Delta-matroids for graph theorists

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Abstract

What happens if you try to develop matroid theory, but start with topological graph theory? This survey provides an introduction to delta-matroids. We aim to illustrate the two-way interaction between graph theory and delta-matroid theory that enriches both subjects. Along the way we shall see intimate connections between delta-matroids and, amongst others, circle graphs, Eulerian circuits, embedded graphs, matchings, pivot-minors, (skew-)symmetric matrices, and vertex-minors.

1 Introduction

> "I lectured on matroids at the first formal conference on them [...] in 1964. To me that was the year of the Coming of the Matroids. Then and there the theory of matroids was proclaimed to the mathematical world. And outside the halls of lecture there arose the repeated cry: 'What the hell is a matroid?'"
>
> — W.T. Tutte [1]

Since that 1964 conference, matroids have become a mainstay of combinatorics (and a regular topic of BCC talks [2, 8, 39, 51, 62, 66, 70, 75, 77]). However, our interest here is in a generalisation of matroids called *delta-matroids*. Delta-matroids, introduced in the mid-1980s, are not nearly so well-known, even among matroid theorists. Here, inspired by Tutte's felicitious phrasing, I aim to answer the question 'what the hell is a delta-matroid?'.

This survey is intended to introduce delta-matroids to readers familiar with graph theory. *No prior knowledge of matroids is assumed.* Delta-matroids were introduced in the mid-1980s, independently, by Bouchet in [9]; Chandrasekaran and Kabadi, under the name of *pseudo-matroids*, in [25]; and Dress and Havel, under the name of *metroids*, in [32]. (Here we follow the terminology and notation of Bouchet.) Our focus here is on how delta-matroids relate to graph theory, and we shall see connections between them and circle graphs, Eulerian circuits, embedded graphs, matchings, pivot-minors, (skew-)symmetric matrices, and vertex-minors. In particular, our aim is to illustrate the two-way interaction between graph theory and delta-matroid theory that enriches both subjects.

The emphasis here is on providing an accessible introduction to delta-matroids that conveys the 'flavour' of the subject. It does not provide a comprehensive account of delta-matroids. In particular, many beautiful results have not been included here, even when they are closely related to those that have been. For example, delta-matroids have applications in theoretical computer science, but here we totally ignore this aspect of delta-matroid theory (although Sections 2.4.6 and 2.4.7

[1] This extract is from Tutte's article, *The Coming of the Matroids*, [75]. It appeared in this *Surveys in combinatorics* series, and is associated with his talk at the 1999 BCC held at the University of Kent at Canterbury.

hint at why they appear in that area). A graph-theoretic topic that we do not touch on is applications of delta-matroid to graph polynomials, including the Tutte [73], Bollobás-Riordan [6, 7], interlace [4, 5], Penrose [1, 36, 63], and transition [43], polynomials (see, for example, [22, 29, 30, 46, 56, 57]). Also, delta-matroids have close connections with several other generalisations of matroids, and other combinatorial structures (see the remark at the end of Section 2.2). Indeed, some delta-matroid results are better understood in terms of more general structures or generalised matroids (such as isotropic systems, jump systems, or multimatroids). We do not discuss these generalisations here: asking a reader to absorb the definition of *one* generalisation of a matroid at a time is quite enough!

A number of exercises can be found throughout the text. These exercises are intended to assist with the digestion of definitions and results, and, as such, they are not hard and mostly require only a few minutes of thought. A similar comment holds for the examples and figures. We provide sketches of some proofs, but not all. At the end, there is a list of frequently used notation.

2 What is a delta-matroid?

2.1 A warm up

Rather than diving straight into the definitions, let us start with an example that shows we have been working with delta-matroids since our undergraduate days.

Suppose we have a finite-dimensional vector space V, and two of its bases $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. From our first courses in linear algebra we know, for each i: (i) there is some y_j such that $(X \backslash \{x_i\}) \cup \{y_j\}$ is a basis for V; and (ii) there is some x_j such that $(X \cup \{y_i\}) \backslash \{x_j\}$ is a basis for V. Knowing that the sets X and Y are of the same size, we can conveniently use the *symmetric difference*, $X \bigtriangleup Y := (X \cup Y) \backslash (X \cap Y)$, to express these two properties as

$$(\forall\, u \in X \bigtriangleup Y)\ (\exists\, v \in X \bigtriangleup Y)\ (X \bigtriangleup \{u, v\} \in \mathcal{F}), \tag{2.1}$$

where $\mathcal{F}$ is the set of all bases of V.

Matroids and delta-matroids are mathematical structures that satisfy the exchange property in (2.1): a *delta-matroid* is a pair $(E, \mathcal{F})$ where E is a set, and $\mathcal{F}$ is a collection of subsets of E that satisfies (2.1) for all $X, Y \in \mathcal{F}$. If every set in $\mathcal{F}$ has the same size, then the delta-matroid $(E, \mathcal{F})$ is said to be a *matroid*.

Thus the set $\mathcal{F}$ of all bases of the vector space V satisfies (2.1) for all $X, Y \in \mathcal{F}$ and so the pair $(V, \mathcal{F})$ (where V is regarded as a set) forms a *delta-matroid*. Moreover, since every member of $\mathcal{F}$ has the same size, which need not be the case for delta-matroids in general, this delta-matroid is a *matroid*.

2.2 The definition

Here we assume all sets (other than, possibly, fields) are finite, and will do so without further comment. Where there is no potential for confusion, we omit the braces when writing single element sets, for example, writing $X \backslash x$ instead of $X \backslash \{x\}$, or $X \cup x$ instead of $X \cup \{x\}$. The *symmetric difference*, $X \bigtriangleup Y$, of sets X and Y is

$$X \bigtriangleup Y := (X \cup Y) \backslash (X \cap Y).$$

Definition 2.1 (Set system) A *set system* is a pair $D = (E, \mathcal{F})$ where E is a set, and $\mathcal{F}$ is a collection of subsets of E. A set system is *proper* if $\mathcal{F}$ is not empty; it is *trivial* if E is empty.

Example 2.2 Let $E = \{a, b, c\}$,

$$\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\},$$

and

$$\mathcal{F}' = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}.$$

Then $D = (E, \mathcal{F})$ and $D' = (E, \mathcal{F}')$ are both set systems.

The Symmetric Exchange Axiom appeared in (2.1).

Definition 2.3 (Symmetric Exchange Axiom) A set system $D = (E, \mathcal{F})$ is said to satisfy the *Symmetric Exchange Axiom* (SEA) if, for all $X, Y \in \mathcal{F}$, if there is an element $u \in X \triangle Y$, then there is an element $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$.

See Figure 1 for an illustration of the Symmetric Exchange Axiom. For ease of reference, here it is in a symbolic form:

$$(\forall\, X, Y \in \mathcal{F})\ (\forall\, u \in X \triangle Y)\ (\exists\, v \in X \triangle Y)\ (X \triangle \{u, v\} \in \mathcal{F})\,. \qquad (\textbf{SEA})$$

It is important to notice that the Symmetric Exchange Axiom allows the possibility that $u = v$.

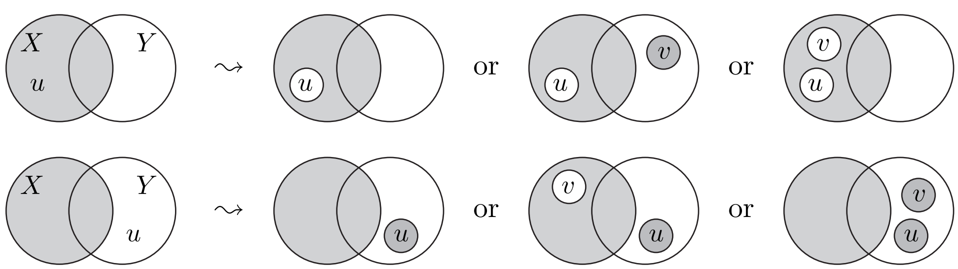


Figure 1: The Symmetric Exchange Axiom, where X, Y, and the shaded parts form feasible sets

Definition 2.4 (Delta-matroid) A *delta-matroid* $D = (E, \mathcal{F})$ is a proper set system that satisfies the Symmetric Exchange Axiom. The set E is called its *ground set*, and the members of $\mathcal{F}$ are called *feasible sets*.

When working with delta-matroids, $E(D)$ is often used to denote the ground set of a delta-matroid D, and $\mathcal{F}(D)$ its collection of feasible sets. Here, although we generally use the letter E for ground sets, for certain classes of delta-matroids we will instead use V. The choice of V or E relates to whether the elements of the ground set correspond most naturally to the vertex set or the edge set of a graph.

Example 2.5 Consider the set systems D and D' in Example 2.2. By examining $\mathcal{F}$, we see that if $X = \{b, c\}$ and $Y = \{a\}$, then $a \in X \triangle Y$, but there there is no $v \in X \triangle Y$ such that $X \triangle \{a, v\} \in \mathcal{F}$. Thus $D = (E, \mathcal{F})$ is *not* a delta-matroid.

On the other hand, it can be checked that $\mathcal{F}'$ satisfies the Symmetric Exchange Axiom and hence $D' = (E, \mathcal{F}')$ is a delta-matroid. Its ground set is $\{a, b, c\}$ and its feasible sets are $\{a\}$, $\{b\}$, $\{a, b\}$, and $\{a, b, c\}$.

Definition 2.6 (Matroid) A delta-matroid is said to be a *matroid* if all of its feasible sets are of the same size.

If a delta-matroid is a matroid, then it is usual to refer to its feasible sets as its *bases*, and to use $\mathcal{B}$, rather than $\mathcal{F}$ to denote its collections of bases.

Remark Introducing matroids as a special type of delta-matroid is somewhat anachronistic. Matroids were introduced by Whitney [78] in the 1930's, while delta-matroids were introduced in the mid-1980's. Furthermore, matroids are much more studied, better known and, better understood than delta-matroids. A reader meeting this topic for the first time should think of a delta-matroid as being a generalisation of a matroid, rather than as a matroid being a special type of delta-matroid, as presented here. Two standard and excellent references for matroid theory are the books [61, 76].

Exercise 2.7 *The standard 'basis definition' of a matroid is as follows: the set system $(E, \mathcal{B})$ is a matroid if (i) $\mathcal{B}$ is non-empty; and (ii) for distinct $A, B \in \mathcal{B}$, if $a \in A \backslash B$, then there exists $b \in B \backslash A$ such that $(A \backslash a) \cup b \in \mathcal{B}$. Verify that this definition of a matroid is equivalent to that given in Definition 2.6.*

Remark The definition of a delta-matroid given here is due to Bouchet and we follow his terminology. As mentioned above, delta-matroids were introduced independently by Bouchet in [9]; Chandrasekaran and Kabadi in [25], under the name of pseudo-matroids; and Dress and Havelin [32], under the name of metroids. Delta-matroids are related to many different matroidal-objects, including the following: Tardos' g-matroids [71], Kung's Pfaffian structures [47], Qi's ditroids [64], Bouchet's symmetric matroids [9], Traldi's transition matroids [72], Bouchet's Isotropic systems [10], jump systems [19], and Bouchet's multimatroids [17]. This list is indicative, not exhaustive.

2.3 Examples of delta-matroids

Having seen the definition of a delta-matroid, we now give a selection of examples of them. Here we provide only constructions and examples, but most, although not all, of these delta-matroids will be discussed in more detail later.

2.3.1 From column spaces Let $\mathbf{A}$ be a matrix with entries in a field $\Bbbk$. Let E be a set of labels for the columns of $\mathbf{A}$, and, define a collection $\mathcal{B}$ of subsets of E by, for each $X \subseteq E$ setting

$$X \in \mathcal{B} \iff X \text{ labels a basis of the column space of } \mathbf{A}.$$

Then the pair $(E, \mathcal{B})$ forms a matroid called the *vector matroid* of $\mathbf{A}$. (This matroid is exactly the example discussed in Section 2.1, and is due to Whitney [78].)

Example 2.8 Working over $\mathbb{R}$, the vector matroid of the matrix

$$\mathbf{A} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

has ground set $E = \{1,2,3,4,5\}$, and its set of bases is

$$\mathcal{B} = \{\{1,3\},\{1,5\},\{3,4\},\{3,5\},\{4,5\}\}.$$

2.4 From (skew-)symmetric matrices

A matrix $\mathbf{A}$ is *symmetric* if $\mathbf{A}^t = \mathbf{A}$, is *skew-symmetric* if $\mathbf{A}^t = -\mathbf{A}$ and the diagonal entries are zero.

Suppose that $\mathbf{A}$ is a symmetric or skew-symmetric matrix over a field $\Bbbk$, and that E labels its rows and columns (in the same order). For $X \subseteq E$, let $\mathbf{A}[X]$ denote the principal submatrix of $\mathbf{A}$ given by the rows and columns indexed by X. Define a collection $\mathcal{F}$ of subsets of E by

$$X \in \mathcal{F} \iff \mathbf{A}[X] \text{ is non-singular},$$

where $\mathbf{A}[\emptyset]$ is considered to be non-singular. Then the pair $(E, \mathcal{F})$ forms a delta-matroid. (This result is due to Bouchet [12].)

Example 2.9 Working over GF(2), consider the matrices

$$\mathbf{A}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \text{and} \quad \mathbf{A}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The matrix $\mathbf{A}_1$ gives rise to a delta-matroid $D(\mathbf{A}_1) = (V, \mathcal{F}_1)$ with ground set $V = \{1,2,3,4\}$ and collection of feasible sets

$$\mathcal{F}_1 = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{1,2,3,4\}\}.$$

The matrix $\mathbf{A}_2$ gives rise to a delta-matroid $D(\mathbf{A}_2) = (V, \mathcal{F}_2)$ with ground set $V = \{1,2,3,4\}$ and collection of feasible sets

$$\mathcal{F}_2 = \{\emptyset, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\}.$$

2.4.1 From simple graphs (with vertices forming the ground set)

Working over some field $\Bbbk$, the *adjacency matrix over* $\Bbbk$ of a graph G is the matrix whose rows and columns correspond to the vertices of G; and whose (u,v)-entry is the number edges between u and v. When the graph has loops, it is usual to take the (v,v)-entry to be twice the number of vv-edges, however, our convention here is to take it to be equal to the number of vv-edges.

Since adjacency matrices are always symmetric, the previous example provides a way to associate a delta-matroid with a graph: given a graph, form its adjacency matrix over some field, and take the delta-matroid of that matrix. Although this construction works for all graphs and over any field, in this survey we shall consider it only for simple graphs and looped simple graphs over the field of two elements, GF(2). To avoid ambiguity, let us give detailed definitions for these cases.

A *simple graph* is a graph with no loops or multiple edges. A *looped simple graph* is a graph obtained from a simple graph by adding (exactly) one loop to some of its vertices.

Definition 2.10 (Adjacency matrix) The *adjacency matrix*, $\mathbf{A}_G$, of a simple graph or a looped simple graph G is the matrix over GF(2) whose rows and columns correspond to the vertices of G; and where, for $u \neq v$, the (u,v)-entry of $\mathbf{A}_G$ is 1 if the corresponding vertices u and v are adjacent in G, and is 0 otherwise; and the (v,v)-entry of $\mathbf{A}_G$ is 1 if there is a loop at the vertex v, and is 0 otherwise.

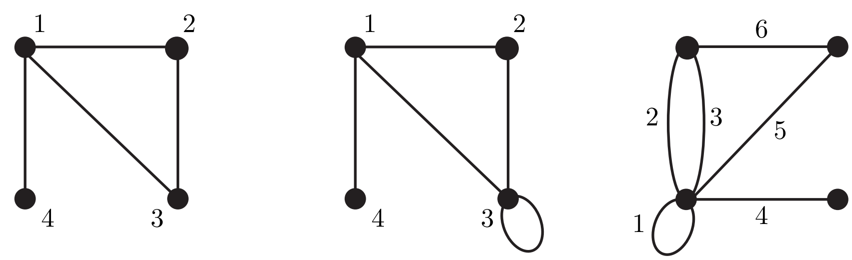
Through its adjacency matrix, a delta-matroid $D(\mathbf{A}_G)$ can be associated with a (looped) simple graph G.

Example 2.11 Let G_1 be the simple graph in Figure 2a, G_2 be the looped simple graph in Figure 2b, and let $\mathbf{A}_1$ and $\mathbf{A}_2$ be the matrices in Example 2.9. Then G_1 has adjacency matrix $\mathbf{A}_{G_1} = \mathbf{A}_1$, and G_2 has adjacency matrix $\mathbf{A}_{G_2} = \mathbf{A}_2$. Thus $D(\mathbf{A}_{G_1}) = (V, \mathcal{F}_1)$ and $D(\mathbf{A}_{G_2}) = (V, \mathcal{F}_2)$ both have ground set $V = \{1,2,3,4\}$ and their collections of feasible sets are

$$\mathcal{F}_1 = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{1,2,3,4\}\},$$

and

$$\mathcal{F}_2 = \{\emptyset, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\}.$$



(a) (b) (c)

Figure 2: Three graphs

2.4.2 From graphs (with edges forming the ground set) Let $G = (V, E)$ be a connected graph. Define a collection $\mathcal{B}(G)$ of subsets of E by, for each $A \subseteq E$ setting

$$A \in \mathcal{B}(G) \iff (V, A) \text{ a spanning tree of } G.$$

(Recall a subgraph H of G is *spanning* if $V(H) = V(G)$.) Then the pair $(E, \mathcal{B}(G))$ forms a matroid called the *cycle matroid* of G, denoted by $C(G)$. (This is due to Whitney [78].)

Example 2.12 Consider the graph G shown in Figure 2c. Its cycle matroid, $C(G)$, has ground set $E = \{1, 2, 3, 4, 5, 6\}$ and set of bases

$$\mathcal{B} = \{\{2, 4, 5\}, \{2, 4, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{4, 5, 6\}\}.$$

2.4.3 From graphs in surfaces Let $G = (V, E)$ be a connected graph (cellularly) embedded in a (connected) surface Σ. (Informally, an embedded graph is a graph drawn on a surface in such a way that edges do not intersect, except for where their ends meet at vertices, as in Figure 3. The cellular condition means that each of its faces, i.e. the components of $\Sigma \backslash G$, is homeomorphic to a disc.) Since G and any subgraph H of it can be regarded as a set of curves and points on the surface, we can take a regular neighbourhood $N(H)$ of each subgraph H of G. (Informally, think of $N(H)$ as a surface with boundary that arises by 'thickening up' the drawing of H, as in Figure 4.)

Each regular neighbourhood $N(H)$ of a subgraph H of the embedded graph G has some number of boundary components. We say that H is a *quasi-tree* if $N(H)$ has exactly one boundary component.

Define a collection $\mathcal{F}$ of subsets of E by, for each $A \subseteq E$ setting

$$A \in \mathcal{F} \iff (V, A) \text{ is a quasi-tree.}$$

Then the pair $(E, \mathcal{F})$ forms a delta-matroid. (This result is implicit in Bouchet's paper [13].)

Example 2.13 Let G be the graph in the torus shown in Figure 3. It has an edge set $E = \{1, 2, \ldots, 6\}$. There are exactly nine subset sets A of E for which (V, A) forms a quasi-tree. Figure 4 gives three of these and their corresponding neighbourhoods $N(V, A)$. The pair $(E, \mathcal{F})$ forms a delta-matroid where

$$\mathcal{F} = \{\{2, 4, 5\}, \{2, 4, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{4, 5, 6\},$$
$$\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 4, 5, 6\}, \{2, 3, 4, 5, 6\}\}.$$

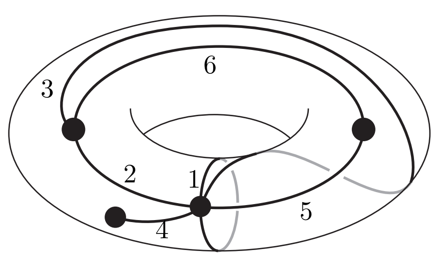


Figure 3: A graph embedded in the torus

Later, we will consider this example in the formalism of ribbon graphs (see Section 5). We will also see that delta-matroids of this type give a topological analogue of cycle matroids.

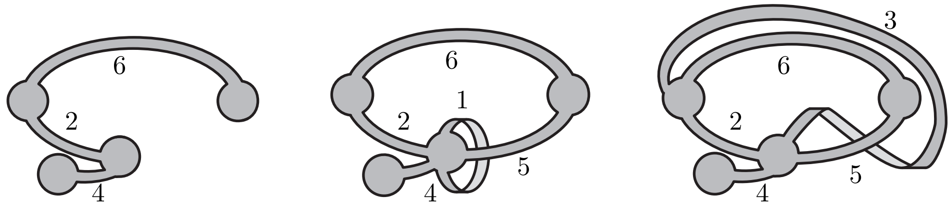


Figure 4: Neighbourhoods of the subgraphs on $\{2,4,6\}$, $\{1,2,4,5,6\}$, and $\{2,3,4,5,6\}$

2.4.4 From Eulerian circuits Let $G = (V, E)$ be a connected 4-regular graph. We are interested in the Eulerian circuits in G. At any vertex v of G there are exactly three possible routes that an Eulerian circuit can take through it. At each vertex, set one choice of route through it as being *forbidden*, and of the other two as *allowed*. Set one allowed route at each vertex as being *preferred*.

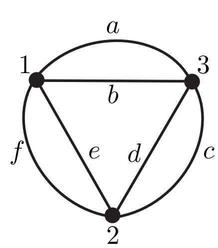
With this information, construct a collection $\mathcal{F}$ of subsets of V by, for each $X \subseteq V$, setting

$$X \in \mathcal{F} \iff \begin{array}{l}\text{there is an Eulerian circuit taking only allowed allowed routes}\\ \text{through vertices, and preferred routes at exactly the vertices in } X.\end{array}$$

Then the pair $(V, \mathcal{F})$ forms a delta-matroid. This type of delta-matroid is known as an *Eulerian delta-matroid*. (This result is due to Bouchet [9].)

Example 2.14 Figure 5 shows a 4-regular graph equipped with a set of preferred and forbidden transitions. It has exactly four Eulerian circuits that avoid forbidden transitions. These are given by *abfcde*, which used preferred transitions at 1; *abdefc*, which used preferred transitions at 3; *acfbde*, which used preferred transitions at 1 and 3. *acdbfe*, which used preferred transitions at 1, 2, and 3; Thus we obtain a delta-matroid on $V = \{1, 2, 3\}$ with the collection of feasible sets

$$\mathcal{F} = \{\{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}.$$



Preferred	
vertex	route
1	$\{ae\}, \{bf\}$
2	$\{cd\}, \{ef\}$
3	$\{ac\}, \{bd\}$

Forbidden	
vertex	route
1	$\{af\}, \{be\}$
2	$\{ce\}, \{df\}$
3	$\{ad\}, \{bc\}$

Figure 5: A 4-regular graph with preferred and forbidden transitions

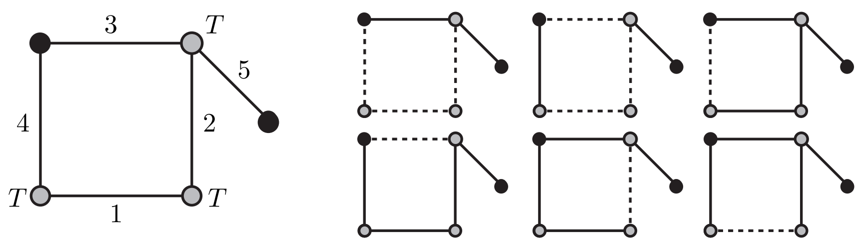
2.4.5 From grafts Let $G = (V, E)$ be a connected graph and $T \subseteq V$ be a non-empty set of its vertices. The pair (G, T) is an example of a *graft*. Define a collection $\mathcal{F}$ of subsets of E by, for each $A \subseteq E$ setting

$$A \in \mathcal{F} \iff \begin{array}{l}(V, A) \text{ a spanning forest of } G \text{ in which each component}\\ \text{has an odd number of vertices in } T.\end{array}$$

Then the pair $(E, \mathcal{F})$ forms a delta-matroid, denoted here by $D(G,T)$. (This result is due to Oum [59].)

Example 2.15 The graft (G,T) shown in Figure 6 has a delta-matroid $D(G,T)$ on ground set $E = \{1,2,3,4,5\}$ and its collection of feasible sets is

$$\mathcal{F} = \{\{3,5\},\{4,5\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}.$$



(a) A graft (G,T) (b) T-spanning subgraphs

Figure 6: A graft and its T-spanning subgraphs

2.4.6 From matchings Let $G = (V,E)$ be a simple graph and, for $U \subseteq V$, let $G[U]$ be its induced subgraph on U. A *perfect matching* on G is a subset A of its edges such that each vertex of G is incident with exactly one edge in A. Define a collection $\mathcal{F}$ of subsets of V by, for each $U \subseteq V$,

$$\mathcal{F} := \{U \subseteq V : G[U] \text{ has a perfect matching}\}.$$

Then the pair $(V,\mathcal{F})$ forms a delta-matroid called the *matching delta-matroid of* G. (This is due to Bouchet [14].)

Example 2.16 Let G be the graph with vertex set $V = \{1,2,3,4\}$ given in Figure 2a. Then G has a perfect matching, and so do its restrictions to any edge, and as does the empty graph. Thus with

$$\mathcal{F} = \{\emptyset,\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{1,2,3,4\}\}.$$

The pair $(V,\mathcal{F})$ is the matching delta-matroid of G.

2.4.7 From the Greedy Algorithm Suppose we have a proper set system $(E,\mathcal{F})$ and a weight function $w : E \to \mathbb{R}$. We want to find a member of $\mathcal{F}$ of maximum weight, that is, we want to find some $F \in \mathcal{F}$ maximising $w(F) := \sum_{x\in F} w(x)$.

Roughly speaking, the *greedy algorithm* runs though the elements of E from largest to smallest and selects an element if, together with the other previously selected elements, it forms a subset of some $F \in \mathcal{F}$ such that F contains no rejected elements. Otherwise it rejects the element.

Formally, suppose that we have a *separation oracle* telling us for each ordered pair (P,Q), where P and Q are disjoint subsets of E, whether there is some $F \in \mathcal{F}$ containing P and disjoint from Q. If such an F exists, (P,Q) is *separable*. The *greedy algorithm* successively examines each element of E according to an ordering $x_1, x_2, \dots, x_n$ such that $|w(x_1)| \geq |w(x_2)| \geq \cdots \geq |w(x_n)|$, putting each x_i in either a set A of selected elements or B of rejected elements:

```
A := ∅
B := ∅
for i := 1 to n do
    if w(x_i) ≥ 0 then
        if (A ∪ x_i, B) is separable then
            A := A ∪ x_i
        else
            B := B ∪ x_i
        end if
    else
        if (A, B ∪ x_i) is separable then
            B := B ∪ x_i
        else
            A := A ∪ x_i
        end if
    end if
end for
```

The greedy algorithm *succeeds* if A is a maximum weight member of $\mathcal{F}$, that is, if $w(A) = \max_{F \in \mathcal{F}} w(F)$.

Bouchet [9], and, independently, Chandrasekaran and Kabadi [25] in the equivalent language of pseudomatroids, characterised delta-matroids as the class of set systems for which the greedy algorithm succeeds:

Theorem 2.17 *The greedy algorithm applied to a set system $(E, \mathcal{F})$ succeeds for every weight function $w : E \to \mathbb{R}$ if and only if $(E, \mathcal{F})$ is a delta-matroid.*

3 Delta-matroid essentials

We now give a brief overview of basic delta-matroid constructions and terminology. The definition of a delta-matroid was given in Section 2.2. Isomorphism is defined in the obvious way: two delta-matroids are *isomorphic* if there is a bijection between their ground sets that induces a bijection between their feasible sets. We use equals signs to denote delta-matroids are isomorphic, although we will generally identify isomorphic delta-matroids.

A delta-matroid is said to be *even* if its feasible sets are either all of odd size, or all of even size. Otherwise it is said to be *odd*. We emphasise that the feasible sets of an even delta-matroid may all be of odd size. A delta-matroid it is said to be *normal* if the empty set is feasible.

Example 3.1 The delta-matroids $D(\mathbf{A}_1)$ and $D(\mathbf{A}_2)$ from Example 2.9 are both normal. $D(\mathbf{A}_1)$ is even but $D(\mathbf{A}_2)$ is not. The delta-matroid in Example 2.13 is even, and is not normal.

The feasible sets of a delta-matroid $D = (E, \mathcal{F})$ are graded by their size. Let $\mathcal{F}_{\min}$ denote the collection of all feasible sets in $\mathcal{F}$ of minimum size, and $\mathcal{F}_{\max}$ the collection of all feasible of maximum size. For $k = 0, 1, 2, \ldots$, let $\mathcal{F}_{\min+k}$ denote the collection of all feasible sets in $\mathcal{F}$ that are of size exactly k larger than a minimum sized feasible set. The *width* of a delta-matroid is the difference between the sizes of its largest and smallest feasible sets.

The maximum gap in the collection of sizes of feasible sets of a delta-matroid is two. That is, if a delta-matroid has a feasible set of size k and a larger feasible set, then it has a feasible set of size $k+1$ or $k+2$. In particular, this means that for an even delta-matroid, all of $\mathcal{F}_{\min}, \mathcal{F}_{\min+2}, \ldots, \mathcal{F}_{\max}$ are non-empty. For odd delta-matroids, if there is a feasible set of size k and one of size greater than k, then, while there will be a feasible set of size $k+1$ or $k+2$, there will not necessarily be both (for example, see the delta-matroids in Theorem 7.11). However, Bouchet proved in [13] that in an odd delta-matroid, there will always be feasible sets of sizes k and $k+1$, for some k.

Exercise 3.2 *Prove (for example, by induction on $|X \triangle Y|$) that if a delta-matroid has a feasible set X of size k and a larger feasible set, then it has a feasible set Y of size $k+1$ or $k+2$.*

When $D = (E, \mathcal{F})$ is a delta-matroid, $D_{\min} := (E, \mathcal{F}_{\min})$ and $D_{\max} := (E, \mathcal{F}_{\max})$ are both matroids. $D_{\min}$ is called the *lower matroid*, and $D_{\max}$ is called the *upper matroid* of D. Bouchet defined these matroids in [13].

Example 3.3 Consider the delta-matroid $D := D(\mathbf{A}_2)$ from Example 2.9. With $E = \{1, 2, 3, 4\}$, we have $D_{\min} = (E, \{\emptyset\})$ and $D_{\max} = (E, \{1, 2, 3, 4\})$. Furthermore, $\mathcal{F}_{\min+2} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}\}$, but the pair $(E, \mathcal{F}_{\min+2})$ does not form a matroid (since the Symmetric Exchange Axiom fails with $X = \{1, 4\}$, $Y = \{2, 3\}$, and $u = 1$).

Exercise 3.4 *Let $D = (E, \mathcal{F})$ be a delta-matroid. Verify that $(E, \mathcal{F}_{\min})$ satisfies the basis definition of a matroid from Exercise 2.7. Conclude that $D_{\min}$ is indeed a matroid.*

A fundamental operation in delta-matroid theory is *twisting* (which is sometimes called *pivoting*). This operation changes a delta-matroid by replacing each feasible set X with its symmetric difference $X \triangle A$, for some fixed set A.

Definition 3.5 (Twist) Let $D = (E, \mathcal{F})$ be a delta-matroid, and $A \subseteq E$. Let

$$\mathcal{F}' := \{X \triangle A : X \in \mathcal{F}\}.$$

Then the *twist* of D by A, denoted $D * A$, is defined as

$$D * A := (D, \mathcal{F}').$$

The *dual* of D, denoted D^*, is defined as $D^* := D * E$.

Bouchet, in [9], showed that the set of delta-matroids is closed under twisting.

Proposition 3.6 *If $D = (E, \mathcal{F})$ is a delta-matroid, then so is $D * A$, for each $A \subseteq E$.*

Example 3.7 If D is the delta-matroid from Example 2.13, then $D * \{3,4\}$ is the delta-matroid on $\{1,\dots,6\}$ with feasible sets

$$\mathcal{F}' = \{\{2,3,5\},\{2,3,6\},\{5\},\{6\},\{3,5,6\},\{1,2,5\},\{1,2,6\},\{1,2,3,5,6\},\{2,5,6\}\}.$$

Exercise 3.8 *Verify the following results about twisting. (1) The twist of a delta-matroid is a delta-matroid (i.e., prove Proposition 3.6). (2) Every delta-matroid is the twist of a normal delta-matroid. (3) The twist of an even delta-matroid is even. (4)* $D_{\max}$ *is a matroid (use Exercise 3.4). (5)* $(D * A) * B = D * (A \triangle B)$.

We now define deletion and contraction for delta-matroids. In defining these, care must be taken in the special cases when an element is in every feasible set, or does not appear in any feasible set. Such elements are called coloops and loops.

Definition 3.9 (Loop and coloop) Let $D = (E, \mathcal{F})$ be a delta-matroid. Then an element $e \in E$ is a *loop* if it is not in any feasible set of D, and a *coloop* if it is in every feasible set of D.

Example 3.10 In Example 2.12, the element 1 is a loop, and 4 is a coloop.

Definition 3.11 (Deletion) Let $D = (E, \mathcal{F})$ be a delta-matroid, and $e \in E$. Then D *delete* by e, denoted $D\backslash e$, is defined as $D\backslash e := (E\backslash e, \mathcal{F}')$, where

1. when e is not a coloop,

$$\mathcal{F}' = \{X : X \in \mathcal{F} \text{ and } e \notin X\};$$

2. and when e is a coloop,

$$\mathcal{F}' = \{X\backslash e : X \in \mathcal{F} \text{ and } e \in X\}.$$

Thus, in words, if e is not a coloop, the feasible sets of $D\backslash e$ are obtained by restricting to feasible sets of D that do not contain e, and when e is a coloop they are obtained by restricting to the feasible sets of D that do contain e, then removing e from them.

Definition 3.12 (Contraction) Let $D = (E, \mathcal{F})$ be a delta-matroid, and $e \in E$. Then D *contract* by e, denoted D/e, is defined as $D/e := (E\backslash e, \mathcal{F}')$, where

1. when e is not a loop,

$$\mathcal{F}' = \{X\backslash e : X \in \mathcal{F} \text{ and } e \in X\};$$

2. and when e is a loop, $\mathcal{F}' = \mathcal{F}$.

Thus if e is not a loop, the feasible sets of D/e are obtained by restricting to the feasible sets of D that contain E, then removing e from them. When e is a loop D and D/e have the same feasible sets.

Example 3.13 Let $D = (E, \mathcal{F})$ be the delta-matroid from Example 2.13. Then $D\backslash 1$ has ground set $\{2, \dots, 6\}$ and its collection of feasible sets is

$$\{\{2,4,5\}, \{2,4,6\}, \{3,4,5\}, \{3,4,6\}, \{4,5,6\}, \{2,3,4,5,6\}\}.$$

$D/1$ has ground set $\{2, \dots, 6\}$ and its collection of feasible sets is

$$\{\{2,3,4,5\}, \{2,3,4,6\}, \{2,4,5,6\}\}.$$

$(D/1)\backslash 4$ has ground set $\{2,3,5,6\}$ and its collection of feasible sets is

$$\{\{2,3,5\}, \{2,3,6\}, \{2,5,6\}\}.$$

Exercise 3.14 *Show that if D is a delta-matroid then so are $D\backslash e$ and D/e. (Deletion and contraction are due to Bouchet and Duchamp [20].)*

An important observation is that the notions of deletion and contraction are 'dual' to each other:

$$D/e = (D * e)\backslash e. \tag{3.1}$$

This identity ties up the three delta-matroid operations of deletion, contraction, and twisting in a fundamental way.

Exercise 3.15 *Verify Equation* (3.1).

Observe that when $e \neq f$, the operations of twisting, deleting, and contracting on e, commute with the operations of twisting, deleting, and contracting on f. In particular, this means that for $D = (E, \mathcal{F})$ and $A \subseteq E$, we can define $D\backslash A$ and D/A as the result of deleting, respectively contracting, every element of A in any order.

Definition 3.16 (Minor) A delta-matroid D' is said to be a *minor* of a delta-matroid D if it can be obtained from D through the operations of deletion, contraction and twisting. Furthermore, D' is said to be a *strong-minor* of D if it can be obtained from D through the operations of deletion and contraction (without twisting).

Note that by (3.1), the operation of contraction is redundant in the definition of a minor. We also note that the term 'strong-minor' used here is not a standard term in the literature, but we need to make a distinction between these two types of minor.

Exercise 3.17 *Prove that a delta-matroid is even if and only if it has no minor isomorphic to the delta-matroid* $(\{a\}, \{\emptyset, \{a\}\})$. *(This result is due to Bouchet [13].)*

4 Graphic matroids

Cycle matroids provide a bridge between graph theory and matroid theory. While there is much to be said about cycle matroids and their role in matroid theory, their importance in terms of the current exposition is that there is a fundamental compatibility between graphs and matroids which means that results in either area can be used to gain insights in the other (Oxley's BCC survey article [62] illustrates this principle well). In Section 5, we shall demonstrate that an analogous connection holds between topological graph theory and delta-matroid theory, and see that many delta-matroid results can be regarded as 'topological' analogues of established matroid results. Our exposition of graphic matroids is tailored towards this aim, and the results mentioned here are standard and can be found in, for example, [61].

The cycle matroid of a connected graph $G = (V, E)$ was described in Section 2.3. The following definition includes the case when G is not connected. Recall that in the context of matroids, a feasible set is called a basis.

Definition 4.1 (Cycle matroid, graphic matroid) Let $G = (V, E)$ be a graph. Let

$$\mathcal{B} := \{F \subseteq E(\mathbb{G}) : F \text{ is the edge set of a maximal spanning forest of } G\},$$

Then $C(\mathbb{G}) := (E, \mathcal{B})$ is the *cycle matroid* of G.

A matroid is *graphic* if it is isomorphic to the cycle matroid of some graph.

Exercise 4.2 *Verify that the bases of $C(G)$ satisfies the Symmetric Exchange Axiom, and hence that $C(G)$ is a matroid.*

Edge and vertex deletion for graphs is denoted $G\backslash e$ and $G\backslash v$, respectively. Edge contraction is denoted G/e. We allow contraction of loops, and it is defined as the graph resulting from deleting the loop. An edge e of a graph G is a *bridge* if $G\backslash e$ has more components than G.

Exercise 4.3 *Let G be a graph with an edge e. Show that*

1. *e is a coloop in $C(G)$ if and only if e is a bridge in G; and*
2. *e is a loop in $C(G)$ if and only if e is a loop in G.*

The usual notion of a *matroid-minor* coincides with strong-minors when the matroid is regarded as a delta-matroid. (Strong-minors are needed as the set of matroids is not closed under twisting.) Graph minors are compatible with matroid-minors, providing a key link between graph and matroid theory.

Theorem 4.4 *Let G be a graph, e be an edge of G, and v be an isolated vertex. Then*

$$C(G\backslash e) = C(G)\backslash e, \quad C(G/e) = C(G)/e, \quad \textit{and} \quad C(G\backslash v) = C(G).$$

Properties of cycle matroids are intimately linked with properties of plane and planar graphs (a graph is *plane* if it *has* been embedded in the plane, and is *planar* if it *can* be embedded in the plane), as exhibited in the following theorems.

Theorem 4.5 *Let G be a plane graph and G^* be its (geometric) dual. Then*

$$C(G^*) = (C(G))^*.$$

Theorem 4.6 *The following are equivalent for a graph G.*

1. *G is planar*
2. *$C(G)^*$ is graphic*
3. *$C(G)$ has no matroid-minor isomorphic to $C(K_5)$ or $C(K_{3,3})$.*

Of course this theorem should be compared with the Kuratowski-Wagner Theorem which states that a graph G is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.

Exercise 4.7 *By considering different embeddings of a graph consisting of one vertex and two loops, show that, in general, Theorem 4.5 does not hold for non-plane embeddings.*

5 Topological graph theory and delta-matroids

It is often productive to think of matroids as 'generalisations of graphs'. In this section we explain how, analogously, delta-matroids can be thought of as being 'generalisations of graphs in surfaces', a point of view that enriches both fields. The usual passage between graphs and matroids is via cycle matroids, as described in the previous section. The passage between embedded graphs and delta-matroids is via ribbon-graphic delta-matroids. These delta-matroids arise by dropping a hidden topological restriction in the definition of a cycle matroid.

Bouchet first constructed delta-matroids from graphs in surfaces in [13]. His approach was very different, but equivalent, to that presented in this section. He associated a transition system to the medial graph of a graph in a surface and considered the Eulerian delta-matroid that arises from it. We instead approach the subject here through the language of ribbon graphs. The connection between ribbon graph theory and delta-matroid theory, as well as the philosophy that delta-matroid theory generalises topological graph theory, is due to Chun, Moffatt, Noble, and Rueckriemen [29, 30]. The equivalence between this approach and Bouchet's is detailed in Section 6, where Eulerian and ribbon-graphic delta-matroids are identified.

5.1 Ribbon graphs

In Section 2.4.3 we saw how a delta-matroid can be associated with a graph in a surface. We now develop this idea. However, to do so it is convenient, and more natural, to work in the language of ribbon graphs, rather than cellularly embedded graphs. This section contains a brief introduction to ribbon graphs. A more comprehensive introduction can be found in [35].

In essence, a ribbon graph is a structure that arises by taking a regular neighbourhood of a graph in a surface, but without throwing away the vertex-edge structure of the graph. See Figure 7. We can think of a ribbon graph informally as 'a graph whose vertices consist of discs, and whose edges consist of ribbons', as in Figure 7c.

Definition 5.1 (Ribbon graph) A *ribbon graph* $\mathbb{G} = (V, E)$ is a (possibly non-orientable) surface with boundary represented as the union of two sets of discs, a set V of *vertices*, and a set of *edges* E such that:

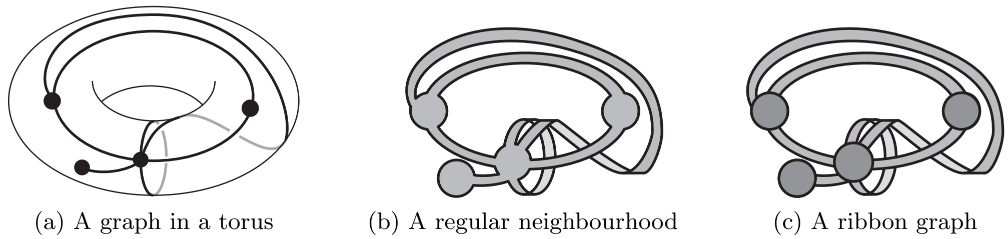


Figure 7: Equivalence of graphs in surfaces and ribbon graphs

1. the vertices and edges intersect in disjoint line segments;
2. each such line segment lies on the boundary of precisely one vertex and precisely one edge;
3. every edge contains exactly two such line segments.

Ribbon graphs describe exactly cellularly embedded graphs (i.e., graphs embedded on a closed surface such that the faces are all discs). We have discussed how a ribbon graph arises from a cellularly embedded graph (Figure 7). In the other direction, given a ribbon graph, the classification of surfaces with boundary ensures there is a unique way (up to homeomorphism) to embed it in a surface by 'filling in the holes'.

In addition to parameters inherited from graph theory, such as numbers of edges, vertices and components, some topological parameters are associated with ribbon graphs. A ribbon graph is *orientable* if it is orientable as a surface, and is *non-orientable* otherwise. The *genus* of a ribbon graph is its genus as a surface. The *Euler genus*, $\gamma(\mathbb{G})$ of a ribbon graph $\mathbb{G}$ equals its genus if it is non-orientable, and equals twice its genus if it is orientable. A connected ribbon graph is *plane* it has Euler genus 0 (i.e., if it corresponds to a graph in a sphere).

Ribbon graph equivalence corresponds to cellularly embedded graph equivalence. Two ribbon graphs are *equivalent* if there is a homeomorphism from one to the other (which should be orientation preserving when the ribbon graph is orientable) that sends vertices to vertices, edges to edges, and preserves the cycle order of half-edges at each vertex. We consider ribbon graphs up to this equivalence. Note that ribbon graphs are not embedded in 3-space, and in drawings ribbon graphs, we can 'push' half-twists of edges around the ribbon graph and 'turn vertices over' as illustrated in Figure 8, as well as 'pushing edges through each other'.

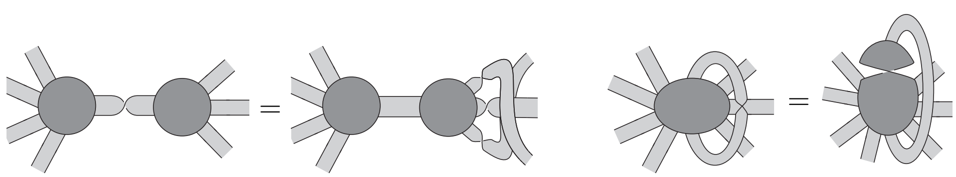


Figure 8: Some equivalent drawings of ribbon graphs

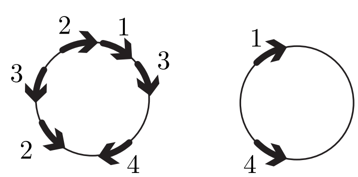
Deletion for ribbon graphs is defined in the obvious way:

Definition 5.2 (Deletion) Let $\mathbb{G}$ be ribbon graph, e be an edge of it, and v a vertex. Then $\mathbb{G}$ *delete* e, written $\mathbb{G}\backslash e$ is the ribbon graph obtained from $\mathbb{G}$ by removing the edge e, and $\mathbb{G}\backslash v$ is the ribbon graph obtained from $\mathbb{G}$ by removing the vertex v and all its incident edges.

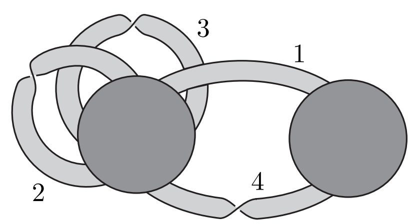
Contraction for ribbon graphs is more tricky to define. The difficulty is that while we would like to define contraction of an edge e to be the result of merging e and its incident vertices into a single vertex, as we do in the case for graphs, applying this operation to a loop in a ribbon graph can result in an object that is no longer a ribbon graph. To obtain a definition of contraction, we move to the language of arrow presentations, which is due to Chmutov [26].

Definition 5.3 (Arrow presentation) An *arrow presentation* is a set of closed curves, each with a collection of disjoint labelled arrows lying on them, and where each label appears on precisely two arrows.

An arrow presentation is shown in Figure 9a.



(a) An arrow presentation



(b) A ribbon graph

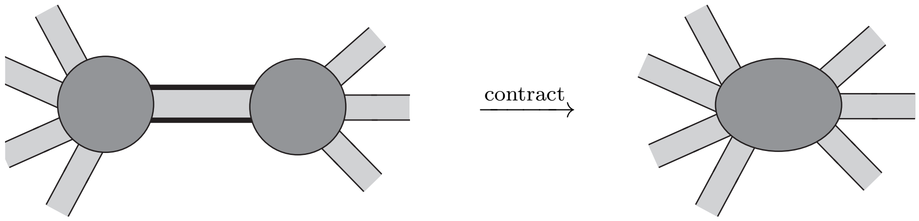
Figure 9: A ribbon graph and its description as an arrow presentation

Arrow presentations describe ribbon graphs. A ribbon graph G can be formed from an arrow presentation by identifying each closed curve with the boundary of a disc (forming the vertex set of G). Then, for each pair of e-labelled arrows, taking a disc (which will form an edge of G), orienting its boundary, placing two disjoint arrows on its boundary that point in the direction of the orientation, and identifying each e-labelled arrow on this edge. See Figure 9.

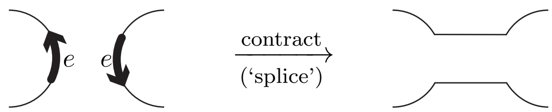
Conversely a ribbon graph can be described as an arrow presentation by arbitrarily labelling and orienting the boundary of each edge disc of G. Then on each arc where an edge disc intersects a vertex disc, place an arrow on the vertex disc, labelling the arrow with the label of the edge it meets and directing it consistently with the orientation of the edge disc boundary. The boundaries of the vertex set marked with these labelled arrows give an arrow presentation.

Now suppose that we have a non-loop edge e of a ribbon graph $\mathbb{G}$. Then the natural contraction operation is illustrated in Figure 10a. Figure 10b shows this operation in terms of a 'splicing' operation on arrow presentations. Notice that in terms of arrow presentation this definition is local and does not see if the edge is a loop or not. Thus it can be applied to *any* edge. This gives our definition of contraction.

Definition 5.4 (Contraction) Let $\mathbb{G}$ be ribbon graph with an edge e. Then $\mathbb{G}$ *contract* e, written $\mathbb{G}/e$ is the ribbon graph obtained from $\mathbb{G}$ by the following pro-



(a) Contracting a non-loop edge in a ribbon graph

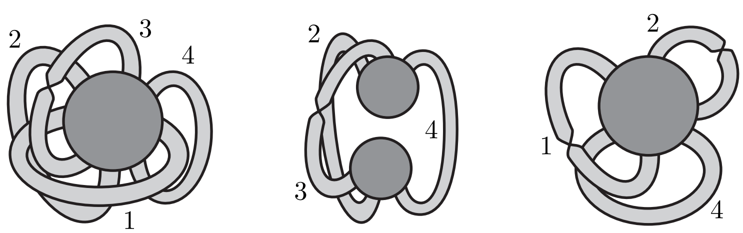


(b) Contracting any edge e in an arrow presentation

Figure 10: Descriptions of contraction

cess: (1) describe $\mathbb{G}$ as an arrow presentation, (2) 'splice' the arrow presentation as indicated in Figure 10b. (That is, delete the two e labelled arrows and the parts of the curves they lie on. Add arcs connecting the two pairs of points that were the tips and tails of the arrow.) (3) The ribbon graph described by this arrow presentation is $\mathbb{G}/e$.

Example 5.5 Figure 11 illustrates the contraction of loops. Notice that the underlying graph of $\mathbb{G}/1$ does not equal the result of contracting the edge 1 in the underlying graph of $\mathbb{G}$, so graph contraction and ribbon graph contraction are not compatible operations.



(a) $\mathbb{G}$ (b) $\mathbb{G}/1$ (c) $\mathbb{G}/3$

Figure 11: Contraction for ribbon graphs

Table 1 shows the local effect of deletion and contraction on a ribbon graph.

Contraction can be defined directly on ribbon graphs as follows. If u_1 and u_2 are the (not necessarily distinct) vertices incident to e, then $\mathbb{G}/e$ denotes the ribbon graph obtained as follows: consider the boundary component(s) of $e \cup u_1 \cup u_2$ as curves on G. For each resulting curve, attach a disc (which will form a vertex of $\mathbb{G}/e$) by identifying its boundary component with the curve. Delete e, u_1 and u_2 from the resulting complex, to get the ribbon graph $\mathbb{G}/e$.

Definition 5.6 (Minor) A ribbon graph $\mathbb{H}$ is a *minor* of a ribbon graph $\mathbb{G}$ if it can be obtained by a sequence of edge deletions, vertex deletions, and contractions.

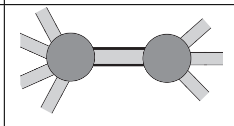
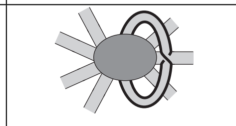
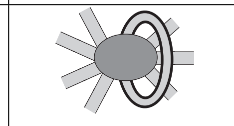
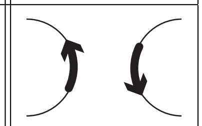
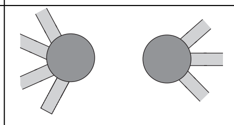
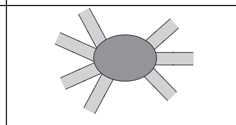
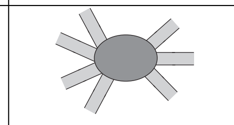
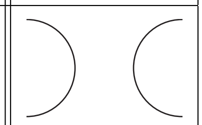
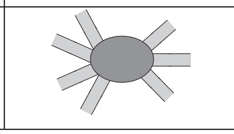
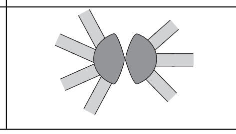
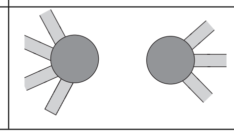
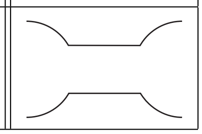
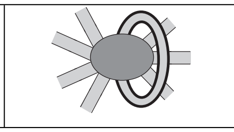
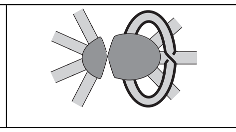
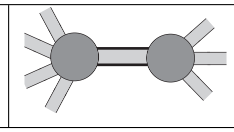
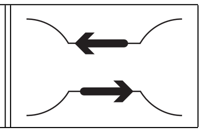
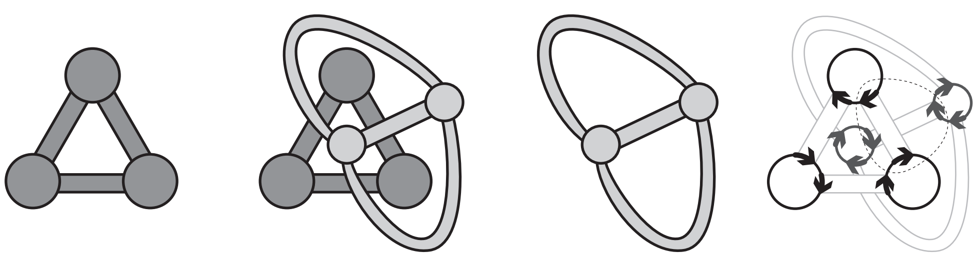
	non-loop	non-orient. loop	orientable loop	arrow pres.
G				
$G\backslash e$				
G/e				
G^e				

Table 1: Operations on an edge e (highlighted in bold) of a ribbon graph. The ribbon graphs are identical outside of the region shown

The other basic operation on ribbon graphs we need here is duality. Recall that the *dual*, G^*, of a graph G in a surface is the graph in the same surface obtained from G by placing one vertex in each of its faces, and embedding an edge of G^* between two of these vertices whenever the faces of G they lie in are adjacent. Edges of G^* are embedded so that they cross the corresponding face boundary (or edge of G) transversally.

Figure 12 shows the construction of a dual, where the plane graphs have been thickened to form ribbon graphs in the plane. We can describe these ribbon graphs as arrow presentations, and Figure 12d shows how the two arrow presentations fit naturally together in the plane with G and G^*. By examining this figure in the locality of an edge (inside the dotted region in the figure) we see that, in terms of arrow presentations, a dual graph can be constructed by using the local change of Figure 13 at each pair of arrows.



(a) $\mathbb{G}$ (b) Forming the dual (c) $\mathbb{G}^*$ (d) Arrow presentations

Figure 12: Dual graphs and their arrow presentations

What we have done is moved from a global construction of the dual G^* of G to a local construction. Since it is local, we can form the edges of a dual graph one at a time. This results in the concept of a *partial dual*, which is due to Chmutov [26].

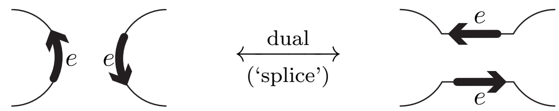


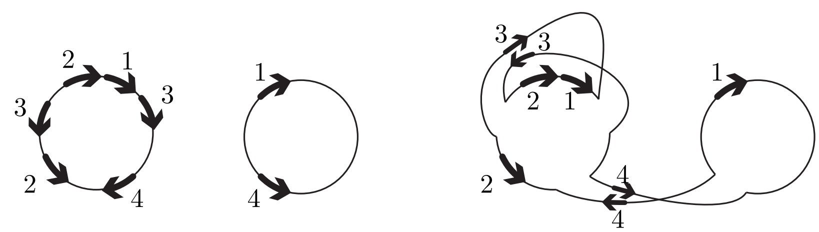
Figure 13: A partial dual in terms of arrow presentations

The partial dual, $\mathbb{G}^A$, of a ribbon graph $\mathbb{G}$ is the result of forming the dual of $\mathbb{G}$, but only at the edges in some set of edges A.

Definition 5.7 (Partial dual) Let $\mathbb{G}$ be ribbon graph with an edge e. Then the *partial dual* of $\mathbb{G}$ with respect to e is the ribbon graph denoted $\mathbb{G}^e$ obtained from $\mathbb{G}$ by the following process: (1) describe $\mathbb{G}$ as an arrow presentation, (2) 'splice' the arrow presentation at the two e-labelled arrows as indicated in Figure 13. (3) The ribbon graph described by this arrow presentation is $\mathbb{G}^e$.

When $e \neq f$ are edges of a ribbon graph $\mathbb{G} = (V, E)$, it is easily seen that $(\mathbb{G}^e)^f = (\mathbb{G}^f)^e$. Thus for $A \subseteq E$, we can define *partial dual* of $\mathbb{G}$ with respect to A, denoted by G^A, to be the ribbon graph obtained from G by forming the partial dual with respect to each edge of A in any order.

Example 5.8 The ribbon graph in Figure 9b can be described by the arrow presentation in Figure 14a. Forming the partial dual with respect to the edges 3 and 4, gives the arrow presentation shown in Figure 14b, which represents the ribbon graph in Figure 11a (so this is $\mathbb{G}^{\{3,4\}}$ when $\mathbb{G}$ is as in Figure 9b).



(a) An arrow presentation

(b) Partial dual with respect to {3, 4}

Figure 14: Forming a partial dual using arrow presentations

Table 1 shows the local effect of forming a partial dual with respect to an edge of a ribbon graph.

It is easy to see that the following properties hold. $\mathbb{G}^* = \mathbb{G}^{E(\mathbb{G})}$, where $\mathbb{G}^*$ is the dual of $\mathbb{G}$; $\mathbb{G}^\emptyset = \mathbb{G}$; $(\mathbb{G}^A)^B = \mathbb{G}^{A \triangle B}$; partial duality acts disjointly on connected components; and $\mathbb{G}^A$ is orientable if and only if $\mathbb{G}$ is. By examining arrow presentations (for example, in Table 1), we immediately see that

$$\mathbb{G}/e = \mathbb{G}^e \backslash e. \tag{5.1}$$

As with contraction, partial duals can be formed without passing through arrow presentations. Let $\mathbb{G} = (V, E)$ be a ribbon graph, $A \subseteq E$, and regard the boundary components of the ribbon subgraph (V, A) as curves on $\mathbb{G}$. Glue a disc to $\mathbb{G}$ along each of these curves by identifying the boundary of the disc with the curve, and remove the interior of all vertices of $\mathbb{G}$. The resulting ribbon graph is $\mathbb{G}^A$.

Exercise 5.9 *Consider the set S of pairs $(\mathbb{G}, e)$ where $\mathbb{G}$ is a ribbon graph and e is one of its edges. Let δ denote the operation $\delta : (\mathbb{G}, e) \mapsto (\mathbb{G}^e, e)$. Let $\tau : (\mathbb{G}, e) \mapsto (\mathbb{G}^{\tau(e)}, e)$ where $\mathbb{G}^{\tau(e)}$ is obtained from $\mathbb{G}$ by adding a 'half-twist' to the edge e (formally, reverse the direction of exactly one e-labelled arrow in an arrow presentation of $\mathbb{G}$). Two ribbon graphs are* twisted duals *if one can be obtained from the other by a sequence of applications of the operations τ and δ to its edges (see [34]). Verify that the operations τ and δ induce an action of the symmetric group $\langle \delta, \tau \mid \delta^2, \tau^2, (\tau\delta)^3 \rangle$ on S.*

5.2 Ribbon-graphic delta-matroids

Thinking of matroid theory as a generalisation of graph theory, where the passage from a graph G to a matroid is through its cycle matroid $C(G)$, suppose we were set the problem of finding the matroid analogue of topological graph theory. We are thus looking for some matroid analogue of a ribbon graph $\mathbb{G}$. We quickly see that cycle matroids do not provide an effective analogue of ribbon graphs, since they do not see any of their topological information (e.g, the two ribbon graphs that are 2-cycles have the same cycle matroid). To progress let us examine the construction of $C(\mathbb{G})$.

For simplicity, suppose $\mathbb{G}$ is connected. Then the bases of $C(\mathbb{G})$ are the edge sets of the spanning trees of $\mathbb{G}$. A spanning tree of $\mathbb{G}$ can be characterised as a ribbon subgraph that is (1) spanning, (2) has exactly one boundary component, and (3) is of genus 0. With this formulation it is apparent why we are seeing no topological information in $C(\mathbb{G})$ — we are only considering subgraphs of genus 0. We immediately see how to adjust the construction to preserves topological information — drop the genus 0 condition.

This takes us to the concept of a *quasi-tree*, which is a ribbon graph with exactly one boundary component. With this, we can obtain a topological version of a cycle matroid by replacing the words "tree" with "quasi-tree" in its definition. It turns out that this results in a delta-matroid, denoted here by $D(\mathbb{G})$, that is a topological counterpart of a cycle matroid.

Definition 5.10 (Quasi-tree) A *quasi-tree* is a ribbon graph with exactly one boundary component. A ribbon subgraph $\mathbb{H}$ of a connected ribbon graph $\mathbb{G}$ is a *spanning quasi-tree* if $\mathbb{H}$ is a quasi-tree and has the same vertex set as $\mathbb{G}$. By an abuse of notation, if $\mathbb{G}$ is not connected then we say a ribbon subgraph $\mathbb{H}$ is a *spanning quasi-tree* of $\mathbb{G}$ if $\mathbb{H}$ induces a spanning quasi-tree of each connected component of $\mathbb{G}$.

We obtain a topological analogue of a cycle matroid by replacing trees with quasi-trees in Definition 4.1.

Definition 5.11 (Ribbon-graphic delta-matroid) Let $\mathbb{G} = (V, E)$ be a ribbon graph, and let

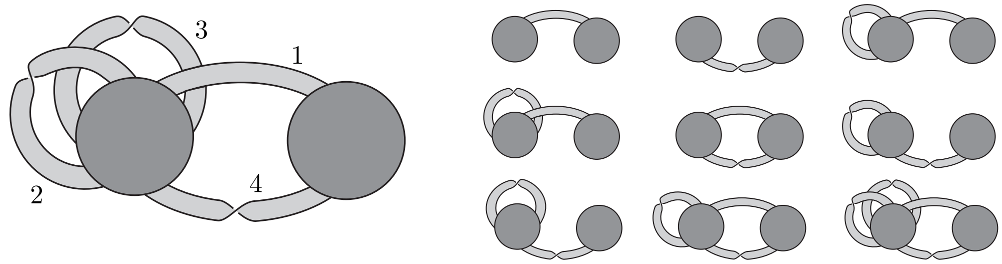
$$\mathcal{F} := \{F \subseteq E : F \text{ is the edge set of a spanning quasi-tree of } \mathbb{G}\}.$$

We call $D(\mathbb{G}) := (E, \mathcal{F})$ the *delta-matroid of* $\mathbb{G}$.

We say a delta-matroid is *ribbon-graphic* if it is isomorphic to the delta-matroid of some ribbon graph.

Example 5.12 Let $\mathbb{G}$ be the ribbon graph shown in Figure 15a. Its spanning quasi-trees are shown in Figure 15b. From this we see that $D(\mathbb{G}) = (E, \mathcal{F})$ where $E = \{1, 2, 3, 4\}$ and

$$\mathcal{F} = \{\{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}.$$



(a) A ribbon graph (b) Spanning quasi-trees

Figure 15: A ribbon graph and its spanning quasi-trees

Example 5.13 The construction of a delta-matroid given in Section 2.4.3 is just Definition 5.11 phrased in terms of graphs in surfaces. Thus Example 2.13 gives the delta-matroid of the ribbon graph in Figure 7.

In [13], Bouchet proved, using the language of Eulerian circuits in medial graphs, that $D(\mathbb{G})$ is a delta-matroid. Figure 16 sketches a proof in terms of the topology of surfaces. A ribbon graphic proof can be found in [29].

Theorem 5.14 *$D(\mathbb{G})$ as constructed in Definition 5.11 is a delta-matroid.*

Exercise 5.15 *Prove that the feasible sets of $D(\mathbb{G})$ of minimum size are exactly the bases of the cycle matroid of $\mathbb{G}$, and hence $D(\mathbb{G})_{\min} = C(\mathbb{G})$.*

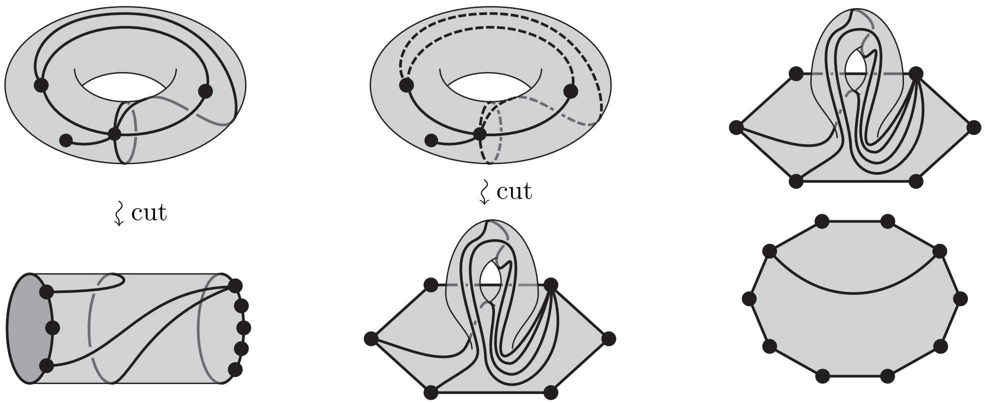
For a ribbon graph $\mathbb{G}$ with k components, it follows by definition that the feasible sets of $D(\mathbb{G})$ are in 1-1 correspondence with the spanning quasi-trees of $\mathbb{G}$. This correspondence can be refined (see [29]) to show that the feasible sets of $D(\mathbb{G})$ with cardinality m are in 1-1 correspondence with the spanning quasi-trees of $\mathbb{G}$ with Euler genus $m - |V| + k$. The following properties of ribbon-graphic delta-matroids follow from this basic result. They were first proved by Bouchet in [13].

Proposition 5.16 *Let $\mathbb{G}$ be a ribbon graph. Then*

1. *the width of $D(\mathbb{G})$ equals the Euler genus $\mathbb{G}$;*

2. *$D(\mathbb{G})$ is even if and only if $\mathbb{G}$ is orientable;*

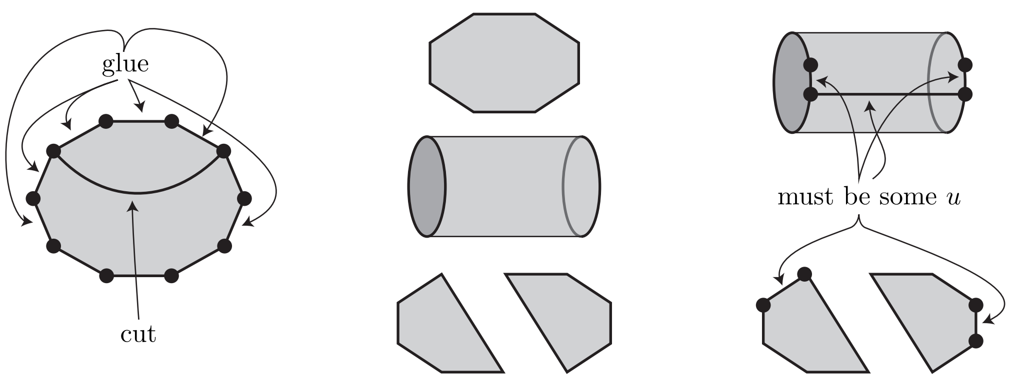
3. *$D(\mathbb{G})_{\min} = C(\mathbb{G})$;*



(a) We can cut a surface open along any spanning subgraph (V, A). Edges in A appear in pairs on the boundary, edges not in A are embedded, vertices are on the boundary

(b) $A \subseteq E$ defines a spanning quasi-tree $\iff$ cutting results in a single boundary component

(c) Suppose $X, Y \subseteq E$ define spanning quasi-trees. Cut along them



(d) Y can be obtained from X by cutting along edges in $Y \backslash X$ and gluing together edges in $X \backslash Y$ that were previously cut open

(e) Pick $u \in X \triangle Y$. Cutting along $X \triangle u$ gives something with one or two boundary components

(f) If it has two boundary components there must be some $v \in X \triangle Y$ such that cutting or gluing will merge these into a single boundary component, otherwise the property in Figure 16d would fail. Thus $X \triangle \{u, v\}$ is a quasi-tree

Figure 16: A sketch of a proof that the Symmetric Exchange Axiom holds for $D(\mathbb{G})$

4. $D(\mathbb{G})_{\max} = C(\mathbb{G}^*)^*$*;*

5. $D(\mathbb{G}) = C(\mathbb{G})$ *if and only if* $\mathbb{G}$ *is the disjoint union of plane ribbon graphs.*

Recall from Exercise 4.3 that loops and coloops in cycle matroids correspond to loops and bridges in graphs. The situation in delta-matroids is a little more complicated since a loop in a ribbon graph can have different topological properties: it can be orientable or non-orientable, and *trivial* or *non-trivial*. A loop edge e incident with a vertex v in a ribbon graph is *non-trivial* if there is some cycle C

in the ribbon graph such that e and C are met in the cyclic order $eCeC$ when following the boundary of the vertex v. It is *trivial* otherwise. For example, in Figure 11c, the loop 2 is trivial, while loops 1 and 4 are non-trivial. Loops 1 and 2 are non-orientable, and loop 4 is orientable.

Exercise 5.17 *Let $\mathbb{G}$ be a ribbon graph, $D(\mathbb{G}) = (E, \mathcal{F})$, and $e \in E(\mathbb{G})$. Show that*

1. *e is a coloop in $D(\mathbb{G})$ if and only if e is a bridge in $\mathbb{G}$; and*
2. *e is a loop in $D(\mathbb{G})$ if and only if e is a trivial orientable loop in $\mathbb{G}$.*

(This result is from Chun et al. [29].)

In fact, each of the four types of loops in ribbon graphs mentioned above can be recognised in their delta-matroids (see [29]). The corresponding four delta-matroid loop types are often used to define cases in induction arguments for delta-matroids, just as loops and coloops do in the matroid case.

5.3 Minors and the interplay with ribbon graphs

From Table 1 it is clear that $\mathbb{G}$ and $\mathbb{G}^e\backslash e (= \mathbb{G}/e)$ have the same numbers of boundary components, as do $\mathbb{G}\backslash e$ and $\mathbb{G}^e$. A consequence of this is that if $\mathbb{H}$ is a spanning quasi-tree of a ribbon graph $\mathbb{G}$, then we can obtain a spanning quasi-tree of its partial dual $\mathbb{G}^A$ by 'toggling' edges in $\mathbb{H}$ that are in A. This sets up a 1-1 correspondence between the spanning quasi-trees of $\mathbb{G}$ and of $\mathbb{G}^A$. Concretely, B is the edge set of a spanning quasi-tree in $\mathbb{G}$ if and only if $B \triangle A$ is the edge set of a spanning quasi-tree in G^A. Phrasing this in terms of delta-matroids gives the following fundamental bridge between delta-matroid theory and ribbon graph theory. The result is from Chun et al. [29].

Theorem 5.18 *Let $\mathbb{G} = (V, E)$ be a ribbon graph and $A \subseteq E$. Then*

$$D(\mathbb{G}^A) = D(\mathbb{G}) * A.$$

As special case, this theorem completes the classical matroid result stated in Theorem 4.5, that, for *plane* graphs, $C(G^*) = C(G)^*$. Taking $A = E$ in Theorem 5.18 gives that for *any* embedded graph,

$$D(\mathbb{G}^*) = D(\mathbb{G})^*.$$

When $\mathbb{G}$ is plane this identity become the matroid one.

Exercise 5.19 *Using that $D(\mathbb{G})_{\min} = C(\mathbb{G})$, deduce from Theorem 5.18 that $D(\mathbb{G})_{\max} = C(\mathbb{G}^*)^*$.*

It was shown in [29] that delta-matroid and ribbon graph deletion and contraction correspond.

Theorem 5.20 *Let $\mathbb{G}$ be a ribbon graph, and $e \in E(\mathbb{G})$. Then*

$$D(\mathbb{G}\backslash e) = D(\mathbb{G})\backslash e \qquad \textit{and} \qquad D(\mathbb{G}/e) = D(\mathbb{G})/e.$$

A proof of the deletion result in this theorem can be obtained by considering a ribbon graph locally at an edge e, the three different ways that boundary components can touch this edge, and how the boundary components change under deletion. The contraction result follows from the deletion result, (5.1), and Theorem 5.18.

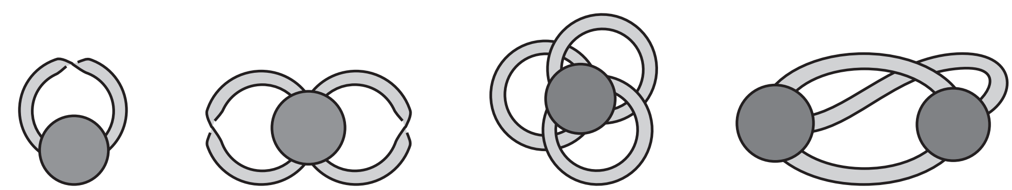
Theorems 5.18 and 5.20 together give a compatibility between delta-matroid minors and ribbon graph minors:

$$\text{Ribbon graph minors} \quad \xleftrightarrow{\text{compatible}} \quad \text{delta-matroid strong-minors,}$$

$$\begin{array}{c}\text{Ribbon graph minors}\\ \text{and partial duals}\end{array} \quad \xleftrightarrow{\text{compatible}} \quad \text{delta-matroid minors.}$$

This means that we can translate results from one setting to another. Of course, ribbon graphs are not identified with delta-matroids so it may be that translating gives a false or partial result, and, even when the result is true, a new proof may be needed. What is important is that intuition developed in either area can provide intuition in the other.

Exercise 5.21 *Let $\mathbb{G}_1$ be the ribbon graph from Figure 17a. Prove that a ribbon graph is orientable if and only if it has no minor equivalent to $\mathbb{G}_1$. Formulate a delta-matroid version of this statement, and compare it to the result in Exercise 3.17.*



(a) $\mathbb{G}_1$ (b) $\mathbb{G}_2$ (c) $\mathbb{G}_3$ (d) $\mathbb{G}_4$

Figure 17: Ribbon graphs appearing in excluded-minor theorems

Let us see how the compatibility between delta-matroids and ribbon graphs can be used in practice. Much of the recent development in ribbon graph theory has been motivated by knot theory. It is a classical and well-known result that alternating knot and link diagrams can be represented by plane graphs. Dasbach et al., in [31], extended this construction to describe *any* (*i.e.*, not only alternating) knot or link diagram as a ribbon graph. Not all ribbon graphs arise from knot and link diagrams. This leads to the problem of characterising the class of ribbon graphs of knots and links. Chmutov showed in [26] that this class consists exactly of ribbon graphs with a plane partial dual (in fact, partial duality was introduced to explain the relationship between the ribbon graphs of knots and links). The following excluded-minor characterisation for this class was given in [52].

Theorem 5.22 *Let $\mathbb{G}_1$, $\mathbb{G}_3$, and $\mathbb{G}_4$ be the ribbon graphs in Figure 17. Then a ribbon graph $\mathbb{G}$ is a partial dual of a plane graph if and only if it has no minor equivalent to $\mathbb{G}_1$, $\mathbb{G}_3$, or $\mathbb{G}_4$.*

Let us translate this into delta-matroids. Partial duality corresponds to twisting in delta-matroids (Theorem 5.18), and plane graphs correspond to delta-matroids of width zero (Proposition 5.16(1)), i.e., to matroids. Thus "$\mathbb{G}$ is a partial dual of a plane graph" becomes "D is a twist of a matroid". By Theorem 5.20, ribbon graph minors correspond to strong delta-matroid minors. So "no (ribbon graph) minor equivalent to $\mathbb{G}_1$, $\mathbb{G}_3$, or $\mathbb{G}_4$" becomes "no strong (delta-matroid) minor isomorphic to $D(\mathbb{G}_1)$, $D(\mathbb{G}_3)$, or $D(\mathbb{G}_4)$". Since $\mathbb{G}_3$, or $\mathbb{G}_4$ are partial duals, $D(\mathbb{G}_3)$ and $D(\mathbb{G}_4)$ are twists, so we can rephrase this as "no (delta-matroid) minor isomorphic to $D(\mathbb{G}_1)$ or $D(\mathbb{G}_3)$". Thus we are led to conjecture that "A delta-matroid D is the twist of a matroid if and only if it does not have a minor isomorphic to $D(\mathbb{G}_1)$ or $D(\mathbb{G}_3)$." This turns out to be a result of Duchamp from [33].

Theorem 5.23 *Let $\mathbb{G}_1$, and $\mathbb{G}_3$ be the ribbon graphs in Figure 17. A delta-matroid D is the twist of a matroid if and only if it does not have a minor isomorphic to $D(\mathbb{G}_1)$ or $D(\mathbb{G}_3)$.*

Just as in the case of graphs and matroids, sometimes delta-matroid versions of ribbon graph results require an 'extra something', as follows.

Theorem 5.22 was extended to graphs in the real projective plane in [54].

Theorem 5.24 *Let $\mathbb{G}_2$, $\mathbb{G}_3$, and $\mathbb{G}_4$ be the ribbon graphs in Figure 17. Then a ribbon graph has a partial dual of Euler genus at most one if and only if it has no ribbon graph minor equivalent to $\mathbb{G}_2$, $\mathbb{G}_3$, or $\mathbb{G}_4$.*

The direct delta-matroid translation of Theorem 5.24 is "a delta-matroid has a twist of width at most one if and only if it has no minor isomorphic to $D(\mathbb{G}_2)$ or $D(\mathbb{G}_3)$". However, this statement is not true (although it does hold for ribbon-graphic and binary delta-matroids). An additional non-ribbon-graphic delta-matroid needs to be included for the correct result, as was found by Chun et al. in [28].

Theorem 5.25 *Let $\mathbb{G}_2$ and $\mathbb{G}_3$ be the ribbon graphs in Figure 17. A delta-matroid has a twist of width at most one if and only if it has no minor isomorphic to $D(\mathbb{G}_2)$ or $D(\mathbb{G}_3)$, or $(\{1,2,3\},\{\emptyset,\{1\},\{2\},\{3\},\{1,2,3\}\})$.*

We have just seen examples of ribbon graph theory informing delta-matroid theory. We now give an example where delta-matroid theory has informed ribbon graph theory.

Proved by Brylawski in [24] and independently by Seymour in [67], the following result says that in a connected matroid M that contains a minor N, it is always possible to delete or contract an element from M to stay connected and keep N as a minor. Results such as this are useful in induction proofs.

Theorem 5.26 *Let M be a connected matroid with a connected minor N. If $e \in E(M)\backslash E(N)$, then $M\backslash e$ or M/e is connected with N as a minor.*

Chun, Chun, and Noble, in [27], extended this result to delta-matroids.

Theorem 5.27 *Let D be a connected even delta-matroid with a connected minor D'. If $e \in E(D)\backslash E(D')$, then $D\backslash e$ or D/e is connected with D' as a minor.*

By translating from delta-matroids to ribbon graphs they obtained the following new result about ribbon graphs.

Theorem 5.28 *Let $\mathbb{G}$ be a 2-connected, orientable ribbon graph. If $\mathbb{H}$ is a 2-connected minor of $\mathbb{G}$ and $e \in E(\mathbb{G})\backslash E(\mathbb{H})$, then $\mathbb{G}\backslash e$ or $\mathbb{G}/e$ is 2-connected with $\mathbb{H}$ as a minor.*

Chun, Chun, and Noble were interested in "Splitter Theorems" for delta-matroids in [27]. Their paper includes other, and more impressive, examples of delta-matroid theory informing ribbon graph theory. About one of their ribbon graph results, they wrote: "It is extremely unlikely that we would have established [the result] without the intuition provided by delta-matroids." Describing these results here would require the introduction of a fairly large amount of terminology, so we will settle with the example just seen.

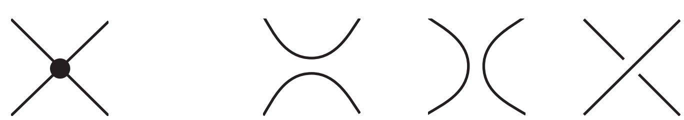
6 Eulerian delta-matroids

In this section we describe a class of delta-matroids arsing from Eulerian circuits, as seen in Section 2.4.4, called *Eulerian* delta-matroids. One of Bouchet's main motivations for introducing delta-matroid was the study of Eulerian circuits through this class.

Our interest here is in the set of Eulerian circuits in a 4-regular graph G. In general, at each vertex there are three ways that an Eulerian circuit can pass though it. Here we want to restrict the set of Eulerian circuits by forbidding, at each vertex, one of these three ways. We then consider the resulting, restricted set of Eulerian circuits.

Let us think how we can record the resulting set of allowed Eulerian circuits. At each vertex there are only two allowed ways an Eulerian circuit may pass through. If we distinguish one of these and call it "preferred" then we can encode each Eulerian circuit by, for each vertex, noting whether or not it follows the preferred route. Thus we can record each allowed Eulerian circuit as a subset U of vertices of G, where we follow the preferred route through a vertex v if and only if $v \in U$. We now formalise this discussion.

Let $G = (V, E)$ be a connected 4-regular graph. Each vertex v of G is incident with exactly four half-edges. (We need to consider half-edges rather than edges as our graphs may have loops.) A *bitransition* at a vertex v is a pairing of its incident half-edges. Each vertex has exactly three bitransitions. A graphical representation of them is given in Figure 18.



(a) A vertex v (b) Bitransitions

Figure 18: A representation of the bitransitions at v in a 4-regular graph

A *transition system* of the graph G is a choice of bitransition at each of its vertices. Notice that transition systems correspond to circuit coverings of G (by "passing through" each vertex in the way specified by its bitransition). We say that a transition system is *Eulerian* if it corresponds to an Eulerian circuit in G.

Definition 6.1 (Eulerian delta-matroid) Let $G = (V, E)$ be a connected 4-regular graph. At each vertex of G specify one bitransition as *forbidden*, and call the other two *allowed*. Specify one of the two allowed bitransitions at each vertex as being *preferred*. A transition system is *allowed* if it does not contain a forbidden transition. Let T_F denote the transition system consisting of all forbidden bitransitions, and T_P denote the transition system consisting of all preferred bitransitions. Set

$$D(G, T_F, T_P) := (V, \mathcal{F}),$$

where

$$\mathcal{F} = \{U \subseteq V : \text{ there exists an allowed Eulerian transition system of } G \text{ with preferred bitransition at exactly the vertices of } U\}.$$

A delta-matroid is said to be *Eulerian* if it is isomorphic to $D(G, T_F, T_P)$ for some choice of G, T_F, and T_P.

An example of $D(G, T_F, T_P)$ can be found in Example 2.14, where G is shown in Figure 5 and T_F and T_P are specified by the tables in that figure.

Bouchet [9] proved that $D(G, T_F, T_P)$ is a delta-matroid.

Theorem 6.2 *$D(G, T_F, T_P)$, as constructed in Definition 6.1, is a delta-matroid.*

A direct proof of Theorem 6.2 can be found in [9], where this class of delta-matroids was introduced. Following [29], we see later that Theorem 6.2 follows from a connection between Eulerian and ribbon-graphic delta-matroids.

There are two situations where a set of forbidden bitransitions arises naturally: graphs in surfaces, and directed graphs. Let us start with the case of graphs in surfaces.

Let $G = (V, E)$ be a connected graph embedded in a surface Σ. The *medial graph* G_m of G is the 4-regular graph embedded in Σ obtained by placing a vertex of degree 4 on each edge of G, and then drawing the edges of the medial graph by following the face boundaries of G. See Figure 19.

(a) A graph G (b) Forming its medial graph (c) The medial graph G_m

Figure 19: Constructing a medial graph

We can obtain a set of forbidden and preferred bitransitions for G_m by allowing only bitransitions that pair half-edges that follow a face boundary of G_m through v,

(a) Edge of G (b) Corresponding vertex in G_m (c) Bitransitions

Figure 20: Bitransitions in medial graphs

and preferring transitions that follow the corresponding edge of G, as illustrated in Figure 20.

This set of forbidden and preferred bitransitions gives rise to the delta-matroid $D(G_m, T_F, T_P)$. Since the vertices of G_m correspond to the edges of G, this can be regarded as a delta-matroid on the edge set E of G, rather than the vertex set of G_m. Let us denote the resulting delta-matroid on E by $D(G \subset \Sigma)$. This class of delta-matroids was introduced by Bouchet in [13], where they were called the *delta-matroids of maps*.

We can recognise $D(G \subset \Sigma)$ as the delta-matroid of a ribbon graph. For this, let $G = (V, E)$ be a connected graph embedded in a surface Σ, and let $\mathbb{G}$ be its description as a ribbon graph. The edges of G and $\mathbb{G}$ correspond to each other, so the ground sets of $D(G \subset \Sigma)$ and $D(\mathbb{G})$ can be identified. Moreover, by considering Figure 21, it is easy to see that, for each $A \subseteq E$, the boundary components of the ribbon subgraph (V, A) of $\mathbb{G}$ correspond to the allowed circuits in G_m that take the preferred bitransition at the vertices of G_m that are in A. This sets up a 1-1 correspondence between the spanning quasi-trees of $\mathbb{G}$ and the allowed Eulerian transition systems of G_m. It follows that $D(G \subset \Sigma) = D(\mathbb{G})$. We note that Bouchet's results for ribbon-graphic delta-matroids stated in Section 5 were phrased and proved in terms of $D(G \subset \Sigma)$ and transition systems. The connection with ribbon graph theory appeared in [29, 30].

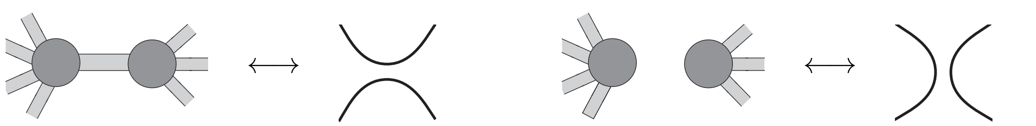


Figure 21: Identifying feasible sets of $D(G \subset \Sigma)$ and $D(\mathbb{G})$

It turns out that *every* Eulerian delta-matroid is the delta-matroid of a ribbon graph. This can be seen by considering a generalisation of partial duality, called twisted duality, that was briefly mentioned in Exercise 5.9. The idea is that every (non-embedded) 4-regular graph F arises as the medial graph of some embedded graph $G \subset \Sigma$, and the twisted duals of $G \subset \Sigma$ give all embedded graphs with a medial graph isomorphic (as a graph) to F. By [34], the forbidden and permitted transitions of the medial graph of one of these must coincide with with the forbidden and permitted transitions of F. A correspondence between Eulerian and ribbon-graphic delta-matroids follows. (The Twisted dual results here are due to Ellis-Monaghan and Moffatt [34]. Further details of the delta-matroid application can be found in [29], and an alternative approach to the result in [13].) This discussion gives the following.

Theorem 6.3 *A delta-matroid D is Eulerian if and only if it is isomorphic to $D(\mathbb{G})$, for some ribbon graph $\mathbb{G}$.*

Theorem 6.3 was proved by Bouchet [13] in the context of delta-matroids of maps $D(G \subset \Sigma)$, the ribbon graph phrasing and approach presented here is from [29].

Taking an apparently different direction for the moment, we consider *Eulerian digraphs*. These are connected digraphs in which the in-degree equals the out-degree at each vertex. We are interested in its *directed Eulerian circuits*, so the circuits must follow the directions of the arcs.

Definition 6.4 (Directed Eulerian delta-matroid) Let $\vec{G}$ be a 4-regular Eulerian digraph. At each vertex there are two bitransitions that agree with the orientation. Take these as the allowed bitransitions, and choose a preferred bitransition at each vertex. Let T_P denote the transition system consisting of all preferred bitransitions. With these choices construct a delta-matroid

$$D(\vec{G}, T_P) := D(G, T_F, T_P).$$

A delta-matroid is said to be *directed Eulerian* if it is isomorphic to $D(\vec{G}, T_P)$ for some $\vec{G}$ and T_P.

From Theorem 6.3 we know every directed Eulerian delta-matroid can be realised as the delta-matroid of a ribbon graph $\mathbb{G}$. However, the directions on the arcs can be used to ensure that we can always construct some such $\mathbb{G}$ that is orientable (see [29] for details). With this we recover the following theorem of Bouchet from [9].

Theorem 6.5 *A delta-matroid D is directed Eulerian if and only if $D = D(\mathbb{G})$, for some orientable ribbon graph $\mathbb{G}$.*

Recalling that $\mathbb{G}$ is orientable if and only if $D(\mathbb{G})$ is even gives the following.

Corollary 6.6 *A delta-matroid is directed Eulerian if and only if it is even and Eulerian.*

As a summary of the identifications of this section,

$$\text{Eulerian delta-matroids} \overset{\text{1-1}}{\longleftrightarrow} \text{delta-matroids of ribbon graphs},$$

$$\text{even Eulerian delta-matroids} \overset{\text{1-1}}{\longleftrightarrow} \text{delta-matroids of orientable ribbon graphs},$$

$$\text{directed Eulerian delta-matroids} \overset{\text{1-1}}{\longleftrightarrow} \text{delta-matroids of orientable ribbon graphs}.$$

Although we have identified ribbon-graphic and Eulerian delta-matroids, it is useful to have both realisations as they provide different insights and applications.

The following natural delta-matroid problems were proposed by Geelen, Iwata, and Murota, in [42]; and Bouchet in [16], respectively. Given a pair of delta-matroids $D = (E, \mathcal{F})$ and $D' = (E, \mathcal{F}')$ on E. The *partition problem* asks if there is some partition of E into two sets F and F' such that $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$. The *delta-covering problem* is to find feasible sets $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$ maximising $|F \triangle F'|$. The delta-covering problem is clearly a generalisation of the partition problem. These problems originate from the theory of Eulerian circuits. (It is worth noting that the delta-covering problem is a generalisation of the matroid parity problem.)

Let $\vec{G}$ be a 4-regular Eulerian digraph. Two directed Eulerian circuits are *compatible* if they use different bitransitions at each vertex (so the two directed Eulerian circuits 'take different routes' through each vertex). The problem is to determine if $\vec{G}$ admits two compatible directed Eulerian circuits. This is exactly the partition problem when $D = D' = D(\vec{G}, T_P)$, for some T_P.

More generally, we could ask for the construction of compatible directed Eulerian circuits (if they exist), or for the construction of two directed Eulerian circuits with the minimum number of common bitransitions. These are special cases of the delta-covering problem with $D = D' = D(\vec{G}, T_P)$.

Geelen, Iwata, and Murota, in [42], gave an efficient solution to the delta covering problem for a class of delta-matroids known as *linear delta-matroids*. This class includes directed Eulerian delta-matroids (by [16]), and hence gives an efficient algorithm for construction pairs of compatible directed Eulerian circuits in a digraph.

The approach taken in [42] was to reformulate the delta-covering problem as a problem called the *delta-parity problem* (its description is more involved than the delta-covering problem so we omit it here). This problem extends the *parity problem for linearly presented matroids*, an extremely general problem that is known to contain NP-hard problems. Geelen, Iwata, and Murota extended Lovász's Minimax Theorem and efficient solution to the parity problem for linearly presented matroids, [48, 49, 50] to solve the delta-matroid problem.

7 Matrices and representability

We revisit the example in Section 2.4. There, given a symmetric or skew-symmetric matrix $\mathbf{A}$ over a field $\Bbbk$, whose rows and columns were labelled (in the same order) by a set E, we formed a delta-matroid $D(\mathbf{A}) := (E, \mathcal{F})$ by taking the labelling set E as the ground set, and, for the collection of feasible sets, we took

$$X \in \mathcal{F} \iff \mathbf{A}[X] \text{ is non-singular.}$$

Recall that $\mathbf{A}[\emptyset]$ is considered to be non-singular, and so $D(\mathbf{A})$ is necessarily normal.

Bouchet proved $D(\mathbf{A})$ is a delta-matroid in [12].

Theorem 7.1 *For every symmetric or skew-symmetric matrix* $\mathbf{A}$ *over a field* $\Bbbk$*, the pair* $D(\mathbf{A}) := (E, \mathcal{F})$ *constructed as above is a normal delta-matroid.*

Remark Bouchet also proved Theorem 7.1 for quasi-symmetric matrices, where $\mathbf{A} = [a_{ij}]$ is *quasi-symmetric* if there is some function $\varepsilon : E \to \{-1, +1\}$ such that $\varepsilon(i)a_{i,j} = \varepsilon(j)a_{j,i}$, for all i, j. (Thus a symmetric matrix is a quasi-symmetric matrix where ε is a constant function.)

The two delta-matroid operations of delete and twist acting on $D(\mathbf{A})$ can be given in terms of operations on $\mathbf{A}$. It is straightforward to see that $D(\mathbf{A})\backslash e$ coincides with the delta-matroid of the matrix obtained from $\mathbf{A}$ by deleting the row and column labelled by e. Thus,

$$D(\mathbf{A})\backslash e = D(\mathbf{A}[E\backslash e]). \tag{7.1}$$

Delta-matroid twisting corresponds to a matrix operation called *pivoting*.

Definition 7.2 (Pivoting for matrices) Let $\mathbf{A}$ be a square matrix over a field $\Bbbk$, whose rows and columns are labelled (in the same order) by a set E. Let $X \subseteq E$. Without loss of generality (reordering if necessary), suppose that X labels the first $|X|$ rows and columns of the matrix. Then $\mathbf{A}$ has a block form

$$\mathbf{A} = \begin{array}{c} \\ X \\ E\backslash X \end{array} \begin{array}{c} \begin{array}{cc} X & E\backslash X \end{array} \\ \left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}\right] \end{array}.$$

Suppose that $\mathbf{A}[X]$ is non-singular. Then the *pivot* of $\mathbf{A}$ with respect to X is the matrix $\mathbf{A} * X$ with block form

$$\mathbf{A} * X = \begin{array}{c} \\ X \\ E\backslash X \end{array} \begin{array}{c} \begin{array}{cc} X & E\backslash X \end{array} \\ \left[\begin{array}{c|c} \alpha^{-1} & \alpha^{-1}\beta \\ \hline -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{array}\right] \end{array}.$$

Example 7.3 Working over GF(2), we have

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \end{array}, \qquad \text{and so} \qquad \mathbf{A} * \{1,2\} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}.$$

Bouchet, in [12], proved that pivoting in a matrix corresponds to twisting in a delta-matroid.

Theorem 7.4 *Let $\mathbf{A}$ be a symmetric or skew-symmetric matrix over a field $\Bbbk$, whose rows and columns are labelled (in the same order) by a set E. Let $X \subseteq E$, be such that $\mathbf{A}[X]$ is non-singular (or, equivalently, let X be a feasible set of $D(\mathbf{A})$). Then $\mathbf{A} * X$ is a symmetric or skew-symmetric matrix (of the same type as $\mathbf{A}$), and*

$$D(\mathbf{A} * X) = D(\mathbf{A}) * X. \tag{7.2}$$

Using (3.1), we can describe contraction $D(\mathbf{A})/e$ in terms of operations on $\mathbf{A}$ in the case when $\{e\}$ is a feasible set of $D(\mathbf{A})$:

$$D(\mathbf{A})/e = D((\mathbf{A} * e)[E\backslash e]), \qquad \text{when } \mathbf{A}[e] \neq [0]. \tag{7.3}$$

While (7.1) gives that for all $X \subseteq E$, $D(\mathbf{A})\backslash X = D(\mathbf{A}[E\backslash X])$, notice that (7.2) and (7.3) require that X is a feasible set of $D(\mathbf{A})$ (or equivalently, $\mathbf{A}[X]$ is non-singular). Of course, this is not surprising since delta-matroids from matrices are always normal, but the set of normal delta-matroids is not closed under twisting or contracting. What this does mean, however, is that care must be taken when representing delta-matroids by matrices, as we shall see presently.

A normal delta-matroid is *representable* if it can be obtained as the delta-matroid of a matrix. Every delta-matroid is a twist of a normal delta-matroid (just twist by any feasible set), and we say that a delta-matroid is *representable* if one of its twists is the delta-matroid of a matrix.

Definition 7.5 (Representable) Let $D = (E, \mathcal{F})$ be a delta-matroid. We say that D is *representable* over $\Bbbk$, if there exists some $X \subseteq E$ and a symmetric or skew-symmetric matrix $\mathbf{A}$ over a field $\Bbbk$ such that

$$D * X = D(\mathbf{A}).$$

We say that $\mathbf{A}$ is a matrix *representing* D.

Example 7.6 Let $D = (E, \mathcal{F})$ be the delta-matroid with $E = \{1, 2, 3, 4\}$ and

$$\mathcal{F} = \{\{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}.$$

Let

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then $D * \{3, 4\} = D(\mathbf{A}_1)$, and $D * \{1, 2, 3, 4\} = D(\mathbf{A}_2)$. Thus $\mathbf{A}_1$ and $\mathbf{A}_2$ are both representing matrices for D.

The definition of representability for delta-matroids requires a choice of a set X to make $D * X$ normal. In general, there are many such sets to choose from (since a necessary and sufficient condition is that X is a feasible set of D), and therefore a delta-matroid D will have many representing matrices. However, it follows readily from the transitivity of twisting and (7.2) that all representing matrices are pivots of one another.

Proposition 7.7 *Working over a fixed field, let $\mathbf{A}_1$ be representing matrix for a delta-matroid D. Then $\mathbf{A}_2$ is a representing matrix for D if and only if $\mathbf{A}_2$ is a pivot of $\mathbf{A}_1$.*

Bouchet and Duchamp proved in [20] that the class of representable delta-matroids is closed under taking minors.

Theorem 7.8 *Let D be a delta-matroid and D' be a minor of it. Then if D is representable by a (skew-)symmetric matrix over $\Bbbk$, so is D'.*

In what will follow, we will mostly focus on representations over the two element field GF(2). Such a representation is called a *binary representation.* Recall that our definition of skew-symmetric matrices requires that the diagonal elements are zero.

Definition 7.9 (Binary) A delta-matroid is *binary* if it is representable over GF(2).

Suppose that we have a delta-matroid $D = (E, \mathcal{F})$ and we know that $D = D(\mathbf{A})$ for some (skew-)symmetric matrix $\mathbf{A}$ over GF(2). Then we know that $\{v\} \in \mathcal{F}$ if and only if $\mathbf{A}[v] = [1]$. This determines the diagonal entries of $\mathbf{A}$. We also know that $\{u, v\} \in \mathcal{F}$ if and only if $\mathbf{A}[\{u, v\}]$ is (skew-)symmetric and non-singular, so, as we know the diagonal entries, the feasible sets of size two determine the off-diagonal entries of $\mathbf{A}$. Specifically, set the (u, v)-entry of $\mathbf{A}$ to be 1 if and only if $\{u\}, \{v\} \in \mathcal{F}$ but $\{u, v\} \notin \mathcal{F}$, or $\{u, v\} \in \mathcal{F}$ but $\{u\}$ and $\{v\}$ are not both in $\mathcal{F}$.

Thus we, over GF(2), when $D = D(\mathbf{A})$, its feasible sets of size at most two completely determine the matrix $\mathbf{A}$, and hence they determine D itself. This leads to the following result of Bouchet and Duchamp from [20].

Theorem 7.10 *Let $D = (E, \mathcal{F})$ be a normal set system (i.e., $\emptyset \in \mathcal{F}$). Then there is exactly one binary delta-matroid $D' = (E, \mathcal{F}')$ such that $\mathcal{F}_{\min + k} = \mathcal{F}'_{\min + k}$, for $k = 0, 1, 2$.*

Observe that the construction above gives a way to read off a representing matrix of a binary delta-matroid D: twist by any feasible set X so that $D * X$ is normal. Construct a matrix $\mathbf{A}$ following the above procedure. Then $D * X = D(\mathbf{A})$.

In [20] Bouchet and Duchamp used Theorem 7.10 to show that the minimal non-binary delta-matroids are of width at most four. Equipped with this bound, they obtained the following excluded-minor characterisation of the class of binary delta-matroids.

Theorem 7.11 (Bouchet and Duchamp [20]) *A delta-matroid is binary if and only if it has no minor isomorphic to one of the following delta-matroids. The delta-matroids on $\{1, 2, 3\}$ with collection of feasible sets*

1. $\{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$,
2. $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$,
3. $\{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$;

or the delta-matroids on $\{1, 2, 3, 4\}$ with collection of feasible sets

4. $\{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$,
5. $\{\emptyset, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4\}\}$.

A notable application of Theorem 7.11 is the recovery of Tutte's excluded-minor characterisation of binary matroids from [74]. Let D_5 denote the delta-matroid described in Item 5 of the theorem. Then $D_5 * \{1, 3\}$ is the only matroid that can be recovered as a twist of any of the delta-matroids in the theorem. The matroid $D_5 * \{1, 3\}$ is known as the *uniform matroid* $U_{2,4}$. Thus restricting the theorem to matroids (and technically using Theorem 7.16 to recover Tutte's form), gives Tutte's theorem.

Theorem 7.12 *A matroid is binary if and only if it has no minor isomorphic to* $U_{2,4}$.

It can be checked by either constructing the delta-matroids of all 1-vertex ribbon graphs on at most four edges, or by an appeal to the topology of ribbon graphs (along the lines of Exercise 7.14 below) that none of the delta-matroids in Theorem 7.11 arise from ribbon graphs. It follows that ribbon-graphic delta-matroids are binary, a result of Bouchet from [12] (where it was phrased in terms of Eulerian delta-matroids).

Theorem 7.13 *Every ribbon-graphic delta-matroid is binary.*

Exercise 7.14 *Consider the delta-matroid in Item 2 of Theorem 7.11. By considering what properties the edges of a 1-vertex ribbon graph must have for the feasible sets to form quasi-trees, give an argument that shows that the delta-matroid cannot come from a ribbon graph.*

Knowing that $D(\mathbb{G})$ is binary, it is straightforward to construct a binary representing matrix for it. For this we say that two loops in a ribbon graph that share a vertex are *interlaced* if their ends are met in an alternating order when travelling round the vertex boundary.

Given a ribbon graph $\mathbb{G} = (V, E)$, construct a representing matrix $\mathbf{A}$ as follows. Choose some spanning quasi-tree of $\mathbb{G}$ (for example a maximal spanning forest). Let X be its edge set. Then each component of a the partial dual $\mathbb{G}^X$ has exactly one vertex. Let the (e, e)-entry of $\mathbf{A}$ be 1 if and only if e is non-orientable in $\mathbb{G}^X$. Let both the (e, f)-entry and (f, e)-entry be 1 if e and f are interlaced in $\mathbb{G}^X$, and 0 otherwise. Its easily seen that the feasible sets of size at most 2 in $D(\mathbb{G}^X)$ and $D(\mathbf{A})$ coincide. By Theorems 7.10 and 5.18, it follows that $D(\mathbb{G}) * X = D(\mathbf{A})$.

Example 7.15 Consider Example 5.12 which gives $D(\mathbb{G})$ for the ribbon graph $\mathbb{G}$ in Figure 15a. The set $\{3, 4\}$ is feasible, and $\mathbb{G}^{\{3,4\}}$ is the ribbon graph shown in Figure 11a. The edge 3 is non-orientable. The pairs of interlaced edges are 12, 13, 14, and 23. This gives the matrix

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

$D(\mathbf{A}) * \{3, 4\}$ was computed in Example 7.6, and we see this is exactly $D(\mathbb{G})$ from Example 5.12. Thus $D(\mathbb{G}) = D(\mathbf{A}) * \{3, 4\}$, and so $D(\mathbb{G}) * \{3, 4\} = D(\mathbb{G}^{\{3,4\}}) = D(\mathbf{A})$.

We have previously seen that ribbon graph results can be used to conjecture results about delta-matroids. Sometimes, the analogues of ribbon graph results hold for binary delta-matroids, but not for delta-matroids in general. (An example of this is in [55], where a canonical form for surfaces with boundary was shown to hold on the level of binary delta-matroids, but not in general.) A similar comment holds for the connection between matroids and graphs via cycle matroids.

We close this section with a remark on matroid representability. Every matroid is a delta-matroid, and so Definition 7.5 provides a definition of representability for matroids. A reader familiar with matroid theory might be worried by the fact that this definition of representability is *not* the standard definition of representability from matroid theory. In matroid theory, a matroid M is *representable* over a field $\Bbbk$ if M equals the vector matroid (see Section 2.3.1) of some matrix over $\Bbbk$. Bouchet proved in [12] that the two notions of matroid representability agree.

Theorem 7.16 *A matroid is representable over* $\Bbbk$ *in the sense of matroid theory if and only if it is representable over* $\Bbbk$ *in the sense of delta-matroid theory by a skew-symmetric matrix.*

A consequence of this is that since not all matroids are representable in the sense of matroid theory, not all delta-matroids are representable.

8 Simple graphs, pivoting and delta-matroids

This section ties the properties of binary delta-matroids to those of simple graphs and looped simple graphs. It is easy to associate a simple graph with an even binary delta-matroid — consider a representing matrix for a delta-matroid D as being the adjacency matrix of a graph, and associate this graph with the delta-matroid. This construction, however, depends upon a choice of representing matrix for D, and different choices can result in different graphs. We need to understand how the resulting graphs are related. For this we need to consider pivots and related graph operations.

We will see that even binary delta-matroids considered up to twists can be identified with simple graphs considered up to edge pivots. Similarly, binary delta-matroids considered up to twists can be identified with looped simple graphs considered up to elementary pivots. This was first written down by Geelen in [41] (see also [40]) although he notes that the graph-theoretical point-of-view was used by both Bouchet and Cunningham in their discussions with him at the time of that paper.

8.1 Simple graphs, pivots and adjacency matrices

Pivoting is a graph operation related to Kotzig's transformations on Eulerian circuits [45]. It was introduced by Bouchet in the context of isotropic systems [11] and multimatroids [18], and rediscovered by Arratia, Bollobás, and Sorkin when they introduced the interlace polynomial in [3, 4].

Definition 8.1 (Pivoting for graphs) Let G be a simple graph, and uv be an edge. Partition the vertices other than u and v into four classes: (1) vertices adjacent to u but not v, (2) vertices adjacent to v but not u, (3) vertices adjacent to both u and v, (4) vertices adjacent to neither u nor v.

The *pivot* of the edge uv is the graph, $G\wedge uv$, constructed from G as follows. For any vertex pair x, y where x is in one of the classes (1)–(3), and y is in a different class (1)–(3), "toggle" the pair xy in the edge set (so if xy was an edge, make it a non-edge; and if xy was a non-edge, make it an edge). Finally, switch the names of the vertices u and v. See Figure 22.

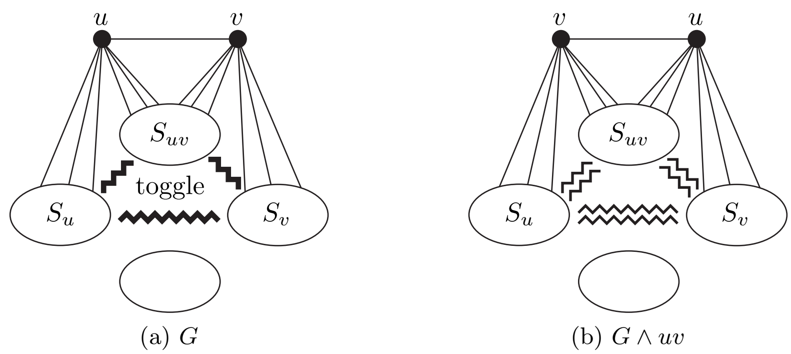


Figure 22: Pivoting (edges between the three sets, S_u, S_v, and $S_{u,v}$, are 'toggled', and the names of u and v are switched)

Definition 8.2 (Pivot-minors for graphs) A *pivot-minor* of a graph is any graph that can be obtained from it by edge pivots and vertex deletions.

A related operation is local complementation, first studied by Kotzig in [45]. We use $N_G(v)$ to denote the set of neighbours of a vertex v in the graph G. Note that $v \notin N_G(v)$.

Definition 8.3 (Local complementation) Let G be simple graph. Then the $G{*}v$ denotes the graph obtained from v by *local complementation* at v. The graph $G{*}v$ is obtained from G by replacing the induced subgraph on $N_G(v)$ with its complement graph. That is, $G * v$ is obtained from G by 'toggling' the edges and non-edges at vertices in $N_G(v)$.

Definition 8.4 (vertex-minor) A *vertex-minor* of a graph is any graph that can be obtained from it by local complementations and vertex deletions.

Exercise 8.5 *Let uv be an edge of a simple graph G. Verify that, after switching the names of vertices u and v, $G \wedge uv = G * u * v * u = G * v * u * v$. (This is due to Bouchet [11].)*

Recall the adjacency matrix over GF(2) of a simple graph G is the matrix $\mathbf{A}_G$ whose rows and columns correspond to the vertices of G, and whose (u, v)-entry is 1 if the corresponding vertices u and v are adjacent in G, and is 0 otherwise. The diagonal entries are 0.

Definition 8.6 (Fundamental graph) Let $\mathbf{A}$ be a skew-symmetric matrix over GF(2). A simple graph G is said to be the *fundamental graph* of $\mathbf{A}$ if $\mathbf{A}$ is its adjacency matrix. The graph G is said to be a *fundamental graph* of an even binary delta-matroid D if it is the fundamental graph of some representing matrix of D.

Over GF(2), every simple graph is the fundamental graph of some skew-symmetric matrix, and every skew-symmetric matrix has a fundamental graph, giving a correspondence:

$$\{\text{skew-symmetric matrices over GF(2)}\} \xleftrightarrow{\text{1-1}} \{\text{simple graphs}\}.$$

It is not hard to determine the effect that deleting a vertex and pivoting an edge in a simple graph G has on its adjacency matrix $\mathbf{A}_G$. For a vertex v,

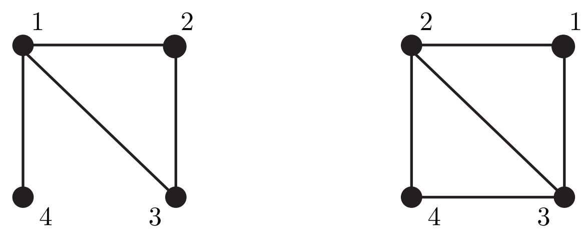
$$\mathbf{A}_{G\backslash v} = \mathbf{A}_G[V\backslash v]. \tag{8.1}$$

Recall that $\mathbf{A}_G[V\backslash v]$ is the matrix obtained from $\mathbf{A}_G$ by deleting the row and column corresponding to v.

For an edge uv of G,

$$\mathbf{A}_G * \{u,v\} = \mathbf{A}_{G\wedge uv}, \tag{8.2}$$

where $\mathbf{A}_G * \{u,v\}$ denotes the pivot of a matrix from Definition 7.2.



(a) G (b) $G \wedge 12$

Figure 23: A simple graph G and its pivot $G \wedge 12$

Example 8.7 Consider the graph G in Figure 23a. Its pivot $G \wedge 12$ is shown in Figure 23b. The adjacency matrices of these graphs are given below, where it can be seen that $\mathbf{A}_{G\wedge 12} = \mathbf{A}_G * \{1,2\}$.

$$\mathbf{A}_G = \begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \text{and} \quad \mathbf{A}_{G\wedge 12} = \begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Exercise 8.8 *Verify Equations* (8.1) *and* (8.2).

8.2 Simple graphs and even binary delta-matroid

Let $G = (V,E)$ be a simple graph and $\mathbf{A}_G$ be its adjacency matrix over GF(2). We can form a delta-matroid $D(\mathbf{A}_G)$ from the adjacency matrix. Thus we have a way to associate a delta-matroid $D(\mathbf{A}_G)$ with any simple graph G. Furthermore, since G is simple, $\mathbf{A}_G$ is has zeros on its diagonal, and it follows that every feasible set of $D(\mathbf{A}_G)$ is of even size. Thus $D(\mathbf{A}_G)$ is an even binary delta-matroid.

On the other hand, suppose that $D = (V, \mathcal{F})$ is an even binary delta-matroid. Then we know, for some symmetric or skew-symmetric matrix $\mathbf{A}$ over GF(2) and some $X \subseteq V$, that $D * X = D(\mathbf{A})$. Since $D(\mathbf{A})$ is even and normal, $\mathbf{A}$ must have zeros on its diagonals, and so is the adjacency matrix of some simple graph $G_{D,X}$. This graph is the fundamental graph of the matrix $\mathbf{A}$, and is a fundamental graph of D.

It is straightforward to construct a fundamental graph $G_{D,X}$ of an even binary delta-matroid $D = (V, \mathcal{F})$ directly. Choose a feasible set $X \in \mathcal{F}$. Then $D * X$ is normal. Take V to be the vertex set of $G_{D,X}$, and add an edge uv if and only if $\{u, v\}$ is a feasible set of $D * X$ (or equivalently, if and only if $\{u, v\} \triangle X$ is a feasible set of D).

Example 8.9 Two simple graphs that are pivots of each other are shown in Figure 23, and their adjacency matrices given in Example 8.7. $D(\mathbf{A}_G) = (E, \mathcal{F})$ and $D(\mathbf{A}_{G\wedge 12}) = (E, \mathcal{F}')$ where $E = \{1, 2, 3, 4\}$,

$$\mathcal{F} = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3, 4\}\},$$

and

$$\mathcal{F}' = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

It is readily checked that $D(\mathbf{A}_G) * \{1, 2\} = D(\mathbf{A}_{G\wedge 12})$.

Thus we have a way to associate an even binary delta-matroid with a simple graph, and a way to associate a simple graph with an even binary delta-matroid. By making use of Theorem 7.10, we see this sets up a 1-1 correspondence between simple graphs and *normal* even binary delta-matroids. However, it does not set up a 1-1 correspondence between simple graphs and even binary delta-matroids since, in general, a delta-matroid will have many different fundamental graphs. (Because there is of the choice of which $X \subseteq V$ we use to make $D * X$ normal). However, all of the fundamental graphs of the delta-matroid are related through pivots.

Theorem 8.10 *Let D be an even binary delta-matroid, and G be a fundamental graph of D. Then a graph H is also a fundamental graph of D if and only if H can be obtained from G by a sequence of edge pivots.*

Before giving a proof of this lemma, let us say a few words about it. Combining (7.2) and (8.2) gives that, when G is simple with an edge uv, and $\mathbf{A}_G$ is its adjacency matrix over GF(2), then

$$D(\mathbf{A}_{G\wedge uv}) = D(\mathbf{A}_G * \{u, v\}) = D(\mathbf{A}_G) * \{u, v\}. \tag{8.3}$$

As G and H are fundamental graphs of the same delta-matroid, we know that $D(\mathbf{A}_H) = D(\mathbf{A}_G) * X$, for some set X. Thus, in light of (8.3), we need to write $D(\mathbf{A}_G) * X$ as $D(\mathbf{A}_G) * \{u_1, v_1\} * \cdots * \{u_k, v_k\}$, where each pair $\{u_k, v_k\}$ is feasible in the relevant delta-matroid, so that the pivots of the fundamental graph are pivots on an edge.

Proof First suppose H can be obtained from G by a series of edge pivots, $H = G \wedge u_1v_1 \wedge \cdots \wedge u_kv_k$. Then, by (8.3), $D(\mathbf{A}_H) = D(\mathbf{A}_G) * \{u_1, v_1\} * \cdots * \{u_k, v_k\} = D(\mathbf{A}_G) * \{u_1, v_1, \cdots, u_k, v_k\}$. Thus the matrix $\mathbf{A}_H$ also represents D, so H is a fundamental graph of D.

Conversely, suppose that H is a fundamental graph of D. Then $D(\mathbf{A}_G)$ and $D(\mathbf{A}_H)$ are both twists of D, and so $D(\mathbf{A}_G) * X = D(\mathbf{A}_H)$, for some set X. Moreover, since these two delta-matroids are normal, X must be a feasible set of $D(\mathbf{A}_G)$, and be of even size (since D is an even delta-matroid). If $X = \emptyset$ we are done, so suppose this is not the case. Applying the Symmetric Exchange Axiom to $X \triangle \emptyset$ gives that there is some $\{u_1, v_1\} \subseteq X$ such that $\{u_1, v_1\}$ is a feasible set of $D(\mathbf{A}_G)$. It follows that $D(\mathbf{A}_G) * X = (D(\mathbf{A}_G) * \{u_1, v_1\}) * (X \backslash \{u_1, v_1\})$, so $D(\mathbf{A}_G) * \{u_1, v_1\}$ is normal with $\{u_1, v_1\}$ feasible in $D(\mathbf{A}_G)$, $X \backslash \{u_1, v_1\}$ feasible in $D(\mathbf{A}_G) * \{u_1, v_1\}$, and $(D(\mathbf{A}_G) * \{u_1, v_1\}) * (X \backslash \{u_1, v_1\}) = D(\mathbf{A}_H)$. We can repeat this argument to write $D(\mathbf{A}_H) = D(\mathbf{A}_G) * \{u_1, v_1\} * \cdots * \{u_k, v_k\}$, and, so by (8.3), $D(\mathbf{A}_H) = G \wedge u_1v_1 \wedge \cdots \wedge u_kv_k$. As each $\{u_i, v_i\}$ is feasible in the relevant delta-matroid, this is a sequence of edge pivots, as required. □

Lemma 8.10 identifies even binary delta-matroids with equivalence classes of simple graphs under pivoting:

$$\{\text{even binary delta-matroids up to twists}\} \overset{\text{1-1}}{\longleftrightarrow} \{\text{simple graphs up to edge pivots}\}. \tag{8.4}$$

Theorem 8.11 *Let D be an even binary delta-matroid, and G be a fundamental graph of D. Then a graph H is a pivot-minor of G if and only if it is a fundamental graph of a minor of D.*

Proof A pivot-minor of G is obtained by pivoting at edges and deleting vertices. By Theorem 8.10, for an edge uv of G, $G \wedge uv$ is also a fundamental graph of D. For a vertex v of G, the adjacency matrix $\mathbf{A}_{G\backslash v}$ of $G\backslash v$ equals $\mathbf{A}_G[V\backslash v]$, and by (7.1), $D(\mathbf{A}_{G\backslash v}) = D(\mathbf{A}_G[V\backslash v]) = D(\mathbf{A}_G)\backslash v$. Thus if G is a fundamental graph of D, then $D * X = D(\mathbf{A}_G)$, for some $X \subseteq V$, and so $G\backslash v$ is a fundamental graph of $(D * X)\backslash v$, which is a minor of D. It follows that if H is a pivot-minor of G, then it is a fundamental graph of a minor of D.

Conversely, since any two fundamental graphs are related by pivots, it is enough to show that there are fundamental graphs of $D * v$ and $D\backslash v$ that are pivot-minors of a fundamental graph of D. For $D * v$, let X be a feasible set of D. Then $D * X$ is normal, and equals $(D * v) * (X \triangle v)$. Reading fundamental graphs from these gives that D and $D * v$ have a common fundamental graph, and hence, by Theorem 8.10, all their fundamental graphs are pivot-minors. For $D\backslash v$, suppose that there is some feasible set X of D that does not contain v (i.e., v is not a coloop). Then $D * X$ and $(D * X)\backslash v$ and are both normal. If G is the fundamental graph read from $D * X$, then $G\backslash v$ is the fundamental graph read from $(D * X)\backslash v$. But $(D * X)\backslash v = (D\backslash v) * X$, since $v \notin X$, so $G\backslash v$ is a fundamental graph of $D\backslash v$. On the other hand, if v is in every feasible set of D (i.e., v is a coloop), and X is a feasible set of D, then $D\backslash v$ and $D * v$ have identical feasible sets. Thus $D * X = (D * v) * (X\backslash v)$ and $(D\backslash v) * (X\backslash v)$ are normal delta-matroids with identical feasible sets and differ only in that v is in the ground set of one but not the other. Reading fundamental graphs from these delta-matroids give that a fundamental graph of $D\backslash v$ can be obtained by deleting v from a fundamental graph of D. This completes the proof. □

Theorems 8.10 and 8.11 gives that binary delta-matroids and their minors correspond to simple graphs and their pivot-minors:

$$\{\text{minors of even binary delta-matroids}\} \leftrightarrow \{\text{pivot-minors of simple graphs}\}. \quad (8.5)$$

Thus results about pivot-minors can be translated into results about delta-matroids.

8.3 Looped simple graphs and binary delta-matroids

Equation (8.4) identified even binary delta-matroids and simple graphs. What if the delta-matroid is not even? To answer this we need to consider looped simple graphs.

Recall that a *looped simple graph* is a graph obtained from a simple graph by adding a loop to some of its vertices. Each vertex has either exactly one loop or no loops.

The following definition provides versions of local complementation and pivots for looped simple graphs. For the definition it is convenient to think of a looped simple graph G as a graft. A *graft* is a pair, (H,T), consisting of a graph H together with a subset T of its vertices. (Grafts will be the topic of Section 10.) A looped simple graph G is then exactly a graft (G_s,T) where G_s is the simple graph obtained from G by deleting all of its loops, and T is the set of vertices of G with loops.

Definition 8.12 (Elementary pivots) Let G be a looped simple graph. Consider G as a graft (G_s,T). Then *local complementation at the looped vertex* v is defined as the operation

$$(G_s,T) \mapsto (G_s * v, T \bigtriangleup N_G(v)), \qquad \text{where } v \in T.$$

(So form the local complement of the underlying simple graph, then 'toggle' the loops and non-loops on the neighbours of v.) We use $G * v$ to denote the looped simple graph resulting from local complementation of G at v,

Pivoting an edge between non-looped vertices is defined as the operation

$$(G_s,T) \mapsto (G_s \wedge uv, T), \qquad \text{where } uv \in E(G_s), u,v \notin T \text{ and } u \neq v.$$

(So form the edge pivot on the underlying simple graph. Do not change the loops.) We use $G \wedge uv$ to denote the looped simple graph resulting from pivoting uv in G.

These two operations on looped simple graphs are collectively called *elementary pivots.*

It is worth emphasising that elementary pivots only act on looped vertices, and on edges incident to two loopless vertices.

Recall from Definition 2.10 that the *adjacency matrix* $\mathbf{A}_G$ of a looped simple graph G is the matrix over GF(2) whose (u,v)-entry is 1 if and only if uv is an edge of G. In particular it has diagonal entry 1 if and only if the corresponding vertex has a loop. Every symmetric matrix over GF(2) can be written as $\mathbf{A}_G$ for some looped simple graph G, giving

$$\{\text{symmetric matrices over GF(2)}\} \overset{\text{1-1}}{\longleftrightarrow} \{\text{looped simple graphs}\}.$$

Versions of (8.1) and (8.2) hold for looped simple graphs. For a vertex v,

$$\mathbf{A}_G[V\backslash v] = \mathbf{A}_{G\backslash v}, \tag{8.6}$$

and if v has a loop,

$$\mathbf{A}_G * \{v\} = \mathbf{A}_{G*v}. \tag{8.7}$$

For an edge uv of G between two loopless vertices,

$$\mathbf{A}_G * \{u, v\} = \mathbf{A}_{G\wedge uv}. \tag{8.8}$$

Exercise 8.13 *Verify Equations* (8.6)–(8.8).

Passing to delta-matroids, and using (7.1), (7.2), and (8.6)–(8.8) gives:

$$D(\mathbf{A}_G)\backslash v = D(\mathbf{A}_{G\backslash v}); \tag{8.9}$$

when $\{v\}$ is feasible in $D(\mathbf{A}_G)$,

$$D(\mathbf{A}_G) * \{v\} = D(\mathbf{A}_{G*v}); \tag{8.10}$$

and when $\{u, v\}$, but not $\{u\}$ nor $\{v\}$, is feasible in $D(\mathbf{A}_G)$,

$$D(\mathbf{A}_G) * \{u, v\} = D(\mathbf{A}_{G\wedge uv}). \tag{8.11}$$

Example 8.14 Consider the looped simple graph G in Figure 24a. Its pivot $G \wedge 12$ is shown in Figure 24b. The adjacency matrices of these graphs are

$$\mathbf{A}_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad \text{and so} \qquad \mathbf{A}_{G\wedge 12} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

It was verified in Example 7.3 that $\mathbf{A}_{G\wedge 12} = \mathbf{A}_G * \{1, 2\}$. We have $D(\mathbf{A}_G) = (E, \mathcal{F})$ and $D(\mathbf{A}_{G\wedge 12}) = (E, \mathcal{F}')$ where $E = \{1, 2, 3, 4\}$,

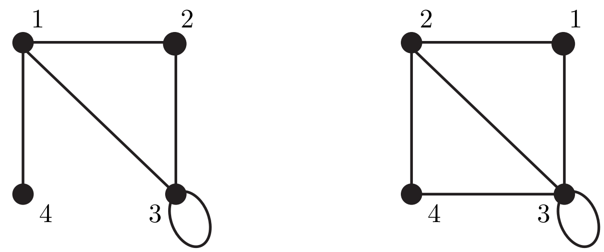
$$\mathcal{F} = \{\emptyset, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\},$$

and

$$\mathcal{F}' = \{\emptyset, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\},$$

from which we can verify that $D(\mathbf{A}_G) * \{1, 2\} = D(\mathbf{A}_{G\wedge 12})$.

Lemma 8.15 *Let D and D' be normal binary delta-matroids on V. Then $D' = D * X$, for some $X \subseteq V$, if and only if there are looped simple graphs G and G' such that $D = D(\mathbf{A}_G)$, $D' = D(\mathbf{A}_{G'})$ and G' can be obtained from G by a sequence of elementary pivots.*



(a) G (b) $G \wedge 12$

Figure 24: A simple graph G and its pivot $G \wedge 12$

Proof One direction follows from (8.10) and (8.11). For the other direction, suppose D and D' are normal binary delta-matroids and $D' = D * X$. Since D' is normal, $\emptyset, X \in \mathcal{F}(D)$. In any delta-matroid D'', if $\emptyset, Y \in \mathcal{F}(D'')$, the symmetric exchange axiom gives that for, each $y \in Y$, either $\{y\} \in \mathcal{F}(D'')$ or $\{y, y'\} \in \mathcal{F}(D'')$, for some $y' \in Y$ (where $\{y'\}$ may or may not be in $\mathcal{F}(D'')$). This means that for any nonempty feasible set Y we can find some $\{y\} \subseteq Y$ that is feasible, or some $\{y, y'\} \subseteq Y$ that is feasible but where neither $\{y\}$ or $\{y'\}$ is. Thus, since D is binary, for some looped simple graph G we can write

$$D' = D * X = D(\mathbf{A}_G) * X = D(\mathbf{A}_G) * \{x_1\} * \cdots * \{x_i\} * \{x_{i+1}, x'_{i+1}\} * \cdots * \{x_k, x'_k\},$$

where each $\{x_j\}$ is feasible in the relevant delta-matroid, or $\{x_j, x'_j\}$ is but neither $\{x_j\}$ or $\{x'_j\}$ are. It follows from (8.10) and (8.11) that $D' = D(\mathbf{A}_{G'})$ where G' is obtained from G by a sequence of elementary pivots. □

Theorem 8.16 *Let $D = (E, \mathcal{F})$ be a binary delta-matroid, and G and H be looped simple graphs. Then $D * X = D(\mathbf{A}_G)$ and $D * Y = D(\mathbf{A}_H)$, for some $X, Y \subseteq E$, if and only if H can be obtained from G by a sequence of elementary pivots.*

Proof If $D * X = D(\mathbf{A}_G)$ and $D * Y = D(\mathbf{A}_H)$, by Lemma 8.15, G and H are related by elementary pivots.

Conversely, if H can be obtained from G by elementary pivots, then, by Lemma 8.15, $D(\mathbf{A}_G) * Y = D(\mathbf{A}_H)$, for some Y. It follows that for some X, $D * (X \triangle Y) = D(\mathbf{A}_H)$. □

Thus we have show a correspondence between Binary delta-matroids and looped simple graphs:

$$\left\{ \begin{array}{c} \text{Binary delta-matroids} \\ \text{up to twists} \end{array} \right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{looped simple graphs} \\ \text{up to elementary pivots} \end{array} \right\}.$$

Note that the correspondence in (8.4) between simple graphs and even binary delta-matroids can be deduced from this since G has a loop if and only if $D(\mathbf{A}_G)$ is an odd delta-matroid.

We say that a looped simple graph H is an *elementary pivot-minor* of G if it can be obtained from G through a sequence of elementary pivots and vertex deletions.

By adapting the proof of Theorem 8.11, it can be shown that minors of binary delta-matroids correspond to elementary pivot-minors of looped simple graphs:

$$\{\text{minors of binary delta-matroids}\} \longleftrightarrow \left\{ \begin{array}{c} \text{elementary pivot-minors} \\ \text{of looped simple graphs} \end{array} \right\}. \qquad (8.12)$$

Stated as a theorem, this is:

Theorem 8.17 *Let D and D' be a binary delta-matroids on E such that $D * X = D(\mathbf{A}_G)$ and $D' * Y = D(\mathbf{A}_H)$, for some $X, Y \subseteq E$. Then a graph H is an elementary pivot-minor of G if and only if D' a minor of D.*

9 Circle graphs, and ribbon-graphic and Eulerian delta-matroids

We have just seen a connection between simple graphs and binary delta-matroids. The case when the graph is a circle graph turns out to be of particular interest in delta-matroid theory as it is related to ribbon-graphic delta-matroids. From Section 6 we know that Eulerian delta-matroids are the delta-matroids of ribbon graphs meaning that we can phrase the ribbon-graphic delta-matroid results in this section in terms of Eulerian delta-matroids.

A *chord diagram* consists of a circle in the plane and a number line segments, called *chords*, whose end-points lie on the circle. The end-points of chords should all be distinct. The *intersection graph* of a chord diagram is the graph $G = (V, E)$ where V is the set of chords, and where $uv \in E$ if and only if the chords u and v intersect. A graph is a *circle graph* if it is the intersection graph of a chord diagram. A *looped circle graph* is a looped graph obtained by adding loops to a circle graph. Figure 25 shows a circle graph and a corresponding chord diagram.

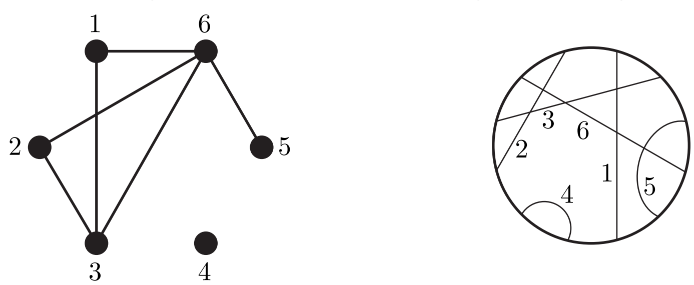


Figure 25: A circle graph and a corresponding chord diagram

Circle graphs are closed under vertex deletion, local-complementation, and edge-pivots. Thus they are closed under taking vertex-minors and pivot-minors. The class of circle graphs has excluded-minor characterisations with respect to both types of minor. Bouchet, in [15], gave the following excluded-vertex-minor characterisation of circle graphs.

Theorem 9.1 *A graph is a circle graph if and only if it has no vertex-minor isomorphic to any of the graphs shown in Figure 26.*

Building upon Bouchet's characterisation, Geelen and Oum, in [40], gave an excluded-pivot-minor characterisation of circle graphs.

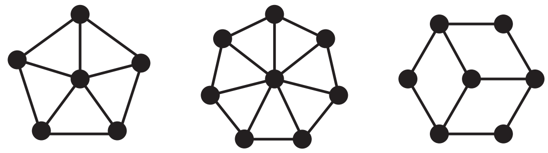


Figure 26: Excluded vertex-minors for circle graphs

Theorem 9.2 *A graph is a circle graph if and only if it has no pivot-minor isomorphic to any of the graphs shown in Figure 27.*

As pivot-minors of simple graphs correspond to minors of even binary delta-matroids, by (8.5), it is reasonable to expect this theorem to find an application to delta-matroids. This is what we find next.

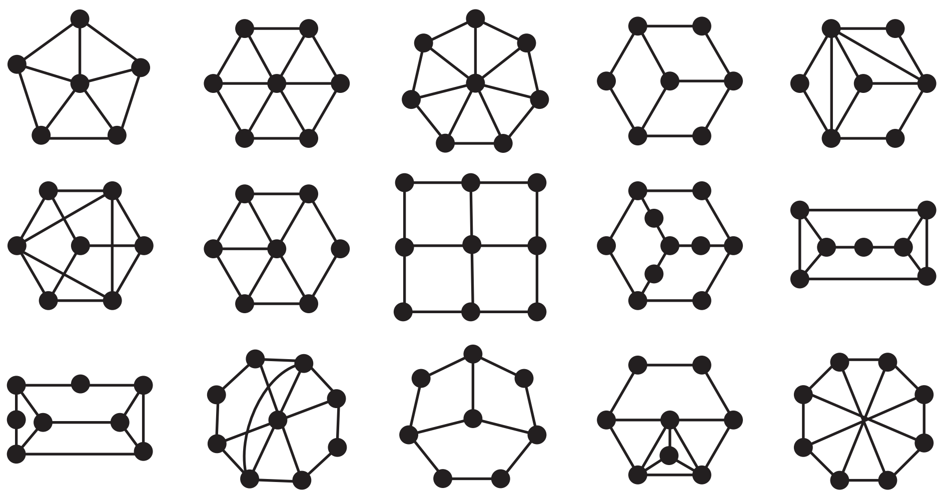


Figure 27: Excluded pivot-minors for circle graphs

There is a natural way to associate a chord diagram with an orientable 1-vertex ribbon graph $\mathbb{G}$: take the boundary of the vertex as the circle, and place a chord between the two ends of each edge of $\mathbb{G}$. By forming the intersection graph of this chord diagram, we have a natural way to associate a circle graph with a ribbon graph. Moreover, this circle graph is a fundamental graph of $D(\mathbb{G})$, and so by (8.5) we obtain the an excluded-minor characterisation of even ribbon-graphic delta-matroids from [40].

Theorem 9.3 *A delta-matroid is even ribbon-graphic if and only if has no minor isomorphic to $D(\mathbf{A}_G)$ where G is one of the graphs shown in Figure 27, or to one of the excluded minors for binary delta-matroids given in Theorem 7.11.*

More generally, by relating the pivot minors of the graph G to circle graphs, in the case where $D = D(\mathbf{A}_G)$ is ribbon-graphic, Geelen and Oum, in were able to find a set of 171 excluded minors for the class of ribbon-graphic delta-matroids.

Theorem 9.4 *A delta-matroid is ribbon-graphic if and only if has no minor isomorphic to $D(\mathbf{A}_G)$ where G is one of the looped simple graphs shown in Figure 28, or to one of the excluded minors for binary delta-matroids given in Theorem 7.11.*

10 Grafts and graphic delta-matroids

A *graft* is a pair (G, T) consisting of a graph G together with a subset T of its vertices. Vertices in T are called *T-vertices.* A graft is shown in Figure 6a. Grafts, introduced by Seymour in [68], are useful in matroid theory. For example, they can be used to give a characterisation of graphic matroids [69]. We do not pursue this classical matroid direction here. Instead we consider a method due to Oum [59] for obtaining a delta-matroid (that need not be a matroid) from a graft, and consider the interaction between grafts, their delta-matroids, and rank-width.

Delta-matroids that arise from grafts are called *graphic* delta-matroids. While circle graphs and bipartite graphs are closed under pivot-minors, line graphs are not. Via their fundamental graphs, the closure of circle and bipartite graphs under pivot-minors corresponds to the closure of even Eulerian delta-matroids, and twists of matroids (see Corollary 11.7) under taking minors. That line graphs are not closed under pivots means that the class of delta-matroids whose fundamental graphs are line graphs is not minor-closed. Graphic delta-matroids were introduced to get around this. They are defined in such a way that pivot-minors of line graphs are exactly fundamental graphs of graphic delta-matroids.

Except where otherwise stated, all of the results in this section are due to Oum and from [59].

Definition 10.1 (T-spanning subgraph) Let (G, T) be a graft. A subgraph H of G is said to be *T-spanning* if $V(H) = V(G)$, and each component of the graft (H, T) has either:

1. an odd number of T-vertices, or
2. spans a component of G that has no T-vertex.

Definition 10.2 (Graphic delta-matroid) Let (G, T) be a graft. Let $D(G, T)$ denote the set system $(E(G), \mathcal{F})$, where, for each $A \subseteq E(G)$

$$A \in \mathcal{F} \iff (V, A) \text{ a } T\text{-spanning forest of } G.$$

We call $D(G, T)$ the *delta-matroid of the graft* (G, T), and a delta-matroid D is said to be *graphic* if there exists a graft (G, T) such that D is a twist of $D(G, T)$.

Oum showed that $D(G, T)$ is indeed a delta-matroid.

Theorem 10.3 *Let (G, T) be a graft. Then the set system $D(G, T)$ defined in Definition 10.2 is a delta-matroid.*

Example 10.4 Example 2.15 shows the delta-matroid of a graft. The delta-matroid on $\{1, 2, 3, 4, 5\}$ with feasible sets

$$\mathcal{F} = \{\emptyset, \{3, 4\}, \{1, 2\}, \{1, 2, 3, 4\}, \{1, 4, \}, \{2, 4\}\}$$

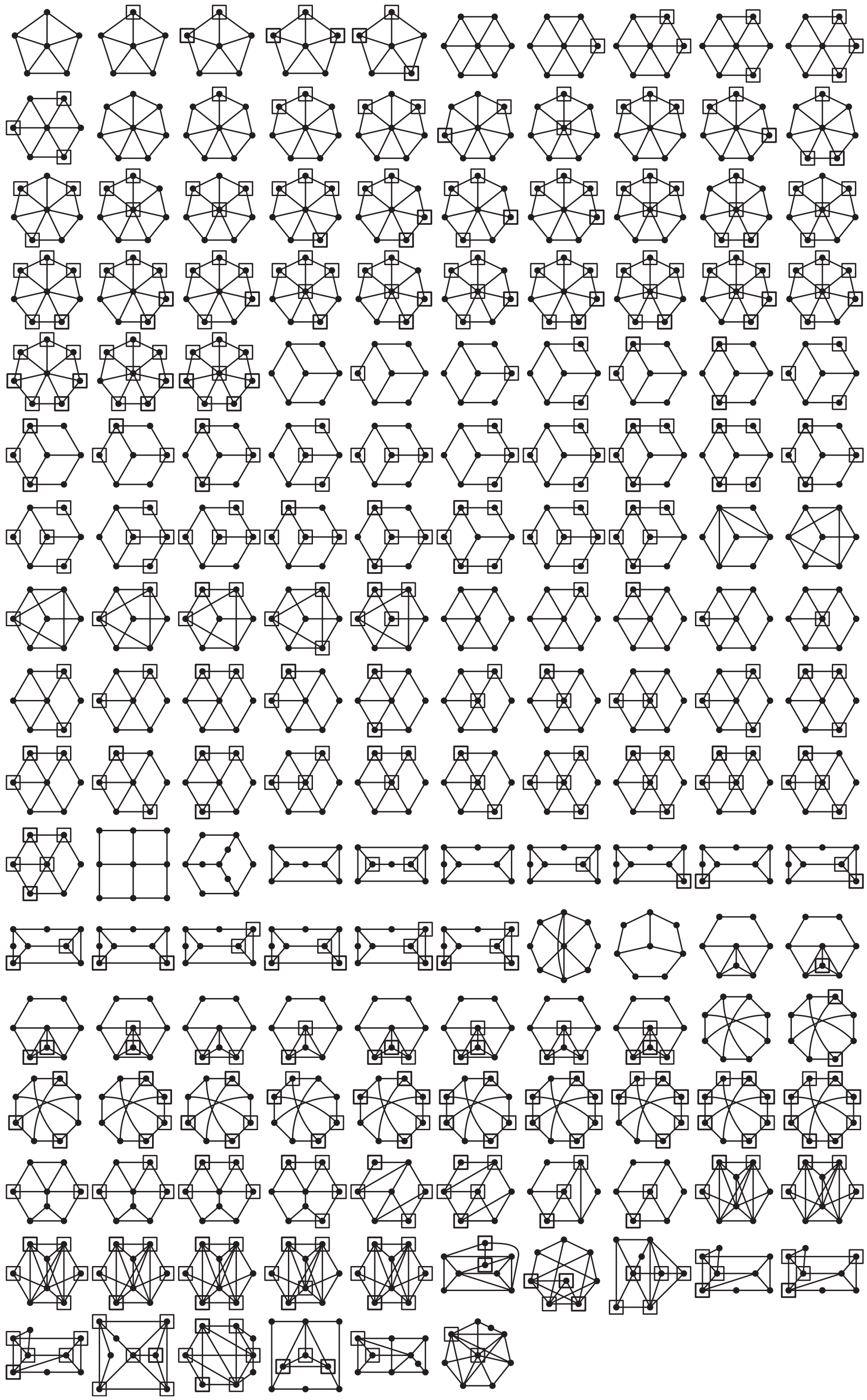


Figure 28: Non-binary excluded minors. Looped vertices are shown in squares. (Image from [40].)

is graphic since it is $D * \{3, 5\}$ where D is the delta-matroid from Example 2.15.

Exercise 10.5 *Consider a connected graph (G, T). Show that if $|T| \leq 1$ then $D(G, T)$ is the cycle matroid of G. Show that if $|T| = 2$ and G' is the graph obtained by identifying the two T-vertices of G, then $D(G, T) = D(G', \emptyset)$. Conclude that if $|T| \leq 2$ then $D(G, T)$ is a graphic matroid. (These results are from [58, 59].)*

Let (G, T) be a graft, e be one of its edges and v one of its vertices. Edge and vertex *deletion* for grafts is defined in the obvious way: $(G, T)\backslash e$ is defined to be the graft $(G\backslash e, T)$, and $(G, T)\backslash v$ is defined to be the graft $(G\backslash v, T\backslash v)$. *Contraction* is defined by setting $(G, T)/e$ as the graft $(G/e, T')$ where, if w is the vertex created by contracting an edge $e = uv$,

$$T' := \begin{cases} (T\backslash\{u, v\}) \cup \{w\} & \text{if exactly one of } u \text{ or } v \text{ is in } T, \\ T\backslash\{u, v\} & \text{otherwise.} \end{cases}$$

See Figure 29.

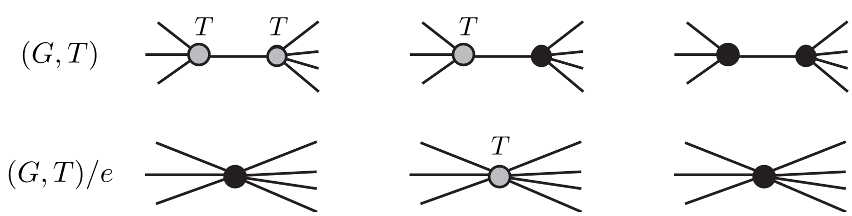
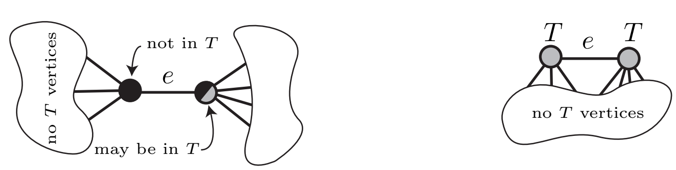


Figure 29: Contracting an edge in a graft

Exercise 10.6 *Let (G, T) be a graft, with a vertex u that is not a T-vertex. Let (G', T') be the graft obtained from (G, T) by adding a vertex v and an edge $e = uv$ to G, then making both u and v into T-vertices. Prove that $D(G, T) = D(G', T')/e$. Deduce that, for any graft (G, T), $D(G, T)$ can be obtained as a minor of $D(G', V(G'))$ for some graft $(G', V(G'))$. (Note this can also be deduced from Theorem 10.8 below.)*

While graphs have bridges and loops, grafts have T-bridges, T-tunnels, and loops. An edge e of a graft (G, T) is a *T-bridge* if $(G, T)\backslash e$ has more components without T-vertices than (G, T). See Figure 30a. An edge $e = uv$ is said to be a *T-tunnel* if u and v are the only T-vertices in the component of G containing them. See Figure 30b.



(a) A T-bridge, e (b) A T-tunnel, e

Figure 30: T-bridges and T-tunnels

Just as loops and coloops in cycle matroids correspond to loops and bridges in graphs, loops and coloops in delta-matroids correspond to their analogues in grafts.

Theorem 10.7 *Let (G,T) be a graft, and e be an edge of G. Then:*

1. *e is a loop in $D(G,T)$ if and only if e is a loop or a T-tunnel in (G,T),*
2. *e is a coloop in $D(G,T)$ if and only if e is a T-bridge in (G,T).*

Theorem 10.8 *Let (G,T) be a graft, and e be an edge of G. Then*

$$D((G,T)\backslash e) = D(G,T)\backslash e \qquad \text{and} \qquad D((G,T)/e) = D(G,T)/e.$$

Deletion and contraction in delta-matroids act differently on coloops and loops, respectively, compared to other types of elements. The proof of Theorem 10.8, proceeds by analysing what it means in terms of the graft when e is a loop or coloop in the delta-matroid, then tracks through how the change in the graft under deletion and contraction changes the delta-matroids.

With Theorem 10.8, it follows trivially that when v is a vertex of G, $D(G,T)\backslash v = D((G,T)\backslash v)$. Thus the theorem shows that minor theory for graphic delta-matroids and for grafts are compatible with one another:

$$\text{Graft minors} \quad \xleftrightarrow{\text{compatible}} \quad \text{delta-matroid strong-minors.}$$

A consequence of this is that the class of graphic delta-matroids is minor-closed.

Theorem 10.9 *A minor of a graphic delta-matroid is graphic.*

Graphic delta-matroids are another example of even binary delta-matroids, and so their properties are tied to simple graphs and pivoting.

Theorem 10.10 *Graphic delta-matroids are even binary.*

The idea behind the proof of this theorem is to show that (i) $D(G,V(G))$ is even binary, (ii) that minors of graphic delta-matroids are graphic (Theorem 10.9), and (iii) that any graft (G,T) can be obtained as a minor of some graft $(G',V(G'))$ (see Exercise 10.6). The proof of (i) depends upon line graphs, using a result of Kishi and Uetake [44] that the adjacency matrix (over GF(2)) of the line graph of a simple graph G is non-singular if and only every component of G is a tree with an odd number of vertices. (This also provides some insight into Definition 10.1.) An alternative approach is to check that the excluded minors for binary delta-matroids from Theorem 7.11 do not arise from grafts.

Oum's interest in graphic delta-matroids arose from his conjecture (also in [59]) that if H is a bipartite circle graph, then every graph G with sufficiently large rank-width must have a pivot-minor isomorphic to H. (Rank-width, is a tree-width-like graph parameter introduced by Oum and Seymour in [60] to investigate clique-width.) This conjecture implies Robertson and Seymour's Grid Theorem [65], as well as its version for binary matroids from [38]. Oum proved that the conjecture holds when G is a line graph. In order to do this he had to navigate the difficulty that line graphs are not closed under pivot-minors. This was done by introducing graphic delta-matroids. With this concept he obtained the following rank-width results in [59].

Theorem 10.11 *Let Γ be the fundamental graph of the delta-matroid $D(G,T)$ of a graft (G,T). If the branch-width of G is k, then the rank-width of Γ is k, $k-1$, or $k-2$.*

Theorem 10.12 *Let H be a bipartite circle graph. Then there is a constant $c(H)$ such that if the fundamental graph Γ of the delta-matroid $D(G,T)$ of a graft (G,T) has rank-width larger than $c(H)$, then Γ has a pivot-minor isomorphic to H.*

11 Matchings and delta-matroids

For a graph $G = (V,E)$, and a subset $U \subseteq V$, let $G[U]$ denote the *induced subgraph* on U (so $G[U]$ is obtained by deleting any vertices of G that are not in U). A *matching* on G is a set of its edges that do not share a vertex. A matching is *perfect* if every vertex is incident with an edge in the matching. A set $U \subseteq V$ is said to be *matchable* if $G[U]$ has a perfect matching.

Definition 11.1 (Matching delta-matroid) Let $G = (V,E)$ be a simple graph. Let $\mathcal{F}$ be the collection of its matchable sets:

$$\mathcal{F} := \{X \subseteq V : G[X] \text{ has a perfect matching}\}.$$

We call $(V,\mathcal{F})$ the *matching delta-matroid of G*.

Example 2.16 gives the matching delta-matroid of a simple graph.

Bouchet, in [14], proved that matching delta-matroids are indeed delta-matroids.

Theorem 11.2 *The matching delta-matroid of a simple graph is a delta-matroid.*

Proof [Sketch] Let $X, X' \in \mathcal{F}$, and let M and M' be perfect matchings of $G[X]$ and $G[X']$, respectively. Any $x \in X \triangle X'$ is incident to an edge in exactly one of the matchings. Let H be the subgraph of G on the edge set $M \triangle M'$, then the component of H that contains x is a chain C with one end equal to x. Let y be the other end C. Then $M \triangle C$ is a perfect matching of $G[X \triangle \{x,y\}]$, and so $X \triangle \{x,y\} \in \mathcal{F}$. □

Exercise 11.3 *Prove that a matching delta-matroid is always even. Realise the delta-matroid of Item 5 of Theorem 7.11 as a matching delta-matroid, and hence show that matching delta-matroids need not be binary. Give an example to show that a matching delta-matroid may be binary.*

In Section 8.2 we met the fundamental graph $G_{D,X}$ of an even binary delta-matroid $D = (E,\mathcal{F})$, where X was a feasible set. For the construction we do not actually need that D is binary, and so we can construct $G_{D,X}$ for any delta-matroid D. The *fundamental graph* $G_{D,X}$ is then the graph with vertex set V, and with an edge uv if and only if $\{u,v\}$ is a feasible set of $D * X$ (or equivalently, if and only if $\{u,v\} \triangle X$ is a feasible set of D).

Exercise 11.4 *Prove that if D is a matroid with a basis X, then $G_{D,X}$ is bipartite. (Hint, suppose that $G_{D,X}$ has an odd cycle.)*

The delta-matroid structure of a matching matroid can be used to gain insight into matchable sets, as in the following theorem from [14].

Theorem 11.5 *Let $D = (V, \mathcal{F})$ be an even delta-matroid and $X, X' \in \mathcal{F}$ be two feasible sets. Then $X \bigtriangleup X'$ is matchable in $G_{D,X}$.*

Remark For those familiar with matroids, it is worth noting that Theorem 11.5 is a generalisation of an theorem of Brualdi's (Theorem 1 of [23]) which states that given two bases F and F' of a matroid, there is a bijection $\sigma : F\backslash F' \to F'\backslash F$ such that $(F\backslash e) \cup \sigma(e)$ is a base for all $e \in F\backslash F'$.

At the end of Section 6 we met the partition problem. A special case (taking $D = D'$) of this asks if the ground set of a delta-matroid can be partitioned into two of its feasible sets. This is related to perfect matchings as follows.

Suppose that F_1 and F_2 are two complementary feasible sets of a delta-matroid D, and that $G = G_{D,X} = G_{D*X,\emptyset}$ is a fundamental graph of D. Then, for $i = 1, 2$, applying Theorem 11.5 to the feasible sets $F_i \bigtriangleup X$ and $\emptyset$ of $D * X$, gives perfect matchings M_i in $G[F_i \bigtriangleup X]$. Since $F_i \bigtriangleup X$ and $F_i \bigtriangleup X$ are complementary, $M_1 \cup M_2$ is a perfect matching for G. Thus we have the following result of Bouchet [16].

Corollary 11.6 *If an even delta-matroid admits two complementary feasible sets then each of its fundamental graphs admits a perfect matching.*

In Exercise 11.4 we saw that the fundamental graphs of matroids were necessarily bipartite. Theorem 11.5 can be used to show the converse.

Corollary 11.7 *A delta-matroid is a twist of a matroid if and only if its fundamental graphs are bipartite. That is, if D is a normal delta-matroid and X is a feasible set, then $D * X$ is a matroid if and only if $G_{D,\emptyset}$ is bipartite.*

The result is from [14], where its (short) proof can be found. The result was extended by Duchamp in [33].

Exercise 11.8 *Let $\mathbb{G} = (V, E)$ be a 1-vertex ribbon graph, and $A \subseteq E$. Use Corollary 11.7 to prove that $\mathbb{G}^A$ is a plane ribbon graph if and only if $\mathbb{G}\backslash A$ and $\mathbb{G}\backslash(E\backslash A)$ are both plane ribbon graphs. (This is a special case of the rough structure theorem for partial duals of plane graph from [53]. A delta-matroid analogue of the rough structure theorem was given in [29].)*

Exercise 11.9 *Use Corollary 11.7 and the results of Section 9 to find a characterisation of the class of Eulerian delta-matroids that are twists of matroids.*

Another family of delta-matroids that is intimately connected with matchings is linking delta-matroids. A *red-blue graph* is a simple graph $G = (V, E)$ in which each edge is coloured either red or blue. A *red-blue path* is a path in it whose edges alternate in colour.

Definition 11.10 (Linking delta-matroid) Let $G = (V, E)$ be a red-blue graph. Let $\mathcal{F}$ be the collection of subsets of V given by

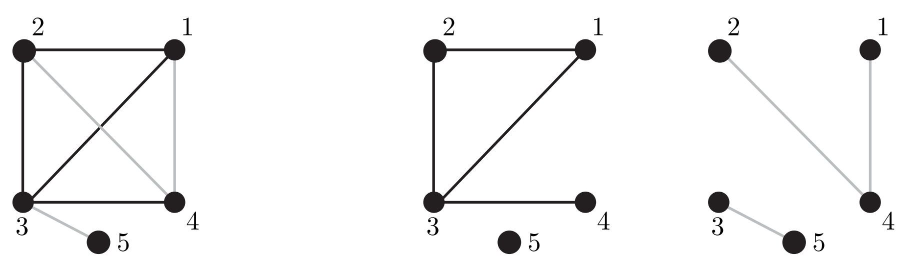
$$\mathcal{F} := \{X : X \text{ is the end vertex set of a collection of} \\ \text{pairwise vertex disjoint red-blue paths}\}$$

Then the pair (V, F) is called the *linking delta-matroid* of G

Linking delta-matroids were shown to be delta-matroids in [16, 21].

Example 11.11 Figure 31a shows a red-blue graph, with the two colour classes indicated as black or grey edges. Then its linking delta-matroid has ground set $V = \{1, 2, 3, 4, 5\}$ and feasible sets

$$\mathcal{F} = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \\ \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}.$$



(a) A red-blue graph (b) The subgraphs on red and blue edges

Figure 31: A red-blue (or black-grey) graph

To see how this relates to matchings we need the notion of the *delta-sum*. The *delta-sum*, $D \mathbin{\underline{\triangle}} D'$, of two delta-matroids $D = (V, \mathcal{F})$ and $D' = (V, \mathcal{F}')$ is defined to be $D \mathbin{\underline{\triangle}} D' := (V, \mathcal{F} \mathbin{\underline{\triangle}} \mathcal{F}')$, where $\mathcal{F} \mathbin{\underline{\triangle}} \mathcal{F}' := \{F \triangle F' : F \in \mathcal{F}, F' \in F'\}$. It was introduced by Duchamp, and while it is cited as 'in preparation' in early delta-matroid papers, he does not appear to have ever published the work. A proof that it does result in a delta-matroid can be found in [21].

Bouchet and Schwärzler, in [21], used the delta-sum to express linking delta-matroids is terms of matching delta-matroids:

Theorem 11.12 *Let G be a red-blue graph. Let D_ℓ be the linking delta-matroid of G, and D_r and D_b be the matching delta-matroids of the subgraph induced by the red and blue edges, respectively. Then*

$$D_\ell = D_r \mathbin{\underline{\triangle}} D_b.$$

Example 11.13 Consider again Example 11.11 which gave the linking delta-matroid of the red-blue graph of Figure 31a. Figure 31b shows the subgraphs induced by the red and blue edges. One has matching delta-matroid $D_r = (V, \mathcal{F}_r)$ where

$$\mathcal{F}_r = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,2,3,4\}\}.$$

The other has matching delta-matroid $D_b = (V, \mathcal{F}_b)$ where

$$\mathcal{F}_b = \{\emptyset, \{1,4\}, \{2,4\}, \{3,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}.$$

Then the feasible sets of $D_r \underline{\triangle} D_b$ are

$$\begin{aligned}\mathcal{F}_r \underline{\triangle} \mathcal{F}_b = \{&\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\},\\ &\{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\},\end{aligned}$$

and we see $D_r \underline{\triangle} D_b$ agrees with the linking delta-matroid from Example 11.11.

Bouchet and Schwärzler found a formula for the polyhedral rank function of a linking delta-matroid. We won't discuss this formula here, although we will point out of one nice graph theoretic corollary from of their delta-matroid work: the recovery of the following result of Gallai [37].

Theorem 11.14 *The maximum number of vertex disjoint paths in a graph $G = (V, E)$ having both end vertices in $U \subseteq V$ is*

$$\min_{S \subseteq V} \left(|S| + \sum_{C} \lfloor |C \cap U|/2 \rfloor \right),$$

where the sum ranges over all components C of $G \backslash S$ with $|C \cap U|$ odd.

Notation

$\mathbf{A}$	A matrix.
$\mathbf{A}[X]$	The principal submatrix of $\mathbf{A}$ on the rows/columns X.
$\mathbf{A}_G$	Adjacency matrix of a graph G.
$\mathbf{A} * X$	Pivot of $\mathbf{A}$ w.r.t. a set of rows/columns X.
$C(G)$	Cycle matroid of a graph G.
D	A delta-matroid.
$D * A$	The twist of delta-matroid D w.r.t. A.
$D(\mathbf{A})$	The delta-matroid of a matrix $\mathbf{A}$.
$D(\mathbb{G})$	The delta-matroid of a ribbon graph $\mathbb{G}$.
$D(\vec{G}, T_P)$	The delta-matroid of an Eulerian 4-regular digraph $\vec{G}$ w.r.t T_P.
$D(G, T)$	The delta-matroid of a graft (G, T).
$D(G, T_F, T_P)$	The Eulerian delta-matroid of 4-regular graph G w.r.t T_F and T_P.
G, H	Graphs.
$G * v$	Local complementation of graph G a graph w.r.t. a vertex v.
$G \wedge uv$	Pivot of graph G a graph w.r.t. an edge uv.
(G, T)	A graft.
$\mathbb{G}$, $\mathbb{H}$	Ribbon graphs.
$\mathbb{G}^A$	The partial dual of a ribbon graph $\mathbb{G}$ w.r.t. A.
$\mathbb{k}$	A field.

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