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d-Transversals of stable sets and vertex covers in weighted bipartite graphs

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ABSTRACT

Let $G = (V, E)$ be a graph in which every vertex $v \in V$ has a weight $w(v) \geqslant 0$ and a cost $c(v) \geqslant 0$. Let $\mathcal{S}_G$ be the family of all maximum-weight stable sets in G. For any integer $d \geqslant 0$, a minimum d-transversal in the graph G with respect to $\mathcal{S}_G$ is a subset of vertices $\mathcal{T} \subseteq V$ of minimum total cost such that $|\mathcal{T} \cap S| \geqslant d$ for every $S \in \mathcal{S}_G$. In this paper, we present a polynomial-time algorithm to determine minimum d-transversals in bipartite graphs. Our algorithm is based on a characterization of maximum-weight stable sets in bipartite graphs. We also derive results on minimum d-transversals of minimum-weight vertex covers in weighted bipartite graphs.

1. Introduction

Problems of safety and reliability occur in many practical contexts and adequate formulations of such issues have opened the way to possible treatments by mathematical optimization procedures.

It is in particular the case in situations where a complex system has to be protected against attacks and for this purpose one may have to identify the "most vital" elements of the system.

Such issues have been presented in [5] and [7] where various examples are sketched and combinatorial optimization models are established for a collection of such situations.

To be concrete assume we have a finite system $\mathcal{S}$ (collection of components) which can be operated in different ways. Each operating mode is characterized by the subsets S of components it uses. In order to find the most vital components of $\mathcal{S}$ we may want to identify a smallest possible subset $\mathcal{T}$ of components in $\mathcal{S}$ which is such that every operating mode S has at least d components in $\mathcal{T}$.

Also in a game-theoretic context, we may have the case in which a player A has a collection of possible actions; each action is represented by a given subset S of a finite set $\mathcal{S}$. An opponent B wants to prevent the moves of A by destroying from $\mathcal{S}$ a smallest possible subset $\mathcal{T}$ of elements in $\mathcal{S}$ in such a way that each possible decision S of A has lost at least d elements. Such subsets $\mathcal{T}$ will be called d-transversals.

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This is the kind of problems which we intend to discuss here. The system may for example be represented by a weighted graph, and this will be the case in the instances which will be considered, where each action (or operating mode) will be associated with a maximum-weight stable set of the graph.

The problem of finding a d-transversal of minimum cardinality with respect to optimal combinatorial structures has been recently studied by several authors [1,3,7,9]. In [3] it was shown that a d-transversal of minimum cardinality with respect to maximum stable sets can be found in polynomial time in (unweighted) bipartite graphs. Further combinatorial structures have been considered in [3,7].

In this paper, we consider weighted bipartite graphs in which a nonnegative cost is assigned to every vertex. We show that a d-transversal of minimum total cost with respect to maximum-weight stable sets can be determined in polynomial time. Notice that finding d-transversals of minimum cardinality with respect to maximum stable sets is $\mathcal{NP}$-hard in line graphs of bipartite graphs. This immediately follows from a result of [9].

Our paper is organized as follows. In Section 2 we present the notions and definitions that will be used throughout the paper. In addition, we give some preliminary results which will be useful in the rest of the paper. In Section 3 we present a characterization of maximum-weight stable sets in bipartite graphs. This characterization is then used in Section 4 to derive a polynomial-time algorithm for finding minimum d-transversals in weighted bipartite graphs. In Section 5 we deal with the case of d-transversals of minimum-weight vertex covers in bipartite graphs. Finally we conclude with Section 6.

For all graph theoretical terms not defined here the reader is referred to [8].

2. Preliminaries

All graphs we consider here are undirected, simple and loopless. Let $G = (V, E)$ be a graph. For any vertex $v \in V$, we denote by $N(v)$ the *neighborhood* of v, i.e., the set of vertices which are adjacent to v. For a subset $V' \subseteq V$, we denote by $G - V'$ the graph obtained from G by deleting all vertices in V' together with all the edges incident to at least one vertex in V'. If V' consists of a single vertex v, we will simply write $G - v$ instead of $G - V'$. For an edge $e \in E$, $G - e$ is the graph obtained from G by deleting the edge e. The subgraph induced by a set $V' \subseteq V$ will be denoted by $G[V']$.

Now consider a graph $G = (V, E)$ in which every vertex $v \in V$ has a weight $w(v) \geqslant 0$. We will refer to such a graph as a *weighted graph*. A *stable set* in G is a set $S \subseteq V$ of pairwise nonadjacent vertices. The weight of a set $V' \subseteq V$ in G is defined as $w(V') = \sum_{v \in V'} w(v)$. We denote by $\mathcal{S}_G$ the family of all maximum-weight stable sets in G. The *weighted stability number* $\alpha_w(G)$ is the maximum weight of a stable set in G. We denote by δ_G the minimum cardinality of a maximum-weight stable set, i.e., $\delta_G = \min\{|S|\colon\ S \in \mathcal{S}_G\}$. A *matching* M in G is a set of pairwise nonadjacent edges. The maximum cardinality of a matching in G is denoted by $\mu(G)$.

A vertex v is called *forced* if every maximum-weight stable set contains v. A vertex v is called *excluded* if no maximum-weight stable set contains v. A vertex which is neither forced nor excluded is called *free*. The set of all forced vertices in G is denoted by $V^f(G)$. Similarly we denote by $V^e(G)$ (resp. by $V^{fr}(G)$) the set of all excluded vertices (resp. free vertices) in G. Clearly $V^f(G)$, $V^e(G)$ and $V^{fr}(G)$ form a partition of the vertex set V. This partition can be obtained in polynomial time for bipartite graphs. This follows from a result of [2] about Kőnig–Egerváry graphs which include bipartite graphs.

Now consider a weighted graph $G = (V, E)$ in which every vertex $v \in V$ has a cost $c(v) \geqslant 0$. The *cost* of a set $V' \subseteq V$ is defined as $c(V') = \sum_{v \in V'} c(v)$. Let $d \geqslant 0$ be an integer. A *d-transversal* in G with respect to $\mathcal{S}_G$ is a subset of vertices $\mathcal{T} \subseteq V$ such that $|\mathcal{T} \cap S| \geqslant d$ for every $S \in \mathcal{S}_G$. A *minimum d-transversal* in G with respect to $\mathcal{S}_G$ is a d-transversal with minimum total cost. Notice that if $c(v) = 1$ for all $v \in V$, a minimum d-transversal in G with respect to $\mathcal{S}_G$ is a d-transversal of minimum cardinality.

A (minimum) d-transversal $\mathcal{T}$ in a graph G with respect to $\mathcal{S}_G$ is called *proper* if for any $v \in \mathcal{T}$, the set $\mathcal{T} \setminus \{v\}$ is not a d-transversal in G with respect to $\mathcal{S}_G$.

In this paper we are interested in the following problem.

TRANS
Input: A weighted graph $G = (V, E)$; a nonnegative cost function c for V; an integer $d \geqslant 0$.
Output: A proper minimum d-transversal $\mathcal{T}$ with respect to $\mathcal{S}_G$.

The following proposition shows that we do not need to care about vertices having a weight equal to zero.

Proposition 2.1. *Consider a weighted graph G. Let G' be the weighted graph obtained from G by removing any vertex v of weight $w(v) = 0$ (together with the edges incident to v). Then $\mathcal{T}$ is a proper d-transversal in G if and only if $\mathcal{T}$ is a proper d-transversal in G'.*

Proof. Consider a vertex v in G with $w(v) = 0$ and an arbitrary maximum-weight stable set S containing v. Since $w(v) = 0$, the set $S' = S \setminus \{v\}$ is a stable set with $w(S') = w(S)$ and hence it is also a maximum-weight stable set.

If $\mathcal{T}$ is a proper d-transversal in G', then $|\mathcal{T} \cap S'| \geqslant d$ and thus $|\mathcal{T} \cap S| \geqslant d$. So $\mathcal{T}$ is a proper d-transversal in G.

Now if $\mathcal{T}$ is a proper d-transversal in G, then $v \notin \mathcal{T}$. Indeed, since $|\mathcal{T} \cap S'| \geqslant d$, it follows that if $v \in \mathcal{T}$, then $\mathcal{T}' = \mathcal{T} \setminus \{v\}$ is a d-transversal in G, a contradiction. Thus $\mathcal{T}$ is a proper d-transversal in G'. $\square$

It follows from Proposition 2.1 that we may assume from now on that for any weighted graph $G = (V, E)$ we have $w(v) > 0$ for all $v \in V$.

3. Maximum-weight stable sets in bipartite graphs

In this section, we will present a characterization of the maximum-weight stable sets in bipartite graphs. This characterization appears in an implicit form in [2], Theorem 9. For this presentation to be self-contained we include a brief derivation based on simple arguments of network flows.

Let $G = (B, W, E)$ be a bipartite graph in which every vertex v has a weight $w(v) > 0$. We construct the following associated network $\hat{G} = (\hat{V}, \hat{E})$: $\hat{V} = B \cup W \cup \{s, t\}$ where s is a source and t is a sink; we add arcs (s, x) with capacity $k(s, x) = w(x)$ for each $x \in B$, arcs (y, t) with capacity $k(y, t) = w(y)$ for each $y \in W$ and arcs (x, y) with capacity $k(x, y) = \infty$ for each edge xy in E.

Proposition 3.1. *Let $G = (B, W, E)$ be a bipartite graph in which every vertex v has a weight $w(v) > 0$. Then all vertices in G are free if and only if in $\hat{G}$ there exists a maximum flow f of value $z(f) = w(B) = w(W)$.*

Proof. Let f^* be a maximum flow in $\hat{G}$. From [4] it follows that

$$\alpha_W(G) = w(B \cup W) - z\big(f^*\big). \tag{3.1}$$

Therefore, if $z(f^*) = w(B) = w(W)$, then $\alpha_W(G) = w(B) = w(W)$. Thus B and W are maximum-weight stable sets and hence all vertices are free.

Conversely, assume that $z(f^*) < w(B)$. Then there is an arc (s, x) such that $f^*(s, x) < k(s, x) = w(x)$. We will prove that $\alpha_W(G - x) < \alpha_W(G)$ which implies that x is forced. Indeed, let f' be the flow obtained from f^* by removing the $f^*(s, x)$ units going through x, and let f'^* be a maximum flow in $\hat{G} - x$. Then $z(f'^*) \geqslant z(f') = z(f^*) - f^*(s, x) > z(f^*) - w(x)$ and from (3.1) we have:

$$\alpha_W(G - x) = w\big(\big(B - \{x\}\big) \cup W\big) - z\big(f'^*\big) = w(B \cup W) - w(x) - z\big(f'^*\big) < w(B \cup W) - z\big(f^*\big) = \alpha_W(G). \qquad \square$$

In $\hat{G} = (\hat{V}, \hat{E})$ we shall say that an arc (x, y) is *forbidden* if $f(x, y) = 0$ for any maximum flow f from s to t. Let $\hat{F} \subseteq \hat{E}$ be the set of forbidden arcs.

Fact 3.2. *In $\hat{G}$ there exists a maximum flow f^* from s to t with $f^*(a) > 0$ for every $a \in \hat{E} \setminus \hat{F}$.*

Proof. Take consecutively each arc $a_i \in \hat{E} \setminus \hat{F}$ and consider any maximum flow f_i from s to t with $f_i(a_i) > 0$; such a flow exists by definition of a_i. Then the flow f^* defined for each arc a by $f^*(a) = (\sum_i f_i(a)/|\hat{E} \setminus \hat{F}|)$ is the required flow. $\square$

Proposition 3.3. *Let $G = (B, W, E)$ be a connected weighted bipartite graph in which all vertices are free. Then B and W are the only two maximum-weight stable sets if and only if in $\hat{G}$ we have $\hat{F} = \emptyset$.*

Proof. Since all vertices in G are free, it follows from Proposition 3.1 that there exists a maximum flow f in $\hat{G}$ saturating all arcs (s, x) and (y, t), for $x \in B$ and $y \in W$. Furthermore, it follows from the proof of Proposition 3.1 that B and W are both maximum-weight stable sets in G.

Assume that $\hat{F} \neq \emptyset$. Then there exists an arc (u, v) with $f(u, v) = 0$ for any maximum flow f in $\hat{G}$. Take a maximum flow f^* with $f^*(a) > 0$ for every arc a in $\hat{E} \setminus \hat{F}$, such as the one given in Fact 3.2. In $\hat{G}$, every cycle C with $V(C) \subseteq B \cup W$ which contains the forbidden forward arc (u, v) must also contain a forbidden backward arc (u', v') otherwise we could transform f^* into another feasible maximum flow f' such that $f'(u, v) > 0$. We take the first such backward arc when following the cycle C starting at vertex u and ending with vertex v. Notice that u' is necessarily reached after following an even number of arcs from u, and that all the backward arcs among these arcs are nonforbidden arcs.

Now we remove the corresponding edge $u'v'$ from G and we repeat this for every such cycle C of $\hat{G}$ containing the arc (u, v) if any. Notice that the resulting graph G' is still connected. Finally, the corresponding edge uv belongs to no cycle anymore in G'. So the graph $G' - uv$ is disconnected. Let G^1, G^2 be the two connected components of $G' - uv$ with vertex sets V^1, V^2 respectively. Without loss of generality, we may assume that $u \in V^1 \cap B$ and $v \in V^2 \cap W$.

Since we only removed edges $u'v'$ in G whose corresponding arcs (u', v') satisfy $f^*(u', v') = 0$, in $\hat{G}$ the flow f^* may be partitioned into two flows f_1^* and f_2^* such that f_i^* corresponds to f^* in $\hat{G}_i = \hat{G}[\{s, t\} \cup V^i]$, for $i = 1, 2$. Clearly f_i^* is a maximum flow in $\hat{G}_i$, for $i = 1, 2$, which saturates all arcs (s, x) and (y, t) with $x \in B \cap V^i$ and $y \in W \cap V^i$. Thus we have $z(f_i^*) = w(B \cap V^i) = w(W \cap V^i)$ for $i = 1, 2$. From Proposition 3.1 all vertices in G^i are free, for $i = 1, 2$. Now (3.1) implies that $B \cap V^i$ and $W \cap V^i$ are maximum-weight stable sets in G^i for $i = 1, 2$. By construction $u' \in V^1 \cap B$. Since $G' - uv$ is disconnected, there is no edge xy with $x \in V^1 \cap W$ and $y \in V^2 \cap B$ in $G' - uv$. This is also the case in G because we only removed edges between $V^1 \cap B$ and W. So $(V^1 \cap W) \cup (V^2 \cap B)$ is a stable set S of G with $w(S) = w(V^1 \cap W) + w(V^2 \cap B) = w(V^1 \cap W) + w(V^2 \cap W) = w(W) = \alpha_W(G)$. This contradicts the fact that W and B are the only maximum-weight stable sets in G.

Conversely, assume that $\hat{G}$ contains no forbidden arc, i.e., $\hat{F} = \emptyset$. We consider a maximum flow f with $f(a) > 0$ for every arc a in $\hat{G}$. From Fact 3.2 such a flow exists. Suppose there is a maximal (inclusionwise) stable set S in G with $S \cap B \neq \emptyset$

and $S \cap W \neq \emptyset$. Then clearly $S \cap B \neq B$ and $S \cap W \neq W$. Since S is inclusionwise maximal, we have $B \setminus S = N(S \cap W)$ and $W \setminus S = N(S \cap B)$. Since G is a connected graph there must be an edge $\bar{x}\bar{y}$ with $\bar{x} \in N(S \cap W) \subseteq B$ and $\bar{y} \in N(S \cap B) \subseteq W$. We have:

$$w(S \cap B) = \sum_{x \in S \cap B} k(s, x) = \sum_{x \in S \cap B;\, y \in N(S \cap B)} f(x, y) < \sum_{x \in S \cap B;\, y \in N(S \cap B)} f(x, y) + f(\bar{x}, \bar{y})$$
$$\leqslant \sum_{y \in N(S \cap B)} k(y, t) = w\big(N(S \cap B)\big).$$

Since $N(S \cap B) \cap (S \cap W) = \emptyset$ we obtain $w(S) = w(S \cap B) + w(S \cap W) < w(N(S \cap B)) + w(S \cap W) = w(W)$. Hence such a set S is not a maximum-weight stable set, so B and W are the only maximum-weight stable sets in G. $\square$

As a consequence, if in a weighted bipartite graph G all vertices are free, one may consider the connected subgraphs $G_1, G_2, \ldots, G_q$ obtained by removing all edges xy of G corresponding to forbidden arcs (x, y) of $\hat{G}$. Set $V_i = V(G_i)$ for each $i = 1, \ldots, q$ and let $\mathcal{V} = (V_1, \ldots, V_q)$ be the partition of V obtained in this way. This partition is clearly unique and there are at least two vertices in each V_i. From Propositions 3.1 and 3.3, it follows that in each G_i all the vertices are free and the only maximum-weight stable sets are $B \cap V_i$ and $W \cap V_i$, for $i = 1, \ldots, q$. Clearly $S = B$ is a maximum-weight stable set with $w(S) = w(B) = \sum_{i=1}^{q} w(B \cap V_i) = \sum_{i=1}^{q} \alpha_w(G_i)$. Since $B \cap V_i$ and $W \cap V_i$ are the only stable sets in G_i with $w(B \cap V_i) = w(W \cap V_i) = \alpha_w(G_i)$ for $i = 1, \ldots, q$, in G the maximum-weight stable sets S are all such that $S \cap V_i = B \cap V_i$ or $S \cap V_i = W \cap V_i$ for all $i = 1, \ldots, q$.

So we obtain the following.

Theorem 3.4. *Let $G = (B, W, E)$ be a weighted bipartite graph containing only free vertices. Then $S \subseteq B \cup W$ is a maximum-weight stable set if and only if S is a stable set and for any $j \in \{1, \ldots, q\}$ either $S \cap V_j = B \cap V_j$ or $S \cap V_j = W \cap V_j$.*

Remark 3.1. Since B and W are disjoint maximum-weight stable sets, we deduce that, for any $d \geqslant 0$, a d-transversal $\mathcal{T}$ in G must satisfy $|\mathcal{T} \cap W| \geqslant d$ and $|\mathcal{T} \cap B| \geqslant d$, and hence $|\mathcal{T}| \geqslant 2d$.

4. Minimum *d*-transversals of maximum-weight stable sets

In this section, we will present a polynomial-time algorithm for finding proper minimum d-transversals in bipartite graphs.

First we need to introduce some additional notions. Let $G = (B, W, E)$ be a weighted bipartite graph and let $S \in \mathcal{S}_G$ be a maximum-weight stable set in G. We say that $V_i \in \mathcal{V}$ is *black with respect to* S if $S \cap V_i = B \cap V_i$, and that V_i is *white with respect to* S if $S \cap V_i = W \cap V_i$.

From the partition $\mathcal{V}$ of V, we define the following auxiliary digraph $G^* = (V^*, A^*)$: $V^* = \{V_1, \ldots, V_q\}$ and $A^* = \{(V_i, V_j) \mid \exists uv \in E,\ u \in V_i \cap B,\ v \in V_j \cap W,\ i \neq j\}$.

Remark 4.1. It follows from Theorem 3.4 and from the definition of G^* that if $S \in \mathcal{S}_G$ and if V_i is black with respect to S, then all successors of V_i in G^* are black with respect to S.

From G^* we define the following relation $(V, \preccurlyeq)$: for $u \in W$ and $v \in B$, $u \preccurlyeq v$ if and only if either $u, v \in V_j$ for some $j \in \{1, \ldots, q\}$ or $u \in V_i$, $v \in V_j$, $i \neq j$, and there exists a directed path from V_i to V_j in G^*.

Remark 4.2. It follows from Theorem 3.4 and Remark 4.1 that if $u \preccurlyeq v$, then for any maximum-weight stable set S we have $S \cap \{u, v\} \neq \emptyset$.

Consider the bipartite graph $\tilde{G} = (B, W, \tilde{E})$ with vertex set $\tilde{V} = B \cup W$ and edge set $\tilde{E} = \{uv:\ u \in W,\ v \in B,\ u \preccurlyeq v\}$ and assign to each edge $uv \in \tilde{E}$ the cost $c(uv) = c(u) + c(v)$. Let M be a matching in $\tilde{G}$. An edge $uv \in M$ such that $u \preccurlyeq v$, $u \in V_i$, $v \in V_j$, $i \neq j$, is called a *cross-edge*. We say that $v \in \tilde{V}$ is *saturated by* M if there exists a vertex $w \in \tilde{V}$ such that $vw \in M$. A vertex $V_i \in V^*$ is *white deficient with respect to* M if every vertex $v \in V_i \cap B$ is saturated by M and there exists $u \in V_i \cap W$ which is not saturated by M. We denote by $\mathcal{W}_M$ the set of white deficient vertices with respect to M in G^*. We define *black deficient vertices with respect to* M in a similar way. The set of black deficient vertices with respect to M in G^* is denoted by $\mathcal{B}_M$. Let $\mathcal{E}_M$ the set of vertices in G^* that are neither white deficient nor black deficient with respect to M. Notice that, if M is a maximum matching, then $\mathcal{B}_M, \mathcal{W}_M, \mathcal{E}_M$ form a partition of V^*.

Furthermore, notice that a matching M in $\tilde{G}$ corresponds to $|M|$ disjoint pairs (u, v) with $u \in W$, $v \in B$ and $u \preccurlyeq v$. We will show that any proper d-transversal $\mathcal{T}$ in a weighted bipartite graph G containing only free vertices, with $0 \leqslant d \leqslant \delta_G$, necessarily consists in d such pairs, and conversely that d such pairs always form a proper d-transversal. This will imply that a proper d-transversal in G is equivalent to a matching of size d in $\tilde{G}$. We first prove the following.

Lemma 4.1. *Let $G = (B, W, E)$ be a weighted bipartite graph containing only free vertices. Then $\delta_G = \mu(\tilde{G})$.*

Proof. It follows from Remark 4.2 that for every set $\mathcal{T}$ of d disjoint pairs of vertices (u, v) such that $u \preccurlyeq v$ and for every maximum-weight stable set S we have $|\mathcal{T} \cap S| \geqslant d$. Since d such pairs correspond to a matching M of size d in $\tilde{G}$, we conclude that $\delta_G \geqslant \mu(\tilde{G})$.

Let us now prove that $\delta_G \leqslant \mu(\tilde{G})$. Consider a maximum matching M in $\tilde{G}$ such that the number of cross-edges in M is minimum. Let $V_i, V_j \in V^*$, $i \neq j$, be such that there exists a path from V_i to V_j in G^*. We observe the following:

(a) We cannot have $V_i \in \mathcal{W}_M$ and $V_j \in \mathcal{B}_M$.
If $V_i \in \mathcal{W}_M$ and $V_j \in \mathcal{B}_M$, then there exist two vertices $u \in V_i \cap W$, $v \in V_j \cap B$ such that $uv \in \tilde{E}$ which are not saturated by M. Thus M would not be maximum, a contradiction.
(b) We cannot have $V_i \in \mathcal{B}_M$ and a cross-edge $uv \in M$ with $u \in V_i$, $v \in V_j$.
If $V_i \in \mathcal{B}_M$, there exists $w \in V_i \cap B$ which is not saturated by M. Let $u \in V_i \cap W$, $v \in V_j \cap B$ be such that $uv \in M$ is a cross-edge. Then $M' = (M \setminus \{uv\}) \cup \{uw\}$ is a maximum matching containing less cross-edges than M, a contradiction.
(c) We cannot have $V_j \in \mathcal{W}_M$ and a cross-edge $uv \in M$ with $u \in V_i$, $v \in V_j$.
We use the same argument as in (b).

We construct the following set S in G: if $V_i \in \mathcal{W}_M$, then add $V_i \cap B$ to S; if $V_i \in \mathcal{B}_M$, then add $V_i \cap W$ to S; if $V_i \in \mathcal{E}_M$ and there exists V_j which is black and a path from V_j to V_i in G^*, then add $V_i \cap B$ to S; if $V_i \in \mathcal{E}_M$ and there exists V_j which is black and a cross-edge uv with $u \in V_i$, $v \in V_j$, then add $V_i \cap B$ to S; for all other $V_i \in \mathcal{E}_M$, add $V_i \cap W$ to S.

We claim that S is a maximum-weight stable set. From Theorem 3.4 it follows that we just have to prove that S is a stable set. Suppose that S is not a stable set. Then it follows from Remark 4.1 that there exists $V_i, V_j \in V^*$ such that V_i is black with respect to S, V_j is white with respect to S and there exists a path from V_i to V_j in G^*. From the construction of S, it follows that $V_j \in \mathcal{B}_M$. We deduce from (a) that $V_i \notin \mathcal{W}_M$, and hence $V_i \in \mathcal{E}_M$. Furthermore, it follows from the construction of S that there exists $V_k \in \mathcal{W}_M$ and a sequence $\Sigma = (V_k = V_{p_1}, V_{p_2}, \ldots, V_{p_r} = V_i, V_{p_{r+1}} = V_j)$ such that for every V_{p_l}, $1 \leqslant l \leqslant r$, either there exists a path from V_{p_l} to $V_{p_{l+1}}$ in G^* or there exists a cross-edge uv with $u \in V_{p_{l+1}}$ and $v \in V_{p_l}$. Let M and Σ be such that $|\Sigma|$ is minimum. This minimality implies that we cannot have $V_{p_{l-1}}, V_{p_l}, V_{p_{l+1}}$, for $2 \leqslant l \leqslant r$, such that there exists a path in G^* from $V_{p_{l-1}}$ to V_{p_l} and a path from V_{p_l} to $V_{p_{l+1}}$. Furthermore, we cannot have $V_{p_{l-1}}, V_{p_l}, V_{p_{l+1}}$, for $2 \leqslant l \leqslant r-1$, such that there exists a cross-edge uv with $u \in V_{p_l}$, $v \in V_{p_{l-1}}$ and there exists a cross-edge $u'v'$ with $u' \in V_{p_{l+1}}$, $v' \in V_{p_l}$. Indeed, in such a case, we necessarily have $u' \preccurlyeq v$ and $u \preccurlyeq v'$. Then we simply replace the edges $uv, u'v' \in M$ by $uv', u'v$. This clearly gives us another maximum matching M' containing less cross-edges than M, a contradiction. Notice that we necessarily have a path from $V_{p_1} = V_k$ to V_{p_2}. Indeed, since $V_k \in \mathcal{W}_M$, there exists a vertex $w \in V_k \cap W$ which is not saturated by M. Thus if we had a cross-edge $w'b'$ with $w' \in V_{p_2}$ and $b' \in V_k$, then $M' = (M \setminus \{w'b'\}) \cup \{wb'\}$ would be a maximum matching containing less cross-edges than M, a contradiction.

Let $C = \{w_2b_2, w_4b_4, \ldots, w_{r-1}b_{r-1}\}$ with $w_l \in V_{p_{l+1}}$ and $b_l \in V_{p_l}$, be the set of cross-edges mentioned above. Let $u \in V_k \cap W$ and $v \in V_j \cap B$ be two vertices which are not saturated by M (recall that $V_k \in \mathcal{W}_M$ and $V_j \in \mathcal{B}_M$). Then $M' = (M \setminus C) \cup \{ub_2, w_2b_4, \ldots, w_{r-3}b_{r-1}, w_{r-1}v\}$ is a matching in $\tilde{G}$ with $|M'| = |M| + 1$, a contradiction. Thus S is a stable set, and hence it follows from Theorem 3.4 that S is a maximum-weight stable set.

It only remains to show that $|S| \leqslant |M|$. First observe that every $v \in S$ is saturated by M (by definition of S). Furthermore, consider a cross-edge uv in M with $u \in V_i, v \in V_j$. It follows from (a), (b) and (c) that either $V_i, V_j \in \mathcal{E}_M$ or $V_i \in \mathcal{E}_M$, $V_j \in \mathcal{B}_M$ or $V_i \in \mathcal{W}_M$, $V_j \in \mathcal{E}_M$. In all three cases, it follows from the construction of S and Remark 4.1 that either V_i, V_j are both black or V_i, V_j are both white. Hence $|S| \leqslant |M|$. Thus we have $\delta_G \leqslant \mu(\tilde{G})$. $\square$

Lemma 4.2. *Let $G = (B, W, E)$ be a weighted bipartite graph containing only free vertices. Then $\mathcal{T} \subseteq V$ is a proper d-transversal if and only if $\mathcal{T}$ is a set of d disjoint pairs of vertices (u, v), with $0 \leqslant d \leqslant \delta_G$, such that $u \preccurlyeq v$.*

Proof. Let $\mathcal{T}$ be a set of d disjoint pairs of vertices (u, v), with $0 \leqslant d \leqslant \delta_G$, such that $u \preccurlyeq v$. From Lemma 4.1, we can always find such pairs. Let S be a maximum-weight stable set in G. It follows from Remark 4.2 that for every such pair (u, v) at least one of u, v belongs to S. Thus $\mathcal{T}$ is a d-transversal. Since $|\mathcal{T}| = 2d$, it follows from Remark 3.1 that $\mathcal{T}$ is a proper d-transversal.

Now let us prove the converse. It follows from the above that if $\mathcal{T}$ is a d-transversal such that either $\mu(\tilde{G}[\mathcal{T}]) > d$ or $\mu(\tilde{G}[\mathcal{T}]) = d, |\mathcal{T}| > 2d$, then $\mathcal{T}$ is not proper. Hence it is enough to show that if $\mathcal{T}$ is a proper d-transversal, then we cannot have $\mu(\tilde{G}[\mathcal{T}]) < d$.

For $d = 1$, let us suppose by contradiction that $\mathcal{T} = \{w_1, w_2, \ldots, w_p\} \cup \{b_1, b_2, \ldots, b_q\}$, $p, q \geqslant 1$, $w_i \in W$, $b_j \in B$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ is a proper 1-transversal such that $\mu(\tilde{G}[\mathcal{T}]) = 0$. For any pair (w_r, b_s), $w_r \in W \cap \mathcal{T} \cap V_i$, $b_s \in B \cap \mathcal{T} \cap V_j$, we have that $i \neq j$ and there is no path from V_i to V_j in T^* (otherwise w_rb_s would be an edge in $\tilde{G}_{\mathcal{T}}$). Now consider the maximum-weight stable set S defined as follows: $V_k \cap S = V_k \cap B$ if there exists $w_r \in W \cap \mathcal{T} \cap V_k$ or there exists $w_r \in W \cap \mathcal{T} \cap V_i$ and a path from V_i to V_k; otherwise $V_k \cap S = V_k \cap W$. We have $\mathcal{T} \cap S = \emptyset$ and thus $\mathcal{T}$ is not a 1-transversal. Hence a proper minimum 1-transversal consists in one pair (w, b) of minimum total cost such that $w \in W$, $b \in B$ and $w \preccurlyeq b$ (i.e. $\tilde{G}[\mathcal{T}] = (\{w, b\}, \{wb\})$).

Suppose that the lemma holds up to $d-1$. By contradiction suppose that $\mathcal{T}$ is a proper d-transversal with $\mu(\tilde{G}[\mathcal{T}]) \leqslant d-1$. Since $\mathcal{T}$ is proper, it follows that $\forall v \in \mathcal{T}$, $\mathcal{T} \setminus \{v\}$ is not a d-transversal, but clearly $\mathcal{T} \setminus \{v\}$ is a $(d-1)$-transversal. Thus by induction there exists a proper $(d-1)$-transversal $\mathcal{T}' \subset \mathcal{T}$ such that $\mu(\tilde{G}[\mathcal{T}']) = \mu(\tilde{G}[\mathcal{T}]) = d-1$ (i.e. $\mathcal{T}'$ corresponds to a matching of size $d-1$ in $\tilde{G}$). Since $|\mathcal{T} \cap W| \geqslant d$ there exists $u \in \mathcal{T} \cap W$ which is not saturated by some maximum matching in $\tilde{G}[\mathcal{T}]$.

Let M be a maximum matching in $\tilde{G}[\mathcal{T}]$ (i.e. $|M| = d-1$) such that the number of cross-edges in M is minimum. Let $u \in \mathcal{T} \cap W \cap V_i$ be a vertex which is not saturated by M. Let $\mathcal{A} \subseteq \mathcal{V}$ be such that: $V_i \in \mathcal{A}$, and $V_k \in \mathcal{A}$ whenever there is a cross-edge wb with $w \in V_k, b \in V_j, V_j \in \mathcal{A}$, or there is a path from $V_j \in \mathcal{A}$ to V_k. Note that if $V_j \in \mathcal{A}$, there is a sequence $\Sigma' = (V_i = V_{p_1}, V_{p_2}, \ldots, V_{p_s} = V_j)$ such that for all $1 \leqslant k < s$, either there exists a path from V_{p_k} to $V_{p_{k+1}}$ in G^* or there exists a cross-edge uv with $u \in V_{p_{k+1}}$ and $v \in V_{p_k}$.

We will show that every $b \in \mathcal{T} \cap B \cap V_j$, $V_j \in \mathcal{A}$, is saturated by M. Suppose that there exists $b \in \mathcal{T} \cap B \cap V_j$, $V_j \in \mathcal{A}$, which is not saturated by M. Let M and Σ' be such that $|\Sigma'|$ is minimum. Notice that we necessarily have a path from $V_{p_{s-1}}$ to $V_{p_s} = V_j$. Indeed, if there was a cross-edge $w'b'$ with $w' \in V_j$ and $b' \in V_{p_{s-1}}$, then $M' = (M \setminus \{w'b'\}) \cup \{w'b\}$ is another maximum matching containing less cross-edges than M, a contradiction. Now using the same arguments as in the proof of Lemma 4.1 for the sequence Σ, we obtain that Σ' has the same properties as Σ. Hence we conclude that M is not maximum, a contradiction. Thus every $b \in \mathcal{T} \cap B \cap V_j$, $V_j \in \mathcal{A}$, is saturated by M. Moreover b is matched with w where $w \in \mathcal{T} \cap W \cap V_k$ and $V_k \in \mathcal{A}$ by definition of $\mathcal{A}$.

We know that there exists a maximum-weight stable set S such that V_i is white with respect to S and $|(\mathcal{T} \setminus \{u\}) \cap S| = d-1$. Let S' be defined as follows: if $V_j \in \mathcal{A}$ then V_j is black with respect to S'; otherwise V_j remains as it was colored with respect to S. Notice that it follows from Remark 4.1 and the definition of $\mathcal{A}$ that S' is a stable set and it follows from Theorem 3.4 that S' has maximum weight. Clearly, we have $|\mathcal{T} \cap S \cap V_j| = |\mathcal{T} \cap S' \cap V_j|$ for every $V_j \notin \mathcal{A}$.

We will show now that $|\mathcal{T} \cap S' \cap \mathcal{A}| < |\mathcal{T} \cap S \cap \mathcal{A}|$. We know that all black vertices in $\mathcal{T} \cap \mathcal{A}$ are matched with vertices in $\mathcal{T} \cap \mathcal{A}$. Let $W_{\bar{M}}$ be the set of white vertices in $\mathcal{T} \cap \mathcal{A}$ which are not saturated by M. Hence $u \in W_{\bar{M}} \cap V_i \cap S$. So we have $|S \cap W_{\bar{M}}| > 0$ and, since $S' \cap \mathcal{A} \subseteq B$, $|S' \cap W_{\bar{M}}| = 0$. Now let us consider an edge $wb \in M$ with $w \in V_k \cap \mathcal{T} \cap W$, $V_k \in \mathcal{A}$ and $b \in V_j \cap \mathcal{T} \cap B$, $V_j \in \mathcal{A}$. Then $\{w, b\} \cap S' = \{b\}$ and $|\{w, b\} \cap S| \geqslant 1$. Indeed, either $V_k \cap S \subseteq B$ and then $V_j \cap S \subseteq B$ so $\{w, b\} \cap S = \{b\}$ or $V_k \cap S \subseteq W$ and then $|\{w, b\} \cap S| \geqslant 1$. Finally, $|S' \cap \mathcal{T} \cap \mathcal{A}| = |S' \cap \mathcal{T} \cap (\mathcal{A} \setminus W_{\bar{M}})| \leqslant |S \cap \mathcal{T} \cap (\mathcal{A} \setminus W_{\bar{M}})| < |S \cap \mathcal{T} \cap \mathcal{A}|$ (since $u \in W_{\bar{M}}$).

Thus we have $|\mathcal{T} \cap S'| < |\mathcal{T} \cap S| = d$, which is a contradiction. $\square$

Let us now prove the main result.

Theorem 4.3. TRANS *is polynomial-time solvable for weighted bipartite graphs.*

Proof. Let $G = (B, W, E)$ be a weighted bipartite graph with $|V| = n$ and let c be a nonnegative cost function on $B \cup W$. The first step consists in determining the sets V^f, V^e, and V^{fr} as shown in [2], Lemma 7. This can be done in $O(|V|^2|E|^2)$ for instance by simply solving $|E|$ flow problems in a bipartite graph.

Then we delete the forced and the excluded vertices from G. Let G' be the resulting graph with connected components $G_1, \ldots, G_p$. Notice that every graph G_i contains only free vertices, for $i = 1, \ldots, p$. Computing $|E|$ maximum flows in a bipartite network, we obtain the set of forbidden arcs $\hat{F}$ in time $O(|V|^2|E|^2)$. This gives us the graph G_i^* for each corresponding G_i, $i = 1, \ldots, p$. Clearly their corresponding graphs $\tilde{G}_i$ can be built in time $O(|V||E|)$. Let $\tilde{G} = \tilde{G}_1 \cup \cdots \cup \tilde{G}_p \cup H$, where H consists of $|V^f|$ isolated edges $e_1, \ldots, e_{|V^f|}$ such that every e_i corresponds to a forced vertex v_i in G and such that $c(e_i) = c(v_i)$, for $i = 1, \ldots, |V^f|$.

Now it follows from Lemmas 4.1 and 4.2 that a proper minimum d-transversal $\mathcal{T}$, with $0 \leqslant d \leqslant \delta_G$, in G corresponds to a matching of size d in $\tilde{G}$ with minimum total cost. Since such a matching can be found in polynomial time by a minimum cost flow algorithm in $O((|E| \log |V|)(|E| + |V| \log |V|))$, (see [6, Chapter 12]), it follows that TRANS is polynomial-time solvable in $O(|V|^2|E|^2)$. $\square$

Remark 4.3. It is easy to see that, in a weighted bipartite graph G, a maximum-weight stable set of minimum size can be obtained in polynomial time by first decreasing the weights in G by a sufficiently small amount $\epsilon > 0$, and then finding a maximum-weight stable set in this new graph. Lemma 4.1 and Theorem 4.3 provide an alternate way to compute the minimum size δ_G of such a maximum-weight stable set.

5. Minimum *d*-transversals of minimum-weight vertex covers

A vertex cover in a graph $G = (V, E)$ is a set $\bar{S} \subseteq V$ such that for every edge $uv \in E$ at least one of u, v belongs to $\bar{S}$. Recall that a vertex cover is the complement of a stable set. We denote by $\bar{\mathcal{S}}_G$ the family of all minimum-weight vertex covers in G, and by $\bar{\delta}_G$ the minimum cardinality of a minimum-weight vertex cover. In this section we will be interested in the following problem.

TRANS-VC
Input: A weighted graph $G = (V, E)$; a nonnegative cost function c for V; an integer $d \geqslant 0$.
Output: A proper minimum d-transversal $\mathcal{T}$ with respect to $\bar{\mathcal{S}}_G$.

In the same way as we defined forced, free and excluded vertices for the maximum-weight stable sets, we may define forced, free and excluded vertices for the minimum-weight vertex covers. Notice that a vertex v which is forced for the maximum-weight stable sets is excluded for the minimum-weight vertex covers, and vice versa. Furthermore a vertex is free for the minimum-weight vertex covers if and only if it is free for the maximum-weight stable sets. Also, Theorem 3.4 gives a characterization of the minimum-weight vertex covers in a bipartite graph G in which all the vertices are free.

Theorem 5.1. *Let $G = (B, W, E)$ be a weighted bipartite graph containing only free vertices. Then $\bar{S} \subseteq B \cup W$ is a minimum-weight vertex cover if and only if $\bar{S}$ is a vertex cover and for any $j \in \{1, \dots, q\}$ either $\bar{S} \cap V_j = B \cap V_j$ or $\bar{S} \cap V_j = W \cap V_j$.*

It immediately follows that any d-transversal $\mathcal{T}$ with respect to $\bar{\mathcal{S}}_G$ contains at least $2d$ vertices.

From the partition $\mathcal{V}$ of V, we define the auxiliary digraph $\bar{G}^* = (V^*, \bar{A}^*)$: $V^* = \{V_1, \dots, V_q\}$ and $\bar{A}^* = \{(V_i, V_j) \mid \exists vu \in E,\ v \in V_i \cap W,\ u \in V_j \cap B,\ i \neq j\}$. Note that $\bar{G}^*$ is obtained from G^* by reversing the direction of each arc. We say that $V_i \in \mathcal{V}$ is white (resp. black) with respect to a minimum-weight vertex cover $\bar{S}$ if $\bar{S} \cap V_i = W \cap V_i$ (resp. $\bar{S} \cap V_i = B \cap V_i$). As a consequence we obtain the following.

Remark 5.1. If $\bar{S} \in \bar{\mathcal{S}}_G$ and if V_i is black with respect to $\bar{S}$, then all successors of V_i in $\bar{G}^*$ are black with respect to $\bar{S}$.

From $\bar{G}^*$ we define the relation $(V, \preccurlyeq)$ in the same way as in Section 4: for $u \in W$ and $v \in B$, $u \preccurlyeq v$ if and only if either $u, v \in V_j$ for some $j \in \{1, \dots, q\}$ or $u \in V_i$, $v \in V_j$, $i \neq j$, and there exists a path from V_i to V_j in $\bar{G}^*$. We define the graph $\bar{\tilde{G}} = (B, W, \bar{\tilde{E}})$ in the same way as $\tilde{G}$, i.e., $\bar{\tilde{E}} = \{uv\colon\ u \in W,\ v \in B,\ u \preccurlyeq v\}$. Given a matching M in $\bar{\tilde{G}}$, the cross-edges, and the sets $\mathcal{B}_M$, $\mathcal{W}_M$, $\mathcal{E}_M$ are defined as in Section 4.

Using exactly the same proofs as for Lemmas 4.1 and 4.2 (changing *maximum-weight stable sets* into *minimum-weight vertex covers*, and so on) we obtain the following.

Lemma 5.2. *Let $G = (B, W, E)$ be a weighted bipartite graph containing only free vertices. Then $\bar{\delta}_G = \mu(\bar{\tilde{G}})$.*

Lemma 5.3. *Let $G = (B, W, E)$ be a weighted bipartite graph containing only free vertices. Then $\mathcal{T} \subseteq V$ is a proper d-transversal with respect to $\bar{\mathcal{S}}_G$ if and only if $\mathcal{T}$ is a set of d disjoint pairs of vertices (u, v), with $0 \leqslant d \leqslant \bar{\delta}_G$, such that $u \preccurlyeq v$.*

Finally, using the same construction as in Section 4 we obtain the following result.

Theorem 5.4. TRANS-VC *is polynomial-time solvable for weighted bipartite graphs.*

Remark 5.2. Notice that in a weighted bipartite graph, the minimum cardinality $\bar{\delta}_G$ of a minimum-weight vertex cover can be determined in polynomial time (see [6, Chapter 17]). Lemma 5.2 and Theorem 5.4 provide an alternate way to compute the minimum size of such a minimum-weight vertex cover. By taking the complements of the subsets of vertices considered, the maximum cardinality of a minimum-weight vertex cover and the maximum cardinality of a maximum-weight stable set can also be determined in polynomial time.

6. Conclusion

In this paper, we considered proper minimum d-transversals with respect to maximum-weight stable sets in bipartite graphs and we gave a polynomial-time algorithm to find such transversals. Our algorithm relies on a characterization of maximum-weight stable sets in bipartite graphs. Exploiting the complementarity between stable sets and vertex covers, we have derived results on d-transversals of minimum-weight vertex covers.

A notion related to d-transversals is the one of d-blockers. A d-blocker with respect to maximum-weight stable sets in a graph G is a subset of vertices $\mathcal{B}$ such that $\alpha_w(G - \mathcal{B}) \leqslant \alpha_w(G) - d$. It is interesting to mention that the problem of finding a d-blocker of minimum cardinality with respect to maximum-weight stable sets has been shown to be $\mathcal{NP}$-hard in bipartite graphs (see [1]).

It would be interesting to determine further classes of graphs in which TRANS can be solved in polynomial time.

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