Doctoral School in Mathematics

Simple objects in the heart of a t-structure

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Introduction

Historically, the study of modules over finite dimensional algebras has started with the study of the finite dimensional modules over these algebras. This is sufficient when dealing with a finite dimensional algebra of finite representation type, where there are only finitely many indecomposable modules of finite length (up to isomorphism). Any module is the direct sum of modules of finite length and, moreover, this decomposition is unique (up to isomorphisms) by means of the famous Krull-Remak-Schmidt-Azumaya Theorem. When dealing with algebras of infinite representation type, it has been proven by Auslander in [8], that indecomposable modules of infinite length occur in this case. The first example is the Kronecker algebra, ie. the path algebra of the quiver:

$$\bullet \rightrightarrows \bullet$$

This is a *tame hereditary* algebra of infinite representation type and its module category has been extensively investigated. We can describe the finite length modules in the following picture:

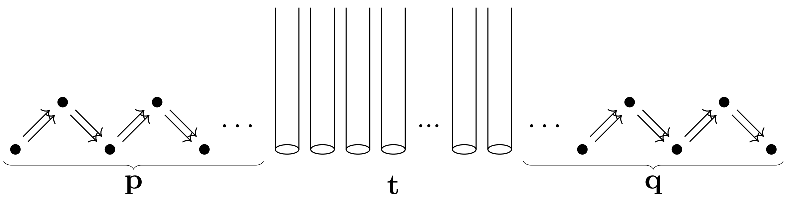


Figure 1: Auslander-Reiten quiver of the category of finite length modules over the Kronecker algebra.

Here **p** and **q** denote, respectively, the *preinjective* and the *preprojective* component, where the indecomposable projective (resp. injective) modules belong to. In the middle, **t** denotes a sincere, stable and separating family of tubes, which are connected and uniserial length categories themselves, consisting of *regular* modules. Examples of infinite dimensional modules occurring in this category are the Prüfer modules, the adic modules and the generic module G. The first two kind of modules somehow resemble the Prüfer and the adic abelian groups and the generic G is defined as the unique indecomposable infinite dimensional module with finite length over its endomorphism ring.
The study of pure-injective modules over a finite dimensional algebra is crucial for the problem of describing infinite dimensional modules. Indeed, the infinite dimensional modules of this type are the first that have been widely studied in literature, see for instance [22, 52, 70]. Pure-injective modules have a wide range of nice properties coming from different point of views: first

of all, they have a cogenerating property, indeed any module can be (purely) embedded in a pure-injective module, furthermore, they give rise to the *Ziegler spectrum*, a topological space which is relevant for studying of the representation theory of a ring from a model-theoretic perspective.

The classification of pure-injective modules over a tubular algebra is the main motivation for the present work. The category of modules over a tubular algebra has been widely studied in literature. In 1984, Ringel gave a detailed description of the structure of the Auslander-Reiten quiver of the category of finite dimensional modules over a tubular algebra Λ (see [56]). Afterwards, the infinite dimensional Λ-modules has been studied by different authors, see eg. [2, 31, 32, 53, 55].

Tubular algebras belong to a wider class of algebras of infinite representation type, called concealed canonical algebras. The category of modules over a concealed canonical algebra, as stated by Lenzing and de La Peña in [47], is characterized by the presence of a family of standard tubes in its Auslander-Reiten quiver, which is stable, sincere and separating.

In the specific case of tubular algebras, there is a countable number of tubular families $\mathbf{t}_w$, $w \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$, and the indecomposable Λ-modules can belong either to a tubular family or to the, so called, *preprojective* or *preinjective* component, respectively denoted by $\mathbf{p}_0$ and $\mathbf{q}_\infty$.

For any rational number $w \in \mathbb{Q}_{\geq 0}$, there is a trisection of the finite dimensional indecomposable Λ-modules, $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$, where $\mathbf{p}_w$ consists of the preprojective component $\mathbf{p}_0$ together with all the tubular families $\mathbf{t}_\alpha$, for $\alpha < w$, and $\mathbf{q}_w$ consists of the preinjective component $\mathbf{q}_\infty$ together with all the tubular families $\mathbf{t}_\beta$, for $\beta > w$. If w is irrational, then we only have a bisection $(\mathbf{p}_w, \mathbf{q}_w)$, with $\mathbf{p}_w$ and $\mathbf{q}_w$ as above.

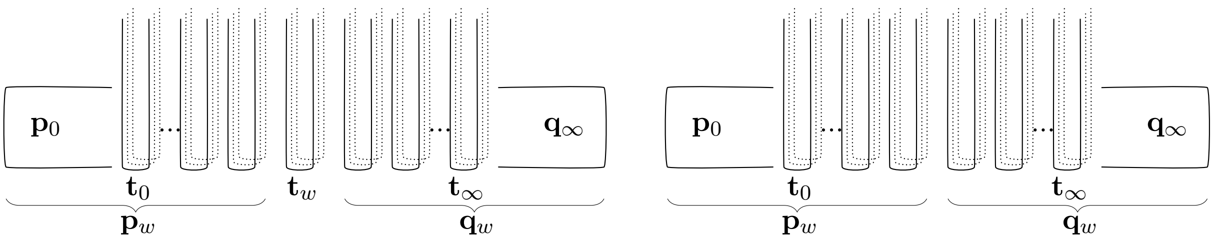


Figure 2: Trisection of mod-Λ for w rational (left) and bisection of mod-Λ for w irrational (right).

Consider the torsion class $\mathcal{B}_w = {}^\circ\mathbf{p}_w$, cogenerated by $\mathbf{p}_w$, and the torsionfree class $\mathcal{C}_w = \mathbf{q}_w{}^\circ$, generated by $\mathbf{q}_w$. We define a new class $\mathcal{M}_w$ as the intersection of $\mathcal{B}_w$ and $\mathcal{C}_w$ and, following [53], modules in $\mathcal{M}_w$ are said of slope w.

If w is rational, all the finite dimensional modules in $\mathbf{t}_w$ belong to $\mathcal{M}_w$ and if w is irrational we only have infinite dimensional modules in $\mathcal{M}_w$.

The problem of classifying pure-injective modules over a tubular algebra has been of particular interest during the last years. Starting from an article by Ringel in 2005, see [54], this subject has been approached from the point of view of tilting theory by Angeleri Hügel and Kussin, in [3], and by Harland and Prest in [32], from a model-theoretic perspective.

In [3], the authors have outlined a partial classification of the pure-injective modules in Mod-Λ, which is satisfactory for modules of rational slope. For the irrational slope case, say w, it is known that there exists a unique cotilting module of slope w, denoted by $\mathbf{W}_w$, and the pure-injective modules of this slope belong to $\mathrm{Prod}(\mathbf{W}_w)$, ie. they are direct summands of product of copies of $\mathbf{W}_w$.

In this thesis, we approach this problem from the point of view of tilting/cotilting theory, more precisely we consider the torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$, for w irrational, where $\mathcal{C}_w = \mathbf{q}_w{}^\circ$, and we look at the category $\mathcal{A}_w$ which is the heart of a t-structure arising from this torsion pair. *t-structures* are the analogous of torsion pairs in a triangulated setting. This concept has been introduced in 1982 by Beilinson, Bernstein and Deligne, in [13]. The *heart* of a t-structure is an abelian subcategory of the initial triangulated category. In the setting of the derived category of a module category, Happel, Reiten and Smalø, in [30], have developed a theory connecting tilting theory and t-structures. Starting from a torsion pair generated by a tilting module over a ring R, they defined a t-structure in $\mathcal{D}^b(\text{Mod-}R)$ whose heart is an abelian category closely related to Mod-R. This theory has had several developments in the last years. Colpi, Gregorio and Mantese, in [17], proved that the heart of a t-structure arising from a torsion pair in Mod-Λ is a Grothendieck category if and only if the torsion pair is cogenerated by a cotilting module, hereafter called *cotilting torsion pair.* Afterwards, this result has been generalized by Čoupek and Šťovíček, in [20], for a general Grothendieck category $\mathcal{G}$.

The heart $\mathcal{A}_w$, considered above, is a locally coherent Grothendieck category and its injective objects correspond to the pure-injective modules of slope w over the tubular algebra Λ. This is true since the torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ is actually a cotilting torsion pair and the cotilting module cogenerating it becomes an injective cogenerator for $\mathcal{A}_w$. This moves our problem from classifying pure-injectives in Mod-Λ to classifying injectives in the category $\mathcal{A}_w$.

In a Grothendieck category $\mathcal{G}$, the direct sum of the injective envelopes of all the simple objects in $\mathcal{G}$ forms an injective cogenerator for $\mathcal{G}$, therefore it is immediate to see that in order to classify the indecomposable injective objects in a Grothendieck category $\mathcal{G}$ one should first focus on the simple objects in $\mathcal{G}$. In this sense, we will use a theorem that relates the simple objects in the heart of a t-structure coming from a torsion pair to some peculiar objects in the original category. Indeed, it has been proven in [1] that the simple objects in the heart $\mathcal{A}$ of a t-structure arising from a torsion pair $(\mathcal{Q}, \mathcal{C})$ in a Grothendieck category are precisely the objects $S \in \mathcal{A}$ of the form $S = Y[1]$ with Y torsionfree, almost torsion, or $S = Q$ with Q torsion, almost torsionfree.

Torsionfree, almost torsion objects in $\mathcal{G}$ are torsionfree objects whose proper quotients are torsion, and torsion, almost torsionfree objects are defined dually. Somehow, one can think about torsionfree almost torsion objects as objects very close to the "border" of the torsion pair.

As a first application of the latter characterization of simples in the heart, we focus our attention to the category of modules over the Kronecker algebra, mentioned at the beginning. In [5], Angeleri Hügel and Sánchez have provided, for this category, a complete classification of all the cotilting torsion pairs, which are parametrized by subsets P of a noncommutative curve of genus zero $\mathbb{X}$. Moreover, this classification actually resembles the classification of cotilting torsion pairs in the category of modules over a commutative noetherian ring (cf. [4]). For each heart arising from a cotilting torsion pair in the category of modules over the Kronecker algebra, we will describe its *atom spectrum.* This spectrum has been first introduced by Kanda in [35]

for a general abelian category and it is a generalization of the prime spectrum for commutative rings. Accordingly, it has a structure of topological space and, for a Grothendieck category $\mathcal{G}$, it is strongly related to the spectrum of the indecomposable injective objects. Indeed, as proven in [35], there is an injective map between the atom spectrum and the spectrum of the indecomposable injectives in $\mathcal{G}$. This correspondence becomes a bijection if $\mathcal{G}$ is a locally noetherian category.
The result we will achieve is the following:

Theorem (Theorem 5.4.4)**.** *Let $\mathcal{G} = \text{Mod-}\Lambda$, with Λ the Kronecker algebra. Consider $P \subseteq \mathbb{X}$ and let C_P be the infinite dimensional cotilting module, together with its corresponding cotilting torsion pair $(\mathcal{Q}_P, \mathcal{C}_P)$ (as in Table 5.1). Consider the heart $\mathcal{A}_P$ of the t-structure arising from the cotilting torsion pair $(\mathcal{Q}_P, \mathcal{C}_P)$. We have the following:*

- *If $P \subsetneq \mathbb{X}$, then:*

$$\text{ASpec}(\mathcal{A}_P) = \overline{G[1]} \cup \{\overline{S_x} \mid S_x \textit{ simple regular in } \bigcup_{x\in P} \mathcal{U}_x\} \cup \\ \cup \{\overline{S_x[1]} \mid S_x \textit{ simple regular in } \bigcup_{x\in \bar{P}} \mathcal{U}_x\}.$$

- *If $P = \mathbb{X}$, then:*

$$\text{ASpec}(\mathcal{A}_{\mathbb{X}}) = \overline{G[1]} \cup \{\overline{S_x} \mid S_x \textit{ simple regular } \Lambda\textit{-module}\}.$$

For all the hearts arising from cotilting torsion pairs in the module category over the Kronecker algebra, there is a bijection between their atom spectrum and the set of indecomposable injectives. This is expected for the case $P = \varnothing$, in which the cotilting module cogenerating the torsion pair is the so called Reiten-Ringel tilting module, ie. $\mathbf{W} = G \oplus \bigoplus_{x\in\mathbb{X}} S_x^{\infty}$, where G is the generic module and S_x^{∞} are Prüfer modules. Contrary to this, the result is somehow surprising for all the other cases; indeed, for the cotilting module $\mathbf{W}$, the associated heart is equivalent to the category of quasi-coherent sheaves over a noncommutative regular projective curve $\mathbb{X}$ and therefore it is a locally noetherian Grothendieck category. In all the other cases, the associated heart is not locally noetherian but it turns out to be a, so called, *Gabriel category*, ie. a category with Gabriel dimension, and it has been proven in [65] that the bijection between the atom spectrum and the spectrum of indecomposable injectives holds for these categories too.
Going back to the problem of classifying pure-injective modules over a tubular algebra Λ, we proceed by setting our approach in a more geometrical environment. The category of modules over a tubular algebra Λ is strictly related to the category of quasi-coherent sheaves over a *tubular curve* $\mathbb{X}$, denoted by Qcoh$\mathbb{X}$.
In [47], Lenzing and de la Peña have proved that there is a derived equivalence between the category Qcoh$\mathbb{X}$, where $\mathbb{X}$ is a noncommutative regular projective curve of genus zero over a field k, and the category Mod-Λ, where Λ is a concealed canonical algebra. Indeed, starting from a tilting sheaf T in Qcoh$\mathbb{X}$, the algebra Λ is isomorphic to End(T), and moreover every concealed canonical algebra arises in this way. In particular, this holds for tubular algebras, and

the curves $\mathbb{X}$ such that the corresponding algebra is tubular are called *tubular curves*.

Noncommutative curves of genus zero over a field k are a generalization of the weighted projective lines introduced by Geigle and Lenzing in 1987 (see [26]). The category of quasi-coherent sheaves over these kind of curves has been widely studied from different point of views, see eg. [2, 42, 39, 48] for some general theory, [40, 44] for the specific case of tubular curves, [41] for a K-theoretical perspective and [67] for the Calabi-Yau properties.

As the derived equivalence mentioned above may suggest, the category of coherent sheaves over a tubular curve $\mathbb{X}$, coh$\mathbb{X}$, has a similar description to the category of modules over a tubular algebra Λ, indeed, also in this case classes of sheaves can be distinguished by a notion of *slope*, which is a rational number or infinity. Passing to the infinite dimensional world, the category of quasi-coherent sheaves over $\mathbb{X}$ is the direct limit closure of the category of coherent sheaves over $\mathbb{X}$, denoted by coh$\mathbb{X}$. In [2], the authors have extended the notion of slope of a sheaf to the non-coherent ones, which, in this case, can be an element of the real numbers or infinity.

The advantages of working in this geometrical framework are numerous. First of all, in coh$\mathbb{X}$ the slope is defined as the ratio of *degree* and *rank*, which are two linear forms over the Grothendieck group of coh$\mathbb{X}$, and this cleary makes computations easier. Moreover, the category of modules over a tubular algebra Λ is not hereditary, but coh$\mathbb{X}$ it is a hereditary category and this simplifies the theory from the homological point of view. Last but not least, all the indecomposable sheaves in coh$\mathbb{X}$ fall into tubular families $\mathbf{t}_w$, $w \in \mathbb{Q} \cap \{\infty\}$, therefore we do not have the preprojective and the preinjective components as in Mod-Λ, simplifying the context of the work.

Furthermore, it follows by a criterion of Jensen and Lenzing (see [51, Theorem 5.4]) that starting with a pure-injective modules over a tubular algebra Λ, we obtain a pure-injective sheaf in Qcoh$\mathbb{X}$. This means that describing pure-injectives in Mod-Λ is the same as describing pure-injectives in Qcoh$\mathbb{X}$.

In Qcoh$\mathbb{X}$ the slope of a sheaf is defined as in Mod-Λ, ie. a sheaf has slope $w \in \mathbb{R} \cup \{\infty\}$ if it belongs to the class $\mathcal{M}_w = \mathcal{B}_w \cap \mathcal{C}_w$, where $\mathcal{B}_w = {}^\circ\mathbf{p}_w$ and $\mathcal{C}_w = \mathbf{q}_w{}^\circ$, as before. If w is rational, then the tubular family $\mathbf{t}_w$ falls entirely into the class $\mathcal{M}_w$ and if w is irrational $\mathcal{M}_w$ consists only of (quasi-coherent) non-coherent sheaves.

As we did for the category of modules over a tubular algebra, we can consider the torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ in Qcoh$\mathbb{X}$. The heart of the t-structure arising from this torsion pair is well known for the rational slope case, indeed it is actually equivalent to the category Qcoh$\mathbb{X}'$, where $\mathbb{X}'$ is another tubular curve (if $\mathbb{X}$ is a curve over an algebraically closed field k, then $\mathbb{X}' = \mathbb{X}$). Dealing with an irrational number w is a totally different story, indeed the heart $\mathcal{A}_w$ in this case is a locally coherent Grothendieck category and not much more is known.

The purpose of the last part of this thesis is to construct a quasi-coherent sheaf over a tubular curve $\mathbb{X}$ of a prescribed irrational slope w such that it becomes simple in the heart $\mathcal{A}_w$. In order to do this, a complex procedure that entwines the properties of continued fractions and universal extensions has been outlined by the author, together with Jan Šťovíček. In more details, the continued fraction expansion of the irrational number w is a way to approximate the number itself via an increasing sequence of rational numbers converging to w. Along with this, the universal extensions provide a functorial way to produce extensions of two fixed coherent sheaves in such a way the slope of the middle sheaf in the short exact sequence is one of the rational numbers in the continued fraction expansion of w. This procedure provides the following result:

Proposition (Proposition 8.1.8)**.** *Let $w = [n_0; n_1, n_2, \dots]$ be a positive irrational number together with its continued fraction form. Let L be the structure sheaf and S_x be simple sheaf in a tube of maximal rank. Set $P_{-2} = L$ and $P_{-1} = S_x$. Then, there exists a sequence of monomorphism:*

$$P_0 \hookrightarrow P_2 \hookrightarrow P_4 \hookrightarrow \dots \hookrightarrow P_{2i} \hookrightarrow P_{2i+2} \hookrightarrow \dots \tag{1}$$

where P_{2i}, for $i \geq 0$, is obtained as the universal extension of P_{2i-2} with respect to P_{2i-1}, iterated n_{2i} times. Moreover, the direct union $P = \varinjlim P_{2i}$ is a quasi-coherent non-coherent sheaf of slope w.

The last statement of the Proposition is a consequence of a result of Reiten and Ringel (see [53]). The crucial part in proving that the slope of this non-coherent sheaf P is precisely w comes from the fact that the slope of the P_{2i}'s in the sequence increases according to the continued fraction approximation of w.

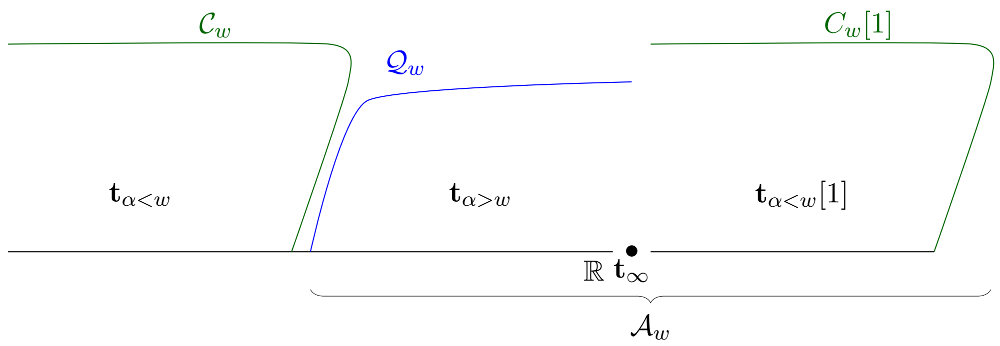


Figure 3: Heart of the t-structure arising from the torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$.

As mentioned above, for an irrational number w we can associate a torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ whose related heart $\mathcal{A}_w$ is a locally coherent Grothendieck category and, inside this heart, we prove the following:

Proposition (Proposition 8.2.1)**.** *The non-coherent sheaf $P = \varinjlim P_{2i}$ defined as in Proposition 8.1.8 becomes a simple object in the category $\mathcal{A}_w$.*

Summary of content

Chapter 1 is a brief overview of the basic notions we are going to use throughout the thesis. First, we give the definition and properties of a Grothendieck category. Passing through the notion of a torsion pair we consider a triangulated category and define t-structures. Afterwards, we focus our attention on purity in Grothendieck categories, defining pure-injective objects and specializing this definition for the case of a module category. The last part of the chapter is

devoted to localizations, together with their link to the theory of torsion pairs, and the properties of Gabriel categories.
In Chapter 2 we move to tilting theory. The definition of tilting and cotilting objects in a Grothendieck category is given. Focusing more on the latter ones, we will describe the properties of the heart of a t-structure when it is obtained from a torsion pair cogenerated by a cotilting object. At the end of the chapter, we describe the simples in this heart as torsionfree, almost torsion and, dually, torsion, almost torsionfree, objects in the original Grothendieck category.
Chapter 3 is dedicated to the description of the category of modules over a concealed canonical algebra. First of all, the category is described in terms of separating tubular families and from the point of view of the morphisms between modules. Subsequently, we specialize the description of the category of modules over a tubular algebra Λ, describing the multitude of torsion pairs in this category, the tilting and cotilting Λ-modules and classifying partially the pure-injective modules in it.
The fourth Chapter is completely dedicated to the description of the atom spectrum of an abelian category, as defined in [35]. We give the definition as set of equivalence classes of *monoform* objects and outline some properties. Afterwards, we focus on the topological properties of this spectrum and on the partial order that arises in it from its topological structure.
Chapter 5 is completely about the Kronecker algebra. The main goal of this chapter is to describe the atom spectrum of the different hearts arising from the cotilting torsion pairs in the category of modules over the Kronecker algebra. Doing so, we give a complete description of the simple objects of these hearts and, consequently, a clear classification of the indecomposable injective objects in these hearts.
In Chapter 6 we give an axiomatic description of the category of quasi-coherent sheaves over a noncommutative curve, specializing in the case of a curve of genus zero, ie. tubular curve. We describe the link between this category and the category of modules over a concealed canonical algebra.
Chapter 7 is devoted to the illustration of the main tools we will use for the construction of the sheaf of irrational slope, that is our candidate for becoming simple in the corresponding heart. Indeed, first we give the definition of a continued fraction and exhibit its properties. Afterwards, we focus on universal and co-universal extensions, following [47].
In Chapter 8 is the main core of the thesis. We fix an irrational number w and construct a quasi-coherent sheaf that becomes simple in the heart $\mathcal{A}_w$ of the t-structure arising from the torsion pair cogenerated by the only cotilting module of slope w. Afterwards, we prove that all the simple objects in the heart $\mathcal{A}_w$ come from quasi-coherent sheaves precisely of slope w in the original category.

Chapter 1

Preliminaries

This chapter is devoted to describing the general theory on which we develop the rest of the thesis. First, we define Grothendieck categories and we describe their most relevant properties, focusing on locally coherent and locally noetherian Grothendieck categories. Afterwards, we outline the theory of torsion pairs in a Grothendieck category $\mathcal{G}$ and we introduce t-structures in triangulated categories. After a quick summary of approximation theory and purity in Grothendieck categories, we focus on localizations, introducing quotient categories and categories with Gabriel dimension.

1.1 Grothendieck categories

Let $\mathcal{C}$ be an abelian category. Let us introduce three axioms defined by Grothendieck in [29]:

(AB3) For every set $\{A_i \mid i \in I\}$ of objects in $\mathcal{C}$, the coproduct $\bigoplus_{i\in I} A_i$ exists in $\mathcal{C}$.
(AB4) $\mathcal{C}$ satisfies (AB3) and the coproduct of a family of monomorphisms is a monomorphism (ie. the coproduct functor is exact).
(AB5) $\mathcal{C}$ satisfies (AB3) and filtered colimits of exact sequences are exact.

and their duals:

(AB3*) For every set $\{A_i \mid i \in I\}$ of objects in $\mathcal{C}$, the product $\prod_{i\in I} A_i$ exists in $\mathcal{C}$.
(AB4*) $\mathcal{C}$ satisfies (AB3*) and the product of a family of epimorphisms is a epimorphism (ie. the product functor is exact).
(AB5*) $\mathcal{C}$ satisfies (AB3*) and filtered limits of exact sequences are exact.

We have the following lemma:

Lemma 1.1.1. *[61, Corollary IV.8.3] $\mathcal{C}$ is* (AB3) *(resp.* (AB3*)*) if and only if $\mathcal{C}$ is cocomplete (resp. complete).*

Definition 1.1.2. An object G in $\mathcal{C}$ is a *generator* in $\mathcal{C}$ if the functor $\mathrm{Hom}_{\mathcal{C}}(G, -)$ is faithful (equivalently, if $\mathrm{Hom}_{\mathcal{C}}(G, C) \neq 0$ for any nonzero $C \in \mathcal{C}$)

we have that:

Proposition 1.1.3. *[61, Proposition IV.6.2] Let $\mathcal{C}$ be an* (AB3) *abelian category. If G is a generator of $\mathcal{C}$, then for any object $C \in \mathcal{C}$ there is an epimorphism $G^{(I)} \to C$, for some index set I.*

Definition 1.1.4. An abelian category is called *Grothendieck category* if it has a generator and satisfies (AB5).

Proposition 1.1.5. *[61, Proposition V.1.1] If $\mathcal{C}$ is a cocomplete abelian category, then the following are equivalent:*

(1) *$\mathcal{C}$ is* (AB5)

(2) *Given $X \in \mathcal{C}$, $Y \subseteq X$ and a directed system $\{A_i \mid i \in I\}$, with $A_i \subseteq X$, we have:*

$$\left(\sum_I A_i\right) \cap Y = \sum_I (A_i \cap Y)$$

(3) *Given a morphism $\varphi\colon X \to X'$ in $\mathcal{C}$ and a directed system $\{B_i \mid i \in I\}$, with $B_i \subseteq X'$, we have:*

$$\varphi^{-1}\left(\sum_I B_i\right) = \sum_I \varphi^{-1}(B_i)$$

From now on, $\mathcal{G}$ will denote a Grothendieck category.

Proposition 1.1.6. *[61, Proposition IV.6.6] Let $X \in \mathcal{G}$. The class of all subobjects of X and the class of all quotient objects of X are actually sets.*

Proof. Let G be a generator of $\mathcal{G}$. For any monomorphism $\beta\colon B \to X$, consider the set $\langle\beta\rangle = \{f \in \mathrm{Hom}(G, X)\colon f \text{ factor through } \beta\}$. Since $\mathrm{Hom}(G, X)$ is a set, it is sufficient to show that if $\beta\colon B \hookrightarrow X$ and $\beta'\colon B' \hookrightarrow X$ represent different subobjects, then $\langle\beta\rangle \neq \langle\beta'\rangle$. Consider the pullback diagram:

$$\begin{array}{ccc} B \cap B' & \xrightarrow{\gamma} & B \\ {\scriptstyle\gamma'}\downarrow & & \downarrow{\scriptstyle\beta} \\ B' & \xrightarrow[\beta']{} & X \end{array}$$

β and β' are monomorphism, and so are γ and γ'. Moreover, since β and β' represent different subobjects, γ and γ' cannot be both isomorphisms, say γ is not. In this case γ is not an epimorphism, hence $\mathrm{Coker}\,\gamma \neq 0$. Call $\nu\colon B \to \mathrm{Coker}\,\gamma$, since the functor $\mathrm{Hom}(G, -)$ is faithful, $\mathrm{Hom}(G, \nu) \neq 0$. Therefore there exists $\alpha \in \mathrm{Hom}(G, B)$ such that $\nu\alpha \neq 0$. Hence $\beta\alpha$ doesn't factor through $\beta\gamma = \beta'\gamma'$ and this means that $\beta\alpha \in \langle\beta\rangle \setminus \langle\beta'\rangle$, therefore $\langle\beta\rangle \neq \langle\beta'\rangle$. □

Definition 1.1.7. An object $X \in \mathcal{G}$ is called *finitely generated* if, whenever there are subobjects X_i, $i \in I$, such that:

$$X = \sum_{i \in I} X_i$$

there exist an index i_0 such that $X = X_{i_0}$.

Lemma 1.1.8. *[61, Lemma V.3.1] Let* $0 \to X' \to X \to X'' \to 0$ *be a short exact sequence in* $\mathcal{G}$*. Then:*

(1) *If* X *is finitely generated, then so is* X''*.*
(2) *If* X' *and* X'' *are finitely generated, then so is* X*.*

Proposition 1.1.9. *[61, Proposition V.3.2] An object* $X \in \mathcal{G}$ *is finitely generated if and only if the functor* $\mathrm{Hom}_{\mathcal{C}}(X, -)$ *commutes with directed colimits.*

We denote the subcategory of finitely generated object of $\mathcal{G}$ by $\mathrm{fg}(\mathcal{G})$. The category $\mathcal{G}$ is *locally finitely generated* if every object $X \in \mathcal{C}$ is a directed union

$$X = \bigcup_{i \in I} X_i$$

of finitely generated subobjects $X_i \subseteq X$. Clearly, the category of modules over a ring R is locally finitely generated.

Lemma 1.1.10. *[61, Lemma V.3.3] Let* $\mathcal{G}$ *be a locally finitely generated Grothendieck category. If* $f \colon Y \to X$ *is an epimorphism with* $X \in \mathrm{fg}(\mathcal{G})$*, then there exists a finitely generated subobject* $Y' \subseteq Y$ *such that* $f(Y') = X$*.*

Definition 1.1.11. An object $X \in \mathcal{G}$ is *finitely presented* if it is finitely generated if every epimorphism $f \colon Y \to X$, where $Y \in \mathrm{fg}(\mathcal{G})$, is such that $\mathrm{Ker}(f) \in \mathrm{fg}(\mathcal{G})$.

Proposition 1.1.12. *[61, Proposition V.3.4] Suppose* $\mathcal{G}$ *is locally finitely generated. An object* $X \in \mathcal{G}$ *is finitely presented if and only if the functor* $\mathrm{Hom}_{\mathcal{G}}(X, -)$ *commutes with direct limits.*

We denote the subcategory of finitely presented object of $\mathcal{G}$ by $\mathrm{fp}(\mathcal{G})$. This subcategory is closed under extension and if $0 \to X' \to X \to X'' \to 0$ is a short exact sequence in $\mathcal{G}$ with $X \in \mathrm{fp}(\mathcal{G})$, then $X'' \in \mathrm{fp}(\mathcal{G})$ if and only if $X' \in \mathrm{fp}(\mathcal{G})$, see [33]. The category $\mathcal{G}$ is *locally finitely presented* if every object $X \in \mathcal{G}$ is a direct limit

$$X = \varinjlim X_i$$

of finitely presented objects X_i. Clearly, in a locally finitely presented category every finitely generated object has an epimorphism from a finitely presented object.

Definition 1.1.13. An object $X \in \mathcal{G}$ is *coherent* if it is finitely presented and every finitely generated subobject $Y \subseteq X$ is finitely presented.

This definition is equivalent to: an object $X \in \mathcal{G}$ is coherent if every epimorphism $f \colon X \to Z$, with $Z \in \mathrm{fp}(\mathcal{G})$, is such that $\mathrm{Ker}(f) \in \mathrm{fp}(\mathcal{G})$. Clearly, every finitely generated subobject of a coherent object is again coherent. We denote the subcategory of coherent objects as $\mathrm{coh}(\mathcal{G})$. This subcategory is exact and closed under extension (see [33, Proposition 1.5]).
Summarizing, we have this chain of subcategories:

$$\mathcal{G} \supseteq \mathrm{fg}(\mathcal{G}) \supseteq \mathrm{fp}(\mathcal{G}) \supseteq \mathrm{coh}(\mathcal{G})$$

and these inclusions become equalities under some condition. First of all, the category $\mathcal{G}$ is called *locally coherent* if every object $X \in \mathcal{G}$ is a direct limit

$$X = \varinjlim X_i$$

of coherent objects X_i. We have:

Theorem 1.1.14. *[33, Theorem 1.6] Let $\mathcal{G}$ be a locally finitely presented Grothendieck category. The following conditions are equivalent:*

(1) *$\mathcal{G}$ is locally coherent.*

(2) $\mathrm{fp}(\mathcal{G}) = \mathrm{coh}(\mathcal{G})$.

(3) $\mathrm{fp}(\mathcal{G})$ *is an exact subcategory of $\mathcal{G}$.*

(4) $\mathrm{fp}(\mathcal{G})$ *is an abelian category.*

Example 1.1.15. Let R be a ring and consider mod-$R = \mathrm{fp}(\text{Mod-}R)$, the category of finitely presented right R-modules. Then, by [33, Proposition 2.1] the category of functors from mod-R to the category **Ab** of abelian groups, denoted by $(\text{mod-}R, \mathbf{Ab})$, is a locally coherent Grothendieck category.

Definition 1.1.16. An object $X \in \mathcal{G}$ is called *noetherian* if for any ascending chain, $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$, of subobjects of X, there exists $i \in \mathbb{Z}_{\geq 0}$ such that $X_i = X_{i+1} = \dots$.

The subcategory of $\mathcal{G}$ consisting of all the noetherian objects of $\mathcal{G}$ is denoted by $\mathrm{noeth}(\mathcal{G})$.

Proposition 1.1.17. *[61, Proposition 4.1] An object $X \in \mathcal{G}$ is noetherian if and only if every subobject of X is finitely generated.*

Moreover, from [61, Propostion 4.2], we have that $\mathrm{noeth}(\mathcal{G})$ is an abelian category. A noetherian object is clearly finitely generated, therefore $\mathrm{fg}(\mathcal{G}) \supseteq \mathrm{noeth}(\mathcal{G})$.
The category $\mathcal{G}$ is called *locally noetherian* if it has a family of noetherian generators, in this case every object is the directed colimit of noetherian subobject and $\mathrm{fg}(\mathcal{G}) = \mathrm{noeth}(\mathcal{G})$. Therefore, for a locally noetherian category we have the following chain of subcategories:

$$\mathcal{G} \supseteq \mathrm{noeth}(\mathcal{G}) \supseteq \mathrm{fp}(\mathcal{G}) \supseteq \mathrm{coh}(\mathcal{G}).$$

Furthermore:

Proposition 1.1.18. *Let $\mathcal{G}$ be a locally noetherian Grothendieck category, then:*

$$\mathrm{noeth}(\mathcal{G}) = \mathrm{fg}(\mathcal{G}) = \mathrm{fp}(\mathcal{G}) = \mathrm{coh}(\mathcal{G}).$$

Proof. We need to prove that $\mathrm{noeth}(\mathcal{G}) \subseteq \mathrm{fp}(\mathcal{G}) \subseteq \mathrm{coh}(\mathcal{G})$. Let $X \in \mathrm{noeth}(\mathcal{G})$, let $f \colon Y \to X$ be an epimorphism with $Y \in \mathrm{fg}(\mathcal{G}) = \mathrm{noeth}(\mathcal{G})$. Complete to the short exact sequence $0 \to$

$K \to Y \xrightarrow{f} X \to 0$. Since Y is noetherian and K is a subobject of Y, by Proposition 1.1.17, $K \in \mathrm{fg}(\mathcal{G})$. Therefore $X \in \mathrm{fp}(\mathcal{G})$.
Now, if $X \in \mathrm{fp}(\mathcal{G})$, consider a finitely generated, ie. noetherian, subobject $Y \subseteq X$. Let $g\colon Z \to Y$ be an epimorphism with $Z \in \mathrm{fg}(\mathcal{G}) = \mathrm{noeth}(\mathcal{G})$. Complete to the short exact sequence $0 \to K \to Z \xrightarrow{g} Y \to 0$. $Z \in \mathrm{noeth}(\mathcal{G})$, therefore, by Proposition 1.1.17, $K \in \mathrm{noeth}(\mathcal{G})$, hence $K \in \mathrm{fp}(\mathcal{G})$. This implies $Y \in \mathrm{fp}(\mathcal{G})$ and therefore $X \in \mathrm{coh}(\mathcal{G})$. □

Clearly, from Proposition 1.1.18 and Theorem 1.1.14, we infer that if $\mathcal{G}$ is locally noetherian, then $\mathcal{G}$ is locally coherent and we have a chain of implications for $\mathcal{G}$:

$$\text{loc. noetherian} \implies \text{loc. coherent} \implies \text{loc. finitely presented} \implies \text{loc. finitely generated}$$

1.2 Torsion pairs

Let $\mathcal{G}$ be a Grothendieck category. Let $\mathcal{M}$ be a class of objects in $\mathcal{G}$ and let $X \in \mathcal{G}$. We say that:

- X is *generated by* $\mathcal{M}$, if X is a quotient object of coproducts of objects in $\mathcal{M}$.
- X is *cogenerated by* $\mathcal{M}$, if X is a subobject of products of objects in $\mathcal{M}$.

Moreover, we denote by:

- $\mathrm{Gen}\,\mathcal{M}$: the class of all objects in $\mathcal{G}$ generated by $\mathcal{M}$.
- $\mathrm{Cogen}\,\mathcal{M}$: the class of all objects in $\mathcal{G}$ cogenerated by $\mathcal{M}$.
- $\mathrm{Add}\,\mathcal{M}$ ($\mathrm{add}\,\mathcal{M}$): the class of objects in $\mathcal{G}$ isomorphic to a direct summand of a (finite) direct sum of objects in $\mathcal{M}$.
- $\mathrm{Prod}\,\mathcal{M}$: the class of objects in $\mathcal{G}$ isomorphic to a direct summand of a direct product of objects in $\mathcal{M}$.

If $\mathcal{M} = \{M\}$ for $M \in \mathcal{G}$, we write $\mathrm{Gen}\,M$, $\mathrm{Cogen}\,M$, $\mathrm{Add}\,M$ and $\mathrm{Prod}\,M$. All these classes are full subcategories of $\mathcal{G}$.
Furthermore, we say that:

- $\mathcal{M}$ *is generating* for $\mathcal{G}$ if $\mathcal{G} = \mathrm{Gen}\,\mathcal{M}$.
- $\mathcal{M}$ *is cogenerating* for $\mathcal{G}$ if $\mathcal{G} = \mathrm{Cogen}\,\mathcal{M}$.

Definition 1.2.1. A *torsion pair* is a pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T}$ and $\mathcal{F}$ are two full subcategories of $\mathcal{G}$, such that:

(1) $\mathrm{Hom}_{\mathcal{G}}(\mathcal{T}, \mathcal{F}) = 0$.
(2) For any $X \in \mathcal{G}$, there is a short exact sequence:

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

$\mathcal{T}$ is called the *torsion class* and $\mathcal{F}$ is called the *torsionfree class*.
We say that a torsion pair $\mathbf{t}$ is:

- *split*: if every short exact sequence $0 \to T \to X \to F \to 0$, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, splits.
- *hereditary*: if the torsion class $\mathcal{T}$ is closed under subobjects.
- *of finite type*: if the torsionfree class $\mathcal{F}$ is closed under direct limits.

Given a class of objects $\mathcal{M} \subset \mathcal{G}$, set:

$$\mathcal{M}^\circ = \operatorname{Ker}\operatorname{Hom}_{\mathcal{G}}(\mathcal{M}, -) = \{B \in \mathcal{G} \mid \operatorname{Hom}_{\mathcal{G}}(M, B) = 0 \text{ for all } M \in \mathcal{M}\}$$
$$\mathcal{M}^\perp = \operatorname{Ker}\operatorname{Ext}^1_{\mathcal{G}}(\mathcal{M}, -) = \{B \in \mathcal{G} \mid \operatorname{Ext}^1_{\mathcal{G}}(M, B) = 0 \text{ for all } M \in \mathcal{M}\}$$
$$\mathcal{M}^{\perp_{>0}} = \bigcap_{i>0} \operatorname{Ker}\operatorname{Ext}^i_{\mathcal{G}}(\mathcal{M}, -) = \{B \in \mathcal{G} \mid \operatorname{Ext}^i_{\mathcal{G}}(M, B) = 0 \text{ for all } M \in \mathcal{M}, i > 0\}$$

The classes ${}^\circ\mathcal{M}$, ${}^\perp\mathcal{M}$ and ${}^{\perp_{>0}}\mathcal{M}$ are defined dually. If $\mathcal{M} = \{M\}$ for $M \in \mathcal{M}$, we write M°, $M^\perp$, $M^{\perp_{>0}}$, ${}^\circ M$, ${}^\perp M$ and ${}^{\perp_{>0}}M$.

Remark 1.2.2. Fixed a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$, it follows from the definition that $\mathcal{F} = \mathcal{T}^\circ$ and $\mathcal{T} = {}^\circ\mathcal{F}$. In particular, $\mathcal{T}$ is closed under extensions, quotient objects and all colimits that exist in $\mathcal{G}$ and, dually, $\mathcal{F}$ is closed under extensions, subobjects and limits.

Let $\mathcal{M}$ be a class of objects in $\mathcal{G}$ and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{G}$. We have that:

- $\mathbf{t}$ *is generated* by $\mathcal{M}$ if $\mathcal{F} = \mathcal{M}^\circ$ and $\mathcal{T} = {}^\circ(\mathcal{M}^\circ)$.
- $\mathbf{t}$ *is cogenerated* by $\mathcal{M}$ if $\mathcal{T} = {}^\circ\mathcal{M}$ and $\mathcal{F} = ({}^\circ\mathcal{M})^\circ$.

If $\mathcal{G}$ is a locally noetherian Grothendieck category, denote by $\mathcal{G}_0 = \operatorname{fp}(\mathcal{G})$. We have the following:

Theorem 1.2.3. *[21, §4.4][20, Lemma 3.11] Let $\mathcal{G}$ be a locally noetherian Grothendieck category. There is a bijective correspondence between as follows:*

$$\begin{aligned} \{\textit{torsion pairs of finite type in } \mathcal{G}\} &\longleftrightarrow \{\textit{torsion pairs in } \mathcal{G}_0\} \\ (\mathcal{T}, \mathcal{F}) &\longmapsto (\mathcal{T} \cap \mathcal{G}_0, \mathcal{F} \cap \mathcal{G}_0) \\ (\varinjlim \mathcal{T}_0, \varinjlim \mathcal{F}_0) &\longleftarrow (\mathcal{T}_0, \mathcal{F}_0) \end{aligned}$$

Moreover, $(\varinjlim \mathcal{T}_0, \varinjlim \mathcal{F}_0)$ *coincides with the torsion pair* $(\operatorname{Gen}\mathcal{T}_0, \mathcal{T}_0^\circ)$ *generated by* $\mathcal{T}_0$.

1.3 t-structures

In a triangulated setting, the notion of torsion pair translates to the notion of t-structure. Here we present some relevant properties.
Consider a triangulated category $\mathcal{D}$, with shift functor [1].

Definition 1.3.1. A pair of full subcategories of $\mathcal{D}$, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, is called a *t-structure* if it satisfies the properties below. We use the following notation: $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.

(1) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$,
(2) $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$
(3) For every object $X \in \mathcal{D}$, there is a triangle:

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

$\mathcal{D}^{\leq 0}$ is called the *aisle* and $\mathcal{D}^{\geq 0}$ is called the *coaisle* of the t-structure.
For a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, the full subcategory defined as:

$$\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$$

is called the *heart* of the t-structure.

Proposition 1.3.2. *[13, Proposition 1.3.3] The inclusion of $\mathcal{D}^{\leq n}$ inside $\mathcal{D}$, for any n, admits a right adjoint $\tau_{\leq n}$, and the inclusion of $\mathcal{D}^{\geq n}$ inside $\mathcal{D}$, for any n, admits a left adjoint $\tau_{\geq n}$. Moreover, for any $X \in \mathcal{D}$, there is a triangle:*

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \longrightarrow \tau_{\leq 0}X[1]$$

which is the unique triangle, up to isomorphism, such that the first term is in $\mathcal{D}^{\leq 0}$ and the third is in $\mathcal{D}^{\geq 1}$.

Let us state some properties of the heart in the following

Proposition 1.3.3. *(1) The heart $\mathcal{A}$ of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is an abelian category, closed under extension in $\mathcal{D}$ (ie. given $X, Z \in \mathcal{A}$ and a triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, then $Y \in \mathcal{A}$.)*
(2) A sequence $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{A}$ if and only if there is a triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$ with $X, Y, Z \in \mathcal{A}$.
(3) For $X, Y \in \mathcal{A}$, there is an isomorphism $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[1])$ which is functorial in both variables.

Proof. (1) Part of [13, Théorème 1.3.6].
(2) It follows from [13, Proposition 1.2.2].
(3) We define a map

$$\varepsilon\colon \mathrm{Ext}^1_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(X, Y[1]),$$

in the following way: let $[\xi] \in \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$, with $\xi\colon 0 \to Y \to E \to X \to 0$, for some $E \in \mathcal{A}$. By (2), the abelian structure on the heart of the t-structure comes from the triangulated structure of $\mathcal{D}$, therefore we have a triangle:

$$Y \longrightarrow E \longrightarrow X \xrightarrow{\alpha} Y[1]$$

and we define $\varepsilon([\xi]) := \alpha$. This map is well defined, indeed: if $\xi'\colon 0 \to Y \to E' \to X \to 0$

is such that $\xi' \in [\xi]$, we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow \wr & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & E' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

and hence:

$$\begin{array}{ccccccc} Y & \longrightarrow & E & \longrightarrow & X & \xrightarrow{\alpha} & Y[1] \\ \| & & \downarrow \wr & & \| & & \| \\ Y & \longrightarrow & E' & \longrightarrow & X & \xrightarrow{\alpha'} & Y[1] \end{array}$$

where the last square commutes. This implies $\alpha = \alpha'$.
Let now $\alpha \in \mathrm{Hom}_{\mathcal{D}}(X, Y[1])$. Let $M(\alpha)$ be the mapping cone of α, then we have the following sequence:

$$Y \longrightarrow M(\alpha)[-1] \longrightarrow X \xrightarrow{\alpha} Y[1] \longrightarrow M(\alpha)$$

Since, by (1), $\mathcal{A}$ is closed under extensions in $\mathcal{D}$, we have $M(\alpha)[-1] \in \mathcal{A}$. We define a map

$$\varepsilon' \colon \mathrm{Hom}_{\mathcal{D}}(X, Y[1]) \to \mathrm{Ext}^1_{\mathcal{A}}(X, Y)$$

which sends $\alpha \in \mathrm{Hom}_{\mathcal{D}}(X, Y[1])$ to the short exact sequence in $\mathcal{A}$:

$$0 \longrightarrow Y \to M(\alpha)[-1] \longrightarrow X \longrightarrow 0$$

which comes from the triangle in $\mathcal{A}$

$$Y \longrightarrow M(\alpha)[-1] \longrightarrow X \xrightarrow{\alpha} Y[1].$$

ε' is a right and left inverse of ε. Hence $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[1])$. □

Example 1.3.4. Let $\mathcal{B}$ be an abelian category, consider $\mathcal{D} = \mathcal{D}(\mathcal{B})$ the derived category of $\mathcal{B}$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defined as:

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i > 0\}$$
$$\mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i < 0\}$$

is a t-structure, called the *standard t-structure*. The heart of this t-structure is

$$\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \{X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i \neq 0\} = \mathcal{G}$$

1.3.1 The t-structure induced by a torsion pair

In [30], the authors have defined a way to construct t-structures in the derived category of an abelian category $\mathcal{B}$ starting from a torsion pair in $\mathcal{B}$. We describe this construction in the setting

of Grothendieck categories.
In the following, we fix a Grothendieck category $\mathcal{G}$ and a torsion pair $\mathbf{T} = (\mathcal{Q}, \mathcal{C})$ on it.

Definition 1.3.5. The *t-structure induced by the torsion pair* $\mathbf{T}$ is the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, of subcategories of $\mathcal{D}^b(\mathcal{G})$, defined by:

$$\mathcal{D}^{\leq 0} = \{X \in \mathcal{D}^b(\mathcal{G}) \mid H^0(X) \in \mathcal{Q}, H^i(X) = 0 \text{ for } i > 0\}$$
$$\mathcal{D}^{\geq 0} = \{X \in \mathcal{D}^b(\mathcal{G}) \mid H^{-1}(X) \in \mathcal{C}, H^i(X) = 0 \text{ for } i < -1\}$$

The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ as in the definition above is indeed a t-structure, by [30, Proposition 2.1], whose heart is the following category:

$$\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \{X \in \mathcal{D}^b(\mathcal{G}) \mid H^0(X) \in \mathcal{Q}, H^{-1}(X) \in \mathcal{C}, H^i(X) = 0 \text{ for } i \neq -1, 0\}$$

In the sequel, we will denote by

$$\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$$

the heart of the t-structure induced by the torsion pair $(\mathcal{Q}, \mathcal{C})$ on the category $\mathcal{G}$.
We know, by Proposition 1.3.3, that $\mathcal{A}$ is an abelian category, whose exact structure is given by the triangles of $\mathcal{D}^b(\mathcal{G})$ and for any two objects $X, Z \in \mathcal{A}$ there are functorial isomorphisms

$$\operatorname{Ext}^i_{\mathcal{A}}(X, Z) \cong \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Z[i]), \text{ for } i = 0, 1.$$

Proposition 1.3.6. *[30, Corollary I.2.2(b), Proposition I.3.2] Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$. The pair $(\mathcal{C}[1], \mathcal{Q})$ is a torsion pair in $\mathcal{A}$.*
Moreover, we have:

(i) *$\mathcal{Q}$ is cogenerating for $\mathcal{G}$ if and only if $\mathcal{Q}$ is generating for $\mathcal{A}$.*
(ii) *$\mathcal{C}$ is generating for $\mathcal{G}$ if and only if $\mathcal{C}[1]$ is cogenerating for $\mathcal{A}$.*

Moreover, we have the following:

Theorem 1.3.7. *[63, Theorem 3.12] Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ such that either $\mathcal{Q}$ is cogenerating or $\mathcal{C}$ is generating for $\mathcal{G}$. Then, there is an equivalence of triangulated categories:*

$$F\colon \mathcal{D}^b(\mathcal{G}) \to \mathcal{D}^b(\mathcal{A})$$

that extends the identity functor on $\mathcal{A}$, ie. $F|_{\mathcal{A}} = id_{\mathcal{A}}$.

Theorem 1.3.8. *[63, Theorem 5.2] Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ such that either $\mathcal{Q}$ is cogenerating for $\mathcal{G}$. Then, $\mathcal{A}$ is hereditary, ie. $\operatorname{Ext}^2_{\mathcal{A}}(-,-) = 0$, if and only if the torsion pair $(\mathcal{Q}, \mathcal{C})$ is split and $\operatorname{pdim}_{\mathcal{A}} Q \leq 1$, for any $Q \in \mathcal{Q}$.*

In the following, we collect some useful facts to compute Hom and Ext groups in the heart $\mathcal{A}$.

Lemma 1.3.9. *Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$. The following statements hold true for $C, Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$.*

(i) $\operatorname{Hom}_{\mathcal{A}}(Q, C[1]) \cong \operatorname{Ext}^1_{\mathcal{G}}(Q, C)$,

(ii) *If either* $\mathcal{Q}$ *is cogenerating or* $\mathcal{C}$ *is generating for* $\mathcal{G}$*,* $\mathrm{Ext}^1_{\mathcal{A}}(Q, C[1]) \cong \mathrm{Ext}^2_{\mathcal{G}}(Q, C)$

(iii) $\mathrm{Hom}_{\mathcal{A}}(Y[1], C[1]) \cong \mathrm{Hom}_{\mathcal{G}}(Y, C)$,

(iv) $\mathrm{Ext}^1_{\mathcal{A}}(Y[1], C[1]) \cong \mathrm{Ext}^1_{\mathcal{G}}(Y, C)$,

(v) $\mathrm{Ext}^1_{\mathcal{A}}(Y[1], Q) \cong \mathrm{Hom}_{\mathcal{G}}(Y, Q)$.

Proof. The statements follow from the isomorphism in Proposition 1.3.3(3) and the fact that for any object $X, Y \in \mathcal{G}$ there is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{G})}(X[i], Y[j]) \cong \mathrm{Ext}^{j-i}_{\mathcal{G}}(X, Y). \qquad \square$$

A natural question that arises in this framework is the following: under which condition on the torsion pair $(\mathcal{Q}, \mathcal{C})$ the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is a Grothendieck category? This problem has been widely treated in literature, see for example [17, 50] and [20].
Being an (AB3) category, $\mathcal{G}$ has coproducts, therefore, also its derived category, $\mathcal{D}^b(\mathcal{G})$, has coproducts. From [50, Proposition 3.2], we have that the heart $\mathcal{A}$ is an (AB3) category. Moreover, the coaisle of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defined above is closed under taking coproducts in $\mathcal{D}^b(\mathcal{G})$. This means, via [50, Proposition 3.3], that the heart $\mathcal{A}$ is an (AB4) category. We have the following:

Theorem 1.3.10. *[50, Corollary 4.10] Let* $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$*. Suppose that either* $\mathcal{C}$ *is generating or* $\mathcal{Q}$ *is cogenerating for* $\mathcal{G}$*. Then* $\mathcal{A}$ *is a Grothendieck category if and only if* $(\mathcal{Q}, \mathcal{C})$ *is of finite type in* $\mathcal{G}$*.*

For the purposes of the present work, we underline the behavior of colimits in the heart. Indeed, in $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$, direct limits are defined via the so called *Milnor colimit* of a sequence in $\mathcal{D}^b(\mathcal{G})$, we refer to [50, Section 3] for the detailes.
What is relevant is the fact that, under the assumption that $\mathcal{C}$ is closed under direct limits in $\mathcal{G}$, colimits of sequences in $\mathcal{Q}$ or $\mathcal{C}[1]$ in the heart behave as colimits of sequences in $\mathcal{Q}$ or $\mathcal{C}$ in $\mathcal{G}$. Indeed:

Proposition 1.3.11. *[50, Proposition 4.2] Let* $(\mathcal{Q}, \mathcal{C})$ *be a torsion pair of finite type and consider* $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$*. The following assertions hold:*

(1) *If* $(C_i)_{i \in I}$ *is a direct system in* $\mathcal{C}$ *then there is an isomorphism in* $\mathcal{A}$*:*

$$(\varinjlim C_i)[1] \cong \varinjlim{}_{\mathcal{A}} (C_i[1]).$$

(2) *If* $(Q_i)_{i \in I}$ *is a direct system in* $\mathcal{Q}$ *then there is an isomorphism in* $\mathcal{A}$*:*

$$\varinjlim Q_i \cong \varinjlim{}_{\mathcal{A}} Q_i.$$

Whenever $\mathcal{G}$ is a locally coherent category, it is possible to say more about the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$.

Theorem 1.3.12. *[59, Theorem 5.2] Let* $\mathcal{G}$ *be a locally coherent Grothendieck category and let* $(\mathcal{Q}, \mathcal{C})$ *be a torsion pair in* $\mathcal{G}$*. The following are equivalent:*

(1) $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ *is a locally coherent Grothendieck category.*
(2) $(\mathcal{Q}, \mathcal{C})$ *is of finite type and* $(\mathcal{Q} \cap \mathrm{fp}(\mathcal{G}), \mathcal{C} \cap \mathrm{fp}(\mathcal{G}))$ *is a torsion pair in* $\mathrm{fp}(\mathcal{G})$.

If, in addition, $\mathcal{G}$ *is locally noetherian, these assertion are equivalent to:*

(3) $(\mathcal{Q}, \mathcal{C})$ *is of finite type.*

1.4 Approximations

Let $\mathcal{G}$ be a Grothendieck category. Consider a full subcategory $\mathcal{S}$ of $\mathcal{G}$ and let $M \in \mathcal{G}$.

Definition 1.4.1. A morphism $f\colon M \to X$ with $X \in \mathcal{S}$ is a $\mathcal{S}$*-preenvelope* of M if the map

$$\mathrm{Hom}_{\mathcal{G}}(f, E)\colon \mathrm{Hom}_{\mathcal{G}}(X, E) \to \mathrm{Hom}_{\mathcal{G}}(M, E)$$

is surjective for any $E \in \mathcal{S}$, in other words, if any map $M \to E$ factors through f for any $E \in \mathcal{S}$.
An $\mathcal{S}$-preenvelope $f\colon M \to X$ of M is called *special* if f is injective and $\mathrm{Coker}(f) \in {}^{\perp}\mathcal{S}$.
An $\mathcal{S}$-preenvelope $f\colon M \to X$ of M is an $\mathcal{S}$*-envelope* if whenever $gf = f$ for $g \in \mathrm{End}_{\mathcal{G}}(X)$ we have that g is an automorphism.
$\mathcal{S}$ is a *preenveloping class* (resp. *special preenveloping*, *enveloping*) if every object in $\mathcal{G}$ has an $\mathcal{S}$-preenvelope (resp. a special $\mathcal{S}$-preenvelope, an $\mathcal{S}$-envelope).
Dually, a morphism $f\colon X \to M$ with $X \in \mathcal{S}$ is a $\mathcal{S}$*-precover* of M if the map

$$\mathrm{Hom}_{\mathcal{G}}(E, f)\colon \mathrm{Hom}_{\mathcal{G}}(E, X) \to \mathrm{Hom}_{\mathcal{G}}(E, M)$$

is surjective for any $E \in \mathcal{S}$, in other words, if any map $E \to M$ factors through f for any $E \in \mathcal{S}$.
An $\mathcal{S}$-precover $f\colon X \to M$ of M is called *special* if f is surjective and $\mathrm{Ker}(f) \in \mathcal{S}^{\perp}$.
An $\mathcal{S}$-precover $f\colon M \to X$ of M is an $\mathcal{S}$*-cover* if whenever $fg = f$ for $g \in \mathrm{End}_{\mathcal{G}}(X)$ we have that g is an automorphism.
$\mathcal{S}$ is a *precovering class* (resp. *special precovering*, *covering*) if every object in $\mathcal{G}$ has an $\mathcal{S}$-precover (resp. a special $\mathcal{S}$-precover, an $\mathcal{S}$-cover).

We have the following:

Theorem 1.4.2. *[24, Theorems 1.2 and 3.2] Let* $\mathcal{S}$ *be a full subcategory of* $\mathcal{G}$.

(1) *If* $\mathcal{S}$ *is closed under direct limits and is precovering, then it is covering.*
(2) *If* $\mathcal{S}$ *is closed under coproducts and direct limits and* $\mathcal{S} = \varinjlim \mathcal{S}_0$ *for a class* $\mathcal{S}_0 \subseteq \mathcal{S}$*, then it is covering.*

1.5 Purity

In this section we describe the notion of purity and pure-injectivity for an object in a Grothendieck category. The concept of purity has been introduced by Prüfer in 1923, in the context of abelian groups, then developed by Cohn in 1959, for modules over a ring.

Here, $\mathcal{G}$ is a locally finitely presented Grothendieck category with products. We define the pure-exact sequences in the sense of Crawley-Boevey (see [21, §3])

Definition 1.5.1. A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{G}$ is called *pure-exact* if, for any $A \in \text{fp}(\mathcal{C})$, the sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{G}}(A, X) \longrightarrow \text{Hom}_{\mathcal{G}}(A, Y) \longrightarrow \text{Hom}_{\mathcal{G}}(A, Z) \longrightarrow 0$$

is exact. In this case, f is called a *pure monomorphism* and g is a *pure epimorphism.*

Accordingly, for an object $Y \in \mathcal{G}$, a subobject $X \subseteq Y$ is a *pure subobject* if the embedding $X \hookrightarrow Y$ is a pure monomorphism.

Definition 1.5.2. An object $E \in \mathcal{G}$ is *pure-injective* if it is injective with respect to pure monomorphisms, ie. if $\iota \colon X \to Y$ is a pure monomorphism and $f \colon X \to E$ is a morphism, then there is a morphism $g \colon Y \to E$ such that $g\iota = f$.

There are many equivalent definitions of pure-injectivity for an object.

Theorem 1.5.3. *[51, Theorem 5.4] Let $E \in \mathcal{G}$. The following are equivalent:*

(i) *E is pure-injective.*
(ii) *Every pure-exact sequence $0 \to E \to Y \to Z \to 0$ splits.*
(iii) *For any index set I, the summation map $\Sigma \colon E^{(I)} \to E$, whose components are identity maps on E, factors through the canonical embedding $E^{(I)} \to E^I$, yielding an extension of the summation map $\overline{\Sigma} \colon E^I \to E$.*

If $G = \text{Mod-}R$ for a ring R, then a further equivalent is:

(iv) *The functor $(E \otimes_R -)$ is an injective object in the functor category $(R\text{-mod}, \mathbf{Ab})$.*

In the context of a general Grothendieck category, the definition of a pure-injective object using pure-exact sequences does not apply, since the existence of finitely presented objects is not ensured. Hence, in this case, we say that an object E in a Grothendieck category is pure-injective if it satisfies condition (iii) of Theorem 1.5.3.

1.6 Localizations

1.6.1 Quotient categories

Definition 1.6.1. A full subcategory $\mathcal{X}$ of $\mathcal{G}$ is called a *Serre subcategory* of $\mathcal{G}$ if, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{G}$, we have $Y \in \mathcal{X}$ if and only if $X \in \mathcal{X}$ and $Z \in \mathcal{X}$.

Given a Serre subcategory $\mathcal{X}$ of $\mathcal{G}$ we can construct the so called *quotient category* $\mathcal{G}/\mathcal{X}$ in the following way.

(1) The objects of $\mathcal{G}/\mathcal{X}$ coincide with the objects of $\mathcal{G}$.

(2) Let $X, Y \in \mathcal{G}$. Define the set of morphisms in $\mathcal{G}/\mathcal{X}$ as:

$$\operatorname{Hom}_{\mathcal{G}/\mathcal{X}}(X, Y) \colon = \varinjlim_{(X', Y')} \operatorname{Hom}_{\mathcal{G}}(X', Y/Y')$$

where the abelian groups $\operatorname{Hom}_{\mathcal{G}}(X', Y/Y')$ define an inductive system for X' and Y' running through all the subobjects of X and Y such that $X/X', Y' \in \mathcal{X}$ (see [25, III.1]).

(3) For $X, Y, Z \in \mathcal{G}$, there is a composition law:

$$\operatorname{Hom}_{\mathcal{G}/\mathcal{X}}(X, Y) \times \operatorname{Hom}_{\mathcal{G}/\mathcal{X}}(Y, Z) \to \operatorname{Hom}_{\mathcal{G}/\mathcal{X}}(X, Z).$$

For the detailed definition of this composition law, we refer to [25, III.1].

In this situation, it is possible to define a canonical functor $\mathbf{Q} \colon \mathcal{G} \to \mathcal{G}/\mathcal{X}$ by $\mathbf{Q}(X) = X$, for each object $X \in \mathcal{G}$, and the canonical map $\operatorname{Hom}_{\mathcal{G}}(X, Y) \mapsto \operatorname{Hom}_{\mathcal{G}/\mathcal{X}}(X, Y)$, for any $X, Y \in \mathcal{G}$.

Lemma 1.6.2. *[25, Lemmes III.1.1, 1.2, 1.3, 1.4] The quotient category $\mathcal{G}/\mathcal{X}$ is an additive category and $\mathbf{Q}$ is an additive functor. Moreover, the following hold for a morphism $f \colon X \to Y$ in $\mathcal{G}$:*

1. *The morphism $\mathbf{Q}(f)$ is the zero morphism (resp. a monomorphism or an epimorphism) if and only if $\operatorname{Im} f \in \mathcal{X}$ (resp. $\operatorname{Ker} f \in \mathcal{X}$ or $\operatorname{Coker} f \in \mathcal{X}$).*
2. *$\mathbf{Q}(f)$ has a kernel and a cokernel and $\mathbf{Q}(\operatorname{Ker} f) \cong \operatorname{Ker} \mathbf{Q}(f)$ and $\mathbf{Q}(\operatorname{Coker} f) \cong \operatorname{Coker} \mathbf{Q}(f)$.*
3. *$\mathbf{Q}(f)$ is an isomorphism if and only if $\operatorname{Ker}(f)$ and $\operatorname{Coker}(f)$ belong to $\mathcal{X}$.*

Proposition 1.6.3. *[25, Proposition III.1.1] For a Serre subcategory $\mathcal{X}$ of $\mathcal{G}$, the quotient category $\mathcal{G}/\mathcal{X}$ is an abelian category. Moreover, the quotient functor $\mathbf{Q} \colon \mathcal{G} \to \mathcal{G}/\mathcal{X}$ is an exact functor.*

The quotient category is a universal construction in the following sense.

Proposition 1.6.4. *[25, Corollaire III.1.2] Let $\mathcal{X}$ be a Serre subcategory of $\mathcal{G}$ and let $F \colon \mathcal{G} \to \mathcal{B}$ an exact functor from $\mathcal{G}$ to an abelian category $\mathcal{B}$. If $F(X) = 0$ for any object $X \in \mathcal{X}$, then there is a unique functor $H \colon \mathcal{G}/\mathcal{X} \to \mathcal{B}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\mathbf{Q}} & \mathcal{G}/\mathcal{X} \\ {\scriptstyle F}\downarrow & \swarrow {\scriptstyle H} & \\ \mathcal{B} & & \end{array}$$

where $\mathbf{Q} \colon \mathcal{G} \to \mathcal{G}/\mathcal{X}$ is the canonical quotient functor.

Definition 1.6.5. A Serre subcategory $\mathcal{X}$ of $\mathcal{G}$ is called *localizing subcategory* if the quotient functor $\mathbf{Q} \colon \mathcal{G} \to \mathcal{G}/\mathcal{X}$ has a right adjoint $\mathbf{S} \colon \mathcal{G}/\mathcal{X} \to \mathcal{G}$. In this case the functor $\mathbf{L} = \mathbf{S} \circ \mathbf{Q}$ is called *localization functor*.

Being a right adjoint functor, $\mathbf{S}$ is left exact and, by [25, Proposition III.2.3(a)], there is a natural equivalence $\mathbf{QS} \cong \mathbf{1}_{\mathcal{G}/\mathcal{X}}$, where $\mathbf{1}_{\mathcal{G}/\mathcal{X}}$ is the identity functor on $\mathcal{G}/\mathcal{X}$. For a localizing subcategory $\mathcal{X}$, we say that an object $X \in \mathcal{G}$ is *$\mathcal{X}$-closed* if $X \cong \mathbf{L}(X)$. Notice that, in [25], this is an a posteriori characterization of the closed objects, indeed, they are defined, in the setting of Serre subcategories, as the objects satisfying the following equivalent conditions.

Lemma 1.6.6. *[25, Lemme III.2.1] Let $X \in \mathcal{G}$. The following are equivalent:*

(1) *For any morphism $f \colon M \to N$ such that* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *belong to $\mathcal{X}$, the morphism* $\operatorname{Hom}_{\mathcal{G}}(f, X)$ *is an isomorphism.*
(2) *No nonzero subobject of X belong to $\mathcal{X}$. Moreover, any short exact sequence $0 \to X \to M \to Y \to 0$ with $Y \in \mathcal{X}$ splits.*
(3) *For any object $Y \in \mathcal{G}$, the morphism* $\operatorname{Hom}_{\mathcal{G}}(Y, X) \to \operatorname{Hom}_{\mathcal{G}/\mathcal{X}}(\mathbf{Q}Y, \mathbf{Q}X)$ *is an isomorphism of abelian groups.*

This notion is useful to describe injective envelopes of objects in the quotient category.

Proposition 1.6.7. *[25, Proposition III.3.6] Let $\mathcal{X}$ be a Serre subcategory of $\mathcal{G}$. Let $E = E(X)$ be the injective envelope of an object $X \in \mathcal{G}$ such that no nonzero subobjects of X belong to $\mathcal{X}$. Then E is $\mathcal{X}$-closed and $\mathbf{Q}E$ is an injective envelope of $\mathbf{Q}X$.*

Localizing subcategories can be characterized in the setting of Grothendieck categories via the following.

Proposition 1.6.8. *[25, Corollaire III.3.1, Proposition III.4.8] Let $\mathcal{X}$ be a Serre subcategory of $\mathcal{G}$. The following are equivalent:*

(1) *$\mathcal{X}$ is a localizing subcategory of $\mathcal{G}$.*
(2) *Every object $X \in \mathcal{G}$ has a maximal subobject in $\mathcal{X}$.*
(3) *Let $(X_i, f_i)_{i \in I}$ be an inductive system in $\mathcal{G}$ such that $X_i \in \mathcal{X}$, for any $i \in I$. Then $\varinjlim X_i \in \mathcal{X}$.*

Theorem 1.6.9. *[25, Lemme III.2.4, Corollaire III.3.2, Proposition III.4.9] Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{G}$. Then $\mathcal{X}$ and $\mathcal{G}/\mathcal{X}$ are Grothendieck categories. More precisely, every injective object in $\mathcal{G}/\mathcal{X}$ is isomorphic to a $\mathbf{Q}E$, for an injective object $E \in \mathcal{G}$ such that no nonzero subobjects of E belong to $\mathcal{X}$. If $(U_i)_{i \in I}$ is a family of generators in $\mathcal{G}$, then $(\mathbf{Q}U_i)_{i \in I}$ is a family of generators in $\mathcal{G}/\mathcal{X}$. Moreover, the canonical functor $\mathbf{Q}$ commutes with colimits.*

Corollary 1.6.10. *[25, Corollaire III.4.1] If $\mathcal{G}$ is locally noetherian and $\mathcal{X}$ be a localizing subcategory of $\mathcal{G}$, then $\mathcal{X}$ and $\mathcal{G}/\mathcal{X}$ are locally noetherian Grothendieck categories. Moreover, the section functor $\mathbf{S}$ commutes with colimits.*

1.6.2 Gabriel categories

Consider $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ a hereditary torsion pair in $\mathcal{G}$. By Remark 1.2.2, we know that $\mathcal{T}$ is closed under extensions, quotient objects and arbitrary direct sums. Moreover, being $\mathbf{t}$ hereditary, $\mathcal{T}$ is closed under subobject. This means that $\mathcal{T}$ is a localizing subcategory of $\mathcal{G}$. Hence, we have the so called localization sequence:

$$\mathcal{T} \underset{\mathbf{T}}{\overset{inc}{\rightleftarrows}} \mathcal{G} \underset{\mathbf{S}}{\overset{\mathbf{Q}}{\rightleftarrows}} \mathcal{G}/\mathcal{T}$$

where *inc* is the inclusion functor, $\mathbf{T}$ is the left-exact torsion radical (see [61, §VI.3]), the Grothendieck category $\mathcal{G}/\mathcal{T}$ is the localization of $\mathcal{G}$ at $\mathcal{T}$ and $\mathbf{Q}$ and $\mathbf{S}$ are the quotient functor and the section functor, respectively, of the localization.
For a set $\mathcal{X}$ of objects in $\mathcal{G}$, we denote by $\langle \mathcal{X} \rangle_{\text{htor}}$ the smallest hereditary torsion class containing $\mathcal{X}$. The *Gabriel filtration* of $\mathcal{G}$ is a transfinite chain of hereditary torsion classes of $\mathcal{G}$

$$\{0\} = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_\alpha \subseteq \dots$$

defined as follows:

- $\mathcal{G}_{-1} = \{0\}$
- suppose that α is an ordinal for which $\mathcal{G}_\alpha$ has already been defined. Let $\mathbf{Q}_\alpha \colon \mathcal{G} \to \mathcal{G}/\mathcal{G}_\alpha$ be the quotient functor. We define $\mathcal{G}_{\alpha+1}$ as:

$$\mathcal{G}_{\alpha+1} = \langle \mathcal{G}_\alpha \cup \{X \in \mathcal{G} \mid \mathbf{Q}_\alpha(X) \text{ is simple in } \mathcal{G}/\mathcal{G}_\alpha\} \rangle_{\text{htor}}$$

- if λ is a limit ordinal, then:

$$\mathcal{G}_\lambda = \langle \bigcup_{\alpha<\lambda} \mathcal{G}_\alpha \rangle_{\text{htor}}$$

Let α be an ordinal. An object X in $\mathcal{G}$ is said to be *α-torsion* if and only if $X \in \mathcal{G}_\alpha$ (it is called *α-torsionfree* if and only if it belongs to the torsionfree class ${\mathcal{G}_\alpha}^\circ$). The torsion class $\mathcal{G}_\alpha$ induces the localization sequence:

$$\mathcal{G}_\alpha \underset{\mathbf{T}_\alpha}{\overset{inc}{\rightleftarrows}} \mathcal{G} \underset{\mathbf{S}_\alpha}{\overset{\mathbf{Q}_\alpha}{\rightleftarrows}} \mathcal{G}/\mathcal{G}_\alpha$$

Definition 1.6.11. Let $\mathbf{\tau} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair on $\mathcal{G}$. An object X of $\mathcal{G}$ is called *$\mathbf{\tau}$-cocritical* if $X \in \mathcal{F}$ and every proper quotient of X is in $\mathcal{T}$.

If we consider the hereditary torsion pair $\mathbf{\tau}_\alpha = (\mathcal{G}_\alpha, {\mathcal{G}_\alpha}^\circ)$ given by a hereditary torsion class in the Gabriel filtration of $\mathcal{G}$, we say that an object X is *α-cocritical* instead of $\mathbf{\tau}_\alpha$-cocritical.

Lemma 1.6.12. *Let $\mathbf{\tau} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair and let $\mathbf{Q} \colon \mathcal{G} \to \mathcal{G}/\mathcal{T}$ be the quotient functor. The following are equivalent for $X \in \mathcal{G}$:*

(i) *X is $\mathbf{\tau}$-cocritical*
(ii) *$\mathbf{Q}(X)$ is simple in $\mathcal{G}/\mathcal{T}$ and $X \in \mathcal{F}$*

Proof. (i) $\Rightarrow$ (ii): Let Y be a nonzero subobject of $\mathbf{Q}(X) \in \mathcal{G}/\mathcal{T}$. Then, applying the section functor $\mathbf{S} \colon \mathcal{G}/\mathcal{T} \to \mathcal{G}$, we have $\mathbf{S}(Y) \subseteq \mathbf{SQ}(X)$. Moreover, $\mathbf{SQ}(X) \subseteq E(X)$, indeed: by Proposition 1.6.7, the injective envelope $E(X)$ is $\mathcal{T}$-closed, hence $\mathbf{SQ}(X) \subseteq \mathbf{SQ}(E(X)) \cong E(X)$. X is essential in $E(X)$, therefore we have that $\mathbf{S}(Y) \neq 0$ if and only if $\mathbf{S}(Y) \cap X \neq 0$, then, since X is $\mathbf{\tau}$-cocritical, $X/(\mathbf{S}(Y) \cap X) \in \mathcal{T}$. Applying the functor $\mathbf{Q}$ to the short exact sequence:

$$0 \to \mathbf{S}(Y) \cap X \to X \to X/(\mathbf{S}(Y) \cap X) \to 0$$

we obtain that $\mathbf{Q}(X) \cong \mathbf{Q}(\mathbf{S}(Y) \cap X) \subseteq \mathbf{QS}(Y) = Y \subseteq \mathbf{Q}(X)$. Therefore $Y \cong \mathbf{Q}(X)$.
(ii) $\Rightarrow$ (i): Let Y be a nonzero subobject of X, then $Y \in \mathcal{F}$ and $\mathbf{Q}(Y)$ is nonzero. Since $\mathbf{Q}$ is an exact functor and $\mathbf{Q}(X)$ is simple, $\mathbf{Q}(Y) = \mathbf{Q}(X)$. Hence, applying $\mathbf{Q}$ to the sequence $0 \to Y \to X \to X/Y \to 0$, we get $\mathbf{Q}(X/Y) = 0$ hence $X/Y \in \mathcal{T}$. □

Notice that, given the class $\mathcal{G}_\alpha$ in the Gabriel filtration, we can define $\mathcal{G}_{\alpha+1}$, using Lemma 1.6.12, as

$$\mathcal{G}_{\alpha+1} = \langle \mathcal{G}_\alpha \cup \{X \in \mathcal{G} \mid X \text{ is } \alpha\text{-cocritical}\} \rangle_{\text{htor}}.$$

Remark 1.6.13. [68, Remark 2.12] A Grothendieck category $\mathcal{G}$ has a generator G. Moreover, G has just a set of subobjects and, equivalently, a set of quotient objects. One can show that:

$$\mathcal{G}_{\alpha+1} = \langle \mathcal{G}_\alpha \cup \{X \in \mathcal{G} \mid X \text{ is a quotient of } G, \mathbf{Q}_\alpha(X) \text{ is simple in } \mathcal{G}/\mathcal{G}_\alpha\} \rangle_{\text{htor}}$$

As a consequence, we obtain that there is a cardinal κ such that $\mathcal{G}_\alpha = \mathcal{G}_\kappa$ for all $\alpha \geq \kappa$, just take $\kappa = \sup\{\alpha \mid \text{ there is } H \subseteq G \text{ such that } \mathbf{Q}_\alpha(G/H) \text{ is simple}\}$.

Consider the union $\bar{\mathcal{G}} = \bigcup_\alpha \mathcal{G}_\alpha$ of all the localizing subcategories in the Gabriel filtration (this makes sense by Remark 1.6.13).

Definition 1.6.14. For an object $X \in \mathcal{G}$, we say that X has *Gabriel dimension* if there is a minimal ordinal δ such that $X \in \mathcal{G}_\delta$, and we write $\text{Gdim}(X) = \delta$. If $\bar{\mathcal{G}} = \mathcal{G}$, we say that $\mathcal{G}$ is a *Gabriel category* with Gabriel dimension $\text{Gdim}(\mathcal{G}) = \kappa$, where κ is the smallest ordinal such that $\mathcal{G}_\kappa = \mathcal{G}$.

Proposition 1.6.15. *Every locally noetherian Grothendieck category is a Gabriel category.*

Proof. Let $\mathcal{G}$ be a locally noetherian Grothendieck category and consider its Gabriel filtration:

$$\{0\} = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_\alpha \subseteq \ldots$$

By Remark 1.6.13 this filtration stabilizes, ie. there is a cardinal κ such that $\mathcal{G}_\alpha = \mathcal{G}_\kappa$ for all $\alpha \geq \kappa$. Let $\mathcal{N}$ be the set of all the noetherian generators of $\mathcal{G}$. We prove that $\mathcal{N} \subseteq \mathcal{G}_\kappa$. Indeed: suppose that there is $N \in \mathcal{N}$ such that $N \notin \mathcal{G}_\kappa$. Consider the set:

$$\mathcal{I} = \{X \subseteq N \mid N/X \notin \mathcal{G}_\kappa\}$$

which is not empty since $0 \in \mathcal{I}$. Since N is noetherian, $\mathcal{I}$ has a maximal element $\bar{X}$. Therefore, for any object Y such that $\bar{X} \subseteq Y \subseteq N$, we have $N/Y \in \mathcal{G}_\kappa$. Moreover, $N/\bar{X}$ is κ-torsionfree, indeed: if it is not, it has a nonzero κ-torsion part $\mathbf{T}_\kappa(N/\bar{X})$ such that $(N/\bar{X})/\mathbf{T}_\kappa(N/\bar{X})$ is κ-torsionfree, but all the proper quotients of $N/\bar{X}$ are κ-torsion, by the maximality of $\bar{X}$. This means that $N/\bar{X}$ is κ-cocritical, since it is κ-torsionfree and any proper quotient of $N/\bar{X}$ is in $\mathcal{G}_\kappa$. Hence, $\mathbf{Q}_\kappa(N/\bar{X})$ is simple in $\mathcal{G}/\mathcal{G}_\kappa$ and then $N/\bar{X} \in \mathcal{G}_{\kappa+1} = \mathcal{G}_\kappa$. Contradiction. $\mathcal{N} \subseteq \mathcal{G}_\kappa$, therefore $\mathcal{G}_\kappa = \mathcal{G}$. □

Chapter 2

Introduction to tilting theory

In this chapter we will define tilting and cotilting objects in a Grothendieck category and we describe their basic properties. Afterwards, we will see that cotilting objects play a crucial role to see whether the heart of a t-structure induced by a torsion pair is a Grothendieck category or not. Finally, we will give some criteria to describe simple objects and their injective envelopes in the heart.

2.1 Tilting and cotilting objects

Let $\mathcal{G}$ be a Grothendieck category.

Definition 2.1.1. An object $T \in \mathcal{G}$ is called *tilting* if:

$$T^{\perp} = \operatorname{Gen} T.$$

In this case, the pair $(\operatorname{Gen} T, T^{\circ})$ is a torsion pair in $\mathcal{G}$, which is called *tilting torsion pair.* The class $\operatorname{Gen} T$ is called *tilting class* and it is cogenerating for $\mathcal{G}$.

Recall that the *projective dimension*, $\operatorname{pdim} X$, of an object $X \in \mathcal{G}$ is the smallest integer number $n \geq -1$ such that $\operatorname{Ext}^{n+1}_{\mathcal{G}}(X, -) = 0$. The *injective dimension* of an object $X \in \mathcal{G}$, $\operatorname{idim} X$, is defined dually, ie. the smallest integer number $n \geq -1$ such that $\operatorname{Ext}^{n+1}_{\mathcal{G}}(-, X) = 0$.
In [16, Proposition 2.1, Remark 2.2], it is proven that the equality $T^{\perp} = \operatorname{Gen} T$ is equivalent to the following three conditions:

(T1) $\operatorname{pdim} T \leq 1$, ie. $\operatorname{Ext}^2_{\mathcal{G}}(T, -) = 0$.
(T2) $\operatorname{Ext}^1_{\mathcal{G}}(T, T^{(\alpha)}) = 0$, for all cardinals α.
(T3) for an object $X \in \mathcal{G}$, if $\operatorname{Hom}_{\mathcal{G}}(T, X) = 0 = \operatorname{Ext}^1_{\mathcal{G}}(T, X)$, then $X = 0$.

If $\mathcal{G} = \text{Mod-}\Lambda$ for some ring Λ, we call T a *tilting module* and condition (T3) can be rephrased as follows:

(T3') There is a short exact sequence:

$$0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \operatorname{Add} T$.

Two tilting objects T and T' are said to be *equivalent* if they induce the same tilting class $\operatorname{Gen} T = \operatorname{Gen} T'$ or, equivalently, $\operatorname{Add} T = \operatorname{Add} T'$.
The dual definition is as follows.

Definition 2.1.2. An object $C \in \mathcal{G}$ is called *cotilting* if:

$$^{\perp}C = \operatorname{Cogen} C.$$

In this case, the pair $({}^{\circ}C, \operatorname{Cogen} C)$ is a torsion pair in $\mathcal{G}$, which is called *cotilting torsion pair*. The class $\operatorname{Cogen} C$ is called *cotilting class* and it is generating for $\mathcal{G}$.

Two cotilting objects C and C' are said to be *equivalent* if they induce the same cotilting class $\operatorname{Cogen} C = \operatorname{Cogen} C'$ or, equivalently, $\operatorname{Prod} T = \operatorname{Prod} T'$.

Example 2.1.3 (Left noetherian ring)**.** For a left noetherian ring Λ with a fixed duality D (eg. $D = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ or $D = \operatorname{Hom}_k(-, k)$ if Λ is a finite dimensional k-algebra) there is a complete classification of all tilting and cotilting modules via resolving subcategories. Recall that a subcategory $\mathcal{S} \subseteq \text{mod-}\Lambda$ is called *resolving* if it contains Λ and it is closed under direct summands, extensions and kernel of epimorphisms. Tilting and cotilting modules are related as follows.

Theorem 2.1.4. *[4, 12] If Λ is a left noetherian ring, there is a bijection between:*

$$\left\{\begin{array}{c}\textit{resolving subcat.}\\ \textit{of } \text{mod-}\Lambda \textit{ with modules}\\ \textit{of proj. dim. } \leq 1\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{equivalence classes of}\\ \textit{tilting modules}\\ \textit{in } \text{Mod-}\Lambda\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{equivalence classes of}\\ \textit{cotilting modules}\\ \textit{in } \Lambda\text{-Mod}\end{array}\right\}$$

$$\mathcal{S} = {}^{\perp}(\operatorname{Gen}(T)) \cap \text{mod-}\Lambda \longleftarrow\!\shortmid T \longmapsto D(T)$$

$$\mathcal{S} \longmapsto \mathcal{S}^{\perp} \qquad \mathcal{S} \longmapsto \mathcal{S}^{\top}$$

where $\mathcal{S}^{\perp}$ is a tilting torsion class and $\mathcal{S}^{\top} = \{X \in \Lambda\text{-Mod} \mid \operatorname{Tor}_1^{\Lambda}(S, X) = 0$ for any $S \in \mathcal{S}\}$ is a cotilting torsionfree class.

A homological description of the cotilting objects in a Grothendieck category $\mathcal{G}$ has been provided in [20]. This description is dual to the one for the tilting case, given above. First of all, we note that the injective dimension of a cotilting object is at most one.

Proposition 2.1.5. *[20, Proposition 2.7] Let $\mathcal{F}$ be a torsionfree class in $\mathcal{G}$ and suppose that $\mathcal{F}$ is generating for $\mathcal{G}$. Then $\operatorname{idim} C \leq 1$ for any $C \in \mathcal{F}^{\perp}$.*

If $C \in \mathcal{G}$ is a cotilting object, then, by [20, Corollary A.3], C^I is also a cotilting object such that $\operatorname{Cogen} C^I = \operatorname{Cogen} C$. In particular, for every $C' \in \operatorname{Prod} C$, $\operatorname{idim} C' \leq 1$ and there is an injective resolution:

$$0 \longrightarrow C' \longrightarrow E_0 \longrightarrow E_1 \longrightarrow 0$$

Moreover, there is a dual resolution for the injective objects in terms of objects in $\operatorname{Prod} C$. This is a generalization of [15, Proposition 1.8] in the setting of Grothendieck categories.

Lemma 2.1.6. *[20, Lemma 2.8] Let* $C \in \mathcal{G}$ *such that* ${}^{\perp}C = \operatorname{Cogen} C$*. Assume that* $K \in \operatorname{Cogen} C$*. Then there is a short exact sequence*

$$0 \longrightarrow K \longrightarrow C^I \longrightarrow L \longrightarrow 0$$

where I *is a set and* $L \in \operatorname{Cogen} C$*.*

Proposition 2.1.7. *[20, Proposition 2.9] Let* C *be a cotilting object in* $\mathcal{G}$*. Given an injective object* $W \in \mathcal{G}$ *(an injective cogenerator, in particular), there is a short exact sequence*

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

with $C_0, C_1 \in \operatorname{Prod} C$*.*

We can now state the homological characterization of cotilting objects in $\mathcal{G}$.

Theorem 2.1.8. *[20, Theorem 2.10] An object* $C \in \mathcal{G}$ *is cotilting if and only if it satisfies the following three conditions.*

(C1) $\operatorname{idim} C \leq 1$, *ie.* $\operatorname{Ext}^2_{\mathcal{G}}(-, C) = 0$.
(C2) $\operatorname{Ext}^1_{\mathcal{G}}(C^{\alpha}, C) = 0$, *for all cardinals* α.
(C3) *For every injective cogenerator* $W \in \mathcal{G}$*, there is a short exact sequence:*

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

where $C_0, C_1 \in \operatorname{Prod} C$*.*

Proof. For the convenience of the reader, we repeat the proof.
If C is cotilting then we use Propositions 2.1.5 and 2.1.7.
Conversely, suppose that C satisfies (C1)-(C3). By (C2) we have that $\operatorname{Prod} C \subseteq {}^{\perp}C$ and, by (C1), ${}^{\perp}C$ is closed under subobjects. Therefore $\operatorname{Cogen} C \subseteq {}^{\perp}C$. We need to prove the reverse inclusion.
We know that $\operatorname{Cogen} C$ is a torsionfree class of a torsion pair in $\mathcal{G}$. Let $A \in {}^{\perp}C$ and consider the short exact sequence:

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

coming from the torsion pair $({}^{\circ}C, \operatorname{Cogen} C)$. Consider now an injective cogenerator $W \in \mathcal{G}$ and the short exact sequence from (C3):

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

Applying $\operatorname{Hom}_{\mathcal{G}}(T, -)$ to this sequence, we obtain:

$$\operatorname{Hom}_{\mathcal{G}}(T, C_0) \longrightarrow \operatorname{Hom}_{\mathcal{G}}(T, W) \longrightarrow \operatorname{Ext}^1_{\mathcal{G}}(T, C_1)$$

where $\operatorname{Hom}_{\mathcal{G}}(T, C_0) = 0$, and since ${}^{\perp}C = {}^{\perp}\operatorname{Prod} C$ is closed under subobjects, $T \in {}^{\perp}\operatorname{Prod} C$,

hence $\mathrm{Ext}^1_{\mathcal{G}}(T, C_1) = 0$. This implies $\mathrm{Hom}_{\mathcal{G}}(T, W) = 0$ and, since W is a cogenerator, $T = 0$. This means that $A \cong F \in \operatorname{Cogen} C$, hence ${}^{\perp}C = \operatorname{Cogen} C$.
We have the generating property of $\operatorname{Cogen} C$ left to prove. Let $X \in \mathcal{G}$ be any object and let W be an injective cogenerator. Consider the monomorphism $\iota\colon X \hookrightarrow W^I$, for a set I. Clearly, also W^I is an injective cogenerator and we can apply (C3), obtaining the exact sequence:

$$0 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\pi} W^I \longrightarrow 0$$

with $C_1, C_0 \in \operatorname{Prod} C$. Take the pullback of π along ι:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & P & \xrightarrow{\pi'} & G & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle \iota'} & \lrcorner & \big\downarrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_0 & \xrightarrow{\pi} & W^I & \longrightarrow & 0 \end{array}$$

Then, since $P \subseteq C_0$ implies $P \in \operatorname{Cogen} C$, we have that X is an epimorphic image of an object in $\operatorname{Cogen} C$. □

2.2 Cotilting objects and Grothendieck hearts

In the context of module theory, Bazzoni proved, in [10], that all cotilting modules are pure-injective. For a general Grothendieck category, this result has been generalized by Čoupek and Šťovíček, in [20]. As seen in Definition 1.5.2, an object is pure-injective if it is injective with respect to pure monomorphisms. A further characterization is the following:

Theorem 2.2.1. *[20, Proposition 3.4] Let $\mathcal{G}$ be a Grothendieck category and let $E \in \mathcal{G}$. The following are equivalent:*

1. *E is pure-injective in $\mathcal{G}$.*
2. *There is a generator $G \in \mathcal{G}$ such that $\mathrm{Hom}_{\mathcal{G}}(G, E)$ is a pure-injective right $\mathrm{End}_{\mathcal{G}}(G)$-module.*
3. *$\mathrm{Hom}_{\mathcal{G}}(G, E)$ is a pure-injective $\mathrm{End}_{\mathcal{G}}(G)$-module for any generator $G \in \mathcal{G}$.*

Using this characterization, Čoupek and Šťovíček proved the following:

Theorem 2.2.2. *[20, Theorem 3.9]. If C is a cotilting object in a Grothendieck category $\mathcal{G}$, then C is pure-injective and the cotilting torsion pair $({}^{\circ}C, \operatorname{Cogen} C)$ is of finite type in $\mathcal{G}$.*

This Theorem, together with Theorem 1.2.3, shows that if C is a cotilting object in a locally noetherian Grothendieck category $\mathcal{G}$, then we can define, in $\mathcal{G}_0 = \mathrm{fp}(\mathcal{G})$, a torsion pair $({}^{\circ}C \cap \mathcal{G}_0, \operatorname{Cogen} C \cap \mathcal{G}_0)$ such that the torsionfree class is generating for $\mathcal{G}_0$. In [20], Čoupek and Šťovíček use approximation theory to prove that this assignment is, de facto, a bijective correspondence. This is a result of Buan and Krause, in [14], and the proof needs also a Theorem from [21].

Theorem 2.2.3. *[14, Theorem 1.13] Let $\mathcal{G}$ be a locally noetherian Grothendieck category and let $\mathcal{G}_0 = \mathrm{fp}(\mathcal{G})$. There is a bijective correspondence:*

$$\{\textit{cotilting torsion pairs in } \mathcal{G}\} \longleftrightarrow \left\{\begin{matrix} \textit{torsion pairs in } \mathcal{G}_0 \textit{ with} \\ \textit{generating torsionfree class} \end{matrix}\right\}$$
$$(\mathcal{Q}, \mathcal{C}) \longmapsto (\mathcal{Q} \cap \mathcal{G}_0, \mathcal{C} \cap \mathcal{G}_0)$$
$$(\varinjlim \mathcal{Q}_0, \varinjlim \mathcal{C}_0) \longleftarrow\!\shortmid (\mathcal{Q}_0, \mathcal{C}_0)$$

Let now $\mathcal{G}$ be a Grothendieck category and $(\mathcal{Q}, \mathcal{C})$ a torsion pair in $\mathcal{G}$. Let us consider the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$, as in Section 1.3.1. A classical property of Grothendieck categories is the existence of an injective cogenerator. When $\mathcal{G}$ is a module category, Colpi, Gregorio and Mantese, in [17], proved that the heart $\mathcal{A}$ has an injective cogenerator if and only if there is a cotilting module $W \in \mathcal{G}$ such that $\mathcal{C} = \operatorname{Cogen} W$. And this injective cogenerator in $\mathcal{A}$ is given by $W[1]$. This result has been generalized in [20] for an arbitrary Grothendieck category.

Proposition 2.2.4. *[20, Proposition 4.4] Let $\mathcal{G}$ be a Grothendieck category, let $(\mathcal{Q}, \mathcal{C})$ be a torsion pair in $\mathcal{G}$ with $\mathcal{C}$ generating for $\mathcal{G}$, and $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$. Then the following hold for an object $W \in \mathcal{G}$.*

(i) *$W[1]$ is injective in $\mathcal{A}$ if and only if $W \in \mathcal{C} \cap \mathcal{C}^{\perp}$.*
(ii) *$W[1]$ is an injective cogenerator of $\mathcal{A}$ if and only if W is a cotilting object in $\mathcal{G}$ with $\mathcal{C} = \operatorname{Cogen} W$.*

Proof. We repeat the proof for the convenience of the reader.

(i) Since $\mathcal{C}$ is generating for $\mathcal{G}$, by Proposition 1.3.6, $\mathcal{C}[1]$ is cogenerating in $\mathcal{A}$. Therefore, if $W[1]$ is injective in $\mathcal{A}$, then it is a summand of an object in $\mathcal{C}[1]$. In particular, $W \in \mathcal{C}$ and, by Lemma 1.3.9(iii), $W \in \mathcal{C}^{\perp}$.
Conversely, if $W \in \mathcal{C} \cap \mathcal{C}^{\perp}$ in $\mathcal{G}$, then the injective dimension of W in $\mathcal{G}$ is at most one by Proposition 2.1.5. Moreover, using Lemma 1.3.9(ii) and (iii), we have:

$$\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{C}[1], W[1]) = 0 \quad \text{and} \quad \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{Q}, W[1]) \cong \operatorname{Ext}^2_{\mathcal{G}}(\mathcal{Q}, W) = 0$$

therefore $W[1]$ is injective in $\mathcal{A}$ since every object in $\mathcal{A}$ is an extension of an object in $\mathcal{C}[1]$ by an object in $\mathcal{Q}$.

(ii) Suppose W is a cotilting object in $\mathcal{G}$ such that $\mathcal{C} = \operatorname{Cogen} W$. By [20, Corollary 2.12], product of copies of W in $\mathcal{G}$ coincides with the corresponding product in $\mathcal{D}^b(\mathcal{G}) \cong \mathcal{D}^b(\mathcal{A})$, hence we have $\operatorname{Prod} W[1] \subseteq \mathcal{A}$ and, in particular, arbitrary product of copies of $W[1]$ exist in $\mathcal{A}$ and agree with the ones in $\mathcal{G}$. By (i), $\operatorname{Prod} W[1]$ consists of injective objects and each object in $\mathcal{C}[1]$ is a subobject in $\mathcal{A}$ of a product of copies of $W[1]$ by Lemmata 1.3.9 and 2.1.6. Since $\mathcal{C}[1]$ is cogenerating in $\mathcal{A}$, $W[1]$ is an injective cogenerator in $\mathcal{A}$.
Conversely, let $W[1]$ be an injective cogenerator in $\mathcal{A}$. We have seen above that $W \in \mathcal{C} \cap \mathcal{C}^{\perp}$ in $\mathcal{G}$ and $\operatorname{idim}(W) \leq 1$. Since $\mathcal{C}$ is torsionfree in $\mathcal{G}$, we have $\operatorname{Cogen} W \subseteq \mathcal{C} \subseteq {}^{\perp}W$ in $\mathcal{G}$. Suppose that $A \in {}^{\perp}W$ in $\mathcal{G}$ and consider the short exact sequence $0 \to T \to A \to F \to 0$ arising from the torsion pair $(\mathcal{Q}, \mathcal{C})$. Then $T \in {}^{\perp}W$, indeed: $\operatorname{idim} W \leq 1$, hence, since

$^{\perp}W$ is closed under subobjects, $\mathrm{Ext}^1_{\mathcal{G}}(T, W) = 0$. But, since $W[1]$ is assumed to be a cogenerator in $\mathcal{A}$, we have that $0 = \mathrm{Ext}^1_{\mathcal{G}}(T, W) \cong \mathrm{Hom}_{\mathcal{A}}(T, W[1])$ implies $T = 0$, hence $A \cong F \in \mathcal{C}$. Hence $^{\perp}W = \mathcal{C}$.
Finally, we know that $\mathrm{Cogen}\, W$ is a torsionfree class in $\mathcal{G}$ in the torsion pair $(^{\circ}W, \mathrm{Cogen}\, W)$. Suppose that $A \in \mathcal{C}$ and consider an exact sequence $0 \to X \to A \to Y \to 0$ arising from the torsion pair $(^{\circ}W, \mathrm{Cogen}\, W)$. Then $X \in \mathcal{C}$ and $\mathrm{Hom}_{\mathcal{A}}(X[1], W[1]) \cong \mathrm{Hom}_{\mathcal{G}}(X, W) = 0$, but since $W[1]$ is assumed to be a cogenerator in $\mathcal{A}$, we have that $X = 0$ and hence $A \in \mathcal{C} \in \mathrm{Cogen}\, W$. Therefore $\mathrm{Cogen}\, W = \mathcal{C}$. $\square$

Now, with the same philosophy of generalizing the result in [17] concerning the relation between cotilting objects and hearts which are Grothendieck categories, we state the following:

Theorem 2.2.5. *[20, Theorem 4.5] Let $\mathcal{G}$ be a Grothendieck category, let $(\mathcal{Q}, \mathcal{C})$ be a torsion pair in $\mathcal{G}$ such that $\mathcal{C}$ is generating for $\mathcal{G}$ and let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$. The following are equivalent:*

(i) *$\mathcal{A}$ is a Grothendieck category.*
(ii) *$\mathcal{A}$ has an injective cogenerator.*
(iii) *$\mathcal{C} = \mathrm{Cogen}\, W = {}^{\perp}C$ for a cotilting object $W \in \mathcal{G}$.*

Proof. Also in this case, we repeat the proof.
(i) $\implies$ (ii): It is well known, for instance see [61, Corollary X.4.3].
(ii) $\implies$ (iii): Follows directly from Proposition 2.2.4(ii).
(iii) $\implies$ (i): Proposition 2.2.4 tells us that the category of injective objects in $\mathcal{A}$ is equivalent to $\mathrm{Prod}\, W$. Furthermore, via the dual argument in [9, Proposition IV.1.2], if two abelian categories with enough injective objects $\mathcal{B}$ and $\mathcal{B}'$ have equivalent corresponding subcategories of injective objects, then $\mathcal{B} \cong \mathcal{B}'$.
We want to construct a Grothendieck category $\mathcal{A}'$ whose full subcategory of injective objects is equivalent to $\mathrm{Prod}\, W$. To this end, let $G \in \mathcal{G}$ be a generator. Let $R = \mathrm{End}_{\mathcal{G}}(G)$ and $W' = \mathrm{Hom}_{\mathcal{G}}(G, W) \in \text{Mod-}R$. By Proposition 2.2.1 and Theorem 2.2.2, W' is a pure-injective R-module. Moreover, it follows from [61, §X.4] that the functor $\mathrm{Hom}_{\mathcal{G}}(G, -)$ induces an equivalence:

$$\mathrm{Prod}\, W \cong \mathrm{Prod}\, W'.$$

Consider now the category $\mathcal{B} = (R\text{-mod}, \mathbf{Ab})$ of all additive functors from R-mod to the category of all abelian groups $\mathbf{Ab}$. $\mathcal{B}$ is a locally coherent Grothendieck category and the functor:

$$\begin{aligned} T \colon \text{Mod-}R &\to \mathcal{B} \\ M &\mapsto (M \otimes_R -)\big|_{R\text{-mod}} \end{aligned}$$

is fully faithful, preserves products and sends pure-injective modules to injective objects of $\mathcal{B}$ (see [34, Theorem B.16]). In particular, set $W'' = T(W') \in \mathcal{B}$, we have an equivalence

$$\mathrm{Prod}\, W \cong \mathrm{Prod}\, W''$$

Since W'' is an injective object in $\mathcal{B}$, we have a hereditary torsion pair $(\mathcal{T}', \mathcal{F}')$ in $\mathcal{B}$, with $\mathcal{T}' = {}^\circ W''$ and $\mathcal{F}' = \operatorname{Cogen} W''$. $\operatorname{Prod} W''$ is, by definition, the subclass of $\mathcal{F}'$ consisting of injective objects in $\mathcal{B}$. Let us consider the quotient category $\mathcal{A}' = \mathcal{B}/\mathcal{T}'$. This is a Grothendieck category whose category of injective objects is equivalent to $\operatorname{Prod} W$ (see [25, Proposition III.4.9 and Corollaire III.3.2]). Then $\mathcal{A} \cong \mathcal{A}'$ and we have the claim. □

Recall that a pure-injective object $E \in \mathcal{G}$ is called Σ-*pure-injective* if $E^{(I)}$ is pure-injective for every set I. In the setting of a module category over a ring, if the cotilting module is Σ-pure-injective, then the heart of t-structure arising from the associated cotilting torsion pair is a locally noetherian Grothendieck category. This fact has been proved by Colpi, Mantese and Tonolo in [18]. Afterwards, this result has been generalized, in [43], in the wider setting of hearts of cosilting t-structures on compactly generated triangulated categories, of which the case of the derived category of a Grothendieck category is an example. We have, therefore:

Proposition 2.2.6. *[43, Proposition 5.6] Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ for a torsion pair $(\mathcal{Q}, \mathcal{C})$ such that $\mathcal{C} = \operatorname{Cogen} W$, with W a cotilting object. Then $\mathcal{A}$ is a locally noetherian Grothendieck category if and only if W is Σ-pure-injective.*

In the particular case when $\mathcal{G} = \text{Mod-}\Lambda$, with Λ a connected artin algebra, and $(\mathcal{Q}, \mathcal{C})$ is a torsion pair in $\mathcal{G}$ with some specific properties, the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ has the following geometric interpretation:

Proposition 2.2.7. *[3, Proposition 2.5] Suppose that the following conditions hold:*

(i) *there is a Σ-pure-injective cotilting Λ-module W such that $\mathcal{C} = \operatorname{Cogen} W$,*
(ii) *the torsion pair $(\mathcal{Q}, \mathcal{C})$ splits,*
(iii) $\Lambda \in \mathcal{C}$ *and* $D({}_\Lambda\Lambda) \in \mathcal{Q}$,
(iv) $\mathcal{Q} \cap {}^\perp\mathcal{Q} = 0$
(v) $\operatorname{pdim} M \leq 1$, *for any* $M \in \mathcal{C}$.

Then the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is equivalent to the category $\operatorname{Qcoh}\mathbb{X}$ *of quasi-coherent sheaves over a noncommutative curve of genus zero $\mathbb{X}$, and the category* $\operatorname{fp}(\mathcal{A})$ *of finitely presented objects in $\mathcal{A}$ corresponds to the category* $\operatorname{coh}\mathbb{X}$ *of coherent sheaves.*

2.3 Simple objects in the heart

Let us consider a Grothendieck category $\mathcal{G}$ and a torsion pair $(\mathcal{Q}, \mathcal{C})$ in it. Notice that the definitions and properties that we are going to present can be rephrased in the more general setting of abelian categories.

Definition 2.3.1. An object Y is said to be *torsionfree, almost torsion* if it satisfies:

(i) $Y \in \mathcal{C}$ and all proper quotients of Y are in $\mathcal{Q}$, and
(ii) for any short exact sequence $0 \to Y \to B \to C \to 0$ with $B \in \mathcal{C}$, then $C \in \mathcal{C}$.

Dually, we say that Y is *torsion, almost torsionfree* if it satisfies the dual properties:

(i') $Y \in \mathcal{Q}$ and all proper subobjects of Y are in $\mathcal{C}$, and

(ii') for any short exact sequence $0 \to B \to C \to Y \to 0$ with $C \in \mathcal{Q}$, then $B \in \mathcal{Q}$.

Remark 2.3.2. Observe that condition (i) in Definition 2.3.1 can be rephrased as follows

(I) $Y \in \mathcal{C}$ and every nonzero morphism $f \colon Y \to B$, with $B \in \mathcal{C}$, is a monomorphism in $\mathcal{G}$.

Proof. Suppose that (i) holds for an object Y, then if f is not a monomorphism, $\operatorname{Ker} f \neq 0$ and $\operatorname{Im} f$ is a proper quotient of Y, therefore $\operatorname{Im} f \in \mathcal{Q}$. But then $\operatorname{Hom}_{\mathcal{G}}(\operatorname{Im} f, B) \neq 0$, contradiction. Suppose now that (I) holds for Y and consider a proper quotient Q of Y. Let $0 \to Q' \to Q \to Q'' \to 0$ be the sequence for Q arising from the torsion pair $(\mathcal{Q}, \mathcal{C})$. The composite $Y \to Q \to Q''$ is a monomorphism by (I), since $Q'' \in \mathcal{C}$. Therefore the epimorphism $Y \to Q$ is a monomorphism and hence an isomorphism, contradicting the fact that Q is a proper quotient of Y. □

Dually, condition (i') has an equivalent characterization as:

(I') $Y \in \mathcal{Q}$ and every nonzero morphism $g \colon B \to Y$, with $B \in \mathcal{Q}$, is an epimorphism in $\mathcal{G}$.

We have the following:

Lemma 2.3.3. *[1, Lemma 3.2] Let $X, X' \in \mathcal{G}$ be both torsionfree, almost torsion, or both torsion, almost torsionfree. If $\operatorname{Hom}_{\mathcal{G}}(X, X') \neq 0$, then $X \cong X'$.*

Proof. Every morphism $0 \neq f : X \to X'$ is a monomorphism by condition (I) for X, hence $\operatorname{Coker} f \in \mathcal{C}$ by condition (ii). By condition (i) for X' it follows $\operatorname{Coker} f = 0$.
The torsion, almost torsionfree case is proven dually. □

Remark 2.3.4. If $\mathcal{G}$ is locally finitely generated and $(\mathcal{Q}, \mathcal{C})$ is of finite type (e.g. when it is a cotilting torsion pair), then all torsion, almost torsionfree objects are finitely generated.

Proof. Let $X \in \mathcal{Q}$ be an almost torsionfree object which is not finitely generated, then we can write $X = \varinjlim X_i$, where X_i are finitely generated subobjects of X. Since X is almost torsion free, all the X_i's are in $\mathcal{C}$ and $\mathcal{C}$ is closed under direct limits, therefore $X \in \mathcal{C}$. This is a contradiction. □

Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ be the heart of the t-structure induced by the torsion pair $(\mathcal{Q}, \mathcal{C})$. The following formulae are useful to compute kernels and cokernels of morphisms in the heart.

Lemma 2.3.5. *[1, Lemma 3.4]*

(1) *Let $f : X \to Y$ be a morphism in $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$, and let Z be the cone of f in $\mathcal{D}^b(\mathcal{G})$. Consider the canonical triangle, given by the truncation functors:*

$$K = \tau_{\leq -1} Z \longrightarrow Z \longrightarrow \tau_{\geq 0} Z \longrightarrow K[1]$$

where $\tau_{\leq -1} Z \in \mathcal{D}^{\leq -1}$ and $\tau_{\geq 0} Z \in \mathcal{D}^{\geq 0}$. Then:

$$\operatorname{Ker}_{\mathcal{A}}(f) = K[-1] \quad \operatorname{Coker}_{\mathcal{A}}(f) = \tau_{\geq 0} Z.$$

(2) *Let* $h\colon Y \to X$ *be a morphism in* $\mathcal{G}$ *with* $Y, X \in \mathcal{C}$. *Then:*

- $h[1]\colon Y[1] \to X[1]$ *is a monomorphism in* $\mathcal{A}$ *if and only if* $\operatorname{Ker} h = 0$ *and* $\operatorname{Coker} h \in \mathcal{C}$.
- $h[1]\colon Y[1] \to X[1]$ *is an epimorphism in* $\mathcal{A}$ *if and only if* $\operatorname{Coker} h \in \mathcal{Q}$.

(3) *Let* $h\colon Y \to X$ *be a morphism in* $\mathcal{G}$ *with* $Y, X \in \mathcal{Q}$. *Then:*

- $h\colon Y \to X$ *is a monomorphism in* $\mathcal{A}$ *if and only if* $\operatorname{Ker} h \in \mathcal{C}$.
- $h\colon Y \to X$ *is an epimorphism in* $\mathcal{A}$ *if and only if* $\operatorname{Coker} h = 0$ *and* $\operatorname{Ker} h \in \mathcal{Q}$.

Proof. Recall that the cone of h has homologies $\operatorname{Ker} h$ in degree -1, $\operatorname{Coker} h$ in degree 0, and zero elsewhere.

(1) See [28, pp. 281].
(2) We know from (1) that $\operatorname{Ker}_{\mathcal{A}}(h[1]) = 0$ if and only if the cone of $h[1]$ belongs to $\mathcal{D}^{\geq 0}$. This means $\operatorname{Ker} h = 0$ and $\operatorname{Coker} h \in \mathcal{C}$. Similarly, $\operatorname{Coker}_{\mathcal{A}}(h[1]) = 0$ if and only if the cone of $h[1]$ belongs to $\mathcal{D}^{\leq -1}$, which means that $\operatorname{Coker} h \in \mathcal{Q}$.
(3) We use the fact that the cone of h belongs to $\mathcal{D}^{\geq 0}$ if and only if $\operatorname{Ker} h \in \mathcal{C}$, and it belongs to $\mathcal{D}^{\leq -1}$ if and only if $\operatorname{Coker} h = 0$ and $\operatorname{Ker} h \in \mathcal{Q}$. □

Now, we can state the main characterization Theorem for simple objects in the heart.

Theorem 2.3.6. *[1, Theorem 3.3] (cf. [69, Lemma 2.2]) The simple objects in* $\mathcal{A}$ *are precisely the objects* S *of the form* $S = Y[1]$ *with* Y *torsionfree, almost torsion, or* $S = Q$ *with* Q *torsion, almost torsionfree.*

Proof. From the torsion pair $(\mathcal{C}[1], \mathcal{Q})$ in $\mathcal{A}$, we have a canonical exact sequence $0 \to Y[1] \to S \to Q \to 0$ with $Y \in \mathcal{C}$ and $Q \in \mathcal{Q}$. From this we see that a simple object S is either of the form $S = Y[1]$ or $S = Q$. Let us show that an object of the form $S = Y[1]$ with $Y \in \mathcal{C}$ is simple if and only if Y is torsionfree, almost torsion. The other case is proven dually.
First, assume that $S = Y[1]$ is simple and $Y \in \mathcal{C}$. Consider a proper subobject U of Y. Then the map $h : U \to Y$ gives rise to an epimorphism $h[1] : U[1] \to Y[1] = S$, hence $Y/U = \operatorname{Coker} h \in \mathcal{Q}$ by Lemma 2.3.5(2). So, (i) in Definition 2.3.1 is verified. To prove (ii), we consider a short exact sequence $0 \to Y \xrightarrow{h} B \to C \to 0$ with $B \in \mathcal{C}$. Here $h[1] : S \to B[1]$ is a monomorphism, and $C = \operatorname{Coker} h \in \mathcal{C}$ again by Lemma 2.3.5(2).
Conversely, we show that (i) and (ii) imply that S is simple. To this end, we claim that every morphism $0 \neq f\colon S \to A$ in $\mathcal{A}$ is a monomorphism. Since $S = Y[1] \in \mathcal{C}[1]$, f factors through the torsion part of A with respect to the torsion pair $(\mathcal{C}[1], \mathcal{Q})$. Hence, we can assume, without loss of generality, that $A = C[1]$ for some $C \in \mathcal{C}$. Then $f = g[1]$ with $g\colon Y \to C$, and by assumption g is a monomorphism with cokernel in $\mathcal{C}$. But then it follows from Lemma 2.3.5(2) that $\operatorname{Ker}_{\mathcal{A}}(f) = 0$, and the claim is proven. □

Suppose that $\mathcal{G}$ is a locally noetherian Grothendieck category. If there is a torsion pair of finite type in $\mathcal{G}$ such that the torsion class is cogenerating for $\mathcal{G}$, then we can prove a more convenient criterion for the simplicity of an object in the heart. Indeed:

Proposition 2.3.7. *Let* $\mathcal{G}$ *be a locally noetherian Grothendieck category and let* $\mathcal{G}_0 = \operatorname{fp}(\mathcal{G})$. *Consider a torsion pair* $(\mathcal{Q}_0, \mathcal{C}_0)$ *in* $\mathcal{G}_0$ *such that:*

(i) $\mathcal{C}_0$ *is generating for* $\mathcal{G}_0$,

(ii) $\mathcal{Q} = \varinjlim \mathcal{Q}_0$ *is cogenerating for* $\mathcal{G}$.

Then, a nonzero object $S \in \mathcal{C} = \varinjlim \mathcal{C}_0$ *becomes simple in the heart* $\mathcal{A}$ *if and only if for any non-split short exact sequence* $0 \to S \to E \to Q \to 0$ *in* $\mathcal{G}$ *with* $Q \in \mathcal{Q}_0$*, we have* $E \in \mathcal{Q}$.

Proof. First of all, notice that the pair $(\mathcal{Q}, \mathcal{C}) = (\varinjlim \mathcal{Q}_0, \varinjlim \mathcal{C}_0)$ is, by Theorem 1.2.3, a torsion pair of finite type in $\mathcal{G}$. Furthermore, by Theorem 1.3.10, the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is a Grothendieck category.

Suppose first that for any non-split short exact sequence $0 \to S \to E \to Q \to 0$ in $\mathcal{G}$ with $Q \in \mathcal{Q}_0$, we have $E \in \mathcal{Q}$. Let $g \colon Y \to S[1]$ be a nonzero morphism in the heart $\mathcal{A}$. By Proposition 1.3.6, $\mathcal{Q}$ is cogenerating for $\mathcal{G}$ if and only if $\mathcal{Q}$ is generating for $\mathcal{A}$, and moreover, by Proposition 1.3.11(2), we have that $\mathcal{Q} = \varinjlim \mathcal{Q}_0 = \varinjlim_{\mathcal{A}} \mathcal{Q}_0$, therefore $\mathcal{Q}_0$ is generating in $\mathcal{A}$. For this reason there exists an epimorphism $\pi = (\pi_i)_i$:

$$\coprod_i Q_i \xrightarrow{\pi} Y \xrightarrow{g} S[1]$$

where $Q_i \in \mathcal{Q}_0$, for any i, and since the composition is nonzero, there is at least a nonzero component in π yielding a nonzero morphism $\alpha \colon Q_j \to S[1]$, with $\alpha = g\pi j$, where $j \colon Q_j \hookrightarrow \coprod Q_i$, for a certain j.

For this reason, we can suppose, without loss of generality, that $Y \in \mathcal{Q}_0$. Using Lemma 1.3.9(i), from the morphism $g \colon Y \to S[1]$ in $\mathcal{A}$ we obtain a non-split short exact sequence in $\mathcal{G}$, $0 \to S \to E \to Y \to 0$, where $E \in \mathcal{Q}$ by hypothesis. Therefore we get a triangle in $\mathcal{D}^b(\mathcal{G})$:

$$S \longrightarrow E \longrightarrow Y \xrightarrow{g} S[1]$$

from which we obtain a short exact sequence in the heart $0 \to E \to Y \xrightarrow{g} S[1] \to 0$, proving that g is an epimorphism. Hence $S[1]$ is simple in $\mathcal{A}$.

Conversely, suppose that $S[1]$ is simple in $\mathcal{A}$. Consider a non-split short exact sequence $0 \to S \to E \to Q \to 0$ in $\mathcal{G}$, with $Q \in \mathcal{Q}_0$. This gives rise to a triangle in $\mathcal{D}^b(\mathcal{G})$:

$$S \longrightarrow E \longrightarrow Q \xrightarrow{g} S[1]$$

where the map g is surjective in the heart since $S[1]$ is simple. Let Z be the cone of g, $Z = E[1]$. Consider the canonical triangle given by the truncation functors:

$$\tau_{\leq -1} Z \longrightarrow Z \longrightarrow \tau_{\geq 0} Z \longrightarrow K[1]$$

By Lemma 2.3.5(1), $\operatorname{Coker} g = \tau_{\geq 0} Z$. Since g surjective, $\tau_{\geq 0} Z = 0$. Therefore $Z \cong \tau_{\leq -1} Z$. This means that $Z \in \mathcal{D}^{\leq -1} = \mathcal{D}^{\leq 0}[1]$, so $E \in \mathcal{D}^{\leq 0}$, hence $H^0(E) = E \in \mathcal{Q}$. □

Assume now that $\mathcal{G} = \Lambda$-Mod and $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is a Grothendieck category, for a torsion theory $(\mathcal{Q}, \mathcal{C})$. We know, from Theorem 2.2.5, that this happens if and only if $\mathcal{C} = \operatorname{Cogen}(W)$ for a cotilting module W. In this case, $W[1]$ is an injective cogenerator of $\mathcal{A}$ and every Λ-module has a special $\mathcal{C}$-cover and a special $\mathcal{C}^{\perp}$-envelope. In the following we prove that the injective envelopes of the objects in the heart come from these special $\mathcal{C}$-covers and special $\mathcal{C}^{\perp}$-envelopes.

Proposition 2.3.8. *[1, Proposition 4.1]*

(1) *Let $Y \in \mathcal{C}$, and let $0 \to Y \overset{f}{\to} B \to C \to 0$ be a special $\mathcal{C}^{\perp}$-envelope. Then $Y[1] \overset{f[1]}{\to} B[1]$ is an injective envelope of $Y[1]$ in $\mathcal{A}$.*

(2) *Let $Q \in \mathcal{Q}$, and let $0 \to B \overset{f}{\to} C \overset{g}{\to} Q \to 0$ be a special $\mathcal{C}$-cover. Then $Q \to B[1]$ is an injective envelope of Q in $\mathcal{A}$.*

Proof. (1) Since Y and C are in $\mathcal{C}$, we have $B \in \mathcal{C} \cap \mathcal{C}^{\perp} = \operatorname{Prod} W$, so $B[1]$ is injective. Moreover, it follows from Lemma 2.3.5 that there is an exact sequence $0 \to Y[1] \overset{f[1]}{\to} B[1] \to C[1] \to 0$ in $\mathcal{A}$. Finally, $f[1]$ is left minimal in $\mathcal{A}$ since so is f in Λ-Mod.

(2) Since $\mathcal{C}$ is closed under submodules, $B \in \mathcal{C} \cap \mathcal{C}^{\perp} = \operatorname{Prod} W$, so $B[1]$ is injective. Moreover, it follows from Lemma 2.3.5 that there is an exact sequence $0 \to Q \overset{h}{\to} B[1] \overset{f[1]}{\to} C[1] \to 0$ in $\mathcal{A}$.

It remains to check that h is left minimal. Consider an endomorphism $\beta[1] \in \operatorname{End}_{\mathcal{A}}(B[1])$ with $\beta[1] \circ h = h$. Then there is $\gamma[1] \in \operatorname{End}_{\mathcal{A}}(C[1])$ yielding a commutative diagram whose rows are given by distinguished triangles:

$$\begin{array}{ccccccc}
Q & \xrightarrow{h} & B[1] & \xrightarrow{f[1]} & C[1] & \xrightarrow{g[1]} & Q[1] \\
\| & & \downarrow{\scriptstyle \beta[1]} & & \downarrow{\scriptstyle \gamma[1]} & & \| \\
Q & \xrightarrow{h} & B[1] & \xrightarrow{f[1]} & C[1] & \xrightarrow{g[1]} & Q[1]
\end{array}$$

and therefore, a commutative diagram with distinguished triangles:

$$\begin{array}{ccccccc}
B & \xrightarrow{-f} & C & \xrightarrow{-g} & Q & \xrightarrow{h} & B[1] \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \| & & \| \\
B & \xrightarrow{-f} & C & \xrightarrow{-g} & Q & \xrightarrow{h} & B[1]
\end{array}$$

Since g is right minimal, γ is an isomorphism. Hence β and $\beta[1]$ are isomorphisms. $\square$

Chapter 3

Concealed-canonical algebras

In this chapter we define concealed-canonical algebras and we illustrate their module categories in terms of their Auslander-Reiten quiver. Mainly following [3] and [53], we describe the torsion pairs in Mod-Λ and some other relevant subclasses. Subsequently, we specialize on tubular algebras, ie. concealed-canonical algebras of tubular representation type, where classes of modules can be distinguished by a notion of *slope*, which is a real number or infinity. We will describe tilting and cotilting modules over tubular algebras and its pure-injective modules.
Let us fix a finite dimensional connected artin algebra Λ over a field k, for simplicity we consider k algebraically closed. We denote by Mod-Λ (mod-Λ) the category of (finitely presented) right Λ-modules.

3.1 The setup

Given a class $\mathcal{X}$ of indecomposable Λ-modules of finite length, we say that an indecomposable Λ-module of finite length M is a *proper predecessor* of $\mathcal{X}$ provided it does not belong to $\mathcal{X}$ and there is a sequence of indecomposable Λ-modules $(M_i)_{i=0}^n$ with $M_0 = M$ and $\mathrm{Hom}_\Lambda(M_{i-1}, M_i) \neq 0$, for all $1 \leq i \leq n$, such that M_n belongs to $\mathcal{X}$. Dually, we say that M is a *proper successor* of $\mathcal{X}$ if it does not belong to $\mathcal{X}$ and there is a sequence of indecomposables $(M_i)_{i=0}^n$ with $M_n = M$ and $\mathrm{Hom}_\Lambda(M_{i-1}, M_i) \neq 0$, for all $1 \leq i \leq n$, such that M_0 belongs to $\mathcal{X}$.

Definition 3.1.1. A *tubular family* $\mathbf{t}$ is a class consisting of all the indecomposable Λ-modules belonging to a set of *tubes*, ie. connected uniserial length categories, in the Auslander-Reiten quiver of Λ.
We say that a tubular family $\mathbf{t}$ in mod-Λ is:

- *sincere*, if every simple module occurs as the composition factor of at least one module from $\mathbf{t}$.
- *stable*, if $\mathbf{t}$ does not contain indecomposable projective or injective modules.
- *separating*, if it is standard, ie. there are no indecomposable modules M of finite length which are both proper predecessors of $\mathbf{t}$ and proper successors of $\mathbf{t}$, and any map from a proper predecessor of $\mathbf{t}$ to a proper successor of $\mathbf{t}$ factors through any of the tubes in $\mathbf{t}$.

Let us denote by $\mathbf{p}$ the class of indecomposable finite length Λ-modules which are proper predecessors of $\mathbf{t}$, called the *preprojective component*, and by $\mathbf{q}$ the class of indecomposable finite length Λ-modules which are proper successors of $\mathbf{t}$, called the *preinjective component*. Then, the separating condition can be rephrased as: any indecomposable module of finite length belongs either to $\mathbf{p}$, $\mathbf{t}$ or $\mathbf{q}$ and we say that $\mathbf{t}$ *separates* $\mathbf{p}$ *from* $\mathbf{q}$, yielding the canonical trisection of mod-Λ, $(\mathbf{p}, \mathbf{t}, \mathbf{q})$, as in the following picture:

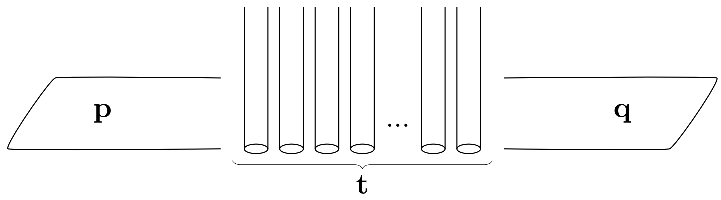


Figure 3.1: Auslander-Reiten quiver of mod-Λ.

Moreover:

$$\mathrm{Hom}_\Lambda(\mathbf{q}, \mathbf{p}) = \mathrm{Hom}_\Lambda(\mathbf{q}, \mathbf{t}) = \mathrm{Hom}_\Lambda(\mathbf{t}, \mathbf{p}) = 0$$

and any map from $\mathbf{p}$ to $\mathbf{q}$ factors through a module in $\mathbf{t}$.

Definition 3.1.2. A *concealed canonical algebra* Λ is a finite dimensional algebra with a sincere, stable and separating tubular family $\mathbf{t}$.

Tame hereditary algebras and canonical algebras are examples of concealed canonical algebras.

Remark 3.1.3. Every concealed canonical algebra is obtained as the endomorphism ring of a tilting module T over a canonical algebra Λ', we refer to [53, Sections 2.2 and 2.3] for the detailed definition of a canonical algebra and the description of the concealed canonical algebra as $\mathrm{End}(T)^{\mathrm{op}}$.

From now on, we fix Λ to be a concealed canonical algebra. All the results we are going to present are proven for canonical algebras, but they can be extended to concealed canonical algebras as shown in [53, Chapter 9].
We denote by τ and τ^- the Auslander-Reiten translations in mod-Λ and by D the classical duality $\mathrm{Hom}_k(-, k)$. We have the following:

Proposition 3.1.4. *[47, (S6) and (S8)(i)] Let Λ be a concealed-canonical algebra. The followings hold:*

(i) $D\,\mathrm{Ext}^1_\Lambda(M, X) \cong \mathrm{Hom}_\Lambda(X, \tau M)$*, for* $M \in \mathrm{add}(\mathbf{p} \cup \mathbf{t})$ *and* $X \in$ mod-Λ.
(i') $D\,\mathrm{Ext}^1_\Lambda(X, N) \cong \mathrm{Hom}_\Lambda(\tau^- N, X)$*, for* $N \in \mathrm{add}(\mathbf{t} \cup \mathbf{q})$ *and* $X \in$ mod-Λ.
(ii) $\mathrm{gl.dim}\,\Lambda \leq 2$.
(iii) $\tau\colon \mathrm{add}(\mathbf{p} \cup \mathbf{t}) \to \mathrm{add}(\mathbf{p} \cup \mathbf{t})$ *and* $\tau^-\colon \mathrm{add}(\mathbf{t} \cup \mathbf{q}) \to \mathrm{add}(\mathbf{t} \cup \mathbf{q})$ *have unique structures as functors making the isomorphisms in* (i) *and* (i') *functorial. Furthermore, restricting* τ *and* τ^- *to* $\mathrm{add}(\mathbf{t})$ *we obtain an autoequivalence:*

$$\tau\colon \mathrm{add}(\mathbf{t}) \longleftrightarrow \mathrm{add}(\mathbf{t})\colon \tau^-$$

(i) *and* (i') *are called* Auslander-Reiten formulae.

Remark 3.1.5. Consider a tubular family $\mathbf{t}$ which is sincere, stable and separating, then all the indecomposable projective modules belong to $\mathbf{p}$ and all the indecomposable injective modules belong to $\mathbf{q}$. This implies that:

(1) If $X \in \mathbf{p} \cup \mathbf{t}$, then $\operatorname{pdim} X \leq 1$.
(2) If $X \in \mathbf{q} \cup \mathbf{t}$, then $\operatorname{idim} X \leq 1$.

The standard tubes, in the tubular family $\mathbf{t}$, are denoted by $\mathcal{U}_x$, for $x \in \mathbb{X}$ an index set. By [47, (S8)(ii)], the tubular family decomposes as:

$$\mathbf{t} = \coprod_{x \in \mathbb{X}} \mathcal{U}_x$$

and $\operatorname{add}(\mathbf{t})$ is an abelian exact subcategory of mod-Λ. The simple objects and the composition factors in $\operatorname{add}(\mathbf{t})$ are called *simple regular* modules and *regular composition factors*, respectively. The set of all simple regular modules in $\mathcal{U}_x$ is called the *clique* of $\mathcal{U}_x$. The order of the clique is called the *rank* of $\mathcal{U}_x$. Moreover, from [47, (S8)(iii)], we have that all the tubes except finitely many are *homogeneous*, ie. of rank 1.

Prüfer and adic Λ-modules

Every simple regular module $S_x \in \mathcal{U}_x$ determines a *ray*, ie. an infinite sequence:

$$S_x \hookrightarrow S_{x,2} \hookrightarrow S_{x,3} \hookrightarrow S_{x,4} \hookrightarrow \dots$$

where $S_{x,n}$ denotes the unique indecomposable module in $\mathcal{U}_x$ of regular length n with socle S_x, the corresponding direct limit is the *Prüfer module* $S_x^{\infty} = \varinjlim S_{x,n}$.
Dually, we define the *coray* ending at S_x as the infinite sequence:

$$\cdots \twoheadrightarrow S_{x,-4} \twoheadrightarrow S_{x,-3} \twoheadrightarrow S_{x,-2} \twoheadrightarrow S_x$$

where $S_{x,-n}$ denotes the unique indecomposable module in $\mathcal{U}_x$ of regular length n with top S_x, the corresponding inverse limit is the *adic module* $S_x^{-\infty} = \varprojlim S_{x,-n}$. The Prüfer and the adic modules are indecomposable, infinite dimensional and pure-injective.

3.2 Torsion pairs in Mod-Λ

Following [53, Section 3.1], we mainly consider three torsion pairs in Mod-Λ.

- The torsion pair $(\mathcal{Q}, \mathcal{C}) = (\operatorname{Gen}(\mathbf{q}), \mathbf{q}^{\circ})$, generated by $\mathbf{q}$. This torsion pair is split by [53, Proposition 1.5]. Moreover, as shown in [53, Chapter 10], there is an infinite dimensional cotilting module $\mathbf{W}$ associated to this torsion pair, whose cotilting class is $\mathcal{C} = \operatorname{Cogen} \mathbf{W}$, and:

$$\mathbf{W} = G \oplus \bigoplus_{x \in \mathbb{X}} S_x^{\infty}$$

where G is the so called *generic module*, that we will introduce later.

- The torsion pair $(\mathcal{D}, \mathcal{R}) = ({}^{\circ}\mathbf{t}, ({}^{\circ}\mathbf{t})^{\circ})$, cogenerated by $\mathbf{t}$. This is a split torsion pair, as shown in [53, Corollary 5.3]. The modules in $\mathcal{D}$ are called *divisible modules* and the ones in $\mathcal{R}$ are called *reduced modules*. Furthermore, the infinite dimensional module $\mathbf{W}$ is a tilting module whose tilting class is $\mathcal{D} = \operatorname{Gen}\mathbf{W}$, see [53, Chapter 10].
- The torsion pair $(\operatorname{Gen}(\mathbf{t}), \mathcal{F})$, with $\mathcal{F} = \mathbf{t}^{\circ}$, generated by $\mathbf{t}$. The modules in $\operatorname{Gen}(\mathbf{t})$ are called *torsion modules* and the ones in $\mathcal{F}$ are called *torsionfree modules*. This is not a split torsion pair and clearly $\mathcal{F} \subseteq \mathcal{C}$, indeed: if $X \in \mathcal{F} = \mathbf{t}^{\circ}$ and there is a nonzero morphism $Q \to X$ with $Q \in \mathbf{q} \subseteq \operatorname{Gen}(\mathbf{t})$, we have $Y \in \operatorname{add}(\mathbf{t})$ and a nonzero map $Y \to X$, contradiction.

We have the following:

Theorem 3.2.1 (Basic splitting result)**.** *[53, Theorem 5.2, Corollary 5.4] For the classes $\mathcal{C}$ and $\mathcal{D}$ defined above, it holds $\operatorname{Ext}^1_\Lambda(\mathcal{C}, \mathcal{D}) = 0$. Moreover, for any $X \in \mathcal{C}$, $\operatorname{pdim} X \leq 1$, and for any $Y \in \mathcal{D}$, $\operatorname{idim} Y \leq 1$.*

The following is an infinite dimensional version of the Auslander-Reiten formulae defined in Proposition 3.1.4.

Lemma 3.2.2. *[62] Let $M, X \in \text{Mod-}\Lambda$. Assume that M is finitely generated without non-zero projective summands. Then:*

(i) *If $\operatorname{pdim} M \leq 1$, then $\operatorname{Hom}_\Lambda(X, \tau M) \cong D\operatorname{Ext}^1_\Lambda(M, X)$.*
(ii) *If $\operatorname{idim} \tau M \leq 1$, then $D\operatorname{Hom}_\Lambda(M, X) \cong \operatorname{Ext}^1_\Lambda(X, \tau M)$.*

Using this Lemma we can easily see that $\mathcal{D} = {}^{\circ}\mathbf{t} = \mathbf{t}^{\perp}$ and $\mathcal{C} = \mathbf{q}^{\circ} = {}^{\perp}\mathbf{q}$.
Consider now the class of torsion modules in $\mathcal{C}$, defined as

$$\mathcal{T} = \mathcal{C} \cap \operatorname{Gen}(\mathbf{t})$$

ie. Λ-modules in $\mathcal{C}$ generated by $\mathbf{t}$. It is clear that any Prüfer module S_x^∞ belongs to $\mathcal{T}$.

Proposition 3.2.3. *[53, Section 3.4] Every module in $\mathcal{T}$ is the direct union of modules in $\operatorname{add}(\mathbf{t})$ and $\mathcal{T} = \varinjlim \mathbf{t}$.*

Proof. If $M = \varinjlim M_i$ with $M_i \in \operatorname{add}(\mathbf{t})$, then clearly $M \in \operatorname{Gen}(\mathbf{t})$. Moreover, for any $Y \in \mathbf{q}$, by Proposition 1.1.12, $\operatorname{Hom}_\Lambda(Y, M) = \varinjlim \operatorname{Hom}_\Lambda(Y, M_i) = 0$. Therefore $M \in \mathcal{C}$.
Conversely, assume that $M \in \mathcal{C}$ is generated by $\mathbf{t}$. There is an epimorphism $g\colon \bigoplus_{i\in I} X_i \to M$, where $X_i \in \operatorname{add}(\mathbf{t})$. Then $M = \varinjlim \operatorname{Im}(g_i)$, where $g_i\colon X_i \to M$ is the i-th component of g. But $\operatorname{Im}(g_i) \in \operatorname{add}(\mathbf{t} \cup \mathbf{q})$, for any i, since they are factor modules of $X_i \in \operatorname{add}(\mathbf{t})$. And also $\operatorname{Im}(g_i) \in \mathcal{C}$, for any i, since they are submodules of M and $\mathcal{C}$ is a torsionfree class, hence closed under submodules. This implies that $\operatorname{Im}(g_i) \in \operatorname{add}(\mathbf{t})$. □

Since $\operatorname{add}(\mathbf{t})$ is an exact abelian subcategory of mod-Λ, we have that $\mathcal{T}$ is an exact abelian subcategory of Mod-Λ. In particular, $\mathcal{T}$ is closed under kernels, images, cokernels and direct sums.

3.2.1 The class $\omega = \mathcal{C} \cap \mathcal{D}$

Let us consider now the class ω defined as:

$$\omega = \mathcal{C} \cap \mathcal{D}$$

It is immediate to see that all the Prüfer modules belong to ω. Indeed: clearly $S_x^\infty \in \mathcal{C}$, for all $x \in \mathbb{X}$. Moreover, S_x^∞ is the injective envelope of any $S_{x,n}$ in the category $\mathcal{T}$, see [55]. Because of this, $\mathrm{Ext}^1_\Lambda(X, S_x^\infty) = 0$ for any $X \in \mathrm{add}(\mathbf{t})$, hence $S_x^\infty \in \mathcal{D}$. Furthermore, still from [55], we have that the Prüfer modules are all the indecomposable injectives in the category $\mathcal{T}$ and every object in $\mathcal{T}$ has an injective envelope. Denote by

$$\omega_0 = \mathcal{T} \cap \mathcal{D}$$

the full subcategory of all the injective objects in $\mathcal{T}$. Thus, ω_0 is the full subcategory of all direct sums of Prüfer modules.
The class ω plays a relevant role from the point of view of approximation theory.

Theorem 3.2.4. *[53, Theorem 4.1] For any $X \in \mathcal{C}$, there is a short exact sequence:*

$$0 \longrightarrow X \xrightarrow{f} M \longrightarrow M' \longrightarrow 0$$

with $M \in \omega$ and $M' \in \omega_0$, such that f is an ω-envelope of X.
If $X \in \mathcal{F}$, then $M \in \mathcal{F}$. If $X \in \mathcal{T}$, then $M \in \mathcal{T}$.

We denote by G the *generic module*. This is the unique indecomposable infinite dimensional module which has finite length over its endomorphism ring. In the notation of [53], G is the unique infinite dimensional module in $\mathcal{F} \cap \mathcal{D}$, ie. it is the only torsionfree divisible module (cf. [55, Theorem 5.3 and p.408]). Moreover, G is the only (up to isomorphism) indecomposable module in ω whose endomorphism ring is a skew field (see [53, Theorem 6.1]). By [53, Corollary 6.2], there exists an embedding from the module G in a direct sum of Prüfer modules.
Modules in ω are described by the following:

Theorem 3.2.5. *[53, Theorem 6.4] Any module in ω is a direct sum of Prüfer modules and of copies of the generic module.*

Notice that, via this characterization, we have that $\mathcal{F} \cap \omega = \mathrm{Add}(G)$. Moreover, all the indecomposables in ω have a local endomorphism ring. As a consequence, it follows from the Theorem of Krull-Remak-Schmidt-Azumaya that the direct sum decomposition in the Theorem above is unique up to isomorphisms.

Corollary 3.2.6. *[53, Corollaries 6.5, 6.6 and 6.7] The following hold:*

1. *The modules in $\mathcal{C}$ are precisely the modules cogenerated by $\mathcal{T}$ and precisely the modules which can be embedded in ω_0.*
2. *If $X \in \mathcal{F}$, then in the ω-envelope of X, we have $M \cong G^{(\alpha)}$, for a cardinal α. If X is also of finite length, then $M \cong G^n$, for an integer $n > 0$.*

(3) *The generic module G is of finite length as an* $\mathrm{End}_\Lambda(G)$*-module.*

Furthermore, ω is a covering class for divisible modules:

Theorem 3.2.7. *[53, Theorem 7.1, Corollary 7.2] For any $Y \in \mathcal{D}$, there is a short exact sequence:*

$$0 \longrightarrow M' \longrightarrow M \stackrel{g}{\longrightarrow} Y \longrightarrow 0$$

with $M \in \omega$ and M' is a direct sum of copies of the generic module, such that g is an ω-cover of X. If $X \in \mathcal{Q}$, then $M \in \omega_0$. If X is also of finite length, then $M' \cong G^n$, for an integer $n > 0$.

As a direct consequence of this Theorem and the Basic splitting result, we have the following description of $\mathcal{C}$ and $\mathcal{D}$ as (see [53, Proposition 7.3]):

$$\mathcal{C} = {}^{\perp}\omega \quad \text{and} \quad \mathcal{D} = \omega^{\perp}$$

and from this it follows:

Lemma 3.2.8. *[53, Lemma 8.4] The classes $\mathcal{C}, \mathcal{D}$ and ω are closed under products.*

In addition, we have:

Proposition 3.2.9. *[53, Corollary 8.1] Any Prüfer module is generated by G and $\mathcal{D} = \mathrm{Gen}(G)$.*

Proof. Following [53, Chapter 8]. Let S_x be a simple regular module. $S_x^\infty \in \mathcal{C}$, hence $\mathrm{pdim}\, S_x^\infty = 1$, and let $p\colon P \to S_x^\infty$ be a projective cover. Take the ω-envelope of P, $P \to M$. By the enveloping property, we can factor p through M and obtain a surjective map $M \to S_x^\infty$. But since $P \in \mathcal{F}$, by Theorem 3.2.4, $M \in \mathcal{F} \cap \omega = \mathrm{Add}(G)$.
For the second statement, clearly $\mathrm{Gen}(G) \subseteq \mathcal{D}$. Indeed: $G \in \mathcal{D}$ and $\mathcal{D}$ is a torsion class, hence closed under direct sums and factor modules. Conversely, the ω-cover of a module in $\mathcal{D}$ is surjective, by Theorem 3.2.7, therefore $\mathcal{D} \subseteq \mathrm{Gen}(\omega)$, but any module in ω is a direct sum of Prüfers and copies of G, thus generated by G. □

Let us consider again the module

$$\mathbf{W} = G \oplus \bigoplus_{x \in \mathbb{X}} S_x^\infty.$$

As we have mentioned above, it is an infinite dimensional cotilting module whose associated cotilting torsion pair is $(\mathcal{Q}, \mathcal{C})$ and it is also an infinite dimensional tilting module whose associated tilting torsion pair is $(\mathcal{D}, \mathcal{R})$. Let $\mathbf{W}_0$ be the torsion part of $\mathbf{W}$, ie. $\mathbf{W}_0 = \bigoplus_{x \in \mathbb{X}} S_x^\infty$. We have:

Proposition 3.2.10. *[53, Proposition 10.1]*

$$\omega = \mathrm{Add}(\mathbf{W}) = \mathrm{Prod}(\mathbf{W}) = \mathrm{Prod}(\mathbf{W}_0).$$

Remark 3.2.11. According to the decomposition of $\mathbf{t}$ as $\coprod_{x \in \mathbb{X}} \mathcal{U}_x$, we can describe $\mathcal{T} = \varinjlim \mathbf{t}$ as the coproduct of categories denoted $\mathcal{T}(x)$ for any $x \in \mathbb{X}$, ie. $\mathcal{T}(x) = \varinjlim \mathcal{U}_x$. Recall that there are finitely many simple regular modules in $\mathcal{U}_x$ and, for almost all $x \in \mathbb{X}$, there is only one simple regular module. This implies that there are finitely many Prüfer modules in $\mathcal{T}(x)$ and, for almost all $x \in \mathbb{X}$, there is only one Prüfer module in $\mathcal{T}(x)$. Let $\omega_0(x)$ be the full subcategory of all the direct sums of copies of the Prüfer modules in $\mathcal{T}(x)$. Therefore, we can divide ω_0 in further subclasses $\omega_0(x)$, for $x \in \mathbb{X}$. As shown in the following picture:

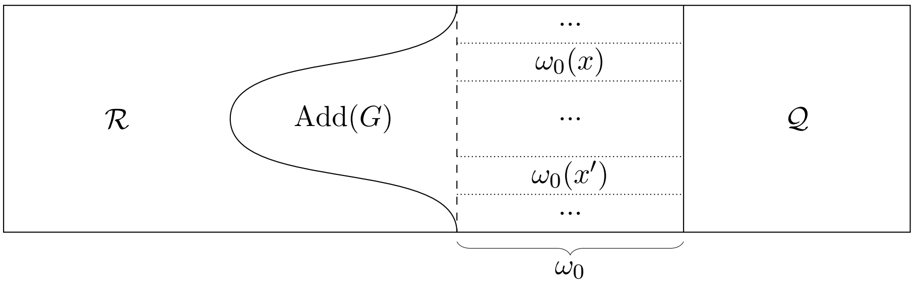


Figure 3.2: Subclasses of Mod-Λ.

ω_0 is separating in the following sense:

$$\begin{aligned}\mathrm{Hom}_\Lambda(\omega_0, \mathcal{R}) = \mathrm{Hom}_\Lambda(\omega_0, \mathrm{Add}(G)) = \mathrm{Hom}_\Lambda(\mathcal{Q}, \mathcal{R}) = \\ = \mathrm{Hom}_\Lambda(\mathcal{Q}, \mathrm{Add}(G)) = \mathrm{Hom}_\Lambda(\mathcal{Q}, \omega_0) = 0\end{aligned}$$

and any map $h\colon X \to Y$, with $X \in \mathcal{R}$ or $X \in \mathrm{Add}(G)$ and $Y \in \mathcal{Q}$, factors through ω_0: indeed, take an ω-cover of Y

$$0 \longrightarrow M' \longrightarrow M \longrightarrow Y \longrightarrow 0$$

with $M \in \omega_0$ and $M' \in \mathrm{Add}(G)$. Apply $\mathrm{Hom}_\Lambda(X, -)$ to obtain:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(X, M') \longrightarrow \mathrm{Hom}_\Lambda(X, M) \longrightarrow \mathrm{Hom}_\Lambda(X, Y) \longrightarrow \mathrm{Ext}^1_\Lambda(X, M')$$

but $\mathrm{Ext}^1_\Lambda(X, M') = 0$ because $(\mathcal{D}, \mathcal{R})$ is a split torsion pair. Therefore $\mathrm{Hom}_\Lambda(X, M) \to \mathrm{Hom}_\Lambda(X, Y)$ is a surjection and the map h factors through $M \in \omega_0$.

3.3 Tubular algebras

A concealed canonical algebra Λ can be of *domestic*, *tubular* or *wild* representation type. This is determined via a numerical invariant called *genus*, as shown in [47, Theorem 7.1].
In the domestic case, Λ is *tame concealed*, ie. $\Lambda \cong \mathrm{End}_{\Lambda'}(T)$, where T is a preprojective or preinjective tilting module over a finite dimensional tame hereditary algebra Λ'.
From now on, Λ is a *tubular algebra*, ie. a concealed canonical algebra of tubular representation type.
The structure of mod-Λ is well known, see for example [47, 56]. According to [56, Theorem 5.2(4)], there is a preprojective component, called $\mathbf{p}_0$, and a preinjective component, called $\mathbf{q}_\infty$. Denote by I_0 the ideal which is maximal with respect to the property that it annihilates all the modules in $\mathbf{p}_0$ and by I_∞ the ideal which is maximal with respect to the property that it

annihilates all the modules in $\mathbf{q}_\infty$. Then, we obtain two factor algebras

$$\Lambda_0 = \Lambda/I_0 \quad \text{and} \quad \Lambda_\infty = \Lambda/I_\infty,$$

which are both tame concealed.

Denote by $\mathbf{t}_0$ the Auslander-Reiten components of mod-Λ which contains regular Λ_0-modules and by $\mathbf{t}_\infty$ the Auslander-Reiten component of mod-Λ which contains regular Λ_∞-modules. Both $\mathbf{t}_0$ and $\mathbf{t}_\infty$ are sincere and separating tubular families, but they are not stable, indeed $\mathbf{t}_0$ contains indecomposable projective Λ-modules and $\mathbf{t}_\infty$ contains indecomposable injective Λ-modules.

Denote now by $\mathbf{q}_0$ the indecomposable Λ-modules which do not belong to $\mathbf{p}_0$ or $\mathbf{t}_0$, then $\mathbf{t}_0$ separates $\mathbf{p}_0$ from $\mathbf{q}_0$. Furthermore, denote by $\mathbf{p}_\infty$ the indecomposable Λ-modules which do not belong to $\mathbf{q}_\infty$ or $\mathbf{t}_\infty$, then $\mathbf{t}_\infty$ separates $\mathbf{p}_\infty$ from $\mathbf{q}_\infty$. Furthermore, the indecomposable modules in $\mathbf{q}_0 \cap \mathbf{p}_\infty$ fall into a countable number of sincere stable separating tubular families indexed by the positive rational numbers, denoted by $\mathbf{t}_\alpha$, $\alpha \in \mathbb{Q}_{>0}$.

All the tubular families $\mathbf{t}_\alpha$, for $\alpha \in \widehat{\mathbb{Q}}_{\geq 0} = \mathbb{Q}_{\geq 0} \cup \{\infty\}$, are such that, for $\alpha, \beta \in \widehat{\mathbb{Q}}_{\geq 0}$ with $\alpha < \beta$, $\mathbf{t}_\alpha$ generates $\mathbf{t}_\beta$ and $\mathbf{t}_\beta$ cogenerates $\mathbf{t}_\alpha$.

Fix a number $w \in \widehat{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, set:

$$\mathbf{p}_w = \mathbf{p}_0 \cup \bigcup_{\alpha < w} \mathbf{t}_\alpha \quad \text{and} \quad \mathbf{q}_w = \bigcup_{w < \beta} \mathbf{t}_\beta \cup \mathbf{q}_\infty$$

If $w \in \widehat{\mathbb{Q}}_{\geq 0}$, then we obtain a trisection $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$, in which $\mathbf{t}_w$ is a sincere tubular family that separates $\mathbf{p}_w$ and $\mathbf{q}_w$, stable if $w \notin \{0, \infty\}$. If w is irrational, then all the indecomposable modules fall into two distinct classes $\mathbf{p}_w$ and $\mathbf{q}_w$ and we have a bisection $(\mathbf{p}_w, \mathbf{q}_w)$.

The category mod-Λ can be depicted as follows:

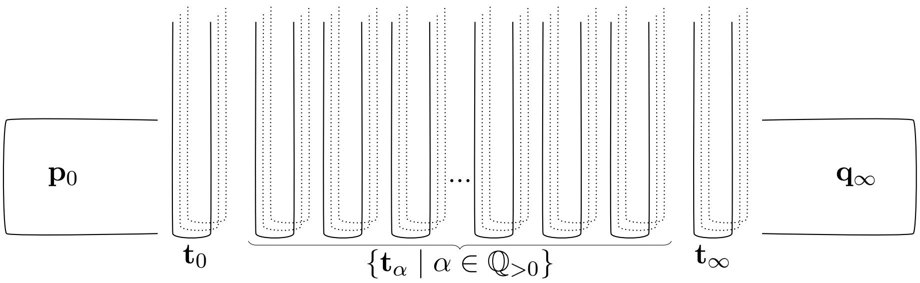


Figure 3.3: Auslander-Reiten quiver of mod-Λ, for a tubular algebra Λ.

3.3.1 Torsion pairs in tubular algebras

For $w \in \widehat{\mathbb{R}}_{\geq 0}$, we consider the following torsion pairs in Mod-Λ.

- The torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$, generated by $\mathbf{q}_w$, where $\mathcal{Q}_w = \text{Gen}(\mathbf{q}_w)$ and $\mathcal{C}_w = \mathbf{q}_w{}^\circ$. If $w \in \widehat{\mathbb{Q}}_{\geq 0}$, this torsion pair is split. Using the Auslander-Reiten formulas, we have $\mathcal{C}_w = \mathbf{q}_w{}^\circ = {}^\perp\mathbf{q}_w$. (See [53, Section 13.1]).

  In this case, the heart $\mathcal{A}_w$, arising from this torsion pair, is equivalent to the category Qcoh$\mathbb{X}_w$ of quasi-coherent sheaves over a non-commutative curve of genus zero $\mathbb{X}_w$

parametrizing the family $\mathbf{t}_w$. We will describe this category in Chapter 6.

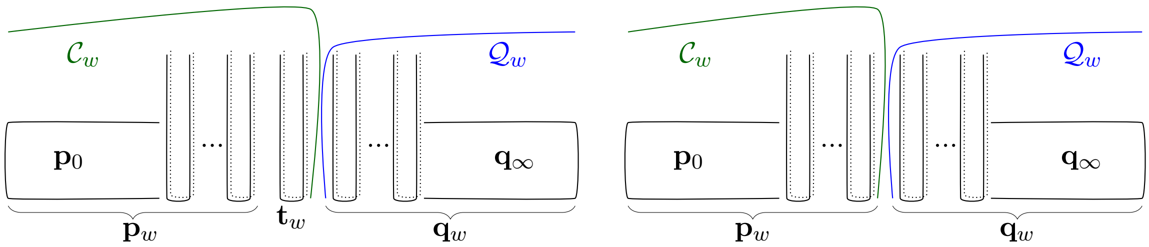


Figure 3.4: Torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ for w rational (left) and w irrational (right)

- The torsion pair $(\mathcal{B}_w, \mathcal{P}_w)$, cogenerated by $\mathbf{p}_w$, where $\mathcal{B}_w = {}^\circ\mathbf{p}_w$ and $\mathcal{P}_w = \mathcal{B}_w{}^\circ$. Moreover, via the separating condition, we have $\mathcal{B}_w = {}^\circ(\bigcup_{\alpha<w} \mathbf{t}_\alpha)$. Using the Auslander-Reiten formulas, we have $\mathcal{B}_w = {}^\circ\mathbf{p}_w = \mathbf{p}_w{}^\perp$. (See [53, Section 13.3]).

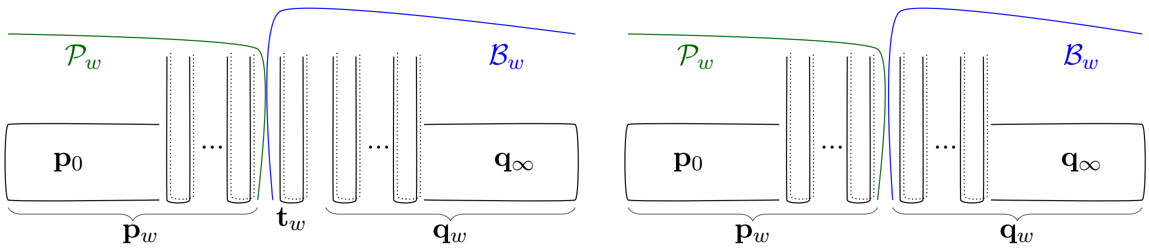


Figure 3.5: Torsion pair $(\mathcal{B}_w, \mathcal{P}_w)$ for w rational (left) and w irrational (right)

- The torsion pair $(\mathrm{Gen}(\mathbf{t}_w), \mathcal{F}_w)$, generated by $\mathbf{t}_w$, where $\mathcal{F}_w = \mathbf{t}_w{}^\circ$. Moreover, via the separating condition, we can state that $\mathcal{F}_w = (\bigcup_{w\leq\gamma} \mathbf{t}_\gamma)^\circ = (\mathbf{t}_w \cup \mathbf{q}_w)^\circ$. Using the Auslander-Reiten formulas, we have $\mathcal{F}_w = \mathbf{t}_w{}^\circ = {}^\perp\mathbf{t}_w$. (See [3, Section 6.2]).

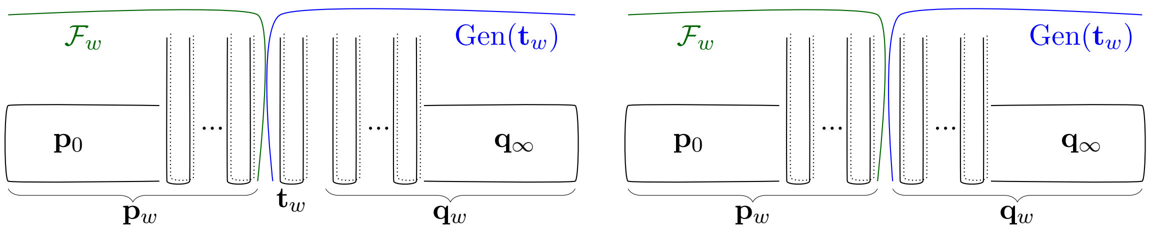


Figure 3.6: Torsion pair $(\mathrm{Gen}(\mathbf{t}_w), \mathcal{F}_w)$ for w rational (left) and w irrational (right)

- The torsion pair $(\mathcal{D}_w, \mathcal{R}_w)$, cogenerated by $\mathbf{t}_w$, where $\mathcal{D}_w = {}^\circ\mathbf{t}_w$ and $\mathcal{R}_w = \mathcal{D}_w{}^\circ$. If $w \in \mathbb{Q}_{\geq 0}$, this torsion pair is split. Moreover, since, for any $\alpha < w$, $\mathbf{t}_\alpha$ is cogenerated by $\mathbf{t}_w$, we have $\mathcal{D}_w = {}^\circ(\bigcup_{\alpha\leq w} \mathbf{t}_\alpha) = {}^\circ(\mathbf{p}_w \cup \mathbf{t}_w)$. Using the Auslander-Reiten formulas, we have $\mathcal{D}_w = {}^\circ\mathbf{t}_w = \mathbf{t}_w{}^\perp$. (See [3, Section 6.2]).

For $\alpha \in \mathbb{Q}_{>0}$, we have the trisection $(\mathbf{p}_\alpha, \mathbf{t}_\alpha, \mathbf{q}_\alpha)$, where $\mathbf{t}_\alpha$ is a sincere stable and separating tubular family, therefore we can rephrase in the setting of tubular algebras many results we have seen for general concealed canonical algebras. In particular, there is a subcategory

$$\omega_\alpha = \mathcal{C}_\alpha \cap \mathcal{D}_\alpha$$

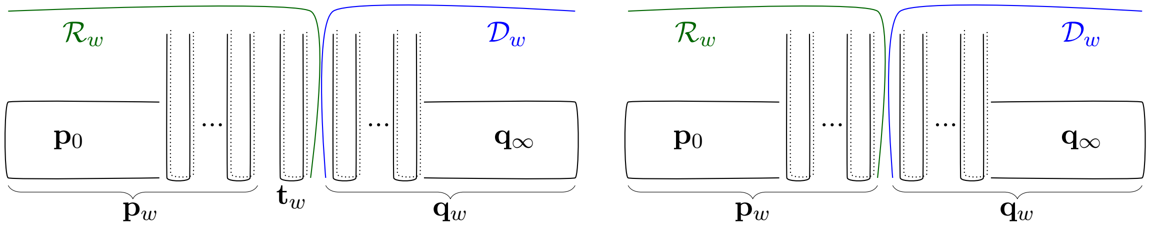


Figure 3.7: Torsion pair $(\mathcal{D}_w, \mathcal{R}_w)$ for w rational (left) and w irrational (right)

for which every property described in Section 3.2.1 holds. Especially, ω_α contains a generic module G_α and Prüfer modules $S^\infty_{x,\alpha}$. Following [53, Section 13.2], there are also generic modules G_0 and G_∞. According to Proposition 3.2.9, for $\alpha \in \mathbb{Q}_{>0}$, we have that $\mathcal{D}_\alpha = {}^\circ\mathbf{t}_\alpha = \mathrm{Gen}(G_\alpha)$.

Lemma 3.3.1. *[53, Lemma 13.2] Let $\alpha, \beta \in \widehat{\mathbb{Q}}_{\geq 0}$. If $\alpha < \beta$, then $\mathbf{t}_\alpha$ generates G_β. If, in addition, $\alpha \neq 0$, then G_α generates $\mathbf{t}_\beta$.*

We have the following properties for the classes defined above:

Lemma 3.3.2. *[3, Lemma 6.3][53, Lemma 13.4] Let $w \in \mathbb{R}_{>0}$.*

(i) *For $v \leq w$, $\mathcal{C}_v \subseteq \mathcal{C}_w$ and $\mathcal{B}_v \supseteq \mathcal{B}_w$.*
(ii) *We have:*

$$\mathcal{C}_w = \bigcap_{w<v\in\mathbb{R}_{>0}} \mathcal{C}_v = \bigcap_{w<\gamma\in\mathbb{Q}_{>0}} \mathcal{F}_\gamma \quad \textit{and} \quad \mathcal{B}_w = \bigcap_{w>v\in\mathbb{R}_{>0}} \mathcal{B}_v = \bigcap_{w>\alpha\in\mathbb{Q}_{>0}} \mathcal{D}_\alpha.$$

(iii) *$\mathcal{Q}_w = \varinjlim \mathbf{q}_w$ and, if $w \notin \mathbb{Q}$, $\mathcal{C}_w = \varinjlim \mathbf{p}_w$.*
(iv) *$\mathcal{P}_w \subset \mathcal{C}_w$ and $\mathcal{Q}_w \subset \mathcal{B}_w$. If $w \in \mathbb{Q}_{>0}$, then $\mathcal{P}_w \subset \mathcal{F}_w \subset \mathcal{C}_w$ and $\mathcal{Q}_w \subset \mathcal{D}_w \subset \mathcal{B}_w$.*
(v) *$(\mathcal{C}_w)^\perp \subset \mathcal{B}_w$ and*

$$\begin{aligned}\mathcal{B}_w = \bigcap_{w>v\in\widehat{\mathbb{R}}} (\mathcal{C}_v)^\perp = \bigcap_{w>v\in\widehat{\mathbb{R}}} \mathcal{Q}_v = \bigcap_{w>\alpha\in\widehat{\mathbb{Q}}} \mathcal{Q}_\alpha = \\ = \{M \in \text{Mod-}\Lambda \mid M \in \mathrm{Gen}(\mathbf{t}_\alpha) \textit{ for any } \alpha \in \mathbb{Q}, 0 < \alpha < w\} \\ = \{M \in \text{Mod-}\Lambda \mid M \in \mathrm{Gen}(G_\alpha) \textit{ for any } \alpha \in \mathbb{Q}, 0 < \alpha < w\}.\end{aligned}$$

Consider now, for any $w \in \widehat{\mathbb{R}}_{\geq 0}$, the class:

$$\mathcal{M}_w = \mathcal{C}_w \cap \mathcal{B}_w$$

Definition 3.3.3. We say that a Λ-module M has *slope* w if $M \in \mathcal{M}_w$.

Clearly, if $\alpha \in \widehat{\mathbb{Q}}_{\geq 0}$, the modules in $\mathbf{t}_\alpha$ and in ω_α have slope α.

Proposition 3.3.4. *[53, Proposition 13.5] The subcategories $\mathcal{C}_w, \mathcal{B}_w$ and $\mathcal{M}_w$ are closed under products and direct limits.*

Theorem 3.3.5. *[53, Theorem 13.1] Any indecomposable Λ-module which does not belong to $\mathbf{p}_0$ or $\mathbf{q}_\infty$ has a slope. For $0 \leq w < w' \leq \infty$, we have $\mathrm{Hom}_\Lambda(\mathcal{M}_{w'}, \mathcal{M}_w) = 0$.*

Proof. We repeat the proof in [53].
$\mathcal{M}_w \subseteq \mathcal{C}_w$ and, from 3.3.2(v), $\mathcal{M}_{w'} \subseteq \mathcal{B}_{w'} \subseteq \mathcal{Q}_w$. Therefore $\mathrm{Hom}_\Lambda(\mathcal{M}_{w'}, \mathcal{M}_w) = 0$, because $(\mathcal{Q}_w, \mathcal{C}_w)$ is a torsion pair.
Let now M be an indecomposable Λ-module which does not belong to $\mathbf{p}_0$ or $\mathbf{q}_\infty$. Because of this, $\mathrm{Hom}_\Lambda(\mathbf{q}_\infty, M) = 0$. Let w be the infimum of all $\alpha \in \widehat{\mathbb{Q}}_{\geq 0}$ such that $\mathrm{Hom}_\Lambda(\mathbf{q}_\alpha, M) = 0$. Since $\mathbf{q}_w = \bigcup_{w>\alpha} \mathbf{q}_\alpha$, we have $\mathrm{Hom}_\Lambda(\mathbf{q}_w, M) = 0$, therefore $M \in \mathbf{q}_w{}^\circ = \mathcal{C}_w$.
Now we need to prove that $M \in \mathcal{B}_w$. First of all, if $w = 0$, this follows immediately from the fact that M is indecomposable and is not in $\mathbf{p}_0$. Let then $w > 0$. Let α be a rational number such that $0 < \alpha < w$. Assume $M \notin \mathcal{Q}_\alpha$. Since $(\mathcal{Q}_\alpha, \mathcal{C}_\alpha)$ is a split torsion pair and M is indecomposable, we infer that $M \in \mathcal{C}_\alpha$. Therefore $\mathrm{Hom}_\Lambda(\mathbf{q}_\alpha, M) = 0$ and this implies, by definition of w, that $w \leq \alpha$. This is a contradiction. Consequently, $M \in \mathcal{Q}_\alpha$, for any rational number α with $0 < \alpha < w$. Hence, via 3.3.2(v), $M \in \mathcal{B}_w = \bigcap_{w>\alpha\in\widehat{\mathbb{Q}}} \mathcal{Q}_\alpha$. □

We have many examples of modules of rational slope but, if w is irrational, not many modules of slope w are known. For w irrational, $\mathcal{M}_w$ does not contain any nonzero module of finite length. In [53], Reiten and Ringel have provided two useful theorems to construct infinite dimensional modules of irrational slope.

Theorem 3.3.6 (First construction)**.** *[53, Section 13.4] Let $\alpha_1 > \alpha_2 > \dots$ be a decreasing sequence of rational numbers converging to $w \in \mathbb{R}_{>0}$. Choose modules $M_i \in \mathrm{add}(\mathbf{t}_{\alpha_i})$, for $i \geq 1$. Then $\prod_i M_i / \bigoplus_i M_i \in \mathcal{M}_w$.*

Theorem 3.3.7 (Second construction)**.** *[53, Section 13.4] Let $\alpha_1 < \alpha_2 < \dots$ be an increasing sequence of rational numbers converging to $w \in \mathbb{R}_{>0}$. Choose modules $M_i \in \mathrm{add}(\mathbf{t}_{\alpha_i})$ such that $M_i \subseteq M_{i+1}$ for any $i \geq 1$, then $M = \varinjlim M_i \in \mathcal{M}_w$.*

3.3.2 Tilting and cotilting Λ-modules

First of all, let us consider the torsion pairs $(\mathcal{B}_w, \mathcal{P}_w)$ and $(\mathcal{Q}_w, \mathcal{C}_w)$. Following [3, Section 6.1], the class $\mathrm{add}(\mathbf{p}_w)$ is a resolving class, ie. closed under direct summands, extensions and kernel of epimorphisms, therefore, by Theorem 2.1.4, $\mathcal{B}_w$ is a tilting class. Hence there is a tilting module $\mathbf{L}_w$ such that $\mathcal{B}_w = \mathrm{Gen}(\mathbf{L}_w)$ and $\mathcal{P}_w = \mathbf{L}_w{}^\circ$. This tilting module is infinite dimensional, otherwise $\mathbf{L}_w \in \mathcal{B}_w \cap {}^\perp\mathcal{B}_w \cap \text{mod-}\Lambda = \mathcal{B}_w \cap \mathrm{add}(\mathbf{p}_w) = {}^\circ\mathbf{p}_w \cap \mathrm{add}(\mathbf{p}_w)$, which is not possible.
Furthermore, the class $\mathbf{q}_w$ is the dual of a resolving class $\mathbf{p}_{\bar{w}} \subset \Lambda\text{-mod}$. It follows that $\mathcal{C}_w = (\mathbf{p}_{\bar{w}})^\top$ is a cotilting class given by a large cotilting module $\mathbf{W}_w$ such that $\mathcal{C}_w = \mathrm{Cogen}(\mathbf{W}_w)$ and $\mathcal{Q}_w = {}^\circ\mathbf{W}_w$.
If $w \in \mathbb{Q}_{>0}$, we have the trisection $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$ and we are in the same setting as in [53, Chapter 10]. Here, the cotilting module $\mathbf{W}_w$, that cogenerates the class $\mathcal{C}_w$, also generates the class $\mathcal{D}_w = {}^\circ\mathbf{t}_w$ and there is an explicit description for it:

$$\mathbf{W}_w = G_w \oplus \bigoplus_{x\in\mathbb{X}} S^\infty_{x,w}$$

where G_w is the generic module in ω_w and the $S^\infty_{x,w}$ are the Prüfer module of slope w, parametrized by the set $\mathbb{X}$, as seen in Section 3.1.

For an irrational number w, we have the following:

Theorem 3.3.8. *[3, Theorem 6.4] Let $w \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$, then:*

(1) $\mathbf{L}_w$ *is the only tilting module of slope w, up to equivalence.*
(2) $\mathbf{W}_w$ *is the only cotilting module of slope w, up to equivalence.*
(3) *The following are equivalent:*

   (i) $M \in \mathcal{M}_w$.
   (ii) M *is a pure submodule of a product of copies of* $\mathbf{W}_w$.
   (iii) M *is a pure epimorphic image of a direct sum of copies of* $\mathbf{L}_w$.

Proof. By definition the modules $\mathbf{L}_w$ and $\mathbf{W}_w$ have slope w. Indeed:

$$\mathbf{L}_w \in \mathcal{B}_w \cap {}^{\perp}(\mathcal{B}_w) \subseteq \mathcal{B}_w \cap \mathcal{C}_w \quad \text{and} \quad \mathbf{W}_w \in \mathcal{C}_w \cap {\mathcal{C}_w}^{\perp} \subseteq \mathcal{C}_w \cap \mathcal{B}_w.$$

(1) Let T be a tilting module of slope w. By Lemma [3, Lemma 5.4], the resolving subcategory of mod-Λ corresponding to T is $\mathcal{S} = {}^{\perp}(T^{\perp}) \cap \text{mod-}\Lambda = \text{add}(\mathbf{p}_w)$. This implies, by Theorem 2.1.4, that T is equivalent to $\mathbf{L}_w$.
(2) Let C be a cotilting module of slope w. Then ${}^{\perp}\mathcal{B}_w \subset {}^{\perp}C$, since $C \in \mathcal{B}_w$, and ${}^{\perp}C = \text{Cogen}(C) \subset \mathcal{C}_w$, since $\mathcal{C}_w$ is a torsionfree class and $C \in \mathcal{C}_w$. Since $\mathbf{p}_w \subset {}^{\perp}\mathcal{B}_w$ and ${}^{\perp}C$ is closed under direct limits, we have $\varinjlim \mathbf{p}_w \subset {}^{\perp}C$. Lemma 6.2.6(iii) tells us that $\varinjlim \mathbf{p}_w = \mathcal{C}_w$, hence $\mathcal{C}_w = {}^{\perp}C$. So C is equivalent to $\mathbf{W}_w$.
(3) By [11, Proposition 4.3(2)], we have that the class $\mathcal{M}_w$ is definable, hence closed under direct sums, direct products, pure submodules and pure epimorphic images. This proves "(ii) implies (i)" and "(iii) implies (i)".
Let us prove "(i) implies (ii)": Consider a module $M \in \mathcal{M}_w$. From the complete cotorsion pair $(\mathcal{C}_w, {\mathcal{C}_w}^{\perp})$ we get a special ${\mathcal{C}_w}^{\perp}$-envelope of M:

$$0 \longrightarrow M \longrightarrow C_0 \longrightarrow C_1 \longrightarrow 0$$

where $C_0 \in {\mathcal{C}_w}^{\perp}$ and $C_1 \in \mathcal{C}_w$. Since $M \in \mathcal{C}_w$, also $C_0 \in \mathcal{C}_w$. Since, by Theorem 3.2.1, the modules in $\mathcal{C}_w$ have projective dimension at most 1, the class ${\mathcal{C}_w}^{\perp}$ is closed under quotient modules, therefore $C_1 \in {\mathcal{C}_w}^{\perp}$. This means that $C_0, C_1 \in \mathcal{C}_w \cap {\mathcal{C}_w}^{\perp} = \text{Prod}(\mathbf{W}_w)$.
Moreover, the sequence defined above is pure-exact, indeed: let $X \in \text{mod-}\Lambda$, we can assume w.l.o.g. that X is indecomposable with $\text{Hom}_\Lambda(X, C_1) \neq 0$, hence $X \in \mathbf{p}_w$. Then, since $M \in \mathcal{B}_w = {\mathbf{p}_w}^{\perp}$, we have $\text{Ext}^1_\Lambda(X, M) = 0$, therefore the sequence

$$0 \longrightarrow \text{Hom}_\Lambda(X, M) \longrightarrow \text{Hom}_\Lambda(X, C_0) \longrightarrow \text{Hom}_\Lambda(X, C_1) \longrightarrow 0$$

is exact, proving the purity of the approximation sequence. Hence, M is a pure submodule of some module in $\text{Prod}(\mathbf{W}_w)$.
The implication from (i) to (iii) is proven dually. □

3.3.3 Pure-injective Λ-modules

A first characterization of pure-injective modules arising from tubular components in the Auslander-Reiten quiver of Artin algebras has been given by Ringel in [54]. The Theorem stated in [54, 2.2] can be reinterpreted in the context of tubular algebras via the following.

Theorem 3.3.9. *[54, 2.2] Let $w \in \mathbb{Q}_{\geq 0}$. A Λ-module M is pure-injective and belongs to $\mathcal{M}_w$ if and only if there is a decomposition $M = M' \oplus M''$ where $M' \in \mathrm{Prod}(\mathbf{t}_w)$ and $M'' \in \mathrm{Add}(\mathbf{W}_w) = \mathrm{Prod}(\mathbf{W}_w)$.*

Let us recall that a Λ-module M is called *superdecomposable* if it has no indecomposable direct summands. In [32], the authors proved that, if the field k is countable, then pure-injective modules of irrational slope can be superdecomposable. We have:

Theorem 3.3.10. *[32, Corollary 7.4] Let w be a positive irrational number. If the field k is countable, then there is a superdecomposable pure-injective Λ-module of slope w.*

The proof of this Theorem uses methods from logic and model theory of modules.
In the case of a cotilting module, which is pure-injective by Theorem 2.2.2, the superdecomposable part of the module is not relevant in the computation of the cotilting class. Indeed, we have the following:

Theorem 3.3.11. *[64, Theorem 3.7] Let W be a cotilting Λ-module. Then $\mathrm{Prod}(W)$ contains a family of indecomposable module $(M_i)_{i\in I}$ such that W is a direct summand in a direct limit of modules in $\mathrm{Prod}\{M_i \mid i \in I\}$ and ${}^{\perp}W = \bigcap_{i\in I} {}^{\perp}M_i$.*

Finally, for the irrational slope case, we have the following characterization of pure-injective Λ-modules, given by Angeleri Hügel and Kussin in [3].

Theorem 3.3.12. *[3, Corollary 6.6] Let $w \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$. Then the cotilting module $\mathbf{W}_w$ is pure-injective, not Σ-pure-injective and the class $\mathrm{Prod}(\mathbf{W}_w)$ is the class of all pure-injective Λ-modules of slope w.*

Proof. $\mathbf{W}_w$ is pure-injective by Theorem 2.2.2 and $\mathrm{Prod}(\mathbf{W}_w)$ is the class of all pure-injective Λ-modules of slope w by Theorem 3.3.8(3).
Assume that $\mathbf{W}_w$ is Σ-pure-injective. Then every product of copies of $\mathbf{W}_w$ and any pure submodule of such product is Σ-pure-injective. This implies, using Theorem 3.3.8(3), that $\mathcal{M}_w = \mathrm{Prod}(\mathbf{W}_w)$. Hence $\mathbf{L}_w \in \mathcal{M}_w \subseteq {\mathcal{C}_w}^{\perp}$, and since $\mathcal{C}_w$ consists of modules of projective dimension at most one, $\mathrm{Gen}(\mathbf{L}_w) \subseteq {\mathcal{C}_w}^{\perp}$, and ${\mathcal{C}_w}^{\perp} = \mathcal{B}_w$ by Lemma 3.3.2(v). Since $\mathcal{Q}_w \subseteq \mathcal{B}_w$, by Lemma 3.3.2(iv), we have that $(\mathcal{Q}_w, \mathcal{B}_w)$ is a split torsion pair satisfying the assumption of Proposition 2.2.7. So, the heart $\mathcal{A}_w$ of the t-structure arising from the previous torsion pair is equivalent to the category $\mathrm{Qcoh}\mathbb{Y}$ of quasi-coherent sheaves over a noncommutative curve of genus zero $\mathbb{Y}$, this category will be described completely in Chapter 6. This is a locally noetherian Grothendieck category whose category of coherent objects is denoted by $\mathrm{coh}\mathbb{Y}$ and $\mathcal{H}_0$ denotes the subcategory of finite length objects in $\mathrm{coh}\mathbb{Y}$. As we will see in Proposition 6.1.2, there is a family of connected uniserial Hom-orthogonal length categories $\mathcal{U}_y$, $y \in \mathbb{Y}$, such that

all $\mathcal{U}_y$ have finite τ-period and $\mathcal{H}_0 = \coprod_{y \in \mathbb{Y}} \mathcal{U}_y$. Therefore, if S is a simple object in $\mathcal{H}_0$, then its injective envelope $E(S)$ has only finitely many non-isomorphic composition factors. Moreover, S is of the form $S = Y[1]$, for $Y \in \mathbf{p}_w$, or $S = Q$, for $Q \in \mathbf{q}_w$.
In the first case, $Y \in \mathbf{p}_\alpha$, for some $\alpha < w$, and there is $\alpha < \beta < w$ such that $E(S)$ has all composition factors in $\mathbf{p}_\beta[1]$ and therefore $\mathrm{Hom}_{\mathcal{A}_w}(\mathbf{t}_\beta[1], E(S)) = 0$. On the other hand, Y is cogenerated by $\mathbf{t}_\beta$, so there is a nonzero morphism $Y \to B$ for some indecomposable module $B \in \mathbf{t}_\beta$. This morphism yield a nonzero map $S \to B[1]$ and therefore a nonzero map $B[1] \to E(S)$, by the injectivity of $E(S)$. This is a contradiction.
This shows that the simples in $\mathcal{H}_0$ are all of the form $S = Q$, for some $Q \in \mathbf{q}_w$, so they belong to the torsionfree class in the torsion pair $(\mathcal{C}_w[1], \mathcal{Q}_w)$ in $\mathcal{A}_w$. But then the noetherian tilting object $V = \Lambda[1] \in \mathcal{C}_w[1]$ cannot have a simple quotient, another contradiction. □

For the sake of completeness, we summarize in the following Theorem the list of all indecomposable pure-injective Λ-modules.

Theorem 3.3.13. *[3, Theorem 6.7] The following is a complete list of the indecomposable pure-injective Λ-modules:*

(i) *the finite dimensional indecomposable Λ-modules,*
(ii) *the Prüfer modules, the adic modules and the generic module of slope $w \in \mathbb{Q}_{\geq 0}$,*
(iii) *the indecomposable modules in $\mathrm{Prod}(\mathbf{W}_w)$, with $w \in \mathbb{R}_{\geq 0} \setminus \mathbb{Q}_{\geq 0}$,*
(iv) *the Prüfer modules, the adic modules and the generic module over Λ_0 and Λ_∞,*
(v) *a finite number of Prüfer Λ-modules of slope 0 and a finite number of adic module of slope ∞ (we refer to [3] for more details about these modules).*

The modules in (iii) are the ones we aim to characterize in greater detail. In the next Chapter we introduce a tool that could provide a better understanding of these modules.

Chapter 4

The Atom Spectrum

In this chapter we introduce the notion of atom spectrum for a Grothendieck category. This is a generalization of the prime spectrum for a commutative ring and, in the same fashion, it is endowed with a topological space structure. The notion of atom spectrum has been introduced by Kanda in [35] in the more general setting of abelian categories, and properties have been developed by Vámos and Virili in [65].

4.1 Definition and properties

Definition 4.1.1. Let $\mathcal{G}$ be an Grothendieck category and let X be an object of $\mathcal{G}$. X is said to be *monoform* if, given any subobject H of X and a morphism $\varphi \in \mathrm{Hom}_{\mathcal{G}}(H, X)$, φ is non-zero if and only if it is a non-zero monomorphism.

Lemma 4.1.2. *[65, Lemma 2.10] Let $X \in \mathcal{G}$. The following conditions are equivalent:*

- (i) *X is monoform.*
- (ii) *for any non-zero subobject $H \subseteq X$, the unique object isomorphic to both a subobject of X and to a subobject of X/H is the zero object.*
- (iii) *X is uniform and, for any non-zero subobject $H \subseteq X$, the unique object isomorphic to both a subobject of H and to a subobject of X/H is the zero object.*

In [35], the author uses (ii) of the Lemma above as a definition of monoform object. Let us collect some useful facts.

Proposition 4.1.3. *Let $X \in \mathcal{G}$. Then:*

- (i) *If X is a simple object, then it is monoform.*
- (ii) *If X is monoform, then any nonzero $Y \subseteq X$ is monoform.*
- (iii) *If X is monoform, then it is uniform.*

Proof. (i) This is clear, since any simple object has only trivial subobjects.

(ii) [35, Proposition 2.2] Let X be monoform and $Y \subseteq X$, then, if Y is not monoform, there is a proper subobject $H \subseteq Y$ such that Y and Y/H have a common nonzero subobject H'. But H' is also a subobject of X and X/H and this is a contradiction.

(iii) [35, Proposition 2.6] Let X be monoform and suppose it is not uniform, then there exist two proper subobjects H and H' of X such that $H \cap H' = 0$, therefore we have:

$$X \supseteq H \cong H \oplus H'/H' \subseteq X/H'$$

and this cannot happen since X is monoform.

□

Theorem 4.1.4. *[35, Theorem 2.9] Let X be a noetherian object in $\mathcal{G}$. There is a filtration:*

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

such that X_i/X_{i-1} is a monoform object, for any $i = 1, \dots, n$. In particular, $X_1 i$ s a monoform subobject if X.

We say that two monoform objects in $\mathcal{G}$ are *atom-equivalent* if they have a common non-zero subobject.

Lemma 4.1.5. *[35, Lemma 5.8][65, Lemma 2.13] Two monoform objects X and X' are atom-equivalent if and only if $E(X) \cong E(X')$.*

Proof. Assume X and X' have a common non-zero subobject Y. Then, since X and X' are uniform, we have $E(X) \cong E(Y) \cong E(X')$.
Conversely, assume $E(X) \cong E(X')$. Then X and X' are, up to isomorphism, both non-zero subobjects of $E(X)$. Since X is uniform, $E(X)$ is uniform. Therefore X and X' have $X \cap X'$ as a common non-zero subobject. □

We denote by $\overline{X}$ the class of all monoform objects atom-equivalent to the monoform object X, these equivalence classes are called *atoms.*

Definition 4.1.6. The *atom spectrum* of an abelian category $\mathcal{G}$, denoted by ASpec($\mathcal{G}$), is the class of all atoms in $\mathcal{G}$.

In the case of a Grothendieck category $\mathcal{G}$, it can be shown that given a monoform object X, the class of all monoform objects equivalent to X has a maximum $\bar{X}$, in the sense that any monoform object equivalent to X is isomorphic to a subobject of $\bar{X}$, which is characterized in the following proposition.

Proposition 4.1.7. *[65, Proposition 2.12] Let $\mathcal{G}$ be a Grothendieck category, let X be an object of $\mathcal{G}$. Consider the torsion pair $\mathbf{T}_X = (\mathcal{T}_X, \mathcal{F}_X)$ cogenerated by $E = E(X)$. The followings are equivalent:*

(i) *X is monoform.*
(ii) *$\mathbf{Q}_X(X)$ is the unique simple object in $\mathcal{G}/\mathcal{T}_X$, up to isomorphisms (where $\mathbf{Q}_X \colon \mathcal{G} \to \mathcal{G}/\mathcal{T}_X$ denotes the quotient functor relative to $\mathbf{T}_X$).*
(iii) *there exists a hereditary torsion pair $\mathbf{T} = (\mathcal{T}, \mathcal{F})$ such that $X \in \mathcal{F}$ and $\mathbf{Q}(X)$ is simple in $\mathcal{G}/\mathcal{T}$.*

Moreover, if the above conditions are verified, a monoform object Y is equivalent to X if and only if Y is isomorphic to a subobject of $\mathbf{S}_X\mathbf{Q}_X(X)$ (where $\mathbf{S}_X\colon \mathcal{G}/\mathcal{T}_x \to \mathcal{G}$ denotes the section functor relative to $\mathbf{\tau}_X$).

Remark 4.1.8. Notice that condition (iii) says precisely that an object X is monoform if and only if there exists a hereditary torsion pair $\mathbf{\tau}$ such that X is $\mathbf{\tau}$-cocritical.

The atom spectrum of an abelian category may not be a set. On the other hand, we have that $\mathrm{ASpec}(\mathcal{G})$ is a set, whenever $\mathcal{G}$ is a Grothendieck category. This is a consequence of the following lemma.

Lemma 4.1.9. *[65, Lemma 2.13] Let $\mathcal{G}$ be a Grothendieck category. There is a well-defined injective map of sets:*

$$\begin{aligned} \mathrm{ASpec}(\mathcal{G}) &\longrightarrow \mathrm{Spec}(\mathcal{G}) \\ \overline{X} &\longmapsto E(X) \end{aligned}$$

Proof. This is clear by Lemma 4.1.5. □

Theorem 4.1.10. *[35, Theorem 5.9] Let $\mathcal{G}$ be a locally noetherian Grothendieck category. The map:*

$$\begin{aligned} \mathrm{ASpec}(\mathcal{G}) &\longrightarrow \mathrm{Spec}(\mathcal{G}) \\ \overline{X} &\longmapsto E(X) \end{aligned}$$

is a well-defined bijection of sets.

Proof. By Lemma 4.1.9 we have injectivity. Since $\mathcal{G}$ is locally noetherian, an indecomposable injective I has a noetherian subobject I'. By Theorem 4.1.4, there is a monoform subobject H of I' such that $E(H) = I$. This proves surjectivity. □

This result has been generalized for Gabriel categories in [65].

Example 4.1.11. Let Λ be the Kronecker algebra. The category Λ-Mod is a locally noetherian Grothendieck category. Therefore $\mathrm{ASpec}(\Lambda\text{-Mod}) \cong \mathrm{Spec}(\Lambda)$ and the only two indecomposable injective modules are the injective envelopes of the two simple Λ-modules, S_1 and S_2, which are monoform. So we have $\mathrm{ASpec}(\Lambda\text{-Mod}) = \{\overline{S_1}, \overline{S_2}\}$.
The generic module G, for example, is not monoform, indeed we have a short exact sequence $0 \to S_1 \to G \to \bigoplus S_\infty \to 0$ (see [53, Theorem 6.1]) and S_1 is a submodule of any Prüfer module.

4.2 Topology and partial order on the atom spectrum

Definition 4.2.1. Let M be an object of $\mathcal{G}$. Define two subsets of the atom spectrum:

$$\mathrm{ASupp}\, M = \{\alpha \in \mathrm{ASpec}(\mathcal{G}) \mid \alpha = \overline{H} \text{ for a monoform subquotient } H \text{ of } M\}$$

called the *atom support* of M, and

$$\mathrm{AAss}\, M = \{\alpha \in \mathrm{ASpec}(\mathcal{G}) \mid \alpha = \overline{H} \text{ for a monoform subobject } H \text{ of } M\}$$

called the set of *associated atoms* of M.

It is clear that the atom support of a simple object S is the set containing only the atom $\overline{S}$.

Proposition 4.2.2. *[35, Propositions 3.3 and 3.5] Let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathcal{G}$. Then:*

(1) $\operatorname{ASupp} M = \operatorname{ASupp} L \cup \operatorname{ASupp} N$
(2) $\operatorname{AAss} L \subset \operatorname{AAss} M \subset \operatorname{AAss} L \cup \operatorname{AAss} N$.

The atom spectrum of a Grothendieck category $\mathcal{G}$ has a structure of topological space in terms of atom support.

Definition 4.2.3. Let Φ be a subset of $\operatorname{ASpec}(\mathcal{G})$. We say that Φ is *open* if for any atom $\alpha \in \Phi$, there exists a monoform object $H \in \mathcal{G}$ such that $\overline{H} = \alpha$ and $\operatorname{ASupp} H \subset \Phi$.

The family of all open subsets of $\operatorname{ASpec}(\mathcal{G})$ satisfies the axioms of topology.

Proposition 4.2.4. *[36, Proposition 3.2] The family $\{\operatorname{ASupp} M \mid M \in \mathcal{G}\}$ is the set of all open subsets of $\operatorname{ASpec} \mathcal{G}$. If $\mathcal{G}$ is locally noetherian, then the family $\{\operatorname{ASupp} M \mid M$ noetherian in $\mathcal{G}\}$ is an open basis of $\operatorname{ASpec}(\mathcal{G})$.*

In [36, Proposition 3.5], the author proved that the atom spectrum of an abelian category is a T_0-*space* (or *Kolmogorov space*), ie. a space in which, for any distinct points x_1 and x_2 in it, there exists an open subset containing exactly one of them. The atom spectrum is a discrete space if the category satisfies some properties.

Proposition 4.2.5. *[36, Proposition 3.7] Let $\mathcal{G}$ be a locally noetherian Grothendieck category:*

(1) *For an atom $\alpha \in \operatorname{ASpec}(\mathcal{G})$, the subset $\{\alpha\}$ is open if and only if there exists a simple object S in $\mathcal{G}$ such that $\overline{S} = \alpha$.*
(2) *For any noetherian object M in $\mathcal{G}$, the subset $\operatorname{ASupp} M$ of $\operatorname{ASpec}(\mathcal{G})$ with the induced topology is discrete if and only if M has finite length.*
(3) *$\operatorname{ASpec}(\mathcal{G})$ is a discrete topological space if and only if any noetherian object in $\mathcal{G}$ has finite length.*

Proof. (1) Any nonzero object in $\mathcal{G}$ has a simple subquotient. Indeed: let $M \in \mathcal{G}$, there exists a nonzero noetherian subobject L of M and L is finitely generated. Since L has a maximal subobject, it also has a simple quotient object which is a subquotient of M. Thus, if $\{\alpha\}$ is open, by definition, there exists a monoform object H in $\mathcal{G}$ such that $\operatorname{ASupp} H \subset \{\alpha\}$. There exists a simple subquotient S of H, hence $\overline{S} \in \operatorname{ASupp} H \subset \{\alpha\}$ implies $\overline{S} = \alpha$. Conversely, for a simple object S in $\mathcal{G}$, $\operatorname{ASupp} S = \{\overline{S}\}$ and this is an open subset of $\operatorname{ASpec}(\mathcal{G})$.

(2) Suppose that $\operatorname{ASupp} M$ is discrete. Then by (1), since the singletons are open, any element of $\operatorname{ASupp} M$ is represented by a simple object in $\mathring{A}$. By [35, Remark 3.6], $\operatorname{AAss} M$ is nonempty, therefore any quotient object of M has a simple subobject. Hence, for any

filtration we can find a refinement and so a composition series for M. Therefore M has finite length.
Conversely, assume that M has finite length. Because of this, any subquotient of M has a simple subobject and so any element of $\operatorname{ASupp} M$ can be represented by a simple object in $\mathcal{G}$. Hence, $\operatorname{ASupp} M$ with the induced topology is discrete.

(3) Assume that $\operatorname{ASpec}(\mathcal{G})$ is discrete. Then by (2), any noetherian object in $\mathcal{G}$ has finite length.
Conversely, assume that any noetherian object has finite length. Since any object in $\mathcal{G}$ has a noetherian subobject, for any $\alpha \in \operatorname{ASpec}(\mathcal{G})$, there exists a noetherian monoform object H such that $\overline{H} = \alpha$. By assumption, H has finite length, hence there exists a simple subobject S of H and so $\overline{S} = \alpha$. By (1), the topological space $\operatorname{ASpec}(\mathcal{G})$ is discrete. □

Let X be a T_0-space. We can define an order on X in this way: for any $x, y \in X$, we have $x \preceq y$ if and only if $x \in \overline{\{y\}}$, where $\overline{\{y\}}$ is the topological closure of $\{y\}$ in X. This is called *specialization order* and it is a partial order. Conversely, a partially ordered set P can be seen as a topological space as follows: a subset Φ of P is open if and only if for any $p, q \in P$ such that $p \preceq q$, $p \in \Phi$ implies $q \in \Phi$. These correspondences defined above are mutually inverse.
$\operatorname{ASpec}(\mathcal{G})$ becomes a partially ordered set defining a specialization order $\leq$ on it. We have the following:

Proposition 4.2.6. *[36, Proposition 4.2] Let $\mathcal{G}$ be a Grothendieck category and $\alpha, \beta \in \operatorname{ASpec}(\mathcal{G})$. Then the following are equivalent:*

(1) *$\alpha \leq \beta$, i.e. $\alpha \in \overline{\{\beta\}}$.*
(2) *If Φ is an open subset of $\operatorname{ASpec}(\mathcal{G})$ such that $\alpha \in \Phi$, then $\beta \in \Phi$. In other words, β belongs to the intersection of all the open subsets containing α.*
(3) *For any object M in $\mathcal{G}$ such that $\alpha \in \operatorname{ASupp} M$, we have $\beta \in \operatorname{ASupp} M$.*
(4) *For any monoform object H in $\mathcal{G}$ such that $\overline{H} = \alpha$, we have $\beta \in \operatorname{ASupp} H$.*

Proof. (1) $\Rightarrow$ (2): Let Φ be an open subset of $\operatorname{ASpec}(\mathcal{G})$ such that $\alpha \in \Phi$. If $\beta \notin \Phi$, then $\{\beta\} \subset X \setminus \Phi$, and since $X \setminus \Phi$ is closed, we have $\{\bar{\beta}\} \subset X \setminus \Phi$, and so $\alpha \notin \Phi$. Contradiction.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) follows from the definition of the topology on $\operatorname{ASpec}(\mathcal{G})$ and Proposition 4.2.4. □

Moreover we have:

Proposition 4.2.7. *[36, Proposition 4.4] Let X be a T_0-space with the specialization order $\preceq$. Let $x \in X$ and Φ a subset of X containing x.*

(1) *x is maximal in Φ if and only if $\{x\}$ is the intersection of some family of open subsets of Φ.*
(2) *x is minimal in Φ if and only if $\{x\}$ is a closed subset of Φ.*

Proof. (1) Suppose that x is maximal in Φ. Let $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ be the family of all open subsets of Φ containing x. It is clear that $\{x\} \subset \bigcap_{\lambda \in \Lambda} \Phi_\lambda$. Let now $y \in \bigcap_{\lambda \in \Lambda} \Phi_\lambda$. We need to show that $y = x$, or equivalently, since x is maximal, $x \preceq y$. Let Ψ an open set such that $x \in \Psi$, then $x \in \Psi \cap \Phi_\lambda \subset \Phi$ and $\Psi \cap \Phi_\lambda$ is open. Therefore $y \in \Phi_\mu = \Psi \cap \Phi_\lambda$ for a certain $\mu \in \Lambda$ and then $y \in \Psi$ and we have the claim.

Conversely, suppose that $\{x\} = \bigcap_{\lambda \in \Lambda} \Phi_\lambda$, where $\{\Phi_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets of Φ. Let $x \preceq y$, then by Proposition 4.2.6, y is contained in all the open subsets of X containing x, in particular $y \in \bigcap_{\lambda \in \Lambda} \Phi_\lambda$. Therefore $y = x$.

(2) Suppose that x is minimal in Φ. Let $y \in \overline{\{x\}}$, then $y \preceq x$, but since x is minimal, $y = x$. This means that $\overline{\{x\}} = \{x\}$, ie. is closed.

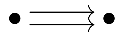
Suppose that $\{x\}$ is closed. Let $y \preceq x$, ie. $y \in \overline{\{x\}}$. Since $\{x\}$ is closed, $\{x\} = \overline{\{x\}}$ implies $y \in \{x\}$, hence $y = x$.

□

Chapter 5

The case of the Kronecker Algebra

Let us now consider the Kronecker algebra Λ, ie. the path algebra of the quiver:



This is a tame hereditary algebra and the Auslander-Reiten quiver of mod-Λ is:

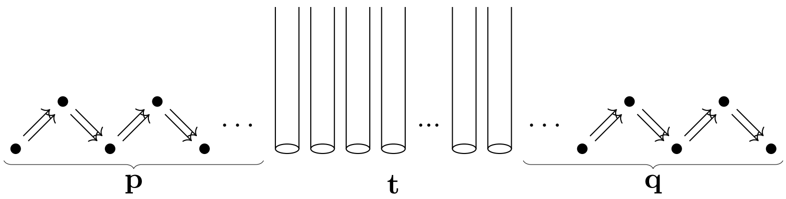


Figure 5.1: Auslander-Reiten quiver of mod-Λ.

As in Chapter 3, $\mathbf{p}$ is the preprojective component, $\mathbf{q}$ is the preinjective component and $\mathbf{t}$ is a sincere stable and separating family of regular homogeneous tubes, $\mathbf{t} = \bigcup_{x \in \mathbb{X}} \mathcal{U}_x$. We denote by S_x^{∞} and $S_x^{-\infty}$ the Prüfer and the adic module, respectively, corresponding to the simple regular S_x. Further, we denote by G the generic module, ie. the unique (up to isomorphism) indecomposable module which has infinite length over Λ, but finite length over its endomorphism ring. Recall, from [53, 55], that $\operatorname{End}_R G$ is a division ring.

5.1 Cotilting torsion pairs in Mod-Λ

We want to describe the torsion pairs in the category of modules over Λ which are cogenerated by infinite dimensional cotilting modules. As we have mentioned in Theorem 2.2.3, cotilting modules in Mod-Λ have been completely classified in [14], we recall the result in this specific setting:

Theorem 5.1.1. *[14, Theorem 1.5, Corollary 3.10] Let C be a cotilting module over the Kronecker algebra Λ. Then C is equivalent to one of the following:*

- *a finitely generated preprojective cotilting module of the form $P_{n+1} \oplus P_{n+2}$, for $n \geq 0$,*
- *a finitely generated preinjective cotilting module of the form $Q_{n+1} \oplus Q_{n+2}$, for $n \geq 0$,*

- *a cotilting module of the form:*

$$C_P = G \oplus \prod_{x\in P} S_x^{-\infty} \oplus \bigoplus_{x\notin P} S_x^{\infty}, \quad \textit{for } P \subseteq \mathbb{X}.$$

Moreover, there is a correspondence between:

$$\left\{\begin{matrix}\textit{equivalence classes}\\ \textit{of cotilting modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{torsion pairs in } \text{mod-}\Lambda\\ \textit{with generating torsionfree class}\end{matrix}\right\}$$

associating to any cotilting module C, the torsionfree class $\mathcal{C} = {}^{\perp}C \cap \text{mod-}\Lambda$ and to any torsion pair $(\mathcal{Q}, \mathcal{C})$ in mod-Λ, a cotilting module C such that $\text{Prod}\, C = \varinjlim \mathcal{F} \cap (\varinjlim \mathcal{F})^{\perp}$.

We summarize all the torsion pairs in mod-Λ with generating torsionfree class in the following table:

cotilting module	torsion pair in mod-Λ
$P_{n+1} \oplus P_{n+2}, (n \geq 0)$	$(\text{add}(\{P_i \mid i > n\} \cup \mathbf{t} \cup \mathbf{q}), \text{add}(P_1 \oplus \cdots \oplus P_n))$
$Q_{n+1} \oplus Q_{n+2}, (n \geq 0)$	$(\text{add}(Q_1 \oplus \cdots \oplus Q_n), \text{add}(\mathbf{p} \cup \mathbf{t} \cup \{Q_i \mid i > n\}))$
$G \oplus \prod_{x\in P} S_x^{-\infty} \oplus \bigoplus_{x\notin P} S_x^{\infty}, (P \subseteq \mathbb{X})$	$(\text{add}(\bigcup_{x\in P} \mathcal{U}_x \cup \mathbf{q}), \text{add}(\mathbf{p} \cup \bigcup_{x\notin P} \mathcal{U}_x))$

Every cotilting module C gives rise to a cotilting torsion pair, $({}^{\circ}C, \text{Cogen}\, C)$, where $\text{Cogen}\, C = \varinjlim({}^{\perp}C \cap \text{mod-}\Lambda)$, by [14, Lemma 1.1].
Using Theorems 2.2.3 and 1.2.3, we have the following table:

subset of $\mathbb{X}$	infinite dimensional cotilting Λ-module	torsion pair of finite type in Mod-Λ with generating torsionfree class
$P = \varnothing$	$C_{\varnothing} = G \oplus \bigoplus_{x\in\mathbb{X}} S_x^{\infty}$	$(\text{Gen}\, \mathbf{q}, \mathbf{q}^{\circ})$
$\varnothing \neq P \subsetneq \mathbb{X}$	$C_P = G \oplus \prod_{x\in P} S_x^{-\infty} \oplus \bigoplus_{x\notin P} S_x^{\infty}$	$(\text{Gen}(\bigcup_{x\in P} \mathcal{U}_x \cup \mathbf{q}), (\bigcup_{x\in P} \mathcal{U}_x \cup \mathbf{q})^{\circ})$
$P = \mathbb{X}$	$C_{\mathbb{X}} = G \oplus \prod_{x\in\mathbb{X}} S_x^{-\infty}$	$(\text{Gen}\, \mathbf{t}, \mathbf{t}^{\circ})$

Table 5.1: Infinite dimensional cotilting Λ-modules and their cotilting torsion pairs.

5.2 Hearts arising from cotilting torsion pairs

To shorten the notation, we denote by $\mathcal{B}$ the category Mod-Λ, where Λ is the Kronecker algebra. Let us analyze more specifically the cotilting torsion pairs described above, giving also a description of the different hearts arising from them. Notice that all the hearts we are going to describe are locally coherent categories.

- If $P = \varnothing$, the cotilting module $C_{\varnothing}$ is the so called Reiten-Ringel tilting module $\mathbf{W}$:

$$\mathbf{W} = G \oplus \bigoplus_{x \in \mathbb{X}} S_x^{\infty}$$

and it cogenerates the torsion pair $(\mathcal{Q}, \mathcal{C}) = ({}^{\circ}\mathbf{W}, \operatorname{Cogen} \mathbf{W})$, which is generated by $\mathbf{q}$. The heart $\mathcal{A} = \mathcal{B}(\mathcal{Q}, \mathcal{C})$ is equivalent to the locally noetherian Grothendieck category $\operatorname{Qcoh}\mathbb{X}$ of quasi-coherent sheaves over a non-commutative curve of genus zero $\mathbb{X}$ (see [3, Section 3.1]). We will describe this category in Chapter 6.

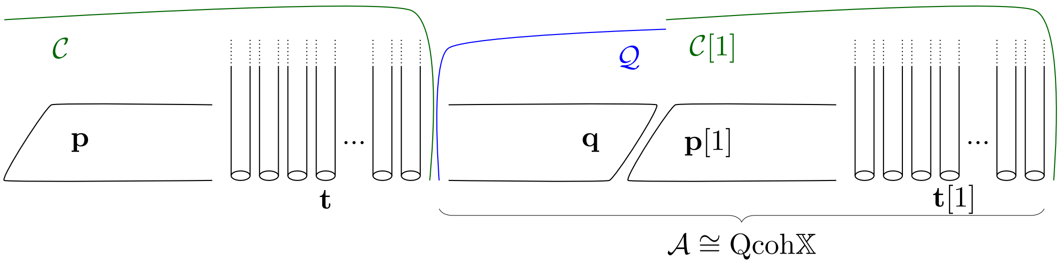


Figure 5.2: Auslander-Reiten quiver of the heart $\mathcal{A} = \mathcal{B}(\mathcal{Q}, \mathcal{C})$, for $P = \varnothing$.

- If $\varnothing \neq P \subsetneq \mathbb{X}$, the cotilting module is

$$C_P = G \oplus \prod_{x \in P} S_x^{-\infty} \oplus \bigoplus_{x \notin P} S_x^{\infty}$$

and it cogenerates the torsion pair $(\mathcal{Q}_P, \mathcal{C}_P) = ({}^{\circ}C_P, \operatorname{Cogen} C_P)$, which is generated by the set $\bigcup_{x \in P} \mathcal{U}_x \cup \mathbf{q}$. This cotilting module is not Σ-pure-injective, therefore, by Proposition 2.2.6, the heart $\mathcal{A}_P = \mathcal{B}(\mathcal{Q}_P, \mathcal{C}_P)$ is not locally noetherian.

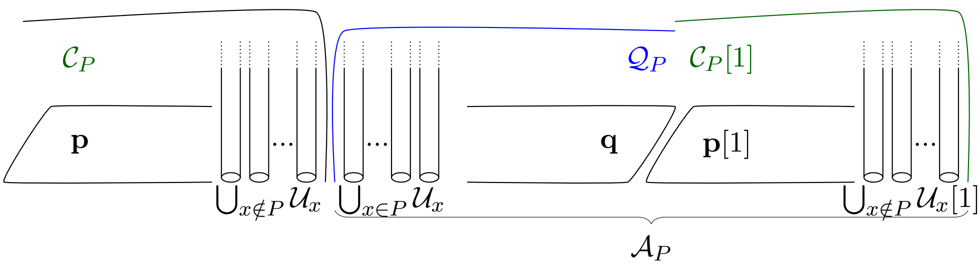


Figure 5.3: Auslander-Reiten quiver of the heart $\mathcal{A}_P = \mathcal{B}(\mathcal{Q}_P, \mathcal{C}_P)$, for $P \subsetneq \mathbb{X}$.

- If $P = \mathbb{X}$, the cotilting module is

$$C_{\mathbb{X}} = G \oplus \prod_{x \in \mathbb{X}} S_x^{-\infty}$$

and it cogenerates the torsion pair $(\operatorname{Gen} \mathbf{t}, \mathcal{F}) = ({}^{\circ}C_{\mathbb{X}}, \operatorname{Cogen} C_{\mathbb{X}})$, which is generated by $\mathbf{t}$. The cotilting module $C_{\mathbb{X}}$ is not Σ-pure-injective, therefore, also in this case, via Proposition 2.2.6, the heart $\mathcal{A}_{\mathbb{X}} = \mathcal{B}(\operatorname{Gen} \mathbf{t}, \mathcal{F})$ is not a locally noetherian category.

Remark 5.2.1. Notice that there are inclusions $\mathcal{F} \subset \mathcal{C}_P \subset \mathcal{C}$ and $\mathcal{Q} \subset \mathcal{Q}_P \subset \operatorname{Gen} \mathbf{t}$.

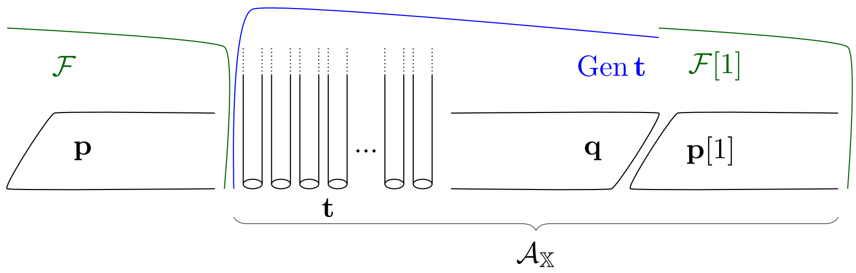


Figure 5.4: Auslander-Reiten quiver of the heart $\mathcal{A}_{\mathbb{X}} = \mathcal{B}(\operatorname{Gen}\mathbf{t}, \mathcal{F})$, for $P = \mathbb{X}$.

Theorem 5.2.2. *[1, Example 6.2] Let $P \subseteq \mathbb{X}$. The complete list of simple objects in the heart $\mathcal{A}_P = \mathcal{B}(\mathcal{Q}_P, \mathcal{C}_P)$ is:*

- $\{S_x \mid S_x$ *simple regular in* $\bigcup_{x\in P}\mathcal{U}_x\} \cup \{S_x[1] \mid S_x$ *simple regular in* $\bigcup_{x\notin P}\mathcal{U}_x\}$, *whenever* $P \neq \mathbb{X}$.
- $\{S_x \mid S_x$ *simple regular*$\} \cup \{G[1]\}$, *whenever* $P = \mathbb{X}$.

Moreover, for $P \subseteq \mathbb{X}$, we have that:

- *The short exact sequence*

$$0 \longrightarrow S_x \longrightarrow S_x^{-\infty}[1] \longrightarrow S_x^{-\infty}[1] \longrightarrow 0$$

  is a minimal injective coresolution in $\mathcal{A}_P$ for a simple regular module $S_x \in \bigcup_{x\in P}\mathcal{U}_x$.
- *The short exact sequence*

$$0 \longrightarrow S_x[1] \longrightarrow S_x^{\infty}[1] \longrightarrow S_x^{\infty}[1] \longrightarrow 0$$

  is a minimal injective coresolution in $\mathcal{A}_P$ for a simple regular module $S_x \in \bigcup_{x\notin P}\mathcal{U}_x$.
- *If $P = \mathbb{X}$, then the object $G[1]$ is simple injective in $\mathcal{A}_{\mathbb{X}}$.*

Proof. To prove the claim, according to Theorem 2.3.6, we show that:

(1) If $P \neq \varnothing$, then $X \in \mathcal{Q}_P$ is almost torsionfree if and only if X is simple regular in $\bigcup_{x\in P}\mathcal{U}_x$.
(2) If $P \neq \mathbb{X}$, then $X \in \mathcal{C}_P$ is almost torsion if and only if X is simple regular in $\bigcup_{x\notin P}\mathcal{U}_x$.
(3) If $P = \varnothing$, there are no torsion, almost torsionfree modules.
(4) If $P = \mathbb{X}$, then $X \in \mathcal{C}_P$ is almost torsion if and only if $X \cong G$.

First of all, observe that the indecomposable modules in $\mathcal{C}_P$ are the modules in $\mathbf{p} \cup \bigcup_{x\notin P}\mathcal{U}_x$ and the indecomposable modules in $\mathcal{Q}_P$ are the modules in $\bigcup_{x\in P}\mathcal{U}_x \cup \mathbf{q}$.
(1): Let S_x be simple regular in $\bigcup_{x\in P}\mathcal{U}_x$. Then $S_x \in \mathcal{Q}_P$ is torsion, almost torsionfree:

(i) All proper subobjects of S_x are preprojective, hence in $\mathcal{C}_P$.
(ii) Let $0 \to A \to B \to S_x \to 0$ be an exact sequence with $B \in \mathcal{Q}_P$. Consider the canonical exact sequence $0 \to A' \to A \to \overline{A} \to 0$ with $A' \in \mathcal{Q}_P$ and $\overline{A} \in \mathcal{C}_P$, and assume that $\overline{A} \neq 0$.

In the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & S_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \overline{A} & \xrightarrow{f'} & B' & \xrightarrow{g'} & S_x & \longrightarrow & 0 \end{array}$$

the map α is surjective and thus $B' \in \mathcal{Q}_P$. Notice that B' cannot have nonzero direct summands in $\mathbf{q}$, because they would be submodules of $\operatorname{Ker} g' \cong \overline{A} \in \mathcal{C}_P$. So we conclude that $B' \in \mathcal{C} \cap \operatorname{Gen} \mathbf{t}$ is a direct sum of regular and Prüfer modules belonging to the tubes $\bigcup_{x \in P} \mathcal{U}_x$, cf. [53, 3.4 and 3.5]. But then also $\operatorname{Ker} g' \cong \overline{A} \in \mathcal{C}_P$ must be of this form, a contradiction. This proves that $A \in \mathcal{Q}_P$.
Conversely, if $X \in \mathcal{Q}_P$ is almost torsionfree, then $X \notin \mathcal{C}_P = (\bigcup_{x \in P} \mathcal{U}_x)^\circ$, so there is a simple regular module $S_x \in \bigcup_{x \in P} \mathcal{U}_x$ with a non-zero map $f : S_x \to X$. But then $S_x \cong X$ by Lemma 2.3.3.

(2): We now turn to the case $P \neq \mathbb{X}$, which is somehow dual to case (1), and pick a simple regular module $S_x \in \bigcup_{x \notin P} \mathcal{U}_x$. First of all, observe that the generic module $G \in \mathbf{t}^\circ \subset \mathcal{C}_P$ is not almost torsion, since the exact sequence $0 \to S_x^{-\infty} \to G \to S_x^{\infty} \to 0$ from [14, 2.4] yields a proper quotient of G in $\mathcal{C}_P$. Then $S_x \in \mathcal{C}_P$ is torsionfree, almost torsion:

(i) All proper quotients of S_x are in $\operatorname{add} \mathbf{q}$.
(ii) Let $0 \to S_x \to B \to C \to 0$ be an exact sequence with $B \in \mathcal{C}_P$. Consider the canonical exact sequence $0 \to C' \to C \to \overline{C} \to 0$ with $C' \in \mathcal{Q}_P$ and $\overline{C} \in \mathcal{C}_P$ and assume $C' \neq 0$. In the pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_x & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ 0 & \longrightarrow & S_x & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

α is injective and then $B' \in \mathcal{C}_P$. Moreover, $B' \in \operatorname{Gen} \mathbf{t}$ indeed, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t(B') & \xrightarrow{i} & B' & \xrightarrow{\pi} & \overline{B'} & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & S_x & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

where the upper row is the canonical short exact sequence coming from the torsion pair $(\operatorname{Gen} \mathbf{t}, \mathcal{F})$. Suppose that $\overline{B'} \neq 0$. The composition $\pi f' = 0$, because $S_x \in \operatorname{Gen} \mathbf{t}$ and $\overline{B'} \in \mathcal{F}$. From this we obtain that $\operatorname{Ker} g' = \operatorname{Im} f' \subseteq \operatorname{Ker} \pi$, therefore there is a nonzero map from C' to $\overline{B'}$. This is a contradiction, since $C' \in \mathcal{Q}_P \subseteq \operatorname{Gen} \mathbf{t}$ and $\overline{B'} \in \mathcal{F}$. Hence $\overline{B'} = 0$ and $B' \in \operatorname{Gen} \mathbf{t}$.
So we have that $B' \in \mathcal{C}_P \cap \operatorname{Gen} \mathbf{t}$ is a direct sum of regular and Prüfer modules belonging to the tubes $\bigcup_{x \notin P} \mathcal{U}_x$ and therefore also $C' \cong B'/S_x \in \mathcal{Q}_P$ must be of this form. This is a contradiction. This proves that $C \in \mathcal{C}_P$.

For the converse implication, it suffices to prove tha,t for any almost torsion module $X \in \mathcal{C}_P$, there is a simple regular module $S_x \in \bigcup_{x \notin P} \mathcal{U}_x$ with $\operatorname{Hom}_\Lambda(S_x, X) \neq 0$. Indeed, this would imply $X \cong S_x$ by Lemma 2.3.3.
So, let us assume that such S_x does not exist. Then $X \in \mathbf{t}^\circ$, and by [53, 6.6] there is a short exact sequence

$$0 \to X \xrightarrow{f} G^{(\alpha)} \to Z \to 0$$

where $G^{(\alpha)} \in \mathcal{C}_P$ and thus, by property (ii), also Z belong to $\mathcal{C}_P$. Moreover, $X \notin \mathbf{p}$, because every $P \in \mathbf{p}$ is the first term of a short exact sequence $0 \to P \to P' \to S_x \to 0$ with $P' \in \mathbf{p} \subset \mathcal{C}_P$ and a simple regular module $S_x \in \bigcup_{x \in P} \mathcal{U}_x \subset \mathcal{Q}_P$. Furthermore, X is indecomposable, since if it is not it would have a proper quotient in $\mathcal{C}_P$. Moreover $X \in {}^\circ\mathbf{p}$, because if there is a nonzero map $X \to P$, with $P \in \mathbf{p}$, then X has a direct summand in $\mathbf{p}$, but X is indecomposable, so $X \in \mathbf{p}$, contradiction. It follows that $X \in {}^\circ\mathbf{t}$. In fact, any $0 \neq h : X \to S_x$ with S_x simple regular would have to be a proper epimorphism with $S_x \in \mathcal{Q}_P$. But then $\operatorname{Ext}^1_\Lambda(Z, S) \cong D\operatorname{Hom}_\Lambda(S_x, Z) = 0$, and h would factor through f, contradicting the fact that $\operatorname{Hom}_\Lambda(G, S_x) = 0$. So we conclude that X belongs to $\mathbf{t}^\circ \cap {}^\circ\mathbf{t} = \operatorname{Add} G$ (see Section 3.2.1). But then $X \cong G$, which is impossible as we have observed above.
(3): Assume $P = \varnothing$. Then $\mathcal{Q}_P = \operatorname{Add}\mathbf{q}$, and every $Q \in \mathbf{q}$ is the end-term of a short exact sequence $0 \to S_x \to Q' \to Q \to 0$, where $Q' \in \mathbf{q}$ and $S_x \notin \mathbf{q}$ is simple regular, so Q is not almost torsionfree. Moreover, all simple regular modules are torsionfree, almost torsion by (2).
(4): It remains to check the case $P = \mathbb{X}$. Then the torsion pair is $(\operatorname{Gen}\mathbf{t}, \mathcal{F})$, where $\mathcal{F} = \mathbf{t}^\circ$ and G is torsionfree, almost torsion. Indeed, $G \in \mathcal{F}$, and

(i) If $g : G \to B$ is a proper epimorphism, and $0 \to B' \to B \to \overline{B} \to 0$ is the canonical exact sequence with $B' \in \operatorname{Gen}\mathbf{t}$ and $0 \neq \overline{B} \in \mathcal{F}$, then $\overline{B} \in \mathcal{F} \cap {}^\circ\mathbf{t} = \operatorname{Add} G$ (see Section 3.2.1). So $G \xrightarrow{g} B \to \overline{B}$ is a morphism over a simple artinian ring Q, which is Morita equivalent to $\operatorname{End}_\Lambda(G)$ (see [23] and [5, 1.7 and 1.8] for the details). Thus, $G \xrightarrow{g} B \to \overline{B}$ a split monomorphism, which is a contradiction. Hence $B \in \operatorname{Gen}\mathbf{t}$.

(ii) If $0 \to G \xrightarrow{f} B \to C \to 0$ is an exact sequence with $B \in \mathcal{F}$, applying $\operatorname{Hom}_\Lambda(S_x, -)$, with S_x a simple regular module, we obtain an exact sequence:

$$\operatorname{Hom}_\Lambda(S_x, B) \to \operatorname{Hom}_\Lambda(S_x, C) \to \operatorname{Ext}^1_\Lambda(S_x, G) \cong D\operatorname{Hom}_\Lambda(G, S_x)$$

where the first and third term are zero, showing that $C \in \mathcal{F}$.

Conversely, if $X \in \mathcal{F}$ is almost torsion, then X is cogenerated by G, hence $\operatorname{Hom}_\Lambda(X, G) \neq 0$, and $X \cong G$ by Lemma 2.3.3.
Finally, to prove that the injective coresolutions have the stated form, we apply Proposition 2.3.8 first to the special $\mathcal{C}_P$-cover $0 \to S_x^{-\infty} \to S_x^{-\infty} \to S_x \to 0$ of $S_x \in \bigcup_{x \in P} \mathcal{U}_x$ and then to the special ${\mathcal{C}_P}^\perp$-envelope $0 \to S_x \to S_x^\infty \to S_x^\infty \to 0$ of $S_x \in \bigcup_{x \notin P} \mathcal{U}_x$. □

Remark 5.2.3. Recall from [63, Theorem 5.2] that $\mathcal{A}_P$ is hereditary only if $(\mathcal{Q}_P, \mathcal{C}_P)$ is a split torsion pair. But if P is infinite and S_x is a simple regular in the tube $\mathcal{U}_x$, then by [58, Proposition 5] there is a non-split exact sequence $0 \to \bigoplus_{x\in P} S_x \to \prod_{x\in P} S_x \to G^{(\alpha)} \to 0$ with $\bigoplus_{x\in P} S_x \in \mathcal{Q}_P$ and $G^{(\alpha)} \in \mathbf{t}^\circ \subset \mathcal{C}_P$.

5.3 Atom spectrum of the hearts

After the description of the simple objects in the different hearts arising from the torsion pairs in Mod-Λ, we compute the atom spectrum of these hearts.
As a notation for the whole section, we denote by $\bar{P}$ the complement of P inside $\mathbb{X}$ and, again, $\mathcal{B} = $ Mod-Λ. We distinguish tho cases: first, when $P \subsetneq \mathbb{X}$ (which includes the case $P = \varnothing$) and second, when $P = \mathbb{X}$.

5.3.1 Case $P \subsetneq \mathbb{X}$

We focus on the torsion pair $\mathbf{t}_P = (\mathcal{Q}_P, \mathcal{C}_P)$ generated by the set $\bigcup_{x\in P} \mathcal{U}_x \cup \mathbf{q}$ and cogenerated by the cotilting module C_P defined in Table 5.1.
By Lemma 4.1.9 there is an injection between ASpec($\mathcal{A}_P$) and Spec($\mathcal{A}_P$) which, by Proposition 2.2.4, is equal to the set of the indecomposable objects in Prod($C_P[1]$). If $P = \varnothing$, then, by Theorem 4.1.10, this injection is actually a bijection.
By Theorem 5.2.2, the simple objects in $\mathcal{A}_P$ are:

$$\{S_x \mid S_x \text{ simple regular in } \bigcup_{x\in P} \mathcal{U}_x\} \cup \{S_x[1] \mid S_x \text{ simple regular in } \bigcup_{x\in \bar{P}} \mathcal{U}_x\}$$

and these are monoform by Proposition 4.1.3(i), clearly not atom-equivalent.
Again by Theorem 5.2.2, the injective envelope of S_x, for $x \in P$ is $S_x^{-\infty}[1]$, and the injective envelope of $S_x[1]$, for $x \in \bar{P}$ is $S_x^{\infty}[1]$.
This is not the complete list of monoform objects in $\mathcal{A}_P$. Recall that, as in Remark 3.2.11, we can decompose the subcategory $\mathcal{T} = \varinjlim \mathbf{t}$ as a coproduct of categories $\mathcal{T}(x) = \varinjlim \mathcal{U}_x$. We have:

Proposition 5.3.1. *$G[1]$ is a monoform object in $\mathcal{A}_P = \mathcal{B}(\mathcal{Q}_P, \mathcal{C}_P)$.*

Proof. By [3, Corollary 5.8], in $\mathcal{A}_P$ there is a hereditary torsion pair $\mathbf{t}_{\bar{P}} = (\mathcal{T}_{\bar{P}}, \mathcal{F}_{\bar{P}})$, where:

$$\mathcal{T}_{\bar{P}} = \coprod_{x\in \bar{P}} \mathcal{T}(x)[1] \quad \text{and} \quad \mathcal{F}_{\bar{P}} = \operatorname{Cogen} C_{\bar{P}}[1].$$

Clearly $G[1] \in \mathcal{F}_{\bar{P}}$. Let Z be a proper quotient of $G[1]$. Hence $Z \in \mathcal{C}_P[1]$, since $\mathcal{C}_P[1]$ is a torsion class in the heart, meaning that $Z = C[1]$ for some $C \in \mathcal{C}_P$. Therefore, by Lemma 2.3.5, the proper epimorphism $h[1]\colon G[1] \to C[1]$ in $\mathcal{A}_P$ comes from a morphism in Mod$-\Lambda$, $h\colon G \to C$, with $\operatorname{Coker} h \in \mathcal{Q}_P$ and $\operatorname{Ker} h \neq 0$. The latter comes from the fact that the exact sequence $0 \to \operatorname{Ker} h \to G \xrightarrow{\bar{h}} \operatorname{Im} h \to 0$ is in $\mathcal{C}_P$, therefore $0 \to \operatorname{Ker} h[1] \to G[1] \xrightarrow{\bar{h}[1]} \operatorname{Im} h[1] \to 0$ is in $\mathcal{C}_P[1]$ and $h[1]$ is a proper epimorphism, so $\operatorname{Ker} h[1] \neq 0$.

In $\mathcal{B}$ we have the sequence:

$$\xymatrix{0 \ar[r] & \operatorname{Ker} h \ar[r] & G \ar[rr]^{h} \ar[dr]_{\bar{h}} & & C \ar[r] & \operatorname{Coker} h \ar[r] & 0 \\ & & & \operatorname{Im} h \ar@{^{(}->}[ur] & & & }$$

Since $G \in \mathcal{D} = {}^{\circ}\mathbf{t}$, which is a torsion class, and $C \in \mathcal{C}_P$, which is a torsionfree class, we have that $\operatorname{Im} h \in \mathcal{C}_P \cap \mathcal{D}$, ie. $\operatorname{Im} h \in \omega$ but its Prüfer summands come only from $\omega_0(x)$, with $x \in \bar{P}$ (see Theorem 3.2.5 and Remark 3.2.11). So we have:

$$\operatorname{Im} h = G^{(\alpha)} \oplus \bigoplus_{x \in \bar{P}} S_x^{\infty\,(\beta_x)},$$

for some cardinals α, β_x. Moreover,

$$\operatorname{Ker} h = \bigcap_{\alpha} \operatorname{Ker} \pi_G \cap \bigcap_{x \in \bar{P}} \bigcap_{\beta_x} \operatorname{Ker} \pi_x,$$

where π_G and π_x are the corestrictions of the map $\bar{h}$ to the different copies of G and S_x^{∞} (for $x \in \bar{P}$) respectively. Now, if G is a direct summand of $\operatorname{Im} h$, π_G is an isomorphism, $\operatorname{Ker} \pi_G = 0$ and $\operatorname{Ker} h = 0$, but this is a contradiction, since $\operatorname{Ker} h \neq 0$.
Therefore, $\operatorname{Im} h$ has no nonzero direct summands from $\operatorname{Add}(G)$, hence:

$$\operatorname{Im} h = \bigoplus_{x \in \bar{P}} S_x^{\infty\,(\beta_x)} \in \coprod_{x \in \bar{P}} \mathcal{T}(x) \subseteq \operatorname{Gen} \mathbf{t}.$$

From the short exact sequence $0 \to \operatorname{Im} h \to C \to \operatorname{Coker} h \to 0$ we have that $C \in \operatorname{Gen} \mathbf{t} \cap \mathcal{C}_P$, therefore $C \in \coprod_{x \in \bar{P}} \mathcal{T}(x)$ and so $C[1] = Z \in \mathcal{T}_{\bar{P}}$.
Therefore, we have seen that any proper quotient of $G[1]$ is in $\mathcal{T}_{\bar{P}}$, which means that $G[1]$ is $\mathbf{T}_{\bar{P}}$-cocritical and so monoform by Proposition 4.1.7. $\square$

Remark 5.3.2. Notice that $G[1]$ is not atom-equivalent to any simple object in $\mathcal{A}_P$, since, if it is atom-equivalent to one of them then, by Lemma 4.1.5, their injective envelopes should be isomorphic, and this is a contradiction.

We can now describe the atom spectrum of $\mathcal{A}_P$, as follows:

$$\begin{aligned} \operatorname{ASpec}(\mathcal{A}_P) = \overline{G[1]} \cup \{\overline{S_x} \mid S_x \text{ simple regular } \Lambda\text{-module in } \bigcup_{x \in P} \mathcal{U}_x\} \cup \\ \cup \{\overline{S_x[1]} \mid S_x \text{ simple regular } \Lambda\text{-module in } \bigcup_{x \in \bar{P}} \mathcal{U}_x\}. \end{aligned}$$

This shows that the injection between $\operatorname{ASpec}(\mathcal{A}_P)$ and $\operatorname{Spec}(\mathcal{A}_P)$ is actually a bijection also when $\varnothing \neq P \subsetneq \mathbb{X}$, and the description of the atom spectrum is complete.
The partial order in $\operatorname{ASpec}(\mathcal{A}_P)$ is the following: the singletons $\{\overline{S_x[1]}\}$ and $\{\overline{S_x}\}$ are open by Proposition 4.2.5(1), and, by Proposition 4.2.7(1), the corresponding atoms are maximal. Moreover, $\overline{G[1]} \leq \overline{S_x[1]}$, for any simple regular in $\bigcup_{x \in \bar{P}} \mathcal{U}_x$, indeed: let H be a monoform object

in $\mathcal{A}_P$ such that $\overline{H} = \overline{G[1]}$, then H and $G[1]$ have a common nonzero subobject Y. For a simple regular module, we have the short exact sequence in $\mathcal{A}_P$: $0 \to S_x^{-\infty}[1] \to G[1] \to S_x^{\infty}[1] \to 0$. Let Z be a pullback in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & Y/Z & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & S_x^{-\infty}[1] & \longrightarrow & G[1] & \longrightarrow & S_x^{\infty}[1] & \longrightarrow & 0 \end{array}$$

where the last vertical arrow is a monomorphism by [61, Proposition IV.5.1]. We have: $Y/Z \twoheadleftarrow Y \hookrightarrow H$ and since $S_x^{\infty}[1]$ is uniserial and $Y/Z \subseteq S_x^{\infty}[1]$, $S_x[1] \subseteq Y/Z$. This means that $\overline{S_x[1]} \in \operatorname{ASupp} Y \subseteq \operatorname{ASupp} H$ (see Proposition 4.2.2(1)). By Proposition 4.2.6, we reach the conclusion. When $P \neq \varnothing$, let us suppose that $\overline{G[1]} \leq \overline{S_x}$, for $S_x \in \bigcup_{x \in P} \mathbf{t}$, then, by Proposition 4.2.6(3), we have that $\overline{S_x} \in \operatorname{ASupp} G[1]$, therefore S_x is atom equivalent to a subobject of a proper quotient object of $G[1]$, but in Proposition 5.3.1 we have seen that any proper quotient of $G[1]$ belongs to the hereditary torsion class $\coprod_{x \in \bar{P}} \mathcal{T}(x)[1]$, and so S_x has to be in there too. This is a contradiction.

5.3.2 Case $P = \mathbb{X}$

Consider now the torsion pair generated by $\mathbf{t}$, $\mathbf{T}_{\mathbb{X}} = (\operatorname{Gen} \mathbf{t}, \mathcal{F})$, in $\mathcal{B}$, which is cogenerated by the cotilting module $C_{\mathbb{X}}$ defined in Table 5.1.

Also here we have an injective map between $\operatorname{ASpec}(\mathcal{A}_{\mathbb{X}})$ and $\operatorname{Spec}(\mathcal{A}_{\mathbb{X}})$, which corresponds, by Proposition 2.2.4, to the set of indecomposable objects in $\operatorname{Prod}(C_{\mathbb{X}}[1])$. From Theorem 5.2.2, we know that the simple objects in $\mathcal{A}_{\mathbb{X}}$ are $G[1]$ and S_x, for any simple regular Λ-module in $\mathbf{t}$, and these are, by Proposition 4.1.3(1), all monoform objects, clearly not atom-equivalent. Their injective envelopes are, respectively, $G[1]$ and $S_x^{-\infty}[1]$, for any $x \in \mathbb{X}$. Therefore we can conclude that the atom spectrum is:

$$\operatorname{ASpec}(\mathcal{A}_{\mathbb{X}}) = \overline{G[1]} \cup \{\overline{S_x} \mid S_x \text{ simple regular } \Lambda\text{-module}\}.$$

Therefore, also in this case, $\operatorname{ASpec}(\mathcal{A}_{\mathbb{X}})$ and $\operatorname{Spec}(\mathcal{A}_{\mathbb{X}})$ are in bijection.

Notice that any singleton $\{\alpha\}$, for $\alpha \in \operatorname{ASpec}(\mathcal{A}_{\mathbb{X}})$, is open by Proposition 4.2.5(1). Therefore, the topology on $\operatorname{ASpec}(\mathcal{A}_{\mathbb{X}})$ is discrete.

5.4 Gabriel dimension of the hearts

In this Section, we aim to prove that $\mathcal{A}_P = \mathcal{B}(\mathcal{Q}_P, \mathcal{C}_P)$ and $\mathcal{A}_{\mathbb{X}} = \mathcal{B}(\operatorname{Gen} \mathbf{t}, \mathbf{t}^{\circ})$ are Gabriel categories.

If $P = \varnothing$, then $\mathcal{A}_{\varnothing} = \mathcal{B}(\mathcal{Q}, \mathcal{C}) = \operatorname{Qcoh}\mathbb{X}$. This is a locally noetherian Grothendieck category, as we will see in Chapter 6, hence by Proposition 1.6.15, $\mathcal{A}_{\varnothing}$ is a Gabriel category and we will compute its Gabriel dimension. In the remaining two cases, ie. when $P \neq \varnothing$, we do not have the locally noetherianness for the hearts $\mathcal{A}_P$ but the bijection between $\operatorname{ASpec}(\mathcal{A}_P)$ and

Spec($\mathcal{A}_P$) still holds. As mentioned in Chapter 4, Theorem 4.1.10 has been generalized for Gabriel categories in [65], therefore it is reasonable to think that also $\mathcal{A}_P$, for $P \neq \varnothing$, is a Gabriel category. We will compute directly its Gabriel dimension.
Recall that, as seen in Definition 1.6.1, a class of objects $\mathcal{S}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if it is closed under subobjects, quotient objects and extensions. $\mathcal{S}$ is a *localizing subcategory* of $\mathcal{A}$ if it is a Serre subcategory and it is closed under arbitrary direct sums, ie. it is a hereditary torsion class.
In order to compute the Gabriel dimension of the hearts, we distinguish three cases: $P = \varnothing$, $\varnothing \neq P \subsetneq \mathbb{X}$ and $P = \mathbb{X}$.

5.4.1 Case $\mathcal{A} = \mathcal{B}(\mathcal{Q}, \mathcal{C})$

As we have seen in Section 5.3.1, $\mathcal{A}$ is equivalent to Qcoh$\mathbb{X}$ and therefore a locally noetherian Grothendieck category. By Proposition 1.6.15, $\mathcal{A}$ is a Gabriel category.
We build the Gabriel filtration step by step. Recall that, for a class of objects $\mathcal{X}$ in $\mathcal{A}$, we denote by $\langle \mathcal{X} \rangle_{\text{htor}}$ the smallest hereditary torsion class containing $\mathcal{X}$.
Set $\mathcal{G}_{-1} = \{0\}$. We have, by definition:

$$\mathcal{G}_0 = \langle \{X \in \mathcal{A} \mid X \text{ is simple in } \mathcal{A}\} \rangle_{\text{htor}}$$

therefore, using Theorem 5.2.2:

$$\mathcal{G}_0 = \langle \{S_x[1] \mid S_x \text{ simple regular } \Lambda\text{-module}\} \rangle_{\text{htor}}.$$

Following [3, Section 5.2], $\mathcal{G}_0 = \mathcal{T}[1]$, where $\mathcal{T} = \varinjlim \mathbf{t}$ (see Proposition 3.2.3), and the corresponding torsionfree class is ${\mathcal{G}_0}^\circ = \varinjlim(\mathbf{q} \cup \mathbf{p}[1])$.
The next step is:

$$\mathcal{G}_1 = \langle \mathcal{G}_0 \cup \{X \in \mathcal{A} \mid \mathbf{Q}_0(X) \text{ is simple in } \mathcal{A}/\mathcal{G}_0\} \rangle_{\text{htor}}$$

where $\mathbf{Q}_0 \colon \mathcal{A} \to \mathcal{A}/\mathcal{G}_0$ is the quotient functor. By Lemma 1.6.12, we have that the objects in $\mathcal{A}$ which become simple objects in $\mathcal{A}/\mathcal{G}_0$ are precisely the 0-cocritical objects, ie. the cocritical objects with respect to the torsion pair $\mathbf{\tau}_0 = (\mathcal{G}_0, {\mathcal{G}_0}^\circ)$. By Proposition 5.3.1, $G[1]$ is monoform, hence it is 0-cocritical via Remark 4.1.8. We claim the following:

Lemma 5.4.1. *If X is a 0-cocritical object in $\mathcal{A}$, then $\mathbf{Q}_0(X) \cong \mathbf{Q}_0(G[1])$.*

Proof. If $X \in \mathcal{A}$ is a 0-cocritical object, then, $\mathbf{Q}_0(X)$ is simple in $\mathcal{A}/\mathcal{G}_0$ and moreover, by Remark 4.1.8, X is monoform in $\mathcal{A}$.
Since we have a complete description of ASpec($\mathcal{A}$), we have that either $X \in \overline{S_x[1]}$ or $X \in \overline{G[1]}$: the first is not possible, because, if it is true, then X and $S_x[1]$ have a common nonzero subobject, but $S_x[1]$ is simple, then $S_x[1] \subseteq X$, which is a contradiction since $\text{Hom}_{\mathcal{A}}(S_x[1], X) = 0$ (recall that $S_x[1] \in \mathcal{G}_0$ and $X \in {\mathcal{G}_0}^\circ$). Then $X \in \overline{G[1]}$, ie. there is an object $Y \in \mathcal{A}$ such that $X \supseteq Y \subseteq G[1]$. This means that, in the quotient category $\mathcal{A}/\mathcal{G}_0$, $\mathbf{Q}_0(X) \supseteq \mathbf{Q}_0(Y) \subseteq \mathbf{Q}_0(G[1])$. But $\mathbf{Q}_0(X)$ is simple in $\mathcal{A}/\mathcal{G}_0$ and, by Lemma 1.6.12, $\mathbf{Q}_0(G[1])$ is simple too, therefore $\mathbf{Q}_0(X) \cong \mathbf{Q}_0(G[1])$. □

So we have $\mathcal{G}_1 = \langle \mathcal{G}_0 \cup G[1] \rangle_{\text{htor}}$.

Theorem 5.4.2. *If* $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$*, then* $\operatorname{Gdim} \mathcal{A} = 1$.

Proof. Consider the algebra Λ as a Λ-module. Since $\Lambda \in \mathbf{p}$ (Λ is the direct sum of the projective Λ-modules), $\Lambda \in \mathcal{F} = \mathbf{t}^{\circ}$. By Theorem 3.2.4, we have the ω-envelope of Λ, which is injective since $\Lambda \in \mathcal{C}$: $0 \to \Lambda \to M \to M' \to 0$, where $M \in \omega$ and $M' \in \omega_0$, ie. M' is a direct sum of Prüfer modules.
Since $\Lambda \in \mathcal{F}$, also $M \in \mathcal{F}$ which means that $M \in \mathcal{F} \cap \omega = \operatorname{Add}(G)$. So we have a short exact sequence entirely lying in $\mathcal{C}$:

$$0 \to \Lambda \to G^{(\alpha)} \to M \to 0$$

that gives rise to a short exact sequence in $\mathcal{A}$, entirely lying in $\mathcal{C}[1]$:

$$0 \to \Lambda[1] \to G^{(\alpha)}[1] \to M[1] \to 0.$$

Now, since $G^{(\alpha)}[1] \in \mathcal{G}_1$ and $\mathcal{G}_1$ is closed under subobjects, we have $\Lambda[1] \in \mathcal{G}_1$. For the same reason, the family $\{Z \mid Z \subseteq (\Lambda[1])^n, n \in \mathbb{N}\}$ is contained in $\mathcal{G}_1$ and this, by [17, Lemma 3.4], is a family of generators for $\mathcal{A}$, therefore $\mathcal{G}_1 = \mathcal{A}$. This means that the Gabriel filtration stops at $\mathcal{G}_1$, therefore $\operatorname{Gdim} \mathcal{A} = 1$. □

5.4.2 Case $\mathcal{A}_P = \mathcal{B}(\mathcal{Q}_P, \mathcal{C}_P)$

Let us move to the category $\mathcal{A}_P$, for $\varnothing \neq P \subsetneq \mathbb{X}$. Set $\mathcal{G}_{-1} = \{0\}$ and, by definition, we have:

$$\mathcal{G}_0 = \langle \{X \in \mathcal{A}_P \mid X \text{ is simple in } \mathcal{A}_P\} \rangle_{\text{htor}}$$

thus, by Theorem 5.2.2:

$$\mathcal{G}_0 = \langle \{S_x \mid S_x \text{ simple regular in } \bigcup_{x \in P} \mathcal{U}_x\} \cup \{S_x[1] \mid S_x \text{ simple regular in } \bigcup_{x \in \bar{P}} \mathcal{U}_x\} \rangle_{\text{htor}}.$$

It is clear that all the modules in the ray of S_x, for $x \in P$, are in $\mathcal{G}_0$, therefore the Prüfer modules S_x^{∞}, for $x \in P$, are in $\mathcal{G}_0$. The same argument can be applied to the ray of $S_x[1]$, for $x \in \bar{P}$, therefore $S_x^{\infty}[1] \in \mathcal{G}_0$, for $x \in \bar{P}$.
The next torsion class in the Gabriel filtration is:

$$\mathcal{G}_1 = \langle \mathcal{G}_0 \cup \{X \in \mathcal{A}_P \mid \mathbf{Q}_0(X) \text{ is simple in } \mathcal{A}_P/\mathcal{G}_0\} \rangle_{\text{htor}}$$

where $\mathbf{Q}_0 \colon \mathcal{A}_P \to \mathcal{A}_P/\mathcal{G}_0$ is the quotient functor. By Lemma 1.6.12, the simple objects in $\mathcal{A}/\mathcal{G}_0$ are precisely the cocritical objects with respect to the torsion pair $\mathbf{T}_0 = (\mathcal{G}_0, \mathcal{G}_0{}^{\circ})$, ie. the 0-cocritical objects. Consider the torsion pair $(\mathcal{T}_{\bar{P}}, \mathcal{F}_{\bar{P}})$ defined in the proof of Proposition 5.3.1, where:

$$\mathcal{T}_{\bar{P}} = \coprod_{x \in \bar{P}} \mathcal{T}(x)[1] \quad \text{and} \quad \mathcal{F}_{\bar{P}} = \operatorname{Cogen} C_{\bar{P}}[1].$$

It is clear that $\mathcal{T}_{\bar{P}} \subseteq \mathcal{G}_0$, hence $G[1]$ is 0-cocritical. Moreover, with the same argument as in Lemma 5.4.1, we can prove that if $X \in \mathcal{A}_P$ is a 0-cocritical object, then $\mathbf{Q}_0(X) \cong \mathbf{Q}_0(G[1])$. In conclusion we have:

$$\mathcal{G}_1 = \langle \mathcal{G}_0 \cup G[1] \rangle_{\text{htor}}.$$

Theorem 5.4.3. *If $\mathcal{A}_P = \mathcal{G}(\mathcal{Q}_P, \mathcal{C}_P)$, then* $\operatorname{Gdim} \mathcal{A}_P = 1$.

Proof. We consider, as in Theorem 5.4.2, the Λ-module Λ. Consider, as before, the ω-envelope of Λ: $0 \to \Lambda \to M \to M' \to 0$ and, since $\Lambda \in \mathcal{F}$, this short exact sequence is actually: $0 \to \Lambda \to G^{(\alpha)} \to M' \to 0$, with $M' \in \omega_0$. Consider the canonical sequence given by the torsion pair $(\mathcal{Q}_P, \mathcal{C}_P)$ for M', $0 \to t(M') \to M' \to M'/t(M') \to 0$, where t is the torsion radical, and denote with Y the pullback of the maps $G^{(\alpha)} \to M'$ and $t(M') \to M'$. We obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & Y & \longrightarrow & t(M') & \longrightarrow & 0 \\
 & & \| & & \downarrow & \lrcorner & \downarrow & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & G^{(\alpha)} & \longrightarrow & M' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M'/t(M') & = & M'/t(M') & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $t(M') \in \mathcal{Q}_P$, $M'/t(M') \in \mathcal{C}_P$ and $Y \in \mathcal{C}_P$ since $G^{(\alpha)} \in \mathcal{C}_P$. Moreover, since $M' \in \omega_0$, we can write M' as a direct sum of copies of Prüfer modules, therefore $t(M')$ is a direct sum of Prüfer modules lying in $\coprod_{x \in P} \mathcal{T}(x)$. So, the upper row becomes

$$0 \to \Lambda \to Y \to \bigoplus_{x \in P} S_x^{\infty(\beta_x)} \to 0$$

and gives rise to a short exact sequence in $\mathcal{A}_P$:

$$0 \to \bigoplus_{x \in P} S_x^{\infty(\beta_x)} \to \Lambda[1] \to Y[1] \to 0$$

with $\bigoplus_{x \in P} S_x^{\infty(\beta_x)} \in \mathcal{G}_0$. From the short exact sequence $0 \to Y \to G^{(\alpha)} \to M'/t(M') \to 0$, which is entirely in $\mathcal{C}_P$, we obtain a short exact sequence in $\mathcal{A}_P$:

$$0 \to Y[1] \to G^{(\alpha)}[1] \to M'/t(M')[1] \to 0$$

showing that $Y[1] \in \mathcal{G}_1$. Therefore, by the extension closure property of $\mathcal{G}_1$, $\Lambda[1] \in \mathcal{G}_1$. From [17, Lemma 3.4] we know that the heart $\mathcal{A}_P$ has a set of generators $\{Z \mid Z \subsetneq (\Lambda[1])^n, n \in \mathbb{N}\}$, which is, therefore, entirely in $\mathcal{G}_1$ and so $\mathcal{G}_1 = \mathcal{A}_P$, showing that $\operatorname{Gdim} \mathcal{A}_P = 1$. $\square$

5.4.3 Case $\mathcal{A}_{\mathbb{X}} = \mathcal{B}(\operatorname{Gen}\mathbf{t}, \mathcal{F})$

Let now $P = \mathbb{X}$. By Theorem 5.2.2, the simple objects in $\mathcal{A}_{\mathbb{X}}$ are $G[1]$ and S_x, for $x \in \mathbb{X}$. Therefore, setting $\mathcal{G}_{-1} = \{0\}$, we obtain:

$$\mathcal{G}_0 = \langle \{S_x \mid S_x \text{ simple regular}\} \cup \{G[1]\} \rangle_{\text{htor}}.$$

It is clear that all the objects in the ray of S_x, for any $x \in \mathbb{X}$, are in $\mathcal{G}_0$, and hence all the Prüfer objects $S_x^\infty \in \mathcal{G}_0$, for any $x \in \mathbb{X}$.

As in the previous section, we can show that the object $\Lambda[1]$ is in $\mathcal{G}_0$. Take the ω-envelope of the regular module $\Lambda \in \mathcal{F}$:

$$0 \to \Lambda \to G^{(\alpha)} \to M \to 0.$$

The first two terms of this short exact sequence are in $\mathcal{F}$ and $M \in \omega_0 \subseteq \operatorname{Gen}\mathbf{t}$, therefore, there is a short exact sequence in the heart $\mathcal{A}_{\mathbb{X}}$:

$$0 \to M \to \Lambda[1] \to G^{(\alpha)}[1] \to 0.$$

Since M is a direct sum of Prüfer objects, $M \in \mathcal{G}_0$, hence $\Lambda[1] \in \mathcal{G}_0$. This means that the set $\{Z \mid Z \subseteq (\Lambda[1])^n, n \in \mathbb{N}\}$ of generators of the heart given by [17, Lemma 3.4] is entirely in $\mathcal{G}_0$ showing that $\mathcal{G}_0 = \mathcal{A}_{\mathbb{X}}$ and $\operatorname{Gdim}\mathcal{A}_{\mathbb{X}} = 0$.

We summarize all the results obtained in Sections 5.3 and 5.4 in the following Theorem.

Theorem 5.4.4. *Let $\mathcal{G} = \text{Mod-}\Lambda$, with Λ the Kronecker algebra. Consider $P \subseteq \mathbb{X}$ and let C_P be the infinite dimensional cotilting module, together with its corresponding cotilting torsion pair $(\mathcal{Q}_P, \mathcal{C}_P)$, as in Table 5.1. Let $\mathcal{A}_P = \mathcal{G}(\mathcal{Q}_P, \mathcal{C}_P)$. We have the following:*

- *If $P \subsetneq \mathbb{X}$, then $\operatorname{Gdim}\mathcal{A}_P = 1$ and*

$$\operatorname{ASpec}(\mathcal{A}_P) = \overline{G[1]} \cup \{\overline{S_x} \mid S_x \textit{ simple regular} \bigcup_{x \in P} \mathcal{U}_x\} \cup \{\overline{S_x[1]} \mid S_x \textit{ simple regular} \bigcup_{x \in \bar{P}} \mathcal{U}_x\}.$$

- *If $P = \mathbb{X}$, then $\operatorname{Gdim}\mathcal{A}_{\mathbb{X}} = 0$ and*

$$\operatorname{ASpec}(\mathcal{A}_{\mathbb{X}}) = \overline{G[1]} \cup \{\overline{S_x} \mid S_x \textit{ simple regular } \Lambda\textit{-module}\}.$$

Chapter 6

Weighted noncommutative regular projective curves

In this chapter we describe the category of quasi-coherent sheaves over a noncommutative curve $\mathbb{X}$. First, we consider only the category of coherent (ie. finitely generated) sheaves over $\mathbb{X}$ and we give an axiomatic description of it. Here, classes of sheaves can be distinguished by a notion of *slope*, which is a rational number or infinity. We define the slope of a sheaf E as the ratio of two integer numbers, degree and rank of E, which are linear forms on the Grothendieck group of the category. Afterwards, we specialize on tubular curves, which are weighted noncommutative curves with *orbifold Euler characteristic* equal to 0. The category $\mathrm{Qcoh}\mathbb{X}$ of quasi-coherent sheaves over a tubular curve $\mathbb{X}$ is derived equivalent to the category of modules over a tubular algebra, seen in Chapter 3. We will describe $\mathrm{Qcoh}\mathbb{X}$ at the very end of the chapter, where we focus on the torsion pairs in it and on the sheaves of irrational slope.
Throughout the whole chapter, we fix an algebraically closed field k.

6.1 Coherent sheaves over a noncommutative curve

Axioms 6.1.1. A *noncommutative curve* $\mathbb{X}$ is given by the category of *coherent sheaves* over $\mathbb{X}$ itself, we call it $\mathcal{H} = \mathrm{coh}\mathbb{X}$. It formally behaves like a category of coherent sheaves over a regular projective curve over k (see [42]):

(NC1) $\mathcal{H}$ is small, connected, abelian and every object in $\mathcal{H}$ is noetherian,

(NC2) $\mathcal{H}$ is a k-category with finite-dimensional Hom- and Ext-spaces,

(NC3) There is an autoequivalence τ on $\mathcal{H}$, called the Auslander-Reiten translation, such that the Serre duality
$$\mathrm{Ext}^1_{\mathcal{H}}(X, Y) = D\,\mathrm{Hom}_{\mathcal{H}}(Y, \tau X)$$
holds, where $D = \mathrm{Hom}_k(-, k)$ is the vector space duality. (In particular $\mathcal{H}$ is then hereditary),

(NC4) $\mathcal{H}$ contains an object of infinite length.

Assume $\mathcal{H}$ satisfies (NC1) to (NC4). We denote by $\mathcal{H}_0$ the class of the indecomposable sheaves of finite length, called the *torsion sheaves*, and by $\mathcal{H}_+$ the class of the indecomposable sheaves without simple subsheaves, called the *vector bundles* (or *torsionfree sheaves*), denoted also by $\operatorname{vect}\mathbb{X}$. The pair $(\operatorname{add}\mathcal{H}_0, \operatorname{add}\mathcal{H}_+)$ is, indeed, a torsion pair.

Proposition 6.1.2. *[49, Proposition 1.1] Each indecomposable coherent sheaf in $\mathcal{H}$ either belongs to $\mathcal{H}_0$ or to $\mathcal{H}_+$. Moreover, we have*

$$\mathcal{H}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x,$$

where each $\mathcal{U}_x$ is a connected uniserial length category.

Assume that $\mathcal{H}$ satisfies also the following condition:

(NC5) $\mathbb{X}$ consists of infinitely many points.

Then we call $\mathbb{X}$ a *weighted noncommutative regular projective curve* over k. It is shown in [42] that, in this case, $\mathbb{X}$ satisfies also the following condition:

(NC6) For any $x \in \mathbb{X}$ there is an integer number $p(x) \geq 1$, called *weight*, that denotes, up to isomorphisms, the number of simple objects in $\mathcal{U}_x$ and we have $p(x) = 1$ for all $x \in \mathbb{X}$ up to finitely many.

Let $\{x_1, \dots, x_t\}$ be finite set of the points in the curve $\mathbb{X}$ such that $p(x_i) > 1$ for any $1 \leq i \leq t$. The *weight type* of the curve $\mathbb{X}$ is a vector $(p_1, \dots, p_t)$, where $p_i = p(x_i)$, for $1 \leq i \leq t$.
For $x \in \mathbb{X}$, the connected uniserial length categories $\mathcal{U}_x$, appearing in Proposition 6.1.2, are called *tubes*. We can classify the tubes with respect to the weight of the corresponding point x.

- **For** $p(x) = 1$**:** $\mathcal{U}_x$ is said of *rank* 1 and it is called *homogeneous*. There is only one simple object S_x in $\mathcal{U}_x$ and it satisfies $\operatorname{Ext}^1_{\mathcal{H}}(S_x, S_x) \neq 0$ (equivalently $\tau S_x \cong S_x$).
- **For** $p(x) > 1$**:** $\mathcal{U}_x$ is said of *rank* $p(x)$ and it is called *exceptional*. The set of all $p(x)$ simple objects in $\mathcal{U}_x$ is given by the Auslander-Reiten orbit, $\{S_x, \tau S_x, \dots, \tau^{p(x)-1} S_x\}$, with $S_x = \tau^{p(x)} S_x$ and $\operatorname{Ext}^1_{\mathcal{H}}(S_x, S_x) = 0$.

More generally, a coherent sheaf $E \in \mathcal{H}$ is called *exceptional* if E is indecomposable and $\operatorname{Ext}^1_{\mathcal{H}}(E, E) = 0$. It follows, from an argument by Happel and Ringel, that in this case $\operatorname{End}(E)$ is a skew field. It is easy to see that the exceptional sheaves in $\mathcal{U}_x$ are only the indecomposable of length smaller than $p(x) - 1$, which makes sense only if $p(x) > 1$. This means that there are only finitely many exceptional sheaves of finite length.
If $p(x) = 1$ for all $x \in \mathbb{X}$, then $\mathbb{X}$ is called a *non-weighted noncommutative regular projective curve* over k. Notice that a non-weighted curve is a particular case of a weighted curve. Indeed, this definition can be rephrased in the following way: $\mathbb{X}$ is a non-weighted noncommutative regular projective curve if the category $\mathcal{H} = \operatorname{coh}\mathbb{X}$ satisfies (NC1) to (NC5) and additionally

(NC6') $\operatorname{Ext}^1_{\mathcal{H}}(S, S) \neq 0$ (equivalently $\tau S \cong S$) for any simple object $S \in \mathcal{H}$.

The condition (NC6') implies the fact that $p(x) = 1$ for all $x \in \mathbb{X}$.
Let us consider another condition:

(g-0) $\mathcal{H}$ admits a tilting object.

It has been shown in [47] that this condition implies the existence of a tilting object $T_{can} \in \mathcal{H}_+$, called *canonical*, whose endomorphism algebra is a canonical algebra, in the sense of [57].
If $\mathcal{H}$ satisfies (NC1) to (NC4) and (g-0), then $\mathbb{X}$ is called a *noncommutative curve of genus zero* (or *exceptional curve*). In this case the condition (NC5) is automatically satisfied (see [39]).

The Grothendieck Group

For an object $X \in \mathcal{H}$, denote by $[X]$ the isomorphism class of X in $\mathcal{H}$. We define $K_0(\mathcal{H})$, the *Grothendieck group* of $\mathcal{H}$, as the free abelian group generated by the set $\{[X] \mid X \in \mathcal{H}\}$, modulo the additivity relation on short exact sequences, ie. for any short exact sequence in $\mathcal{H}$, $0 \to X \to Y \to Z \to 0$, we have $[Y] = [X] + [Z]$.
As mentioned above, for $\mathbb{X}$ of genus zero, $\mathcal{H}$ admits a tilting object whose endomorphism ring is a canonical algebra, therefore the Grothendieck group $K_0(\mathcal{H})$ is a finitely generated free abelian group (see [41, 47]).
$K_0(\mathcal{H})$ is equipped with the *Euler form*, which is a bilinear form over k, defined on isoclasses of objects $X, Y \in \mathcal{H}$ by:

$$\langle [X], [Y] \rangle = \dim_k \operatorname{Hom}_{\mathcal{H}}(X, Y) - \dim_k \operatorname{Ext}^1_{\mathcal{H}}(X, Y).$$

To ease the notation we write $\langle X, Y \rangle$, without the brackets. It is easy to see from the Serre duality that $\langle X, Y \rangle = -\langle Y, \tau X \rangle$ and since τ is an autoequivalence, we have $\langle X, Y \rangle = \langle \tau X, \tau Y \rangle$.

Rank and line bundles

Since $\mathcal{H}_0$ is a Serre subcategory of $\mathcal{H}$, we consider the quotient category $\mathcal{H}/\mathcal{H}_0$ and let $\pi \colon \mathcal{H} \to \mathcal{H}/\mathcal{H}_0$ be the quotient functor, which is an exact functor. In [49, Proposition 3.4], it is proven that $\mathcal{H}/\mathcal{H}_0$ is equivalent to the category mod$-k(\mathcal{H})$ of finite dimensional modules over the *function field* $k(\mathcal{H})$ of $\mathbb{X}$, which is a skew field.
For a coherent sheaf F, we define the *rank* of F by the following formula:

$$\operatorname{rk}(F) := \dim_{k(\mathcal{H})}(\pi F).$$

The rank is additive on short exact sequences, meaning that for any short exact sequence $0 \to E \to F \to G \to 0$, we have:

$$\operatorname{rk}(F) = \operatorname{rk}(E) + \operatorname{rk}(G).$$

Therefore, it induces a linear form $\operatorname{rk} \colon K_0(\mathcal{H}) \to \mathbb{Z}$ on the Grothendieck group of $\mathcal{H}$.
The finite length objects are precisely the objects of rank zero, ie. $\operatorname{rk}(\mathcal{H}_0) = 0$, and every object in $\mathcal{H}_+$ has positive rank. Moreover, by [45, §10.2(H 5), Remark 10.2(ii)], the rank is τ-invariant.
We call *line bundles* the vector bundles of rank one. A line bundle L is called *special* if for each $x \in \mathbb{X}$ there is (up to isomorphism) precisely one simple sheaf S_x with $\operatorname{Ext}^1_{\mathcal{H}}(S_x, L) \neq 0$.

Remark 6.1.3. Notice that, since S_x is in a tube of rank $p(x)$ and it is such that $S_x \cong \tau^{p(x)} S_x$, using Serre duality, we can say that a line bundle is special if, for any $x \in \mathbb{X}$, there is precisely one $0 \leq j \leq p(x) - 1$ such that $\mathrm{Hom}_{\mathcal{H}}(L, \tau^j S_x) \neq 0$. Moreover, it is clear that, for the j as above, $0 \neq \mathrm{Hom}_{\mathcal{H}}(L, \tau^j S_x) = \mathrm{Hom}_{\mathcal{H}}(L, \tau^{j+np(x)} S_x)$ for any $n \in \mathbb{Z}$.

If $\mathbb{X}$ is a noncommutative curve of genus zero, then every line bundle is an exceptional sheaf (see [39, Chapter 8]). Furthermore:

Proposition 6.1.4. *[49, Lemma 1.3, Proposition 1.6] Every non-zero morphism from a line bundle L to a vector bundle is a monomorphism, and* $\mathrm{End}_{\mathcal{H}}(L) \cong k$*. Every vector bundle has a line bundle filtration.*

From now on, we always consider a noncommutative regular projective curve $\mathcal{H} = \mathrm{coh}\mathbb{X}$ together with a fixed special line bundle L, which is considered as the *structure sheaf.*

Remark 6.1.5. Without loss of generality, we can suppose that, for the structure sheaf L and for any $x \in \mathbb{X}$, $\mathrm{Hom}_{\mathcal{H}}(L, \tau^j S_x) \neq 0$ if and only if $j \equiv 0 \mod p(x)$. Clearly, $\mathrm{Hom}_{\mathcal{H}}(L, S_x)$ can be regarded as a left $\mathrm{End}(S_x)$-module and a right $\mathrm{End}(L)$-module. If the field k is algebraically closed, then:

$$\dim_{\mathrm{End}(L)} \mathrm{Hom}_{\mathcal{H}}(L, S_x) = 1 \quad \text{and} \quad \dim_{\mathrm{End}(S_x)} \mathrm{Hom}_{\mathcal{H}}(L, S_x) = 1.$$

For the details, we refer to [39, 46].

The average Euler form

Let $\mathbb{X}$ be a weighted noncommutative regular projective curve and let $(p_1, \ldots, p_t)$ be the weight type of $\mathbb{X}$. Denote by $\bar{p}$ the least common multiple of $p_1, \ldots, p_t$. We define the *average Euler form* as

$$\langle\langle E, F \rangle\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j E, F \rangle = \sum_{j=0}^{\bar{p}-1} \dim \mathrm{Hom}(\tau^j E, F) - \dim \mathrm{Ext}^1(\tau^j E, F).$$

Remark 6.1.6. Consider a simple sheaf S_x in a tube $\mathcal{U}_x$ of rank $p(x)$. Then, for a coherent sheaf $E \in \mathcal{H}$ and since τ is an autoequivalence, we have:

$$\langle\langle E, S_x \rangle\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j E, S_x \rangle = \sum_{j=0}^{\bar{p}-1} \langle E, \tau^{-j} S_x \rangle.$$

Recall that the τ-orbit of S_x is periodic with period $p(x)$ and that $\bar{p}$ is a positive integer multiple of $p(x)$. Therefore, we can rewrite the last sum as:

$$\langle\langle E, S_x \rangle\rangle = \sum_{j=0}^{\bar{p}-1} \langle E, \tau^{-j} S_x \rangle = \sum_{j=0}^{\bar{p}-1} \langle E, \tau^j S_x \rangle.$$

Clearly, this result holds also for $\langle\langle E, F \rangle\rangle$, with $F \in \mathcal{H}_0$.

Numerical invariants and representation type

An important numerical invariant for the weighted curve $\mathbb{X}$ is the so called *orbifold* (or *virtual*) *genus* of $\mathbb{X}$, which is defined as (when k is an algebraically closed field):

$$g_{orb}(\mathbb{X}) = 1 + \frac{\bar{p}}{2}\left(\sum_{i=1}^{t}\left(1 - \frac{1}{p_i}\right) - 2\right).$$

See [42, Corollary 13.16]. The orbifold genus of $\mathbb{X}$ is strictly related to another important invariant, called the *orbifold Euler characteristic*, $\chi'_{orb}(\mathbb{X})$, defined via the average Euler form, which is used to determine the representation type of the category $\mathcal{H} = \mathrm{coh}\mathbb{X}$, we refer to [42, Chapter 13] for the precise definition. Following [42, Chapter 13], this classification holds:

- $\chi'_{orb}(\mathbb{X}) > 0$ (equivalently: $g_{orb}(\mathbb{X}) < 1$): $\mathbb{X}$ is domestic.
- $\chi'_{orb}(\mathbb{X}) = 0$ and $\mathbb{X}$ is non-weighted: $\mathbb{X}$ is elliptic.
- $\chi'_{orb}(\mathbb{X}) = 0$ and $\mathbb{X}$ is weighted (equivalently: $g_{orb}(\mathbb{X}) = 1$): $\mathbb{X}$ is tubular.
- $\chi'_{orb}(\mathbb{X}) < 0$ (equivalently: $g_{orb}(\mathbb{X}) > 1$): $\mathbb{X}$ is wild.

Degree and slope

Let $\mathbb{X}$ be a noncommutative regular projective curve and let $(p_1, \ldots, p_t)$ be the weight type of $\mathbb{X}$, when weighted. Denote by $\bar{p} = \mathrm{l.c.m.}\{p_i\}_{i=1}^{t}$.

Definition 6.1.7. For an object $F \in \mathcal{H}$, we define the *degree* function $\deg : K_0(\mathcal{H}) \to \mathbb{Z}$ as:

$$\deg(F) = \langle\langle L, F\rangle\rangle - \mathrm{rk}(F)\langle\langle L, L\rangle\rangle$$

Clearly, $\deg(L) = 0$ and deg is positive and τ-invariant on sheaves in $\mathcal{H}_0$.

Remark 6.1.8 (Degree of simple sheaves)*.* If S_x is a simple sheaf in $\mathcal{U}_x$, then $\bar{p} = \mathrm{l.c.m.}\{p_i\}_{i=1}^{t}$ is an integer multiple of $p(x)$. Set $\bar{p} = mp(x)$ for $m \in \mathbb{Z}_{>0}$. We have:

$$\begin{aligned}
\deg(S_x) &= \langle\langle L, S_x\rangle\rangle - \mathrm{rk}(S_x)\langle\langle L, L\rangle\rangle = \langle\langle L, S_x\rangle\rangle = \sum_{j=0}^{\bar{p}-1}\langle L, \tau^j S_x\rangle = \\
&= \sum_{j=0}^{\bar{p}-1}\dim_k \mathrm{Hom}(L, \tau^j S_x) - \dim_k \mathrm{Ext}^1(L, \tau^j S_x) = && (\mathrm{Hom}(\mathcal{H}_0, \mathcal{H}_+) = 0) \\
&= \sum_{j=0}^{\bar{p}-1}\dim_k \mathrm{Hom}(L, \tau^j S_x) = m\sum_{j=0}^{p(x)-1}\dim_k \mathrm{Hom}(L, \tau^j S_x) = && (\text{Remarks 6.1.3 and 6.1.5}) \\
&= m.
\end{aligned}$$

Therefore:

$$\deg(S_x) = \frac{\bar{p}}{p(x)}.$$

Definition 6.1.9. Let F be a non-zero coherent sheaf in $\mathcal{H}$. The *slope* of F, denoted by $\mu(F)$, is defined as the ratio:

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)}.$$

The slope of a sheaf is an element in $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ and the sheaves of slope ∞ are the ones in $\mathcal{H}_0$ (ie. of rank 0). Moreover, F is called *semistable* (resp. *stable*) if for every non-zero proper subsheaf $F' \subseteq F$ we have $\mu(F') \leq \mu(F)$ (resp. $\mu(F') < \mu(F)$).

Proposition 6.1.10. *Let $0 \to E \to F \to G \to 0$ be a short exact sequence in $\mathcal{H}$, then:*

(1) $\mu(E) \leq \mu(F)$ if and only if $\mu(F) \leq \mu(G)$,

(2) if $\mu(E) \leq \mu(G)$ then $\mu(E) \leq \mu(F) \leq \mu(G)$.

These properties hold too if we replace "$\leq$" with "$<$".

Proof. It is clear using the definition of slope and the fact that degree and rank are additive on short exact sequences. □

Using the notion of (semi)stability for a sheaf, we can classify vector bundles in $\mathcal{H}$ (we refer to [26, Proposition 5.5], [42], [39, Proposition 8.1.6], [49])

Theorem 6.1.11. *Let $\mathcal{H} = \mathrm{coh}\mathbb{X}$ be a weighted noncommutative regular projective curve over k.*

(1) *If $\chi'_{orb}(\mathbb{X}) > 0$ (domestic type), then every indecomposable vector bundle is stable and exceptional. Moreover, $\mathcal{H}$ admits a tilting bundle.*
(2) *If $\chi'_{orb}(\mathbb{X}) = 0$ (elliptic or tubular type), then every indecomposable coherent sheaf is semistable. If $\mathbb{X}$ is tubular (ie. $\bar{p} > 1$), then $\mathcal{H}$ admits a tilting bundle. If $\mathbb{X}$ is elliptic (ie. $\bar{p} = 1$), then every indecomposable coherent sheaf E is non-exceptional and satisfies $\tau E \cong E$.*
(3) *If $\chi'_{orb}(\mathbb{X}) < 0$ (wild type), then every Auslander-Reiten component in $\mathcal{H}_+$ is of type $\mathbb{Z}A_\infty$. ($\mathcal{H}$ may or may not admit a tilting bundle).*

Tubular curves

In this section, $\mathcal{H}$ will denote the category of coherent sheaves over a noncommutative regular projective curve of **tubular type** $\mathbb{X}$ over an algebraically closed field k. We can completely describe the shape of this category using again the notion of semistability. Notice that the next result gives a similar description of the category of coherent sheaves over an elliptic curve, due to Atiyah (see [7]).
Denote by $\mathbf{t}_\alpha$ the class of all indecomposable semistable sheaves in $\mathcal{H}$ of slope $\alpha \in \widehat{\mathbb{Q}}$. In particular, $\mathbf{t}_\infty$ coincides with $\mathcal{H}_0$. We have:

Theorem 6.1.12. *Let $\mathbb{X}$ be a noncommutative regular projective curve of tubular type. Then the following holds:*

(i) *For each* $\alpha \in \widehat{\mathbb{Q}}$*, the category* $\operatorname{add} \mathbf{t}_\alpha$ *is an exact abelian subcategory of* $\mathcal{H}$*, closed under extensions, under the Auslander-Reiten translation* τ *and it is equivalent to the category* $\mathcal{H}_0$*, ie. every* $F \in \operatorname{add} \mathbf{t}_\alpha$ *is of finite length in* $\mathbf{t}_\alpha$ *and there is a decomposition:*

$$\mathbf{t}_\alpha = \coprod_{x \in \mathbb{X}} \mathcal{U}_{\alpha,x}$$

where $\mathcal{U}_{\alpha,x}$ *is a connected uniserial category with* $p(x)$ *simple objects. Moreover, the simple objects in* $\operatorname{add} \mathbf{t}_\alpha$ *are the stable sheaves and if* F *is stable,* $\operatorname{End}_k(F) \cong k$*.*

(ii) *Let* $F \in \operatorname{add} \mathbf{t}_\alpha$ *and* $F' \in \operatorname{add} \mathbf{t}_{\alpha'}$*, for* $\alpha, \alpha' \in \widehat{\mathbb{Q}}$*. If* $\alpha < \alpha'$*, then* $\operatorname{Hom}_{\mathcal{H}}(F', F) = 0$*.*

(iii) *We have:*

$$\mathcal{H} = \operatorname{add}\left(\bigvee_{\alpha \in \widehat{\mathbb{Q}}} \mathbf{t}_\alpha\right)$$

meaning that $\mathcal{H}$ *is the additive closure of all the tubular families* $\mathbf{t}_\alpha$ *and* $\operatorname{Hom}_{\mathcal{H}}(\mathbf{t}_\alpha, \mathbf{t}_\beta) = 0$ *for* $\alpha > \beta$*.*

Proof. (i) See [26, Proposition 5.2(i)(ii)], [26, Theorem 5.6(ii)(iii)] and [48, Theorem 4.4].

(ii) See [26, Proposition 5.2(iii)].

(iii) See [44, Theorem 2.2].

□

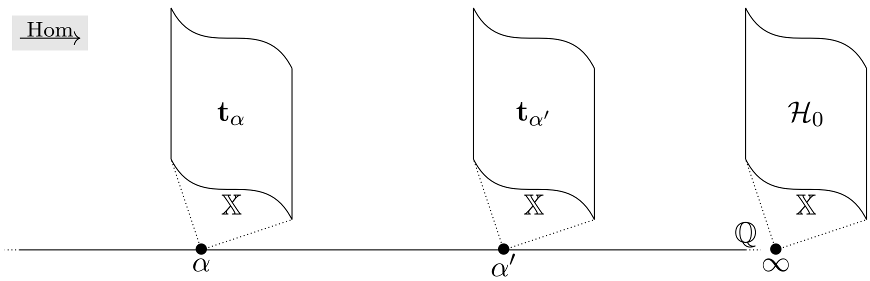


Figure 6.1: Tubular families in coh$\mathbb{X}$.

Let $(p_1, \ldots, p_t)$ be the weight type of $\mathbb{X}$ and let $\bar{p} = \text{l.c.m.}\{p_i\}$. Since $\mathbb{X}$ is tubular, the orbifold genus of this curve is $g_{orb} = 1$. Hence:

$$1 + \frac{\bar{p}}{2}\left(\sum_{i=1}^{t}\left(1 - \frac{1}{p_i}\right) - 2\right) = 1.$$

By direct computation, it is possible to see that this happens if and only if the weight type of $\mathbb{X}$ is (2,2,2,2), (3,3,3), (2,4,4) or (2,3,6). As one can notice, any weight type is completely determined by the integer $\bar{p}$, which can be $2, 3, 4$ or 6, and every divisor of $\bar{p}$ appears in the weight type.

Remark 6.1.13. As we have seen, for each $\alpha \in \widehat{\mathbb{Q}}$, $\mathbf{t}_\alpha \cong \mathcal{H}_0$. Therefore, an indecomposable sheaf in $\mathcal{U}_{\alpha,x}$ is τ-periodic with period $p(x)$. This allows us to extend the result in Remark 6.1.6 in such a way that, for any $E, F \in \mathcal{H}$:

$$\langle\!\langle E, F\rangle\!\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j E, F\rangle = \sum_{j=0}^{\bar{p}-1} \langle E, \tau^j F\rangle.$$

Proposition 6.1.14. *If $\mathbb{X}$ is of tubular type, the degree function* $\deg\colon K_0(\mathcal{H}) \to \mathbb{Z}$ *is τ-invariant on $\mathcal{H}$.*

Proof. Let $F \in \mathcal{H}$ be an indecomposable in a tube $\mathcal{U}_{\alpha,x}$, therefore F is τ-periodic with period $p(x)$. We have:

$$\deg(\tau F) = \langle\!\langle L, \tau F\rangle\!\rangle - \operatorname{rk}(F)\langle\!\langle L, L\rangle\!\rangle$$

but:

$$\langle\!\langle L, \tau F\rangle\!\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^j L, \tau F\rangle = \sum_{j=0}^{\bar{p}-1} \langle \tau^{j-1} L, F\rangle = \langle\!\langle L, F\rangle\!\rangle$$

since $\bar{p}$ is a multiple of the τ-orbit of F. Hence:

$$\deg(\tau F) = \langle\!\langle L, \tau F\rangle\!\rangle - \operatorname{rk}(F)\langle\!\langle L, L\rangle\!\rangle = \langle\!\langle L, F\rangle\!\rangle - \operatorname{rk}(F)\langle\!\langle L, L\rangle\!\rangle = \deg(F). \qquad \square$$

Proposition 6.1.15. *If $\mathbb{X}$ is of tubular type, any special line bundle $L \in \mathcal{H}$ belongs to a tube of maximal rank in its tubular family, ie. in the tube of rank $\bar{p} = \text{l.c.m.}\{p_i\}$.*

Proof. Suppose $L \in \mathbf{t}_\alpha$ is not in a tube of maximal rank, so in a tube of rank $p < \bar{p}$. This means that L is τ-periodic with period p, ie. $L \cong \tau^p L$. By Remark 6.1.3, for any $x \in \mathbb{X}$, there exists precisely one j with $0 \leq j \leq p(x) - 1$ such that $\operatorname{Hom}_{\mathcal{H}}(L, \tau^j S_x) \neq 0$. This happens in particular for $x \in \mathbb{X}$ such that the rank of the tube $\mathcal{U}_x$ is maximal, ie. $p(x) = \bar{p}$, and, in this case, let j be exactly the integer such that $\operatorname{Hom}_{\mathcal{H}}(L, \tau^j S_x) \neq 0$. Applying τ, we get:

$$0 \neq \operatorname{Hom}_{\mathcal{H}}(L, \tau^j S_x) \cong \operatorname{Hom}_{\mathcal{H}}(\tau L, \tau^{j+1} S_x) \cong \ldots \cong \operatorname{Hom}_{\mathcal{H}}(\tau^p L, \tau^{j+p} S_x)$$

but $\operatorname{Hom}_{\mathcal{H}}(\tau^p L, \tau^{j+p} S_x) \cong \operatorname{Hom}_{\mathcal{H}}(L, \tau^{j+p} S_x) = 0$, since S_x is τ-periodic with period $\bar{p}$ which is strictly larger than p. This leads to a contradiction. $\square$

Proposition 6.1.16 (Riemann-Roch formula)**.** *[42, Theorem 13.8] For any $E, F \in \mathcal{H}$, we have:*

$$\langle\!\langle E, F\rangle\!\rangle = \operatorname{rk}(E)\deg(F) - \deg(E)\operatorname{rk}(F).$$

Remark 6.1.17. Using the Riemann-Roch formula, it is easy to see that, for any $E, F \in \mathcal{H}$:

$$-\langle\!\langle E, F\rangle\!\rangle = \langle\!\langle F, E\rangle\!\rangle.$$

Notice that, if $\operatorname{rk}(E) \neq 0 \neq \operatorname{rk}(F)$, we have that $\langle\!\langle E, F\rangle\!\rangle = \operatorname{rk}(E)\operatorname{rk}(F)(\mu(F) - \mu(E))$. From this, it follows directly that:

Proposition 6.1.18. *[2, Lemma 7.2] If $E, F \in \mathcal{H}$ are indecomposable coherent sheaves such that $\mu(E) < \mu(F)$, then there exists j with $0 \leq j \leq \bar{p} - 1$ such that* $\operatorname{Hom}_{\mathcal{H}}(E, \tau^j F) \neq 0$.

Proof. If E, F are indecomposables such that $\mu(E) < \mu(F)$, then, by definition of slope, $\mathrm{rk}(E)\deg(F) - \deg(E)\,\mathrm{rk}(F) > 0$. Hence, using the Riemann-Roch formula:

$$\langle\!\langle E, F \rangle\!\rangle = \sum_{j=0}^{\bar{p}-1} \dim_k \mathrm{Hom}_{\mathcal{H}}(E, \tau^j F) - \dim_k \mathrm{Ext}^1_{\mathcal{H}}(E, \tau^j F) > 0$$

which means:

$$\sum_{j=0}^{\bar{p}-1} \dim_k \mathrm{Hom}_{\mathcal{H}}(E, \tau^j F) > \sum_{j=0}^{\bar{p}-1} \dim_k \mathrm{Ext}^1_{\mathcal{H}}(E, \tau^j F) \geq 0$$

hence:

$$\sum_{j=0}^{\bar{p}-1} \dim_k \mathrm{Hom}_{\mathcal{H}}(E, \tau^{-j} F) > 0$$

therefore, there is at least one j such that $\mathrm{Hom}_{\mathcal{H}}(E, \tau^j F) \neq 0$ □

Corollary 6.1.19. *Let $E, F \in \mathcal{H}$ be non-exceptional indecomposable coherent sheaves such that $\mu(E) < \mu(F)$, then* $\mathrm{Hom}_{\mathcal{H}}(E, F) \neq 0$.

Bounded derived category of $\mathcal{H}$

Since $\mathcal{H}$ is hereditary, the bounded derived category $\mathcal{D}^b(\mathcal{H})$ is the repetitive category:

$$\mathcal{D}^b(\mathcal{H}) = \mathrm{add}\left(\bigvee_{n \in \mathbb{Z}} \mathcal{H}[n]\right) = \mathrm{add}\left(\bigvee_{(n,\alpha) \in \mathbb{Z} \times \widehat{\mathbb{Q}}} \mathbf{t}_\alpha[n]\right)$$

ie, $\mathcal{D}^b(\mathcal{H})$ is the additive closure of copies of $\mathcal{H}$ and $[n]$ denotes the shift functor. For all $X, Y \in \mathcal{H}$ and all $m, n \in \mathbb{Z}$, we have:

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{H})}(X[n], Y[m]) \cong \mathrm{Ext}^{m-n}_{\mathcal{H}}(X, Y)$$

We define the *Grothendieck group* $K_0(\mathcal{D}^b(\mathcal{H}))$ of $\mathcal{D}^b(\mathcal{H})$, in a similar fashion to $K_0(\mathcal{H})$, as the abelian group generated by the isomorphism classes of complexes in $\mathcal{D}^b(\mathcal{H})$ together with the relations $[X] + [Z] = [Y]$ for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}^b(\mathcal{H})$. There is a canonical isomorphism between $K_0(\mathcal{H})$ and $K_0(\mathcal{D}^b(\mathcal{H}))$ which maps a class $[X] \in K_0(\mathcal{H})$ to the class $[X] \in K_0(\mathcal{D}^b(\mathcal{H}))$ and assigns to the class $[C] \in K_0(\mathcal{D}^b(\mathcal{H}))$, for $C = (C_n)_{n \in \mathbb{Z}} \in \mathcal{D}^b(\mathcal{H})$, the element $\sum_{n \in \mathbb{Z}} (-1)^n [C_n] \in K_0(\mathcal{H})$.
We can extend the definition of the Euler form in $K_0(\mathcal{H})$ to stalk complexes in $\mathcal{D}^b(\mathcal{H})$. Indeed, let $X, Y \in \mathcal{H}$, using the bilinearity of the Euler form, we have:

$$\langle [X[n]], [Y[m]] \rangle = \langle (-1)^n [X], (-1)^m [Y] \rangle = (-1)^{n+m} \langle [X], [Y] \rangle.$$

We can also extend the definition of degree and rank to stalk complexes in $\mathcal{D}^b(\mathcal{H})$. Indeed, for

$X \in \mathcal{H}$, using linearity:

$$\deg([X[n]]) = \deg([(-1)^n X]) = (-1)^n \deg([X])$$
$$\mathrm{rk}([X[n]]) = \mathrm{rk}([(-1)^n X]) = (-1)^n \mathrm{rk}([X])$$

Interval categories

The *interval category* $\mathcal{H}\langle\alpha\rangle$, for $\alpha \in \widehat{\mathbb{Q}}$, is the full subcategory of $\mathcal{D}^b(\mathcal{H})$ defined by:

$$\mathcal{H}\langle\alpha\rangle = \mathrm{add}\left(\bigvee_{\beta>\alpha} \mathbf{t}_\beta \vee \bigvee_{\gamma\leq\alpha} \mathbf{t}_\gamma[1]\right)$$

This is the heart of the t-structure arising from the split torsion pair $(\mathcal{T}_\alpha, \mathcal{F}_\alpha)$ in $\mathcal{H}$ given by:

$$\mathcal{T}_\alpha = \bigvee_{\beta>\alpha} \mathbf{t}_\beta \quad \text{and} \quad \mathcal{F}_\alpha = \bigvee_{\gamma\leq\alpha} \mathbf{t}_\gamma$$

From [39, Theorem 8.1.6], we have that $\mathcal{H}\langle\alpha\rangle = \mathrm{coh}\mathbb{X}$ and clearly $\mathcal{H}\langle\alpha\rangle_0 = \mathbf{t}_\alpha[1]$ and $\mathcal{D}^b(\mathcal{H}) \cong \mathcal{D}^b(\mathcal{H}\langle\alpha\rangle)$ (if the field k is not algebraically closed, $\mathcal{H}\langle\alpha\rangle = \mathrm{coh}\mathbb{X}_\alpha$, for some tubular curve $\mathbb{X}_\alpha$). There is a rank function defined in $K_0(\mathcal{H}\langle\alpha\rangle)$, namely $\mathrm{rk}_\alpha\colon K_0(\mathcal{H}\langle\alpha\rangle) \to \mathbb{Z}$ such that, for $F \in \mathcal{H}\langle\alpha\rangle$, $\mathrm{rk}_\alpha(F) = r\deg(F) - d\,\mathrm{rk}(F)$, where $d, r \in \mathbb{Z}$ are coprime and $\alpha = d/r$, see [39, Proposition 8.1.6]. It is easy to see that:

$$\mathrm{rk}_\alpha(F) = \begin{cases} 0, & \text{if and only if } F \in \mathbf{t}_\alpha[1] \\ n > 0, & \text{if and only if } F \in \mathcal{H}\langle\alpha\rangle_+ \end{cases}$$

This rank function induces a *normalized rank function* in $\mathcal{H}$, $\mathrm{rk}_\alpha\colon K_0(\mathcal{H}) \to \mathbb{Z}$.
Moreover, a sequence $0 \to E' \to E \to E'' \to 0$ with objects in $\mathcal{H} \cap \mathcal{H}\langle\alpha\rangle$ is exact in $\mathcal{H}$ if and only if it is exact in $\mathcal{H}\langle\alpha\rangle$, indeed both condition are equivalent to $E' \to E \to E'' \to E'[1]$ being a triangle in $\mathcal{D}^b(\mathcal{H})$.

6.2 Quasi-coherent sheaves over a noncommutative curve

Let us consider a noncommutative regular projective curve $\mathbb{X}$, so a category $\mathcal{H} = \mathrm{coh}\mathbb{X}$ satisfying (NC1) to (NC6).
From [42, Lemma 3.5], $\mathcal{H}$ can be seen as a noncommutative noetherian projective scheme in the sense of Artin-Zhang (see [6]) and it satisfies Serre's Theorem. This means that there is a positively H-graded noetherian ring R (where H is an ordered abelian group of rank one) such that:

$$\mathcal{H} \cong \frac{\mathrm{mod}^H(R)}{\mathrm{mod}_0^H(R)},$$

ie. $\mathcal{H}$ is the quotient category of the category of finitely generated H-graded R-modules modulo the Serre subcategory of those modules which are finite-dimensional over k.

With this description we define $\vec{\mathcal{H}} = \mathrm{Qcoh}\mathbb{X}$, the category of *quasi-coherent sheaves over* $\mathbb{X}$, as the quotient category:

$$\vec{\mathcal{H}} \cong \frac{\mathrm{Mod}^H(R)}{\mathrm{Mod}_0^H(R)},$$

where $\mathrm{Mod}_0^H(R)$ denotes the localizing subcategory of $\mathrm{Mod}^H(R)$ of all the H-graded torsion, ie. locally finite dimensional, modules. The category $\vec{\mathcal{H}}$ is a hereditary and locally noetherian Grothendieck category, every object of $\vec{\mathcal{H}}$ is a direct limit of objects in $\mathcal{H}$, ie. $\vec{\mathcal{H}} = \varinjlim \mathcal{H}$. The category $\mathcal{H}$ is the full subcategory of coherent objects in $\vec{\mathcal{H}}$, ie. $\mathcal{H} = \mathrm{fp}(\vec{\mathcal{H}})$, by Proposition 1.1.18.
Moreover, we remark that $\vec{\mathcal{H}}$ can also be recovered from $\mathcal{H}$ as the category of left-exact (covariant) k-linear functors from $\mathcal{H}^{\mathrm{op}}$ to Mod$-k$ (see [25, Théorème 1]).
Let τ be the Auslander-Reiten translation on $\mathcal{H}$ and let τ^- be its (quasi-)inverse. It follows from [38, Theorem 4.4] that in $\vec{\mathcal{H}}$ there is a *generalized Serre duality* in the following sense. For all $E \in \mathcal{H}$ and all $X \in \vec{\mathcal{H}}$ there are isomorphisms:

$$D\,\mathrm{Ext}^1_{\vec{\mathcal{H}}}(E, X) \cong \mathrm{Hom}_{\vec{\mathcal{H}}}(X, \tau E) \quad \text{and} \quad \mathrm{Ext}^1_{\vec{\mathcal{H}}}(X, E) \cong D\,\mathrm{Hom}_{\vec{\mathcal{H}}}(\tau^- E, X),$$

where $D = \mathrm{Hom}_k(-, k)$ denotes the vector space duality.

Tilting bundles and concealed-canonical algebras

Let $\mathbb{X}$ be a noncommutative curve of genus zero and consider $\mathcal{H} = \mathrm{coh}\mathbb{X}$ and $\vec{\mathcal{H}} = \mathrm{Qcoh}\mathbb{X}$. Fix a tilting bundle $T \in \mathcal{H}$.
The endomorphism ring $\Lambda = \mathrm{End}(T)$ is a concealed-canonical algebra and every concealed-canonical algebra arises in this way (see [47]).
As an object in $\vec{\mathcal{H}}$, T is a noetherian tilting object and it is a compact generator of $\mathcal{D} = \mathcal{D}^b(\vec{\mathcal{H}})$. The right derived functor of $\mathrm{Hom}_{\mathcal{H}}(T, -)$ induces an equivalence of derived categories (see [26, Theorem 3.2]):

$$\mathbf{R}\mathrm{Hom}_{\mathcal{D}}(T, -) \colon \mathcal{D}^b(\mathrm{coh}\mathbb{X}) \longleftrightarrow \mathcal{D}^b(\text{mod-}\Lambda) \colon - \otimes^{\mathbf{L}}_{\Lambda} T$$

And therefore an equivalence of derived categories:

$$\mathbf{R}\mathrm{Hom}_{\mathcal{D}}(T, -) \colon \mathcal{D}^b(\mathrm{Qcoh}\mathbb{X}) \longleftrightarrow \mathcal{D}^b(\text{Mod-}\Lambda) \colon - \otimes^{\mathbf{L}}_{\Lambda} T \tag{6.1}$$

Moreover, the tilting torsion pair $(\mathcal{T}, \mathcal{F}) = (T^{\perp}, T^{\circ})$ in $\mathcal{H}$ induces a torsion pair $(\mathcal{F}[1], \mathcal{T})$ in mod-Λ, which is actually the torsion pair $(\mathcal{Q}, \mathcal{C})$ defined in Section 3.2. This torsion pair is split by [26, Corollary 3.10]. Therefore, it is possible to identify Mod-Λ with the full subcategory $\mathrm{Add}(\mathcal{T} \cup \mathcal{F}[1])$ of $\mathcal{D}^b(\mathcal{H})$.

Prüfer and adic sheaves

Let S_x be a simple sheaf in a tube $\mathcal{U}_x$. The *ray* starting at S_x is the infinite sequence:

$$S_x \hookrightarrow S_{x,2} \hookrightarrow S_{x,3} \hookrightarrow S_{x,4} \hookrightarrow \dots$$

where $S_{x,n}$ denotes the unique indecomposable coherent sheaf in $\mathcal{U}_x$ of length n with socle S_x, the corresponding direct limit is the *Prüfer sheaf* S_x^∞. Dually, we define the *coray* ending at S_x as the infinite sequence:

$$\cdots \twoheadrightarrow S_{x,-4} \twoheadrightarrow S_{x,-3} \twoheadrightarrow S_{x,-2} \twoheadrightarrow S_x$$

where $S_{x,-n}$ denotes the unique indecomposable coherent sheaf in $\mathcal{U}_x$ of length n with top S_x, the corresponding inverse limit is the *adic sheaf* $S_x^{-\infty}$.
From Theorem 6.1.12, we know that there is an equivalence between $\mathcal{H}_0$ and $\mathbf{t}_\alpha$, for $\alpha \in \widehat{\mathbb{Q}}$, therefore we can define the Prüfer and adic sheaves arising from rays and corays in $\mathbf{t}_\alpha$. Indeed, let $S_{x,\alpha}$ be a simple object in $\mathbf{t}_\alpha$. The *ray* starting at $S_{x,\alpha}$ is the infinite sequence:

$$S_{x,\alpha} \hookrightarrow S_{x,\alpha,2} \hookrightarrow S_{x,\alpha,3} \hookrightarrow S_{x,\alpha,4} \hookrightarrow \dots$$

where $S_{x,\alpha,n}$ denotes the unique indecomposable coherent sheaf in $\mathbf{t}_\alpha$ of length n with socle $S_{x,\alpha}$, the corresponding direct limit is the *Prüfer sheaf* $S_{x,\alpha}^\infty$. Dually we define the *coray* ending at $S_{x,\alpha}$ and the *adic sheaf* $S_{x,\alpha}^{-\infty}$.

The sheaf of rational functions

Definition 6.2.1. A quasi-coherent sheaf G is called *generic* (in the sense of [44]) if it is non-coherent and $\mathrm{Hom}_{\vec{\mathcal{H}}}(T, G)$ and $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(T, G)$ have finite $\mathrm{End}(G)$-length, where T is the tilting module that gives the equivalence 6.1.

The sheaf $\mathcal{K}$ *of rational functions* is the injective envelope of the structure sheaf L of $\vec{\mathcal{H}}$. This is another indecomposable quasi-coherent, non-coherent sheaf. It is torsionfree, ie. $\mathrm{Hom}_{\vec{\mathcal{H}}}(\mathcal{H}_0, \mathcal{K}) = 0$, by [40, Lemma 14], and it is a generic sheaf in the sense of [44]. Moreover, its endomorphism ring is the function field, $\mathrm{End}_{\vec{\mathcal{H}}}(\mathcal{K}) \cong \mathrm{End}_{\mathcal{H}/\mathcal{H}_0}(\pi L) \cong k(\mathcal{H})$.

Injective sheaves

A complete classification of injective sheaves has been done in [2]. Indeed we have the following:

Proposition 6.2.2. *[2, Proposition 3.6] The indecomposable injective sheaves in $\vec{\mathcal{H}}$ are (up to isomorphism) the sheaf of rational functions $\mathcal{K}$ (or generic sheaf) and the Prüfer sheaves S_x^∞, for any simple $S_x \in \mathcal{H}_0$.*

It is clear that the Prüfer sheaves are the injective envelopes of the torsion sheaves in the corresponding uniserial category in $\mathcal{H}_0$ and the generic sheaf $\mathcal{K}$ is the injective envelope of the vector bundles.

Bounded derived category of $\vec{\mathcal{H}}$

$\vec{\mathcal{H}} = \mathrm{Qcoh}\mathbb{X}$ is a hereditary category, therefore, also in this case, the derived category $\mathcal{D}^b(\vec{\mathcal{H}})$ is the repetitive category:

$$\mathcal{D}^b(\vec{\mathcal{H}}) = \mathrm{add}\left(\bigvee_{n\in\mathbb{Z}} \vec{\mathcal{H}}[n]\right).$$

Every object in $\mathcal{D}^b(\vec{\mathcal{H}})$ can be written as $\bigoplus_{i\in I} X_i[i]$, for a finite subset $I \subseteq \mathbb{Z}$ and $X_i \in \vec{\mathcal{H}}$ for all i. Moreover, for any coherent sheaf $E, F \in \mathcal{H}$ and any $n, m \in \mathbb{Z}$:

$$\mathrm{Hom}_{\mathcal{D}^b(\vec{\mathcal{H}})}(E[n], F[m]) \cong \mathrm{Ext}^{m-n}_{\vec{\mathcal{H}}}(E, F)$$

Torsion pairs and slope of a quasi-coherent sheaf

Consider a noncommutative curve of genus zero $\mathbb{X}$. Using the equivalence 6.1:

$$\mathbf{R}\mathrm{Hom}_{\mathcal{D}}(T, -) \colon \mathcal{D}^b(\mathrm{Qcoh}\mathbb{X}) \longleftrightarrow \mathcal{D}^b(\mathrm{Mod}\text{-}\Lambda) \colon - \otimes^{\mathbf{L}}_{\Lambda} T$$

between $\mathrm{Qcoh}\mathbb{X}$ and Mod-Λ, given by a tilting sheaf $T \in \mathcal{H}$, it is possible to transfer many properties that hold for concealed-canonical algebras to the category of quasi-coherent sheaves over $\mathbb{X}$.
Let $w \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We define:

$$\mathbf{p}_w = \bigcup_{\alpha<w} \mathbf{t}_\alpha \qquad \mathbf{q}_w = \bigcup_{w<\beta} \mathbf{t}_\beta,$$

where $\alpha, \beta \in \widehat{\mathbb{Q}}$. Notice that, if w is rational, we have a trisection in $\mathcal{H}$, $(\mathbf{p}_w, \mathbf{t}_w, \mathbf{q}_w)$, such that

$$\mathrm{Hom}_{\vec{\mathcal{H}}}(\mathbf{q}_w, \mathbf{t}_w) = \mathrm{Hom}_{\vec{\mathcal{H}}}(\mathbf{t}_w, \mathbf{p}_w) = \mathrm{Hom}_{\vec{\mathcal{H}}}(\mathbf{q}_w, \mathbf{p}_w) = 0.$$

If w is irrational, we have a bisection in $\mathcal{H}$, $(\mathbf{p}_w, \mathbf{q}_w)$, such that $\mathrm{Hom}_{\vec{\mathcal{H}}}(\mathbf{q}_w, \mathbf{p}_w) = 0$.
We define two classes:

$$\mathcal{C}_w = \mathbf{q}_w{}^{\circ} \quad \text{and} \quad \mathcal{B}_w = {}^{\circ}\mathbf{p}_w$$

Remark 6.2.3. It is immediate to see that $\mathcal{C}_w$ is closed under direct limits, indeed: for any direct system $(E_i)_{i\in I}$ in $\mathcal{C}_w$ and for any $Q \in \mathbf{q}_w$, $\mathrm{Hom}_{\vec{\mathcal{H}}}(Q, \varinjlim E_i) \cong \varinjlim \mathrm{Hom}_{\vec{\mathcal{H}}}(Q, E_i) = 0$, by Proposition 1.1.12. Therefore $\varinjlim E_i \in \mathbf{q}_w{}^{\circ} = \mathcal{C}_w$.

Lemma 6.2.4. *[2, Lemma 7.3][53, Proposition 1.5] For every $w \in \widehat{\mathbb{R}}$ the pair $(\mathcal{Q}_w, \mathcal{C}_w)$, where $\mathcal{Q}_w = \mathrm{Gen}(\mathbf{q}_w)$, is a torsion pair of finite type, which is split in case $w \in \widehat{\mathbb{Q}}$.*

Proof. For the convenience of the reader, we repeat the proof.
It follows from [53, Lemma 1.4] that $\mathrm{Gen}(\mathbf{q}_w)$ is closed under extension, indeed in the locally noetherian setting the same proof works replacing "finite length" by "finitely presented". Then, by [53, Lemma 1.3], $\mathrm{Gen}(\mathbf{q}_w) = {}^{\circ}(\mathbf{q}_w{}^{\circ}) = {}^{\circ}\mathcal{C}_w$, therefore the pair $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$ is a torsion pair, which is of finite type by Remark 6.2.3.
Let now $w \in \widehat{\mathbb{Q}}$. Consider a short exact sequence $\eta : 0 \to X \to Y \to Z \to 0$, with $X \in \mathrm{Gen}(\mathbf{q}_w)$ and $Z \in \mathcal{C}_w$. We can assume that X is a subobject if Y and that $Z = Y/X$. If Z is finitely presented, then, by Serre duality, we have $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(Z, X) \cong D\,\mathrm{Hom}_{\vec{\mathcal{H}}}(X, \tau Z)$ and hence it is zero by the property of the torsion pair. Hence η splits.
If $Z \in \mathcal{C}_w$ not finitely presented, let us consider the set of subobjects U of Y such that $U \cap X = 0$ and $Y/(U + X) \in \mathcal{C}_w$. As shown in [53, Proposition 1.5], there is a maximal object $\overline{U}$ in this set and $Y = X \oplus \overline{U}$ and so η splits. □

Remark 6.2.5. The torsion pair $(\operatorname{add}\mathbf{q}_w, \operatorname{add}(\mathbf{t}_w \cup \mathbf{p}_w))$ in $\mathcal{H}$ has a generating torsionfree class for $\mathcal{H}$ by [45, Proposition 10.7], therefore, since $\vec{\mathcal{H}}$ is a locally noetherian Grothendieck category, we can use Theorem 2.2.3 to prove that $(\mathcal{Q}_w, \mathcal{C}_w)$ is a cotilting torsion pair and $\mathcal{C}_w = \operatorname{Cogen}\mathbf{W}_w$, for a cotilting sheaf $\mathbf{W}_w$.

The following is similar to Lemma 3.3.2 for concealed canonical algebras:

Lemma 6.2.6. *[2, §7] Let $w \in \widehat{\mathbb{R}}$.*

(i) *For $v \leq w$, $\mathcal{C}_v \subseteq \mathcal{C}_w$ and $\mathcal{B}_v \supseteq \mathcal{B}_w$.*
(ii) $\mathcal{C}_w = \bigcap_{w<v\in\widehat{\mathbb{R}}} \mathcal{C}_v$ *and* $\mathcal{B}_w = \bigcap_{\widehat{\mathbb{R}}\ni v<w} \mathcal{B}_v$.
(iii) $\mathcal{Q}_w = \varinjlim \mathbf{q}_w$ *and, if $w \notin \mathbb{Q}$,* $\mathcal{C}_w = \varinjlim \mathbf{p}_w$.
(iv) $\mathcal{B}_w = \bigcap_{\widehat{\mathbb{R}}\ni v<w} \mathcal{Q}_v$.
(v) $\bigcup_{w\in\widehat{\mathbb{R}}} \mathcal{C}_w = \mathcal{C}_\infty = \vec{\mathcal{H}}$ *and* $\bigcap_{w\in\widehat{\mathbb{R}}} \mathcal{C}_w = 0$.
(vi) $\bigcap_{w\in\widehat{\mathbb{R}}} \mathcal{C}_w = 0$ *and* $\bigcup_{w\in\widehat{\mathbb{R}}} \mathcal{C}_w = \mathcal{C}_\infty = \vec{\mathcal{H}}$.
(vii) $\bigcap_{w\in\widehat{\mathbb{R}}} \mathcal{B}_w = \mathcal{B}_\infty = {}^\circ\mathcal{H}_+$ *and* $\mathcal{H} \cap \bigcup_{w\in\widehat{\mathbb{R}}} \mathcal{B}_w = \mathcal{H}$.

Definition 6.2.7. Define the class $\mathcal{M}_w$ as:

$$\mathcal{M}_w = \mathcal{B}_w \cap \mathcal{C}_w.$$

A quasi-coherent sheaf $E \in \vec{\mathcal{H}}$ has *slope* w if $E \in \mathcal{M}_w$.

Clearly the slope of a quasi-coherent sheaf is an element of $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For coherent sheaves this definition agrees with the definition of slope as the quotient of degree and rank and if w is irrational we have only non-coherent sheaves in $\mathcal{M}_w$.

Theorem 6.2.8. *[2, Theorem 7.6] [53, Theorem 13.1] Every indecomposable sheaf has a well-defined slope $w \in \widehat{\mathbb{R}}$. Moreover, if $w < w'$, we have* $\operatorname{Hom}_{\vec{\mathcal{H}}}(\mathcal{M}_{w'}, \mathcal{M}_w) = 0$.

Proof. Also here, we repeat the proof.
Let $X \in \vec{\mathcal{H}}$ be indecomposable. Then $0 \neq X \in \bigcup_{w\in\widehat{\mathbb{R}}} \mathcal{C}_w \setminus \bigcap_{w\in\widehat{\mathbb{R}}} \mathcal{C}_w$, by Lemma 6.2.6(v). Consider $w \in \widehat{\mathbb{R}}$ to be infimum of all $\alpha \in \widehat{\mathbb{Q}}$ such that $X \in \mathcal{C}_\alpha$. Since $\mathbf{q}_w = \bigcup_{\alpha>w} \mathbf{q}_\alpha$, we have $\operatorname{Hom}_{\vec{\mathcal{H}}}(\mathbf{q}_w, X) = 0$, which means $X \in \mathcal{C}_w$.
Now, observe that:

$$\mathcal{B}_w = \bigcap_{\alpha<w} {}^\circ\mathbf{t}_\alpha$$

and $\operatorname{Gen}(\mathbf{q}_\alpha) \subseteq {}^\circ\mathbf{t}_\alpha$. Hence, if $X \notin \mathcal{B}_w$, then there is a rational $\beta < w$ with $X \notin \operatorname{Gen}(\mathbf{q}_\beta)$. But $(\operatorname{Gen}(\mathbf{q}_\beta), \mathcal{C}_\beta)$ is a split torsion pair, and since X is indecomposable, we get $X \in \mathcal{C}_\beta$. Since $\beta < w$, this is a contradiction to the minimality of w.
For the second part of the statement, we refer to Theorem 3.3.5. Indeed, notice that $\mathcal{M}_w \subseteq \mathcal{C}_w$ and $\mathcal{M}_{w'} \subseteq \mathcal{B}_{w'} \subseteq \mathcal{Q}_w$, by Lemma 6.2.6(iv), and since $(\mathcal{Q}_w, \mathcal{C}_w)$ is a torsion pair we have $\operatorname{Hom}_{\vec{\mathcal{H}}}(\mathcal{M}_{w'}, \mathcal{M}_w) = 0$. □

We can define, as in Section 3.2.1, the class ω_w for any $w \in \widehat{\mathbb{Q}}$ as:

$$\omega_w = \mathcal{C}_w \cap \mathcal{D}_w,$$

where $\mathcal{D}_w$ is the class of *divisible* sheaves with respect to $\mathbf{t}_w$:

$$\mathcal{D}_w = {}^{\circ}\mathbf{t}_w = \mathbf{t}_w{}^{\perp}.$$

It is clear that Prüfer sheafes of slope w belong to ω_w. Moreover, for any $w \in \widehat{\mathbb{Q}}$ there is an infinite dimensional module of slope w, G_w, which is generic in the sense of Definition 6.2.1 (see [44, Theorem 4.4]). We have also an analogue of Theorem 3.2.5, indeed any sheaf in ω_w is a direct sum of Prüfer sheaves of slope w and of copies of the generic sheaf G_w.

Moreover, we can prove the left-approximation property of the class ω_w, as in Theorem 3.2.4. For every $F \in \mathcal{C}_w$, we obtain short exact sequences:

$$0 \longrightarrow F \xrightarrow{f} X \longrightarrow X' \longrightarrow 0$$

where $X \in \omega_w$ and X' is a direct sum of Prüfer sheaves in ω_w. In particular, if F is torsionfree with respect to the tubular family $\mathbf{t}_w$, ie. $F \in \mathcal{F}_w = \mathbf{t}_w{}^{\circ}$, then also $X \in \mathcal{F}_w$.

Notice that if $w = \infty$, then we can identify ω_∞ with the class of all the injective sheaves in $\mathrm{Qcoh}\mathbb{X}$.

The class of all Prüfer sheaves in ω_w is separating in the following sense:

Proposition 6.2.9. *Let $F_1, F_2 \in \mathcal{H}$ be indecomposable coherent sheaves of slopes $\mu(F_1) = w_1$ and $\mu(F_2) = w_2$ and suppose $w_1 < w_2$. For any $w \in \mathbb{Q}$ such that $w_1 < w < w_2$, any map $f \colon F_1 \to F_2$ factors through a direct sum of Prüfer sheaves of slope w.*

Proof. Consider the torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ as in Lemma 6.2.4 with heart $\mathcal{A}_w = \mathrm{Qcoh}\mathbb{X}_w$. Clearly $F_1 \in \mathcal{C}_w$ and $F_2 \in \mathcal{Q}_w$ and $\mathrm{Hom}_{\mathcal{H}}(F_1, F_2) \cong \mathrm{Ext}^1_{\mathcal{A}_w}(F_1[1], F_2)$, by Lemma 1.3.9(v).

We have the following diagram in the heart $\mathcal{A}_w$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_2 & \longrightarrow & E & \longrightarrow & F_1[1] & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & F_2 & \longrightarrow & G_w^{(\alpha)}[1] & \longrightarrow & \bigoplus_{x\in\mathbb{X}} (S^{\infty}_{x,w}[1])^{(\beta_x)} & \longrightarrow & 0 \end{array}$$

for some cardinal numbers α, β_x. Here, the upper short exact sequence comes from the map $f \colon F_1 \to F_2$ via the isomorphism above and the lower one is the ω_w-left-approximation of F_2. It is clear that $F_2 \in \mathcal{F}_w$, hence also the middle term of the approximation is in $\mathcal{F}_w$, therefore it is a coproduct of copies of the generic sheaf of slope w.

In $\mathcal{A}_w = \mathrm{Qcoh}\mathbb{X}_w$, $G_w[1]$ is an injective sheaf and since we are in a locally noetherian category

$G_w^{(\alpha)}[1]$ is injective too. We can complete the diagram as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_2 & \longrightarrow & E & \longrightarrow & F_1[1] & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_2 & \longrightarrow & G_w^{(\alpha)}[1] & \longrightarrow & \bigoplus_{x\in\mathbb{X}}(S_{x,w}^{\infty}[1])^{(\beta_x)} & \longrightarrow & 0 \end{array}$$

Which translates in $\mathcal{D}^b(\mathcal{A}_w)$ as a diagram of triangles

$$\begin{array}{ccccccc} F_2 & \longrightarrow & E & \longrightarrow & F_1[1] & \xrightarrow{f[1]} & F_2[1] \\ \| & & \downarrow & & \downarrow & & \| \\ F_2 & \longrightarrow & G_w^{(\alpha)}[1] & \longrightarrow & \bigoplus_{x\in\mathbb{X}}(S_{x,w}^{\infty}[1])^{(\beta_x)} & \longrightarrow & F_2[1] \end{array}$$

Therefore $f[1]$ factors through the direct sum of the shifts of Prüfer sheaves, which means, shifting back to $\vec{\mathcal{H}}$, that we have a factorization:

$$\begin{array}{ccc} F_1 & \xrightarrow{\quad f \quad} & F_2 \\ & \searrow \quad \nearrow & \\ & \bigoplus_{x\in\mathbb{X}}(S_{x,w}^{\infty})^{(\beta_x)} & \end{array}$$

□

6.2.1 Sheaves of irrational slope

If w is rational, we have seen plenty of examples of sheaves of slope w, such as the Prüfer sheaves, the adics and the generic G_w.

If w is not rational, we provide a tool to construct a quasi-coherent sheaf of slope w. This is a modification of the Second Construction in [53, Section 13.4].

Theorem 6.2.10 (Construction). *Let $\alpha_1 < \alpha_2 < \dots$ be a sequence of rational numbers converging to w and choose coherent sheaves $E_i \in \operatorname{add}(\mathbf{t}_{\alpha_i})$ such that $E_i \subseteq E_{i+1}$ for all $i \geq 1$, then:*

$$E = \varinjlim E_i \in \mathcal{M}_w.$$

Proof. It is clear that all sheaves E_i belong to $\mathcal{C}_w$ and since $\mathcal{C}_w$ is closed under direct limits, we have that $E \in \mathcal{C}_w$. Now, for any rational number $\alpha < w$ we can fix an i such that $\alpha < \alpha_i$. Therefore, any $E_j \in \operatorname{add}(\mathbf{q}_\alpha)$ for all $j \geq i$. This means that $E \in \operatorname{Gen}(\mathbf{q}_\alpha)$ for all rationals $\alpha < w$. It is obvious that $\operatorname{Gen}(\mathbf{q}_\alpha) \subseteq \mathcal{B}_\alpha$, so $E \in \bigcap_{\alpha<w,\alpha\in\mathbb{Q}} \mathcal{B}_\alpha$ implies $E \in \mathcal{B}_w$. Therefore $E \in \mathcal{B}_w \cap \mathcal{C}_w = \mathcal{M}_w$. □

Remark 6.2.11. As seen in Lemma 6.2.4, we have, for w irrational, a torsion pair $(\mathcal{Q}_w, \mathcal{C}_w)$ of finite type, whose corresponding heart $\mathcal{A}_w = \vec{\mathcal{H}}(\mathcal{Q}_w, \mathcal{C}_w)$ is, by Theorem 1.3.12, a locally coherent Grothendieck category. Very few things are known about this category.

Since the torsionfree class $\mathcal{C}_w \cap \mathcal{H}$ is generating for $\mathcal{H}$, by Theorem 2.2.3, $(\mathcal{Q}_w, \mathcal{C}_w)$ is cogenerated by a cotilting sheaf $\mathbf{W}_w$ which becomes an injective cogenerator of $\mathcal{A}_w$ by Proposition 2.2.4(ii). The explicit description of this cotilting sheaf is unknown and, therefore, the study of injective objects in $\mathcal{A}_w$ would be useful for this purpose. To this end, we focus on the description of the simple objects in $\mathcal{A}_w$ and we will develop this topic in Chapter 8.

Chapter 7

Continued fractions and universal extensions

In this chapter we introduce continued fractions and universal extensions. We will need these tools in Chapter 8 to describe the simples in the heart $\mathcal{A}_w$ of the t-structure arising from the torsion pair $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$, where $w \in \widehat{\mathbb{R}} \setminus \widehat{\mathbb{Q}}$.

7.1 General facts on continued fractions

A *continued fraction* is an expression of the form

$$n_0 + \cfrac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$$

where n_0 is an arbitrary integer number and $n_1, n_2, \dots$ are positive integers. The terms n_i can be finitely or infinitely many. We use the following notation for a continued fraction with a finite (resp. infinite) number of terms:

$$[n_0; n_1, n_2, \dots, n_d] = n_0 + \cfrac{1}{n_1 + \frac{1}{\dots + \frac{1}{n_d}}}, \quad [n_0; n_1, n_2, \dots] = n_0 + \cfrac{1}{n_1 + \frac{1}{n_2 + \dots}}.$$

For a finite continued fraction $[n_0; n_1, n_2, \dots, n_d]$, we say that its *order* is $d + 1$. Every finite continued fraction represents a finite number of rational operations on its elements. Therefore, it can be represented as an ordinary number in $\mathbb{Q}$.

Example 7.1.1. Let us consider the following finite continued fraction $\alpha = [-2; 1, 3, 5]$. Expanding it, we obtain:

$$[-2; 1, 3, 5] = -2 + \cfrac{1}{1 + \frac{1}{3 + \frac{1}{5}}} = -2 + \cfrac{1}{1 + \frac{5}{16}} = -2 + \frac{16}{21} = -\frac{26}{21}.$$

Definition 7.1.2. Given an infinite continued fraction $\alpha = [n_0; n_1, n_2, \dots]$, we define, for $k \in \mathbb{N}$, the *k-th convergent* of the continued fraction as the finite continued fraction $\alpha_k = [n_0; n_1, \dots, n_k]$.

Clearly this definition works also if the continued fraction α is finite of order n and provided that $k < n$. We call *even-convergent* a k-convergent with k an even number and, similarly, we define the *odd-convergent*. An arbitrary k-convergent of a continued fraction is a continued fraction itself, therefore it can be expressed as a rational number:

$$\alpha_k = \frac{p_k}{q_k} \in \mathbb{Q}$$

where p_k and q_k are coprime. This will be more clear using the recursion in Proposition 7.1.5.

Example 7.1.3. Consider the continued fraction as in Example 7.1.1. The convergents are: $\alpha_0 = [-2] = -2$, $\alpha_1 = [-2; 1] = -1$, $\alpha_2 = [-2; 1, 3] = -5/4$ and $\alpha_4 = \alpha = -26/21$.

Remark 7.1.4. Consider the finite continued fraction $[n_0; n_1, \ldots, n_{d-1}, 1]$. Expanding it, it is clearly visible that this is the same as the continued fraction $[n_0; n_1, \ldots, n_{d-1} + 1]$. For this reason, from now on we can suppose, without loss of generality, that the last term of all the finite continued fractions is different from 1.

The following result is a fundamental tool in the theory of continued fractions.

Proposition 7.1.5 (Recursion for convergents)**.** *[37, Theorem 1] Let* $\alpha = [n_0; n_1, n_2, \ldots]$ *be a continued fraction. Consider*

$$p_{-2} = 0, p_{-1} = 1,$$
$$q_{-2} = 1, q_{-1} = 0.$$

For arbitrary $k \geq 0$*, the convergents of* α *satisfy the following equalities:*

$$p_k = n_k p_{k-1} + p_{k-2},$$
$$q_k = n_k q_{k-1} + q_{k-2}.$$

Moreover, we have the following properties for a continued fraction $\alpha = [n_0; n_1, n_2, \ldots]$:

Proposition 7.1.6. *[37, Theorem 2, Corollary, Theorem 3, Corollary]*

(1) *For all* $k \geq 0$*:*
$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

(2) *For all* $k \geq 1$*:*
$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1} q_k}.$$

(3) *For all* $k \geq 1$*:*
$$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} n_k.$$

(4) *For all* $k \geq 2$*:*
$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} n_k}{q_{k-2} q_k}.$$

Remark 7.1.7. By Proposition 7.1.6(2), we have that:

$$\left|\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}\right| = \frac{1}{q_{k-1}q_k}$$

and this is the smallest nonzero distance between p_{k-1}/q_{k-1} and any rational number whose denominator is less or equal q_k. Indeed: let $a/b \neq p_{k-1}/q_{k-1}$ be such that $b \leq q_k$, we have:

$$\left|\frac{p_{k-1}}{q_{k-1}} - \frac{a}{b}\right| = \left|\frac{p_{k-1}b - aq_{k-1}}{q_{k-1}b}\right| \geq \frac{1}{q_{k-1}b} \geq \frac{1}{q_{k-1}q_k}.$$

Theorem 7.1.8. *[37, Theorem 14] To every $\alpha \in \mathbb{R}$ corresponds a unique continued fraction, whose value is α. If α is rational, the continued fraction is finite. If α is irrational, the continued fraction is infinite.*

Using these properties one can prove the following.

Proposition 7.1.9. *[37, Theorem 4] Consider a continued fraction $\alpha = [n_0; n_1, n_2, \dots]$. Even-convergents form an increasing sequence of rational numbers. Odd-convergents form a decreasing sequence of rational numbers. Moreover, every odd-convergent is greater than any even-convergent.*

In particular, every even-convergent is smaller than α and every odd-convergent is greater than α and we can state the following.

Proposition 7.1.10. *[37, Theorem 8] Consider an infinite continued fraction $\alpha = [n_0; n_1, n_2, \dots]$. Let $\alpha_k = p_k/q_k$ be a k-convergent, for $k \geq 0$. Then p_k/q_k is greater than any even-convergent of α_k and it is smaller than any odd-convergent of α_k.*

Lemma 7.1.11. *Let a/b and c/d be two rational numbers such that $b \neq d$ or $b \neq -d$. If $a/b \leq c/d$, then:*

$$\frac{a-c}{b-d} \leq \frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d} \leq \frac{c-a}{d-b}.$$

The same holds if we replace "$\leq$" by "$<$" or "$=$".

Proof. Straightforward. □

Remark 7.1.12. Fix now $\alpha = [n_0; n_1, n_2, \dots]$, for any $k \geq 0$ we can define a sequence:

$$\frac{p_k}{q_k}, \frac{p_k + p_{k+1}}{q_k + q_{k+1}}, \frac{p_k + 2p_{k+1}}{q_k + 2q_{k+1}}, \dots, \frac{p_k + n_k p_{k+1}}{q_k + n_k q_{k+1}} = \frac{p_{k+2}}{q_{k+2}},$$

whose elements are called *intermediate convergents*, which is increasing if k is even and decreasing if k is odd.

7.2 Universal and co-universal extensions

In this section we are going to introduce the notion of universal extension, which has been described mainly in [47, Ch.3]. In [39], Kussin has developed this concept in the setting of noncommutative curves. In literature, universal extensions are also called *tubular mutations*, *tubular shifts* or *twist functors* (cf. for instance [48, 60, 66]).

Let $\mathbb{X}$ be a noncommutative curve of tubular type and consider $\mathcal{H} = \text{coh}\mathbb{X}$. Consider a tube $\mathcal{U}_{x,\alpha}$, of rank $p(x)$, in a fixed tubular family $\mathbf{t}_\alpha$ in $\mathcal{H}$. Let S_x be a sheaf in the mouth of $\mathcal{U}_{x,\alpha}$, hence this is a simple object in $\mathcal{U}_{x,\alpha}$, considered as a connected uniserial category. Let $\mathcal{S}_x$ be the additive closure of the Auslander-Reiten orbit of S_x, ie. $\mathcal{S}_x = \text{add}(\{\tau^j S_x \mid 1 \le j \le p(x)\})$, which is a semisimple abelian category.

Lemma 7.2.1. *Every k-linear functor $F\colon \mathcal{S}_x \to \text{mod}-k$ is representable by the following object in $\mathcal{S}_x$:*

$$Z = \bigoplus_{j=1}^{p(x)} F(\tau^j S_x) \otimes \tau^j S_x,$$

where the tensor product is taken over $\text{End}(S_x) \cong k$ (see Theorem 6.1.12(i)).

Proof. Let F be contravariant (the covariant case is dual), an object $S \in \mathcal{S}_x$ is of the form $S = \bigoplus_{j=1}^{p(x)} (\tau^j S_x)^{m_j}$, for some $m_j \in \mathbb{Z}_{\ge 0}$, then:

$$F(S) = F\left(\bigoplus_{j=1}^{p(x)} (\tau^j S_x)^{m_j}\right) = \bigoplus_{j=1}^{p(x)} F((\tau^j S_x)^{m_j}) = \bigoplus_{j=1}^{p(x)} F(\tau^j S_x)^{m_j},$$

therefore it is enough to check how F works for a single $\tau^i S_x$, with $1 \le i \le p(x)$.
We claim $F(\tau^i S_x) \cong \text{Hom}(\tau^i S_x, Z)$. Now:

$$Z = \bigoplus_{j=1}^{p(x)} F(\tau^j S_x) \otimes \tau^j S_x \cong \bigoplus_{j=1}^{p(x)} (\tau^j S_x)^{d_j},$$

where $d_j = \dim_{\text{End}(S_x)} F(\tau^j S_x)$, hence:

$$\text{Hom}(\tau^i S_x, Z) \cong \text{Hom}\left(\tau^i S_x, \bigoplus_{j=1}^{p(x)} (\tau^j S_x)^{d_j}\right) \cong \bigoplus_{j=1}^{p(x)} \text{Hom}(\tau^i S_x, (\tau^j S_x)^{d_j}) =$$
$$= \text{Hom}(\tau^i S_x, (\tau^i S_x)^{d_i}) \cong \text{End}(\tau^i S_x)^{d_i} \cong \text{End}(S_x)^{d_i} \cong F(\tau^i S_x). \qquad \square$$

Let us apply this argument to the functor $F = \text{Ext}^1(-, E)\big|_{\mathcal{S}_x}$, for $E \in \mathcal{H}$. We get a natural equivalence of functors $\eta_E\colon \text{Hom}(-, E_x)\big|_{\mathcal{S}_x} \cong \text{Ext}^1(-, E)\big|_{\mathcal{S}_x}$, where:

$$E_x = \bigoplus_{j=1}^{p(x)} \text{Ext}^1_{\mathcal{H}}(\tau^j S_x, E) \otimes \tau^j S_x \in \mathcal{S}_x.$$

Via Yoneda lemma, we know that:

$$\text{Hom}_{(\mathcal{H}^{\text{op}}, \mathbf{Ab})}(\text{Hom}_{\mathcal{H}}(-, E_x)\big|_{\mathcal{S}_x}, \text{Ext}^1_{\mathcal{H}}(-, E)\big|_{\mathcal{S}_x}) \cong \text{Ext}^1_{\mathcal{H}}(E_x, E),$$

hence the natural equivalence η_E corresponds to an exact sequence:

$$\eta_E\colon 0 \longrightarrow E \longrightarrow T_{S_x}E \longrightarrow E_x \longrightarrow 0$$

such that the Yoneda composition $\mathrm{Hom}(U, E_x) \to \mathrm{Ext}^1(U, E)$ sending f to $\eta_E \cdot f$ is an isomorphism for each $U \in \mathcal{S}_x$. We call $T_{S_x}E$ an S_x-*universal extension* of E (here, S_x is taken as a representative of the Auslander-Reiten orbit which $\mathcal{S}_x$ is built from).

Via the identification $\mathrm{Ext}^1(-, E)\big|_{\mathcal{S}_x} = \mathrm{Hom}(-, E_x)\big|_{\mathcal{S}_x}$, the assignment $E \mapsto E_x$ extends into a functor for which a map $u\colon E \to E'$, in $\mathcal{H}$, is sent to a map $u_x\colon E_x \to E'_x$ satisfying $u{\cdot}\eta_E = \eta_{E'}{\cdot}u_x$.

Dually, consider the functor $F = \mathrm{Ext}^1(E, -)\big|_{\mathcal{S}_x}$, for $E \in \mathcal{H}$. Applying the argument, we get a natural equivalence of functors $\eta^E\colon \mathrm{Hom}({}^xE, -)\big|_{\mathcal{S}_x} \cong \mathrm{Ext}^1(E, -)\big|_{\mathcal{S}_x}$, where:

$${}^xE = \bigoplus_{j=1}^{p(x)} \mathrm{Ext}^1_{\mathcal{H}}(E, \tau^j S_x) \otimes \tau^j S_x \in \mathcal{S}_x.$$

By the Yoneda lemma, we have:

$$\mathrm{Hom}_{(\mathcal{H}^{\mathrm{op}}, \mathbf{Ab})}(\mathrm{Hom}_{\mathcal{H}}({}^xE, -)\big|_{\mathcal{S}_x}, \mathrm{Ext}^1_{\mathcal{H}}(E, -)\big|_{\mathcal{S}_x}) \cong \mathrm{Ext}^1_{\mathcal{H}}(E, {}^xE),$$

hence the natural equivalence η^E corresponds to an exact sequence:

$$\eta^E\colon 0 \longrightarrow {}^xE \longrightarrow T^\star_{S_x}E \longrightarrow E \longrightarrow 0$$

such that the Yoneda composition $\mathrm{Hom}({}^xE, U) \to \mathrm{Ext}^1(E, U)$ sending f to $f \cdot \eta^E$ is an isomorphism for each $U \in \mathcal{S}_x$. We call $T^\star_{S_x}E$ an S_x-*co-universal extension* of E (here, as above, S_x is taken as a representative of the Auslander-Reiten orbit which $\mathcal{S}_x$ is built from).

In a similar fashion, via the identification $\mathrm{Ext}^1(E, -)\big|_{\mathcal{S}_x} \cong \mathrm{Hom}({}^xE, -)\big|_{\mathcal{S}_x}$, the assignment $E \mapsto {}^xE$ extends into a functor for which a map $u\colon E \to E'$, in $\mathcal{H}$, is sent to a map ${}^xu\colon {}^xE \to {}^xE'$ satisfying $\eta^{E'} \cdot u = {}^xu \cdot \eta^E$.

Similarly we define the universal morphisms with respect to S_x considering the functors $\mathrm{Hom}(-, E)\big|_{\mathcal{S}_x}$ and $\mathrm{Hom}(E, -)\big|_{\mathcal{S}_x}$. We get natural equivalences:

$$\gamma_E\colon \mathrm{Hom}(-, E)\big|_{\mathcal{S}_x} \cong \mathrm{Hom}(-, {}_xE)\big|_{\mathcal{S}_x} \quad \text{and} \quad \gamma^E\colon \mathrm{Hom}(E, -)\big|_{\mathcal{S}_x} \cong \mathrm{Hom}(E^x, -)\big|_{\mathcal{S}_x}$$

where:

$${}_xE = \bigoplus_{j=1}^{p(x)} \mathrm{Hom}_{\mathcal{H}}(\tau^j S_x, E) \otimes \tau^j S_x \in \mathcal{S}_x,$$

and:

$$E^x = \bigoplus_{j=1}^{p(x)} \mathrm{Hom}_{\mathcal{H}}(E, \tau^j S_x) \otimes \tau^j S_x \in \mathcal{S}_x.$$

These natural equivalences, by means of the Yoneda lemma, give rise to two morphisms:

$$\gamma_E\colon {}_xE \longrightarrow E \quad \text{and} \quad \gamma^E\colon E \longrightarrow E^x.$$

which are, respectively, S_x*-universal* and S_x*-co-universal morphisms* of E.
Let us describe some properties of the universal construction we have outlined above. First of all, let us fix a tube $\mathcal{U}_{x,\alpha}$ of rank $p(x)$ in a tubular family $\mathbf{t}_\alpha$ in $\mathcal{H}$. Let $\mathcal{S}_x$ be the additive closure of the Auslander-Reiten orbit of a simple object S_x in $\mathcal{U}_{x,\alpha}$, ie.

$$\mathcal{S}_x = \operatorname{add}(\{\tau^j S_x \mid 1 \leq j \leq p(x)\})$$

Let $E, G \in \mathcal{H}$ and consider:

$$\eta_E\colon 0 \longrightarrow E \xrightarrow{\alpha_E} T_{S_x}E \xrightarrow{\beta_E} E_x \longrightarrow 0 \quad \text{and} \quad \eta^G\colon 0 \longrightarrow {}^xG \xrightarrow{\alpha^G} T^\star_{S_x}G \xrightarrow{\beta^G} G \longrightarrow 0$$

where η_E is the S_x-universal extension of E and η^G is the S_x-co-universal extension of G.
Let us consider the trisection of $\mathcal{H}$, defined in Section 6.2, given by $(\mathbf{p}_\alpha, \mathbf{t}_\alpha, \mathbf{q}_\alpha)$, where:

$$\mathbf{p}_\alpha = \bigcup_{\beta<\alpha} \mathbf{t}_\beta \qquad \mathbf{q}_\alpha = \bigcup_{\alpha<\gamma} \mathbf{t}_\gamma,$$

such that

$$\operatorname{Hom}_{\mathcal{H}}(\mathbf{q}_\alpha, \mathbf{t}_\alpha) = \operatorname{Hom}_{\mathcal{H}}(\mathbf{t}_\alpha, \mathbf{p}_\alpha) = \operatorname{Hom}_{\mathcal{H}}(\mathbf{q}_\alpha, \mathbf{p}_\alpha) = 0.$$

It is clear that $\operatorname{add}(\mathbf{p}_\alpha) = {\mathbf{t}_\alpha}^\circ$ and $\operatorname{add}(\mathbf{q}_\alpha) = {}^\circ\mathbf{t}_\alpha$, where the Hom-orthogonal is taken inside $\mathcal{H}$. Denote by ρ_α the subclass of $\operatorname{add}(\mathbf{p}_\alpha)$ consisting of coherent sheaves whose S_x-co-universal morphism is surjective. Dually, denote by λ_α the subclass of $\operatorname{add}(\mathbf{q}_\alpha)$ consisting of coherent sheaves whose S_x-universal morphism is injective.
We have the following:

Proposition 7.2.2. *[47, (S10)(i)] For E, G, η_E, η^G as above, the following properties hold:*

(i) *If $E \in \operatorname{add}(\mathbf{p}_\alpha)$, then $T_{S_x}E \in \rho_\alpha$ and β_E is the S_x-co-universal morphism for $T_{S_x}E$.*
(i') *If $G \in \operatorname{add}(\mathbf{q}_\alpha)$, then $T^\star_{S_x}G \in \lambda_\alpha$ and α^G is the S_x-universal morphism for $T^\star_{S_x}G$.*

Proof. (i) Let $E \in \operatorname{add}(\mathbf{p}_\alpha)$ and consider the long exact sequence of functors on $\mathcal{S}_x$:

$$\begin{aligned} 0 = \operatorname{Hom}(-, E)\big|_{\mathcal{S}_x} &\longrightarrow \operatorname{Hom}(-, T_{S_x}E)\big|_{\mathcal{S}_x} \xrightarrow{-\circ\beta_E} \operatorname{Hom}(-, E_x)\big|_{\mathcal{S}_x} \xrightarrow{\eta_E} \\ \xrightarrow{\eta_E} \operatorname{Ext}^1(-, E)\big|_{\mathcal{S}_x} &\longrightarrow \operatorname{Ext}^1(-, T_{S_x}E)\big|_{\mathcal{S}_x} \xrightarrow{\operatorname{Ext}^1(-,\beta_E)} \operatorname{Ext}^1(-, E_x)\big|_{\mathcal{S}_x} \longrightarrow 0 \end{aligned}$$

By definition of S_x-universal extension, the map η_E is an isomorphism. This means that $T_{S_x}E \in {\mathcal{S}_x}^\circ$ and that $\operatorname{Ext}^1(-, \beta_E)$ is an isomorphism. By Serre duality, also $\operatorname{Hom}(\beta_E, -)$ is an isomorphism. This proves that the map $\beta_E\colon T_{S_x}E \to E_x$ is a S_x-co-universal morphism. Since $E \in \operatorname{add}(\mathbf{p}_\alpha)$, then it is obvious that there are no nonzero morphisms from $\mathcal{U}_{y,\alpha}$, with $y \neq x$, to $T_{S_x}E$, hence $T_{S_x}E \in \rho_\alpha$.

(i') We proceed dually. Let $G \in \operatorname{add}(\mathbf{q}_\alpha)$ and consider the long exact sequence of functors on $\mathcal{S}_x$:

$$0 = \operatorname{Hom}(G,-)\big|_{\mathcal{S}_x} \longrightarrow \operatorname{Hom}(T^\star_{S_x}G,-)\big|_{\mathcal{S}_x} \xrightarrow{\alpha^G\circ -} \operatorname{Hom}({}^xG,-)\big|_{\mathcal{S}_x} \xrightarrow{\eta^G}$$
$$\xrightarrow{\eta^G} \operatorname{Ext}^1(G,-)\big|_{\mathcal{S}_x} \longrightarrow \operatorname{Ext}^1(T^\star_{S_x}G,-)\big|_{\mathcal{S}_x} \xrightarrow{\operatorname{Ext}^1(\alpha^G,-)} \operatorname{Ext}^1({}^xG,-)\big|_{\mathcal{S}_x} \longrightarrow 0$$

By definition of S_x-co-universal extension, the map η^G is an isomorphism. This means that $T^\star_{S_x}G \in {}^\circ\mathcal{S}_x$ and that $\operatorname{Ext}^1(\alpha^G,-)$ is an isomorphism. By Serre duality, also $\operatorname{Hom}(-,\alpha^G)$ is an isomorphism. This proves that the map $\alpha^G\colon {}^xG \to T^\star_{S_x}$ is a S_x-universal morphism. Since $G \in \operatorname{add}(\mathbf{q}_\alpha)$, then it is obvious that there are no nonzero morphisms from $T^\star_{S_x}G$ to $\mathcal{U}_{y,\alpha}$, with $y \neq x$, hence $T^\star_{S_x}G \in \lambda_\alpha$. □

The assignments $E \mapsto T_{S_x}E$ and $G \mapsto T^\star_{S_x}G$ turn out to be functors, indeed:

Proposition 7.2.3. *[39, 0.4.2(2)] [47, (S10)(ii)] For E, G, η_E, η^G as above, the following properties hold:*

(i) *If $\varepsilon\colon 0 \to E \to E' \to C \to 0$ is such that $E' \in \operatorname{add}(\mathbf{p}_\alpha)$ and $C \cong E_x$, then there is a commutative diagram:*

$$\begin{array}{ccccccccc}
\varepsilon\colon 0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & C & \longrightarrow & 0\\
 & & \| & & \downarrow\cong & & \downarrow\cong & & \\
\eta_E\colon 0 & \longrightarrow & E & \xrightarrow{\alpha_E} & T_{S_x}E & \xrightarrow{\beta_E} & E_x & \longrightarrow & 0
\end{array}$$

(i') *If $\varepsilon\colon 0 \to K \to G' \to G \to 0$ is such that $G' \in \operatorname{add}(\mathbf{q}_\alpha)$ and $K \cong {}^xG$, then there is a commutative diagram:*

$$\begin{array}{ccccccccc}
\eta^G\colon 0 & \longrightarrow & {}^xG & \xrightarrow{\alpha^G} & T^\star_{S_x}G & \xrightarrow{\beta^G} & G & \longrightarrow & 0\\
 & & \downarrow\cong & & \downarrow\cong & & \| & & \\
\varepsilon\colon 0 & \longrightarrow & K & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

(ii) *For any morphism $u\colon E \to F$ in $\operatorname{add}(\mathbf{p}_\alpha)$, there exists a unique morphism $T_{S_x}u\colon T_{S_x}E \to T_{S_x}F$ yielding a commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & T_{S_x}E & \longrightarrow & E_x & \longrightarrow & 0\\
 & & \downarrow u & & \downarrow T_{S_x}u & & \downarrow u_x & & \\
0 & \longrightarrow & F & \longrightarrow & T_{S_x}F & \longrightarrow & F_x & \longrightarrow & 0
\end{array}$$

(ii') *For any morphism $w\colon F \to G$ in $\operatorname{add}(\mathbf{q}_\alpha)$, there exists a unique morphism $T^\star_{S_x}w\colon T^\star_{S_x}F \to$*

$T^\star_{S_x} G$ yielding a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}^x\!F & \longrightarrow & T^\star_{S_x} F & \longrightarrow & F & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle w^x} & & \Big\downarrow {\scriptstyle T^\star_{S_x} w} & & \Big\downarrow {\scriptstyle w} & & \\
0 & \longrightarrow & {}^x\!G & \longrightarrow & T^\star_{S_x} G & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

Therefore, the assignments $E \mapsto T_{S_x} E$ and $G \mapsto T^\star_{S_x} G$ are functors, respectively:

$$T_{S_x} \colon \operatorname{add}(\mathbf{p}_\alpha) \longrightarrow \rho_\alpha \quad \textit{and} \quad T^\star_{S_x} \colon \operatorname{add}(\mathbf{q}_\alpha) \longrightarrow \lambda_\alpha.$$

Proof. (i) Since η_E is an S_x-universal extension and $C \in \mathcal{S}_x$, the isomorphism $\operatorname{Hom}(C, E_x) \cong \operatorname{Ext}^1(C, E)$ implies that there is a map $f \in \operatorname{Hom}(C, E_x)$ such that $\varepsilon = \eta_E \cdot f$. Therefore we obtain a pullback diagram:

$$\begin{array}{rcccccccc}
\varepsilon \colon 0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow & & \Big\downarrow {\scriptstyle f} & & \\
\eta_E \colon 0 & \longrightarrow & E & \xrightarrow{\alpha_E} & T_{S_x} E & \xrightarrow{\beta_E} & E_x & \longrightarrow & 0
\end{array}$$

where $\operatorname{Ker} f \in \mathcal{S}_x$ since $E_x, C \in \mathcal{S}_x$ and $\mathcal{S}_x$ is an abelian category. Moreover, by Snake lemma, $\operatorname{Ker} f$ is a subobject of $E' \in \operatorname{add}(\mathbf{p}_\alpha)$, therefore $\operatorname{Ker} f \in \operatorname{add}(\mathbf{p}_\alpha)$. Hence f is a monomorphism.

Since C and E_x are of the same length, f is an isomorphism.

(i') The proof is dual. Since η^G is an S_x-co-universal extension and $K \in \mathcal{S}_x$, the isomorphism $\operatorname{Hom}({}^x\!G, K) \cong \operatorname{Ext}^1(G, K)$ implies that there is a map $f \in \operatorname{Hom}({}^x\!G, K)$ such that $\varepsilon = f \cdot \eta^G$. Therefore we obtain a pushout diagram:

$$\begin{array}{rcccccccc}
\eta^G \colon 0 & \longrightarrow & {}^x\!G & \xrightarrow{\alpha^G} & T^\star_{S_x} G & \xrightarrow{\beta^G} & G & \longrightarrow & 0 \\
 & & {\scriptstyle f} \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
\varepsilon \colon 0 & \longrightarrow & K & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

where $\operatorname{Coker} f \in \mathcal{S}_x$ since ${}^x\!G, K \in \mathcal{S}_x$ and $\mathcal{S}_x$ is an abelian category. Moreover, by Snake lemma, $\operatorname{Coker} f$ is a quotient object of $G' \in \operatorname{add}(\mathbf{q}_\alpha)$, therefore $\operatorname{Coker} f \in \operatorname{add}(\mathbf{q}_\alpha)$. Hence f is an epimorphism.

Since K and ${}^x\!G$ are of the same length, f is an isomorphism.

(ii) Let $u \colon E \to F$ be a nonzero morphism in $\operatorname{add}(\mathbf{p}_\alpha)$. Since $u \cdot \eta_E = \eta_F \cdot u_x$, we have the following commutative diagram:

$$\begin{array}{rccccccccc}
\eta_E\colon & 0 & \longrightarrow & E & \longrightarrow & T_{S_x}E & \longrightarrow & E_x & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle u} & & \downarrow & & \| & & \\
u\cdot\eta_E = \eta_F\cdot u_x\colon & 0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & E_x & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle u_x} & & \\
\eta_F\colon & 0 & \longrightarrow & F & \longrightarrow & T_{S_x}F & \longrightarrow & F_x & \longrightarrow & 0
\end{array}$$

We define $T_{S_x}u\colon T_{S_x}E \to T_{S_x}F$ as the composition of the vertical arrows in the middle. The uniqueness of $T_{S_x}u$ follows from the commutativity of the left rectangle and from the fact that $E_x \in \mathcal{S}_x$.

(ii') Let $w\colon F \to G$ be a nonzero morphism in $\mathrm{add}(\mathbf{q}_\alpha)$. Since $\eta^G \cdot w = {}^x w \cdot \eta^F$, we have the following commutative diagram:

$$\begin{array}{rccccccccc}
\eta^F\colon & 0 & \longrightarrow & {}^xF & \longrightarrow & T^\star_{S_x}F & \longrightarrow & F & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle {}^xw} & & \downarrow & & \| & & \\
\eta^G\cdot w = {}^xw\cdot\eta^F\colon & 0 & \longrightarrow & {}^xG & \longrightarrow & X & \longrightarrow & F & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle w} & & \\
\eta^G\colon & 0 & \longrightarrow & {}^xG & \longrightarrow & T^\star_{S_x}G & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

We define $T^\star_{S_x}w\colon T^\star_{S_x}F \to T^\star_{S_x}G$ as the composition of the vertical arrows in the middle. The uniqueness of $T^\star_{S_x}w$ follows from the commutativity of the right rectangle and from the fact that ${}^xG \in \mathcal{S}_x$.

□

Proposition 7.2.4. *[47, (S10)(iii,iv)] For E, G, η_E, η^G as above, the following properties hold:*

(i) *The functor $T_{S_x}\colon \mathrm{add}(\mathbf{p}_\alpha) \to \rho_\alpha$ is an equivalence, it is exact on short exact sequences with terms from $\mathrm{add}(\mathbf{p}_\alpha)$ and induces an isomorphism:*

$$\mathrm{Ext}^1_{\mathcal{H}}(E,F) \cong \mathrm{Ext}^1_{\mathcal{H}}(T_{S_x}E, T_{S_x}F).$$

(i') *The functor $T^\star_{S_x}\colon \mathrm{add}(\mathbf{q}_\alpha) \to \lambda_\alpha$ is an equivalence, it is exact on short exact sequences with terms from $\mathrm{add}(\mathbf{q}_\alpha)$ and induces an isomorphism:*

$$\mathrm{Ext}^1_{\mathcal{H}}(F,G) \cong \mathrm{Ext}^1_{\mathcal{H}}(T^\star_{S_x}F, T^\star_{S_x}G).$$

Proof. (i) Let $E \in \rho_\alpha$, consider the surjective $\mathcal{S}_x$-co-universal morphism $\gamma^E\colon E \to E^x$. Denote the kernel of γ^E by $T^-_{S_x}E$, which is clearly an object in $\mathrm{add}(\mathbf{p}_\alpha)$. The assignment $E \mapsto T^-_{S_x}E$ extends to a functor $T^-_{S_x}\colon \rho_\alpha \to \mathrm{add}(\mathbf{p}_\alpha)$, which is by Proposition 7.2.2(i) a left inverse to T_{S_x}.

To see that it is also a right inverse, start with $E \in \rho_\alpha$ and consider the sequence

$$\eta^E \colon 0 \to T^-_{S_x} E \longrightarrow E \xrightarrow{\gamma^E} E^x \longrightarrow 0$$

which gives us a long exact sequence:

$$0 \longrightarrow \operatorname{Hom}(-, E^x)\big|_{\mathcal{S}_x} \xrightarrow{\eta^E} \operatorname{Ext}^1(-, T^-_{S_x} E)\big|_{\mathcal{S}_x} \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1(-, E)\big|_{\mathcal{S}_x} \xrightarrow{\operatorname{Ext}^1(-,\gamma^E)} \operatorname{Ext}^1(-, E^x)\big|_{\mathcal{S}_x} \longrightarrow 0$$

Since γ^E is a co-universal morphism, $\operatorname{Hom}(\gamma^E, -)$ is an isomorphism. Therefore, by Serre duality, $\operatorname{Ext}^1(-, \gamma^E)$ is an isomorphism, proving that η^E is an S_x-universal extension.

Let $0 \to E' \to E \to E'' \to 0$ be a short exact sequence in $\operatorname{add}(\mathbf{p}_\alpha)$, consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\eta_{E'} \colon 0 & \longrightarrow & E' & \xrightarrow{\alpha_{E'}} & T_{S_x} E & \xrightarrow{\beta_{E'}} & E'_x & \longrightarrow & 0 \\
 & & \downarrow u & & \downarrow T_{S_x} u & & \downarrow u_x & & \\
\eta_E \colon 0 & \longrightarrow & E & \xrightarrow{\alpha_E} & T_{S_x} E & \xrightarrow{\beta_E} & E_x & \longrightarrow & 0 \\
 & & \downarrow v & & \downarrow T_{S_x} v & & \downarrow v_x & & \\
\eta_{E''} \colon 0 & \longrightarrow & E'' & \xrightarrow{\alpha_{E''}} & T_{S_x} E'' & \xrightarrow{\beta_{E''}} & E''_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose rows and the left column are exact. The right column is exact since the functor sending $E \to E_x$ is exact because $\operatorname{Ext}^1(U, -)$, with $U \in \mathcal{S}_x$, is exact on short exact sequences with terms in $\operatorname{add}(\mathbf{p}_\alpha)$. This implies that also the middle column is exact.

Since T_{S_x} and $T^-_{S_x}$ are mutually inverse equivalences, it is clear that, for $E, F \in \operatorname{add}(\mathbf{p}_\alpha)$, $\operatorname{Ext}^1_{\mathcal{H}}(E, F) \cong \operatorname{Ext}^1_{\mathcal{H}}(T_{S_x} E, T_{S_x} F)$.

(i') We proceed dually. Let $G \in \lambda_\alpha$, consider the injective $\mathcal{S}_x$-universal morphism $\gamma_G \colon {}_xG \to G$. Denote the cokernel of γ_G by $T^{-\star}_{S_x} G$, which is clearly an object in $\operatorname{add}(\mathbf{q}_\alpha)$. The assignment $G \mapsto T^{-\star}_{S_x} G$ extends to a functor $T^{-\star}_{S_x} \colon \rho_\alpha \to \operatorname{add}(\mathbf{p}_\alpha)$, which is by Proposition 7.2.2(i') a left inverse to $T^\star_{S_x}$.

To see that it is also a right inverse, start with $G \in \operatorname{add}(\mathbf{q}_\alpha)$ and consider the sequence

$$\eta_G \colon 0 \to {}_xG \xrightarrow{\gamma_G} G \longrightarrow T^{-\star}_{S_x} G \longrightarrow 0$$

which gives us a long exact sequence:

$$0 \longrightarrow \operatorname{Hom}({}_xG, -)\big|_{\mathcal{S}_x} \xrightarrow{\eta_E} \operatorname{Ext}^1(T^{-\star}_{S_x} G, -)\big|_{\mathcal{S}_x} \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1(G, -)\big|_{\mathcal{S}_x} \xrightarrow{\operatorname{Ext}^1(\gamma_G,-)} \operatorname{Ext}^1({}_xG, -)\big|_{\mathcal{S}_x} \longrightarrow 0$$

Since γ_G is a universal morphism, $\mathrm{Hom}(-, \gamma_G)$ is an isomorphism. Therefore, by Serre duality, $\mathrm{Ext}^1(\gamma_G, -)$ is an isomorphism too, proving that η_G is an S_x-co-universal extension.

Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence in $\mathrm{add}(\mathbf{q}_\alpha)$, consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta^{G'}: & 0 & \longrightarrow & {}^{x}G' & \xrightarrow{\alpha^{G'}} & T^{-\star}_{S_x}G' & \xrightarrow{\beta^{G'}} & G' & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle {}^{x}u} & & \downarrow{\scriptstyle T^{-\star}_{S_x}u} & & \downarrow{\scriptstyle u} & & \\
\eta^{G}: & 0 & \longrightarrow & {}^{x}G & \xrightarrow{\alpha^{G}} & T^{-\star}_{S_x}G & \xrightarrow{\beta^{G}} & G & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle {}^{x}v} & & \downarrow{\scriptstyle T^{-\star}_{S_x}v} & & \downarrow{\scriptstyle v} & & \\
\eta^{G''}: & 0 & \longrightarrow & {}^{x}G'' & \xrightarrow{\alpha^{G''}} & T^{-\star}_{S_x}G'' & \xrightarrow{\beta^{G''}} & G'' & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

whose rows and the right column are exact. The left column is exact since the functor sending $G \to {}^{x}G$ is exact because $\mathrm{Ext}^1(-, U)$, with $U \in \mathcal{S}_x$, is exact on short exact sequences with terms in $\mathrm{add}(\mathbf{q}_\alpha)$. This implies that also the middle column is exact.

Since $T^{\star}_{S_x}$ and $T^{-\star}_{S_x}$ are mutually inverse equivalences, it is clear that, for $F, G \in \mathrm{add}(\mathbf{q}_\alpha)$, $\mathrm{Ext}^1_{\mathcal{H}}(F, G) \cong \mathrm{Ext}^1_{\mathcal{H}}(T^{\star}_{S_x}F, T^{\star}_{S_x}G)$. $\square$

Chapter 8

Sheaves of irrational slope

Let $\mathbb{X}$ be a noncommutative curve of tubular type. As mentioned in Remark 6.2.11, very few things are known about the category $\mathcal{A}_w = \vec{\mathcal{H}}(\mathcal{Q}_w, \mathcal{C}_w)$, when w is an irrational number. By Theorem 1.3.12, $\mathcal{A}_w$ is a locally coherent Grothendieck category and in this chapter we give a first characterization of the simple objects in it.

First, we fix a positive irrational number w and we use the methods described in Chapter 7 to construct a quasi-coherent sheaf over $\mathbb{X}$ of slope w via a sequence of indecomposable coherent sheaves of smaller slope. Second, we prove that the sheaf obtained in this way actually becomes a simple object in the heart $\mathcal{A}_w$ of the t-structure arising from the torsion pair $(\mathrm{Gen}\,\mathbf{q}_w, \mathbf{q}_w{}^{\circ})$.

Subsequently, we prove that any simple object in the heart $\mathcal{A}_w$ of the t-structure arising from the torsion pair $(\mathrm{Gen}(\mathbf{q}_w), \mathcal{C}_w)$, where $w \in \widehat{\mathbb{R}} \setminus \widehat{\mathbb{Q}}$, comes from a quasi-coherent sheaf of slope w.

8.1 Construction of a sheaf of irrational slope

Let $\mathbb{X}$ be a noncommutative curve of tubular type over an algebraically closed field k, as described in Chapter 6 and let $\mathcal{H} = \mathrm{coh}\mathbb{X}$. Consider the weight type $(p_1, \ldots, p_t)$ of $\mathbb{X}$ and denote by $\bar{p} = \mathrm{l.c.m.}\{p_i\}_{1 \leq i \leq t} = \max\{p_i\}_{1 \leq i \leq t}$.

Definition of the process

Setting. Consider a positive irrational number w together with its continued fraction form $[n_0; n_1, n_2, \ldots]$. Let L be the structure sheaf in $\mathcal{H}$, which is, by Proposition 6.1.4, an endo-simple object. Recall that L is in a tube of maximal rank, ie. of rank $\bar{p}$, as seen in Proposition 6.1.15 and:

$$\deg(L) = 0 \quad \text{and} \quad \mathrm{rk}(L) = 1. \tag{8.1}$$

Let $S_x \in \mathcal{H}_0$ be a simple sheaf in a tube of maximal rank $\bar{p}$. This is a stable sheaf and therefore endo-simple, by Theorem 6.1.12(i). Recall that we have (see Remark 6.1.8):

$$\deg(S_x) = \frac{\bar{p}}{\bar{p}} = 1 \quad \text{and} \quad \mathrm{rk}(S_x) = 0. \tag{8.2}$$

Since $L \in \mathcal{H}_+$, we can apply the S_x-universal extension functor T_{S_x} to L. Set:

$$P_0 = T_{S_x}^{n_0} L$$

where $T_{S_x}^k L = T_{S_x} T_{S_x}^{k-1} L$, for $k \geq 1$. This is well defined, since every $T_{S_x}^k L \in {\mathcal{H}_0}^{\circ}$, for $1 \leq k \leq n_0$, by Proposition 7.2.2(i).
P_0 is an indecomposable and endo-simple coherent sheaf, since L is such and T_{S_x} is an equivalence. We have the following:

Lemma 8.1.1. *Let w, L and S_x be as in Setting. Then, for any $0 \leq n \leq n_0$, we have:*

$$\deg(T_{S_x}^n L) = n \quad \textit{and} \quad \operatorname{rk}(T_{S_x}^n L) = 1.$$

Proof. We proceed by induction on n. If $n = 0$, then the Lemma holds for (8.1).
Suppose that the Lemma holds for a certain n. We obtain $T_{S_x}^{n+1} L$ as the S_x-universal extension of $T_{S_x}^n L$, therefore there is a short exact sequence:

$$0 \longrightarrow T_{S_x}^n L \longrightarrow T_{S_x}^{n+1} L \longrightarrow \bigoplus_{j=1}^{\bar{p}} \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L) \otimes \tau^j S_x \longrightarrow 0$$

Recall that degree and rank are additive on short exact sequence, by definition, and τ-invariant, by Proposition 6.1.14 and [45, §10.2(H 5), Remark 10.2(ii)]. So, we get:

$$\begin{aligned} \deg(T_{S_x}^{n+1} L) &= \deg(T_{S_x}^n L) + \sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L) \deg(\tau^j S_x) = \\ &= \deg(T_{S_x}^n L) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L) \right) \deg(S_x), \\ \operatorname{rk}(T_{S_x}^{n+1} L) &= \operatorname{rk}(T_{S_x}^n L) + \sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L) \operatorname{rk}(\tau^j S_x) = \\ &= \operatorname{rk}(T_{S_x}^n L) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L) \right) \operatorname{rk}(S_x), \end{aligned}$$

Recall that, by definition of average Euler form, we have:

$$\langle\!\langle S_x, T_{S_x}^n L \rangle\!\rangle = \sum_{j=1}^{\bar{p}} \dim \operatorname{Hom}(\tau^j S_x, T_{S_x}^n L) - \dim \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L).$$

Using Riemann-Roch formula (see Proposition 6.1.16), we obtain:

$$\sum_{j=1}^{\bar{p}} \dim \operatorname{Hom}(\tau^j S_x, T_{S_x}^n L) - \dim \operatorname{Ext}^1(\tau^j S_x, T_{S_x}^n L) = \deg(S_x) \operatorname{rk}(T_{S_x}^n L) - \operatorname{rk}(S_x) \deg(T_{S_x}^n L)$$

By Proposition 7.2.2(i), $\operatorname{Hom}(\tau^j S_x, T_{S_x}^n L) = 0$, for any $1 \leq j \leq \bar{p}$. Therefore, by (8.2) and inductive hypothesis, we get:

$$\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T^n_{S_x} L) = \deg(S_x)\operatorname{rk}(T^n_{S_x} L) - \operatorname{rk}(S_x)\deg(T^n_{S_x} L) = 1$$

Therefore, using again (8.2) and the inductive hypothesis:

$$\begin{aligned}
\deg(T^{n+1}_{S_x} L) &= \deg(T^n_{S_x} L) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T^n_{S_x} L) \right) \deg(S_x) = \\
&= \deg(T^n_{S_x} L) + \deg(S_x) = n+1, \\
\operatorname{rk}(T^{n+1}_{S_x} L) &= \operatorname{rk}(T^n_{S_x} L) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j S_x, T^n_{S_x} L) \right) \operatorname{rk}(S_x) = \\
&= \operatorname{rk}(T^n_{S_x} L) + \operatorname{rk}(S_x) = 1.
\end{aligned}$$

□

In particular, if $n = n_0$ in Lemma 8.1.1, we obtain:

$$\deg(P_0) = \deg(T^{n_0}_{P_{-1}} L) = n_0 \quad \text{and} \quad \operatorname{rk}(P_0) = \operatorname{rk}(T^{n_0}_{P_{-1}} L) = 1. \tag{8.3}$$

Moreover:

$$\mu(P_0) = \frac{\deg(P_0)}{\operatorname{rk}(P_0)} = n_0. \tag{8.4}$$

Definition 8.1.2. Let w, L and S_x be as in Setting. Define $P_{-2} = L$ and $P_{-1} = S_x$.
For $i \in \mathbb{Z}_{\geq 0}$, define the *$w$-convergent sheaves* with respect to L and S_x, as the coherent sheaves P_i, obtained by the following recursion:

$$P_i = \begin{cases} T^{n_i}_{P_{i-1}} P_{i-2}, & \text{if } i \text{ is even} \\ T^{\star n_i}_{P_{i-1}} P_{i-2}, & \text{if } i \text{ is odd.} \end{cases}$$

Moreover, for $1 \leq k \leq n_i - 1$, we call $T^k_{P_{i-1}} P_{i-2}$ and $T^{\star k}_{P_{i-1}} P_{i-2}$ the *k-th intermediate w-convergent sheaves after P_{i-2}* with respect to L and S_x.

It is clear that P_0, as it is defined above, is the 0-th w-convergent sheaf with respect to L and S_x.

Remark 8.1.3. Notice that, since T_{P_i} and $T^\star_{P_i}$ are equivalences, all the (intermediate) w-convergent sheaves are indecomposable and endo-simple since L and S_x are such. Moreover, since L and S_x are exceptional sheaves and, by Proposition 7.2.4, T_{P_i} and $T^\star_{P_i}$ preserve extension groups, all the (intermediate) w-convergent sheaves are exceptional.

Proposition 8.1.4. *Let w, L and S_x be as in Setting. Let P_i be the w-convergent sheaves with respect to L and S_x as in Definition 8.1.2.*
For all even $i \geq 0$ and for all $0 \leq n \leq n_i$, we have:

$$\deg(T^n_{P_{i-1}} P_{i-2}) = n \deg(P_{i-1}) + \deg(P_{i-2})$$

$$\operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) = n \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}). \tag{8.5}$$

For all odd $i \geq 0$ and for all $0 \leq n \leq n_i$, we have:

$$\begin{aligned} \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) &= n \deg(P_{i-1}) + \deg(P_{i-2}) \\ \operatorname{rk}(T^{\star^n}_{P_{i-1}} P_{i-2}) &= n \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}). \end{aligned} \tag{8.6}$$

Moreover, for all $i \geq 0$:

$$\langle\langle P_i, P_{i-1} \rangle\rangle = (-1)^i. \tag{8.7}$$

Proof. We prove (8.7) by induction on $i \geq 0$. For the inductive step of the proof, we consider two cases: first, for i odd, we prove the formulas in (8.6), and then for i even, we prove the formulas in (8.5). Both cases are proven using induction on n.
Base case $i = 0$. This case follows from Lemma 8.1.1. Indeed, for all $0 \leq n \leq n_0$, we have:

$$\begin{aligned} \deg(T^n_{S_x} L) &= n = n \deg(S_x) + \deg(L) \\ \operatorname{rk}(T^n_{S_x} L) &= 1 = \operatorname{rk}(S_x) + \operatorname{rk}(L) \end{aligned}$$

and, by the Riemann-Roch formula:

$$\langle\langle P_0, P_{-1} \rangle\rangle = \langle\langle P_0, S_x \rangle\rangle = \operatorname{rk}(P_0) \deg(S_x) - \deg(P_0) \operatorname{rk}(S_x) = 1.$$

Therefore (8.7) holds for $i = 0$.
Inductive step. Fix a positive integer number i. Suppose that (8.7) holds for $i - 1$.
Suppose first that i **is even**. We prove the identities (8.5) by induction on n.
If $n = 0$, then (8.5) clearly holds.
Suppose (8.5) holds for a certain $n > 0$, then we obtain $T^{n+1}_{P_{i-1}} P_{i-2}$ as the P_{i-1}-universal extension of $T^n_{P_{i-1}} P_{i-2}$. Therefore, we have the following short exact sequence:

$$0 \longrightarrow T^n_{P_{i-1}} P_{i-2} \longrightarrow T^{n+1}_{P_{i-1}} P_{i-2} \longrightarrow \bigoplus_{j=1}^{\bar{p}} \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \otimes \tau^j P_{i-1} \longrightarrow 0$$

Degree and rank are τ-invariant and additive on short exact sequences, so:

$$\begin{aligned} \deg(T^{n+1}_{P_{i-1}} P_{i-2}) &= \deg(T^n_{P_{i-1}} P_{i-2}) + \sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \deg(\tau^j P_{i-1}) = \\ &= \deg(T^n_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \right) \deg(P_{i-1}), \\ \operatorname{rk}(T^{n+1}_{P_{i-1}} P_{i-2}) &= \operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) + \sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \operatorname{rk}(\tau^j P_{i-1}) = \\ &= \operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \right) \operatorname{rk}(P_{i-1}). \end{aligned}$$

By definition of average Euler form and using the Riemann-Roch formula (see Proposition 6.1.16), we get:

$$\sum_{j=1}^{\bar{p}} \dim \operatorname{Hom}(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) - \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) =$$
$$= \deg(P_{i-1}) \operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) - \operatorname{rk}(P_{i-1}) \deg(T^n_{P_{i-1}} P_{i-2}).$$

By Proposition 7.2.2(i), $\operatorname{Hom}(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) = 0$, for any integer j. Moreover, by inductive hypothesis, the identities in (8.5) hold for $T^n_{P_{i-1}} P_{i-2}$ and, since (8.7) holds for $i-1$, which is an odd number, we obtain:

$$\begin{aligned}
\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) &= \deg(P_{i-1}) \operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) - \operatorname{rk}(P_{i-1}) \deg(T^n_{P_{i-1}} P_{i-2}) = \\
&= \deg(P_{i-1})(n \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2})) - \operatorname{rk}(P_{i-1})(n \deg(P_{i-1}) + \deg(P_{i-2})) = \\
&= \deg(P_{i-1}) \operatorname{rk}(P_{i-2}) - \operatorname{rk}(P_{i-1}) \deg(P_{i-2}) = \\
&= -\langle\!\langle P_{i-1}, P_{i-2} \rangle\!\rangle = -(-1)^{i-1} = 1.
\end{aligned}$$

Hence, using again the inductive hypothesis for n, we get:

$$\begin{aligned}
\deg(T^{n+1}_{P_{i-1}} P_{i-2}) &= \deg(T^n_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \right) \deg(P_{i-1}) = \\
&= \deg(T^n_{P_{i-1}} P_{i-2}) + \deg(P_{i-1}) = \\
&= n \deg(P_{i-1}) + \deg(P_{i-2}) + \deg(P_{i-1}) = \\
&= (n+1) \deg(P_{i-1}) + \deg(P_{i-2}), \\
\operatorname{rk}(T^{n+1}_{P_{i-1}} P_{i-2}) &= \operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(\tau^j P_{i-1}, T^n_{P_{i-1}} P_{i-2}) \right) \operatorname{rk}(P_{i-1}) = \\
&= \operatorname{rk}(T^n_{P_{i-1}} P_{i-2}) + \operatorname{rk}(P_{i-1}) = \\
&= n \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}) + \operatorname{rk}(P_{i-1}) = \\
&= (n+1) \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}).
\end{aligned}$$

Proving that (8.5) holds for $T^{n+1}_{P_{i-1}} P_{i-2}$. In particular, if $n = n_i$, we have:

$$\deg(P_i) = \deg(T^{n_i}_{P_{i-1}} P_{i-2}) = n_i \deg(P_{i-1}) + \deg(P_{i-2})$$
$$\operatorname{rk}(P_i) = \operatorname{rk}(T^{n_i}_{P_{i-1}} P_{i-2}) = n_i \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}).$$

Suppose now that i **is odd**. The proof is dual to the even case and we prove the identities in (8.6) by induction on n.

If $n = 0$, then it is clear that (8.6) holds.

Suppose (8.6) holds for a certain $n > 0$, then we obtain ${T^{\star}_{P_{i-1}}}^{n+1} P_{i-2}$ as the P_{i-1}-co-universal extension of $T^{\star^n}_{P_{i-1}} P_{i-2}$. Therefore, we have the following short exact sequence:

$$0 \longrightarrow \bigoplus_{j=1}^{\bar{p}} \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) \otimes \tau^j P_{i-1} \longrightarrow T^{\star^{n+1}}_{P_{i-1}} P_{i-2} \longrightarrow T^{\star^n}_{P_{i-1}} P_{i-2} \longrightarrow 0$$

Degree and rank are τ-invariant and additive on short exact sequences, so:

$$\begin{aligned}
\deg(T^{\star^{n+1}}_{P_{i-1}} P_{i-2}) &= \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) + \sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) \deg(\tau^j P_{i-1}) = \\
&= \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) \right) \deg(P_{i-1}), \\
\operatorname{rk}(T^{\star^{n+1}}_{P_{i-1}} P_{i-2}) &= \operatorname{rk}(T^{\star^n}_{P_{i-1}} P_{i-2}) + \sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) \operatorname{rk}(\tau^j P_{i-1}) = \\
&= \operatorname{rk}(T^{\star^n}_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) \right) \operatorname{rk}(P_{i-1}).
\end{aligned}$$

By definition of average Euler form and using the Riemann-Roch formula (see Proposition 6.1.16), we get:

$$\begin{aligned}
\sum_{j=1}^{\bar{p}} \dim \operatorname{Hom}(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) - \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) = \\
= \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) \operatorname{rk}(P_{i-1}) - \operatorname{rk}(T^{\star^n}_{P_{i-1}} P_{i-2}) \deg(P_{i-1}).
\end{aligned}$$

By Proposition 7.2.2(i'), $\operatorname{Hom}(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) = 0$, for any integer j. Moreover, by inductive hypothesis, the identities in (8.6) hold for $T^{\star^n}_{P_{i-1}} P_{i-2}$ and, since (8.7) holds for $i-1$, which is an even number, we obtain:

$$\begin{aligned}
\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) &= \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) \operatorname{rk}(P_{i-1}) - \operatorname{rk}(T^{\star^n}_{P_{i-1}} P_{i-2}) \deg(P_{i-1}) = \\
&= (n \deg(P_{i-1}) + \deg(P_{i-2})) \operatorname{rk}(P_{i-1}) - (n \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2})) \deg(P_{i-1}) = \\
&= \deg(P_{i-2}) \operatorname{rk}(P_{i-1}) - \operatorname{rk}(P_{i-2}) \deg(P_{i-1}) = \\
&= -\langle\!\langle P_{i-2}, P_{i-1} \rangle\!\rangle = \langle\!\langle P_{i-1}, P_{i-2} \rangle\!\rangle = (-1)^{i-1} = 1.
\end{aligned}$$

Hence, using again the inductive hypothesis for n, we get:

$$\begin{aligned}
\deg(T^{\star^{n+1}}_{P_{i-1}} P_{i-2}) &= \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^n}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1}) \right) \deg(P_{i-1}) = \\
&= \deg(T^{\star^n}_{P_{i-1}} P_{i-2}) + \deg(P_{i-1}) = \\
&= n \deg(P_{i-1}) + \deg(P_{i-2}) + \deg(P_{i-1}) =
\end{aligned}$$

$$= (n+1)\deg(P_{i-1}) + \deg(P_{i-2}),$$

$$\operatorname{rk}(T^{\star^{n+1}}_{P_{i-1}} P_{i-2}) = \operatorname{rk}(T^{\star^{n}}_{P_{i-1}} P_{i-2}) + \left(\sum_{j=1}^{\bar{p}} \dim \operatorname{Ext}^1(T^{\star^{n}}_{P_{i-1}} P_{i-2}, \tau^j P_{i-1})\right) \operatorname{rk}(P_{i-1}) =$$
$$= \operatorname{rk}(T^{\star^{n}}_{P_{i-1}} P_{i-2}) + \operatorname{rk}(P_{i-1}) =$$
$$= n\operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}) + \operatorname{rk}(P_{i-1}) =$$
$$= (n+1)\operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}).$$

Proving that (8.6) holds for $n+1$. In particular, for $n = n_i$, we have:

$$\deg(P_i) = \deg(T^{\star^{n_i}}_{P_{i-1}} P_{i-2}) = n_i \deg(P_{i-1}) + \deg(P_{i-2}),$$
$$\operatorname{rk}(P_i) = \operatorname{rk}(T^{\star^{n_i}}_{P_{i-1}} P_{i-2}) = n_i \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}).$$

Moreover, in both cases, (8.7) holds, indeed:

$$\langle\langle P_i, P_{i-1}\rangle\rangle = \operatorname{rk}(P_i)\deg(P_{i-1}) - \deg(P_i)\operatorname{rk}(P_{i-1}) =$$
$$= (n_i \operatorname{rk}(P_{i-1}) + \operatorname{rk}(P_{i-2}))\deg(P_{i-1}) - (n_i \deg(P_{i-1}) + \deg(P_{i-2}))\operatorname{rk}(P_{i-1}) =$$
$$= \operatorname{rk}(P_{i-2})\deg(P_{i-1}) - \deg(P_{i-2})\operatorname{rk}(P_{i-1}) =$$
$$= -\langle\langle P_{i-1}, P_{i-2}\rangle\rangle = (-1)(-1)^{i-1} = (-1)^i.$$

This concludes the proof. □

From now on, to ease the notation, we set for $k \geq -2$:

$$\mu_k = \mu(P_k) = \frac{\deg(P_k)}{\operatorname{rk}(P_k)}.$$

8.1.1 Relation with the continued fractions

The recursion described in Proposition 8.1.4 is a two-term recursion and it is closely related to the recursion for the convergents of a continued fraction defined in Proposition 7.1.5.
Let us set, for any integer $k \geq -2$:

$$\deg(P_k) = p_k \text{ and } \operatorname{rk}(P_k) = q_k.$$

Via this identification, we can translate the recursions (8.5) and (8.6) in Proposition 8.1.4 to the language of continued fractions, indeed:

$$\deg(P_{-2}) = \deg(L) = 0 = p_{-2} \quad \text{and} \quad \deg(P_{-1}) = \deg(S_x) = 1 = p_{-1}$$
$$\operatorname{rk}(P_{-2}) = \operatorname{rk}(L) = 1 = q_{-2} \quad \text{and} \quad \operatorname{rk}(P_{-1}) = \operatorname{rk}(S_x) = 0 = q_{-1}$$

and for any $k \geq 0$:

$$\deg(P_k) = n_k \deg(P_{k-1}) + \deg(P_{k-2}) = n_k p_{k-1} + p_{k-2} = p_k$$

$$\text{rk}(P_k) = n_k \,\text{rk}(P_{k-1}) + \text{rk}(P_{k-2}) = n_k q_{k-1} + q_{k-2} = q_k$$

Hence, the slope of the k-th w-convergent sheaf P_k is:

$$\mu_k = \frac{\deg(P_k)}{\text{rk}(P_k)} = \frac{p_k}{q_k} = w_k.$$

where w_k is the k-th convergent of the continued fraction $[n_0; n_1, n_2, \dots]$ representing the real number w, fixed at the beginning.
Using this correspondence, we can translate many properties of the continued fractions to the w-convergent sheaves setting. For example, it is immediate to see that (8.7) in Proposition 8.1.4 corresponds to Proposition 7.1.6(1), indeed:

$$\langle\langle P_k, P_{k-1} \rangle\rangle = \text{rk}(P_k)\deg(P_{k-1}) - \deg(P_k)\,\text{rk}(P_{k-1}) = q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

Remark 8.1.5. It follows from Proposition 7.1.9 that the slopes of the even w-convergent sheaves form an increasing sequence in $\mathbb{Q}_{\geq 0}$, ie.

$$\mu_0 < \mu_2 < \dots < \mu_k < \mu_{k+2} < \dots$$

(with k even), that converges to w.
Dually, the slopes of the odd w-convergent sheaves form a decreasing sequence in $\mathbb{Q}_{\geq 0}$, ie.

$$\mu_1 > \mu_3 > \dots > \mu_k > \mu_{k+2} > \dots$$

(with k odd), that converges to w.
Moreover, $\mu_k < w$ for k even, $\mu_k > w$ for k odd and this implies that $\mu_i < \mu_j$, for any i even and j odd.

As seen in Remark 7.1.7, the distance between two consecutive convergents, p_{k-1}/q_{k-1} and p_k/q_k, of w is minimal in the sense that the interval $(p_{k-1}/q_{k-1}, p_k/q_k)$ (or the interval $(p_k/q_k, p_{k-1}/q_{k-1})$) does not contain any rational number whose denominator is less or equal than q_k. This translates in the language of w-convergent sheaves as:

Lemma 8.1.6. *Let $F \in \mathcal{H}$ such that $\text{rk}(F) \leq \text{rk}(P_k)$ for a certain $k \geq -1$, then, if k is odd, we have $\mu(F) \leq \mu_{k-1}$ or $\mu(F) \geq \mu_k$ and, if k is even, we have $\mu(F) \leq \mu_k$ or $\mu(F) \geq \mu_{k-1}$.*

Proof. By Remark 7.1.7, we know that $|\mu_{k-1} - \mu_k|$ is smaller than the distance between μ_{k-1} and any other rational with denominator less or equal than $\text{rk}(P_k)$. So $|\mu_{k-1} - \mu(F)| \geq |\mu_{k-1} - \mu_k|$, which means that, if k is odd, $\mu(F)$ cannot be in the interval (μ_{k-1}, μ_k) therefore $\mu(F) \leq \mu_{k-1}$ or $\mu(F) \geq \mu_k$ and, if k is even, $\mu(F)$ cannot be in the interval (μ_k, μ_{k-1}) therefore $\mu(F) \leq \mu_k$ or $\mu(F) \geq \mu_{k-1}$. □

Remark 8.1.7. Notice that, for any $i \geq 0$, the k-th intermediate w-convergent sheaves after P_{i-2}, as in Definition 8.1.2, are related to the intermediate convergents of the continued fraction representing w. Indeed, setting $\deg(P_k) = p_k$ and $\text{rk}(P_k) = q_k$ as above, we get by Proposition

8.1.4 (we show it for i even, but it clearly holds also for i odd):

$$\mu(T^k_{P_{i-1}}P_{i-2}) = \frac{\deg(T^k_{P_{i-1}}P_{i-2})}{\mathrm{rk}(T^k_{P_{i-1}}P_{i-2})} = \frac{k\deg(P_{i-1}) + \deg(P_{i-2})}{k\,\mathrm{rk}(P_{i-1}) + \mathrm{rk}(P_{i-2})} = \frac{kp_{i-1} + p_{i-2}}{kq_{i-1} + q_{i-2}}.$$

These are the intermediate convergent of w as in Remark 7.1.12. Hence, we obtain, for $0 \leq k \leq n_i$, a sequence of slopes of k-th intermediate w-convergent sheaves after P_{i-2}. This sequence is increasing for i even:

$$\mu(P_{i-2}) < \mu(T_{P_{i-1}}P_{i-2}) < \cdots < \mu(T^k_{P_{i-1}}P_{i-2}) < \cdots < \mu(T^{n_i}_{P_{i-1}}P_{i-2}) = \mu(P_i)$$

and it is decreasing for i odd:

$$\mu(P_{i-2}) > \mu(T^\star_{P_{i-1}}P_{i-2}) > \cdots > \mu(T^{\star^k}_{P_{i-1}}P_{i-2}) > \cdots > \mu(T^{\star^{n_i}}_{P_{i-1}}P_{i-2}) = \mu(P_i).$$

Notation: From now on, we denote by P_{2i} the even w-convergent sheaves and by P_{2i+1} the odd w-convergent sheaves, for $i \in \mathbb{Z}_{\geq 0}$.

Proposition 8.1.8. *Let w, L and S_x be as in Setting. Let P_{2i}, for $i \in \mathbb{Z}_{\geq 0}$, be the even w-convergent sheaves with respect to L and S_x. There exists a sequence of monomorphism:*

$$P_0 \hookrightarrow P_2 \hookrightarrow P_4 \hookrightarrow \ldots \hookrightarrow P_{2i} \hookrightarrow P_{2i+2} \hookrightarrow \ldots \qquad (\star)$$

such that the direct union $P = \varinjlim P_{2i}$ is a quasi-coherent non-coherent sheaf of slope w.

Proof. Fix $i \in \mathbb{Z}_{\geq 0}$. For any $0 \leq k < n_{2i+2}$, $T^{k+1}_{P_{2i+1}}P_{2i} = T_{P_{2i+1}}(T^k_{P_{2i+1}}P_{2i})$. Therefore, from the iterated P_{2i+1}-universal extensions, we obtain a sequence of monomorphisms:

$$P_{2i} \hookrightarrow T_{P_{2i+1}}P_{2i} \hookrightarrow T^2_{P_{2i+1}}P_{2i} \hookrightarrow \ldots \hookrightarrow T^{n_{2i+1}-1}_{P_{2i+1}}P_{2i} \hookrightarrow T^{n_{2i+1}}_{P_{2i+1}}P_{2i} = P_{2i+2}$$

whose composition $P_{2i} \hookrightarrow P_{2i+2}$ is a monomorphism. By the generality of the argument we obtain a sequence of monomorphisms:

$$P_0 \hookrightarrow P_2 \hookrightarrow P_4 \hookrightarrow \ldots \hookrightarrow P_{2i} \hookrightarrow P_{2i+2} \hookrightarrow \ldots$$

By Remark 8.1.5, $\mu_{2i} < \mu_{2i+2}$, for any $i \in \mathbb{Z}_{\geq 0}$. Therefore, by Theorem 6.2.10, $P = \varinjlim P_{2i}$ is a quasi-coherent non-coherent sheaf of slope w. □

8.1.2 Properties of the quotients

Let w be a positive irrational number and consider its continued fraction form $[n_0; n_1, n_2, \ldots]$. For $i \in \mathbb{Z}_{\geq 0}$, let P_i be the w-convergent sheaves with respect to the structure sheaf L and a simple sheaf S_x in a tube of maximal rank.
For any integer $i \geq 0$, let us denote by Q_{2i+1} the cokernel of the map $P_{2i} \to P_{2i+2}$ in $(\star)$.

Proposition 8.1.9. *For any $i \in \mathbb{Z}_{\geq 0}$, $\mu(Q_{2i+1}) = \mu_{2i+1}$.*

Proof. For any $i \in \mathbb{Z}_{\geq 0}$, we have a short exact sequence

$$0 \longrightarrow P_{2i} \longrightarrow P_{2i+2} \longrightarrow Q_{2i+1} \longrightarrow 0$$

Degree and rank are additive on short exact sequences, hence, using Proposition 8.1.4, we infer that:

$$\begin{aligned}\mu(Q_{2i+1}) &= \frac{\deg(Q_{2i+1})}{\operatorname{rk}(Q_{2i+1})} = \frac{\deg(P_{2i+2}) - \deg(P_{2i})}{\operatorname{rk}(P_{2i+2}) - \operatorname{rk}(P_{2i})} = \\ &= \frac{n_{2i+2}\deg(P_{2i+1}) + \deg(P_{2i}) - \deg(P_{2i})}{n_{2i+2}\operatorname{rk}(P_{2i+1}) + \operatorname{rk}(P_{2i}) - \operatorname{rk}(P_{2i})} = \frac{\deg(P_{2i+1})}{\operatorname{rk}(P_{2i+1})} = \mu_{2i+1}\end{aligned}$$ □

Lemma 8.1.10. *For any integer $i, j \geq 0$ with $j > i$, we have:*

$$P_{2j}/P_{2i} \cong \bigoplus_{k=i}^{j-1} Q_{2k+1}.$$

Proof. Fix an integer $i \geq 0$. We prove it by induction on $j > i$.

If $j = i + 1$ then $P_{2i+2}/P_{2i} = Q_{2i+1}$.

Suppose that for $j > i$, we have:

$$P_{2j}/P_{2i} \cong \bigoplus_{k=i}^{j-1} Q_{2k+1}.$$

Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{2i} & \longrightarrow & P_{2j} & \longrightarrow & \bigoplus_{k=i}^{j-1} Q_{2k+1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{2i} & \longrightarrow & P_{2j+2} & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q_{2j+1} & = & Q_{2j+1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the bottom equality comes from the Snake lemma. For every odd ℓ such that $2i < \ell < 2j$, $\mu(Q_\ell) = \mu_\ell > \mu_{2j+1} = \mu(Q_{2j+1})$, by Remark 8.1.5. Therefore, using Theorem 6.1.12(ii), we have:

$$\operatorname{Ext}^1\left(Q_{2j+1}, \bigoplus_{k=i}^{j-1} Q_{2k+1}\right) \cong D\operatorname{Hom}\left(\bigoplus_{k=i}^{j-1} Q_{2k+1}, \tau Q_{2j+1}\right) \cong D\left(\bigoplus_{k=i}^{j-1} \operatorname{Hom}(Q_{2k+1}, \tau Q_{2j+1})\right) = 0.$$

Proving that the right column in the diagram splits, hence:

$$K \cong Q_{2j+1} \oplus \bigoplus_{k=i}^{j-1} Q_{2k+1} = \bigoplus_{k=i}^{j} Q_{2k+1}. \qquad \square$$

Proposition 8.1.11. *For any integer $i \geq 0$:*

$$P/P_{2i} \cong \bigoplus_{k \geq i} Q_{2k+1}.$$

Proof. By Lemma 8.1.10, we get a sequence of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{2i} & \longrightarrow & P_{2i+2} & \longrightarrow & Q_{2i+1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{2i} & \longrightarrow & P_{2i+4} & \longrightarrow & Q_{2i+1} \oplus Q_{2i+3} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{2i} & \longrightarrow & P_{2i+6} & \longrightarrow & Q_{2i+1} \oplus Q_{2i+3} \oplus Q_{2i+5} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

and since direct limits are exact in $\vec{\mathcal{H}}$, we get the short exact sequence:

$$0 \longrightarrow P_{2i} \longrightarrow P \longrightarrow \bigoplus_{k \geq i} Q_{2k+1} \longrightarrow 0. \qquad \square$$

8.2 On simples in $\mathcal{A}_w$

As in Section 6.2, for every irrational number w, we define:

$$\mathcal{C}_w = \mathbf{q}_w{}^{\circ} = \left(\bigcup_{w<\beta} \mathbf{t}_\beta \right)^{\circ} \quad \text{and} \quad \mathrm{Gen}(\mathbf{q}_w) = \mathcal{Q}_w = {}^{\circ}\mathcal{C}_w.$$

By Lemma 6.2.4, $(\mathcal{Q}_w, \mathcal{C}_w)$ is a torsion pair of finite type and, by Theorem 1.3.12, the heart $\mathcal{A}_w = \vec{\mathcal{H}}(\mathcal{Q}_w, \mathcal{C}_w)$ is a locally coherent Grothendieck category, whose injective cogenerator comes from a cotilting sheaf $\mathbf{W}_w$ such that $\mathcal{C}_w = \mathrm{Cogen}\,\mathbf{W}_w$ (as seen in Remark 6.2.11). In this Section we want to describe the behavior of simple objects in $\mathcal{A}_w$.
Consider w a positive irrational number, L the structure sheaf in $\vec{\mathcal{H}}$ and S_x a simple sheaf in a tube of maximal rank. Let P_i be the w-convergent sheaf with respect to L and S_x as in Definition 8.1.2.
We have seen in Proposition 8.1.8, that the direct limit $P = \varinjlim P_{2i}$ of the sequence $(\star)$ is a quasi-coherent sheaf of slope w. It is clear that $P \in \mathcal{C}_w$, so $P[1] \in \mathcal{A}_w$ is in $\mathcal{C}_w[1]$ and moreover, by Proposition 1.3.11, $P[1] = (\varinjlim P_{2i})[1] \cong \varinjlim_{\mathcal{A}_w} (P_{2i}[1])$. We have the following.

Proposition 8.2.1. $P[1]$ *is a simple object in* $\mathcal{A}_w$.

Proof. Using Proposition 2.3.7, we need to prove that for any non-split short exact sequence $0 \to P \to E \to Q \to 0$, with $Q \in \mathbf{q}_w$ we have $E \in \mathrm{Gen}(\mathbf{q}_w)$.

Consider a non-split short exact sequence $0 \to P \to E \to Q \to 0$, with $Q \in \mathbf{q}_w$. From the isomorphism $\mathrm{Ext}^1_{\vec{\mathcal{H}}}(Q, P) \cong \mathrm{Hom}_{\mathcal{A}_w}(Q, P[1])$, we get a nonzero map $g \colon Q \to P[1]$ in $\mathcal{A}_w$. Since Q is coherent, from Proposition 1.1.12, we get $\mathrm{Hom}_{\mathcal{A}_w}(Q, \varinjlim P_{2i}[1]) \cong \varinjlim \mathrm{Hom}_{\mathcal{A}_w}(Q, P_{2i}[1])$. Therefore there exists a nonzero map $g_i \colon Q \to P_{2i}[1]$, for a certain i, such that g factors through g_i.

Notice that, in the construction described in Section 8.1, the sequence of the ranks of the P_i's is strictly increasing. Therefore, since Q is fixed, we can choose, without loss of generality, i in such a way that $\mathrm{rk}(Q) \leq \mathrm{rk}(P_{2i-1})$.

From the map g_i we get a non-split short exact sequence in $\vec{\mathcal{H}}$:

$$0 \longrightarrow P_{2i} \longrightarrow E' \longrightarrow Q \longrightarrow 0$$

where P_{2i} and Q are coherent, therefore E' is coherent. Since $P_{2i} \in \mathcal{C}_w$ and $Q \in \mathbf{q}_w$, we have $\mu_{2i} < \mu(Q)$, therefore by Proposition 6.1.10, $\mu_{2i} < \mu(E') < \mu(Q)$.

By the additivity of the rank function and from the fact that $\mathrm{rk}(Q) \leq \mathrm{rk}(P_{2i-1})$, we have:

$$\mathrm{rk}(E') = \mathrm{rk}(P_{2i}) + \mathrm{rk}(Q) \leq \mathrm{rk}(P_{2i}) + \mathrm{rk}(P_{2i-1}) \leq n_{2i+1}\,\mathrm{rk}(P_{2i}) + \mathrm{rk}(P_{2i-1}) = \mathrm{rk}(P_{2i+1}).$$

Where the last equality comes from Proposition 8.1.4. By Lemma 8.1.6 with $k = 2i + 1$, we can conclude that $\mu(E') \leq \mu_{2i}$ or $\mu(E') \geq \mu_{2i+1}$. The first case is not possible, hence $\mu(E') \geq \mu_{2i+1} > w$, which implies $E' \in \mathrm{Gen}(\mathbf{q}_w)$.

From the short exact sequences in $\vec{\mathcal{H}}$, $0 \to P_{2i} \to E' \to Q \to 0$ and $0 \to P \to E \to Q \to 0$, we get the following diagram in $\mathcal{D}^b(\vec{\mathcal{H}})$:

$$\begin{array}{ccccccc}
P_{2i} & \longrightarrow & E' & \longrightarrow & Q & \xrightarrow{g_i} & P_{2i}[1] \\
\downarrow & & \vdots & & \| & & \downarrow \\
P & \longrightarrow & E & \longrightarrow & Q & \xrightarrow{g} & P[1]
\end{array}$$

where the map $E' \to E$ comes from the triangulated structure of $\mathcal{D}^b(\vec{\mathcal{H}})$. So, in $\vec{\mathcal{H}}$ we obtain the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{2i} & \longrightarrow & E' & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

By the Snake lemma, we get a short exact sequence:

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \overline{Q} \longrightarrow 0$$

where $\overline{Q} = P/P_{2i} \in \mathrm{Add}\,\mathbf{q}_w$ by Proposition 8.1.11. Therefore $E \in \mathrm{Gen}(\mathbf{q}_w)$. □

Recall that, as we have seen in Theorem 2.3.6, the simple objects in the heart of a t-structure induced by a torsion pair can come either from a torsionfree almost torsion object or from a torsion almost torsionfree object in the original category. In the following we prove that sheaves becoming simple in $\mathcal{A}_w$ can only be torsionfree almost torsion and, moreover, they are all of slope w. We first prove a lemma.

Lemma 8.2.2. *Let $S \in \mathcal{A}_w$ be a simple object, then:*

(i) *If $S = Y[1]$ for $Y \in \mathcal{C}_w$, then $Y \notin \mathbf{p}_w$.*
(ii) *If $S = Q$ for $Q \in \mathcal{Q}_w$, then $Q \notin \mathbf{q}_w$.*

Proof. (i) Suppose $Y \in \mathbf{p}_w$ is coherent and let P_i be an w-convergent sheaf, as in Definition 8.1.2, of slope $\mu_i = p_i/q_i$ with an even i such that $\mu(Y) < \mu_i$. This is possible since $\mu(Y)$ is fixed and we have an infinite strictly increasing sequence of even convergents of w, converging to w. Then, by Proposition 7.1.9, $\mu(Y) < \mu_i < w < \mu_{i-1}$.
Using Proposition 6.1.18, we obtain a map $g\colon Y \to P_i$, which is a monomorphism by Remark 2.3.2(I). Let $C = \operatorname{Coker} g$ and by the short exact sequence $0 \to Y \xrightarrow{g} P_i \to C \to 0$ we infer that $\operatorname{rk}(C) < \operatorname{rk} P_i = q_i$. Therefore, by Lemma 8.1.6, we have two possibilities: $\mu(C) \leq \mu_i$ or $\mu(C) \geq \mu_{i-1}$. But the first one is not possible, indeed: if $\mu(C) < \mu_i$, then there is a nonzero map from $P_i \in \mathbf{t}_{\mu_i}$ to a sheaf of smaller slope, which contradicts Theorem 6.1.12(ii). Furthermore, if $\mu(C) = \mu_i$, then Y has slope μ_i too, since $\operatorname{add}(\mathbf{t}_{\mu_i})$ is an abelian subcategory of $\mathcal{H}$ (see Theorem 6.1.12(i)), contradicting the fact that $\mu(Y) < \mu_i$. Hence, from $\mu(C) \geq \mu_{i-1}$, we infer that $C \in \mathbf{q}_w$.
Therefore, the short exact sequence $0 \to Y \xrightarrow{g} P_i \to C \to 0$ induces a short exact sequence in $\mathcal{A}_w$:

$$0 \longrightarrow C \longrightarrow Y[1] \xrightarrow{g[1]} P_i[1] \longrightarrow 0$$

where C and $P_i[1]$ are nonzero. But, since $Y[1] = S$ is simple in $\mathcal{A}_w$, this is a contradiction.

(ii) The proof is dual. Suppose that Q is coherent, ie. $Q \in \mathbf{q}_w$. Let P_i be an w-convergent sheaf, as in Definition 8.1.2, of slope $\mu_i = p_i/q_i$ with an odd i such that $\mu_i < \mu(Q)$. This is possible since $\mu(Q)$ is fixed and we have an infinite strictly decreasing sequence of odd convergents of w, converging to w. Then, using Proposition 7.1.9, $\mu_{i-1} < w < \mu_i < \mu(Q)$. Consider the nonzero map $g\colon P_i \to Q$, which exists by Proposition 6.1.18. By Remark 2.3.2(I'), g is an epimorphism. Set $K = \operatorname{Ker} g$ and by the short exact sequence $0 \to K \to P_i \xrightarrow{g} Q \to 0$ we infer that $\operatorname{rk}(K) < \operatorname{rk} P_i = q_i$. Using Lemma 8.1.6, we have that either $\mu(K) \leq \mu_{i-1}$ or $\mu(K) \geq \mu_i$. But the latter is not possible, indeed: if $\mu(K) > \mu_i$, then there is a nonzero map from K to P_i which is a sheaf of slope smaller than $\mu(K)$, contradicting Theorem 6.1.12(ii). Furthermore, if $\mu(K) = \mu_i$, then by the abelianity of $\operatorname{add}(\mathbf{t}_{\mu_i})$ as a subcategory of $\mathcal{H}$ (see Theorem 6.1.12(i)), we have $\mu(Q) = \mu_i$. This contradicts the fact that $\mu_i < \mu(Q)$. Hence, from $\mu(K) \leq \mu_{i-1}$, we infer that $K \in \mathbf{p}_w$.
Therefore, the short exact sequence $0 \to K \to P_i \xrightarrow{g} Q \to 0$ induces a short exact sequence in $\mathcal{A}_w$:

$$0 \longrightarrow P_i \xrightarrow{g} Q \longrightarrow K[1] \longrightarrow 0$$

where both P_i and $K[1]$ are nonzero. But, since $Q = S$ is simple in $\mathcal{A}_w$, this is a contradiction. □

Theorem 8.2.3. *If $S \in \mathcal{A}_w$ is a simple object, then $S = Y[1]$ for a quasi-coherent sheaf $Y \in \mathcal{M}_w$.*

Proof. First of all, let us prove that S comes from a torsionfree almost torsion sheaf in $\vec{\mathcal{H}}$. If not, then S comes from a sheaf $Q \in \vec{\mathcal{H}}$ which is torsion almost torsion free. By Remark 2.3.4, every torsion almost torsionfree object in $\vec{\mathcal{H}}$ is coherent, but this contradicts Lemma 8.2.2(ii). So $S = Y[1]$ for a $Y \in \mathcal{C}_w$, torsionfree almost torsion.
Let us prove now that $Y \in \mathcal{B}_w$. Consider a nonzero map $f : Y \to E$ with $E \in \mathbf{p}_w$. Then, since $Y \in \mathcal{C}_w$ and by Remark 2.3.2(I), f is a monomorphism. Y is, then, a subsheaf of a coherent sheaf and therefore coherent, contradicting Lemma 8.2.2(i). So $Y \in {}^{\circ}\mathbf{p}_w = \mathcal{B}_w$. $\square$

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