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Note

A note on chromatic properties of threshold graphs

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ABSTRACT

In threshold graphs one may find weights for the vertices and a threshold value t such that for any subset S of vertices, the sum of the weights is at most the threshold t if and only if the set S is a stable (independent) set. In this note we ask a similar question about vertex colorings: given an integer p, when is it possible to find weights (in general depending on p) for the vertices and a threshold value t_p such that for any subset S of vertices the sum of the weights is at most t_p if and only if S generates a subgraph with chromatic number at most $p-1$? We show that threshold graphs do have this property and we show that one can even find weights which are valid for all values of p simultaneously.

1. Introduction

Threshold graphs have been introduced by Chvátal and Hammer [2] with a characterization based on stable sets, namely the existence of weights for the vertices and of a threshold value t such that the sum of the weights of vertices in a subset S of vertices is at most t if and only if S is a stable set. First, we observe that a stable set can be viewed as a set of vertices which induce no clique K_2. Considering that a stable set in a graph G is also a 1-colorable set in the sense of classical vertex coloring, we may envisage extensions of the above properties by considering subsets S of vertices in G which induce no clique K_p and also subsets S which would be p-colorable for some specific $p \geq 1$. Based on results obtained for this generalized notion of threshold graphs, we will show that the class of threshold graphs does indeed have some remarkable properties of chromatic flavor which have to our knowledge not been made explicit yet.

In Section 2, we will present these extensions and use the corresponding properties to give new characterizations of threshold graphs. Section 3 deals with some further properties of threshold graphs. Finally, in Section 4 we consider the special case $p = 3$.

All graph-theoretical terms not defined here can be found in [7]. For more properties of threshold graphs, the reader is referred to [4,5]. We recall that threshold graphs belong to the class of split graphs, i.e., graphs in which the vertex set can be partitioned into a clique and a stable set. These graphs are perfect and they have been extensively studied by various authors, see for instance [1]. It follows that threshold graphs are perfect.

2. New characterizations of threshold graphs

All graphs in this paper are finite, undirected, loopless and without multiple edges. Let $G = (V, E)$ be a graph. An edge joining two vertices u and v is denoted by uv. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ and

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$N_G[v] = N_G(v) \cup \{v\}$. Let $H \subseteq V$. The subgraph of G induced by H is denoted by $G[H]$. As usual, a clique on n vertices is denoted by K_n.

In what follows, we introduce some graph classes that either can be seen as a generalization of threshold graphs, or are defined by seemingly stronger conditions. Our main theorem (Theorem 6) will present an interesting link between these graph classes.

Definition 1. Let $p \geq 2$ be an integer. A graph $G = (V, E)$ is a **p**-*threshold graph* if there exists a value t_p and nonnegative weights $w_p(v)$ for all vertices $v \in V$ such that for any $S \subseteq V$, $w_p(S) = \sum_{v \in S} w_p(v) \leq t_p$ if and only if $G[S]$ contains no clique K_p.

Remark 2. Notice that p-threshold graphs are a generalization of threshold graphs. Indeed 2-threshold graphs are equivalent to threshold graphs. We also notice that the property of being p-threshold is hereditary, i.e., any induced subgraph of a p-threshold graph is also p-threshold. Furthermore, the set of p-threshold graphs is clearly closed under removal of any edges that do not belong to any p-clique. For brevity, we call a class of graphs p-hereditary if it is closed under taking induced subgraphs and removing edges not belonging to any p-clique.

Definition 3. A graph $G = (V, E)$ is *totally threshold* if there exist weights $w(v)$ for each $v \in V$ and values $t_2, t_3, \ldots, t_p, \ldots$ such that for any $p \geq 2$ and any subset S of vertices $\sum_{v \in S} w(v) \leq t_p$ if and only if $G[S]$ contains no clique K_p.

Hence, totally threshold graphs are p-threshold for any $p \geq 2$. But even more strongly, there is a uniform set of weights (with different thresholds) that certifies p-thresholdness for $p \geq 2$. Whereas at first sight totally thresholdness looks like a significantly stronger property than thresholdness, we will prove later that these notions actually coincide.

Notice that for any given weights $w: V \to \mathbb{R}_+$ and any p that is larger than the size of a maximum clique in G, one can simply choose the threshold for p to be $t_p = w(V)$ to obtain a threshold for p that certifies that G is p-threshold.

The above definitions can naturally be extended to chromatic numbers, by replacing the property of p-threshold graphs that an induced subgraph $G[S]$ whose weights are below the threshold value contains no p-clique with the property that $G[S]$ is $p-1$-colorable. This leads to the classes of p-chromishold and totally chromishold graphs as defined below.

Definition 4. A graph $G = (V, E)$ is said to be **p**-*chromishold* for some integer $p \geq 2$, there exists a value t_p and nonnegative weights $w_p(v)$ for all vertices $v \in V$ such that for any $S \subseteq V$, $w_p(S) = \sum_{v \in S} w_p(v) \leq t_p$ if and only if $G[S]$ has chromatic number $\chi(G[S]) \leq p - 1$.

Notice that 2-chromishold is the same as 2-threshold.

Definition 5. A graph G is called *totally chromishold* if there exist weights $w(v)$ for each vertex v and threshold values $t_2 < t_3 < \cdots < t_p < \cdots$ such that for any $p \geq 2$ and any subset S of vertices, $\sum_{v \in S} w(v) \leq t_p$ if and only if $G[S]$ has chromatic number at most $p - 1$.

We are now ready to prove our main result of this section, which shows in particular that the property of a graph being threshold is equivalent to the seemingly stronger property of being totally threshold.

Theorem 6. *For a graph G the following statements are equivalent:*

(a) *G is a threshold graph;*
(b) *G is totally threshold;*
(c) *G is totally chromishold.*

Proof. It immediately follows from the definitions above that (b) $\Rightarrow$ (a). Furthermore, G being totally chromishold implies that G is 2-chromishold which in turn is equivalent to G being threshold; therefore (c) $\Rightarrow$ (a).

We now show (a) $\Rightarrow$ (b). We shall show that we can assign to each vertex v a single weight $w(v)$ and find a collection of threshold values $t_2 < t_3 < \cdots < t_p < \cdots$ such that for any subset S of vertices and any value of p, $\sum_{v \in S} w(v) \leq t_p$ if and only if $G[S]$ contains no induced K_p.

In order to assign the weights to the vertices of a given threshold graph G, that certify that G is totally threshold, we will use a well-known property of threshold graphs (see [5]): a graph G is a threshold graph if and only if it can be constructed by introducing consecutively vertices in an order $v_1, v_2, \ldots, v_n$ such that for any i, either v_i is linked to all vertices $v_1, \ldots, v_{i-1}$ or to none (v_i is called *universal* in the first case or *isolated* in the second case).

If $n = |V|$, we set $t_p = (p - 1)2^n$ for $p = 2, 3, \ldots$. For $i = 1$ we set $w(v_1) = 2^{n-1}$ and for $i = 2, \ldots, n$ we set

$$w(v_i) = \begin{cases} 2^n - 2^{n-i} & \text{if } v_i \text{ is universal} \\ 2^{n-i} & \text{if } v_i \text{ is isolated.} \end{cases}$$

Claim 1. *Assume we have assigned weights as above to the vertices $v_1, v_2, \ldots, v_i$; then for any subset $S \subseteq \{v_1, \ldots, v_i\}$ we have $\min_{k \in \mathbb{N}} |\sum_{v \in S} w(v) - k2^n| \geq 2^{n-i}$.*

Proof of the Claim. At iteration 1, we have $w(v_1) = 2^{n-1}$ and the result holds. If the result holds at iteration $i-1$, then by adding a vertex v_i with weight $w(v_i) = 2^{n-i}$ or $w(v_i) = 2^n - 2^{n-i}$, it is immediate to observe that any subset S will have a weight such that $\min_{k\in\mathbb{N}} |\sum_{v\in S} w(v) - k2^n| \geq 2^{n-i+1} - 2^{n-i} = 2^{n-i}$. This proves the claim. □

It remains to show that the assigned weights are indeed a certificate of G being totally threshold. We prove the statement by induction on the size of G. The statement trivially holds for a single-vertex graph. Hence, assume that G is of size at least 2 and that the suggested weights and thresholds certify that any graph of size strictly smaller than G is totally threshold. We have to show that for any $p \in \mathbb{Z}_{\geq 2}$, the given weights and threshold certify that G is p-threshold. Notice that for $p = 2$, the result follows from the fact that the concepts of threshold and 2-threshold coincide (for this, we do not need induction on the size of G). Hence, let $p \in \mathbb{Z}_{\geq 3}$.

Consider the vertex v_n, and let $G' = G[V \setminus \{v_n\}]$. Notice that the restriction of the weights w, which were assigned to G, to the graph G', corresponds to the weights w' we would assign with our scheme to G' multiplied by a factor of 2, because G has one more vertex than G'. Also the thresholds assigned to G are twice as large as those that would be assigned to G'. Hence by induction, the weights w and the threshold value t_p, when restricted to G', certify that G' is k-threshold for any $k \in \mathbb{Z}_{\geq 2}$, since the definition of being k-threshold is invariant with respect to scaling all weights and thresholds by a same positive factor. We distinguish two cases depending on whether v_n is isolated or universal.

1. *v_n is an isolated vertex.*
   Let $S \subseteq V$. If $G[S]$ contains an induced K_p, then so does $G[S \setminus \{v_n\}]$, and therefore we obtain by induction $w(S) \geq w(S \setminus \{v_n\}) > t_p$. If $G[S]$ does not contain an induced K_p, then $G[S \setminus \{v_n\}]$ does as well not contain an induced K_p and we have $w(S \setminus \{v_n\}) \leq t_p = (p-1)2^n$. Together with Claim 1, this implies $w(S \setminus \{v_n\}) \leq (p-1)2^n - 2$, and hence $w(S) = w(S \setminus \{v\}) + \underbrace{w(v_n)}_{=1} \leq t_p$.
2. *v_n is a universal vertex.*
   Let $S \subseteq V$. If $G[S]$ contains an induced K_p, then $G[S \setminus \{v_n\}]$ contains an induced K_{p-1}, and by induction we have $w(S \setminus \{v_n\}) > t_{p-1}$. Thus, $w(S) = w(S \setminus \{v_n\}) + w(v_n) > t_{p-1} + 2^n - 1 = t_p - 1$, which implies $w(S) \geq t_p$. Claim 1 implies $w(S) \neq t_p$, and therefore $w(S) > t_p$. If $G[S]$ does not contain an induced K_p, then $G[S \setminus \{v_n\}]$ does not contain an induced K_{p-1}. Hence, by induction we have $w(S \setminus \{v_n\}) \leq t_{p-1}$, and thus, $w(S) = w(S \setminus \{v_n\}) + w(v_n) \leq t_{p-1} + 2^n - 1 \leq t_p$.

We complete the proof by showing (b) $\Rightarrow$ (c). Consider a totally threshold graph G. Since G is perfect, its chromatic number $\chi(G)$ is equal to the maximum clique size $\omega(G)$. Thus, for any subgraph G' of G we have $\chi(G') \leq p$ if and only if G' contains no induced clique K_{p+1}. Hence, the property of being totally chromishold is equivalent to totally threshold and the result follows. □

We will now observe a few simple consequences of Theorem 6. First, in a similar spirit as the definition of threshold graphs, one could as well have considered a class of graphs strongly related to p-threshold graphs, where the property of not containing a p-clique in an induced subgraph whose weight is below the threshold is replaced by not containing a stable set of size p. Calling such graphs *p-cliqueshold*, we have that a graph is *(totally)* p-cliqueshold if its complement is *(totally)* p-threshold. An interesting fact about threshold graphs is that they are closed under taking complements. This follows easily from a characterization of Chvátal and Hammer [3], showing that a graph G is threshold if and only if it does not contain any induced subgraph isomorphic to $2K_2$, P_4 or C_4, and by observing that complementing each graph in the family $\{2K_2, P_4, C_4\}$ leads again to the same family of graphs. Alternatively, one can observe that the complement of a threshold graph G can be obtained by exchanging the roles of universal and isolated vertices in the characterization used in the proof of Theorem 6. Hence, together with Theorem 6 this implies the following:

Corollary 7. *A graph is threshold if and only if it is totally cliqueshold.*

Furthermore, for completeness we would like to observe that since 2-threshold, 2-chromishold and 2-cliqueshold are equivalent, we obtain the following corollary as a simple consequence of Theorem 6 and Corollary 7.

Corollary 8. *For a graph G the following statements are equivalent:*

(a) *G is threshold;*
(b) *G is 2-chromishold;*
(c) *G is 2-cliqueshold;*
(d) *G is totally threshold;*
(e) *G is totally chromishold;*
(f) *G is totally cliqueshold.*

3. More properties of threshold graphs

In this section we present some additional properties of threshold graphs which are linked to the properties given in the previous section.

Definition 9. Let $G = (V, E)$ be a graph and let $p \geq 2$ be an integer. Then G has *property* $\mathbf{P_p}$ if the following holds: Let $V_1 \cup V^*, V_2 \cup V^*$ be vertex sets of two p-cliques with $V_1 \cap V_2 = V_1 \cap V^* = V_2 \cap V^* = \emptyset$; then for any partition V_1', V_2' of $V_1 \cup V_2$, at least one of $V_1' \cup V^*, V_2' \cup V^*$ induces a graph containing a p-clique.

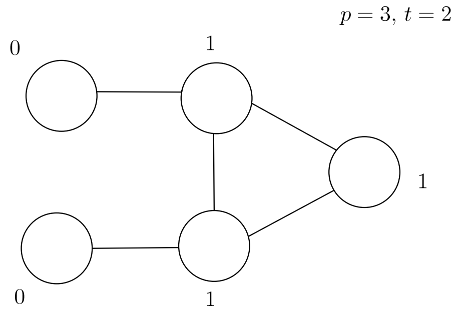

Fig. 1. Graph G showing that the statement of Lemma 13 cannot be extended to hold for the graph G itself, instead of only for its p-core $C_p(G)$.

Notice that threshold graphs must have property P_2.

Definition 10. Let $G = (V, E)$ be a graph and $p \geq 2$ an integer. The **p**-*core of* G, denoted by $C_p(G) = (V_p, E_p)$, is the graph obtained from G by removing all vertices and edges not contained in any p-clique.

Fact 11. *It follows from the above definitions that $C_p(G)$ is a p-threshold graph if and only if G is a p-threshold graph. Indeed if G is p-threshold, then so is $C_p(G)$ because the property of being p-threshold is p-hereditary. Conversely, weights and threshold values certifying that $C_p(G)$ is a p-threshold graph can be extended to G by assigning weights $w_p(v) = 0$ to all vertices v in $V \setminus V(C_p(G))$.*

Fact 12. *A p-threshold graph G has property P_p.*

Proof. Let $w_p(v)$ be the weights assigned to the vertices of G and let t_p be the threshold value. We have $a = \sum_{v \in V_1 \cup V^*} w_p(v) > t_p$ and $b = \sum_{v \in V_2 \cup V^*} w_p(v) > t_p$. If G does not have property P_p, there is a partition V'_1, V'_2 of $V_1 \cup V_2$ such that $a' = \sum_{v \in V'_1 \cup V^*} w_p(v) \leq t_p$ and $b' = \sum_{v \in V'_2 \cup V^*} w_p(v) \leq t_p$. But then we have $2t_p < a + b = a' + b' \leq 2t_p$, which is a contradiction. Hence G has property P_p. □

We can now state the following.

Lemma 13. *Let $G = (V, E)$ be a graph and $p \geq 2$ an integer. If $C_p(G)$ has property P_p, then $\forall u, v \in V_p$ we have either $N_{C_p(G)}(u) \subseteq N_{C_p(G)}[v]$ or $N_{C_p(G)}(v) \subseteq N_{C_p(G)}[u]$.*

Proof. Assume that in $H = C_p(G)$ there are two vertices u, v for which we have neither $N_H(u) \subseteq N_H[v]$ nor $N_H(v) \subseteq N_H[u]$. This means that there is a vertex $x \in N_H(u) - N_H(v)$ and a vertex $y \in N_H(v) - N_H(u)$. Since we are in $H = C_p(G)$, the edge xu belongs to some p-clique K^1 and the edge yv belongs to some p-clique K^2; these cliques are necessarily different since yu and xv do not exist in H. But now in the subgraph H induced by $K^1 \cup K^2$, for the partition $(K^1 - x) \cup \{y\}, (K^2 - y) \cup \{x\}$ none of the sets of vertices $(K^1 - x) \cup \{y\}$ and $(K^2 - y) \cup \{x\}$ does induce a clique on p vertices and so H does not have property P_p, a contradiction. □

Example 14. Notice that if G is a p-threshold graph which is different from its p-core $C_p(G)$, Lemma 13 may not hold for G itself as shown in Fig. 1. The weights are indicated besides each vertex and the two vertices of weight 0 have non-nested neighborhoods.

It follows from Lemma 13 that if $H = C_p(G)$ has property P_p, the neighborhoods of all vertices are nested. This is a characteristic property of threshold (i.e., 2-threshold) graphs (see Chapter 1 in [4]): a graph G is a threshold graph if and only if for any two vertices u, v we have $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$. So we obtain the following.

Corollary 15. *Let $G = (V, E)$ be a graph satisfying P_p for some $p \geq 2$; then $C_p(G)$ is a 2-threshold graph.*

Example 16. The graph G in Fig. 1 is a 3-threshold graph (use the threshold value $t_3 = 2$ and the weights shown in Fig. 1). Notice that G is not 2-threshold. However $C_3(G)$, which is the triangle in G, is a 2-threshold graph.

Corollary 17. *A graph G is p-threshold if and only if its core $C_p(G)$ is a threshold graph.*

Proof. It follows from Fact 12 that any p-threshold graph has property P_p. So $C_p(G)$ also has property P_p and it follows from Corollary 15 that $C_p(G)$ is a 2-threshold graph. Conversely, if $C_p(G)$ is a 2-threshold graph, it is a p-threshold graph from Theorem 6 and Fact 11 establishes that G is p-threshold. □

Example 18. Notice that Corollary 17 does not hold if we replace p-threshold by p-chromishold. Indeed, consider the graph $G = K_3 \cup C_5$ (i.e., the disjoint union of a triangle and an induced cycle of length five with no edges between them). This graph is 3-threshold and its core $C_3(G)$ consists in K_3, thus it is a threshold graph. But clearly G is not 3-chromishold.

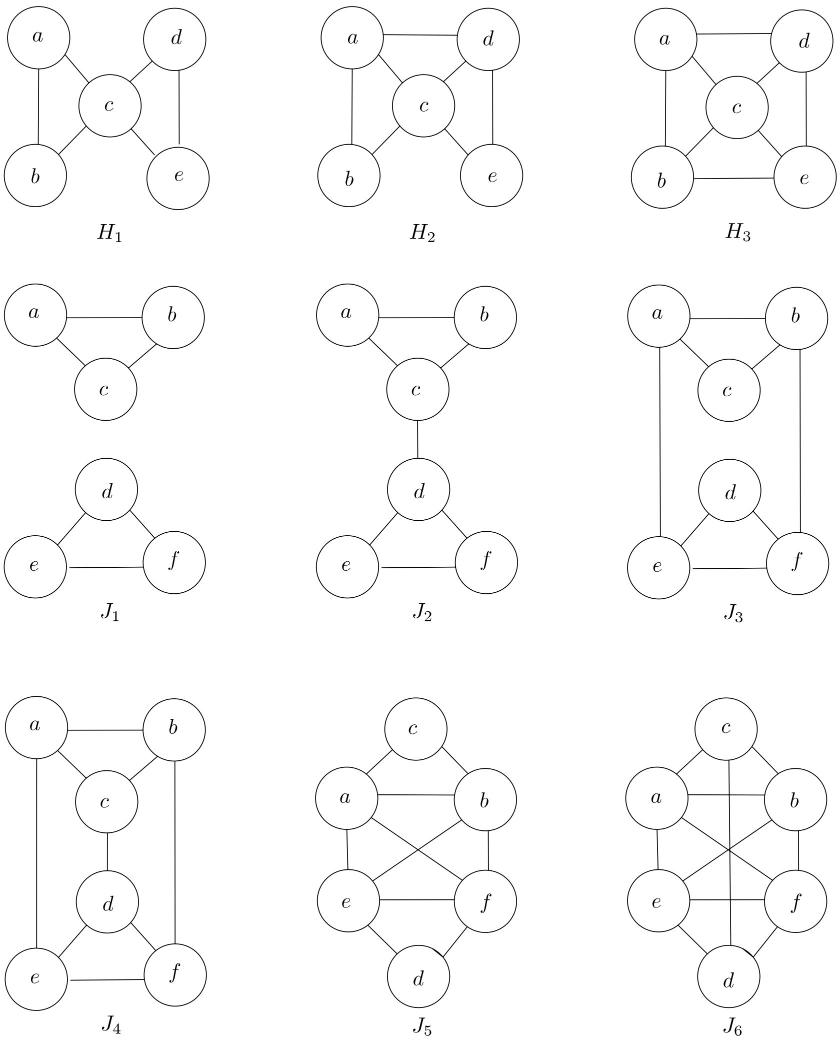

Fig. 2. Forbidden induced subgraphs for graphs satisfying property P_3.

Corollary 19. *Let $G = (V, E)$ be a graph and let $p \geq 2$ be an integer. Then G is p-threshold if and only if G satisfies property P_p.*

Proof. We only have to show that if G satisfies property P_p then it is p-threshold. From Corollary 15, its core $C_P(G)$ is a 2-threshold graph and from Corollary 17 G is p-threshold. □

Remark 20. The above corollary implies that the property of being p-threshold can be tested locally because it is equivalent to property P_p which can be tested by looking at all induced subgraphs of G with at most $2p$ vertices. Hence, the property of being p-threshold can be characterized by providing a list of forbidden induced subgraphs of size at most $2p$.

As an example, we provide the list of forbidden induced subgraphs for the property P_3 in Section 4.

4. The special case $p = 3$

In this section we focus on property P_p with $p = 3$. We will first give a characterization of all graphs that satisfy property P_3 by a family of forbidden induced subgraphs. Let $G = (V, E)$ be a graph. We say that G is *H-free*, for some graph H, if no induced subgraph of G is isomorphic to H. If $\mathcal{H}$ is a family of graphs, we say that G is *$\mathcal{H}$-free*, if no induced subgraph of G is isomorphic to some graph of $\mathcal{H}$.

Now consider the graphs shown in Fig. 2. Let $\mathcal{F}$ be the family of these graphs, i.e., let $\mathcal{F} = \{H_1, H_2, H_3, J_1, \ldots, J_6\}$. One can easily verify that $\mathcal{F}$ is the list of forbidden induced subgraphs for property P_3 as discussed in Remark 20. Hence we have the following.

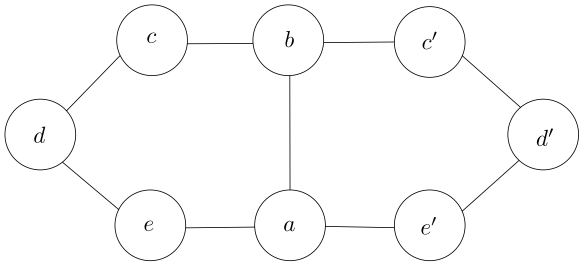

Fig. 3. Graph showing that P_3 does not imply $\mathcal{O}$.

Proposition 21. *A graph $G = (V, E)$ satisfies property P_3 if and only if G is $\mathcal{F}$-free.*

Definition 22. $G = (V, E)$ has *property* $\mathcal{O}$ if the following holds: let $V_1 \cup V^*$, $V_2 \cup V^*$ be the vertex sets of two odd cycles in G with $V_1 \cap V_2 = V_1 \cap V^* = V_2 \cap V^* = \emptyset$. Then for any partition V_1', V_2' of $V_1 \cup V_2$ at least one of $V_1' \cup V^*$, $V_2' \cup V^*$ contains an odd cycle.

The following immediately follows from the definition of property $\mathcal{O}$ and property P_3.

Lemma 23. *Let $G = (V, E)$ be a graph having property $\mathcal{O}$. Then G has property P_3.*

Remark 24. Notice that P_3 does not imply $\mathcal{O}$ as shown by the graph G in Fig. 3. G has property P_3 since it contains no triangles. But if $V_1 = \{c, d, e\}$, $V_2 = \{c', d', e'\}$, $V^* = \{a, b\}$ the choice $V_1' = \{c, d, c', d'\}$, $V_2' = \{e, e'\}$ shows that $\mathcal{O}$ does not hold.

We recall the following definition. A graph G is *line-perfect* if its line graph $L(G)$ is perfect. These graphs are characterized by the following (see [6]).

Proposition 25. *G is line-perfect if and only if G contains no odd elementary cycle of length at least five.*

We can derive immediately the following statement.

Proposition 26. *For a line-perfect graph, properties $\mathcal{O}$ and P_3 are equivalent.*

Finally we have the following.

Proposition 27. *A line-perfect graph is* 3*-threshold if and only if it does not contain any of H_1, J_1, J_2 as an induced subgraph.*

Proof. Combining Corollary 19 and Proposition 21 we have that a graph G is 3-threshold if and only if it is $\mathcal{F}$-free. However, if G is line-perfect, then it cannot contain any of the graphs $H_2, H_3, J_3, J_4, J_5, J_6$ as an induced subgraph since all of those graphs contain an elementary cycle of length 5. Hence, for line-perfect graphs the condition of being $\mathcal{F}$-free is equivalent to the condition of not containing any of H_1, J_1, J_2 as an induced subgraph. □

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