PHYSICAL REVIEW A **89**, 062109 (2014)

Discrete spacetime and relativistic quantum particles

Terence C. Farrelly[1,*] and Anthony J. Short[2,†]
[1] *DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[2] *H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, United Kingdom*
(Received 24 February 2014; published 16 June 2014)

We study a single quantum particle in discrete spacetime evolving in a causal way. We see that in the continuum limit, any massless particle with a two-dimensional internal degree of freedom obeys the Weyl equation, provided that we perform a simple relabeling of the coordinate axes or demand rotational symmetry in the continuum limit. It is surprising that this occurs regardless of the specific details of the evolution: it would be natural to assume that discrete evolutions giving rise to relativistic dynamics in the continuum limit would be very special cases. We also see that the same is not true for particles with larger internal degrees of freedom, by looking at an example with a three-dimensional internal degree of freedom that is not relativistic in the continuum limit. In the process, we give a formula for the Hamiltonian arising from the continuum limit of massless and massive particles in discrete spacetime.

DOI: 10.1103/PhysRevA.89.062109 PACS number(s): 03.65.Pm

I. INTRODUCTION

Approximating physical systems in continuous spacetime by discrete systems is an important challenge in physics. For example, to simulate physics in the continuum, one typically discretizes spacetime and other degrees of freedom. Also, it is often useful to define quantum field theories in continuous spacetime as the continuum limit of quantum field theories in discrete spacetime [1]. Furthermore, it is tempting to speculate that spacetime might be discrete at some small scale. A prominent example of this is causal set theory [2]. Whatever the motivation, if discrete spacetime models are to be useful, they must approximate the dynamics of continuous physical systems at low energies.

Here, we will study a single quantum particle evolving in a causal and translationally invariant way in discrete spacetime, where causal means that there is a maximum speed of propagation of information. In fact, for this to be possible in discrete space, we *must* take time to be discrete [3]. Furthermore, in order to obtain nontrivial dynamics, we must give the particle an internal "spin" degree of freedom [4]. Such single-particle evolutions are examples of discrete-time quantum walks, which are useful in quantum computing [5].

To show that such discrete dynamics approximate physical systems in continuous spacetime, we take the continuum limit of the discrete evolution. Our main result is that the continuum limit of the evolution of a discrete massless particle with an additional two-dimensional degree of freedom is always equivalent to a particle obeying the relativistic Weyl equation $[d\psi(t)/dt = \pm\vec{\sigma}\cdot\vec{P}\psi(t)]$ if we relabel the coordinate axes in a simple way (by rotating, rescaling, and removing a constant velocity shift). Alternatively, if such a discrete evolution is chosen to have rotational symmetry in the continuum limit, then it must obey the Weyl equation. Discrete models can not have continuous spacetime symmetries, so it is surprising that the emergence of Lorentz symmetry is generic for these models. Note that the resulting continuum particles are noninteracting.

The Weyl and Dirac equations describe the free evolution of spin-half fermions in the continuum, which, together with bosonic fields, are the basic constituents of nature. Finding discrete causal models that may reproduce these systems in the continuum limit is a useful endeavor, particularly because such models may be well suited to simulation by quantum computers [3,6].

Examples of quantum particles in discrete spacetime have been studied in connection with relativistic dynamics in [7–14]. In particular, [8] has examples of discrete quantum particles that obey the three-dimensional Weyl and Dirac equations in the continuum limit. In fact, by making some requirements on how the evolutions transform under rotations, [8,15] show that discrete evolutions with a body-centered-cubic neighborhood and two-dimensional extra degrees of freedom obey the Weyl equation in the continuum limit.

After introducing notation, we discuss causal quantum particles in discrete spacetime in Sec. III. Then, in Sec. IV we take their continuum limit. In Sec. V, we show that, if the discrete particle is massless and has a two-dimensional extra degree of freedom, then it obeys the Weyl equation in the continuum limit. In Sec. V A, we reproduce the discrete evolution given in [8] that becomes a particle obeying the Weyl equation in the continuum limit. In Sec. V B, we see that the continuum limit of massless systems with more than two extra degrees of freedom may have rotational but not necessarily Lorentz symmetry. In Sec. VI, we look at the continuum dynamics with mass included. We conclude with a discussion in Sec. VII.

II. SETUP

We label discrete space coordinates by vectors $\vec{n}$, where each of the d components of $\vec{n}$ takes integer values. Then, the orthornormal basis $|\vec{n}\rangle$ of the Hilbert space $\mathcal{H}_P$ describes the particle's position. The particle also has a finite-dimensional extra degree of freedom described by states in $\mathcal{H}_S$, so its total state space is $\mathcal{H}_P \otimes \mathcal{H}_S$. The extra degree of freedom will often correspond to spin or chirality in the continuum limit.

*tcf24@cam.ac.uk
†tony.short@bristol.ac.uk

We are assuming time translation invariance, so the evolution operator U_D is the same for every time step. We denote the identity on a Hilbert space $\mathcal{H}_X$ by $\mathbb{1}_X$. And, if, for example, A is an operator on $\mathcal{H}_Y$ and ψ is a vector in $\mathcal{H}_X \otimes \mathcal{H}_Y$, then we write $A\psi$ to mean $(\mathbb{1}_X \otimes A)\psi$.

We will mostly be interested in particles that obey the Weyl equation in the continuum limit, which is the equation of motion of massless chiral fermions. This means that they evolve via the Weyl Hamiltonian $H = \pm\vec{\sigma}\cdot\vec{P}$, with $c = \hbar = 1$. The components of $\vec{\sigma}$ are the three Pauli operators, which act on the particle's spin, and $\vec{P}$ is the momentum operator. The plus sign corresponds to right-handed particles and the minus sign corresponds to left-handed particles.[1]

III. PROPERTIES OF QUANTUM PARTICLES IN DISCRETE SPACETIME

To get some intuition, it is useful to look at a simple example. Suppose we have a particle on a discrete line of points, with an extra degree of freedom described by the orthonormal states $|r\rangle$ and $|l\rangle$. One possible evolution is

$$U_D = S|r\rangle\langle r| + S^\dagger|l\rangle\langle l|, \tag{1}$$

where S is the unitary shift operator that takes the position state $|n\rangle$ to $|n+1\rangle$. But, this evolution is not terribly interesting: U_D merely shifts all $|l\rangle$ states to the left and all $|r\rangle$ states to the right. Instead, we can consider the new evolution

$$U_D = W(S|r\rangle\langle r| + S^\dagger|l\rangle\langle l|), \tag{2}$$

where W is a unitary operator on $\mathcal{H}_S$.

With initial state $|r\rangle|0\rangle$, U_D first shifts the position from $|0\rangle$ to $|1\rangle$, and then W takes $|r\rangle$ to a superposition of $|r\rangle$ and $|l\rangle$. Over the next time step, because the state now has overlap with both $|l\rangle$ and $|r\rangle$, the particle spreads out and is effectively slowed down. This is a simple discrete analog of how mass mixes chiralities in the Dirac equation.

Let us now consider a general causal quantum particle on a lattice. Translational invariance allows us to write the evolution operator in a simple form. First,

$$U_D = \sum_{\vec{n},\vec{q}} A^{\vec{n}}_{\vec{q}}|\vec{n}+\vec{q}\rangle\langle\vec{n}|, \tag{3}$$

where $A^{\vec{n}}_{\vec{q}} = \langle\vec{n}+\vec{q}|U_D|\vec{n}\rangle$ is an operator on $\mathcal{H}_S$. Translational invariance means $A^{\vec{n}}_{\vec{q}}$ does not depend on $\vec{n}$. With $A_{\vec{q}} = A^{\vec{n}}_{\vec{q}}$, and defining $S_{\vec{q}}$ to be the operator that shifts a position state by $\vec{q}$, we have

$$U_D = \sum_{\vec{q}} A_{\vec{q}} S_{\vec{q}}. \tag{4}$$

We also impose causality, so that $A_{\vec{q}}$ will only be nonzero for some finite set of vectors $\vec{q}$. Note that an extra degree of freedom is required for these particles to have nontrivial evolution, where trivial means U_D is just proportional to a shift operator [4].

Finally, before we take the continuum limit, we will define massive and massless evolution. Unitarity implies that

$$U_D^\dagger U_D = \sum_{\vec{q}} A^\dagger_{\vec{q}} S^\dagger_{\vec{q}} \sum_{\vec{p}} A_{\vec{p}} S_{\vec{p}} = \mathbb{1}_D. \tag{5}$$

But, terms such as $S^\dagger_{\vec{q}} S_{\vec{p}}$ with $\vec{q} \neq \vec{p}$ must vanish, so it follows that

$$\sum_{\vec{q}\neq\vec{p}} A^\dagger_{\vec{q}} A_{\vec{p}} = 0 \quad \text{and} \quad \sum_{\vec{q}} A^\dagger_{\vec{q}} A_{\vec{q}} = \mathbb{1}_S, \tag{6}$$

which implies that $\sum_{\vec{q}} A_{\vec{q}}$ is a unitary operator on $\mathcal{H}_S$. This allows us to write

$$U_D = W \sum_{\vec{q}} A'_{\vec{q}} S_{\vec{q}}, \tag{7}$$

where $W = \sum_{\vec{q}} A_{\vec{q}}$ is a unitary on $\mathcal{H}_S$ and $A'_{\vec{q}} = W^\dagger A_{\vec{q}}$ such that $\sum_{\vec{q}} A'_{\vec{q}} = \mathbb{1}_S$. Then, analogously to the example at the beginning of this section, if $W = \mathbb{1}_S$, we say that the particle is massless.

For now, we will focus on massless evolutions, but later in Sec. VI we will look at continuum limits of massive evolutions. In the massive case, one way to ensure that the dynamics will have a continuum limit is to let W tend to $\mathbb{1}_S$ as the length of the time step, δt, goes to zero.

In a sense, massless evolutions seem more natural because to take the continuum limit, we need only shrink the lattice spacing and the length of the time step; the evolution on the lattice remains the same. On the other hand, for massive evolutions we need to make the discrete evolution dependent on the lattice scale to get a continuum limit.[2]

IV. TAKING THE CONTINUUM LIMIT

Now, we will take the continuum limit of these discrete evolutions. The discrete evolution operator is

$$U_D = \sum_{\vec{q}} A_{\vec{q}} S_{\vec{q}}, \tag{8}$$

which has the corresponding continuum Hamiltonian

$$H = \left(\frac{a}{\delta t}\right) \sum_{\vec{q}} A_{\vec{q}} (\vec{q}\cdot\vec{P}), \tag{9}$$

where a is the lattice spacing and δt is the discrete time step. To see this, we look at states that are smooth over many lattice sites, which is equivalent to looking at the subspace of states with low momentum.

Discrete momentum states are

$$|\vec{p}\rangle = \frac{1}{a^{d/2}} \sum_{\vec{n}} e^{i\vec{p}\cdot\vec{n}a}|\vec{n}\rangle, \tag{10}$$

where the components of $\vec{p}$ take values in $(-\frac{\pi}{a}, \frac{\pi}{a}]$.

[1] We can rewrite the right- and left-handed Weyl equations in a form that makes their Lorentz invariance more obvious: $i\sigma^\mu\partial_\mu\psi(x) = 0$ and $i\overline{\sigma}^\mu\partial_\mu\psi(x) = 0$ are the right- and left-handed Weyl equations, respectively, where $\sigma^\mu = (\mathbb{1},\vec{\sigma})$ and $\overline{\sigma}^\mu = (\mathbb{1},-\vec{\sigma})$. Lorentz invariance follows because σ^μ and $\overline{\sigma}^\mu$ transform like four vectors under Lorentz transformations.

[2] Although this is necessary to get the Dirac equation as the continuum limit of a discrete evolution (see Sec. VI), it is reassuring to note that in the standard model, fermions are fundamentally massless and only acquire mass through the Higgs mechanism.

Continuum momentum states are

$$|\vec{p}\rangle = \int_{-\infty}^{\infty} d^d x\, e^{i\vec{p}\cdot\vec{x}} |\vec{x}\rangle, \tag{11}$$

where the components of $\vec{p}$ take values in $\mathbb{R}$.

Now, we identify the discrete particle's momentum states with those of a continuum particle with the same value of $\vec{p}$. When acting on states with high momentum, the continuum and discrete evolutions will be very different. But, the two evolutions will be similar if we restrict to low-momentum states. Let us define $\mathcal{H}_\Lambda$ as the space spanned by states with $|\vec{p}| \leqslant \Lambda \ll \frac{\pi}{a}$, and define $\tilde{U}_D$ and $\tilde{H}$ to be the restriction of U_D and H to $\mathcal{H}_\Lambda$.

Consider a discrete evolution for n time steps of length δt, corresponding to a total evolution time $t = n\delta t$. To compare the discrete and continuum evolution, with the latter given by e^{-iHt}, on the low-momentum subspace, we evaluate

$$\left\| e^{-iHt}|\psi_\Lambda\rangle - U_D^n|\psi_\Lambda\rangle \right\|_2 \leqslant \left\| e^{-i\tilde{H}t} - \tilde{U}_D^n \right\|, \tag{12}$$

where $|\psi_\Lambda\rangle \in \mathcal{H}_\Lambda$ and $\|\ldots\|$ is the operator norm on $\mathcal{H}_\Lambda$. Next, we use the inequality for unitaries U and V: $\|U^n - V^n\| \leqslant n\|U - V\|$ [16]. It follows that

$$\left\| e^{-i\tilde{H}t} - \tilde{U}_D^n \right\| \leqslant n\| e^{-i\tilde{H}\delta t} - \tilde{U}_D \|. \tag{13}$$

To bound the right-hand side, note that the evolution operator for a discrete particle can be written as

$$U_D = \sum_{\vec{q}} A_{\vec{q}} S_{\vec{q}} \equiv \sum_{\vec{q}} A_{\vec{q}} \exp[-i(\vec{q}\cdot\vec{P})a], \tag{14}$$

where $\vec{P}$ is the momentum operator. By taking the Taylor expansions of both $e^{-i\tilde{H}\delta t}$ and $\tilde{U}_D$, we show in Appendix A that for sufficiently small values of Λa,

$$\| e^{-i\tilde{H}\delta t} - \tilde{U}_D \| \leqslant C(\Lambda a)^2, \tag{15}$$

where C is a constant. (The bound for the massive case is slightly different. See Sec. VI for details.) Then,

$$\left\| e^{-iHt}|\psi_\Lambda\rangle - U_D^n|\psi_\Lambda\rangle \right\|_2 \leqslant Ct\Lambda^2 \frac{a^2}{\delta t}. \tag{16}$$

To get a continuum limit, we fix t and let $a,\delta t \to 0$ in such a way that $a/\delta t$ is constant. Because t is fixed, the number of time steps n must tend to infinity. We also take $\Lambda \to \infty$ at a slower rate than $a \to 0$, such that $\Lambda^2 a \to 0$. As the right-hand side of (16) tends to zero and the momentum cutoff tends to infinity, this tells us that the discrete evolution defined by U_D converges to the continuum evolution generated by the Hamiltonian H.

V. CONTINUUM HAMILTONIAN

In this section, we will look at the continuum Hamiltonian. For now we will suppose that these particles live in three spatial dimensions. At the end of the section, we will comment on what changes when $d \neq 3$.

First, we will see that, if we can construct a massless evolution with a two-dimensional extra degree of freedom that has the rotational symmetries of the lattice in the continuum limit, it must also have Lorentz symmetry. Suppose that the continuum Hamiltonian has the rotational symmetries of the lattice. The Hamiltonian is

$$H = \vec{B}\cdot\vec{P}, \tag{17}$$

where $\vec{B} = (\frac{a}{\delta t})\sum_{\vec{q}} A_{\vec{q}}\,\vec{q}$. As each B_i is Hermitian, we have $B_i = c_i \mathbb{1}_S + \vec{n}_i\cdot\vec{\sigma}$, with c_i and $\vec{n}_i$ real. That the evolution has the rotational symmetries of the lattice implies that there is a subgroup G of SU(2) whose action on $\{B_i : i = 1,2,3\}$ is a representation of these symmetries. Now, for a three-dimensional lattice and a given i and $j \neq i$ there must be a $V \in G$ such that $VB_iV^\dagger = -B_i$ and $VB_jV^\dagger = B_j$. This implies that $c_i = 0$, and also that $\mathrm{tr}[B_i^\dagger B_j] = 0$, which in turn means that $\vec{n}_i\cdot\vec{\sigma}$ form an orthogonal set. Furthermore, for any i and j there must exist a $V \in G$ such that $VB_iV^\dagger = B_j$, so we must have $|\vec{n}_i| = |\vec{n}_j|$. It follows that B_i are proportional to a representation of σ_i or $-\sigma_i$. We can modify the constant of proportionality by rescaling a or δt. If we embed the lattice in the continuum with $a/\delta t$ chosen such that the constant of proportionality is one, the Hamiltonian will be equal to either the left- or right-handed Weyl Hamiltonian, which describes a Lorentz-invariant evolution.

Now, we will show that requiring rotational symmetry of H is not quite necessary, meaning any massless discrete particle obeys the Weyl equation in the continuum limit if it has a two-dimensional extra degree of freedom.

We can rewrite the Hamiltonian [in Eq. (17)] as

$$H = \sigma_1\tilde{P}_1 + \sigma_2\tilde{P}_2 + \sigma_3\tilde{P}_3 + \vec{\beta}\cdot\vec{P}, \tag{18}$$

where $\vec{\beta}$ is a real vector and $\tilde{P}_i$ are real linear combinations of components of the momentum vector operator $\vec{P}$. Now, the overall shift term $\vec{\beta}\cdot\vec{P}$ is physically meaningless, so we remove it by changing to coordinates that are moving with a constant velocity $\vec{\beta}$. This gets us closer to the Weyl Hamiltonian, but $\tilde{P}_i$ are not necessarily momentum operators in orthogonal directions. To fix this we should think of $\sigma_1\tilde{P}_1 + \sigma_2\tilde{P}_2 + \sigma_3\tilde{P}_3$ as a sum of tensor products of vectors since σ_i and $\tilde{P}_j$ both span vector spaces. Now, we use the singular value decomposition (Chap. 7 of [17]) to rewrite H as

$$H = \gamma_1\sigma_1'P_1' + \gamma_2\sigma_2'P_2' + \gamma_3\sigma_3'P_3',$$

where σ_i' are spin operators along orthogonal axes, P_i' are momentum operators along orthogonal spatial axes, and γ_i are real numbers. Note that we can choose σ_i' and P_i' to be real combinations of σ_i and P_j, respectively [17]. This is necessary so that P_i' and σ_i' have the right physical interpretation. If all the γ_i are nonzero, we can rescale the spatial axes so that $\gamma_i P_i' \to P_i'$. Then, dropping primes, we get

$$H = \sigma_1 P_1 + \sigma_2 P_2 + \sigma_3 P_3 \equiv \vec{\sigma}\cdot\vec{P}, \tag{19}$$

where σ_i are a representation of the Pauli operators.[3] If any of the $\gamma_i = 0$, then the Hamiltonian is that of a lower-dimensional Weyl equation. This means that all massless discrete quantum particles with a two-dimensional extra degree of freedom obey the Weyl equation in the continuum limit. In the next section,

[3]We do not get a representation of $-\sigma_i$ because we may have done a reflection when going from P_i to P_i'.

we reproduce an example of a discrete evolution that has this property.

In the argument above, we had to relabel the coordinate axes to get the right answer. Only if we had different particles with evolutions whose continuum limits could not be made into the same form by the *same* relabeling of the coordinate axes would there be any physical significance to the different forms of evolution in the continuum limit.

If the number of spatial dimensions is fewer than three, the same results apply but the particle obeys a lower-dimensional Weyl equation. If the number of spatial dimensions is greater than three, the particle still obeys the Weyl equation in at most three dimensions, meaning it does not move in the remaining directions.

A. Reproducing the Weyl equation in three space dimensions

A discrete evolution in three-dimensional space that becomes a Weyl particle in the continuum limit was first presented in [8]. It works by performing conditional shifts in each direction:

$$U_D = T_x T_y T_z, \tag{20}$$

with

$$T_b = S_b|{\uparrow_b}\rangle\langle{\uparrow_b}| + S_b^\dagger|{\downarrow_b}\rangle\langle{\downarrow_b}|, \tag{21}$$

where $b \in \{x,y,z\}$, S_b shifts one lattice site in the b direction, and $|{\uparrow_b}\rangle$ and $|{\downarrow_b}\rangle$ are spin up and spin down along the b axis. So, for example, T_z shifts a particle in the state $|\vec{n}\rangle|{\uparrow_z}\rangle$ one step in the $+\hat{z}$ direction.

It is interesting that this discrete evolution essentially uses a body-centered-cubic neighborhood. In fact, the most obvious choice, the cubic neighborhood, can not give the three-dimensional Weyl equation in the continuum limit [8].

B. More than two extra degrees of freedom

Unfortunately, it is not true that discrete evolutions with more than two extra degrees of freedom become relativistic evolutions in the continuum limit. Following is a simple example with a three-dimensional extra degree of freedom, with basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$. In the continuum limit it becomes a single particle evolving via the Hamiltonian

$$H = \vec{J}\cdot\vec{P}, \tag{22}$$

where $J_i = -i\sum_{jk}\varepsilon_{ijk}|j\rangle\langle k|$ are a three-dimensional representation of the generators of the lie algebra of SO(3) acting on $\mathcal{H}_S$. Although this has rotational symmetry, it does not have Lorentz symmetry.[4] To see this, note that $H^2 - P^2$ is not Lorentz invariant.[5]

[4]Note that we can not add a term like $\vec{\beta}\cdot\vec{P}$ to H as we did in Sec. V because this would break rotational symmetry, as would rescaling coordinate axes.

[5]To see this, look at $\sum_i \langle i|H^2 - \vec{P}^2|i\rangle = -\vec{P}^2$. If we are to have Lorentz invariance, $\sum_i \langle i|U_\Lambda(H^2 - \vec{P}^2)U_\Lambda^\dagger|i\rangle$ should be independent of the boost operator U_Λ. As we are talking about free particles, the effect of a Lorentz transformation is $U_\Lambda|\vec{p}\rangle|k\rangle = \sqrt{E_{\Lambda\vec{p}}/E_{\vec{p}}}|\Lambda\vec{p}\rangle D(\Lambda,\vec{p})|k\rangle$ where $D(\Lambda,\vec{p})$ is a unitary on the extra degree of freedom [18]. But, it follows from this that $\sum_i\langle i|U_\Lambda(H^2 - \vec{P}^2)U_\Lambda^\dagger|i\rangle = U_\Lambda(-\vec{P}^2)U_\Lambda^\dagger$, which is not independent of Λ.

The discrete evolution is a product of conditional shifts in each spatial direction:

$$U_D = T_x T_y T_z, \tag{23}$$

but now with

$$T_b = \exp(-iaP_bJ_b), \tag{24}$$

where P_b is the momentum operator in the b direction, with $b \in \{x,y,z\}$. Also, we have relabeled J_i by x, y, and z in the usual way: $J_1 = J_x$, $J_2 = J_y$, and $J_3 = J_z$.

To see the analogy with Eq. (21), we can rewrite T_b as

$$T_b = S_b|{+1_b}\rangle\langle{+1_b}| + |0_b\rangle\langle 0_b| + S_b^\dagger|{-1_b}\rangle\langle{-1_b}|, \tag{25}$$

where $|\lambda_b\rangle$ is the eigenvector of J_b with eigenvalue λ and S_b is a shift by one lattice site in the b direction.

VI. MASS AND THE DIRAC EQUATION

Now we turn to evolutions with mass. Recall that the evolution operator can be written as

$$U_D = W\sum_{\vec{q}} A'_{\vec{q}} S_{\vec{q}}, \tag{26}$$

where W is a unitary on $\mathcal{H}_S$ and $\sum_{\vec{q}} A'_{\vec{q}} = \mathbb{1}_S$. To get a continuum limit, we will let W tend to $\mathbb{1}_S$ as $\delta t \to 0$ in the following way:

$$W = e^{-iM\delta t}, \tag{27}$$

with M a fixed self-adjoint operator on $\mathcal{H}_S$.

The resulting continuum Hamiltonian is

$$H = \left(\frac{a}{\delta t}\right)\sum_{\vec{q}} A'_{\vec{q}}(\vec{q}\cdot\vec{P}) + M. \tag{28}$$

To see this, we proceed exactly as in Sec. IV, with the only difference being a different upper bound for $\|e^{-i\tilde{H}\delta t} - \tilde{U}_D\|$, which is derived in Appendix B. As in Sec. IV we let $a,\delta t \to 0$ to see that the discrete evolution agrees with the Hamiltonian above in the continuum limit.

As in the massless case, we can relabel coordinates so that the Hamiltonian becomes

$$H = \vec{\sigma}\cdot\vec{P} + M. \tag{29}$$

In one space dimension, taking $M = m\sigma_x$, we get the Dirac Hamiltonian in one dimension:

$$H = \sigma_z P_z + m\sigma_x. \tag{30}$$

This is not generic, however. For example, the choice $M = m_1\sigma_z + m_2\sigma_x$ is not a Lorentz-invariant evolution [14]. That said, had we required emergent symmetry under a parity transformation, this Hamiltonian would not be allowed.

A discrete evolution that becomes a particle evolving via the Dirac equation in three spatial dimensions is given in [8]. This works by taking two evolutions that give the left- and right-handed Weyl equations in the continuum limit and then mixing between them with a mass term.

VII. DISCUSSION

We looked at the continuum limit of the evolution of a causal quantum particle in discrete spacetime. In the massless case, when the particle had a two-dimensional extra degree of freedom, we saw that the continuum limit evolution was essentially equivalent to that of a Weyl particle in three or fewer dimensions. That such relativistic evolutions emerge generally in the continuum limit from discrete systems is exciting: it would have been reasonable to assume that discrete evolutions that are relativistic in the continuum limit would be very special cases.

These results for single particles naturally apply to free-fermion fields in discrete spacetime evolving in a causal way. The main challenge for the future is to find physically relevant *interacting* field theories evolving causally in discrete spacetime that have a continuum limit. (One example that becomes the Thirring model in one spatial dimension is given in [19].)

The evolutions we examined are discrete-time quantum walks, which first arose in quantum computation. Also, causal (and potentially interacting) quantum systems in discrete spacetime can be viewed as Quantum Cellular Automata (a type of quantum computer) [3,6,20,21]. So, it is interesting to consider that applying ideas from quantum computation may help to understand the continuum limits of discrete quantum field theories [22].

ACKNOWLEDGMENTS

A.J.S. acknowledges support from the Royal Society. T.C.F. acknowledges support from the Robert Gardiner Memorial Scholarship, CQIF, DAMTP, EPSRC, and St. John's College, Cambridge.

APPENDIX A: BOUNDING THE NORM

Here, we bound $\|e^{-i\tilde{H}\delta t} - \tilde{U}_D\|$ for a massless evolution. After Taylor expanding both terms, $\|e^{-i\tilde{H}\delta t} - \tilde{U}_D\|$ becomes

$$\left\| \sum_{m\geqslant 2} \frac{(-i\tilde{H}\delta t)^m}{m!} - \sum_{\vec{q}} A_{\vec{q}} \sum_{l\geqslant 2} \frac{(-i\vec{q}\cdot\vec{P}a)^l}{l!} \right\| \tag{A1}$$

$$\leqslant \sum_{m\geqslant 2} \frac{1}{m!} \left\| (-i\tilde{H}\delta t)^m - \sum_{\vec{q}} A_{\vec{q}} (-i\vec{q}\cdot\vec{P}a)^m \right\| \tag{A2}$$

$$\leqslant \sum_{m\geqslant 2} \frac{a^m}{m!} \left(\left\| \sum_{\vec{q}} A_{\vec{q}}\, \vec{q}\cdot\vec{P} \right\|^m + \left\| \sum_{\vec{q}} A_{\vec{q}} (\vec{q}\cdot\vec{P})^m \right\| \right) \tag{A3}$$

$$\leqslant \sum_{m\geqslant 2} \frac{a^m}{m!} \left[\left(\sum_{\vec{q}} \|A_{\vec{q}}\| \|\vec{q}\cdot\vec{P}\| \right)^m + \sum_{\vec{q}} \|A_{\vec{q}}\| \|\vec{q}\cdot\vec{P}\|^m \right] \tag{A4}$$

$$\leqslant \sum_{m\geqslant 2} \frac{a^m}{m!} [(Kq\Lambda)^m + K(q\Lambda)^m] \tag{A5}$$

$$\leqslant 2 \sum_{m\geqslant 2} \frac{(Kq\Lambda a)^m}{m!} \tag{A6}$$

$$\leqslant C(\Lambda a)^2, \tag{A7}$$

where K is the number of $A_{\vec{q}} \neq 0$, q is the largest value of $|\vec{q}|$ for which $A_{\vec{q}} \neq 0$, and the fifth line follows from $\|A_{\vec{q}}\| \leqslant 1$, which itself follows from $\sum_{\vec{q}} A_{\vec{q}}^\dagger A_{\vec{q}} = \mathbb{1}_S$. The last line applies when $\Lambda a \leqslant \frac{1}{Kq}$ and follows from the fact that, when $\alpha \leqslant 1$, $\sum_{m\geqslant 2} \frac{\alpha^m}{m!} \leqslant \alpha^2 \sum_{m\geqslant 2} \frac{1}{m!} = (e-2)\alpha^2 = C'\alpha^2$.

APPENDIX B: BOUNDING THE NORM WITH MASS

Here, we bound $\|e^{-i\tilde{H}\delta t} - \tilde{U}_D\|$ for a massive evolution. We omit tildes now to simplify notation. Define $U'_D = W^{-1}U_D$, which is a massless discrete evolution with corresponding continuum Hamiltonian $H' = H - M$. It follows from the triangle inequality that

$$\begin{aligned} \|e^{-iH\delta t} - U_D\| &\leqslant \|e^{-iH\delta t} - e^{-iM\delta t}e^{-iH'\delta t}\| \\ &\quad + \|e^{-iM\delta t}e^{-iH'\delta t} - e^{-iM\delta t}U'_D\|. \end{aligned} \tag{B1}$$

The second term is $\|e^{-iH'\delta t} - U'_D\|$ because the operator norm is unitarily invariant. We bounded this expression from above by $C(\Lambda a)^2$ in the previous section, so it remains to bound the first term. To do this, note that the order one and order δt terms cancel. Then, by expanding in power series and using the triangle inequality, it follows that for sufficiently small a (and hence δt),

$$\|e^{-iH\delta t} - e^{-iM\delta t}e^{-iH'\delta t}\| \leqslant C_1(\Lambda\, a)^2 + C_2\Lambda a\delta t + C_3\delta t^2, \tag{B2}$$

where C_i are constants and Λ is the momentum cutoff. It follows that

$$\|e^{-iH\delta t} - U_D\| \leqslant (C + C_1)(\Lambda a)^2 + C_2\Lambda a\delta t + C_3\delta t^2. \tag{B3}$$

And so, $\|e^{-iHt} - U_D^n\| \leqslant n\|e^{-iH\delta t} - U_D\| \to 0$ as a tends to zero, provided we choose the momentum cutoff to grow sufficiently slowly with a.

[1] M. Creutz, *Quarks, Gluons and Lattices*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, UK, 1983).

[2] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Phys. Rev. Lett. **59**, 521 (1987).

[3] T. C. Farrelly and A. J. Short, Phys. Rev. A **89**, 012302 (2014).

[4] D. A. Meyer, Phys. Lett. A **223**, 337 (1996).

[5] J. Kempe, Contemp. Phys. **44**, 307 (2003).

[6] G. M. D'Ariano, *Foundations of Probability and Physics- 6*, AIP Conference Proceedings Vol. 371 (AIP, Melville, NY, 2012), p. 1424.

[7] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), p. 35.

[8] I. Bialynicki-Birula, Phys. Rev. D **49**, 6920 (1994).

[9] D. A. Meyer, J. Stat. Phys. **85**, 551 (1996).

[10] D. A. Meyer, Phys. Rev. E **55**, 5261 (1997).
[11] F. W. Strauch, Phys. Rev. A **73**, 054302 (2006).
[12] A. J. Bracken, D. Ellinas, and I. Smyrnakis, Phys. Rev. A **75**, 022322 (2007).
[13] F. W. Strauch, J. Math. Phys. **48**, 082102 (2007).
[14] P. Kurzyński, Phys. Lett. A **372**, 6125 (2008).
[15] G. M. D'Ariano and P. Perinotti, arXiv:1306.1934.
[16] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000), p. 195.
[17] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, UK, 1985).
[18] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1 (Cambridge University Press, Cambridge, UK, 1995).
[19] C. Destri and H. J. de Vega, Nucl. Phys. B **290**, 363 (1987).
[20] G. M. D'Ariano, Phys. Lett. A **376**, 697 (2012).
[21] A. Bisio, G. M. D'Ariano, and A. Tosini, arXiv:1212.2839.
[22] S. P. Jordan, K. S. M. Lee, and J. Preskill, Science **336**, 1130 (2012).