification of all typical representations. In this article, we achieve this

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classification for certain non-cuspidal inertial classes of $\mathrm{GL}_n(F)$–including all non-cuspidal inertial classes of $\mathrm{GL}_3(F)$–which give the classification of the typical representations for all inertial classes of $\mathrm{GL}_3(F)$, when $q_F > 2$.

Henniart (see [BM02]) classified typical representations for all inertial classes of $\mathrm{GL}_2(F)$. Later Paškūnas (see [Pas05]) classified typical representations occurring in the cuspidal representations of $\mathrm{GL}_n(F)$, for $n \geq 3$. It turns out that there exists a unique typical representation occurring in each cuspidal representation. Typical representations for depth-zero inertial classes of $\mathrm{GL}_n(F)$ are classified by the author in the article [Nad17]. We refer to the articles of Latham [Lat16], [Lat18], and [Lat17] on typical representations for cuspidal representations of $\mathrm{SL}_2(F)$ (the tame case), cuspidal representations of $\mathrm{SL}_n(F)$ (the tame case) and depth-zero cuspidal representations respectively. We also refer to the article [LN18] for some results on the typical representations for the toral cuspidal representations. The classification of the typical representations for the non-cuspidal inertial classes remains an open question even for $\mathrm{GL}_n(F)$ in the higher depth case. In this article we prove the following results.

Theorem 1.0.1. *Let $n > 2$ be an integer and $q_F > 2$. Let s be an inertial class of the form $[\mathrm{GL}_{n-1}(F) \times F^\times, \sigma \otimes \eta]$, where σ is a cuspidal representation of $\mathrm{GL}_{n-1}(F)$ and η is a character of $F^\times$. Any typical representation τ_s for s is isomorphic to $\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathfrak{o}_F)} \lambda_s$, where (J_s, λ_s) is a Bushnell–Kutzko type for s.*

Theorem 1.0.2. *Let $n > 2$ be an integer, and let $q_F > 3$ if $n \neq 3$ and let $q_F > 2$ if $n = 3$. Let s be an inertial class of the form $[T, \chi]$, where T is a maximal F-split torus contained in $\mathrm{GL}_n(F)$ and χ is a smooth character of T. Any typical representation τ_s for s is a subrepresentation of $\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathfrak{o}_F)} \lambda_s$, where (J_s, λ_s) is a Bushnell–Kutzko type for s.*

Any non-cuspidal inertial class of $\mathrm{GL}_3(F)$ is of the above form. Combined with the result of Paškūnas on the unicity of typical representation for cuspidal inertial classes, we prove the following theorem.

Theorem 1.0.3. *Let $q_F > 2$. Let s be any inertial class of $\mathrm{GL}_3(F)$. Any typical representation τ_s for s occurs as a subrepresentation of $\mathrm{ind}_{J_s}^{\mathrm{GL}_3(\mathfrak{o}_F)}(\lambda_s)$, where (J_s, λ_s) is a Bushnell–Kutzko type for s.*

In our analysis we will also obtain a certain multiplicity result on the typical representations τ_s.

We briefly explain the method of proof. Let M be a Levi subgroup of an F-parabolic subgroup P of $\mathrm{GL}_n(F)$. Let σ be a cuspidal representation of M. Let τ be the unique $M \cap \mathrm{GL}_n(\mathfrak{o}_F)$ typical representation contained in σ. The uniqueness of τ is a result of Paškūnas in the article [Pas05]. In order to classify typical representations, we begin by decomposing the representation

$$\mathrm{res}_{\mathrm{GL}_n(\mathfrak{o}_F)}\, i_P^{\mathrm{GL}_n(F)}(\sigma).$$

It follows from the results of the article [Nad17] that

$$\mathrm{res}_{\mathrm{GL}_n(\mathfrak{o}_F)}\, i_P^{\mathrm{GL}_n(F)}(\sigma) = \mathrm{ind}_{\mathrm{GL}_n(\mathfrak{o}_F)\cap P}^{\mathrm{GL}_n(\mathfrak{o}_F)} \tau \oplus \Gamma,$$

where any irreducible $\mathrm{GL}_n(\mathfrak{o}_F)$-subrepresentation of Γ is not a typical representation.

We then construct compact open subgroups of $\mathrm{GL}_n(\mathfrak{o}_F)$, denoted by H_m, for $m \geq 1$ such that

$$H_{m+1} \subset H_m, \text{ for all } m \geq 1.$$

and

$$\bigcap_{m\geq 1} H_m = P \cap \mathrm{GL}_n(\mathfrak{o}_F).$$

We will also show that τ extends as a representation of H_m, for $m \geq 1$. We will then show that any $\mathrm{GL}_n(\mathfrak{o}_F)$ subrepresentation of the representation $\mathrm{ind}_{H_{m+1}}^{\mathrm{GL}_n(\mathfrak{o}_F)} \tau / \mathrm{ind}_{H_m}^{\mathrm{GL}_n(\mathfrak{o}_F)} \tau$ is not typical. The group H_1 will be close enough to the compact subgroup J_s in a Bushnell–Kutzko type (J_s, λ_s). With some more additional work, similar to the above procedure, we complete the classification of typical representations. This requires the analysis of the induced representation $\mathrm{ind}_{H_{m+1}}^{H_m} \mathrm{id}$. We will also require some subtle aspects in the theory of Bushnell–Kutzko types. In fact, the monumental theory of Bushnell–Kutzko is the fundamental basis for this article.

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2. Preliminaries

For any ring R with unity, we denote by $\mathrm{Mat}_{n\times m}(R)$ the set of $n \times m$ matrices with entries in R. For any matrix X, we denote by X^{T} the transpose of X. The identity matrix in $\mathrm{Mat}_{n\times n}(R)$ is denoted by id_n or by 1_n.

Let F be any non-Archimedean local field with its ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$, and ϖ_F be an uniformiser of F. Let k_F be the residue field of F. We denote by q_F the cardinality of k_F. For any character χ of $F^\times$, we denote by $l(\chi)$ the level of χ, i.e., the least positive integer m such that $1+\mathfrak{p}_F{}^m$ is contained in the kernel of χ. Note that the level $l(\chi)$ of an unramified character χ is still 1. We denote by $\nu_F : F^\times \to \mathbb{Z}$ the normalised valuation of F.

All representations in this article are defined over complex vector spaces.

Let G be the F-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over F. Let $\mathcal{R}(G)$ be the category of smooth representations of G. For any closed subgroup H of G, we denote by ind_H^G the compact induction functor from $\mathcal{R}(H)$ to $\mathcal{R}(G)$. Let P be the group of F-rational points of any F-parabolic subgroup of $\mathbf{G}$. Let M be a Levi subgroup of P. We denote by i_P^G the normalised parabolic induction functor from $\mathcal{R}(M)$ to $\mathcal{R}(G)$.

Let H_1 and H_2 be two groups and τ_1 and τ_2 be any representations of H_1 and H_2 respectively. We denote by $\tau_1 \boxtimes \tau_2$ the tensor product representation of $H_1 \times H_2$. If $H_1 = H_2$, then the representation $\tau_1 \otimes \tau_2$ is the tensor product representation of H_1.

For any positive integer n, the group $\mathrm{GL}_n(F)$ is denoted by G_n and the group $\mathrm{GL}_n(\mathfrak{o}_F)$ is denoted by K_n. The principal congruence subgroup of K_n of level m is denoted by $K_n(m)$, for $m \geq 1$. Let I be a sequence of positive integers $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$. For any ring R with unity, we denote by $P_I(R)$ (resp. $\bar{P}_I(R)$) the group of invertible block upper (resp. lower) triangular matrices of the type I. Let $U_I(R)$ (resp. $\bar{U}_I(R)$) be the group of block upper (lower) unipotent matrices of the type I. Let $M_I(R)$ be the block diagonal matrices of the type I. If $I = (1, 1, \ldots, 1)$, the groups $P_I(R)$, $M_I(R)$ and $U_I(R)$ and $\bar{U}_I(R)$ are denoted by $B_n(R)$, $T_n(R)$ and $U_n(R)$ and $\bar{U}_n(R)$ respectively. When $R = F$, we drop the symbol R, i.e., $P_I(R)$ will be denoted by P_I etc. We have $P_I = M_I U_I$ and $\bar{P}_I = M_I \bar{U}_I$.

As an example, when $I = (n-1, 1)$, the group P_I in the block form is given by:

$$\begin{pmatrix} G_{n-1} & \mathrm{Mat}_{n-1\times 1}(F) \\ 0 & F^\times \end{pmatrix}.$$

In the block form the groups M_I and U_I are given by

$$M_I = \begin{pmatrix} G_{n-1} & 0 \\ 0 & F^\times \end{pmatrix} \text{ and } U_I = \begin{pmatrix} 1_{n-1} & \mathrm{Mat}_{n-1\times 1}(F) \\ 0 & 1 \end{pmatrix}.$$

We identify the group $M_{(n-1,1)}$ with the group $G_{n-1} \times G_1$. Any irreducible smooth representation of $M_{(n-1,1)}$ is identified with $\sigma \boxtimes \chi$, where σ is an irreducible smooth representation of G_{n-1} and χ is a character of G_1.

We briefly recall the theory of Bernstein decomposition. Let $B(G)$ be the set of pairs (M, σ), where M is a Levi subgroup of an F-parabolic subgroup P of G, and σ is an irreducible cuspidal representation of M. The pairs (M_1, σ_1) and (M_2, σ_2) in $B(G)$ are said to be inertially equivalent if and only if there exist an element $g \in G$ and an unramified character χ of M_2 such that

$$M_1 = g M_2 g^{-1} \text{ and } \sigma_1^g \simeq \sigma_2 \otimes \chi.$$

We denote by $\mathcal{B}_G$ the set of equivalence classes, called *inertial classes.*

Any irreducible smooth representation π of G occurs as a sub-representation of a parabolic induction $i_P^G(\sigma)$, where σ is an irreducible cuspidal representation of a Levi subgroup M of P. The pair (M, σ) is well determined up to G-conjugation. We call the class $s = [M, \sigma]$ the *inertial support* of π. We denote by $\mathcal{I}(\pi)$ the inertial support of π. For any inertial class $s = [M, \sigma]$, we denote by $\mathcal{R}_s(G)$ the full sub-category of $\mathcal{R}(G)$ consisting of smooth representations all of whose irreducible sub-quotients have inertial support s. It is shown by Bernstein in [Ber84] that the category $\mathcal{R}(G)$ decomposes as a direct product of $\mathcal{R}_s(G)$. The category $\mathcal{R}_s(G)$ is called a *Bernstein component* associated to the inertial class s. In particular, every smooth representation can be written as a direct sum of objects in the categories $\mathcal{R}_s(G)$. We denote by $\mathcal{A}_n(s)$ the set of isomorphism classes of irreducible representations in $\mathcal{R}_s(G_n)$.

Definition 2.0.1. *Let s be an inertial class for G_n. An irreducible smooth representation τ of K_n is called a typical representation for s, if for any irreducible smooth representation π of G, we have*

$$\mathrm{Hom}_{K_n}(\tau, \pi) \neq 0 \implies \mathcal{I}(\pi) = s.$$

A non typical representation is called an atypical representation.

For any inertial class s of G_n, the existence of a typical representation can be deduced from the theory of types developed by Bushnell and Kutzko in the book [BK93] and the article [BK99]. Bushnell and Kutzko constructed explicit pairs (called types) (J_s, λ_s), where J_s is a compact open subgroup of $\mathrm{GL}_n(F)$, and λ_s is an irreducible smooth representation of J_s. The pair (J_s, λ_s) satisfies the condition that, for any irreducible smooth representation π of G, we have

$$\mathrm{Hom}_{J_s}(\pi, \lambda_s) \neq 0 \ \Leftrightarrow \ \mathcal{I}(\pi) = s.$$

The group J_s can be arranged to be a subgroup of $\mathrm{GL}_n(\mathfrak{o}_F)$ by conjugating with an element of $\mathrm{GL}_n(F)$. Hence we assume that $J_s \subseteq \mathrm{GL}_n(\mathfrak{o}_F)$. It follows from Frobenius reciprocity that any irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathfrak{o}_F)}(\lambda_s) \tag{2}$$

is a typical representation. The irreducible sub representations of (2) are classified by Schneider and Zink in [SZ99, Section 6, $T_{K,\lambda}$ functor].

For $s = [G_n, \sigma]$, Paškūnas in the article [Pas05, Theorem 8.1] showed that up to isomorphism there exists a unique typical representation for s. More precisely,

Theorem 2.0.2 (Paškūnas)**.** *Let $n \geq 1$ be an integer and σ be an irreducible cuspidal representation of G_n. Let (J_s, λ_s) be a Bushnell-Kutzko type for the inertial class $s = [G_n, \sigma]$ with $J_s \subseteq K_n$. The representation*

$$\mathrm{ind}_{J_s}^{K_n}(\lambda_s)$$

is the unique typical representation for the inertial class $[G_n, \sigma]$. The representation $\mathrm{ind}_{J_s}^{K_n}(\lambda_s)$ occurs with a multiplicity one in σ.

In this article, we classify typical representations for $\mathrm{GL}_3(F)$ in terms of Bushnell–Kutzko types. We first obtain a classification of typical representations for the inertial classes $[M_{(n-1,1)}, \sigma \boxtimes \eta]$ and $[T_n, \chi]$, where η and χ are characters of $F^\times$ and T_n respectively. We will use some basic results from the article [Nad17] and we recall some of these results.

Lemma 2.0.3. *Let χ be a character of G_n and let τ be a typical representation for an inertial class $s = [M, \sigma]$ of G_n. The representation $\tau \otimes \chi$ is a typical representation for the inertial class $[M, \sigma \otimes \chi]$.*

Proof. We refer to [Nad17, Lemma 2.7] for a proof. □

Let P be any parabolic subgroup of G_n with a Levi subgroup M and U be the unipotent radical of P. Let $\bar{U}$ be the unipotent radical of the opposite parabolic subgroup of P with respect to M. Let J_1 and J_2 be two compact open subgroups of K_n such that J_1 contains J_2. Suppose J_1 and J_2 both satisfy the Iwahori decomposition with respect to P and M. With $J_1 \cap U = J_2 \cap U$ and $J_1 \cap \bar{U} = J_2 \cap \bar{U}$. Let λ be an irreducible smooth representation of J_2 which admits an Iwahori decomposition i.e. $J_2 \cap U$ and $J_2 \cap \bar{U}$ are contained in the kernel of λ.

Lemma 2.0.4. *The representation $\mathrm{ind}_{J_2}^{J_1}(\lambda)$ is the extension of the representation $\mathrm{ind}_{J_2 \cap M}^{J_1 \cap M}(\lambda)$ such that $J_1 \cap U$ and $J_1 \cap \bar{U}$ are contained in the kernel of the extension.*

Proof. The lemma is well known and frequently used when dealing with the formalism of G-covers. We refer to [Nad17, Lemma 2.6] for a proof. □

Let $t_i = [M_i, \Theta_i]$ be an inertial class of G_{n_i}, for $1 \leq i \leq r$. Let σ_i be a smooth representation from $\mathcal{R}_{t_i}(G_{n_i})$. We suppose

$$\mathrm{res}_{K_{n_i}} \sigma_i = \tau_i^0 \oplus \tau_i^1,$$

for $1 \leq i \leq r$, such that irreducible K_{n_i}-subrepresentations of τ_i^1 are atypical. We denote by t the inertial class

$$[M_1 \times M_2 \times \cdots \times M_r, \Theta_1 \boxtimes \Theta_2 \boxtimes \cdots \boxtimes \Theta_r]$$

of G_n. The inertial class t is independent of the choice of representatives (M_i, Θ_i). Let $\tau_I^0 = \boxtimes_{i=1}^r \tau_i^0$ and $\sigma_I = \boxtimes_{i=1}^r(\sigma_i)$.

Lemma 2.0.5. *The representation* $\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau_I^0)$ *admits a complement in* $\mathrm{res}_{K_n} i_{P_I}^{G_n}(\sigma_I)$ *with all its irreducible sub-representations atypical.*

Proof. We refer to [Nad17, Proposition 2.3] for a proof. □

In particular, if $\Theta_i = \sigma_i$ is a cuspidal representation, then from Theorem 2.0.2 we have $\mathrm{res}_{K_{n_i}} \sigma_i = \tau_i^0 \oplus \tau_i^1$, where τ_i^0 is the unique typical representation for the inertial class $[G_{n_i}, \sigma_i]$. Hence, any typical representation for the inertial class t occurs as a sub-representation of $\mathrm{ind}_{P_I \cap K_n}^{K_n} \sigma_I$.

Lemma 2.0.6. *Let $s = [M, \sigma]$ be any inertial class of G_n. Then there exists a partition I of n and a cuspidal representation σ_I of M_I such that $s = [M_I, \sigma_I]$.*

Proof. We refer to [Nad17, Section 2.2, page no. 5] for a proof. □

The following result is useful in understanding some stabilisers in the later part of this article. The space $\mathrm{Mat}_{n\times m}(k_F)$ is equipped with an action of $M_{(m,n)}(k_F) = \mathrm{GL}_m(k_F) \times \mathrm{GL}_n(k_F)$ given by $(g_1, g_2)U = g_2 U g_1^{-1}$, for $U \in \mathrm{Mat}_{n\times m}(k_F)$. We also have a $M_{(m,n)}(k_F)$ action on the set of matrices $\mathrm{Mat}_{m\times n}(k_F)$ by setting $(g_1, g_2)V = g_1 V g_2^{-1}$, for $V \in \mathrm{Mat}_{n\times m}(k_F)$. Let ψ be a non-trivial character of the additive group k_F. We define a pairing B between $\mathrm{Mat}_{m\times n}(k_F)$ and $\mathrm{Mat}_{n\times m}(k_F)$ by defining $B(V, U) = \psi \circ \mathrm{tr}(VU)$. Let T be the map from $\mathrm{Mat}_{m\times n}(k_F)$ and $\mathrm{Mat}_{n\times m}(k_F)^\wedge$ defined by

$$T(V)(U) = B(V, U).$$

Lemma 2.0.7. *The map T is an $M_{(m,n)}(k_F)$-equivariant isomorphism.*

Let s be any depth-zero inertial class $[M_I, \sigma_I]$ of G_n. The group $K_n \cap M_I$ acts on the space

$$\sigma_I^{K_n(1)\cap M_I},$$

and we denote this representation of $K_n \cap M_I$ by τ_I. The pair $(K_n \cap M_I, \tau_I)$ is a Bushnell–Kutzko type for the inertial class $[M_I, \sigma_I]$ of M_I. Let $P_I(1)$ be the group $K_n(1)(P_I \cap K_n)$ and observe that $P_I(1) \cap M_I$ is equal to $K_n \cap M_I$. The representation τ_I extends as a representation of $P_I(1)$ such that $P_I(1) \cap U_I$ and $P_I(1) \cap \bar{U}_I$ are contained in the kernel of this extension.

Theorem 2.0.8. *Let $s = [M_I, \sigma_I]$ be any depth-zero inertial class of G_n. Any typical representation τ_s for s occurs as a subrepresentation of* $\mathrm{ind}_{P_I(1)}^{K_n} \tau_I$. *Moreover, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_n}(\tau_s, i_{P_I}^{G_n} \sigma_I) = \dim_{\mathbb{C}} \mathrm{Hom}_{K_n}(\tau_s, \mathrm{ind}_{P_I(1)}^{K_n} \tau_I).$$

Proof. We refer to [Nad17, Theorem 3.2] for the proof. □

3. The inertial class with Levi subgroup of type $(n-1,1)$

Let $n>1$ be any positive integer. In this section we assume that $I=(n-1,1)$. Let V and V_1 be two F-vector spaces of dimensions $n-1$ and 1 respectively. Let P be the parabolic subgroup of $GL(V\oplus V_1)$ fixing the flag $V\subseteq V\oplus V_1$. We denote by M its Levi subgroup fixing the decomposition $V\oplus V_1$. Hence, we have $M=GL(V)\times GL(V_1)$. In this section, we are interested in the classification of typical representations for inertial classes $[M,\sigma\boxtimes\chi]$, where σ is a cuspidal representation of $GL(V)$, and χ is a character of $GL(V_1)$. We will use the language of the book [BK93] freely in this section. Let $(J(\mathfrak{A},\beta),\lambda)$ be a maximal simple (Bushnell–Kutzko) type contained in the representation σ. We recall certain important features of this type for our purpose.

3.1. **Bushnell-Kutzko semi-simple type.** We denote by A the algebra $\mathrm{End}_F(V)$. Let $[\mathfrak{A},l,0,\beta]$ be a simple stratum in A defining the maximal simple type $(J(\mathfrak{A},\beta),\lambda)$. We denote by B the commutant of $E=F[\beta]$ in A. Let $\mathfrak{B}=\mathfrak{A}\cap B$. Let $\mathfrak{P}$ and $\mathfrak{D}$ be the radicals of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Given any hereditary order $\mathfrak{A}$, we define the filtration $U^i(\mathfrak{A})$ by setting

$$U^i(\mathfrak{A})=\mathrm{id}+\mathfrak{P}^i,$$

for all $i\geq 1$, and $U^0(\mathfrak{A})$ is the set of units of $\mathfrak{A}$. The type $(J(\mathfrak{A},\beta),\lambda)$ is called maximal if $\mathfrak{B}$ is a maximal hereditary order in B.

The group $J(\mathfrak{A},\beta)$ contains $U^0(\mathfrak{B})$. There is a normal subgroup $J^1(\mathfrak{A},\beta)$ of $J(\mathfrak{A},\beta)$ such that $J^1(\mathfrak{A},\beta)\cap U^0(\mathfrak{B})=U^1(\mathfrak{B})$ and

$$\frac{U^0(\mathfrak{B})}{U^1(\mathfrak{B})}\simeq\frac{J(\mathfrak{A},\beta)}{J^1(\mathfrak{A},\beta)}.$$

The group $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ is a general linear group of a vector space over a finite field. The representation λ is an irreducible representation which is given by a tensor product representation $\kappa\otimes\rho$, where κ is a representation of $J(\mathfrak{A},\beta)$, called a β-extension (see [BK93, Chapter 5, Definition 5.2.1]), and ρ is a cuspidal representation of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ (considered as a representation of $J(\mathfrak{A},\beta)$ through its quotient $J(\mathfrak{A},\beta)/J^1(\mathfrak{A},\beta)$). We refer to [BK93, Chapter 5] for complete details of these constructions. For the precise definition and description see [BK93, chapter 5, Definition 5.5.10]. For simplicity, we will denote by J^0 and J^1 the groups $J(\mathfrak{A},\beta)$ and $J^1(\mathfrak{A},\beta)$ respectively.

We fix the following notations. Let e and f be the ramification index and inertial index of E respectively. We fix an $\mathfrak{o}_E$-lattice chain $\mathcal{L}$ defining the hereditary $\mathfrak{o}_E$-order $\mathfrak{B}$. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order defined by the lattice chain $\mathcal{L}$, considering $\mathcal{L}$ as an $\mathfrak{o}_F$-lattice chain. We fix a $\mathfrak{o}_E$-basis $(w_1,w_2,\dots w_{(n-1)/ef})$ for the lattice chain $\mathcal{L}$ (see [BK93, Chapter 1, 1.1.7]) and then a $\mathfrak{o}_F$-basis for $\mathfrak{o}_E w_i$ for $1\leq i\leq ef$; hence, we obtain an F-basis $(v_1,v_2,\dots,v_{n-1},v_n)$ for the vector space $V\oplus V_1$, where $v_n\in V_1$. In this basis, we write all our endomorphisms as matrices of $\mathrm{Mat}_{n\times n}(F)$. With this basis we have $J^0\subseteq K_{n-1}$.

We are interested in the classification of typical representations for the inertial class $[M_I,\sigma\boxtimes\chi]$. By twisting with a character if necessary we may (and do) assume that $\chi=\mathrm{id}$ (see Lemma 2.0.3). Let τ be any typical representation for the above inertial class. The representation $\mathrm{ind}_{K_n}^{G_n}\tau$ is a finitely generated representation of G_n and hence admits an irreducible quotient π. Using the definition of a typical representation we see that π occurs as a sub quotient of $i_{P_I}^{G_n}(\sigma\chi_1\boxtimes\chi_2)$, where χ_1 and χ_2 are unramified characters of G_{n-1} and G_1 respectively.

Hence, in order to classify typical representations for the inertial class $s=[M_I,\sigma\boxtimes\mathrm{id}]$, it is enough to examine which K_n-irreducible sub representations of

$$\mathrm{res}_{K_n}i_{P_I}^{G_n}(\sigma\boxtimes\mathrm{id})$$

are typical for the inertial class s. Let τ be the unique typical representation contained in the representation σ. The representation

$$\mathrm{ind}_{P_I\cap K_n}^{K_n}(\tau\boxtimes\mathrm{id})$$

has a complement in

$$\mathrm{res}_{K_n}i_{P_I}^{G_n}(\sigma\boxtimes\chi)$$

whose irreducible sub representations are atypical (see Lemma 2.0.5).

Now we have to look for typical representations occurring in the representation

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau \boxtimes \mathrm{id}).$$

For this purpose, we will define compact subgroups $H_m \subseteq K_n$, for $m \geq N_0$ (for some positive integer N_0), such that

Hypothesis 3.1.

(1) *$H_{m+1} \subseteq H_m$, for $m \geq N_0$ and $\bigcap_{m \geq N_0} H_m = P_I(\mathfrak{o}_F)$,*
(2) *the group H_m has the Iwahori decomposition with respect to P_I and its Levi subgroup M_I,*
(3) *the representation $\tau \boxtimes \mathrm{id}$ admits an extension to H_{N_0} such that $H_{N_0} \cap \bar{U}_I$ and $H_{N_0} \cap U_I$ are contained in the kernel of this extension.*

For any such sequence $\{H_m, m \geq N_0\}$ as above we have:

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_0} \mathrm{ind}_{H_m}^{K_n}(\tau \boxtimes \mathrm{id}).$$

Before we start this construction it is instructive to first examine the Bushnell-Kutzko semi-simple type for the inertial class $[M_I, \sigma \boxtimes \mathrm{id}]$.

Let us recall some standard material required from [BK99]. First, let us begin with lattice sequences. An $\mathfrak{o}_F$-lattice sequence in a F-vector space, say V, is a function Λ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in V with the following conditions on Λ:

$$\Lambda(n+1) \subseteq \Lambda(n), \text{ for all } n \in \mathbb{Z};$$

and there exists an $e(\Lambda) \in \mathbb{Z}$ such that

$$\Lambda(n + e(\Lambda)) = \mathfrak{p}_F \, \Lambda(n), \text{ for all } n \in \mathbb{Z}.$$

An $\mathfrak{o}_F$-lattice chain is an $\mathfrak{o}_F$-lattice sequence with the strict inclusion between $\Lambda(n+1)$ and $\Lambda(n)$, for all $n \in \mathbb{Z}$. One extends the function Λ to the set of real numbers by setting

$$\Lambda(r) = \Lambda(-[-r]),$$

for all $r \in \mathbb{R}$. Here, $[x]$ is the greatest integer less than or equal to x.

Given two $\mathfrak{o}_F$-lattice sequences Λ_1 and Λ_2 in the vector spaces V_1 and V_2 over F, Bushnell and Kutzko defined the notion of direct sum of Λ_1 and Λ_2. Let $e = \mathrm{lcm}(e(\Lambda_1), e(\Lambda_2))$. The direct sum of Λ_1 and Λ_2, denoted by Λ, is an $\mathfrak{o}_F$-lattice sequence in the vector space $V_1 \oplus V_2$ given by

$$\Lambda(er) = \Lambda_1(e_1 r) \oplus \Lambda(e_2 r),$$

for any $r \in \mathbb{R}$. Given an $\mathfrak{o}_F$-lattice sequence Λ in a vector space V one can define a filtration $\{a_r(\Lambda) \mid r \in \mathbb{R}\}$ on the algebra $\mathrm{End}_F(V)$ given by the equation

$$a_r(\Lambda) = \{x \in \mathrm{End}_F(V) | \; x\Lambda(i) \subseteq \Lambda(i+r) \; \forall \; i \in \mathbb{Z}\}.$$

Let $u_0(\Lambda)$ be the group of units in the order $a_0(\Lambda)$ and, for $r > 0$ and $r \in \mathbb{Z}$, we set $u_r(\Lambda)$ to be $1 + a_r(\Lambda)$.

Let (J_s, λ_s) be a Bushnell-Kutzko type for the inertial class

$$s = [M_I, \sigma \boxtimes \mathrm{id}].$$

The group J_s satisfies the Iwahori decomposition with respect to the parabolic subgroup P_I and the Levi subgroup M_I. Let $[\mathfrak{A}, l, 0, \beta]$ be a simple stratum defining the type (J^0, λ) for the inertial class $[G_{n-1}, \sigma]$. The order $\mathfrak{A}$ is defined by a lattice chain Λ_1 with values in sub-lattices of $\mathfrak{o}_F^n$. We denote by Λ_2 the lattice chain defined by $\Lambda_2(i) = \mathfrak{p}_F^i$, for all $i \in \mathbb{Z}$. We have

(1) $J_s \cap U_I = u_0(\Lambda_1 \oplus \Lambda_2) \cap U_I$,
(2) $J_s \cap M_I = J^0 \times \mathfrak{o}_F^\times$,
(3) $J_s \cap \bar{U}_I = u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \bar{U}_I$,
(4) the restriction of λ_s to $J_s \cap M_I$ is isomorphic to $\lambda \boxtimes \mathrm{id}$, and the groups $J_s \cap \bar{U}_I$ and $J_s \cap U_I$ are contained in the kernel of λ_s.

We refer to [BK99, Section 8, paragraph 8.3.1] for the construction of the above Bushnell–Kutzko type.

Now we make an explicit calculation of the groups $u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \bar{U}_I$ and $u_0(\Lambda_1 \oplus \Lambda_2) \cap U_I$. Note that the period of the direct sum $\Lambda_1 \oplus \Lambda_2$ is the least common multiple of the period of the two lattice sequences Λ_1 and Λ_2. Hence, the period of the lattice sequence Λ is e, where e is the period of the lattice chain Λ_1. Let t be an integer such that $0 \leq t \leq e-1$. Let L_0 be the free $\mathfrak{o}_F$ module $\mathfrak{o}_F^{(n-1)/e}$. The lattice $\Lambda_1(t)$ is given by :

$$\Lambda_1(t) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\varpi_F L_0 \oplus \varpi_F L_0 \oplus \cdots \oplus \varpi_F L_0),$$

where the L_0 is repeated $e-t$ times, and $\varpi_F L_0$ is repeated t times. Hence, the lattice chain Λ is given by

$$\Lambda(0) = \Lambda_1(0) \oplus \Lambda_2(0) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus \mathfrak{o}_F$$

and

$$\Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t/e) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\varpi_F L_0 \oplus \varpi_F L_0 \oplus \cdots \oplus \varpi_F L_0) \oplus \mathfrak{p}_F,$$

for $0 \leq t \leq e-1$.

We note that $u_0(\Lambda) \cap U_I = U_I(\mathfrak{o}_F)$. We denote by $\bar{\mathfrak{n}}_I$ the lower nilpotent matrices of the type $(n-1,1)$, i.e. the Lie algebra of $\bar{U}_I$. We then have:

$$u_{l+1}(\Lambda) \cap \bar{U}_I = \mathrm{id} + (a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I).$$

Let $l+1 = el' + r$, where $0 \leq r < e$. Since Λ is a lattice chain of period e, we deduce that

$$a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}}_I).$$

Finally, it remains to calculate the group $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$. We note that $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is the following set

$$\{x \in \mathrm{Mat}_{n \times n}(F) \cap \bar{\mathfrak{n}}_I \mid x\Lambda(i) \subseteq \Lambda(i+r) \ \forall \ i \in \mathbb{Z}\}.$$

For $r \geq 1$, the n^{th} row (in block form) of an element in $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is of the form $A = [M_1, M_2, \ldots, M_e, 0]$, where M_i is a matrix of type $1 \times (n-1)/e$, for $1 \leq i \leq e$ and:

(1) $M_i \in \varpi_F^2 \mathrm{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$, for $i \leq r-1$,
(2) $M_i \in \varpi_F \mathrm{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$, for $i > r-1$.

If $r = 0$ and $e > 1$, then we know that $M_i \in \varpi_F \mathrm{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$, for $1 \leq i \leq e-1$, and $M_e \in \mathrm{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$. If $r = 0$ and $e = 1$, then we have $A \in \mathrm{Mat}_{1 \times n}(\mathfrak{o}_F)$. This description is enough for the present purposes.

3.2. **Some auxiliary groups.** Let m be a positive integer and $P_I(m)$ be the inverse image of the group $P_I(\mathfrak{o}_F / \mathfrak{p}_F{}^m)$ under the mod-$\mathfrak{p}_F{}^m$ reduction map

$$K_n \to \mathrm{GL}_n(\mathfrak{o}_F/\mathfrak{p}_F^m).$$

There exists a positive integer N_1 such that the principal congruence subgroup of level N_1 is contained in the kernel of the representation τ. The representation $\tau \boxtimes \mathrm{id}$ of $M_I(\mathfrak{o}_F / \mathfrak{p}_F{}^{N_1})$ now extends to a representation of $P_I(N_1)$ by inflation. We note that $P_I(N_1) \cap \bar{U}_I$ and $P_I(N_1) \cap U_I$ are both contained in the kernel of this extension. Now the sequence of groups $H_m = P_I(m)$ and the representation $\tau \boxtimes \mathrm{id}$, for $m \geq N_1$ satisfy the conditions in Hypothesis 3.1. Hence, we get that

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_1} \mathrm{ind}_{P_I(m)}^{K_n}(\tau \boxtimes \mathrm{id}).$$

We conclude that typical representations occur as sub representations of

$$\mathrm{ind}_{P_I(m)}^{K_n}(\tau \boxtimes \mathrm{id}),$$

for some positive integer $m \geq N_1$.

For making Mackey decompositions easier and other reasons, it is convenient to work with a smaller subgroup $P_I^0(m)$ of $P_I(m)$. We begin by rewriting the representation

$$\mathrm{ind}_{P_I(m)}^{K_n}(\tau \boxtimes \mathrm{id}).$$

We also require to make N_1 explicit. We recall that $K_{n-1}(m)$ is the principal congruence subgroup of level m of G_{n-1}. The group J^0 contains the group $U^{[l/2]+1}(\mathfrak{A})$ and the representation λ restricted to the group $U^{[l/2]+1}(\mathfrak{A})$ is a direct sum of copies of the same character ψ_β. The character ψ_β is trivial on the group $U^{l+1}(\mathfrak{A})$. We also recall the notation that $l+1 = el' + r$, where $0 \leq r \leq e-1$. We note that $U^{l+1}(\mathfrak{A}) = \mathrm{id}_{n-1} + \varpi_F^{l'}\mathfrak{P}_{\mathfrak{A}}^r$. If $r=0$, then $K_{n-1}(1) \subseteq \mathfrak{P}_{\mathfrak{A}}^r$. If $r>1$, then from the formulas [BK93, 2.5.2] we get that $K_{n-1}(2) \subseteq \mathfrak{P}_{\mathfrak{A}}^r$, for $0 \leq r < e$. This shows that the representation λ is trivial on $K_{n-1}(N_s)$, where N_s is given by:

Notation 3.2.1. *From now we fix $N_s = [(l+1)/e]+1$ if $r=0$ and $e>1$. If $r=0$ and $e=1$, then $N_s = l+1$. Finally, $N_s = [(l+1)/e]+2$ if $r \geq 1$.*

Let π be the projection map
$$P_I(\mathfrak{o}_F) \to M_I(\mathfrak{o}_F).$$
For $m \geq N_s$, we denote by $P_I^0(m)$ the group $K_n(m)\pi^{-1}(J^0 \times \mathfrak{o}_F^\times)$. Since $K_n(m) \cap P_I \subseteq \pi^{-1}(J^0 \times \mathfrak{o}_F^\times)$, the group $P_I^0(m)$ satisfies the Iwahori decomposition with respect to the subgroup P_I and the Levi subgroup M_I. In particular, we have
$$P_I^0(m) = (P_I^0(m) \cap U_I)(P_I^0(m) \cap M_I)(P_I^0(m) \cap \bar{U}_I).$$
Here, $P_I^0(m) \cap U_I$ is equal to $U_I(\mathfrak{o}_F)$, $P_I^0(m) \cap M_I$ is equal to $J^0 \times \mathfrak{o}_F^\times$, and $(P_I^0(m) \cap \bar{U}_I)$ is equal to $K_n(m) \cap \bar{U}_I$.

We observe that $\lambda \boxtimes \mathrm{id}$ extends as a representation of $P^0(m)$, for all $m \geq N_s$; the groups $P^0(m) \cap U_I$ and $P^0(m) \cap \bar{U}_I$ are contained in the kernel of this extension. Now the representation $\tau \boxtimes \mathrm{id}$ of $K_{n-1} \times \mathfrak{o}_F^\times$ is isomorphic to
$$\{\mathrm{ind}_{J^0}^{K_{n-1}}(\lambda)\} \boxtimes \mathrm{id}.$$
Hence, we get that
$$\mathrm{ind}_{P_I(m)}^{K_n}(\tau \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{K_n}(\lambda \boxtimes \mathrm{id}),$$
for all $m \geq N_s$ (we apply Lemma 2.0.4 to the groups $J_1 = P_I(m)$ and $J_2 = P_I^0(m)$ and $\lambda = \lambda \boxtimes \mathrm{id}$). We get that
$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau \boxtimes \mathrm{id}) \simeq \bigcup_{m \geq N_s} \mathrm{ind}_{P_I^0(m)}^{K_n}(\lambda \boxtimes \mathrm{id}).$$
Hence, any typical representation occurs as a sub-representation of
$$\mathrm{ind}_{P_I^0(m)}^{K_n}(\lambda \boxtimes \mathrm{id}),$$
for some $m \geq N_s$.

We first have to understand the representation
$$\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}),$$
for $m \geq N_s$. It is convenient to define a normal subgroup $R_I(m)$ of $P_I^0(m)$ such that $P_I^0(m)$ is equal to $R_I(m)P_I^0(m+1)$, and $R_I(m) \cap P_I^0(m+1) = R_I(m+1)$, for $m \geq N_s$. For any integer $m \geq N_s$, we define $R_I(m)$ to be the group $K_n(m)\pi^{-1}(K_{n-1}(N_s) \times (1+\mathfrak{P}_F^{N_s}))$. The group $R_I(m)$ has the Iwahori decomposition with respect to the parabolic subgroup P_I and its Levi subgroup M_I.

Lemma 3.2.1. *The group $R_I(m)$ is a normal subgroup of $P_I^0(m)$. The group $R_I(m+1)$ is a normal subgroup of $R_I(m)$, for all $m \geq N_s$.*

Proof. By definition of the groups $R_I(m)$, we have $R_I(m) \cap U_I = P_I^0(m) \cap U_I$, and $R_I(m) \cap \bar{U}_I = P_I^0(m) \cap \bar{U}_I$. To show the normality of $R_I(m)$ in $P_I^0(m)$, we have to verify that $P_I^0(m) \cap M_I$ normalize the group $R_I(m)$. But, $P_I^0(m) \cap M_I$ normalizes the group $R_I(m) \cap U_I = U_I(\mathfrak{o}_F)$ and $R_I(m) \cap \bar{U}_I = \bar{U}_I(\varpi_F^m \mathfrak{o}_F)$. The group $K_n(m) \cap M_I$ is a normal subgroup of $M_I(\mathfrak{o}_F)$. Hence, $P_I^0(m) \cap M_I$ normalizes $R_I(m) \cap M_I$. This shows the first part of the lemma.

Since $R_I(m) \cap P_I = R_I(m+1) \cap P_I$, we have to check that $R_I(m) \cap \bar{U}_I$ normalizes the group $R_I(m+1)$. We note that $\bar{U}_I$ is abelian. Hence, we have to check that the conjugations $u^- j (u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to the group $R_I(m+1)$, for all $u^- \in R_I(m) \cap \bar{U}_I$, $j \in R_I(m+1) \cap M_I = R_I(m) \cap M_I$, and $u^+ \in R_I(m+1) \cap U_I = U_I(\mathfrak{o}_F)$. Let us begin with the element $u^- j (u^-)^{-1}$. We have $u^- j (u^-)^{-1} = j\{j^{-1} u^- j (u^-)^{-1}\}$. Let
$$j = \begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \quad u^- = \begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}$$

be the block diagonal form of j and u^-; $J_1 \in K_{n-1}(N_s), j_1 \in 1+\mathfrak{p}_F{}^{N_s}$ and $U^- \in \varpi_F^m \mathrm{Mat}_{1\times(n-1)}(\mathfrak{o}_F)$. The element $j^{-1}u^-j(u^-)^{-1}$ is of the form

$$\begin{pmatrix} 1_{n-1} & 0 \\ j_1^{-1}U^-J_1 - U^- & 1 \end{pmatrix}.$$

We note that the matrix $j_1^{-1}U^-J_1 - U^-$ belongs to $\varpi_F^{m+1}\mathrm{Mat}_{1\times(n-1)}(\mathfrak{o}_F)$. This shows that $j^{-1}u^-j(u^-)^{-1} \in R_I(m+1) \cap \bar{U}_I$. Hence we get that

$$u^-j(u^-)^{-1} = j\{j^{-1}u^-j(u^-)^{-1}\} \in R_I(m+1).$$

We now consider the conjugation $u^-u^+(u^-)^{-1}$. We write u^+ in its block matrix form as

$$\begin{pmatrix} 1_{n-1} & U^+ \\ 0 & 1 \end{pmatrix}$$

where $U^+ \in \mathrm{Mat}_{(n-1)\times 1}(\mathfrak{o}_F)$. Now the conjugation $u^-u^+(u^-)^{-1}$ in the block matrix from is as follows

$$\begin{pmatrix} 1_{n-1} - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1 \end{pmatrix}.$$

Since $U^-U^+U^- \in \varpi^{m+1}\mathrm{Mat}_{1\times(n-1)}(\mathfrak{o}_F)$, we conclude that $u^-u^+(u^-)^{-1} \in R_I(m+1)$. This ends the proof of this lemma. □

Note that $P_I^0(m) = R_I(m)P_I^0(m+1)$. Using Mackey decomposition, we get that

$$\mathrm{res}_{R_I(m)} \mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) \simeq \mathrm{ind}_{R_I(m+1)}^{R_I(m)}(\mathrm{id}).$$

It follows from the Iwahori decomposition with respect to P_I and M_I that the inclusion of $R_I(m) \cap \bar{U}_I$ in $R_I(m)$ induces an isomorphism between $R_I(m)/R_I(m+1)$ and the abelian group

$$\frac{R_I(m) \cap \bar{U}_I}{R_I(m+1) \cap \bar{U}_I}.$$

Hence, the representation $\mathrm{ind}_{R_I(m+1)}^{R_I(m)}(\mathrm{id})$ decomposes as a direct sum of characters η_k, for $1 \leq k \leq p$, where η_k is trivial on $R_I(m+1)$. The group $P_I^0(m)$ acts on these characters, and let $\{\eta_{n_k} \mid n_k \in \{1, 2, \ldots, p\}\}$ be a set of representatives for the orbits under this action. Let $Z(\eta_k)$ be the $P_I^0(m)$-stabiliser of the character η_k, for $1 \leq k \leq p$. Now Clifford theory gives us the isomorphism

$$\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) = \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{P_I^0(m)}(U_{\eta_{n_k}}), \tag{3}$$

where $U_{\eta_{n_k}}$ is any irreducible representation of $Z(\eta_{n_k})$ such that $\mathrm{res}_{R_I(m)} U_{\eta_{n_k}}$ contains η_{n_k}.

We have to bound the group $Z(\eta_k)$. We note that $P_I^0(m)$ is equal to $(P_I^0(m) \cap M_I)R_I(m)$. Hence, we have $Z(\eta_k) = (Z(\eta_k) \cap M_I)R_I(m)$. To bound the group $Z(\eta_k)$ we can only need to control $Z(\eta_k) \cap M_I$. Let $u^- \in R_I(m) \cap \bar{U}_I$ and

$$\begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}$$

be the block form of u^-, where U^- is a matrix in $\varpi_F^m \mathrm{Mat}_{1\times(n-1)}(\mathfrak{o}_F)$. The map $u^- \mapsto \varpi_F^{-m}U^-$ induces an $M_I(\mathfrak{o}_F)$-equivariant isomorphism between $\mathrm{Mat}_{1\times(n-1)}(k_F)$ and the quotient

$$\frac{R_I(m) \cap \bar{U}_I}{R_I(m+1) \cap \bar{U}_I}.$$

We also have an $M_I(\mathfrak{o}_F)$-equivariant isomorphism between $\mathrm{Mat}_{(n-1)\times 1}(k_F)$ and $\widehat{\mathrm{Mat}_{1\times(n-1)}}(k_F)$ (see Lemma 2.0.7). We note that $P_I^0(m) \cap M_I = J^0 \times \mathfrak{o}_F^\times$.

Let η be a non-trivial character of $R_I(m)$ which is trivial on $R_I(m+1)$. For the present purposes, it is enough to bound the subgroup $Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathfrak{o}_F^\times)$, for $\eta \neq \mathrm{id}$. Since we have a $M_I(\mathfrak{o}_F)$-equivariant isomorphism between the group of characters on the quotient $R_I(m)/R_I(m+1)$ with $\mathrm{Mat}_{n-1\times 1}(k_F)$, we can as well study the group $Z(A) \cap (U^0(\mathfrak{B}) \times \mathfrak{o}_F^\times)$, where $Z(A)$ is the $M_I(\mathfrak{o}_F)$-stabiliser of a non-zero matrix

$A \in \mathrm{Mat}_{(n-1)\times 1}(k_F)$. The action of the group $M_I(\mathfrak{o}_F)$ factorizes through $K_{n-1}(1) \times (1+\mathfrak{p}_F)$. Hence, the group $(\mathrm{id}_{n-1} + \mathfrak{D}^e) \times (1+\mathfrak{p}_F)$ is contained in the kernel of the action of $U^0(\mathfrak{B}) \times \mathfrak{o}_F^\times$ obtained by restriction. Recall that $\mathfrak{D}$ is the radical of $\mathfrak{B}$.

This reduces our situation to the following setting. The group $\mathrm{GL}_{n-1}(k_F) \times k_F^\times$ acts on $\mathrm{Mat}_{n-1\times 1}(k_F)$ by setting

$$(g_1, g_2)A = g_1 A g_2^{-1},$$

where $g_1 \in \mathrm{GL}_{n-1}(k_F)$, $g_2 \in k_F^\times$, and $A \in \mathrm{Mat}_{n-1\times 1}(k_F)$. The basis $(v_1, v_2, \dots, v_{n-1})$ we fixed for the vector space V at the beginning of this section, gives a basis of the k_F-vector space

$$(\mathfrak{o}_E/\varpi_F\mathfrak{o}_E)^{(n-1)/ef} = (\mathfrak{o}_E/\mathfrak{p}_E^e)^{(n-1)/ef}.$$

Such a basis gives the inclusion

$$\mathrm{GL}_{(n-1)/ef}(\mathfrak{o}_E \,/\, \mathfrak{p}_E{}^e) = U^0(\mathfrak{B})/U^e(\mathfrak{B}) \hookrightarrow \mathrm{GL}_{n-1}(k_F).$$

Recall that $\lambda = \kappa \otimes \rho$, where ρ is a cuspidal representation of $\mathrm{GL}_{(n-1)/ef}(k_E) = U^0(\mathfrak{B})/U^1(\mathfrak{B})$. Hence, we are interested in the mod $\mathfrak{p}_E$ reduction of the first projection of

$$Z_{\mathrm{GL}_{(n-1)/ef}(\mathfrak{o}_E \,/\, \mathfrak{p}_E{}^e)\times k_F^\times}(A),$$

for some non-zero matrix A in $\mathrm{Mat}_{(n-1)\times 1}(k_F)$. We set $n_0 = (n-1)/ef$.

Let ϖ_E be an uniformiser of $\mathfrak{o}_E$. Let N be the operator on the k_E-vector space $W := (\mathfrak{o}_E/\mathfrak{p}_E^e)^{n_0}$ given by

$$N(w) = \varpi_E.w \text{ for all } w \in W.$$

Since $\mathfrak{o}_E/\mathfrak{P}_E^e = k_E \oplus k_E\overline{\varpi_E} \oplus k_E\overline{\varpi_E}^2 \oplus \cdots \oplus k_E\overline{\varpi_E}^{e-1}$, we obtain a decomposition of $W = W_1 \oplus W_2 \oplus \cdots \oplus W_e$ such that N restricted to W_i is an isomorphism onto W_{i+1} for $i < e$, and N acts trivially on W_e. The mod $\mathfrak{p}_E$-reduction of W is the projection onto the first factor W_1.

Any $k_E[N]$-linear map T is determined by its restriction to the space W_1. Given a map $T \in \mathrm{Hom}_{k_E}(W_1, W)$ we obtain an extension $\tilde{T} \in \mathrm{End}_{k_E[N]}(W)$ by setting

$$\tilde{T}(w) = N^{(i-1)}T(N^{-(i-1)}w),$$

for all $w \in W_i$ and $1 \le i \le e$. The map $T \mapsto \tilde{T}$ gives us an isomorphism of vector spaces

$$\mathrm{Hom}_{k_E}(W_1, W) \simeq \mathrm{End}_{k_E[N]}(W, W). \tag{4}$$

We may write $W = W_1 \oplus NW$. This shows that the mod $\mathfrak{p}_E$ reduction map, denoted by π_E, is given by sending $\tilde{T}$ to $p_1 \circ \tilde{T}_{|W_1}$, where p_1 is the projection onto the first factor of the direct sum $W_1 \oplus W_2 \oplus \cdots \oplus W_e$. Now $\mathrm{End}_{k_E}(V_1)$ is a subspace of $\mathrm{Hom}_{k_E}(W_1, W)$ and mod $\mathfrak{p}_E$ reduction of $\widetilde{\mathrm{End}_{k_E}(W_1)}$ (the image of $\widetilde{\mathrm{End}_{k_E}(W_1)}$ under the map $T \mapsto \tilde{T}$) is identity on $\mathrm{End}_{k_E}(W_1)$. Hence $\mathrm{Aut}_{k_E[N]}(W)$ is the semi-direct product $\widetilde{\mathrm{Aut}_{k_E}(W_1)} \ker(\pi_E)$.

Let Q be a parabolic subgroup fixing the flag $\mathcal{F}^i = \oplus_{j=1}^i W_i$, for $1 \le i \le e$, and L be its Levi subgroup fixing the decomposition $W_1 \oplus W_2 \oplus \cdots \oplus W_e$. Now $\widetilde{\mathrm{Aut}_{k_E}(W_1)}$ diagonally embeds in L and $\ker(\pi_E)$ is a subgroup of the radical of Q. The group $\mathrm{GL}_{n-1}(k_F) \times k_F^\times$ acts on $\mathrm{Mat}_{n-1\times 1}(k_F)$ by the map $(g_1, g_2)A \mapsto g_1 A g_2^{-1}$, where $g_1 \in \mathrm{GL}_{n-1}(k_F)$, $g_2 \in k_F^\times$, and $A \in \mathrm{Mat}_{n-1\times 1}(k_F)$. We now have the action of $\mathrm{GL}_{n_0}(\mathfrak{o}_E/\mathfrak{P}_E^e) \times k_F^\times$ on $\mathrm{Mat}_{n-1\times 1}(k_F)$ by restriction from the action of $\mathrm{GL}_{n-1}(k_F) \times k_F^\times$. We are interested in

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathfrak{o}_E/\mathfrak{P}_E^e)\times k_F^\times}(A)\},$$

for some $A \in \mathrm{Mat}_{n-1\times 1}(k_F) \backslash \{0\}$.

We first look at $Z_{Q\times k_F^\times}(A)$. Let (A_{ij}) be an element of Q in its block form. Let $(A_1, A_2, \dots, A_e)^{\mathrm{T}}$ be the block form of A, where A_j is a block of size $1 \times n_0 f$. If k is the largest positive integer such that $A_k \neq 0$ and $A_k = 0$, then we get that

$$A_{kk}A_k a^{-1} = A_k,$$

for all $((A_{ij}),a) \in Z_{Q\times k_F^\times}(A)$. Hence $\{A_{kk} \mid ((A_{ij}),a) \in Z_{Q\times k_F^\times}(A)\}$ is contained in a proper parabolic subgroup of $\mathrm{Aut}_{k_F}(W_k)$. If $n_0f > 1$, we get that

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathfrak{o}_E/\mathfrak{P}_E^e)\times k_F^\times}(A)\}$$

is a subgroup of $H \times k_F^\times$, where H is a subgroup of $\mathrm{Aut}_{k_E}(W_1)$ whose image under the inclusion map $\mathrm{Aut}_{k_E}(W_1) \hookrightarrow \mathrm{Aut}_{k_F}(W_1)$ is contained in a proper k_F-parabolic subgroup of $\mathrm{Aut}_{k_F}(W_1)$, since $A_k \neq 0$.

We recall the following proposition due to Paškūnas (see [Pas05, Definition 6.2, lemma 6.5, Proposition 6.8]).

Proposition 3.2.2 (Paškūnas). *Let W be a k_E-vector space with (finite) dimension greater than one. Let ρ be a cuspidal representation of $\mathrm{Aut}_{k_E}(W)$. Let H be a subgroup of $\mathrm{Aut}_{k_E}(W)$ such that the image of H under the inclusion map $\mathrm{Aut}_{k_E}(W) \hookrightarrow \mathrm{Aut}_{k_F}(W)$ is contained in a proper parabolic subgroup of $\mathrm{Aut}_{k_F}(W)$. For every H-irreducible sub-representation ξ of $\mathrm{res}_H(\rho)$ there exists an irreducible representation ρ' of $\mathrm{Aut}_{k_E}(W)$ such that $\rho' \not\simeq \rho$ and $\mathrm{Hom}_H(\xi,\rho') \neq 0$.*

Going back to $Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathfrak{o}_F^\times)$, for $n_0 > 1$, we get that for every irreducible sub representation ξ of

$$\mathrm{res}_{Z(\eta)\cap(U^0(\mathfrak{B})\times\mathfrak{o}_F^\times)}((\kappa\otimes\rho)\boxtimes\mathrm{id}),$$

there exists an irreducible representation ρ' of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that

$$\mathrm{Hom}_{Z(\eta)\cap(U^0(\mathfrak{B})\times\mathfrak{o}_F^\times)}(\xi,(\kappa\otimes\rho')\boxtimes\mathrm{id}) \neq 0.$$

For the case $n_0 = 1$ and $q_F > 2$, we have to look at

$$(\pi_E \times \mathrm{id})\{Z_{\mathfrak{o}_E/\mathfrak{P}_E^e{}^\times\times k_F^\times}(A)\} \tag{5}$$

for some nonzero matrix $A \in \mathrm{Mat}_{n-1\times 1}(k_F)$. We notice that the group (5) is of the form $\{(a,a) | a \in k_F^\times\}$ if $k_E = k_F$. Let k_E be a proper extension of k_F. If (a,b) is an element of the centraliser (5) then $aA_kb^{-1} = A_k$ (A_k is defined in the previous paragraph). This shows that a lies in a proper parabolic subgroup of $\mathrm{GL}_f(k_F)$. This shows that the group (5) is of the form $\{(a,b) \mid a \in \mathbb{F}^\times, b \in k_F^\times\}$ where $\mathbb{F}$ is a proper sub-field of k_E.

In the case where $n_0 = 1$ and $k_E = k_F$, we consider a non-trivial character ϕ of $U^0(\mathfrak{B})/U^1(\mathfrak{B}) = k_F^\times$. We observe that

$$\mathrm{res}_{Z_{J^0\times\mathfrak{o}_F^\times}(A)}(\lambda\phi\boxtimes\phi^{-1}) \simeq \mathrm{res}_{Z_{J^0\times\mathfrak{o}_F^\times}(A)}(\lambda\boxtimes\mathrm{id}).$$

Moreover, $[M_I, \sigma\boxtimes\mathrm{id}]$ and $[M_I, \sigma'\boxtimes\phi^{-1}]$ are two distinct inertial classes for any cuspidal representation σ' containing $(J^0, \lambda\otimes\phi)$. Here,we will use the same notation ϕ for the inflation of ϕ to the group $\mathfrak{o}_F^\times$.

In the case where $n_0 = 1$ and k_E is a proper extension of k_F, we consider a non-trivial character ϕ of $k_E^\times$ which is trivial on $\mathbb{F}^\times$. We note that

$$\mathrm{res}_{Z_{J^0\times\mathfrak{o}_F^\times}(A)}(\lambda\phi\boxtimes\mathrm{id}) \simeq \mathrm{res}_{Z_{J^0\times\mathfrak{o}_F^\times}(A)}(\lambda\boxtimes\mathrm{id})$$

and moreover $[M_I, \sigma\boxtimes\mathrm{id}]$ and $[M_I, \sigma'\boxtimes\mathrm{id}]$ are two distinct inertial classes for any cuspidal representation σ' containing $(J^0, \lambda\otimes\phi)$. With this we finish our preliminaries.

3.3. **Uniqueness of typical representations.** In this part, we will prove the uniqueness of typical representations for $[M_I, \sigma\boxtimes\mathrm{id}]$. By Frobenius reciprocity, we get that $\lambda\boxtimes\mathrm{id}$ occurs with multiplicity one in $\mathrm{ind}_{P_I^0(m)}^{P_I^0(N_s)}(\lambda\boxtimes\mathrm{id})$, for all $m > N_s$. We denote by $U_m^0(\lambda\boxtimes\mathrm{id})$ the complement of $\lambda\boxtimes\mathrm{id}$ in $\mathrm{ind}_{P_I^0(m)}^{P_I^0(N_s)}(\lambda\boxtimes\mathrm{id})$. We use the notation $U_m(\lambda\boxtimes\mathrm{id})$ for the representation

$$\mathrm{ind}_{P_I^0(N_s)}^{K_n}\{U_m^0(\lambda\boxtimes\mathrm{id})\}.$$

Theorem 3.3.1. *Let $\#k_F > 2$. The K_n-irreducible sub representations of $U_m(\lambda\boxtimes\mathrm{id})$ are atypical, for all $m > N_s$.*

Proof. We prove the theorem by induction on the positive integer $m > N_s$. We suppose the theorem is true for some positive integer $m > N_s$. We will show the same for $m+1$.

We first note that

$$\mathrm{ind}_{P_I^0(m+1)}^{K_n}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{K_n}\{\mathrm{ind}_{P_I^0(m+1)}^{P_I^0(m)}(\mathrm{id}) \otimes (\lambda \boxtimes \mathrm{id})\}.$$

From the decomposition 3, we get that

$$\mathrm{ind}_{P_I^0(m+1)}^{K_n}(\lambda \boxtimes \mathrm{id}) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

Note that the above sum is taken over the orbits for the action of $P_I(m)$ on the set of characters of $R_I(m)/R_I(m+1)$. Since there is a unique orbit, among the characters $\{\eta_k \mid 1 \leq k \leq p\}$, consisting the identity character, we get that

$$\mathrm{ind}_{P_I^0(m+1)}^{K_n}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I^0(m)}^{K_n}(\lambda \boxtimes \mathrm{id}) \oplus \bigoplus_{\eta_{n_k} \neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{6}$$

Let Γ be an irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{7}$$

We have two cases $n_0 = 1$ and $n_0 > 1$. If $n_0 = 1$ we have seen that we can find a non-trivial character ϕ of $k_E^\times = U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda\phi \boxtimes \phi^{-1}) \otimes U_{\eta_{n_k}}\}$$

or

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda\phi \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

Hence in this case, the irreducible subrepresentations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{(\lambda \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\} \tag{8}$$

occur as subrepresentations of $\mathrm{res}_{K_n} i_{P_I}^{G_n}(\sigma' \boxtimes \chi')$, where σ' is a cuspidal representation of G_{n-1} containing the type $(J^0, \lambda \otimes \phi))$. The inertial classes $[G_{n-1}, \sigma]$ and $[G_{n-1}, \sigma']$ are distinct. Hence, any irreducible subrepresentation of (8) is atypical.

Now consider the case $n_0 > 1$. In this case, there exists an irreducible representation ξ of $(\pi_E \times \mathrm{id})\{Z(\eta) \cap (U^0(\mathfrak{B}) \times \mathfrak{o}_F^\times)\}$ such that Γ is a sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{((\xi \otimes \kappa) \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}. \tag{9}$$

Now Proposition 3.2.2 gives us an irreducible representation $\rho' \not\simeq \rho$ of $U^0(\mathfrak{B})$ obtained by inflation of an irreducible representation of $U^0(\mathfrak{B})/U^1(\mathfrak{B})$ such that ξ is contained in ρ'. Now the representation (9) is a sub-representation of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{((\rho' \otimes \kappa) \boxtimes \mathrm{id}) \otimes U_{\eta_{n_k}}\}.$$

The above representation is contained in

$$\mathrm{ind}_{P_I^0(m+1)}^{K_n}((\rho' \otimes \kappa) \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{P_I(m+1)}^{K_n}(\tau' \boxtimes \mathrm{id}), \tag{10}$$

where τ' is isomorphic to $\mathrm{ind}_{J^0}^{K_{n-1}}(\rho' \otimes \kappa)$. The representation $\rho' \otimes \kappa$ is still irreducible (see [BK93, Chapter 5, Proposition 5.3.2(3)]).

We will show that irreducible subrepresentations of (10) are atypical for the inertial class $[M_I, \sigma \boxtimes \mathrm{id}]$.

Any irreducible sub-representation of (10) occurs as a sub-representation of

$$\mathrm{ind}_{P_I(m)}^{K_n}(\gamma \boxtimes \mathrm{id}),$$

where γ is a K_{n-1}-irreducible subrepresentation of τ'. Now γ is contained in an irreducible smooth representation say σ_0 of G_{n-1}. By Frobenius reciprocity this is possible only if the representation $\rho' \otimes \kappa$ of J^0

is contained in σ_0. We have two possible situations either ρ' is cuspidal or otherwise. If ρ' is cuspidal, then the representation σ_0 is a cuspidal representation such that $\sigma_0 \not\simeq \sigma$. Hence the representation

$$\mathrm{ind}_{P_I(m+1)}^{K_n}(\gamma \boxtimes \mathrm{id})$$

occurs in

$$\mathrm{res}_{K_n}\, i_{P_I}^{G_n}(\sigma_0 \boxtimes \mathrm{id})$$

with $[G_{n-1} \times F^\times, \sigma_0 \boxtimes \mathrm{id}] \neq [G_{n-1} \times F^\times, \sigma \boxtimes \mathrm{id}]$. This shows that irreducible subrepresentations of (10) are atypical representations.

Consider the case where ρ' is not cuspidal. If $(J^0, \rho' \otimes \kappa)$ is contained in an smooth irreducible representation σ_0, then σ_0 either contains a non-maximal simple-type $(J_1^0, \rho_1 \otimes \kappa_1)$ or contains a split type (see [BK93, Chapter 8, Theorem 8.3.5]). We also refer to the article [BH13, Lemma 2, Proposition 1] for quick reference. From this we conclude that σ_0 is not a cuspidal representation. Hence, the representation (10) is contained in

$$\mathrm{res}_{K_n}\, i_{P'}^{G_n}(\sigma'),$$

where P' is a parabolic subgroup G_n properly contained in P_I, and σ' is a cuspidal representation of a Levi subgroup of P'. Since $\mathcal{I}(i_{P'}^{G_n}(\sigma')) \neq [M_I, \sigma \boxtimes \mathrm{id}]$, we get that the irreducible subrepresentations of (10) are atypical. $\square$

Recall the definition of the integer N_s from (3.2.1). Now, any typical representation for s occurs as a subrepresentation of $\mathrm{ind}_{P_I^0(m)}^{K_n}(\lambda \boxtimes \mathrm{id})$, for some $m \geq N_s$. For $m > N_s$, we have

$$\mathrm{ind}_{P_I^0(m)}^{K_n}(\lambda \boxtimes \mathrm{id}) = \mathrm{ind}_{P_I^0(N_s)}^{K_n}(\lambda \boxtimes \mathrm{id}) \oplus U_m(\lambda \boxtimes \mathrm{id}).$$

From the above theorem we get that the typical representations for s occur as subrepresentations of

$$\mathrm{ind}_{P_I^0(N_s)}^{K_n}(\lambda \boxtimes \mathrm{id}).$$

The above representation may still contain atypical representations. We will indeed show that this is the case and complete the classification.

The first observation is that the group J_s in a semisimple Bushnell-Kutzko type $(J_s, \lambda \boxtimes \mathrm{id})$, for $s = [M_I, \sigma \boxtimes \mathrm{id}]$, contains the group $P_I^0(N_s)$. Hence we will try to decompose the representation

$$\mathrm{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}). \tag{11}$$

We also note that $P_I^0(N_s) \cap P_I = J_s \cap P_I$. Let $l + 1 = el' + r$, where $0 \leq r < e$. If $r \leq 1$, then $J_s = P_I^0(N_s)$ and hence, we have nothing further to analyse and Theorem 3.3.1 completes the classification of typical representations. **From now we assume that** $e > 2$ **and** $r > 1$. Note that the depth-zero case is already handled in [Nad17] (see Theorem 2.0.8). We will first verify that the group $U_I(\mathfrak{o}_F)$ acts trivially on the representation (11).

Let u^+ and u^- be two matrices from $J_s \cap U_I = U_I(\mathfrak{o}_F)$ and $J_s \cap \bar{U}_I$ respectively. Let u^+ and u^- in block form be written as:

$$\begin{pmatrix} 1_{n-1} & U^+ \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}$$

respectively. The block form of the conjugation $u^- u^+ (u^-)^{-1}$ is given by

$$\begin{pmatrix} 1_{n-1} - U^+U^- & U^+ \\ -U^-U^+U^- & U^-U^+ + 1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 0 & 0 \\ U^- & 0 \end{pmatrix} \in a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I = \varpi_F^{l'}(a_r(\Lambda) \cap \bar{\mathfrak{n}}_I).$$

If $r \geq 1$, then the valuation of each entry of a matrix in $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is at least one. This shows that the valuation of each entry in $U^-U^+U^-$ is at least $l' + 2$. From which the conjugation $u^- u^+ (u^-)^{-1}$ lies in the group $P^0(N_s)$. If $r = 0$ and $l' = 0$, then we are in the case where σ is a level-zero cuspidal representation and in this case $J_s = P_I^0(N_s)$. If $r = 0$ and $l' > 0$, then valuation of each entry in $U^-U^+U^-$ has valuation

$2l' > l'+1$ and hence $u^-u^+(u^-)^{-1} \in P_I^0(N_s)$. Hence, the group $U_I(\mathfrak{o}_F)$ acts trivially on the representation (11).

From the Iwahori decomposition of the group J_s, we get that J_s is equal to $(J_s \cap \bar{P}_I)P_I^0(N_s)$. Hence we have:

$$\mathrm{res}_{J_s\cap\bar{P}_I} \mathrm{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}) \simeq \mathrm{ind}_{P_I^0(N_s)\cap\bar{P}_I}^{J_s\cap\bar{P}_I}(\mathrm{id}).$$

Note that $J_s \cap \bar{P}_I$ is a semi-direct product of the groups $(J_s \cap M_I)$ and $(J_s \cap \bar{U}_I)$. Let η_k, for $1 \leq k \leq t$, be all the characters of the group $J_s \cap \bar{U}_I$ which are trivial on the group $P_I^0(N_s) \cap \bar{U}_I$. The group $J_s \cap \bar{P}_I$ acts on these characters. Let $\{\eta_{k_p}\}$ be a set of representatives for the orbits under this action. We denote by $Z(\eta_{k_p})$ the $J_s \cap \bar{P}_I$ stabiliser of the character η_{k_p}. Let $U_{\eta_{k_p}}$ be the isotypic component of the character η_{k_p} in the representation

$$\mathrm{ind}_{P_I^0(N_s)\cap\bar{P}_I}^{J_s\cap\bar{P}_I}(\mathrm{id}).$$

The space $U_{\eta_{k_p}}$ has a natural action of $Z(\eta_{k_p})$. Now Clifford theory gives the decomposition

$$\mathrm{ind}_{P_I^0(N_s)\cap\bar{P}_I}^{J_s\cap\bar{P}_I}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})}^{J_s\cap\bar{P}_I}(U_{\eta_{k_p}}).$$

We note that the character id occurs with a multiplicity one in the list of characters η_k.

If K_s is the kernel of the representation (11), then $K_s \cap Z(\eta_{k_p})$ acts trivially on $U_{\eta_{k_p}}$. Hence we can extend the representation $U_{\eta_{k_p}}$ to the group $Z(\eta_{k_P})K_s$ such that K_s acts trivially on the extension. Now consider the representation

$$\pi = \mathrm{ind}_{Z(\eta_{k_P})K_s}^{J_s} U_{\eta_{k_p}}.$$

Note that $K_s \cap \bar{P}_I$ is contained in the group $Z(\eta_{k_p}) \cap \bar{P}_I$ and moreover, $U_I(\mathfrak{o}_F)$ is contained in K_s. Hence we have $J_s = (J_s \cap \bar{P}_I)Z(\eta_{k_p})K_s$. From Mackey decomposition, we have

$$\mathrm{res}_{J_s\cap\bar{P}_I} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}} \simeq \mathrm{ind}_{Z(\eta_{k_p})K_s\cap(J_s\cap\bar{P}_I)}^{J_s\cap\bar{P}_I}(U_{\eta_{k_p}}) \simeq \mathrm{ind}_{Z(\eta_{k_p})}^{J_s\cap\bar{P}_I}(U_{\eta_{k_p}}).$$

We hence have

$$\mathrm{ind}_{P_I^0(N_s)}^{J_s}(\mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{J_s} U_{\eta_{k_p}}. \tag{12}$$

Now using the decomposition (12) we get the decomposition

$$\mathrm{ind}_{P_I^0(N_s)}^{K_n}(\lambda \boxtimes \mathrm{id}) \simeq \bigoplus_{\eta_{k_p}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{K_n}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}.$$

Note that the character id occurs with multiplicity one among the characters η_k. Moreover, we have $Z(\mathrm{id})K_s = (J_s \cap \bar{P}_I)K_s = J_s$ and we get that

$$\mathrm{ind}_{P_I^0(N_s)}^{K_n}(\lambda \boxtimes \mathrm{id}) \simeq \mathrm{ind}_{J_s}^{K_n}(\lambda \boxtimes \mathrm{id}) \oplus \bigoplus_{\eta_{k_p}\neq\mathrm{id}} \mathrm{ind}_{Z(\eta_{k_p})K_s}^{K_n}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}. \tag{13}$$

Lemma 3.3.2. *Let $\#k_F > 2$ and η_{k_p} be a non-trivial character. The irreducible sub representations of*

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{K_n}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}$$

are atypical.

Proof. We observe that $Z(\eta_{k_p}) = (Z(\eta_{k_p}) \cap M_I)(J_s \cap \bar{U}_I)$. This shows that we have to bound the group $Z(\eta_{k_P}) \cap M_I$, for $\eta_{k_p} \neq \mathrm{id}$. Recall that η_k, for $1 \leq k \leq t$, are the characters of the quotient group

$$\frac{(J_s \cap \bar{U}_I)}{(P_I^0(N_s) \cap \bar{U}_I)}. \tag{14}$$

Now let u^- be a matrix from the group $J_s \cap \bar{U}_I$. In the block form the matrix u^- is of the form

$$\begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}$$

where $U^- = [M_1, M_2, \ldots, M_e]$, M_i is a matrix of size $(1 \times (n-1)/e)$. Let $\delta = N_s - 1$. The map Φ

$$[M_1, M_2, \ldots, M_e] \mapsto [\varpi_F^{\delta} M_1, \varpi_F^{\delta} M_2, \ldots, \varpi_F^{\delta} M_e]$$

identifies the quotient (14) with a subspace $\mathfrak{t}_1$ of $\mathrm{Mat}_{1\times n-1}(k_F)$. This identification commutes with the action of $M_I \cap J_s$, since Φ is none other than conjugation by an element from the $Z(M_I)$ (The centre of M_I).

Let $\mathfrak{t}_2$ be the following space of column matrices:

$$\mathfrak{t}_2 = \{(0, 0, \ldots, 0, M_r, \ldots, M_e)^{\mathrm{T}} \mid M_j \in \mathrm{Mat}_{(n-1)/e\times 1}(k_F) \ \forall\ r \leq j \leq e\}.$$

The group $M_I(\mathfrak{o}_F) \simeq K_{n-1} \times K_1$ acts on the space $\mathrm{Mat}_{(n-1)\times 1}(k_F)$ via the conjugation action of M_I on U_I. The space $\mathfrak{t}_2$ is stable under the action of $U^0(\mathfrak{A}) \times \mathfrak{o}_F^{\times} \subseteq M_I$. The pairing XY, where $X \in \mathfrak{t}_2$ and $Y \in \mathfrak{t}_1$, gives a perfect pairing between $\mathfrak{t}_1$ and $\mathfrak{t}_2$. This pairing is equivariant for the action of $U^0(\mathfrak{A}) \times \mathfrak{o}_F^{\times} \subseteq M_I$. This gives an identification of the space of characters of $\mathfrak{t}_1$ with the space $\mathfrak{t}_2$ in a $U^0(\mathfrak{A}) \times \mathfrak{o}_F^{\times}$ equivariant way.

The group $M_I(\mathfrak{o}_F)$ acts on $\mathrm{Mat}_{(n-1)\times 1}(k_F)$ through its quotient $M_I(k_F)$. Now, the action of the group $(U^0(\mathfrak{B}) \times \mathfrak{o}_F^{\times}) \subseteq J_s \cap M_I$ on $\mathfrak{t}_2$ factors through its quotient by its subgroup $(1_n + \mathfrak{D}^e) \times (1 + \mathfrak{p}_F)$. Let A be a non-zero matrix in $\mathfrak{t}_2$.

Recall that $n_0 = (n-1)/ef$. Now recall that we denote by π_E by mod $\mathfrak{p}_E$ reduction map. We have seen that (the paragraph above the proposition 3.2.2)

$$(\pi_E \times \mathrm{id})\{Z_{\mathrm{GL}_{n_0}(\mathfrak{o}_E/\mathfrak{P}_E^e)\times k_F^{\times}}(A)\}$$

is a subgroup of $H \times k_F^{\times}$, where H is a subgroup of $\mathrm{GL}_{n_0}(k_E)$ whose image under the inclusion map $\mathrm{GL}_{n_0}(k_E) \hookrightarrow \mathrm{GL}_{n-1}(k_F)$ is contained in a proper k_F-parabolic subgroup of $\mathrm{GL}_{n-1}(k_F)$. From the result of Paškūnas, stated as Proposition 3.2.2, we get that for every irreducible representation ξ of

$$\mathrm{res}_{Z(\eta_{k_p})}\{U_{k_p} \otimes ((\kappa \otimes \rho) \boxtimes \mathrm{id})\}$$

we can find an irreducible representation $\rho' \not\simeq \rho$ such that ξ occurs in the representation

$$\mathrm{res}_{Z(\eta_{k_p})}\{U_{k_p} \otimes ((\kappa \otimes \rho') \boxtimes \mathrm{id})\}.$$

Hence irreducible subrepresentations of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{K_n}\{U_{\eta_{k_p}} \otimes (\lambda \boxtimes \mathrm{id})\}$$

occur as subrepresentations of

$$\mathrm{ind}_{Z(\eta_{k_p})K_s}^{K_n}\{U_{\eta_{k_p}} \otimes ((\kappa \otimes \rho') \boxtimes \mathrm{id})\}.$$

Now the above representation occurs as a sub-representation of

$$\mathrm{ind}_{P_I^0(N_s)}^{K_n}\{(\kappa \otimes \rho') \boxtimes \mathrm{id}\} \simeq \mathrm{ind}_{P_I(N_s)}^{K_n}(\tau' \boxtimes \mathrm{id})\},$$

where τ' is given by

$$\mathrm{ind}_{J^0}^{K_{n-1}}(\kappa \otimes \rho').$$

Any irreducible sub representation γ of τ' occurs in an irreducible smooth representation σ_0 of $\mathrm{GL}_{n-1}(F)$. Assume that ρ' is cuspidal. The representation $\kappa \otimes \rho'$ is contained in the representation $\mathrm{res}_{J^0}\gamma$ and hence is contained in σ_0. This implies that σ_0 is cuspidal but not inertially equivalent to σ. If ρ' is not cuspidal, then the representation σ_0 is not cuspidal. Hence, in every case, σ_0 is not inertially equivalent to σ. This shows that irreducible sub representations of

$$\mathrm{ind}_{P_I(N_s)}^{K_n}(\gamma \boxtimes \mathrm{id})$$

are atypical. We conclude the lemma. □

Theorem 3.3.3. *Let $n > 2$ and $q_F > 2$. Let Γ be any typical representation for the inertial class $s = [M_I, \sigma \boxtimes \chi]$. The representation Γ is isomorphic to the representation*

$$\mathrm{ind}_{J_s}^{K_n}(\lambda_s),$$

where (J_s, λ_s) is any Bushnell-Kutzko semi-simple type for the inertial class s. If P is a parabolic subgroup containing M_I as a Levi subgroup, then Γ occurs with a multiplicity one in the representation

$$\mathrm{res}_{K_n} i_P^{G_n}(\sigma \boxtimes \chi).$$

Proof. Let G be the group of F-rational points of a connected reductive group defined over F. For any inertial class $t = [L, \Theta]$ of G, and for $g \in N_G(L)$ we define t^g to be the inertial class $[L, \Theta^g]$. The map sending t to t^g is well defined. We denote by $N_G(t)$ the group $\{g \in N_G(L) \mid t^g = t\}$. The group $N_G(t)$ clearly contains the group L and the quotient $W_t = N_G(t)/L$ is finite. The cardinality of W_t does not depend on the choice of L. We return to the case where $G = G_n$ and $t = s$. The intertwining of the representation $\mathrm{ind}_{J_s}^{K_n}(\lambda_s)$ is bounded by the cardinality of W_s. We have $|W_s| = 1$ since $n > 2$. Hence, the representation $\mathrm{ind}_{J_s}^{K_n}(\lambda_s)$ is irreducible. We refer to [BK98][Lemma 11.5] for these results. Hence the uniqueness of the typical representation. The multiplicity follows from the results 3.3.1 and 3.3.2. □

4. Principal series components

Let I be the partition $(1, 1, \ldots, 1)$ of n. Recall that we denote by B_n the group P_I, U_n the group U_I, and T_n the group M_I respectively. In this section, we will classify typical representations for the inertial classes $s = [T_n, \chi]$, where χ is a character of T_n. Let τ be a typical representation for the inertial class s. The compact induction $\mathrm{ind}_{K_n}^{G_n} \tau$ is a finitely generated representation. Let π be an irreducible quotient of $\mathrm{ind}_{K_n}^{G_n} \tau$. By Frobenius reciprocity, the K_n-representation τ occurs in the G_n representation π.

Let B be any Borel subgroup of G_n, T be a maximal split torus of G_n contained in B, and χ' be a character of T. If (T, χ') and (T_n, χ) are inertially equivalent, then the representation π occurs as a sub-quotient of $i_B^{G_n}(\chi'')$, where χ'' is obtained from χ' by twisting with an unramified character of T. For classifying typical representations it is enough to say which K_n-irreducible sub representations of $i_B^{G_n}(\chi')$ are typical for the inertial class $[T_n, \chi]$.

Let σ be a permutation of the set $\{1, 2, \ldots, n\}$. Let $\chi = \boxtimes_{i=1}^n \chi_i$ be any character of $T_n = \prod_{i=1}^n F^\times$. We denote by χ^σ the character $\boxtimes_{i=1}^n \chi_{\sigma(i)}$ of T_n. We observe that the pairs (T_n, χ^σ) and (T_n, χ) are inertially equivalent. This implies that for a classifying typical representations we can classify typical representation occurring in $i_{B_n}^{G_n} \chi^\sigma$, for any σ. We will use a convenient permutation σ which satisfies the condition in the following lemma.

Lemma 4.0.1. *Given any sequence of characters $x_i = \chi_i$ of $\mathfrak{o}_F^\times$, there exists a permutation $\{y_i \mid 1 \le i \le n\}$ of $\{x_i \mid 1 \le i \le n\}$ such that*

$$l(y_i y_k^{-1}) \ge \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\},$$

for all $1 \le i \le j \le k \le n$.

Proof. For any ultrametric space (X, d) and given any n points $x_1, x_2, x_3, \ldots, x_n$ in X we may choose a permutation $y_1, y_2, \ldots, y_n$ of the sequence $\{x_i | 1 \le i \le n\}$ such that

$$d(y_i, y_k) \ge \max\{d(y_i, y_j), d(y_j, y_k)\}$$

for all $i \le j \le k$. Now apply this fact to the space X consisting of characters of $\mathfrak{o}_F^\times$ and the distance function $d(\chi_1, \chi_2)$ is defined as the level $l(\chi_1 \chi_2^{-1})$ if $\chi_1 \ne \chi_2$ and 0 otherwise. We point out that this ordering is not unique in general. We refer to [How73, Lemma 1]for a proof of these results. □

Remark 4.0.2. *We note that the condition $l(y_i y_k^{-1}) \ge \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}$ is equivalent to an equality since we always have*

$$l(y_i y_k^{-1}) \le \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}.$$

Given an inertial class $[T_n, \chi]$ we choose the representative (T_n, χ^σ) where σ is a permutation such that

$$l(\chi_{\sigma(i)} \chi_{\sigma(k)}^{-1}) \ge \max\{l(\chi_{\sigma(i)} \chi_{\sigma(j)}^{-1}), l(\chi_{\sigma(j)} \chi_{\sigma(k)}^{-1})\}.$$

From now on we assume that the pair $(T_n, \boxtimes_{i=1}^n \chi_i)$ satisfies the condition

$$l(\chi_i \chi_k^{-1}) \ge \max\{l(\chi_i \chi_j^{-1}), l(\chi_j \chi_k^{-1})\}, \tag{15}$$

for all $i \leq j \leq k$.

In the following subsection we construct subgroups H_m, for $m \geq 1$ such that

(1) $H_1 = J_s$, where J_s is the compact open subgroup of a Bushnell–Kutzko type (J_s, χ) of s,
(2) $H_{m+1} \subset H_m$, for all $m \geq 1$ and $\bigcap_{m\geq 1} H_m = B_n \cap K_n$,
(3) The representation χ of $T_n \cap K_n$ extends to a representation of H_m such that $H_m \cap U_n$ and $H_m \cap \bar{U}_n$ are contained in the kernel of this extension.

Such a construction gives the following equality:

$$\mathrm{ind}_{K_n\cap B_n}^{K_n} \chi = \bigcup_{m\geq 1} \mathrm{ind}_{H_m}^{K_n} \chi.$$

Later, we show that any K_n-irreducible sub representation of $\mathrm{ind}_{H_{m+1}}^{K_n} \chi / \mathrm{ind}_{H_m}^{K_n} \chi$ is atypical.

4.1. **Construction of compact open subgroups H_m.** Let $\mathcal{A} = (a_{ij})$ be a lower nilpotent matrix of size $n \times n$ such that a_{ij} is non-negative, for $i > j$, and

$$a_{ki} = \max\{a_{ji}, a_{kj}\}, \tag{16}$$

for $1 \leq i < j < k \leq n$. We denote by $J(\mathcal{A})$ the set of $n \times n$ matrices (m_{pq}) such that $m_{pq} \in \mathfrak{o}_F$, for $p < q$, and $m_{pq} \in \mathfrak{P}_F^{a_{pq}}$, for $p \geq q$. As a consequence of the condition $a_{ki} = \max\{a_{ji}, a_{kj}\}$ we get two important inequalities

$$a_{i1} \geq a_{i2} \geq \cdots \geq a_{ii-1} \tag{17}$$

and

$$a_{j+1j} \leq a_{j+2j} \leq \cdots \leq a_{nj}. \tag{18}$$

The first is a consequence of $a_{ik-1} = \max\{a_{kk-1}, a_{ik}\}$, for $k < i$, and the second is a consequence of $a_{k+1j} = \max\{a_{k+1k}, a_{kj}\}$, for $j < k$.

Lemma 4.1.1. *The set $\mathcal{J}(\mathcal{A})$ is an order in* $\mathrm{Mat}_{n\times n}(\mathfrak{o}_F)$

Proof. The set $\mathcal{J}(\mathcal{A})$ is an additive group. We now check that the set $\mathcal{J}(\mathcal{A})$ is closed under multiplication. Let (m_{ij}) and (m'_{ij}) be two matrices from $J(\mathcal{A})$. If $i > j$, then the $i \times j$ term in the product matrix $(m_{ij})(m'_{ij})$ can be split into three terms:

$$t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ji},$$

$$t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ii}m'_{ij}$$

and

$$t_3 := m_{ii+1}m'_{i+1j} + \cdots + m_{in}m'_{nj}.$$

Observe that $\nu_F(m_{ik}m'_{kj}) \geq a_{i1}$, for $k \leq j$. This shows that $\nu_F(t_1) \geq \min\{a_{i1}, a_{i2}, \ldots, a_{ij}\}$ and

$$\min\{a_{i1}, \ldots, a_{ij}\} \geq a_{ij}.$$

The valuation $\nu_F(m_{ik}m'_{kj}) \geq a_{ik} + a_{kj}$, for all $j \leq k \leq i$, and $a_{ik} + a_{kj}$ is greater or equal to a_{ij}. We get that $\nu_F(t_2) \geq a_{ij}$. Finally the valuation $\nu_F(m_{ik}m_{kj}) \geq a_{kj}$, for $k > i$. The valuation $\nu_F(t_3) \geq \min\{a_{i+1j}, \ldots, a_{nj}\}$ and $\min\{a_{i+1j}, \ldots, a_{nj}\} \geq a_{ij}$. Hence the additive group $\mathcal{J}(\mathcal{A})$ is closed under multiplication. Since $\mathcal{J}(\mathcal{A})$ is an $\mathfrak{o}_F$ lattice in $\mathrm{Mat}_{n\times n}(F)$ we get that $\mathcal{J}(\mathcal{A})$ is an order in $\mathrm{Mat}_{n\times n}(\mathfrak{o}_F)$. □

We denote by $J(\mathcal{A})$ the set of invertible elements of $\mathcal{J}(\mathcal{A})$. The following are examples of $J(\mathcal{A})$.

(1) If $\mathcal{A} = 0$ then the group $J(\mathcal{A})$ is K_n.
(2) If $\mathcal{A} = (a_{ij})$ with $a_{ij} = 1$, for $i > j$, then $J(\mathcal{A})$ is the Iwahori subgroup with respect to the standard Borel subgroup B_n.

The examples (1) and (2) satisfy Iwahori decomposition with respect to the standard Borel subgroup B_n. The next lemma concerns the Iwahori decomposition of $J(\mathcal{A})$ in general.

Let $\mathcal{A} = (a_{ij})$ be a lower nilpotent matrix such that $a_{ki} = \max\{a_{kj}, a_{ji}\}$, for $1 \leq i < j < k \leq n$. We define an ordered partition I of n by induction on the set of positive integers $m \leq n$. Let $I_1 := (1)$ and if we know $I_m = (n_1, n_2, \dots, n_r)$, for some $m \leq n-1$, then I_{m+1} is the following partition

$$I_{m+1} = \begin{cases} (n_1, n_2, \dots, n_r, 1) \text{ if } a_{m+1m} \neq 0 \\ (n_1, n_2, \dots, n_r + 1) \text{ otherwise.} \end{cases} \tag{19}$$

We denote by $I(\mathcal{A})$ the partition I_n.

Lemma 4.1.2. *The group $J(\mathcal{A})$ satisfies Iwahori decomposition with respect to the parabolic subgroup $P_{I(\mathcal{A})}$ and the Levi subgroup $M_{I(\mathcal{A})}$. We have $J(\mathcal{A}) \cap M_{I(\mathcal{A})} = M_{I(\mathcal{A})}(\mathfrak{o}_F)$, $J(\mathcal{A}) \cap U_{I(\mathcal{A})} = U_{I(\mathcal{A})}(\mathfrak{o}_F)$.*

Proof. We use induction on the positive integer n. If $n = 1$ then $J(\mathcal{A})$ is $\mathfrak{o}_F^{\times}$ and the lemma is vacuously true. We assume that the lemma is true for all positive integers less than n. Let $I(\mathcal{A})$ be the ordered partition $(n_1, n_2, \dots n_r)$. If $r = 1$ then the lemma is true by default. We suppose $r > 1$. We will show below that every element $j \in J(\mathcal{A})$ can be written as a product $u_1 j_1$ with $u_1 \in J(\mathcal{A}) \cap \bar{U}_{(n_1, n-n_1)}$ and $j_1 \in J(\mathcal{A}) \cap P_{(n_1, n-n_1)}$. Now j_1 can be written as $j_2 u_1^+$ where $u_1^+ \in U_{(n_1, n-n_1)}(\mathfrak{o}_F)$ and $j_2 \in M_{(n_1, n-n_1)} \cap J(\mathcal{A})$. Now j_2 can be written as $j_3 u_2^+$ where $j_3 \in J(\mathcal{A}) \cap M_{I(\mathcal{A})}$ and $u_2^+ \in U_{I(\mathcal{A})}(\mathfrak{o}_F)$. The group $J(\mathcal{A}) \cap M_{(n_1, n-n_1)}$ is equal to $K_{n_1} \times J(\mathcal{A}')$ where the nilpotent matrix $\mathcal{A}' = (a'_{ij})$ is given by $a'_{ij} = a_{i+n_1 j+n_1}$. By induction hypothesis $J(\mathcal{A}')$ satisfies Iwahori decomposition with respect to the standard parabolic subgroup $P_{I(\mathcal{A}')}$ and its Levi subgroup $M_{I(\mathcal{A}')}$ and $I(\mathcal{A}') = (n_2, n_3, \dots, n_r)$. Let $j_3 = (j_3^0, j_3^1)$ where $j_3^0 \in K_{n_1}$ and $j_3^1 \in J(\mathcal{A}')$. Now $j_3^1 = u_3^- j_4 u_3^+$ where $u_3^- \in \bar{U}_{I(\mathcal{A}')} \cap J(\mathcal{A}')$, $u_3^+ \in U_{I(\mathcal{A}')} \cap J(\mathcal{A}')$ and $j_4 \in M_{I(\mathcal{A}')} \cap J(\mathcal{A}')$. Hence $j = u_1 u_3^- (j_3^0, j_4) u_3^+ u_2^+$ (with a slight abuse of notation the elements u_3^- and u_3^+ are considered as elements of $\bar{U}_{I(\mathcal{A})}$ and $U_{I(\mathcal{A})}$ respectively and (j_3^0, j_4) is an element of $J(\mathcal{A}) \cap M_{I(\mathcal{A})} = K_{n_1} \times (J(\mathcal{A}') \cap M_{I(\mathcal{A}')})$).

We now prove that $j \in J(\mathcal{A})$ can be written as a product $u_1 j_1$ with $u_1 \in J(\mathcal{A}) \cap \bar{U}_{(n_1, n-n_1)}$ and $j_1 \in J(\mathcal{A}) \cap P_{(n_1, n-n_1)}$. Let $j = (j_{pq})$. Let C_i^1 be the i^{th}-column of the first diagonal block (of size $n_1 \times n_1$) on the diagonal. If every entry of C_i^1 has positive valuation then, we claim that the all the entries of the i^{th} column C_i have positive valuation. Suppose the k^{th} entry j_{ki} of C_i is an unit for some $k > n_1$. This shows that a_{ki} the ki^{th}-entry of $\mathcal{A}$ is zero. Now the inequality (17) gives $a_{ki} \geq a_{kn_1}$ and this implies that $a_{kn_1} = 0$. Now note that $a_{kn_1} \geq a_{n_1+1n_1}$ from the inequality (18). This shows that $a_{n_1+1n_1}$ is zero which gives a contradiction from the definition of $I(\mathcal{A})$. We now deduce that j_{ki} is not invertible. This shows the claim. Since j is invertible we conclude that at least one entry of C_i^1 is an unit. Let $E_{ij}(c) = I_n + e_{ij}(c)$ where $e_{ij}(c)$ is the matrix with its ij entry c and all other entries 0. The left multiplication of $E_{ij}(c)$ results in the row operation $R_j + cR_i$. Since at least one entry of C_i^1 is an unit we assume that its q^{th}-entry is an unit. We can perform row operations $R_p + cR_q$ for all $p \geq n_1$ to make the p^{th}-entry trivial. We also note that the elementary matrix corresponding to this row operation also belongs to the group $J(\mathcal{A})$ (note that $q \leq n_1 \leq p$). This completes the task of making j as the product $u_1 j_1$. The uniqueness of the Iwahori decomposition is standard. $\square$

Let $s = [T_n, \chi]$ be an inertial equivalence class. Let m be a positive integer and $\mathcal{A}_\chi(m)$ be the lower nilpotent matrix $(a_{ij}^m) \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$, where

$$a_{ij}^m = l(\chi_i \chi_j^{-1}) + m - 1,$$

for $n \geq i > j \geq 1$. As shown earlier, the representative $(T_n, \chi = \boxtimes_{i=1}^n \chi_i)$ for s, can be chosen such that

$$a_{ik} = \max\{a_{ij}, a_{jk}\},$$

for all $i < j < k$. We denote by $J_\chi(m)$ the group $J(\mathcal{A}_\chi(m))$. Note that $J_\chi(m') \subseteq J_\chi(m)$, for all $m' \geq m$. In our situation we have $I(\mathcal{A}_\chi(m))$ is $(1, 1, \dots, 1)$, since none of a_{ii+1}^m are zero. Hence, using lemma 4.1.2 the group $J_\chi(m)$ satisfies the Iwahori decomposition with respect to B_n and T_n.

Lemma 4.1.3. *The character $\chi = \boxtimes_{i=1}^n \chi_i$ of $T_n \cap K_n$ extends to a character of $J_\chi(1)$ such that $J_\chi(1) \cap U_n$ and $J_\chi(1) \cap \bar{U}_n$ are contained in the kernel of the extension.*

Proof. Let $m = (m_{ij})$ be an element of $J_\chi(1)$. We define $\tilde{\chi}(m) = \prod_{i=1}^n \chi_i(m_{ii})$. The verification that $\tilde{\chi}$ is a character of the group $J_\chi(1)$ is very computational in nature and we sketch the proof here and for complete details see [Roc98, Section 3, Lemma 3.1, Lemma 3.2] or [How73, Pg 278-279]. The idea is to get an open normal subgroup U of $J_\chi(1)$ such that $J_\chi(1)/U$ is isomorphic to $T(\mathfrak{o}_F)/T_\chi$ where T_χ is an open subgroup of $T(\mathfrak{o}_F)$ which is contained in the kernel of χ. The subgroup U is generated by $J_\chi(1) \cap \bar{U}_n$ and $J_\chi(1) \cap U_n = U_n(\mathfrak{o}_F)$. One shows that U satisfies Iwahori decomposition with respect to the Borel subgroup B_n and $U \cap T_n$ is given by $\prod_{\alpha \in \Phi} \alpha^\vee(1 + \mathfrak{P}_F^{l(\chi\alpha^\vee)})$ where Φ is the set of roots of GL_n with respect to T_n and $\alpha^\vee$ stands for the dual root. We observe that $U \cap T_n$ is contained in the kernel of χ. □

We are now ready to define the sequence of groups H_m. We set $H_m = J_\chi(m)$, for $m \geq 1$. We now get the equality

$$\mathrm{res}_{K_n} i_{B_n}^{G_n}(\chi) = \bigcup_{m \geq 1} \mathrm{ind}_{J_\chi(m)}^{K_n}(\chi).$$

For the purposes of proofs by induction, we need to construct some more compact open subgroups of K_n.

We denote by $\mathcal{A}_\chi(1, m)$ the lower nilpotent matrix (a_{ij}) where $a_{ij} = l(\chi_i \chi_j^{-1})$ for $j < i < n$, $a_{nj} = l(\chi_n \chi_j^{-1}) + m - 1$, for $1 \leq j < n - 1$. Given a lower nilpotent matrix $\mathcal{A} = (a_{ij})$ such that $a_{ki} = \max\{a_{kj}, a_{ji}\}$ we associated a compact subgroup $J(\mathcal{A})$. The matrix $\mathcal{A}_\chi(1, m)$ need not satisfy this condition but, we can still associate the group $J(\mathcal{A}_\chi(1, m))$ to the matrix $\mathcal{A}_\chi(1, m)$. We will prove this in the next Lemma.

Lemma 4.1.4. *Let $\mathcal{J}(\mathcal{A}_\chi(1, m))$ be the set consisting of matrices $(m_{ij}) \in \mathrm{Mat}_{n\times n}(\mathfrak{o}_F)$ such that $m_{ij} \in \mathfrak{P}_F^{a_{ij}}$ for all i, j. The set $\mathcal{J}(\mathcal{A}_\chi(1, m))$ is an order in $\mathrm{Mat}_{n\times n}(\mathfrak{o}_F)$.*

Proof. The set $\mathcal{J}(\mathcal{A}_\chi(1, m))$ is a lattice in $\mathrm{Mat}_{n\times n}(F)$ and we have to verify that $\mathcal{J}(\mathcal{A}_\chi(1, m))$ is closed under multiplication. Let (m_{ij}) and (m'_{ij}) be two elements of the set $\mathcal{J}(\mathcal{A}_\chi(1, m))$. We suppose $i > j$. The ij^{th}-term of the product $(m_{ij})(m'_{ij})$ is the sum of the terms:

$$t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ji},$$

$$t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ii}m'_{ij}$$

and

$$t_3 := m_{ii+1}m'_{i+1j} + \cdots + m_{in}m'_{nj}.$$

Note that

$$\nu_F(t_1) \geq \min\{\nu_F(m_{ik}m'_{kj}) \mid \text{for all } 1 \leq k \leq j\},$$

and $\nu_F(m_{ik}m'_{kj}) = a_{ik}$. If $i < n$, then $a_{ik} = l(\chi_i\chi_k^{-1})$ and $a_{ij} \leq a_{ik}$, for all $k \leq j < i$. This shows that $\nu_F(t_1) \geq a_{ij}$. If $i = n$, then we have

$$a_{ik} = a_{nk} = l(\chi_n\chi_k^{-1}) + m - 1 \geq l(\chi_i\chi_j^{-1}) + m - 1 = a_{ij},$$

for all $k \leq j < n$. We conclude that in every possibility $\nu_F(t_1) \geq a_{ij}$.

Consider the term t_2. We have

$$\nu_F(t_2) \geq \{\nu_F(m_{ik}m'_{kj}) \mid \text{for all } j < k \leq i\}$$

and $\nu_F(m_{ik}m'_{kj}) = a_{ik} + a_{kj}$, for $j < k \leq i$. If $i < n$, then we have $a_{ik} = l(\chi_i\chi_k^{-1})$ and $a_{kj} = l(\chi_k\chi_j^{-1})$. From our assumption on the arrangement of characters χ_i, for $1 \leq i \leq n$, we get that

$$l(\chi_i\chi_j^{-1}) = \max\{l(\chi_i\chi_k^{-1}), l(\chi_k\chi_j^{-1})\}.$$

At the same time $i < n$ implies that $a_{ij} = l(\chi_i\chi_j^{-1})$. This shows that $\nu_F(m_{ik}m'_{kj})$ is equal to $a_{ik} + a_{kj}$ and $a_{ik} + a_{kj} \geq a_{ij}$. Consider the case $i = n$ and in this case, $a_{nk} = l(\chi_n\chi_k^{-1}) + m - 1$. Now $a_{kj} = l(\chi_k\chi_j^{-1})$ and $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1$. From the equality $l(\chi_i\chi_j^{-1}) = \max\{l(\chi_i\chi_k^{-1}), l(\chi_k\chi_j^{-1})\}$ we deduce that

$$l(\chi_i\chi_k^{-1}) + l(\chi_k\chi_j^{-1}) > l(\chi_i\chi_j^{-1}).$$

Now, adding $m - 1$ on both sides we get $a_{ij} > a_{ik} + a_{kj}$. We conclude that $\nu_F(t_2) \geq a_{ij}$.

We observe that $\nu_F(m_{ik}m'_{kj})$, for $i<k<n$ is equal to $a_{kj}=l(\chi_k\chi_j^{-1})$ and we have $l(\chi_k\chi_j^{-1})\geq a_{ij}$. Note that $a_{nj}=l(\chi_n\chi_j^{-1})+m-1\geq a_{ij}$ from which we conclude that $\nu_F(t_3)\geq a_{ij}$. This shows that $\nu_F(t_1+t_2+t_3)\geq a_{ij}$ and we prove our lemma. □

Let $J_\chi(1,m)$ be the group of units of $\mathcal{J}(\mathcal{A}_\chi(1,m))$. We will need the structure of the representation

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}).$$

We follow a similar strategy to that from previous section. Let (a_{ij}) be the matrix $\mathcal{A}_\chi(1,m)$. Let $K_\chi(1,m)$ be the set of matrices (m_{ij}) such that $m_{ij}\in\mathfrak{p}_F$, for $i<j<n$, $m_{in}\in\mathfrak{o}_F$, for $i<n$, $m_{ii}\in 1+\mathfrak{p}_F$, for $i\leq n$, $m_{ij}\in{\mathfrak{p}_F}^{a_{ij}}$, for $i>j$. In the block form the group $K_\chi(1,m)$ is given by:

$$\begin{pmatrix}\mathcal{A}_\chi(1)\cap K_{n-1}(1) & \mathrm{Mat}_{(n-1)\times 1}(\mathfrak{o}_F)\\ \mathfrak{n} & 1+\mathfrak{p}_F\end{pmatrix},$$

where $\mathfrak{n}$ is the lattice $(\mathfrak{p}_F^{l(\chi_1\chi_n^{-1})},\dots,\mathfrak{p}_F^{l(\chi_{n-1}\chi_n^{-1})})$.

Lemma 4.1.5. *The set $K_\chi(1,m)$ is a normal subgroup of $J_\chi(1,m)$.*

Proof. We first check that $K_\chi(1,m)$ is closed under matrix multiplication. Let (m_{ij}) and (m'_{ij}) be two matrices from the set $K_\chi(1,m)$. Let $i<j<n$ the ij^{th} term is the sum of

$$t_1=m_{i1}m'_{1j}+m_{i2}m'_{2j}+\cdots+m_{ii}m'_{ij},$$

$$t_2=m_{ii+1}m'_{i+1j}+m_{ii+2}m'_{i+2j}+\cdots+m_{ij}m'_{jj}$$

and

$$t_3=m_{ij+1}m'_{j+1j}+\cdots+m_{in}m'_{nj}.$$

Observe that $\nu_F(m'_{kj})>0$, for $1<k\leq i$ and hence, $\nu_F(t_1)>0$. Now $\nu_F(m_{ik})>0$, for $i<k\leq j$ and we get that $\nu_F(t_2)>0$. Note that $\nu_F(m'_{kj})>0$, for $j<k\leq n$ and hence, the valuation $\nu_F(t_3)>0$. This shows that ij^{th}-term of the matrix product has positive valuation. The verifications on congruence conditions for the in^{th}-term are exactly the same as in Lemma 4.1.4. The existence of inverse for an element in $K_\chi(1,m)$ follows from Gaussian elimination.

Now we establish the normality of $K_\chi(1,m)$. The group $K_\chi(1,m)$ satisfies the Iwahori decomposition with respect to the subgroups $P_{(n-1,1)}$ and $M_{(n-1,1)}$. We also note that $K_\chi(1,m)\cap U_{(n-1,1)}$ is equal to $J_\chi(1,m)\cap U_{(n-1,1)}$ and $K_\chi(1,m)\cap\bar{U}_{(n-1,1)}$ is equal to $J_\chi(1,m)\cap\bar{U}_{(n-1,1)}$. To check the normality of $K_\chi(1,m)$ we have to check that $J_\chi(1,m)\cap M_{(n-1,1)}$ normalizes $K_\chi(1,m)$. This is equivalent to checking that $K_\chi(1,m)\cap M_{(n-1,1)}$ is a normal subgroup of $J_\chi(1,m)\cap M_{(n-1,1)}$.

We note that $J_\chi(1,m)\cap M_{(n-1,1)}=J_{\chi'}(1)\times\mathfrak{o}_F^\times$ where $\chi'=\boxtimes_{i=1}^{n-1}\chi_i$. Let p_1 be the projection of $J_\chi(1,m)\cap M_{(n-1,1)}$ onto $J_{\chi'}(1)$ and π_1 be the reduction mod $\mathfrak{p}_F$ map. Note that $K_\chi(1,m)\cap M_{(n-1,1)}$ is the kernel of $\pi_1\circ p_1$. □

From the above lemma, the group $K_\chi(1,m)$ is a normal subgroup of $J_\chi(1,m)$. We also note that $J_\chi(1,m)\cap\bar{U}_{(n-1,1)}$ is contained in $K_\chi(1,m)$. From this we conclude that $J_\chi(1,m)=K_\chi(1,m)J_\chi(1,m+1)$. From the Mackey decomposition we get that

$$\mathrm{res}_{K_\chi(1,m)}\,\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})\simeq\mathrm{ind}_{K_\chi(1,m)\cap J_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id}).$$

From the definition of $K_\chi(1,m)$ we get that $K_\chi(1,m)\cap J_\chi(1,m+1)=K_\chi(1,m+1)$ and

$$\mathrm{res}_{K_\chi(1,m)}\,\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})\simeq\mathrm{ind}_{K_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id}). \tag{20}$$

Lemma 4.1.6. *The group $K_\chi(1,m+1)$ is a normal subgroup of $K_\chi(1,m)$.*

Proof. The group $K_\chi(1,m)$ has the Iwahori decomposition with respect to $P_{(n-1,1)}$ and $M_{(n-1,1)}$, $K_\chi(1,m)\cap U_{(n-1,1)}$ is equal to $K_\chi(1,m+1)\cap U_{(n-1,1)}$ and $K_\chi(1,m)\cap M_{(n-1,1)}$ is equal to $K_\chi(1,m+1)\cap M_{(n-1,1)}$. We have to check that $u^-j(u^-)^{-1}$ and $u^-u^+(u^-)^{-1}$ belong to $K_\chi(1,m+1)$, for $u^-\in K_\chi(1,m)\cap \bar{U}_{(n-1,1)}$, $j\in K_\chi(1,m)\cap M_{(n-1,1)}$ and $u^+\in K_\chi(1,m)\cap U_{(n-1,1)}$.

We first consider the case $u^-j(u^-)^{-1}$. We can rewrite $u^-j(u^-)^{-1}$ as $j\{j^{-1}u^-j(u^-)^{-1}\}$. Since $j\in K_\chi(1,m)\cap M_{(n-1,1)} = K_\chi(1,m+1)\cap M_{(n-1,1)}$, it is enough to show that $j^{-1}u^-j(u^-)^{-1}$ belongs to the group $K_\chi(1,m+1)$. Let j and u^- be written in their block matrix form as follows.

$$j=\begin{pmatrix} J_1 & 0\\ 0 & j_1\end{pmatrix} \quad u^-=\begin{pmatrix} 1_{n-1} & 0\\ U^- & 1\end{pmatrix}$$

The conjugation $j^{-1}u^-j(u^-)^{-1}$ in its block form is given by

$$\begin{pmatrix} 1_{n-1} & 0\\ j_1^{-1}U^-J_1-U^- & 1\end{pmatrix}$$

Let $U^-=[u_1,u_2,\dots,u_{n-1}]$ and $J_1=(j_{ij})$. The k^{th} entry of the matrix U^-J_1 is the sum of

$$t_1=u_1j_{1k}+u_2j_{2k}+\cdots+u_{k-1}j_{k-1k},$$

$$t_2=u_kj_{kk},$$

and

$$t_3=u_{k+1}j_{k+1k}+\cdots+u_{n-1}j_{n-1k}.$$

If $l(\chi_k\chi_n^{-1})>1$, then valuation $\nu_F(u_tj_{tk})$, for $t<k$, is at least $l(\chi_t\chi_n^{-1})+m-1+1\geq l(\chi_k\chi_n^{-1})+m$. Hence, we have

$$\nu_F(t_1)\geq l(\chi_k\chi_n^{-1})+m-1.$$

The valuation $\nu_F(u_ta_{tk})$, for $k<t$, is at least $l(\chi_t\chi_n^{-1})+l(\chi_t\chi_k^{-1})+m-1>l(\chi_k\chi_n^{-1})+m-1$. This shows that $t_1+t_2+t_3\equiv t_2=u_kj_{kk}=u_k \mod {\mathfrak{p}_F}^{l(\chi_k\chi_n^{-1})+m}$. We note that $j_1^{-1}u-u\in\mathfrak{p}_F^{x+1}$, for any $u\in\mathfrak{p}_F^{x}$. hence the matrix

$$\begin{pmatrix} 1_{n-1} & 0\\ j_1^{-1}U^-J_1-U^- & 1\end{pmatrix}$$

is contained in $K_\chi(1,m+1)\cap\bar{U}_{(n-1,1)}$

Let us consider the conjugation $u^-u^+(u^-)^{-1}$. We write u^+ in the block form as

$$\begin{pmatrix} 1_{n-1} & U^+\\ 0 & 1\end{pmatrix}$$

The conjugated matrix $u^-u^+(u^-)^{-1}$ is given by

$$\begin{pmatrix} 1_{n-1}-U^+U^- & U^+\\ -U^-U^+U^- & U^-U^++1\end{pmatrix}.$$

Let $1_{n-1}-U^+U^-=(u_{ij})$. The valuation $\nu_F(u_{ij})\geq l(\chi_n\chi_j^{-1})$, for $i>j$, and $l(\chi_n\chi_j^{-1})$ is greater or equal to $l(\chi_i\chi_j^{-1})$. From this we conclude that $u^-u^+(u^-)^{-1}\in K_\chi(1,m+1)$. □

4.2. **Calculation of some stabilisers.** The inclusion map of $K_\chi(1,m)\cap\bar{U}_n$ in $K_\chi(1,m)$ induces an isomorphism of the quotient $K_\chi(1,m)/K_\chi(1,m+1)$ with the abelian group

$$\frac{K_\chi(1,m)\cap\bar{U}_{(n-1,1)}}{K_\chi(1,m+1)\cap\bar{U}_{(n-1,1)}}. \tag{21}$$

Hence the representation $\mathrm{ind}_{K_\chi(1,m+1)}^{K_\chi(1,m)}(\mathrm{id})$ splits into direct sum of characters say $\{\eta_k \mid \text{for } 1\leq k\leq p\}$. The group $J_\chi(1,m)$ acts on these characters and let $Z(\eta_k)$ be the $J_\chi(1,m)$-stabiliser of the character η_k. From Clifford theory we get that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id})\simeq\bigoplus_{\eta_{n_k}}\mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1,m)}(U_{\eta_{n_k}}), \tag{22}$$

where η_{n_k} is a representative for an orbit under the action of $J_\chi(1,m)$ and $U_{\chi_{n_k}}$ is an irreducible representation of the group $Z(\eta_{n_k})$. Since

$$J_\chi(1,m) = (J_\chi(1,m) \cap M_{(n-1,1)})K_\chi(1,m)$$

we get that $Z(\eta_k) = (Z(\eta_k) \cap M_{(n-1,1)})K_\chi(1,m)$.

The final step in our preliminaries is to understand the mod $\mathfrak{p}_F$ reduction of the group

$$Z(\eta_k) \cap M_{(n-1,1)}$$

for some non-trivial character η_k. The group $J_\chi(1,m) \cap M_{(n-1,1)}$ (which is $J_{\chi'}(1) \times \mathfrak{o}_F^\times$, for $\chi' = \boxtimes_{i=1}^{n-1}\chi_i$) acts on the quotient

$$\frac{K_\chi(1,m) \cap \bar{U}_{(n-1,1)}}{K_\chi(1,m+1) \cap \bar{U}_{(n-1,1)}} \tag{23}$$

by conjugation. Let j and u^- be two elements from $J_\chi(1,m) \cap M_{(n-1,1)}$ and $K_\chi(1,m) \cap \bar{U}$ respectively. We write the elements j and u^- written in their block diagonal form as

$$\begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \text{ and } \begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}$$

respectively. The map $u^- \mapsto \varpi_F^{-(m-1)}U^-$ induces an isomorphism between the group (23) and $\mathrm{Mat}_{1\times(n-1)}(k_F)$.

The map $u^- \mapsto \varpi_F^{-(m-1)}U^-$ gives an $J_\chi(1) \times \mathfrak{o}_F{}^\times$-equivariant map between $M_{1\times(n-1)}(k_F)$ and the group (23). We also have a $M_{(n-1,1)}(k_F)$-equivariant map between the group of characters of $M_{1\times(n-1)}(k_F)$ and the group $M_{(n-1)\times 1}(k_F)$ (see Lemma 2.0.7). Hence we obtain a $J_\chi(1) \times \mathfrak{o}_F{}^\times$ equivariant map between the group of characters of the quotient (23) and the group $M_{(n-1)\times 1}(k_F)$, where $J_\chi(1)$ acts through its (mod $\mathfrak{p}_F$) quotient $B_{(n-1)}(k_F) \times k_F^\times$ and the action is $(b,x)A = bAx^{-1}$. Hence to understand the group $Z(\eta_k) \cap M_{(n-1,1)}$ for non-trivial η_k, we first look at $Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ for some non-zero matrix A in $M_{(n-1)\times 1}(k_F)$.

Let p be the projection of $B_{n-1}(k_F) \times k_F^\times$ onto the diagonal torus

$$T_{n-1}(k_F) \times k_F^\times = T_n(k_F),$$

let p_i be the i^{th} projection of $T_n(k_F)$ onto $k_F^\times$. The centraliser $Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ of a non-zero matrix $A = [u_1, u_2, \ldots, u_{n-1}]^{\mathrm{T}}$ satisfies the following property: there exists a $j < n$ such that $p_j(p(t)) = p_n(p(t))$, for all $t \in Z_{B_{n-1}(k_F)\times k_F^\times}(A)$ (see [Nad17, Lemma 3.8]). This shows that for any non-trivial character η_{n_k}, $Z(\eta_{n_k}) \cap T_n$ satisfies the property that

$$p_j(t) \equiv p_n(t) \bmod \mathfrak{p}_F .$$

The character $\chi = \boxtimes_{i=1}^n \chi_i$ of $J_\chi(1)$ occurs with multiplicity one in the representation

$$\mathrm{ind}_{J_\chi(m)}^{J_\chi(1)}(\chi).$$

We denote by $U_m^0(\chi)$ the complement of χ in $\mathrm{ind}_{J_\chi(m)}^{J_\chi(1)}(\chi)$. We denote by $U_m(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(1)}^{K_n}\{U_m^0(\chi)\}.$$

4.3. **Elimination of atypical representations.**

Theorem 4.3.1. *Let $q_F > 3$. The irreducible sub representations of $U_m(\chi)$ are atypical. If $n = 3$ and $k_F > 2$, then the irreducible sub representations of $U_m(\chi)$ are atypical.*

Proof. We prove the theorem by using induction on the positive integers n and m. For $n = 1$ the representation $U_m(\chi)$ is trivial and the theorem is vacuously true. Let n be a positive integer greater than one. We assume that the theorem is proved for all positive integers less than n. We will use the induction hypothesis to show the theorem for n.

We note that $J_\chi(1,m)$ and $J_\chi(m)$ satisfy the Iwahori decomposition with respect to the parabolic subgroup $P_{(n-1,1)}$ and its Levi subgroup $M_{(n-1,1)}$; we have $J_\chi(1,m) \cap U_{(n-1,1)} = J_\chi(m) \cap U_{(n-1,1)}$ and $J_\chi(1,m) \cap$

$\bar{U}_{(n-1,1)} = J_\chi(m) \cap \bar{U}_{(n-1,1)}$. Hence, the representation $\mathrm{ind}_{J_\chi(m)\cap M_{(n-1,1)}}^{J_\chi(1,m)\cap M_{(n-1,1)}}(\chi)$ extends to a representation of $J_\chi(1,m)$ and this extension is given by

$$\mathrm{ind}_{J_\chi(m)}^{J_\chi(1,m)}(\chi).$$

If we denote by χ' the character $T_{n-1} = \boxtimes_{i=1}^{n-1}\chi_i$ of $\prod_{i=1}^{n-1} F^\times$, then we have

$$\mathrm{ind}_{J_\chi(m)\cap M_{(n-1,1)}}^{J_\chi(1,m)\cap M_{(n-1,1)}}(\chi) \simeq \mathrm{ind}_{J_{\chi'}(m)}^{J_{\chi'}(1)}(\chi') \boxtimes \chi_n.$$

We also have

$$\mathrm{ind}_{J_{\chi'}(m)}^{J_{\chi'}(1)}(\chi') \boxtimes \chi_n \simeq U_m^0(\chi') \boxtimes \chi_n \oplus \chi.$$

Combining the above isomorphisms we get that

$$\mathrm{ind}_{J_\chi(m)}^{K_n}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{K_n}\{U_m^0(\chi') \boxtimes \chi_n\} \bigoplus \mathrm{ind}_{J_\chi(1,m)}^{K_n}(\chi). \tag{24}$$

We will use the induction hypothesis to show that K_n-irreducible sub representations of

$$\mathrm{ind}_{J_\chi(1,m)}^{K_n}\{U_m^0(\chi') \boxtimes \chi_n\} \tag{25}$$

are atypical representations. By induction hypothesis any K_{n-1}-irreducible sub-representation of $U_m(\chi')$ occurs as sub-representation of some

$$i_{P_I}^{G_{n-1}}(\sigma)$$

where $[T_{n-1}, \chi']$ and $[M_I, \sigma]$ are two distinct inertial classes. We now get that irreducible sub representations of (25) occur as sub representations of

$$i_{P_{I'}}^{G_n}(\sigma \boxtimes \chi_n)$$

where I' is obtained from I by adding 1 at the end of the ordered partition I of $n-1$. If $I \neq (1, 1, \dots, 1)$ then the Levi sub-groups $M_{I'}$ and T_n are not conjugate and hence the inertial classes $[M_{I'}, \sigma \boxtimes \chi_n]$ and $[T_n, \chi]$ are distinct inertial classes and this proves our claim in this case.

Now, we assume that $M_I = T_{n-1}$ and $\sigma = \boxtimes_{i=1}^{n-1}\sigma_i$ be the tensor factorisation of the character σ of T_{n-1}. Since the inertial classes $[T_{n-1}, \chi']$ and $[T_{n-1}, \sigma]$ are distinct we get a character χ_t occurring with non-zero multiplicity in the multi-set $\{\chi_1, \chi_2, \dots, \chi_{n-1}\}$ but with a different multiplicity in the multi-set $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$. Adding the character χ_n to both multi-sets above keeps the multiplicities of the character χ_t distinct and this shows that $[T_n, \chi]$ and $[T_n, \sigma \boxtimes \chi_n]$ are different inertial classes.

This shows that any typical representation must occur as a sub-representation of

$$\mathrm{ind}_{J_\chi(1,m)}^{K_n}(\chi).$$

The character χ occurs with multiplicity one in the representation $\mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi)$. We denote by $U_{1,m}^0(\chi)$ the complement of the character χ in $\mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi)$. We denote by $U_{1,m}(\chi)$ the representation

$$\mathrm{ind}_{J_\chi(1)}^{K_n}\{U_{1,m}^0(\chi)\}.$$

We first note that

$$U_m(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{K_n}\{U_m^0(\chi') \boxtimes \chi_n\} \oplus U_{1,m}(\chi). \tag{26}$$

We already showed that the K_n-irreducible sub representations of the first summand on the right-hand side of the equation (26) are atypical. We now show that K_n-irreducible sub representations of $U_{1,m}(\chi)$ are atypical and this proves the main theorem.

We first note that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}\{\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1,m)}(\mathrm{id}) \otimes \chi\}.$$

Using the decomposition (22) we get that

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \bigoplus_{\eta_{n_k}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}.$$

Recall that η_{n_k} is a representative for the orbit under the action of the group $J_\chi(1,m)$ on the set of characters $\{\eta_k|\ 1 \leq k \leq p\}$ of the group $K_\chi(1,m)/K_\chi(1,m+1)$, and $Z(\eta_{n_k})$ is the $J_\chi(1,m)$-stabiliser of the character η_{n_k}. There is exactly one orbit consisting of the identity character and hence

$$\mathrm{ind}_{J_\chi(1,m+1)}^{J_\chi(1)}(\chi) \simeq \mathrm{ind}_{J_\chi(1,m)}^{J_\chi(1)}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}. \tag{27}$$

Consider the representation

$$\mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}$$

for some representative $\eta_{n_k} \neq \mathrm{id}$. Now, recall that $Z(\eta_{n_k}) \cap T_n$ is a subgroup of $T_n(\mathfrak{o}_F) = \prod_{i=1}^n \mathfrak{o}_F{}^\times$, and there exists a positive integer $j < n$ such that $p_j(t) \equiv p_n(t) \bmod \mathfrak{p}_F$, for all $t \in Z(\eta_{n_k})$. Let κ be a character of $F^\times$ such that κ is ramified and $1+\mathfrak{P}_F$ is contained in the kernel of κ. Let χ^κ be the character

$$\chi_1 \boxtimes \chi_2 \boxtimes \chi_j\kappa \boxtimes \cdots \boxtimes \chi_n\kappa^{-1}.$$

We observe that $\mathrm{res}_{Z(\eta_{n_k})}(\chi) = \mathrm{res}_{Z(\eta_{n_k})}(\chi^\kappa)$ and hence

$$\mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi^\kappa\}. \tag{28}$$

From the above paragraph we get that

$$U^0_{1,m+1}(\chi) \simeq U^0_{1,m}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{J_\chi(1)}\{U_{\eta_{n_k}} \otimes \chi\}.$$

and from the above identity we conclude that

$$U_{1,m+1}(\chi) \simeq U_{1,m}(\chi) \bigoplus_{\eta_{n_k}\neq \mathrm{id}} \mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{U_{\eta_{n_k}} \otimes \chi\}. \tag{29}$$

From the equation (28) we get that

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{U_{\eta_{n_k}} \otimes \chi\} \simeq \mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{U_{\eta_{n_k}} \otimes \chi^\kappa\}.$$

If we choose κ such that $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ are two distinct inertial classes, then we can conclude that irreducible sub representations of

$$\mathrm{ind}_{Z(\eta_{n_k})}^{K_n}\{U_{\eta_{n_k}} \otimes \chi\}$$

are atypical. Hence, using the identity (29) recursively we get that irreducible sub representations of $U_{1,m}(\chi)$ are atypical representations, for all positive integers m.

To prove the theorem we have to justify that we can choose a character κ as in the previous paragraph. Now for any character κ non-trivial on $\mathfrak{o}_F{}^\times$ (such a character exists since $q_F > 2$) and trivial on $1+\mathfrak{p}_F$, the equality of the inertial classes $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ implies the equality of multiplicities of χ_j in the multi-sets $\{\chi_1, \chi_2, \dots, \chi_n\}$ and $\{\chi_1, \chi_2, \dots, \chi_j\kappa, \dots, \chi_n\kappa^{-1}\}$. The equality of multiplicities implies $\chi_j\chi_n^{-1} = \kappa$. If $q_F > 3$, then we have at least two non-trivial tame characters and hence we can choose κ distinct from a possibly tame character $\chi_j\chi_n^{-1}$.

If $l(\chi_i) = 1$, for $1 \leq i \leq n-1$, and $l(\chi_n) > 1$, then we can always find κ such that $[T, \chi]$ and $[T, \chi^\kappa]$ are distinct inertial classes. We note that the induction hypothesis here is supplied by depth-zero case stated as Theorem 2.0.8.

Consider the case where $n = 3$, $q_F = 3$ and η is the non-trivial character of $k_F^\times$. We have the character $\chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3$ of T_3. Assume that there exists $i \neq j$ and $i, j \in \{1,2,3\}$ such that $\chi_i\chi_j^{-1} = \eta$, with $l(\eta) = 1$. If such a pair (i, j) does not exist, then our present proof goes through. Now, twisting with the character χ_j if necessary, and permuting the characters χ_1, χ_2 and χ_3 if necessary, we may assume that $\chi_1 = \mathrm{id}$, $\chi_2 = \eta$. This arrangement still satisfies the condition (15). If $l(\chi_3) = 1$, then we are depth-zero case and we refer to Theorem 2.0.8 for a proof of this result. If $l(\chi_3) > 1$, then K_2 irreducible subrepresentations of $U_m(\chi_1 \boxtimes \chi_2)$ are atypical, for $m \geq 1$ (we refer to 2.0.8 or [BM02, Appendix] for a proof). Now, the above proof shows that irreducible subrepresentations of $U_{1,m}(\chi)$ are atypical, for $m \geq 1$. This shows the theorem in the present case. □

The pair $(J_\chi(1), \chi)$ is a Bushnell-Kutzko type for the inertial class s (see [BK99, Section 8]). From the above theorem we deduce the following result:

Theorem 4.3.2. *Let $q_F > 3$ if $n > 3$ and $q_F > 2$ if $n \in \{2, 3\}$. Let τ be a typical representation for the inertial class $s = [T_n, \chi]$ then τ is a subrepresentation of $\mathrm{ind}_{J_\chi(1)}^{K_n}(\chi)$. Moreover we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_n}(\tau, i_{B_n}^{G_n}(\chi)) = \dim_{\mathbb{C}} \mathrm{Hom}_{K_n}(\tau, \mathrm{ind}_{J_\chi(1)}^{K_n}(\chi))$$

Remark 4.3.3. *When $\#k_F = 2$ and $n = 2$ Henniart showed in* [BM02]*[A.2.6, A.2.7] that the Bushnell-Kutzko type for the inertial class $s = [T_2, \chi_1 \boxtimes \chi_2]$, $\chi_1\chi_2^{-1} \neq \mathrm{id}$ has two typical representations one given by*

$$\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_2(\mathfrak{o}_F)}(\chi)$$

and the other representation turns out to be the complement (it follows from [Cas73, Proposition 1(b)] *that there is a unique complement) of $\mathrm{ind}_{J_\chi(1)}^{\mathrm{GL}_2(\mathfrak{o}_F)}(\chi)$ in $\mathrm{ind}_{J_\chi(2)}^{\mathrm{GL}_2(\mathfrak{o}_F)}(\chi)$. But for $\#k_F > 2$ and $n > 3$ we expect that typical representations are precisely the irreducible sub representations of*

$$\mathrm{ind}_{J_\chi(1)}^{K_n}(\chi).$$

For $\#k_F = 2$ and $n > 2$ a typical representation may not be contained in the above representation as shown by Henniart for the case of $\mathrm{GL}_2(F)$.

5. Typical representations for $\mathrm{GL}_3(F)$

Any inertial class of the group G_3 belongs to one of the following classes (see Lemma 2.0.6):

1 $[G_3, \sigma]$, where σ is a cuspidal representation of G_3
2 $[G_2 \times G_1, \sigma \boxtimes \chi]$, where σ is a cuspidal representation of G_2 and χ is any character of $F^\times$.
3 $[T_3, \chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3]$, where χ_1, χ_2 and χ_3 are three characters of $F^\times$.

Typical representations for any inertial class of the form $s = [G_3, \sigma]$ are classified in the work of Paškūnas [Pas05]. Up to isomorphism there exists a unique typical representation for s. Similarly, the theorem 3.3.3 shows that, if $q_F > 2$, then up to isomorphism there exists a unique typical representation for any inertial class of the form $[G_2 \times G_1, \sigma \boxtimes \chi]$. If $q_F > 2$, then for any inertial class $s = [T_3, \chi]$ we showed that any typical representation occurs as a sub representation of $\mathrm{ind}_{J_\chi(1)}^{K} \chi$. The pair $(J_\chi(1), \chi)$ is a Bushnell–Kutzko type for s. Moreover, we also have the multiplicity result Theorem 4.3.2. We conclude the following result for $\mathrm{GL}_3(F)$.

Theorem 5.0.1. *Let $q_F > 2$ and s be any inertial class of G_3. Typical representations for s are precisely the irreducible subrepresentations of $\mathrm{ind}_{J_s}^{K_3} \lambda_s$, where (J_s, λ_s) is a Bushnell–Kutzko type for s.*

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