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The Mathematics of Continuous Multiplicities: The Role of Riemann in Deleuze's Reading of Bergson

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Abstract: A central claim of Deleuze's reading of Bergson is that Bergson's distinction between space as an extensive multiplicity and duration as an intensive multiplicity is inspired by the distinction between discrete and continuous manifolds found in Bernhard Riemann's 1854 thesis on the foundations of geometry. Yet there is no evidence from Bergson that Riemann influences his division, and the distinction between the discrete and continuous is hardly a Riemannian invention. Claiming Riemann's influence, however, allows Deleuze to argue that quantity, in the form of 'virtual number', still pertains to continuous multiplicities. This not only supports Deleuze's attempt to redeem Bergson's argument against Einstein in *Duration and Simultaneity*, but also allows Deleuze to position Bergson against Hegelian dialectics. The use of Riemann is thereby an important element of the incorporation of Bergson into Deleuze's larger early project of developing an anti-Hegelian philosophy of difference.

This article will review the role of discrete and continuous multiplicities or manifolds in Riemann's *Habilitationsschrift*, and how Riemann uses it to establish the foundations of an intrinsic geometry. It will then outline how Deleuze reinterprets Riemann's thesis to make it a credible resource for Deleuze's Bergsonism. Finally, it will explore the limits of this move, and how Deleuze's later move away from Bergson turns on the rejection of an assumption of Riemann's thesis, that of 'flatness in smallest parts', which Deleuze challenges with the idea, taken from Riemann's contemporary, Richard Dedekind, of the irrational cut.

Keywords: Deleuze; Bergson; Riemann; Dedekind; philosophy of mathematics

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The Mathematics of Continuous Multiplicities: The Role of Riemann in Deleuze's Reading of Bergson[1]

Central to Deleuze's early reading of Bergson's philosophy of duration is the claim that Bergson's distinction between space as an extensive multiplicity and duration as an intensive multiplicity is an adaptation of the distinction between discrete and continuous manifolds in Bernhard Riemann's 1854 *Habilitationsschrift* on the foundations of geometry.[2] Bergson does identify two types of multiplicity in his early work, *Time and Free Will*, which superficially supports Deleuze's assertion. The first involves numerically distinct discrete objects in extended space, while the second involves a continuous succession of interpenetrating but heterogeneous qualitative states in consciousness that compose durational time and that cannot be quantified without being symbolically represented in space (see Bergson 1910: 85–7). Nevertheless, there is no direct evidence that Riemann inspires this division. Bergson never mentions Riemann in his writings, leaving Deleuze to speculate that Bergson's interest in mathematics must have made him 'well aware of Riemann's general problems' (Deleuze 1991a: 39). Moreover, the distinction between the discrete and the continuous is not itself particularly Riemannian, and goes back at the very least to Aristotle's division of quantity into plurality and magnitude – that is, into countable and measurable quantity (Aristotle 1933–35: 5.13). Finally, Deleuze's reading of Riemann is hardly comprehensive or faithful, focussing on a brief passage at the conclusion of Riemann's thesis in order to connect what Deleuze himself admits to be Bergson and Riemann's otherwise radically different understanding of continuous multiplicities.

Bergson's early philosophy insists on an absolute separation of time and space, and thus of quality and quantity. In *Time and Free Will*, he maintains that 'the fact is that there is no point of contact between the unextended and the extended, between quality and quantity'

(Bergson 1910: 70) and that ‘if magnitude, outside you, is never intensive, intensity, within you, is never magnitude’ (225). However, from *Matter and Memory* onwards Bergson proposes various rapprochements between the terms he initially considered entirely different in kind, building on the idea that ‘the interval between quantity and quality [might] be lessened by considerations of *tension*’ (Bergson 1991: 183). This leads him, on the one hand, to propose an inverse relation between quantitative space and qualitative duration in which the latter constitutes the former as an effect of its force of becoming (*élan vital*), and, on the other hand, to suggest the possibility of multiple levels or paces of duration. The two ideas follow from the same implications Bergson draws from the way modern physics understands extensive movements as composed of microscopic continuous vibrations of forces (196–201). From this, he argues that the reality of the numbers attached to the frequencies of these forces – in the case of red light, some 400 billion vibrations per second – presupposes a duration that is sufficiently relaxed so as to be able to tally them (204–6), and thus to the conclusion that ‘we must distinguish here between our own duration and time in general’ (206). Yet, as Deleuze notes, these developments in Bergson’s thought open him to the criticism that he reintroduces extensity into duration. It also leaves unexplained Bergson’s apparent regression back to psychologism and the absolute separation of time and space in his argument against Einstein in *Duration and Simultaneity* (Bergson 1999), where he declares that the frequencies by which physics interprets the universe to be a mathematical fiction and insists on there being only one pace of lived duration (Bergson 1999: 25, 32).

Claiming Riemann’s influence allows Deleuze to respond to all of these issues by holding that an unextended or intensive form of quantity still pertains to duration as a qualitative continuous multiplicity. This supports Deleuze’s attempt in *Bergsonism* (Deleuze 1991a) to redeem Bergson’s argument in *Duration and Simultaneity* by holding that it is directed only towards what Bergson sees as Einstein's misuses of extensive quantity to understand time. It also supports Deleuze's more general attempt to present Bergsonian duration as offering a concept of difference within a monistic ontology. This latter side of Deleuze's reading also allows him to claim that duration involves an internal difference superior to the internal difference conceived by Hegel as speculative contradiction, and thus to employ Bergson to advance his own anti-Hegelian philosophy.

Deleuze's early critiques of Hegel, implicit in his reading of Hume (Deleuze 1991b) and explicit in several readings of Bergson, turn on the idea that contradiction or opposition is an external relation applying only to already actualised terms, leaving unexplained the constitution of both these terms and the extensity they occupy. Treating the external relation of opposition as though it were a constitutive internal difference leaves dialectics abstract, as it is 'always composing movement from points of view, as a relation between actual terms instead of seeing in it the actualization of something virtual' (Deleuze 2004: 28). The virtual here is a domain of constitutive differences that are fully real but not susceptible to representation, differences that for Bergson relate to the temporality that makes things what they are. For Deleuze, duration, in contrast to dialectical opposition, is a virtual difference, unextended and immanent to the entities it infuses, that grasps 'the thing itself, according to what it is, in its difference from everything it is not, in other words, in its *internal difference*' (32); but it also, as Deleuze argues with regards to Bergson's late works, accounts for the extensity and spatiality in which actual things come to be seen as distinct and opposed. In this way, duration, understood as a continuous multiplicity, constitutes an extensity that is, in Riemannian terms, a discrete manifold.

Conceptualising such an intensive continuous domain and how it constitutes extensity, however, has nothing to do with Riemann's project. Rather, Riemann seeks to outline the general foundations of domains of multiply-extended magnitudes, whose constituent components may be either discrete or continuous, and he is agnostic about the real structure of space, which indeed is only one such domain of extended magnitudes. In these ways and others, his project differs substantially from Bergson's – and Deleuze's – even while Deleuze marshals it to support both Bergson's project and his own. Despite these differences, Deleuze finds a positive way to adapt Riemann's thought to support his reading and defence of Bergson. Yet it is also from the product of this Bergson–Riemann combination that Deleuze plots a course beyond both of them through the idea of immanent intensive discontinuities within continuous multiplicities.

The following will first review Riemann's concept of the manifold, its use in establishing the foundations of an intrinsic geometry, and how Riemann's concluding discussion of space sets the stage both for the development of Einstein's general theory of relativity and Deleuze's adaption of Riemann in his reading of Bergson. It will then outline how Deleuze uses Riemann's thesis to support his early Bergsonism. Finally, it will explore the limits of this move, and how Deleuze's later turn from Bergson relates to a rejection of a crucial assumption of Riemann's analysis, that of 'flatness in smallest parts', developing these final thoughts by examining Deleuze's deployment of the 'irrational cut', an idea he takes from Riemann's contemporary, the mathematician Richard Dedekind.

I. Riemann on Discrete and Continuous Manifolds

Riemann's thesis concerns 'the general concept of multiply extended magnitudes [*der Begriff mehrfach ausgedehnter Grössen*], in which spatial magnitudes are comprehended', and seeks to construct 'the concept of a multiply extended magnitude [*Grösse*] out of general notions of quantity [*Grössenbegriffen*]' (Riemann 1929: 411).[3] Enquiry into the general conditions for the construction of such magnitudes, he contends, demonstrates

> <EXT> that a multiply extended magnitude is susceptible of various metric relations and that space accordingly constitutes only a particular case of a triply extended magnitude. A necessary sequel of this is that the propositions of geometry are not derivable from general concepts of quantity, but that those properties by which space is distinguished from other conceivable triply extended magnitudes can be gathered only from experience. (411–412)

Whilst experience indicates the high probability that Euclidean propositions are valid within the observable domain, it is a separate matter to decide 'the admissibility of protracting them outside the limits of observation, not only toward the immeasurably large, but also toward the immeasurably small' (412). Riemann, as already noted, is agnostic about whether physical space is discrete or continuous, holding this to be an empirical question belonging to the science of physics (424–5). Regardless, his subject is magnitudes that are extended, divisible and quantifiable.

The general notions of quantity from which the concept of multiply extended magnitudes is constructed presuppose 'a general concept which allows various modes of determination' (Riemann 1929: 412). This is the concept of the manifold (*Manningfaltigkeit*), which denotes a domain of forms or elements organised along multiple dimensions. Riemann introduces this as a philosophical and not merely mathematical concept.[4] Depending on whether one can progress from one of the manifold's constituents to another by way of an unbroken path or not, the mode of determination will yield a manifold that is either continuous or discrete, and the manifold will be quantified accordingly. A discrete manifold comprises a collection of elements to be enumerated, like leaves on a tree or animals on a farm – or, in the case of space, if it were held to be composed of atomic units, as some of the ancients believed, or to be constituted so as to be isomorphic with a compact, well-ordered infinity of mathematical points, as Russell (1926) maintains. In ‘the doctrine of discrete quantities’, mathematicians can ‘set out without scruple from the postulate that given things are to be considered as all of one kind’ (Riemann 1929: 413). Conversely, a continuous manifold, examples of which include ‘the positions of objects of sense, and the colors’ (413), involves one or more extended continua – such as length, width, and height in the everyday conception of space – where quantity involves measurements that determine the distance, area, or volume between points, these points not constituting the continuum but rather marking the limits or transitions from one location to another.[5]

Quantities within a discrete manifold are compared simply by counting – the farm has more sheep than cattle, for example. Measurement or comparison in a continuous manifold, however,

> <EXT> consists in superposition of the magnitudes to be compared; for measurement there is requisite some means of carrying forward one magnitude as a measure for the other. In default of this, one can compare two magnitudes only when the one is a part of the other, and even then one can only decide upon the question of more and less, not upon the question of how many. (Riemann 1929: 413)

A great deal is contained in this short statement. Within a regular and ‘flat’ manifold, a magnitude taken from any location can serve as a unit of measure for any part of the rest, just as, in the space of our everyday lives, a straight edge marked off in centimetres can be superposed onto a straight line between any two points to measure the shortest distance between them. Such a manifold can be gridded as a Euclidean plane or space is, with the distance between any two points able to be the independent unit of measure. The same holds true for regular but curved manifold, such as the surface of a geometrical sphere, where a curved line taken from any part of the surface can be superposed cleanly onto any other part because the curvature in all directions is everywhere uniform. In both cases, measurements are independent of location, and 'the figures lying in them [the manifolds] can be moved without stretching' (420). Such a universal standard of measure, however, would be absent in cases where a manifold's dimensions were stretched or curved irregularly. Consider instead of the surface of a perfect sphere that of the rough and irregular surface of the Earth, where no straight or curved line could simply be taken from one part and overlaid onto any another because the curvatures of the two magnitudes would not necessarily be the same. Uniform measurement would be possible only locally, in regions where curvature or stretch was consistent, and even then it would ultimately depend – as Riemann goes on to assume – on there being lines of sufficiently small magnitude to be superposable onto any part of the manifold. In the absence even of this uniformity, however, there would be no way to carry forward one magnitude as the measure of another, meaning that two magnitudes could only be compared if one were already part of the other, and, with no consistency in the manifold to allow the larger merely to be a multiple of the smaller, the determination of the two magnitudes could only be of more or less without any specification of by how much. Investigations of these last cases, Riemann holds, 'form a general part of the doctrine of quantity independent of metric determinations, where magnitudes are thought of not as existing independent of position and not as expressible by a unit, but only as regions in a manifold' (413). Rather than exploring the matter further, however, he states that he will only draw from these considerations answers to the questions of how to conceive the construction of multiply extended magnitudes and how to move from the determination of positions in them to determinations of quantity (413). In this way, Riemann restricts himself to cases of manifolds where metric determinations are possible.

Riemann goes on first to investigate determinations of measure within a regular manifold or region of manifold. Such determinations 'require magnitude to be independent of location' (Riemann 1929: 415) and thus assume that 'the length of lines be independent of their situation, that therefore every line be measurable by every other' (415–16). They further assume that at an 'indefinitely small' (419) level there obtains a 'flatness in smallest parts' (419). In other words, at an infinitesimal level, the distances between points are rendered by straight lines, so that they equal the square root of the sum of squares of the distances from one point to the other along each of the manifolds dimensions, just as the length of a straight line on a Euclidean plane is the square root of the sum of squares of its X- and Y-lengths, and in a Euclidean space of its X-, Y- and Z-lengths.[6] Flatness in smallest parts therefore entails that every manifold is infinitesimally Euclidean and thus uniform in itself even if at a larger scale it is non-Euclidean. If the place of each point in the regular manifold or region thereof is identified by the numbers assigned to each of its dimensions – the way a point in a Euclidean space is determined by the number lines setting out its X, Y, and Z coordinates, and in this way all possible relations of place form a discrete manifold even while relations of measurement might form a continuous one (423) – then measurements of distance, area, volume and so forth can be rendered as infinite sums of differentials following the procedures of integral calculus. These formulations, again, assume that the manifold or region being considered is flat, which means that it has a curvature of 0, but while further determinations are required if the manifold or region is curved or stretched, this remains compatible with measurement through integration of differentials as long as ‘flatness in smallest parts is assumed’ (422). For certain manifolds with a positive curvature, fragments can move within them – the way bodies can move in space – without any bending, just as a geometric figure projected onto the surface of a perfect sphere would retain the same shape regardless of where it moved on the surface. However, only with a manifold whose curvature is 0 is the direction of movement also independent of position (421–2).

Riemann then considers the implications of his analysis for space as a particular three-dimensional manifold, again under the assumptions of lines, bodies, and their directions being independent of place, of flatness at an infinitesimal level, and of discreteness in location even if there is continuousness in measurement. These assumptions hold for observable space, but their extension beyond the maximum and minimum of observability is uncertain. Concerning extension beyond the maximum, Riemann maintains that immeasurably great space has unlimited extension but not necessarily infinite magnitude, since the observed independence of bodies from their positions indicates that space must have a constant curvature even if this is 0, but if space has even the slightest positive curvature the universe will close back upon itself and its magnitude will therefore be limited (Riemann 1929: 423). However, concerning what is beyond the minimum – that is, beyond the ‘spatially small’ into which ‘the microscope permits’ (424) the natural sciences to pursue phenomena, where the exactness given by the mathematics of Analysis is merely assumed – the situation is even more problematic. For if the observable independence of bodies from position is only apparent, then ‘one cannot conclude to relations of measure in the indefinitely small from those in the large’ (424). Beyond the observable minimum, real space may contain irregular curvatures with arbitrary values even while its overall curvature at the observable and measurable level remains at or close to zero (424). Moreover, it may even be the case that ‘the line element is not representable, as has been premised, by the square root of a differential expression of the second degree’ (424) – in other words, it may be that in real space flatness in smallest parts does not obtain, in which case consistency in measurement at the infinitesimal level would not hold. For this reason,

> <EXT> the empirical notions on which spatial measurements are based appear to lose their validity when applied to the indefinitely small, namely, namely the concept of a fixed body and that of a light-ray; accordingly it is entirely conceivable that in the indefinitely small the spatial relations of size are not in accord with the postulates of geometry, and one would indeed be forced to this assumption as soon as it would permit a simpler explanation of the phenomena. (424)

This would imply, however, that whatever measurements might obtain in the observable domain would not be grounded in the procedures of integration, and other considerations would be needed to account for this reality.

The possibility that flatness in smallest parts does not hold, the implications of which Riemann had set aside in order to examine manifolds for which metric relations not only of more or less but also of how much are possible, thus leads on to ‘the question concerning the ultimate basis of relations of size in space’ (Riemann 1929: 424). In this respect,

> <EXT> while in a discrete manifold the principle of metric relations is implicit in the notion of this manifold, it must come from somewhere else in

> the case of a continuous manifold. Either then the actual things forming the groundwork of a space must constitute a discrete manifold, or else the basis of metric relations must be sought for outside that actuality, in colligating forces that operate upon it. (424–5).

If space were really composed of discrete elements, then all measurements would be reduced to matters of counting, and thus the principle of space's metric relations would be found in the very notion of its extension. If space were really continuous, however, then this principle would have to be imposed from the outside and with regard to the forces that bind space together. The latter case could entail forces composing space in a way that allowed for division into arbitrary but standard units of measure – that is, these forces might constitute space in a continuous but regular form that would accord with the postulates of Euclidean geometry – or in a way that made all units of measure vary locally. Contrary to the position Riemann draws from observation, wherein Euclidean assumptions hold, Einstein's general theory of relativity uses Riemann's analysis to maintain that the metric principle for space comes from the bodies that occupy it, with the gravitational forces of each body curving space and thus determining its metrical principle within the region of its movement. It is notable that this answer applies only to macroscopic reality and not the infinitesimal world that prompted Riemann to ask his final question, and Einstein of course was never able to marry general relativity to the rules of the quantum world of discrete energy states, non-locality, and indeterminacy, to whose discovery he also contributed so much.

II. Deleuze's reading of Bergson through Riemann

When Deleuze links Bergson to Riemann, he does so in reference to Riemann's final point on the metric principle in discrete versus continuous manifolds:

<EXT> Riemann … distinguished *discrete multiplicities* and *continuous multiplicities*. The former contain the principle of their own metrics (the measure of one of their parts being given by the number of elements they contain). The latter found a metrical principle in something else, even if only in phenomena unfolding in them or in the forces acting in them. (Deleuze 1991a: 39)

However, Deleuze contends, Bergson 'profoundly changed the direction of the Riemannian distinction' (39–40) by the way in which 'continuous multiplicities seemed to him to belong essentially to the sphere of duration' (40) and so to the domain of quality. Bergson consistently portrays what he considers to be both the scientific and the philosophical conception of space as involving a real infinity of discrete points, holding it further to engender a spatialized misconception of time as an infinity of instants.[7] Statements he makes that seem manifestly to confuse countable and measurable quantity – for example, that 'the very possibility of arranging the numbers in ascending order arises from their having to each other relations of container and contained' (Bergson 1910: 2) and that 'every clear idea of number implies a visual image in space; and the direct study of the units which go to form a discrete multiplicity will lead us to the same conclusion on this point as the examination of number itself' (79) – would be absurd unless they meant to follow Riemann's thesis that within a discrete manifold the principle of measure is grounded in the numerable units constituting its extension. Bergson further, as already noted, insists upon an absolute separation of quantitative space and qualitative duration in *Time and Free Will*, and even in his later proposals to relate quantity and quality he does so only by maintaining the absolute priority of quality over quantity.[8] However, by assuming that space is a discrete manifold and duration a continuous one both in Riemann's sense, Deleuze is able to argue that although Bergson treats continuous multiplicities as qualitative, there is a kind of quantity that still pertains to them. Based on the assumed connection to Riemann, Deleuze contends that: 'for Bergson, duration was not simply the indivisible, nor was it the nonmeasurable. Rather, it was that which divided only by changing in kind, that which was susceptible to measurement only by varying its metrical principle at each stage of the division' (Deleuze 1991a: 40). This allows Deleuze to hold that, for Bergson, 'the multiplicity proper to duration had … a "precision" as great as that of science' (40), thus lending credibility to Bergson's later challenge to Einstein in *Duration and Simultaneity*. By holding that Bergson's continuous multiplicities – which, contra Riemann, are intensive rather than extensive, virtual rather than actual, related to duration rather than space, and which are constitutive of discrete multiplicities – are still directly indebted to Riemann, Deleuze is able to argue that what is at issue between Einstein and Bergson is the nature of time as a continuous multiplicity. Bergson's critique, Deleuze contends, comes out of his Riemann-influenced idea that Einstein misinterprets time as extended and numerable in a way suitable only to discrete multiplicities (78–89). Yet, for all that, duration still involves metric principles.[9]

Duration as a continuous multiplicity, then, is unextended, virtual, constitutive, qualitative, and does not divide without changing in kind, but constitutes a new immanent metrical principle with each division. This is how it is an internal difference, by which a thing differs from everything it is not. A thing is unique by virtue of the past and memory in which it participates, by virtue, that is, of the continuous recording and synthesis of qualitatively distinct states. Although originally a psychological thesis, Deleuze contends that from *Matter and Memory* onwards duration becomes an ontological recording without which not only could the past and memory not be lived but the very passage of the present would be impossible. The present can only pass if it becomes different from what it is, yet it cannot constitute the past into which it passes if it is not already passing away itself, nor can it ‘wait’ for the past to be constituted as it would then not be possible for a future present to arrive to replace it; these impasses can only be resolved if the present is *virtually past* ‘at the same time’ as it is *actually present* (Deleuze 1991a: 59–60). Every present is also virtually past in a way that contains the entire past within it, with the virtual past of one present being a layer within the virtual pasts of subsequent presents; and the past as a whole comprises infinite layers that resonate and repeat one another up to the actual present, as Bergson famously portrays in *Matter and Memory* with the image of the cone (Bergson 1991: 152). Each present is what it is by virtue of all that comes before it, and the present present, contracting the entire past into it, is driven by this virtual past into an open future, actualising the past, but in the form of a difference – a creative evolution.

Virtual internal difference thus constitutes an actuality that, while extended, discrete, and quantifiable, remains characterised by difference and novelty. Actualisation is responsible for both sides of this actuality because it gives rise to the matter that opposes creative duration. Deleuze here draws particularly on *Creative Evolution*, where Bergson explains the relation between duration and matter through the image of the jet of stream thrust into the air, with the droplets forming through condensation being likened to actualised matter that falls back against and hinders the virtual steam jet even while the latter’s force continues to drive both them and itself upwards (Bergson 1998: 247); in this way, duration and matter are portrayed as being inversely related (201). For Deleuze, duration is internal difference, while matter is this same difference relaxed and extended until it becomes a difference outside itself, and so part of a discrete multiplicity. This might seem to contradict Deleuze’s use of relaxation and contraction to describe the relation between the virtual past and actual present, since there the actual present is the virtual at its most contracted whereas here actual matter is the virtual at its most relaxed. But these refer to different processes: in its *actualisation*, the present is the most contracted form of the virtual past, compressed to the point where it burst forward in creative evolution; while in its *actuality*, as matter and space, it is duration and difference externalised and then coming into opposition not only with other actual entities but with duration’s own *élan vital*. Actuality seems to be nothing more than a discrete world of entities in extensity only when the contracted force of actualisation is forgotten. For Deleuze, these ways of relating virtual and actual in terms of contraction and relaxation resolve the seemingly irreconcilable dualisms Bergson establishes between differences in kind and difference of degree, quality and quantity, perception and recollection, past and present, memory and matter. The dualisms reflect a method that identifies differences in kind by separating them from differences of degree, carving up the given according to lines of articulation or tendencies (see Deleuze 1991a: 44–5; 2004: 33–4). But this external difference in kind between differences in kind and differences of degree – that is, between internal difference in itself and internal difference externalised into space – ultimately becomes a monism as one tendency is identified as the condition of possibility of the other: ‘if there is a privileged half in the division [into tendencies]; it must be that this half contains in itself the secret of the other’ (Deleuze 2004: 27). Thus both sides of the initial dualism turn out to be durational, each differing from its apparent contrary in terms of relative contraction and relaxation, so that the dualism of actual tendencies becomes a monism at the level of their virtual unity (Deleuze 1991a: 93).

Contraction and relaxation also differentiate orders of lived duration above and below the pace lived by human consciousness – a thesis Bergson advances particularly in *Matter and Memory* but notably rejects in *Duration and Simultaneity* – with Deleuze contending that these different rhythms of duration nevertheless belong to a single Time (Deleuze 1991a: 76–8, 82–5). However, as Deleuze acknowledges, everything Bergson criticises now seems to return to the heart of his philosophy, as the differences of contraction and relaxation appear to be nothing more than the quantitative differences of degree – that is, intensive magnitudes – of the very sort Bergson associates with abstract and faulty discrete multiplicities being used to interpret qualitative internal difference, but now embedded in the virtual differentiation meant as their replacement (75–6). Deleuze responds by holding that duration's levels of contraction and dilation amount to 'degrees of difference' rather than differences of degree (93), and as such introduce a new notion of quantity specific to the virtual. Here, the claim that Bergson's continuous multiplicities, being Riemannian in character, find their metric principles in the forces binding them together, is cashed out by Deleuze in the notion that 'one of Bergson's more curious ideas is that difference itself has a number, a virtual number, a numbering number' (Deleuze 2004: 34), and that for Bergson, 'there are numbers enclosed in qualities, intensities included in duration' (Deleuze 1991a: 92). Unlike differences of degree, which involve extensive and discrete magnitudes and thus fixed numerical differences between them, the virtual number associated with degrees of difference would change with each difference made as the virtual is creatively actualised. Virtual intensive quantity, then, would accord with Riemann's situations where, in the absence of a fixed standard of measure, two magnitudes can be compared only when one is part of the other, and even then only in terms of more or less but not how much[10] – situations, in short, where magnitudes are not independent of position but rather depend upon the forces that act upon and hold together this multiplicity. Duration expresses such a situation as it immediately differs from itself, changing continuously and qualitatively with the syntheses of the past that compose it and the *élan vital* that actualises it. Virtual quantity refers to the tendencies or forces of differentiation correlated with this actualisation.

III. Moving beyond Bergson and Riemann: the 'irrational cut'

If duration as a continuous multiplicity invokes an intensive quantity involving a more or less but not how much, then it would seem to entail the corollary suggested by Riemann of a situation where flatness in smallest parts cannot be assumed, and so a domain of quantity that is 'independent of metric determinations' and 'not … expressible by a unit' (Riemann 1929: 413). Such a quantity would accord with what Deleuze and Guattari call a 'nomad science' characterised by a special form of 'numbering number' and a 'geometry of the trait' (Deleuze and Guattari 1987: 361–74; 387–94), and also with what commentators see as a post-Riemannian patchwork space developed in Deleuze's later works generally (see Calamari 2015: 77–83; Duffy 2013: 107–15; Plotnitsky 2009: 202–7). Nonetheless, it is not clear that Deleuze finds Bergsonian duration sufficient to speak fully to this form of quantity. At stake here is the nature of intensity and the resources Deleuze feels can be mustered to give it an adequate conceptualisation. When Deleuze declares in *Difference and Repetition* that Bergson's critique of intensive quantity 'seems unconvincing' because Bergson has 'already attributed to quality everything that belongs to intensive quantities' (Deleuze 1994: 239), he quickly offers degrees of difference as a way to recover Bergson's position, holding these really to be differences of intensity within a synthesis of duration that engenders both quality and extensity (239–40). But the Bergsonian account of intensity Deleuze provides here is limited on his own broader terms, as Bergson's is a 'great synthesis of Memory' (239) and is thereby associated with only the second of three syntheses that Deleuze employs to explain the constitution of both time and space (70–128; 229–30). This second synthesis concerns the implication of intensity in the extensities that explicate it (230) and how this implication grounds the extensity in which intensive quantities seem to disappear. But intensity, Deleuze contends, is implicated first of all in itself, whereby it 'includes the unequal in itself' (232) and announces a 'universal "ungrounding"' (230). This intensity is moreover linked to a pre-quantitative and pre-qualitative power of individuation and dramatisation that determines the actualisation of the virtual.[11] It is defined by three aspects: 'the enveloping difference, the enveloped distances, and the unequal in itself which testifies to the existence of a natural "remainder" which provides the material for a change of nature [that is, a difference in kind]' (238). To my mind, Deleuze never even tries to associate Bergson with this in-itself of intensity, but consistently turns to others such as Nietzsche to theorise it as part of a third synthesis of time that ungrounds duration.

With respect to Deleuze's Bergson–Riemann construct, in this final section I will consider the 'irrational cut' as the idea that most directly challenges 'flatness in smallest parts', and so offers the mathematical correlate to the path Deleuze traces that moves him from Bergsonian duration to Nietzsche's eternal return. The irrational cut is explicitly developed in *Cinema 2* (1989), but is also found implicitly in *The Fold* (1993). It makes an early appearance, but for a very different use, in *Difference and Repetition* (1994: 172), when Deleuze refers to the 'Dedekind cut'. Richard Dedekind is the author of the idea, and coincidentally both a fellow student with Riemann at the University of Berlin and the person responsible for the posthumous publication of Riemann's *Habilitationsschrift*.

Dedekind's thesis concerns continuity and the notion of irrational numbers needed to complete a mathematical account of it. These are crucial for the infinitesimal analysis upon which Riemann's own thesis depends, 'and yet an explanation of this continuity is nowhere given' (Dedekind 1963: 2). Furthermore, the introduction of irrational numbers 'is based directly upon the conception of extensive magnitudes – which itself is nowhere carefully defined – and explains number as the result of measuring such a magnitude by another of the same kind' (9–10). Against this, Dedekind demands that both the continuum and irrational numbers are 'established in a purely arithmetic manner' (2), whereby 'arithmetic shall be developed out of itself' (10).

Deriving rational numbers by arithmetic means is straightforward, as Dedekind regards 'the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetic act, that of counting' (Dedekind 1963: 4). Although the limitations arising from subtraction and division each require 'a new creative act' (4) in the form of the inventions of negative numbers and fractions respectively, ultimately 'the system of all rational numbers … denote[d] by ***R***' (4–5)[12] follows directly from 'the four fundamental operations of arithmetic' (4). The completeness of this system is confirmed by the fact that, with the exception of division by 0, the four operations can be performed using any two rational numbers and will always yield a rational number (5). But the system is further characterised by the property of forming 'a well-arranged domain of one dimension extending to infinity on two opposite sides' (5). This means that each rational number separates the entire system into two exclusive classes, one containing all rational numbers less than the given number and the other containing all rational numbers greater than it, with the number itself being freely assignable as either the highest of the first class or the lowest of the second (6). Each such division corresponds to one number only, as the classes formed by any two distinct numbers cannot be the same regardless of their proximity to each other.

Dedekind then compares the rational number system to the points of a straight line, declaring that the analogy between them 'becomes a real correspondence when we select upon the straight line a definite origin or zero-point *0* and a definite unit of length for the measurement of segments' (Dedekind 1963: 7–8). Extensive magnitude and measurement by superposition of magnitudes are thereby introduced, but the geometric image is derived from the arithmetic deduction of the number system, not the reverse. But this is not the case for the introduction of irrational numbers. The fact that between any two rational numbers lies infinitely many other rational numbers (6) might seem to indicate a complete correspondence of that rational number system to the continuous straight line, but for the ancient Greeks' discovery of the incommensurability of the square with its diagonal, which entails that an arc drawn from the end point of the diagonal onto a horizontal number line extending from and measuring the square's base will intersect that straight line at a point corresponding to no rational number (8–9). This demonstrates that 'in the straight line *L* there are infinitely many points which correspond to no rational number' (8), meaning that 'the straight line *L* is infinitely richer in point-individuals than the domain ***R*** of rational numbers in number-individuals' (9). It is therefore necessary to construct a new domain of numbers that 'shall gain the same completeness, or as we may say at once, the same *continuity*, as the straight line' (9). This is 'the system $\mathfrak{R}$ of all real numbers' (19), which includes all rational and irrational numbers. Despite geometric considerations providing 'the immediate occasion' for this extension, they are not themselves 'sufficient ground for introducing these foreign notions into arithmetic, the science of numbers' (10). Dedekind thus declares: 'we must endeavor completely to define irrational numbers by means of the rational numbers alone' (10).

This context of defining irrationals leads Dedekind to introduce the idea of 'a *cut* [Schnitt]' (Dedekind 1963: 13) in a number system. It was prefigured by the idea that each rational number separates its system ***R*** into two distinct infinite classes, with the separating number assignable to either class, but importantly that property did not define the rational numbers themselves, as they were already derived from the basic arithmetic operations. In contrast, although once introduced the property of cutting applies to both rational and irrational numbers, it is the basis for defining the latter as an extension of the former. Moreover, it is used to define the continuity the real number system must demonstrate, Dedekind summing this up, again by way of geometric reference, as follows: ‘If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions’ (11). Each rational number is seen to enact two cuts, one in which it is the highest of the lower class it creates and the other in which it is the lowest of the other class, but these two cuts are ‘only unessentially different’ (17). In contrast, two distinct numbers, be they rational or irrational, are considered ‘*different* or *unequal* always and only when they correspond to essentially different cuts’ (15), which means when, in addition to the cutting number, there is at least one other number that belongs in the higher class for the cuts made by one number and the lower class for the cuts made by the other (17).

Dedekind proceeds to demonstrate ‘that there are infinitely many cuts not produced by rational numbers’ (13). The numbers corresponding to these cuts are specified through the proposition of a positive integer D whose square root is not an integer; it follows that the root falls between two consecutive integers, λ and $\lambda+1$, whose squares will be less than and greater than D respectively. The root of D will divide the number line such that all positive rational numbers whose squares are greater than D will be greater than D’s square root, while all other rational numbers, be they positive or negative, will by implication be less than the root. The root itself, however, will not be a rational number, as demonstrated by indirect proof (13–15).[13] These irrational numbers, Dedekind holds, fill the gaps ‘between’ rational numbers, and the classes of rational numbers greater and less than the irrational number now said to cut between them will approach but never reach this irrational. In this way, irrational numbers serve as limits of convergent series of rational numbers, but this threatens the idea of continuity inasmuch as the irrational number itself will be unassignable to either class. Nevertheless, the unique place of each irrational number, required to demonstrate that the real number system is both well-ordered on one dimension and continuous, follows from there necessarily being an infinity of rational numbers lying between any irrational number and any other rational or irrational number, so that no two numbers, rational or irrational, will cut the real number system into the same two classes; with that, every real number will be freely assignable to either of the classes their cuts create (20–1).[14] Just as arithmetic operations with rational numbers yield definitive results, so too with real numbers, conceived as cuts: the sum of two numbers, for example, will be another definitive cut, whether rational or irrational (21–4). Moreover, the continuity of the real number system ensures the findings of infinitesimal analysis by confirming the existence of definitive limit values, so that, for example, where a magnitude $\boldsymbol{x}$ grows continually but not beyond all limits, there will be a definitive value $\boldsymbol{a}$ that is the lowest value of a class $\boldsymbol{U_2}$ formed by a cut of $\mathfrak{R}$ and towards which the value of $\boldsymbol{x}$ will approach by passing through the infinite values of the lower class $\boldsymbol{U_1}$ without ever reaching it (24–7).

Despite claiming to have done otherwise, it is unclear that Dedekind ever provides his sought after rigorous arithmetic demonstration for the continuity and well-ordered character of real numbers. Irrational numbers are not derived from the four fundamental arithmetic operations, but instead are defined by reference to the cut, and only once they are so defined is it argued that the operations can be performed with them to yield the definitive results in the same way as numbers derived from those operations. The geometric crutch persists in the very notion of the cut, which not only derives from the figure of the diagonal of the square cutting the number line, but also borrows from the pictures associated with it in everyday language. It also remains a mere assumption on Dedekind's part that irrationals are indeed numbers, as this is not proven by the fact that operations performed on numbers can also be performed upon them, and it is only again with reference to the geometric image that a new number seems to be required wherever a point on the straight line corresponds to an incommensurable length.[15] Without this assumption, the well-ordered place of irrationals within the continuum rests solely on the image of the precise and well-placed cut of the straight line, as the well-ordered rational numbers approach but never reach it, while, conversely, the endless and never repeating arithmetic expansion of each irrational means none are ever completely given but only approximated to rational numbers: even when π is calculated to two billion decimals, the result is still merely a rational number. An irrational number would then no longer be freely assignable as the definition of continuity demands. These considerations suggest the possibility of seeing irrational cuts as something other than well-ordered supplements that complete the continuity of the one-dimensional number line.

When Deleuze references Dedekind in *Difference and Repetition*, he attributes to the cut the power of constituting for the virtual 'the next genus of number, the ideal cause of continuity or the pure element of quantitatability' (Deleuze 1994: 172), so that the virtual is a continuous multiplicity to which quantity nevertheless pertains.[16] When he later invokes the idea of the cut, however, focussing solely on the irrational cut, it plays a very different role of expressing a power of discontinuity and incommensurability. In *The Fold*, the irrational and the differential relation are invoked together, with the first being 'the common limit of two convergent series [of rational numbers], of which one has no maximum and the other no minimum [since no rational number is "nearest" to its irrational limit]', and the second being 'the common limit of the relation between two quantities that are vanishing [as *dy* and *dx* each approach 0]' (Deleuze 1993: 17). What the irrational and the differential quotient share, however, is ‘the presence of a curved element [that] acts as a cause’ (17), this being illustrated by the figure Dedekind himself uses of the arc drawn from the diagonal descending onto and cutting the number line. But rather than demonstrating the complete and well-ordered nature of the straight line, Deleuze maintains that it demonstrates its discontinuity and disparateness:

> <EXT> The irrational number implies the descent of a circular arc on the straight line of rational points, and exposes the latter as a false infinity, a simple undefinite that includes an infinity of lacunae; that is why the continuous is a labyrinth that cannot be represented by a straight line. The straight line always has to be intermingled with curved lines. (17)

In *Difference and Repetition*, Deleuze associates this same image with the eternal return understood not as a return of identical events in time but rather as the discontinuous structure of time itself: it is a ‘straight-line labyrinth … “invisible, incessant”’ (Deleuze 1994: 111), but this disjointed line also ‘reconstitutes an eternally decentred circle’ (115) that ensures that only difference returns in time.

When Deleuze introduces the irrational cut as a modern film technique that ‘determines the non-commensurable relations between images’, and, as such, is ‘no longer a lacuna that the associated images would be assumed to cross’ (Deleuze 1989: 213), he directly associates it with the mathematics of cuts:

> <EXT> Cinema and mathematics are the same here: sometimes the cut, so-called *rational*, forms part of one of the two sets which it separates (end of one

> or beginning of the other). … Sometimes, as in modern cinema, the cut has become the interstice, *it is irrational and does not form part of either set, one of which has no more an end than the other has a beginning*: false continuity is such an irrational cut. (181)

These cuts also relate directly to a Nietzschean 'power of the false' that is absent in Bergsonian duration – it is unsurprising that Bergson disappears from Deleuze's analysis of cinema once the powers of the false are introduced – inasmuch as it 'poses the simultaneity of incompossible presents, or the coexistence of not-necessarily true pasts … inexplicable differences to the present and alternatives which are undecidable between true and false to the past' (131). While duration's continuity of virtual past and actual present 'concerned the *order of time*, that is, the coexistence of relations or the simultaneity of the elements internal to time', the power of the false 'concerns the *series of time*, which brings together the before and the after in a becoming, instead of separating them; its paradox is to introduce an enduring interval in the moment itself' (155). The irrational cut achieves this because even while the images it brings together are held in the same moment without any extensive separation, they remain out of sync with each other and thus never simultaneous. Recast here as a temporal determination, the irrational cut dovetails with the eternal return conceived by Deleuze as the structure of an out-of-sync time.

The irrational cut thus replaces flatness in smallest parts with the constitution of a 'disjunction of the two images, at the same time as their new type of relation, a relation of very precise incommensurability' (Deleuze 1989: 256). It installs within rational continuous manifolds a principle of foundational irrational discontinuity. Its time as series expresses 'the intrinsic quality of that which becomes in time' (275). In this respect, while particular irrational cuts such as cinematic cuts may introduce difference, the irrational cut as a structure is not itself the production of the new but rather the guarantor of the possibility of novelty – it is what ensures that creative evolution is in fact creative. In this way, it is the internal, intensive difference and power on which duration depends.

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[1] Earlier versions of this paper were presented at the 2016 Society for European Philosophy – Forum for European Philosophy annual conference at Reagents College, London, and the 2017 Deleuze Studies annual conference in Toronto.

[2] Riemann’s *Manningfaltigkeit* is translated directly into English as ‘manifold’ but into French as ‘*multiplicité*’ and from there ‘multiplicity’ in English translations of the French. I will therefore use manifold when addressing Riemann’s work but multiplicity when dealing with Bergson’s and Deleuze’s work.

[3] I use the Henry S. White translation found in Smith’s *A Source Book in Mathematics* (Riemann 1929) rather than the original William Kingdon Clifford translation of 1873 (Riemann 1882) used by other Deleuze scholars

who have engaged Riemann's work such as Calamari (2015), Duffy (2013), Durie (2004), Plotnitsky (2006 and 2009), and Voss (2013). Not only is White's translation generally clearer and more comprehensible, but White's variable rendering of *Grösse* as 'magnitude' or 'quantity' – it can be translated as either, and in its mathematical sense it refers to quantity in general, to anything that can be increased or diminished ('*als fachausdruck der mathematik viel gebraucht, und zwar als verdeutschung von quantitas: quantitas, eine grösze, heiszet in der mathematik alles, was sich vermehren und vermindern lässet*' [Grimm and Grimm 1935; thanks to Julia Ng for pointing me to this usage]) – and particularly of *Grössenbegriff* primarily as 'notion of quantity' (it could also be 'concept') and of *Grössenbestimmung* as 'determination of quantity' or 'determination of magnitude' as appropriate to the context, allows his text to position magnitude as a subset of quantity and to treat the latter as dividing into discrete and continuous forms. Such nuances make White's translation consistent with mathematical and philosophical distinctions that Riemann, acknowledged to be well-trained philosophically (see Plotnitsky 2009: 191), would have known well. Clifford, in contrast, almost always uses 'magnitude' in places where 'quantity' would be more appropriate, though he does translate *Grössenbestimmung* in largely the same ways as White. Compare, for example, his translation of the last quote above, which has Riemann describing his project in the rather circular and unilluminating way as 'constructing the notion of a multiply extended magnitude out of general notions of magnitude' (Riemann 1882: 55–6).

The peculiarities of the translation used may not matter for the work done by these other authors, who, with the exception of Durie, neither engage in close readings of Riemann's text nor address its relation to Deleuze's early writings on Bergson (this is true of Calamari's article even though Deleuze's *Bergsonism* features in the title), but are generally focused instead on later Riemann-inspired developments in the history of mathematics and how these are taken up in Deleuze's later writings. Nevertheless, I suspect that certain problematic statements that seem to attribute to Riemann the view that continuous multiplicities are qualitative and non-metrical rather than quantitative, or that Euclidean geometry applies to discrete but not to continuous space (see Duffy 2013: 103–7; Durie 2004: 65 and 67n16; Plotnitsky 2006: 191 and 2009: 200), may reflect the translation they are using.

The French translation of Riemann referenced by Deleuze (Riemann 1898: 280–299) seems in all cases to translate *Grösse* and its related terms with '*grandeur*', which ordinarily refers to magnitude but whose mathematical usage, similarly to *Grösse*, denotes '*Quantité, tout ce qui est susceptible d'augmentation ou de diminution*' (Littré 1957).

[4] 'While I now attempt in the first place to solve the first of these problems, the development of the concept of manifolds multiply extended, I think myself the more entitled to ask considerate judgment inasmuch as I have had little practise in such matters of a philosophical nature, where the difficulty lies more in the concepts than in the construction' (Riemann 1929: 412). It should be noted that Deleuze's many invocations of mathematics almost always involve texts or ideas that expressly concern the philosophical foundations of mathematics. As such, his references to mathematics do not serve to provide a ground for philosophical concepts or offer ideas analogous to those concepts, nor do they serve as a form of speculative construction of such concepts. Rather, they articulate philosophical problems that appear alongside or subjacent to the mathematical domain.

[5] Hence the individual modes of a continuous manifold are 'points' and those of a discrete manifold are 'elements' (Riemann 1929: 412), and the determinate parts of a manifold can be distinguished either by a superficial 'mark' or a substantive 'boundary' (413).

[6] Thus, in a Euclidean space, given two points positioned at X_0, Y_0, Z_0 and X_1, Y_1, Z_1, the distance S between them is $\sqrt{(X_1 - X_0)^2 + (Y_1 - Y_0)^2 + (Z_1 - Z_0)^2}$, and similarly with a higher number of dimensions. To assume the flatness of the smallest parts is to assume that this formulation holds at an infinitesimal level even though the manifold's curvature or stretch may preclude this at greater magnitudes – in short, that $ds = \sqrt{dx^2 + dy^2 + dz^2 + \ldots}$.

[7] See, for example, Bergson (1983: 142–58; 1991: 206; 1998: 154–57), and also Bergson's assertion in his Latin thesis that for Aristotle the division of a body into parts containing other parts 'will go on into infinity' (Bergson 1970: 70).

[8] On this point see Widder (2012).

[9] Bergson's critique of Einstein, however, is directed at the special theory of relativity, which does not consider gravity, whereas it is Einstein's general theory of relativity that builds on Riemann's thought.

[10] Compare with Bergson's early critique of the concept of intensive quantities as being one for which, while 'not admitting of measure … it can nevertheless be said that it is greater or less than another intensity' (Bergson 1910: 3). Bergson's response is that this is self-contradictory, since 'as soon as a thing is acknowledged to be capable of increase and decrease, it seems natural to ask by how much it decreases or by how much it increases' (72). Against this, Deleuze contends that Bergson's critique is ambiguous, as it is unclear that it is 'directed against the very notion of intensive quantity' rather than 'merely against the idea of an intensity of psychic states' (Deleuze 1991a: 91–2). It is from this purported ambiguity that Deleuze suggests that some form of quantity remains in Bergson's notion of qualitative duration (92–4).

[11] Determining how the intensive is not the same as the difference constituting the virtual is a central aspect of Deleuze (1994: ch. 5).

[12] Dedekind's essay predates Cantor's introduction of set theory into mathematics by two years.

[13] The indirect proof first assumes that the square root in question is a rational number that can be put as a ratio of two integers in the form $\sqrt{(a^2/b^2)}$ before demonstrating that the assumption cannot hold for one of the variables. This proof of necessity leaves out negative real numbers, since considerations of the square root of any negative number requires the introduction of the complex number system that includes imaginary units (i). But negative real numbers would follow afterwards from the application of basic operations to the positive real numbers derived from the argument.

[14] The translation incorrectly concludes the proof by holding that the real number cuts the rational number system $\boldsymbol{R}$ rather than the real number system $\mathfrak{R}$. Compare with the original German (Dedekind 1872: 26)

[15] For these and related criticisms of Dedekind see Widder (2008: 22–33).

[16] On the role of this reference to Dedekind in *Difference and Repetition* see Voss (2013: 236–41).