Parameterized Complexity of Conflict-Free Set Cover

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Abstract. Set Cover is one of the well-known classical NP-hard problems. Following some recent trends, we study the *conflict-free* version of the Set Cover problem. Here we have a universe $\mathcal{U}$, a family $\mathcal{F}$ of subsets of $\mathcal{U}$ and a graph $G_{\mathcal{F}}$ on the vertex set $\mathcal{F}$ and we look for a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of minimum size that covers $\mathcal{U}$ and also forms an independent set in $G_{\mathcal{F}}$. Here we initiate a systematic study of the problem in parameterized complexity by restricting the focus to the variants where Set Cover is fixed-parameter tractable (FPT). We give upper bounds and lower bounds for conflict-free version of the Set Cover with and without duplicate sets along with restrictions to the graph classes of $G_{\mathcal{F}}$.

1 Introduction and Previous Work

Covering problems are problems in combinatorics that ask whether a certain structure "covers" another. Covering problems are very well-studied in theoretical computer science. Examples include Vertex Cover, Feedback Vertex Set, Cluster Vertex Deletion among others.

Several of these covering problems can be encapsulated by a a problem called Set Cover which is one of the well-studied classical NP-hard problems. In the Set Cover problem, we have a universe $\mathcal{U}$, a family $\mathcal{F}$ of subsets of $\mathcal{U}$ and an integer k and the goal is to find a subfamily $\mathcal{F}'$ of size at most k such that $\bigcup_{S \in \mathcal{F}'} S = \mathcal{U}$.

Set Cover is very well studied in a variety of algorithmic settings, especially in the realm of approximation algorithms and parameterized complexity. Unfortunately, Set Cover when parameterized by solution size k is $W[2]$-hard [5] and hence is unlikely to be fixed-parameter-tractable (FPT).

It has been seen in computational problems where some pairs of elements in the problem are in conflict with each other and hence cannot go in the solution together. This can be modeled by defining a graph on the elements and an edge (u, v) is added if elements u and v do not go into the solution together or in other words form a conflict. Hence a solution without conflicts will form an independent set in this graph. Conflict-free versions of classical problems in P like Maximum Flow [21], Maximum Matching [7], Shortest Path [14] have been studied. Conflict-free version of problems like Vertex Cover [13], Feedback Vertex

Set [1], Interval Covering [2,3] are studied from the parameterized point of view very recently.

We look at the conflict-free version of Set Cover defined as follows:

> Conflict-Free Set Cover
> **Input:** An universe $\mathcal{U}$, a family $\mathcal{F}$ of subsets of $\mathcal{U}$, a graph $G_{\mathcal{F}}$ with vertex set $\mathcal{F}$ and an integer k.
> **Goal:** Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k such that $\cup_{F \in \mathcal{F}'} F = \mathcal{U}$ and $\mathcal{F}'$ forms an independent set in G?

We assume that there are no duplicate sets in the family $\mathcal{F}$. Hence $|\mathcal{F}| \leq 2^{|\mathcal{U}|}$.

Note that if $G_{\mathcal{F}}$ is edgeless, the problem is equivalent to Set Cover as every subset of vertices of $G_{\mathcal{F}}$ forms an independent set. Hence Conflict-Free Set Cover is $W[2]$-hard whenever Set Cover is $W[2]$-hard and the only interesting cases of Conflict-Free Set Cover are those special instances or parameterizations where Set Cover is FPT. Banik et al. [3] introduced Conflict-Free Set Cover and considering restrictions on $G_{\mathcal{F}}$ showed that

- Conflict-Free Set Cover is W[1]-hard when $G_{\mathcal{F}}$ is from those classes of graphs where Independent Set is W[1]-hard, and
- when $G_{\mathcal{F}}$ has bounded arboricity, Conflict-Free Set Cover is FPT parameterized by k whenever Set Cover is FPT parameterized by k.

Our results: We note that the reduction instance of Conflict-Free Set Cover in the W[1]-hardness result above [3] contains duplicate sets.

- Our first result is an $f(k)|\mathcal{F}|^{o(k)}$ lower bound for Conflict-Free Set Cover without any duplicate sets assuming the Exponential Time Hypothesis (ETH) even when the sets of $\mathcal{F}$ pairwise intersect in at most one element. The lower bound holds even when $G_{\mathcal{F}}$ is restricted to bipartite graphs where Independent Set is polynomial-time solvable. Hence the result can be seen as generalizing the previous W[1]-hardness result.
- On the positive side, for Conflict-Free Set Cover we give FPT algorithms parameterized by k whenever Set Cover is FPT and $G_{\mathcal{F}}$ belongs to graph classes which are sparse like graphs with bounded degeneracy or nowhere dense graphs using the recently introduced independence covering family [18]. On the other hand if $G_{\mathcal{F}}$ is a dense graph like split or co-chordal, we give an FPT algorithm. This algorithm works for a large class of graphs where the number of maximal independent sets is polynomial in the number of vertices (that are sets in the family in our case).

  Next we consider the problem parameterized by the universe size $|\mathcal{U}|$. Here, since the number of sets $|\mathcal{F}| \leq 2^{|\mathcal{U}|}$, Conflict-Free Set Cover is FPT as the trivial brute-force algorithm of choosing at most k sets from $\mathcal{F}$ is of complexity bounded by $\binom{|\mathcal{F}|}{k} \leq \binom{2^{|\mathcal{U}|}}{|\mathcal{U}|} \leq 2^{|\mathcal{U}|^2}$.
- We give a matching lower bound of $2^{o(|\mathcal{U}| \log |F|)}$ for any value of $|\mathcal{F}|$ as well assuming the ETH.

We note that the problem does not have a polynomial kernel as when $G_{\mathcal{F}}$ is an empty graph, the problem becomes Set Cover parameterized by universe size which does not have a polynomial kernel unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$ [4].

Unlike the Set Cover problem, in Conflict-Free Set Cover duplicate sets do play an important role as the neighbourhood sets in the graph can vary which matters in the independence requirement of the solution.

- For Conflict-Free Set Cover with duplicate sets we give an $f(|\mathcal{U}|)|\mathcal{F}|^{o(|\mathcal{U}|)}$ lower bound assuming the ETH even when all pairs of sets intersect in at most one element and even when $G_{\mathcal{F}}$ is restricted to bipartite graphs where the independent set problem can be solved in polynomial time.
- In addition, we give FPT algorithms when we restrict $G_{\mathcal{F}}$ to interval graphs via a dynamic programming algorithm using the perfect elimination ordering of graphs. We extend this idea and give an FPT algorithm for chordal graphs which is a superclass of interval graphs via dynamic programming on the clique-tree decomposition of the graph.
  We also study the Conflict-Free Set Cover problem where there is an underlying (linearly representable) matroid on the family of subsets, and we want the solution to be an independent set in the matroid. Banik et al. [3] studied this version for a specialization of Set Cover where the sets are intervals on a real line.
- We show that even the more general problem (where the sets in the family are arbitrary) is FPT when parameterized by the universe size, using the idea of dynamic programming over representative families [10].
  We note that this result can be obtained as a corollary of a result by Bevern et al. [23] where they studied generalization called uncapacitated facility location problem with multiple matroid constraints. But our algorithm is simpler and has a better running time.

2 Preliminaries

We use $[n]$ to denote the set $\{1, \ldots, n\}$. We use the standard terminologies of the graph theory book by Diestel [8]. For a graph $G = (V, E)$ we denote n as the number of vertices and m as the number of edges. For $S \subseteq V(G)$, we denote $G[S]$ to be the subgraph induced on S. A *complement* of graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G. A set $S \subseteq V(G)$ is called an *independent set* if for all $u, v \in S, (u, v) \notin E(G)$.

An *interval* graph is an undirected graph formed from a family of intervals in the real line $\mathcal{I}$ with the vertex set as $\mathcal{I}$ and for intervals $u, v \in \mathcal{I}$, add edge (u, v) if the intervals u and v intersect. A chord in a cycle is an edge between two non-adjacent vertices of the cycle. A *chordal* graph is a graph in which any cycle of four or more vertices has a chord. A graph G is said to be *d-degenerate* if every subgraph of G has a vertex of degree at most d. The *arboricity* of an undirected graph is the minimum number of forests into which its edges can be partitioned. We use the following conjecture to prove lower bounds.

Conjecture 1 (Exponential Time Hypothesis (ETH)([12])). 3-CNF-SAT cannot be solved in $\mathcal{O}^*(2^{o(n)})$ [3] time where the input formula has n variables and m clauses.

3 Conflict-Free Set Cover parameterized by k

3.1 Hardness results

Theorem 1. Conflict-Free Set Cover *where every pair of sets in $\mathcal{F}$ intersect in at most one element is* W[1]*-hard with respect to solution size k when $G_{\mathcal{F}}$ is bipartite.*

Proof. We give a reduction from the W[1]-hard problem Multicolored Biclique [6] defined as follows:

Multicolored Biclique	**Parameter:** k
Input: A bipartite graph $G = (A \cup B, E)$, an integer k, a partition of A into k sets $A_1, A_2, \ldots, A_k$ and a partition of B into k sets $B_1, B_2, \ldots, B_k$.	
Question: Does there exist a subgraph of G isomorphic to the biclique $K_{k,k}$ with one vertex from each of the sets A_i and B_i?	

We note that while the $W[1]$-hardness of Multicolored Biclique can be easily shown from a reduction from Multicolored clique, the complexity of the normal k-biclique problem was open for a long time and was only shown recently to be $W[1]$-hard [15].

Given an instance of $(G, A_1, \ldots, A_k, B_1, \ldots, B_k)$ of Multicolored Biclique with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we construct an instance of Conflict-Free Set Cover $(\mathcal{U}, \mathcal{F}, G_{\mathcal{F}}, k)$ without duplicates as follows:

We define the universe $\mathcal{U} = [2k] \cup V(G) \cup \{x\}$. Let S_{v_j} denote the set corresponding to vertex v_j. For $i \in [k]$, if $v_j \in A_i$, define $S_{v_j} = \{v_j, i\}$. For $i \in [2k] \setminus [k]$, if $v_j \in B_{i-k}$, define $S_{v_j} = \{v_j, i\}$. Define a set $D = V(G) \cup \{x\}$. We have $\mathcal{F} = \bigcup_{v \in V(G)} S_v \cup \{D\}$. The graph $G_{\mathcal{F}}$ is obtained by taking the complement of the graph G, making the sets A and B independent and making D as an isolated vertex. Note that the graph $G_{\mathcal{F}}$ remains bipartite.

Note that $\mathcal{F}$ is defined in such a way that all pairs of sets intersect in at most one element. Also there are no duplicate sets.

We claim that there is a multicolored biclique of size k in G if and only if there is a Conflict-Free Set Cover of size $2k+1$ in the instance $(\mathcal{U}, \mathcal{F}, G_{\mathcal{F}})$.

Let $S = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ be the vertices in G that form a multicolored biclique. Then $\mathcal{F}' = \{D, S_{a_1}, \ldots, S_{a_k}, S_{b_1}, \ldots, S_{b_k}\}$ covers $\mathcal{U}$ as D covers $V(G) \cup \{x\}$ and $i \in S_{a_i}$ for $i \in [k]$ and $i \in S_{b_{i-k}}$ for $i \in [2k] \setminus [k]$. Since the edges across A and B in G are non-edges in $G_{\mathcal{F}}$ and D is an isolated vertex, $\mathcal{F}'$ forms an independent set in $G_{\mathcal{F}}$. In the reverse direction, let $\mathcal{F}' = \{S_1, \ldots, S_{2k+1}\}$ be a solution of size $2k+1$ covering $\mathcal{U}$. The set D has to be part of the solution $\mathcal{F}'$ as

[3] $\mathcal{O}^*$ notation ignores polynomial factors of input

only D contains element x. Now note that an element $i \in [k]$ can be covered only by sets S_v where $v \in A_i$. Similarly an element $i \in [2k] \setminus [k]$ can be covered only by sets S_v where $v \in B_{i-k}$. Hence the vertices of the sets in $\mathcal{F}'$ are such that there is at least one vertex from each of the sets A_i and B_i. Since the budget is limited to $2k$ after picking D, exactly one vertex from each of the sets A_i and B_i is contained in $\mathcal{F}'$. Since the vertices $\mathcal{F}' \setminus D$ form an independent set in $G_{\mathcal{F}}$, the corresponding vertices form a biclique in G. □

Since Multicolored Biclique cannot be solved in time $f(k)|\mathcal{F}|^{o(k)}$ for solution size k assuming ETH [20], we have the following corollary.

Corollary 1. Conflict-Free Set Cover *where every pair of sets in $\mathcal{F}$ intersect in at most one element cannot be solved in time $f(k)|\mathcal{F}|^{o(k)}$ for solution size k in bipartite graphs for any computable function f assuming the ETH.*

3.2 Upper Bounds

In the following results, we restrict the graph $G_{\mathcal{F}}$.

Graphs with bounded number of maximal independent sets

Theorem 2. *When $G_{\mathcal{F}}$ is restricted to a graph where the number of maximal independent sets is polynomial in $|\mathcal{F}|$ and can be enumerated in time polynomial in $|\mathcal{F}|$, if* Set Cover *can be solved in $\mathcal{O}^*(f(k))$ time,* Conflict-Free Set Cover *can be solved in $\mathcal{O}^*(f(k))$ time.*

Proof. For each maximal independent set I of G, we run the $\mathcal{O}^*(f(k))$ algorithm for Set Cover with the family $\mathcal{F}$ containing sets corresponding to the vertices in I. Since the solution X of Conflict-Free Set Cover is an independent set, $X \subseteq I'$ for some maximal independent set I'. So if the Set Cover algorithm returns YES for any I, return YES, else return NO.

Note that although Set Cover problem is W-hard when parameterized by k, there are variants of Set Cover like when the size of the intersection of sets in $\mathcal{F}$ is bounded [22] where the problem can be solved in $\mathcal{O}^*(f(k))$ time.

As the number of maximal independent sets in split graphs (since at most one vertex of the clique can be in the independent set), co-chordal graphs [11] and $2K_2$-free graphs [9] are polynomial in the number of vertices, we have the following corollary.

Corollary 2. *If* Set Cover *can be solved in $\mathcal{O}^*(f(k))$ time,* Conflict-Free Set Cover *can be solved in $\mathcal{O}^*(f(k))$ time when $G_{\mathcal{F}}$ is restricted to split graphs, co-chordal graphs or $2K_2$-free graphs.*

Graphs with bounded degeneracy

We use the notion of k-Independence Covering Family introduced by [18] defined as follows:

Definition 1 (k-Independence Covering Family). *For a graph G and integer k, a family of independent sets of G is called an independence covering family for (G, k), denoted by $\mathscr{F}(G, k)$, if for any independent set X in G of size at most k, there exists an independent set $Y \in \mathscr{F}(G, k)$ such that $X \subseteq Y$.*

Lemma 1. *(Deterministic Independence Covering Lemma [18]) Given a d-degenerate graph G and an integer k, there is an algorithm that runs in time $O^*(\binom{k(d+1)}{k} \cdot 2^{o(k(d+1)} \cdot (n+m)\log n)$ and outputs a k-independence covering family for (G, k) of size at most $O^*(\binom{k(d+1)}{k} \cdot 2^{o(k(d+1))} \cdot \log n)$.*

Theorem 3. Conflict-Free Set Cover *has an algorithm with running time $\mathcal{O}^*(f(k)\binom{k(d+1)}{k} \cdot 2^{o(k(d+1))})$ with solution size k when $G_{\mathcal{F}}$ is a d-degenerate graph if* Set Cover *can be solved in $\mathcal{O}^*(f(k))$ time.*

Proof. We use Lemma 1 on $G_{\mathcal{F}}$ to get a k-independence covering family $\mathscr{F}(G_{\mathcal{F}}, k)$. For each independent set $Y \in \mathscr{F}(G_{\mathcal{F}}, k)$, we run the algorithm for Set Cover for the instance $(\mathcal{U}, Y, k)$ in $\mathcal{O}^*(f(k))$ time. If for any of the sets Y, $(\mathcal{U}, Y, k)$ is a yes instance, we return yes. Otherwise we return no.

Let X be the solution of size k. There is a set Y in $\mathscr{F}(G_{\mathcal{F}}, k)$ such that $X \subseteq Y$. Hence when we run the algorithm for Set Cover in instance $(\mathcal{U}, Y, k)$, since $G[Y]$ is an independent set, the algorithm will return X. □

Note that graphs with bounded degeneracy contain many other graph classes such as planar graphs, graphs with bounded arboricity and graphs with bounded treewidth. We note that a similar result as Theorem 3 has been proven in graphs with bounded arboricity [3] (through a different argument) from which a result for graphs with bounded degeneracy follows as the degeneracy of a graph is also bounded when the arboricity is bounded.

Nowhere Dense graphs

Nowhere dense graphs contains a number of classes of graphs that are not contained in the class of graphs with bounded degeneracy, including graphs with bounded local treewidth and graphs that locally exclude a fixed minor. In [18], the authors construct a k-independence covering family for nowhere dense graphs.

Lemma 2 ([18]). *Let G be a nowhere dense graph and k be an integer. There is a deterministic algorithm that runs in time*

$$\mathcal{O}\Big(f(k, \frac{1}{k}) \cdot n^{1+o(1)} + g(k) \cdot \binom{k^2}{k} \cdot 2^{o(k^2)} \cdot n(n+m)\log n\Big)$$

and outputs a k-independence covering family for (G, k) of size $\mathcal{O}(g(k)\binom{k^2}{k} \cdot 2^{o(k^2)} \cdot n \log n)$ where f is a computable function and $g(k) = (f(k, \frac{1}{k}))^k$.

We get the following theorem whose proof is similar to Theorem 3.

Theorem 4. Conflict-Free Set Cover *has an algorithm with running time $\mathcal{O}^*(h(k)g(k)\binom{k^2}{k} \cdot 2^{o(k^2)})$ for nowhere dense graphs with solution size k and a computable function g if* Set Cover *can be solved in $\mathcal{O}^*(h(k))$ time.*

4 Conflict-Free Set Cover parameterized by $|\mathcal{U}|$

4.1 Lower Bounds when $\mathcal{F}$ has no duplicates

We define the following variant of Multicolored Biclique.

Small Multicolored Biclique	**Parameter:** k

Input: A bipartite graph $G = (A \cup B, E)$, an integer k, a partition of A into k sets $A_1, A_2, \ldots, A_k$ and a partition of B into k sets $B_1, B_2, \ldots, B_k$ such that $|A_i| = |B_i| = s$ where $k \leq s \leq 2^k/2k$.
Question: Does there exist a subgraph of G isomorphic to the biclique $K_{k,k}$ with one vertex from each of the sets A_i and B_i?

We first note that the reduction from 3-coloring used in [16] can be modified so that we get the following lower bound for Small Multicolored Biclique.

Theorem 5. [4] Small Multicolored Biclique *cannot be solved in time* $2^{o(k \log s)}$ *under the ETH*

Theorem 6. Conflict-Free Set Cover *without duplicates when* $G_{\mathcal{F}}$ *is bipartite cannot be solved in time* $2^{o(|\mathcal{U}| \log |\mathcal{F}|)}$ *under ETH.*

Proof. Given an instance of $(G, A_1, \ldots, A_k, B_1, \ldots, B_k)$ of Small Multicolored Biclique with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we construct an instance of Conflict-Free Set Cover $(\mathcal{U}, \mathcal{F}, G_{\mathcal{F}}, 2k+1)$ without duplicates as follows:

Let us define sets $Z = \{z_1, z_2, \ldots z_{\lceil \log n \rceil}\}$ and $O = \{o_1, o_2, \ldots o_{\lceil \log n \rceil}\}$.

We define the universe $\mathcal{U} = [2k] \cup Z \cup O \cup \{x\}$.

Let us look at vertex $v_j \in V$ and construct sets $S_{v_j} \in \mathcal{F}$. Let us map j to its binary representation $b_1, b_2, \ldots, b_{\lceil \log n \rceil}$. We create a set T_j as follows: for all $i \in [\lceil \log n \rceil]$, when $b_i = 0$, add z_i to T_j, else add o_i to T_j. For $i \in [k]$, if $v_j \in A_i$, define $S_{v_j} = \{i\} \cup T_j$. For $i \in [2k] \setminus [k]$, if $v_j \in B_{i-k}$, define $S_{v_j} = \{i\} \cup T_j$. Define the extra set $D = Z \cup O \cup \{x\}$. The graph $G_{\mathcal{F}}$ is obtained by taking the complement of the graph G, making the sets A and B independent and making D as an isolated vertex. Note that the graph $G_{\mathcal{F}}$ remains bipartite.

Note that the construction is almost exactly the same as in Theorem 1 but the vertices are encoded in binary form. The correctness of the reduction can then easily be seen after noting that the set D has to go in the solution as it is the only set containing element x.

Note that in the Small Multicolored Biclique instance, $n = 2k \cdot s \leq 2^k$. Since $\log n \leq k$, $|U| \leq 4k + 1$.

Now suppose Conflict-Free Set Cover has an algorithm with running time $2^{o(|\mathcal{U}| \log |\mathcal{F}|)}$. Since $s = \frac{|\mathcal{F}|}{2k}$ and $|\mathcal{U}| \leq 4k+1$, we have a running time of $2^{o(4k \log(2k \cdot s))} = 2^{o(k(\log s + \log k)} = 2^{o(k \log s)}$ for Small Multicolored Biclique violating the ETH. ◻

[4] Proof in full version

4.2 Lower bounds when $\mathcal{F}$ has duplicates

We have the following hardness result [3].

Theorem 7 ([3]). *If for a subclass of graphs $\mathscr{G}$, finding an independent set of size k is* W[1]*-hard, then* Conflict-Free Set Cover *parameterized by $|\mathcal{U}|$ is* W[1]*-hard when $G_{\mathcal{F}}$ is restricted to the class $\mathscr{G}$.*

Bipartite Graphs

Bipartite graphs is one class of graphs where the Independent Set problem can be solved in polynomial time. In contrast to Theorem 7, we show that Conflict-Free Set Cover on bipartite graphs is W[1]-hard. Note that in Theorem 1 proven previously, the size of the universe can be much larger than the solution size k and hence the hardness result does not follow from it.

Lemma 3. Conflict-Free Set Cover *parameterized by $|\mathcal{U}|$ is* W[1]*-hard on bipartite graphs.*

Proof. We again give a reduction from the W[1]-hard problem Multicolored Biclique. The construction is very similar to that in Theorem 1, the difference being the vertex v is not added to sets S_v.

Given an instance of Multicolored Biclique, we construct an instance of Conflict-Free Set Cover as follows: $\mathcal{U} = [2k]$. Let S_v denote the set corresponding to vertex v we add to $\mathcal{F}$. For $i \in [k], i \in S_v$ if $v \in A_i$. For $i \in [2k] \setminus [k], i \in S_v$ if $v \in B_{i-k}$. The graph G' is obtained by complementing the graph G and making the sets A and B independent. The graph G' remains bipartite.

The correctness proof easily follows. □

4.3 Upper Bounds when $\mathcal{F}$ has duplicates

Interval Graphs

Interval graphs have the property that its vertices can be ordered as $v_1, \ldots, v_n$ such that for each v_i, $N[v_i] \cap \{v_i, \ldots, v_n\}$ is present consecutively in the ordering where $N[v_i]$ is the closed neighbourhood set of v_i. Such an ordering is actually the *perfect elimination ordering* of the graph and can be obtained in time polynomial in $|V(G)|$ by arranging the corresponding intervals in order of their leftmost endpoint. We make use of this ordering to give a dynamic programming algorithm for Conflict-Free Set Cover with duplicates on interval graphs.

Theorem 8. [5] Conflict-Free Set Cover *with duplicate sets when $G_{\mathcal{F}}$ is restricted to interval graphs can be solved in $\mathcal{O}^*(2^{|\mathcal{U}|})$ time.*

Now we give a $\mathcal{O}^*(3^{|\mathcal{U}|})$-time dynamic programming algorithm for chordal graphs which is a superclass of interval graphs.

[5] Proof in full version

Chordal Graphs

A *clique tree decomposition* is a nice tree decomposition T where for all nodes $i \in V(T)$, the vertices of in the bag X_i are such that $G[X_i]$ forms a clique. All chordal graphs have clique-tree decompositions and can be found in polynomial time [11]. Given a clique tree decomposition, it can be converted to a nice clique tree decomposition in polynomial time as well [6].

In the theorem below, we give an algorithm for Conflict-Free Set Cover with duplicates on chordal graphs using dynamic programming on the nice clique tree decomposition of the graph.

Theorem 9. [6] Conflict-Free Set Cover *with duplicates on chordal graphs can be solved in* $O^*(3^{|\mathcal{U}|})$ *running time.*

5 Matroidal Conflict-free Set Cover

Let us define the Matroidal Conflict-Free Set Cover problem.

Matroidal Conflict-Free Set Cover
Input: A universe $\mathcal{U}$, a family $\mathcal{F}$ of subsets of $\mathcal{U}$, a linear matroid $M = (\mathcal{F}, \mathcal{I})$ and an integer k.
Goal: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k such that $\bigcup_{F \in \mathcal{F}'} F = \mathcal{U}$ and $\mathcal{F}'$ forms an independent set in M?

We give a dynamic programming algorithm for Matroidal Conflict-Free Set Cover containing duplicate sets using computation of representative sets noting that the similar ideas used in [3] for Interval Covering can be extended to Matroidal Conflict-Free Set Cover .

For $W \subseteq \mathcal{U}$, let $\mathcal{B}^W$ denote the collection of subfamilies X of $\mathcal{F}$ of size at most k such that X covers W and forms a independent set in the matroid M.

$$\mathcal{B}^W = \{X \subseteq \mathcal{F} \;\Big|\; |X| \leq k, W \subseteq \bigcup_{S \in X} S \text{ and } X \in \mathcal{I}\}$$

Note that $\mathcal{B}^{\mathcal{U}}$ contains all the solutions of size at most k of Matroidal Conflict-Free Set Cover . Hence we solve the Matroidal Conflict-Free Set Cover problem by checking whether $\mathcal{B}^{\mathcal{U}}$ is empty or not.

Definition 2 (q-representative family [19]). *Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{A}$ be a family of sets of size p in M. For sets $A, B \subseteq E$, we say that A fits B if $A \cap B = \phi$ and $A \cup B \in \mathcal{I}$. A subfamily $\hat{\mathcal{A}} \subseteq \mathcal{A}$ is said to q-represent $\mathcal{A}$ if for every set B of size q such that there is an $A \in \mathcal{A}$ that fits B, there is an $\hat{A} \in \hat{\mathcal{A}}$ that also fits B. We use $\hat{\mathcal{A}} \subseteq^q_{rep} \mathcal{A}$ to denote that $\hat{\mathcal{A}}$ q-represents $\mathcal{A}$.*

Lemma 4 ([10]). *For a matroid $M = (E, \mathcal{I})$ and $\mathcal{S} \subseteq E$, if $\mathcal{S}_1 \subseteq^q_{rep} \mathcal{S}$ and $\mathcal{S}_2 \subseteq^q_{rep} \mathcal{S}_1$, then $\mathcal{S}_2 \subseteq^q_{rep} \mathcal{S}$.*

[6] Proof in full version

Note that $\mathcal{B}^{\mathcal{U}}$ is nonempty if and only if $\hat{\mathcal{B}}^{\mathcal{U}} \subseteq^0_{rep} \mathcal{B}^{\mathcal{U}}$ is nonempty. Let us define $\mathcal{B}^{Wj}$ as the subset of $\mathcal{B}^W$ containing sets of size exactly j. We denote $\hat{\mathcal{B}}^W \subseteq^{1,\dots,k}_{rep} \mathcal{B}^W$ to denote that $\hat{\mathcal{B}}^W$ contains the union of all the i-representative families of $\mathcal{B}^W$ where $1 \leq i \leq k$. In other words,

$$\hat{\mathcal{B}}^W = \bigcup_{j=1}^{k} \left(\hat{\mathcal{B}}^{Wj} \subseteq^{k-j}_{rep} \mathcal{B}^{Wj}\right)$$

Lemma 5 ([17]). *Let $M = (E, \mathcal{I})$ be a linear matroid of rank n and $\mathcal{S}$ be a family of t independent sets of size p. Let A be a $n \times |E|$ matrix representation of M over a field $\mathbb{F}$ where $\mathbb{F} = \mathbb{F}_{p^\ell}$ or $\mathbb{F}$ is $\mathbb{Q}$. Then there is a deterministic algorithm to compute $\hat{\mathcal{S}} \subseteq^q_{rep} \mathcal{S}$ size $np\binom{p+q}{p}$ in $\mathcal{O}\big(\binom{p+q}{p}tp^3n^2 + t\binom{p+q}{p}^{\omega-1}(pn)^{\omega-1}\big) + (n + |E|)^{O(1)})$ operations over $\mathbb{F}$ where ω is the matrix multiplication exponent.*

Theorem 10. Matroidal Conflict-Free Set Cover *can be solved in $\mathcal{O}^*(2^{(\omega+1)\cdot|\mathcal{U}|})$ time where ω is the matrix multiplication exponent.*

Proof. Let $\mathcal{D}$ be an array of size $2^{|\mathcal{U}|}$ with $\mathcal{D}[W]$ storing the family $\hat{\mathcal{B}}^W \subseteq^{1,\dots,k}_{rep} \mathcal{B}^W$. We compute the entries of $\mathcal{D}$ in the increasing order of subsets of $\mathcal{U}$. To do so we compute the following:

$$\mathcal{N}^W = \bigcup_{S_i \in \mathcal{F}} (\mathcal{D}[W \setminus S_i] \bullet S_i) \cap \mathcal{I} \tag{1}$$

where $\mathcal{A} \bullet \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}$ and $B \in \mathcal{B}$ and $A \cap B = \phi\}$.

We show that $\mathcal{N}^W \subseteq^{1\dots k}_{rep} \mathcal{B}^W$. Let $S \in \mathcal{B}^{Wj}$ and Y be a set of size $k - j$ such that $S \cap Y = \phi$ and $S \cup Y \in \mathcal{I}$. We give a set $\hat{S} \in \mathcal{N}^{Wj}$ such that $\hat{S} \cap Y = \phi$ and $\hat{S} \cup Y \in \mathcal{I}$.

Let $S = \{S_1, S_2, \dots, S_j\}$. Let $S' = S \setminus S_j$. Let $Y' = Y \cup S_j$. $|S'| = j - 1$ and $|Y'| = k-j+1$. Since S' covers $W \setminus S_j$, $S' \in \mathcal{B}^{(W \setminus S_j)(j-1)}$. By definition, $D[W \setminus S_j]$ contains $\hat{\mathcal{B}}^{(W \setminus S_j)(j-1)} \subseteq^{k-j+1}_{rep} \mathcal{B}^{(W \setminus S_j)(j-1)}$ and hence a set $S^* \in D[W \setminus S_j]$ such that $S^* \cap Y' = \phi$ and $S^* \cup Y' \in \mathcal{I}$. From equation (1), $S^* \cup S_j \in \mathcal{N}^W$. The set $\hat{S} = S^* \cup S_j$ is such that $\hat{S} \cap Y = \phi$ and $\hat{S} \cup Y \in \mathcal{I}$. Hence $\mathcal{N}^W \subseteq^{1\dots k}_{rep} \mathcal{B}^W$.

We store $\hat{\mathcal{N}}^W \subseteq^{1\dots k}_{rep} \mathcal{N}^W$ in $\mathcal{D}[W]$. The sets $\hat{\mathcal{N}}^{Wj}$ are computed using Lemma 5. We have $\hat{\mathcal{N}}^{Wj} \subseteq^{k-j}_{rep} \mathcal{N}^{Wj} \subseteq^{k-j}_{rep} \mathcal{B}^{Wj}$ for all $1 \leq j \leq k$. Hence from Lemma 4, we have $\mathcal{D}[W] = \hat{\mathcal{N}}^W \subseteq^{1\dots k}_{rep} \mathcal{B}^W$.

We now focus on the running time to compute $\mathcal{D}[W]$ and the size of $\mathcal{D}[W]$. Assume that $\mathcal{D}[Y]$ is precomputed for all subsets $Y \subseteq W$. We have $|\mathcal{D}[Y]| = |\hat{\mathcal{N}}^Y| = \sum_{j=1}^{k} |\hat{\mathcal{N}}^{Yj}|$. From Lemma 5, $|\hat{\mathcal{N}}^{Yj}| \leq |\mathcal{F}| \cdot k \cdot \binom{k}{j}$. Hence from equation (1), putting $Y = W \setminus S_i$, we have $|\mathcal{N}^{Wj}| \leq |\mathcal{F}|^2 \cdot k \cdot \binom{k}{j}$. Using Lemma 5, the time to compute $\hat{\mathcal{N}}^{Wj} \subseteq^{k-j}_{rep} \mathcal{N}^{Wj}$ is $\mathcal{O}^*\big(\binom{k}{j}^2 + \binom{k}{j}^\omega\big)$ where ω is the exponent for matrix

multiplication. Hence the total time to compute $\mathcal{D}[W]$ is $\sum_{j=1}^{k} \mathcal{O}^*(\binom{k}{j}^{\omega}) = \mathcal{O}^*(2^{\omega k})$.
The size of $\mathcal{D}[W]$ is $\mathcal{O}(|\mathcal{F}| \cdot k \cdot \sum_{j=1}^{k} \binom{k}{j}) = \mathcal{O}(2^k \cdot k \cdot |\mathcal{F}|)$.

The overall running time to check if $\mathcal{D}[U]$ is empty or not is bounded by $\mathcal{O}^*(2^{|\mathcal{U}|} \cdot 2^{\omega k}) = \mathcal{O}^*(2^{\omega|\mathcal{U}|+|\mathcal{U}|}) = \mathcal{O}^*(10.361^{|\mathcal{U}|})$. □

6 Conclusion

We have initiated a systematic study of Conflict-Free Set Cover with various parameterizations with restrictions to $G_{\mathcal{F}}$. One open question is to identify a general characterization for the graph classes of $G_{\mathcal{F}}$ when Conflict-Free Set Cover becomes FPT parameterized by k or by $|\mathcal{U}|$.

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