Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and L^p-L^q Fourier multipliers on compact homogeneous manifolds

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Abstract

In this paper we prove new inequalities describing the relationship between the "size" of a function on a compact homogeneous manifold and the "size" of its Fourier coefficients. These inequalities can be viewed as noncommutative versions of the Hardy-Littlewood inequalities obtained by Hardy and Littlewood [HL27] on the circle. For the example case of the group SU(2) we show that the obtained Hardy-Littlewood inequalities are sharp, yielding a criterion for a function to be in $L^p(\mathrm{SU}(2))$ in terms of its Fourier coefficients. We also establish Paley and Hausdorff-Young-Paley inequalities on general compact homogeneous manifolds. The latter is applied to obtain conditions for the L^p-L^q boundedness of Fourier multipliers for $1 < p \leq 2 \leq q < \infty$ on compact homogeneous manifolds as well as the L^p-L^q boundedness of general (non-invariant) operators on

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compact Lie groups. We also record an abstract version of the Marcinkiewicz interpolation theorem on totally ordered discrete sets, to be used in the proofs with different Plancherel measures on the unitary duals.

Keywords: Hardy-Littlewood inequality, Paley inequality, Hausdorff-Young inequality, Lie groups, homogeneous manifolds, Fourier multipliers, Marcinkiewicz interpolation theorem

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1. Introduction

A fundamental problem in Fourier analysis is that of investigating the relationship between the "size" of a function and the "size" of its Fourier transform.

The aim of this paper is to give necessary conditions and sufficient conditions for the L^p-integrability of a function on an arbitrary compact homogeneous space G/K by means of its Fourier coefficients. The obtained inequalities provide a noncommutative version of known results of this type on the circle $\mathbb{T}$ and the real line $\mathbb{R}$.

To explain this briefly, we recall that in [HL27], Hardy and Littlewood have shown that for $1 < p \leq 2$ and $f \in L^p(\mathbb{T})$, the following inequality holds true:

$$\sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p \leq C \|f\|^p_{L^p(\mathbb{T})}, \tag{1}$$

arguing this to be a suitable extension of the Plancherel identity to L^p-spaces. Hewitt and Ross [HR74] generalised this to the setting of compact abelian groups. While we refer to Section 2 and particularly to Theorem 2.1 for more details on this, to give a flavour of our results, our analogue for this on compact homogeneous manifolds G/K of dimension $n = \dim G/K$ is the inequality

$$\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p} - \frac{1}{2})} \langle \pi \rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}} \leq C \|f\|^p_{L^p(G/K)}, \quad 1 < p \leq 2, \tag{2}$$

which for $p = 2$ gives the ordinary Plancherel identity on G/K, see (16). Briefly, here $\widehat{G}_0$ stands for class I representations of a compact Lie group G with respect to the subgroup K, $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ is the Fourier coefficient of f at the representation π of degree d_π, k_π is the number of invariant vectors of the representation π with respect to K, and $\langle \pi \rangle$ are the eigenvalues of the operator $(I - \Delta_{G/K})^{1/2}$ corresponding to π for a Laplacian $\Delta_{G/K}$ on the compact homogeneous space G/K. We refer to Theorem 2.2 for this statement and to Section 2.1 for precise definitions.

In particular, in this paper we establish the following results, that we now summarise and briefly discuss:

- *Hardy-Littlewood inequality*: The Hardy-Littlewood type inequality (2) holds on arbitrary compact homogeneous manifolds. In particular, we can also rewrite it as

$$\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi \langle \pi \rangle^{n(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}} \right)^p \leq C \|f\|^p_{L^p(G/K)}, \quad 1 < p \leq 2, \tag{3}$$

  interpreting

$$\mu(Q) = \sum_{\pi \in Q} d_\pi k_\pi \tag{4}$$

  as the Plancherel measure on the set $\widehat{G}_0$, the 'unitary dual' of the homogeneous manifold G/K, and k_π the maximal rank of Fourier coefficients matrices $\widehat{f}(\pi)$, so that e.g. $\|\widehat{\delta}(\pi)\|_{\mathtt{HS}} = \sqrt{k_\pi}$ for the delta-function δ on G/K and $\pi \in \widehat{G}_0$.

  Using the Hilbert-Schmidt norms of Fourier coefficients in (2) rather than Schatten norms (leading to a different version of ℓ^p-spaces on the unitary dual) leads to the sharper estimate – this is shown in (33) and (34).

- *Differential/Sobolev space interpretations*: The exact form of (2) or (3) is justified in Section 2.1 by comparing the differential interpretations (14) and (28) of the classical Hardy-Littlewood inequality (1) and of (2), respectively. In fact, it is exactly from these differential interpretations is how we arrive at the desired expression in (2). Roughly, both are saying that for $1 < p \leq 2$,

$$g \in L^p_{2n(\frac{1}{p} - \frac{1}{2})}(G/K) \Longrightarrow \widehat{g} \in \ell^p(\widehat{G}_0) \tag{5}$$

with the corresponding norm estimate $\|\widehat{g}\|_{\ell^p(\widehat{G}_0)} \leq C\|g\|_{L^p_{2n(\frac{1}{p}-\frac{1}{2})}(G/K)}$, where $L^p_{2n(\frac{1}{p}-\frac{1}{2})}$ is the Sobolev space over L^p of order $2n(\frac{1}{p}-\frac{1}{2})$, and $\ell^p(\widehat{G}_0)$ is an appropriately defined Lebesgue space ℓ^p on the unitary dual $\widehat{G}_0$ of representations relevant to G/K, with respect to the corresponding Plancherel measure. In particular, as a special case we have the original Hardy-Littlewood inequality (1), which can be reformulated as

$$g \in L^p_{2(\frac{1}{p}-\frac{1}{2})}(\mathbb{T}) \Longrightarrow \widehat{g} \in \ell^p(\mathbb{Z}), \quad 1 < p \leq 2,$$

see (14), since $\ell^p(\widehat{\mathbb{T}}_0) \simeq \ell^p(\mathbb{Z})$, and the Plancherel measure is the counting measure on $\mathbb{Z}$ in this case.

- *Duality*: By duality, the inequality (2) remains true (with the reversed inequality) also for $2 \leq p < \infty$.

- *Sharpness*: The inequality (2) is sharp in the following sense: if the Fourier coefficients are positive and monotone (in a suitable sense), and a certain non-oscillation condition holds, the inequality in (2) becomes an equivalence. In the case of the circle $G = \mathbb{T}$, this was shown by Hardy and Littlewood (see Theorem 2.6) – here, positivity and monotonicity are understood classically, and the oscillation condition is automatically satisfied (see Remark 2.11). While we conjecture this equivalence to be true for general compact homogeneous manifolds, we make this precise in the example of the group $G = \mathrm{SU}(2)$.

- *Paley inequality*: We propose (10) as a Paley-type inequality that holds on general compact homogeneous manifolds. On one hand, our inequality (10) extends Hörmander's Paley inequality on $\mathbb{R}^n$. On the other hand, combined with the Weyl asymptotic formula for the eigenvalue counting function of elliptic differential operators on the compact manifold G/K, it implies the Hardy-Littlewood inequality (2) as a special case (and this is how we prove it too).

- *Hausdorff-Young-Paley inequality*: The Paley inequality (10) and the Hausdorff-Young inequalities on G/K in a suitable scale of spaces $\ell^p(\widehat{G}_0)$ on the unitary dual of G/K imply the Hausdorff-Young-Paley inequality. This is given in Theorem 2.5.

- *L^p-L^q Fourier multipliers.* The established Hausdorff-Young-Paley inequality becomes instrumental in obtaining L^p-L^q Fourier multiplier theorems on G/K for indices $1 < p \leq 2 \leq q < 2$. In Section 3 we give such results for Fourier multipliers on G/K: for a Fourier multiplier A acting by $\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi)$ and $1 < p \leq 2 \leq q < 2$ we have

$$\|A\|_{L^p(G/K)\to L^q(G/K)} \lesssim \sup_{s>0}\left\{s\mu(\pi \in \widehat{G}_0 : \|\sigma_A(\pi)\|_{\mathtt{op}} > s)^{\frac{1}{p}-\frac{1}{q}}\right\},$$

where μ is the Plancherel measure as in (4), see Theorem 3.1. Consequently, in Theorem 3.3 we also give a general $L^p(G)$-$L^q(G)$ boundedness result for general (not necessarily invariant) operators A on a compact Lie group G in terms of their matrix symbols $\sigma_A(x, \xi)$.

We now discuss some of these results, their relevance, and motivation behind them in more detail.

In [HL27], Hardy and Littlewood established the necessary condition for f to be in $L^p(\mathbb{T})$ in terms of its Fourier coefficients for $1 < p \leq 2$, and by duality the sufficient conditions for f to be in $L^p(\mathbb{T})$ for $2 \leq p < \infty$ (we recall these statements in Theorem 2.1). We discuss how to extend these results to the noncommutative setting of general compact homogeneous manifolds. This is done in Section 2.1 and in Theorem 2.2.

On the circle, Hardy and Littlewood have shown that for $1 < p < \infty$, if the Fourier coefficients $\widehat{f}(m)$ are monotone, then one also has the converse to (1), namely,

$$f \in L^p(\mathbb{T}) \quad \text{if and only if} \quad \sum_{m\in\mathbb{Z}}(1+|m|)^{p-2}|\widehat{f}(m)|^p < \infty. \tag{6}$$

To show that our Hardy-Littlewood inequalities in Theorem 2.2 are sharp, in Section 2.3 we introduce the notion of 'monotonicity' for sequences of matrix Fourier coefficients for functions on SU(2), and in Theorem 2.10 we show that for $\frac{3}{2} < p \leq 2$ and $G = \mathrm{SU}(2)$ the Hardy-Littlewood inequalities in Theorem 2.2 can be also strengthened to provide a criterion: if the Fourier coefficients of a central function $f \in L^{3/2}(\mathrm{SU}(2))$ are 'general monotone' and a certain (natural) non-oscillation condition is satisfied, then

$$f \in L^p(\mathrm{SU}(2)) \quad \text{if and only if} \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5p}{2}-4} \|\widehat{f}(l)\|_{\mathtt{HS}}^p < \infty. \tag{7}$$

The equivalence in (7) can be thought of as the analogue of (6) on the circle: indeed, on the circle, the mentioned non-oscillation condition is automatically satisfied, all functions are central, and the power $\frac{5p}{2} - 4$ in (7) has a natural interpretation (in particular, for $p = 2$, it boils down to the Plancherel formula on $\mathrm{SU}(2)$, see (43)).

The restriction on p to satisfy $\frac{3}{2} < p < \frac{5}{2}$ in Theorem 2.10 (and above in (7), but we are interested in $p \leq 2$ since $p > 2$ will be covered by the dual part of the Hardy-Littlewood inequality) is a particular instance of the fact that on compact simply connected semisimple Lie groups, the polyhedral Fourier partial sums of (a central function) f converge to f in L^p if and only if $2 - \frac{1}{s+1} < p < 2 + \frac{1}{s}$. Here the number s depends on the root system $\mathcal{R}$ of the compact Lie group G (see Stanton [Sta76], Stanton and Tomas [ST76], and Colzani, Giulini and Travaglini [CGT89] for the only if statement), see Appendix Appendix A for precise definitions and review. It can be shown that for $G = \mathbb{T}$ and $G = \mathrm{SU}(2)$, we have $s = 0$ and $s = 1$ respectively. Thus, Theorem 2.10 can be considered as a natural counterpart on $\mathrm{SU}(2)$ to the criterion (6) of Hardy and Littlewood on the circle. In order to prove the above statements, we need to develop several things which are of interest on their own:

- In Proposition 4.2 we prove an estimate for the Dirichlet kernel on the group $\mathrm{SU}(2)$. This estimate appears to be sharp because its application yields a sharp criterion for the L^p-integrability of functions on $\mathrm{SU}(2)$ in Theorem 2.10.

- In Appendix Appendix B, we establish an abstract version of the Marcinkiewicz interpolation theorem on totally ordered discrete sets. Consequently, it is applied in proofs in the paper for different choices of the measure on the discrete unitary dual $\widehat{G}$ and on the discrete set $\widehat{G}_0 \subset \widehat{G}$ of class I representations of G.

In Section 2.2 we establish Paley-type inequalities on compact homogeneous manifolds. Recall briefly that in [Hör60] Lars Hörmander has shown that if a positive function $\varphi \geq 0$ satisfies

$$|\{\xi \in \mathbb{R}^n \colon \varphi(\xi) \geq t\}| \leq \frac{C}{t} \quad \text{for } t > 0, \tag{8}$$

then

$$\left(\int\limits_{\mathbb{R}^n} |\widehat{u}|^p \, \varphi^{2-p} \, d\xi \right)^{\frac{1}{p}} \lesssim \|u\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq 2. \tag{9}$$

We note that condition (8) is equivalent to

$$M_\varphi := \sup_{t>0} t|\{\xi \in \mathbb{R}^n \colon \varphi(\xi) \geq t\}| < \infty.$$

Our analogue for this is the inequality

$$\left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}} \, \varphi(\pi)^{2-p} \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \, \|f\|_{L^p(G/K)}, \quad 1 < p \leq 2, \tag{10}$$

where $\varphi(\pi)$ is a positive sequence over $\widehat{G}_0$ such that

$$M_\varphi := \sup_{t>0} t \sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \geq t}} d_\pi k_\pi < \infty.$$

Here, as well as in other results of this paper, the measure $\mu(Q) = \sum\limits_{\pi \in Q} d_\pi k_\pi$ appears as an analogue of the Plancherel measure on sets $Q \subset \widehat{G}_0$.

The sum over an empty set in the definition of M_φ is assumed to be zero. With $\varphi(\pi) = \langle \pi \rangle^{-n}$, using the asymptotic formula for the Weyl eigenvalue counting function for the Laplacian on G/K to show that $M_\varphi < \infty$, inequality (10) gives inequality (2). In this sense, the Paley inequality (10) is an extension of one of the Hardy-Littlewood inequalities.

We prove such Paley-type inequality in Theorem 2.3. Consequently, we can use the weighted interpolation between the Paley inequality and a suitable version of the noncommutative Hausdorff-Young inequality (37) on the homogeneous manifolds. This yields what we then call the Hausdorff-Young-Paley inequality in Theorem 2.5. This inequality is very useful for obtaining the L^p-L^q multiplier theorems for Fourier multipliers on compact Lie groups and compact homogeneous spaces. This application is given in Section 3 to provide conditions for the L^p-L^q boundedness of Fourier multipliers for $p \leq q$. A special case on SU(2) has been done by the authors in [ANR16]. For $p = q$, the Fourier multipliers have been analysed in [RW13], with the Hörmander-Mikhlin theorem on general compact Lie groups established in [RW15], extending the results for Fourier multipliers on SU(2) by Coifman-de Guzman [CdG71] and Coifman and Weiss [CW71b, CW71a], to the general setting of compact Lie groups.

The paper is organised as follows. In Section 2 we fix the notation for the representation theory of compact Lie groups and formulate estimates relating functions to the behaviour of their Fourier coefficients: the version of the Hardy–Littlewood inequalities on arbitrary compact homogeneous manifold G/K and further extensions. In Section 2.3 we give a criterion for the p^{th} power integrability of a function on SU(2) in terms of its Fourier coefficients. In Section 3 we obtain L^p-L^q Fourier multiplier theorem on G/K and the L^p-L^q boundedness theorem for general operators on G. In Section 4 we complete the proofs of the results presented in previous sections. In Section 4.4 we give an interesting estimate for the Dirichlet kernel on SU(2) which is instrumental in the proof of the inverse to the Hardy-Littlewood inequality on the case of the group being SU(2). In Appendix Appendix A we briefly review the topic of polyhedral sums for Fourier series. In Appendix Appendix B we discuss a matrix-valued version of the Marcinkiewicz interpolation theorem that will be instrumental for our proofs.

Main inequalities in this paper are established on general compact homogeneous manifolds of the form G/K, where G is a compact Lie group and K is a compact subgroup. Important examples are compact Lie groups themselves when we take the trivial subgroup $K = \{e\}$ in which case $k_\pi = d_\pi$, or spaces like spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or complex spheres (projective spaces)

$\mathbb{CS}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ in which cases the subgroups are massive and so $k_\pi = 1$ for all $\pi \in \widehat{G}_0$. We briefly describe such spaces and their representation theory in Section 2.1. When we want to show the sharpness of the obtained inequalities, we may restrict to the case of semisimple Lie groups G. As another special case, we consider the group SU(2), in which case in Theorem 2.10 we obtain an analogue of the Hardy-Littlewood criterion for integrability of functions in $L^p(\mathrm{SU}(2))$ in terms of their Fourier coefficients. This provides the converse to Hardy-Littlewood inequalities on SU(2) previously obtained by the authors in [ANR16].

We shall use the symbol C to denote various positive constants, and $C_{p,q}$ for constants which may depend only on indices p and q. We shall write $x \lesssim y$ for the relation $|x| \leq C|y|$, and write $x \cong y$ if $x \lesssim y$ and $y \lesssim x$.

2. Main results

In this section we introduce the necessary notation and formulate main results of the paper. Along the exposition, we provide references to the relevant literature.

2.1. Notation and Hardy-Littlewood inequalities

In [HL27, Theorems 10 and 11], Hardy and Littlewood proved the following generalisation of the Plancherel's identity on the circle $\mathbb{T}$.

Theorem 2.1 (Hardy–Littlewood [HL27])**.** *The following holds.*

1. *Let $1 < p \leq 2$. If $f \in L^p(\mathbb{T})$, then*

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2} |\widehat{f}(m)|^p \leq C_p \|f\|^p_{L^p(\mathbb{T})}, \tag{11}$$

where C_p is a constant which depends only on p.

2. *Let $2 \leq p < \infty$. If $\{\widehat{f}(m)\}_{m\in\mathbb{Z}}$ is a sequence of complex numbers such that*

$$\sum_{m\in\mathbb{Z}} (1+|m|)^{p-2} |\widehat{f}(m)|^p < \infty, \tag{12}$$

then there is a function $f \in L^p(\mathbb{T})$ with Fourier coefficients given by $\widehat{f}(m)$, and

$$\|f\|^p_{L^p(\mathbb{T})} \leq C'_p \sum_{m\in\mathbb{Z}} (1+|m|)^{p-2} |\widehat{f}(m)|^p.$$

Hewitt and Ross [HR74] generalised this theorem to the setting of compact abelian groups. We note that if $\Delta = \partial_x^2$ is the Laplacian on $\mathbb{T}$, and $\mathscr{F}_{\mathbb{T}}$ is the Fourier transform on $\mathbb{T}$, the Hardy-Littlewood inequality (11) can be reformulated as

$$\|\mathscr{F}_{\mathbb{T}}\Big((1-\Delta)^{\frac{p-2}{2p}} f\Big)\|_{\ell^p(\mathbb{Z})} \leq C_p \|f\|_{L^p(\mathbb{T})}. \tag{13}$$

Denoting $(1-\Delta)^{\frac{p-2}{2p}} f$ by f again, this becomes also equivalent to the estimate

$$\|\widehat{f}\|_{\ell^p(\mathbb{Z})} \leq C_p \|(1-\Delta)^{-\frac{p-2}{2p}} f\|_{L^p(\mathbb{T})} \equiv C_p \|(1-\Delta)^{\frac{1}{p}-\frac{1}{2}} f\|_{L^p(\mathbb{T})},\ 1<p\leq 2. \tag{14}$$

The first purpose of this section is to argue what could be a noncommutative version of these estimates and then to establish an analogue of Theorem 2.1 in the setting of compact homogeneous manifolds. To motivate the formulation, we start with a compact Lie group G. Identifying a representation π with its equivalence class and choosing some bases in the representation spaces, we can think of $\pi \in \widehat{G}$ as a unitary matrix-valued mapping $\pi : G \to \mathbb{C}^{d_\pi \times d_\pi}$. For $f \in L^1(G)$, we define its Fourier transform at $\pi \in \widehat{G}$ by

$$(\mathscr{F}_G f)(\pi) \equiv \widehat{f}(\pi) := \int_G f(u)\pi(u)^* du,$$

where du is the normalised Haar measure on G. This definition can be extended to distributions $f \in \mathcal{D}'(G)$, and the Fourier series takes the form

$$f(u) = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\Big(\pi(u)\widehat{f}(\pi)\Big). \tag{15}$$

The Plancherel identity on G is given by

$$\|f\|^2_{L^2(G)} = \sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|^2_{\mathtt{HS}} =: \|\widehat{f}\|^2_{\ell^2(\widehat{G})}, \tag{16}$$

yielding the Hilbert space $\ell^2(\widehat{G})$. Thus, Fourier coefficients of functions and distributions on G take values in the space

$$\Sigma = \left\{\sigma = (\sigma(\pi))_{\pi\in\widehat{G}} : \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}\right\}. \tag{17}$$

The ℓ^p-spaces on the unitary dual of a compact Lie group can be defined, for example, motivated by the Hausdorff–Young inequality in the form

$$\left(\sum_{\pi\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{S^{p'}}^{p'}\right)^{1/p'} \leq \|f\|_{L^p(G)} \text{ for } 1<p\leq 2, \tag{18}$$

with an obvious modification for $p=1$, with $\frac{1}{p}+\frac{1}{p'}=1$, and where $S^{p'}$ is the p'-Schatten class on the space of matrices $\mathbb{C}^{d_\pi\times d_\pi}$. For the inequality (18) see [Kun58]. Thus, for any $1\leq p<\infty$ we can define the (Schatten-based) spaces $\ell^p_{sch}(\widehat{G})\subset\Sigma$ by the norm

$$\|\sigma\|_{\ell^p_{sch}(\widehat{G})} := \left(\sum_{\pi\in\widehat{G}} d_\pi \|\sigma(\pi)\|_{S^p}^{p}\right)^{1/p}, \ \sigma\in\Sigma. \tag{19}$$

The Hausdorff-Young inequality (18) can be then reformulated as

$$\|\widehat{f}\|_{\ell^{p'}_{sch}(\widehat{G})} \leq \|f\|_{L^p(G)} \text{ for } 1<p\leq 2.$$

We refer to Hewitt and Ross [HR70, Section 31] or to Edwards [Edw72, Section 2.14] for a thorough analysis of these spaces.

At the same time, another scale of ℓ^p-spaces on the unitary dual $\widehat{G}$ has been developed in [RT10] based on fixing the Hilbert-Schmidt norms, and this scale will actually provide sharper results in our problem. In view of subsequently established converse estimates using the same expressions, it appears that this scale of spaces is the correct one for extending the Hardy-Littlewood inequalities to the noncommutative setting. Thus, for $1\leq p<\infty$, we define the space $\ell^p(\widehat{G})$ by the norm

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{\pi\in\widehat{G}} d_\pi^{p\left(\frac{2}{p}-\frac{1}{2}\right)} \|\sigma(\pi)\|_{\mathtt{HS}}^{p}\right)^{1/p}, \ \sigma\in\Sigma,\ 1\leq p<\infty, \tag{20}$$

where $\|\cdot\|_{\mathtt{HS}}$ denotes the Hilbert-Schmidt matrix norm i.e.

$$\|\sigma(\pi)\|_{\mathtt{HS}} := \left(\mathrm{Tr}(\sigma(\pi)\sigma(\pi)^*)\right)^{\frac{1}{2}}.$$

It was shown in [RT10, Section 10.3] that, among other things, these are interpolation spaces, and that the Fourier transform $\mathscr{F}_G$ and its inverse $\mathscr{F}_G^{-1}$ satisfy the Hausdorff-Young inequalities in these spaces.

The power of d_π in (20) can be naturally interpreted if we rewrite it in the form

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{\pi\in\widehat{G}} d_\pi^2 \left(\frac{\|\sigma(\pi)\|_{\mathtt{HS}}}{\sqrt{d_\pi}} \right)^p \right)^{1/p}, \ \sigma \in \Sigma,\ 1 \le p < \infty, \tag{21}$$

and think of $\mu(Q) = \sum_{\pi\in Q} d_\pi^2$ as the Plancherel measure on $\widehat{G}$, and of $\sqrt{d_\pi}$ as the normalisation for matrices $\sigma(\pi) \in \mathbb{C}^{d_\pi\times d_\pi}$, in view of $\|I_{d_\pi}\|_{\mathtt{HS}} = \sqrt{d_\pi}$ for the identity matrix $I_{d_\pi} \in \mathbb{C}^{d_\pi\times d_\pi}$.

We note that for a matrix $\sigma(\pi) \in \mathbb{C}^{d_\pi\times d_\pi}$, for $1 \le p \le 2$, by Hölder inequality we have

$$\|\sigma(\pi)\|_{S^p} \le d_\pi^{\frac{1}{p}-\frac{1}{2}} \|\sigma(\pi)\|_{\mathtt{HS}}.$$

Consequently, for $1 \le p \le 2$, one can show the embedding $\ell^p(\widehat{G}) \subset \ell^p_{sch}(\widehat{G})$, with the inequality

$$\|\sigma\|_{\ell^p_{sch}(\widehat{G})} \le \|\sigma\|_{\ell^p(\widehat{G})}, \ \forall\sigma \in \Sigma,\ 1 \le p \le 2. \tag{22}$$

We now describe the setting of Fourier coefficients on a compact homogeneous manifold M following [DR14] or [NRT14], and referring for further details with proofs to Vilenkin [Vil68] or to Vilenkin and Klimyk [VK91].

Let G be a compact motion group of M and let K be the stationary subgroup of some point. Alternatively, we can start with a compact Lie group G with a closed subgroup K, and identify $M = G/K$ as an analytic manifold in a canonical way. We normalise measures so that the measure on K is a probability one. Typical examples are the spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or complex spheres $\mathbb{CS}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$.

Let us denote by $\widehat{G}_0$ the subset of $\widehat{G}$ of representations that are class I with respect to the subgroup K. This means that $\pi \in \widehat{G}_0$ if π has at least one non-zero invariant vector a with respect to K, i.e. that

$$\pi(h)a = a \ \text{ for all } \ h \in K.$$

Let $\mathcal{B}_\pi$ denote the space of these invariant vectors and let

$$k_\pi := \dim \mathcal{B}_\pi.$$

Let us fix an orthonormal basis in the representation space of π so that its first k_π vectors are the basis of B_π. The matrix elements $\pi(x)_{ij}$, $1 \le j \le k_\pi$, are invariant under the right shifts by K.

We note that if $K = \{e\}$ so that $M = G/K = G$ is the Lie group, we have $\widehat{G} = \widehat{G}_0$ and $k_\pi = d_\pi$ for all π. As the other extreme, if K is a massive subgroup of G, i.e., if for every such π there is precisely one invariant vector with respect to K, we have $k_\pi = 1$ for all $\pi \in \widehat{G}_0$. This is, for example, the case for the spheres $M = \mathbb{S}^n$. Other examples can be found in Vilenkin [Vil68].

We can now identify functions on $M = G/K$ with functions on G which are constant on left cosets with respect to K. Then, for a function $f \in C^\infty(M)$ we can recover it by the Fourier series of its canonical lifting $\widetilde{f}(g) := f(gK)$ to G, $\widetilde{f} \in C^\infty(G)$, and the Fourier coefficients satisfy $\widehat{\widetilde{f}}(\pi) = 0$ for all representations with $\pi \notin \widehat{G}_0$. Also, for class I representations $\pi \in \widehat{G}_0$ we have $\widehat{\widetilde{f}}(\pi)_{ij} = 0$ for $i > k_\pi$.

With this, we can write the Fourier series of f (or of $\widetilde{f}$, but we identify these) in terms of the spherical functions π_{ij} of the representations $\pi \in \widehat{G}_0$, with respect to the subgroup K. Namely, the Fourier series (15) becomes

$$f(x) = \sum_{\pi \in \widehat{G}_0} d_\pi \sum_{i=1}^{d_\pi} \sum_{j=1}^{k_\pi} \widehat{f}(\pi)_{ji} \pi(x)_{ij} = \sum_{\pi \in \widehat{G}_0} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\pi(x)), \tag{23}$$

where, in order to have the last equality, we adopt the convention of setting $\pi(x)_{ij} := 0$ for all $j > k_\pi$, for all $\pi \in \widehat{G}_0$. With this convention the matrix $\pi(x)\pi(x)^*$ is diagonal with the first k_π diagonal entries equal to one and others equal to zero, so that we have

$$\|\pi(x)\|_{\mathtt{HS}} = \sqrt{k_\pi} \text{ for all } \pi \in \widehat{G}_0, \ x \in G/K. \tag{24}$$

Following [DR14], we will say that *the collection of Fourier coefficients* $\{\widehat{f}(\pi)_{ij} : \pi \in \widehat{G}, 1 \le i,j \le d_\pi\}$ *is of class I with respect to* K if $\widehat{f}(\pi)_{ij} = 0$ whenever $\pi \notin \widehat{G}_0$ or $i > k_\pi$. By the above discussion, if the collection of Fourier coefficients is of class I with respect to K, then the expressions (15) and (23) coincide and yield a function f such that $f(xh) = f(h)$ for all $h \in K$, so that this function becomes a function on the homogeneous space G/K.

For the space of Fourier coefficients of class I we define the analogue of the set Σ in (17) by

$$\Sigma(G/K) := \{\sigma : \pi \mapsto \sigma(\pi) : \ \pi \in \widehat{G}_0, \ \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}, \ \sigma(\pi)_{ij} = 0 \text{ for } i > k_\pi\}. \tag{25}$$

In analogy to (20), we can define the Lebesgue spaces $\ell^p(\widehat{G}_0)$ by the following norms which we will apply to Fourier coefficients $\widehat{f} \in \Sigma(G/K)$ of $f \in \mathcal{D}'(G/K)$. Thus, for $\sigma \in \Sigma(G/K)$ we set

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p} - \frac{1}{2})} \|\sigma(\pi)\|_{\mathtt{HS}}^p \right)^{1/p}, \ 1 \le p < \infty. \tag{26}$$

In the case $K = \{e\}$, so that $G/K = G$, these spaces coincide with those defined by (20) since $k_\pi = d_\pi$ in this case. Again, by the same argument as that in [RT10], these spaces are interpolation spaces and the Hausdorff-Young inequality holds for them. We refer to [NRT14] for some more details on these spaces.

Similarly to (21), the power of k_π in (26) can be naturally interpreted if we rewrite it in the form

$$\|\sigma\|_{\ell^p(\widehat{G}_0)} := \left(\sum_{\pi \in \widehat{G}_0} d_\pi k_\pi \left(\frac{\|\sigma(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}} \right)^p \right)^{1/p}, \ \sigma \in \Sigma(G/K), \ 1 \le p < \infty, \tag{27}$$

and think of $\mu(Q) = \sum_{\pi \in Q} d_\pi k_\pi$ as the Plancherel measure on $\widehat{G}_0$, and of $\sqrt{k_\pi}$ as the normalisation for matrices $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ under the adopted convention on their zeros in (25).

Let $\Delta_{G/K}$ be the differential operator on G/K obtained by the Laplacian Δ_G on G acting on functions that are constant on right cosets of G, i.e., such that $\widetilde{\Delta_{G/K} f} = \Delta_G \widetilde{f}$ for $f \in C^\infty(G/K)$.

Recalling, that the Hardy-Littlewood inequality can be formulated as (13) or (14), we will show that the analogue of (14) on a compact homogeneous manifold G/K becomes

$$\|\widehat{f}\|_{\ell^p(\widehat{G}_0)} \le C_p \|(1 - \Delta_{G/K})^{n(\frac{1}{p} - \frac{1}{2})} f\|_{L^p(G/K)}, \tag{28}$$

where $n = \dim G/K$. This yields sharper results compared to using the Schatten-based space $\ell^p_{sch}(\widehat{G}_0)$ in view of the inequality

$$\|\widehat{f}\|_{\ell^p_{sch}(\widehat{G}_0)} \leq \|\widehat{f}\|_{\ell^p(\widehat{G}_0)}.$$

For more extensive analysis and description of Laplace operators on compact Lie groups and on compact homogeneous manifolds we refer to e.g. [Ste70] and [Pes08], respectively. We note that every representation $\pi(x) = (\pi_{ij}(x))_{i,j=1}^{d_\pi} \in \widehat{G}_0$ is invariant under the right shift by K. Therefore, $\pi(x)_{ij}$ for all $1 \leq i, j \leq d_\pi$ are eigenfunctions of $\Delta_{G/K}$ with the same eigenvalue, and we denote by $\langle\pi\rangle$ the corresponding eigenvalue for the first order pseudo-differential operator $(1 - \Delta_{G/K})^{1/2}$, so that we have

$$(1 - \Delta_{G/K})^{1/2}\pi(x)_{ij} = \langle\pi\rangle\pi(x)_{ij} \ \text{ for all } 1 \leq i, j \leq d_\pi.$$

We now formulate the analogue of the Hardy-Littlewood Theorem 2.1 on a compact homogeneous manifolds G/K as the inequality (28) and its dual:

Theorem 2.2 (Hardy-Littlewood inequalities)**.** *Let G/K be a compact homogeneous manifold of dimension n. Then the following holds.*

1. *Let $1 < p \leq 2$. If $f \in L^p(G/K)$, then $\mathscr{F}_{G/K}\Big((1 - \Delta_{G/K})^{n(\frac{1}{2}-\frac{1}{p})} f\Big) \in \ell^p(\widehat{G}_0)$, and*

$$\|\mathscr{F}_{G/K}\Big((1 - \Delta_{G/K})^{n(\frac{1}{2}-\frac{1}{p})} f\Big)\|_{\ell^p(\widehat{G}_0)} \leq C_p\|f\|_{L^p(G/K)}. \tag{29}$$

   Equivalently, we can rewrite this estimate as

$$\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})}\langle\pi\rangle^{n(p-2)}\|\widehat{f}(\pi)\|^p_{\mathtt{HS}} \leq C_p\|f\|^p_{L^p(G/K)}. \tag{30}$$

2. *Let $2 \leq p < \infty$. If $\{\sigma(\pi)\}_{\pi\in\widehat{G}_0} \in \Sigma(G/K)$ is a sequence of complex matrices such that $\langle\pi\rangle^{n(p-2)}\sigma(\pi)$ is in $\ell^p(\widehat{G}_0)$, then there is a function $f \in L^p(G/K)$ with Fourier coefficients given by $\widehat{f}(\pi) = \sigma(\pi)$, and*

$$\|f\|_{L^p(G/K)} \leq C'_p\|\langle\pi\rangle^{\frac{n(p-2)}{p}}\widehat{f}(\pi)\|_{\ell^p(\widehat{G}_0)}. \tag{31}$$

Using the definition of the norm on the right hand side we can write this as

$$\|f\|^p_{L^p(G/K)} \le C'_p \sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \langle\pi\rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}}. \tag{32}$$

For $p=2$, both of these statements reduce to the Plancherel identity (16).

We note that in view of the inequality (22) the formulations in terms of the space $\ell^p(\widehat{G}_0)$ are sharper than if we used the space $\ell^p_{sch}(\widehat{G}_0)$. Indeed, for example, for $1<p\le 2$, the inequality (22) means that

$$\sum_{\pi\in\widehat{G}_0} d_\pi \langle\pi\rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{S^p} \le \sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \langle\pi\rangle^{n(p-2)} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}}, \tag{33}$$

which in turn implies

$$\begin{aligned} &\|\mathscr{F}_{G/K}\Big((1-\Delta_{G/K})^{n(\frac{1}{2}-\frac{1}{p})} f\Big)\|_{\ell^p_{sch}(\widehat{G}_0)} \\ &\qquad \le \|\mathscr{F}_{G/K}\Big((1-\Delta_{G/K})^{n(\frac{1}{2}-\frac{1}{p})} f\Big)\|_{\ell^p(\widehat{G}_0)} \le C_p \|f\|_{L^p(G/K)}. \end{aligned} \tag{34}$$

2.2. Paley and Hausdorff-Young-Paley inequalities

In [Hör60], Lars Hoӱmander proved a Paley-type inequality for the Fourier transform on $\mathbb{R}^n$, see (9). Here we give an analogue of this inequality on compact homogeneous manifolds.

Theorem 2.3 (Paley-type inequality)**.** *Let G/K be a compact homogeneous manifold. Let $1<p\le 2$. If $\varphi(\pi)$ is a positive sequence over $\widehat{G}_0$ such that*

$$M_\varphi := \sup_{t>0} t \sum_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\ge t}} d_\pi k_\pi < \infty \tag{35}$$

is finite, then we have

$$\left(\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})} \|\widehat{f}(\pi)\|^p_{\mathtt{HS}}\ \varphi(\pi)^{2-p}\right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G/K)}. \tag{36}$$

As usual, the sum over an empty set in (35) is assumed to be zero.

With $\varphi(\pi)=\langle\pi\rangle^{-n}$, where $n=\dim G/K$, using the asymptotic formula (64) for the Weyl eigenvalue counting function, we recover the first part of Theorem

2.2 (see the proof of Theorem 2.2). In this sense, the Paley inequality is an extension of one of the Hardy-Littlewood inequalities.

Now we recall the Hausdorff-Young inequality:

$$\left(\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{p'(\frac{1}{p'}-\frac{1}{2})}\|\widehat{f}(\pi)\|_{\mathtt{HS}}^{p'}\right)^{\frac{1}{p'}} \equiv \|\widehat{f}\|_{\ell^{p'}(\widehat{G}_0)} \lesssim \|f\|_{L^p(G/K)}, \quad 1\leq p\leq 2, \tag{37}$$

where, as usual, $\frac{1}{p}+\frac{1}{p'}=1$. The inequality (37) was argued in [NRT14] in analogy to [RT10, Section 10.3], so we refer there for its justification. Further, we recall a result on the interpolation of weighted spaces from [BL76]:

Theorem 2.4 (Interpolation of weighted spaces)**.** *Let $d\mu_0(x)=\omega_0(x)d\mu(x)$, $d\mu_1(x)=\omega_1(x)d\mu(x)$, and write $L^p(\omega)=L^p(\omega d\mu)$ for the weight ω. Suppose that $0<p_0,p_1<\infty$. Then*

$$(L^{p_0}(\omega_0),L^{p_1}(\omega_1))_{\theta,p}=L^p(\omega),$$

where $0<\theta<1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, and $\omega=\omega_0^{p\frac{1-\theta}{p_0}}\omega_1^{p\frac{\theta}{p_1}}$.

From this, interpolating between the Paley-type inequality (36) in Theorem 2.3 and Hausdorff-Young inequality (37), we obtain:

Theorem 2.5 (Hausdorff-Young-Paley inequality)**.** *Let G/K be a compact homogeneous manifold. Let $1<p\leq b\leq p'<\infty$. If a positive sequence $\varphi(\pi)$, $\pi\in\widehat{G}_0$, satisfies condition*

$$M_\varphi:=\sup_{t>0} t\sum_{\substack{\pi\in\widehat{G}_0\\ \varphi(\pi)\geq t}} d_\pi k_\pi<\infty, \tag{38}$$

then we have

$$\left(\sum_{\pi\in\widehat{G}_0} d_\pi k_\pi^{b(\frac{1}{b}-\frac{1}{2})}\left(\|\widehat{f}(\pi)\|_{\mathtt{HS}}\ \varphi(\pi)^{\frac{1}{b}-\frac{1}{p'}}\right)^b\right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}}\|f\|_{L^p(G/K)}. \tag{39}$$

This reduces to the Hausdorff-Young inequality (37) when $b=p'$ and to the Paley inequality in (36) when $b=p$.

Proof of Theorem 2.5. We consider a sub-linear operator A which takes a function f to its Fourier transform $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ divided by $\sqrt{k_\pi}$, i.e.

$$L^p(G/K) \ni f \mapsto Af = \left\{ \frac{\widehat{f}(\pi)}{\sqrt{k_\pi}} \right\}_{\pi \in \widehat{G}_0} \in \ell^p(\widehat{G}_0, \omega),$$

where the spaces $\ell^p(\widehat{G}_0, \omega)$ is defined by the norm

$$\|\sigma(\pi)\|_{\ell^p(\widehat{G}_0,\omega)} := \left(\sum_{\pi \in \widehat{G}_0} \|\sigma(\pi)\|^p_{\mathtt{HS}}\, \omega(\pi) \right)^{\frac{1}{p}},$$

and $\omega(\pi)$ is a positive scalar sequence over $\widehat{G}_0$ to be determined. Then the statement follows from Theorem 2.4 if we regard the left-hand sides of inequalities (36) and (37) as $\|Af\|_{\ell^p(\widehat{G}_0,\omega)}$-norms in weighted sequence spaces over $\widehat{G}_0$, with the weights given by $\omega_0(\pi) = d_\pi k_\pi \varphi(\pi)^{2-p}$ and $\omega_1(\pi) = d_\pi k_\pi$, $\pi \in \widehat{G}_0$, respectively. □

2.3. Integrability criterion for functions in terms of the matrix Fourier coefficients

In this section we show that the results of Section 2.1 are in general sharp, by looking at the specific example of the group SU(2) in detail.

Imposing more conditions on matrix Fourier coefficients, we make a criterion out of the Hardy-Littlewood inequalities in Theorem 2.2. In fact, here we aim at obtaining a noncommutative version of the following criterion that Hardy and Littlewood proved in [HL27]:

Theorem 2.6. *Let $1 < p < \infty$. Suppose $f \in L^1(\mathbb{T})$, $f \sim \sum \widehat{f}_m e^{2\pi i m x}$, and its Fourier coefficients $\{\widehat{f}_m\}_{m \in \mathbb{Z}}$ are monotone. Then we have*

$$f \in L^p(\mathbb{T}) \tag{40}$$

if and only if

$$\sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}_m|^p < \infty. \tag{41}$$

In this section we extend this result to $G = \mathrm{SU}(2)$ and formulate a necessary and sufficient condition for $f \in L^1(\mathrm{SU}(2))$ to belong to $L^p(\mathrm{SU}(2))$. The criterion is given in terms of the matrix Fourier coefficients. It argues that the powers chosen in the Hardy-Littlewood inequality are in general sharp.

First we propose a notion of general monotonicity for a sequence of matrices extending the usual notion of monotonicity of scalars (of scalar Fourier coefficients).

Definition 2.7. *A sequence of matrices* $\{\sigma(\pi)\}_{\pi\in\widehat{G}_0} \in \Sigma(G/K)$ *will be called almost scalar if the following conditions hold:*

1. *For any* $\pi \in \widehat{G}_0$ *the matrix* $\sigma(\pi)$ *is normal.*
2. *There are constants* $C_1 > 0$ *and* $C_2 > 0$ *such that for any* $\pi \in \widehat{G}_0$ *we have*

$$C_1 \leq \frac{|\lambda_i(\pi)|}{|\lambda_j(\pi)|} \leq C_2,$$

for every $\lambda_i(\pi) \neq 0$ *and* $\lambda_j(\pi) \neq 0$, *where* $\lambda_i(\pi) \in \mathbb{C}$, $i = 1, \ldots, d_\pi$, *denote the eigenvalues of* $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$.

As our main interest in this subsection is the group $\mathrm{SU}(2)$, we specify the following definition to its setting, with the specific notation for $\mathrm{SU}(2)$ explained after the following definition. This specification is done for simplicity; the following notion of monotonicity can be naturally extended to the setting of general compact Lie groups as well.

Definition 2.8. *A sequence of matrices* $\{\sigma(l)\}_{l\in\frac{1}{2}\mathbb{N}_0}$ *is said to be monotone if the following conditions hold:*

1. *The sequence* $\{\sigma(l)\}_{l\in\frac{1}{2}\mathbb{N}_0}$ *is almost scalar and every matrix* $\sigma(l)$ *is non-negative definite.*
2. *Denoting by* σ_l *any non-zero eigenvalue of the almost scalar matrix* $\sigma(l) \in \mathbb{C}^{(2l+1)\times(2l+1)}$, *the sequence* $(2l+1)\sigma_l$ *is decreasing, i.e.*

$$(2l+1)\sigma_l - (2l+2)\sigma_{l+\frac{1}{2}} \geq 0 \tag{42}$$

for all $l \in \frac{1}{2}\mathbb{N}_0$.

In terms of general compact Lie groups, condition (42) means that the sequence $\{d_\pi \sigma_\pi\}_\pi$ is decreasing along some specified ordering on the representation lattice. In the case of the torus we have $d_\pi \equiv 1$, so this corresponds to the usual notion of monotonicity on $\widehat{\mathbb{T}} \cong \mathbb{Z}$.

We now give a criterion for f to be in L^p also for $p < 2$, for central functions on the compact Lie group SU(2). In this case it is common to simplify the notation, since we have the identification of the dual $\widehat{\mathrm{SU}(2)} \cong \frac{1}{2}\mathbb{N}_0$ with non-negative half-integers. Following Vilenkin [Vil68] it is customary to denote the representations by $T^l \in \widehat{\mathrm{SU}(2)}$ for $l \in \frac{1}{2}\mathbb{N}_0$. Then we have $d_l := d_{T^l} = 2l + 1$, and we abbreviate $\widehat{f}(T^l) = \widehat{f}(l)$. The Plancherel identity on SU(2) can then be written as

$$\|f\|^2_{L^2(\mathrm{SU}(2))} = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \|\widehat{f}(l)\|^2_{\mathtt{HS}}. \tag{43}$$

We can refer to [RT13, RT10] for explicit calculations of representations and difference operators on SU(2).

Remark 2.9. *The range $\frac{3}{2} < p \leq 2$ appearing in Theorem 2.10 has a natural interpretation and is related to the convergence properties of the polyhedral Fourier partial sums. It corresponds exactly to the range $1 < p < \infty$ on the circle. We refer to Appendix Appendix A for the detailed explanation of these properties in terms of an auxiliary number s that can be expressed in terms of the root system of the group.*

In the following theorem, we denote by $L^p_*(G)$ space of central functions on G. The restriction on p to satisfy $2 - \frac{1}{s+1} < p < 2 + \frac{1}{s}$ in the setting of compact Lie groups comes from the fact that on compact simply connected semisimple Lie groups, the polyhedral Fourier partial sums of (a central function) f converge to f in L^p if and only if $2 - \frac{1}{s+1} < p < 2 + \frac{1}{s}$, with s defined as in (A.3) *in terms of the root system of G, see Stanton [Sta76], Stanton and Tomas [ST76], and Colzani, Giulini and Travaglini [CGT89] for the only if statement. We recall one of such statements in Theorem Appendix A.1. In the case of* SU(2) *the number s is $s = 1$, so that the range $2 - \frac{1}{1+s} < p \leq 2$ that we are interested in* *becomes $\frac{3}{2} < p \leq 2$ appearing in Theorem 2.10. We note that such restriction of $p > \frac{3}{2}$ already appeared in the literature on* SU(2) *also in other contexts, for example also for questions related to Fourier multipliers (extending the results of Coifman and Weiss [CW71b]), see Clerc [Cle71].*

Theorem 2.10. *Let $\frac{3}{2} < p \leq 2$. Suppose $f \in L^{3/2}_*(\mathrm{SU}(2))$ and the sequence of its Fourier coefficients $\{\widehat{f}(l)\}_{l \in \frac{1}{2}\mathbb{N}_0}$ is monotone. Assume that there is a constant $C > 0$ such that for any $\xi \in \frac{1}{2}\mathbb{N}_0$ the following inequality holds true*

$$\sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ l \geq \xi}} (d_{l+\frac{1}{2}} - d_l) \widehat{f}_l \leq C d_\xi \widehat{f}_\xi, \tag{44}$$

where d_l are the dimensions of the irreducible representations $\{T^l\}_{l \in \frac{1}{2}\mathbb{N}_0}$ of the group SU(2)*, and $\widehat{f}_l$ are obtained from $\widehat{f}(l)$ as in Definition 2.8. Then we have*

$$f \in L^p(\mathrm{SU}(2)) \tag{45}$$

if and only if

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5p}{2}-4} \|\widehat{f}(l)\|^p_{\mathtt{HS}} < \infty. \tag{46}$$

Moreover, in this case we have

$$\|f\|^p_{L^p(\mathrm{SU}(2))} \cong \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5p}{2}-4} \|\widehat{f}(l)\|^p_{\mathtt{HS}}. \tag{47}$$

Remark 2.11. *The non-oscillation type condition* (44) *always holds for compact abelian groups, since in that case all the irreducible representations are* 1-*dimensional, so that the expression on the left hand side of* (44) *would be zero. Here, in the setting of SU(2), since $d_l = 2l+1$,* (44) *boils down to assuming*

$$\sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ l \geq \xi}} \widehat{f}_l \leq C(2\xi + 1)\widehat{f}_\xi.$$

From this point of view this kind of assumption may be viewed as rather natural in some sense because it does measure how fast the sequence of Fourier coefficients decreases compared to the dimensions of representations. We formulate this condition in the form (44) *to emphasise its geometric meaning: it becomes* *clear how it can be extended to more general groups*[2] *and it is very clear that it is trivially satisfied on the torus.*

Therefore, Theorem 2.10 can be regarded as the direct extension of the Hardy-Littlewood criterion in Theorem 2.6 from the circle $\mathbb{T}$ to SU(2). Indeed, the condition that functions on SU(2) are central is rather natural since (all) functions on $\mathbb{T}$ are also central. Moreover, the indices of p correspond to each other as well, both coming from the condition $2 - \frac{1}{1+s} < p \leq 2$, which on $\mathbb{T}$ becomes $1 < p \leq 2$ since $s = 0$, and on SU(2) it is $\frac{3}{2} < p \leq 2$ since $s = 1$. We also note that the assumption for functions to be central functions is rather natural since their behaviour is very different from that of general functions, as we now briefly explain also from a more general perspective.

For example, if G is a compact connected semisimple Lie group and $p \neq 2$, there is a function $f \in L^p(G)$ such that the polyhedral Fourier partial sum of f does not converge to f in L^p, for the dilations of any open convex polyhedron in the Lie algebra of the maximal torus centred at the origin, see Stanton and Tomas [ST76, ST78]. Such negative results are closely related with multiplier problems for the ball and for multiple Fourier series, see Fefferman [Fef71b]. The same negative results hold also for spherical sums, see Fefferman [Fef71a] on the torus, and on more general groups Clerc [Cle72] and [Cle73]. In this paper we are using the polyhedral Fourier sums, in which case positive results become possible if we restrict to considering central functions. Thus, on a compact semisimple Lie group G, for $2 - \frac{1}{s+1} < p < 2 + \frac{1}{s}$, polyhedral Fourier partial sums of a central function f converge to f in $L^p(G)$, see Stanton [Sta76, Theorem 4.1]. If G is a simple simply connected compact Lie group and p falls outside of the above interval, there are central functions in $L^p(G)$ such that their polyhedral Fourier partial sums do not converge to f in the L^p-norm, see Stanton

[2] We conjecture an analogue of Theorem 2.10 to hold for general compact Lie groups, or even for compact homogeneous manifolds. At the moment we can not prove it in full generality since currently we can prove in Proposition 4.2 the estimate for the Dirichlet kernel that is needed for our proof only in the setting of SU(2).

and Tomas [ST76, ST78] and Colzani, Giulini and Travaglini [CGT89]. Such restrictions are not surprising as they also appear naturally in the multiplier problems already on $\mathbb{R}^n$ with $n \geq 2$: while the characteristic function of the ball is not a multiplier on $L^p(\mathbb{R}^n)$ for any $p \neq 2$ [Fef71a], it does become an L^p-multiplier on radial functions if and only if $2 - \frac{2}{n+1} < p < 2 + \frac{2}{n-1}$, see Herz [Her54]. We refer to Appendix Appendix A for further precise statements.

3. L^p-L^q boundedness of operators

In this section we use the Hausdorff-Young-Paley inequality in Theorem 2.5 to give a sufficient condition for the L^p-L^q boundedness of Fourier multipliers on compact homogeneous spaces. It extends the condition that was obtained by a different method in [NT00] on the circle $\mathbb{T}$. In the case of compact Lie groups, we extend the criterion for Fourier multipliers in a rather standard way, to derive a condition for the L^p-L^q boundednes of general operators, all for the range of indices $1 < p \leq 2 \leq q < \infty$.

In the case of a compact Lie group G, the Fourier multipliers correspond to left-invariant operators, and these can be characterised by the condition that their symbols do not depend on the space variable. Thus, we can write such operators A in the form

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \tag{48}$$

with the symbol $\sigma_A(\pi)$ depending only on $\pi \in \widehat{G}$. The Hörmander-Mihlin type multiplier theorem for such operators to be bounded on $L^p(G)$ for $1 < p < \infty$ was obtained in [RW15].

Now, in the context of compact homogeneous spaces G/K we still want to keep the formula (48) as the definition of Fourier multipliers, now for all $\pi \in \widehat{G}_0$. Indeed, due to properties of zeros of the Fourier coefficients, we have that both sides of (48) are zero for $\pi \notin \widehat{G}_0$. Also, for $\pi \in \widehat{G}_0$, we have $\widehat{f}(\pi) \in \Sigma(G/K)$ with the set $\Sigma(G/K)$ defined in (25), which means that

$$\widehat{f}(\pi)_{ij} = \widehat{Af}(\pi)_{ij} = 0 \ \text{ for } \ i > k_\pi.$$

Therefore, we can assume that the symbol σ_A of a Fourier multiplier A on G/K satisfies

$$\sigma_A(\pi) = 0 \text{ for } \pi \notin \widehat{G}_0; \text{ and } \sigma_A(\pi)_{ij} = 0 \text{ for } \pi \in \widehat{G}_0, \text{ if } i > k_\pi \text{ or } j > k_\pi. \quad (49)$$

Therefore, only the upper-left block in $\sigma_A(\pi)$ of the size $k_\pi \times k_\pi$ may be non-zero. Thus, we will say that A is a Fourier multiplier on G/K if conditions (48) and (49) are satisfied.

Theorem 3.1. *Let $1 < p \leq 2 \leq q < \infty$ and suppose that A is a Fourier multiplier on the compact homogeneous space G/K. Then we have*

$$\|A\|_{L^p(G/K)\to L^q(G/K)} \lesssim \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma_A(\pi)\|_{\mathsf{op}}>s}} d_\pi k_\pi \right)^{\frac{1}{p}-\frac{1}{q}} . \quad (50)$$

We note that if $\mu(Q) = \sum_{\pi\in Q} d_\pi k_\pi$ denotes the Plancherel measure on $\widehat{G}_0$, then (50) can be rewritten as

$$\|A\|_{L^p(G/K)\to L^q(G/K)} \lesssim \sup_{s>0} \left\{ s\mu(\pi \in \widehat{G}_0 : \|\sigma_A(\pi)\|_{\mathsf{op}} > s)^{\frac{1}{p}-\frac{1}{q}} \right\}.$$

Remark 3.2. *Inequality* (50) *is sharp for $p = q = 2$.*

Proof. First, we have the estimate

$$\|A\|_{L^2(G/K)\to L^2(G/K)} \leq \sup_{\pi\in\widehat{G}_0} \|\sigma_A(\pi)\|_{\mathsf{op}}.$$

Since the set

$$\{\pi \in \widehat{G}_0 \colon \|\sigma_A(\pi)\|_{\mathsf{op}} \geq s\}$$

is empty for $s > \|A\|_{L^2(G/K)\to L^2(G/K)}$ and a sum over the empty set is set to be zero, we have by (50)

$$\begin{aligned}\|A\|_{L^2(G/K)\to L^2(G/K)} &\leq \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma_A(\pi)\|_{\mathsf{op}}\geq s}} d_\pi k_\pi \right)^0 \\ &= \sup_{0<s\leq\|A\|_{L^2(G/K)\to L^2(G/K)}} s \cdot 1 = \|A\|_{L^2(G/K)\to L^2(G/K)}.\end{aligned}$$

Thus, for $p = q = 2$ we attain equality in (50). $\square$

Proof of Theorem 3.1. Recall that A is a Fourier multiplier on G/K, i.e.

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi),$$

with σ_A satisfying (49). Since the application of [ANR16, p. 14, Theorem 4.2] with $X = G/K$ and $\mu = \{\text{Haar measure on } G\}$ yields

$$\|A\|_{L^p(G/K)\to L^q(G/K)} = \|A^*\|_{L^{q'}(G/K)\to L^{p'}(G/K)}, \tag{51}$$

we may assume that $p \leq q'$, for otherwise we have $q' \leq (p')' = p$ and $\|\sigma_{A^*}(\pi)\|_{\mathsf{op}} = \|\sigma_A(\pi)\|_{\mathsf{op}}$. When $f \in C^\infty(G/K)$ the Hausdorff-Young inequality gives, since $q' \leq 2$,

$$\|Af\|_{L^q(G/K)} \lesssim \|\widehat{Af}\|_{\ell^{q'}(\widehat{G}_0)} = \|\sigma_A\widehat{f}\|_{\ell^{q'}(\widehat{G}_0)}.$$

We set $\sigma(\pi) := \|\sigma_A(\pi)\|_{\mathsf{op}}^r I_{d_\pi}$. It is obvious that

$$\|\sigma(\pi)\|_{\mathsf{op}} = \|\sigma_A(\pi)\|_{\mathsf{op}}^r. \tag{52}$$

Now, we are in a position to apply the Hausdorff-Young-Paley inequality in Theorem 2.5. With $\sigma(\pi) = \|\sigma_A\|^r I_{d_\pi}$ and $b = q'$, the assumption of Theorem 2.5 are then satisfied and since $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, we obtain

$$\|\sigma_A\widehat{f}\|_{\ell^{q'}(\widehat{G}_0)} \lesssim \left(\sup_{s>0} s \sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma(\pi)\|_{\mathsf{op}}\geq s}} d_\pi k_\pi \right)^{\frac{1}{r}} \|f\|_{L^p(G/K)}, \quad f \in L^p(G/K).$$

Further, it can be easily checked that

$$\left(\sup_{s>0} s \sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma(\pi)\|_{\mathsf{op}}>s}} d_\pi k_\pi \right)^{\frac{1}{r}} = \left(\sup_{s>0} s \sum_{\substack{\pi\in\widehat{G}_0 \\ \|\sigma_A(\pi)\|_{\mathsf{op}}^r>s}} d_\pi k_\pi \right)^{\frac{1}{r}} = \left(\sup_{s>0} s^r \sum_{\substack{\pi\in\widehat{G} \\ \|\sigma_A(\pi)\|_{\mathsf{op}}>s}} d_\pi k_\pi \right)^{\frac{1}{r}}$$
$$= \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\sigma_A(\pi)\|_{\mathsf{op}}>s}} d_\pi k_\pi \right)^{\frac{1}{r}}.$$

This completes the proof. $\square$

A standard addition to the proof of the preceding theorem extends Theorem 3.1 to the non-invariant case. For the simplicity in the formulation and in the understanding a variant of (48) in the non-invariant case, the following result is given in the context of general compact Lie groups. To fix the notation, we note that according to [RT10, Theorem 10.4.4] any linear continuous operator A on $C^\infty(G)$ can be written in the form

$$Af(g) = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\pi(g)\sigma_A(g,\pi)\widehat{f}(\pi)\right)$$

for a symbol σ_A that is well-defined on $G\times\widehat{G}$ with values $\sigma_A(g,\pi)\in\mathbb{C}^{d_\pi\times d_\pi}$.

Theorem 3.3. *Let $1<p\leq 2\leq q<\infty$. Suppose that $l>\frac{p}{\dim(G)}$ is an integer. Let A be a linear continuous operator on $C^\infty(G)$. Then we have*

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sum_{|\alpha|\leq l} \sup_{u\in G}\sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G}\\ \|\partial_u^\alpha \sigma_A(u,\pi)\|_{\mathrm{op}}\geq s}} d_\pi k_\pi \right)^{\frac{1}{p}-\frac{1}{q}}. \qquad (53)$$

In other words, if the expression on the right hand side of (53) is finite, the operator A extends to a bounded operator from $L^p(G)$ to $L^q(G)$. The derivatives ∂_u^α are derivatives with respect to a basis of left-invariant vector fields on the Lie algebra $\mathfrak{g}$ of G.

Proof. Let us define

$$A_u f(g) := \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\pi(g)\sigma_A(u,\pi)\widehat{f}(\pi)\right)$$

so that $A_g f(g) = Af$. Then

$$\|Af\|_{L^q(G)} = \left(\int_G |Af(g)|^q\,dg\right)^{\frac{1}{q}} \leq \left(\int_G \sup_{u\in G}|A_u f(g)|^q\,dg\right)^{\frac{1}{q}}. \qquad (54)$$

By an application of the Sobolev embedding theorem we get

$$\sup_{u\in G}|A_u f(g)|^q \leq C\sum_{|\alpha|\leq l}\int_G |\partial_u^\alpha A_u f(g)|^q\ dy.$$

Therefore, using the Fubini theorem to change the order of integration, we obtain

$$\begin{aligned}
\|Af\|^q_{L^q(G)} &\leq C \sum_{|\alpha|\leq l} \int_G \int_G |\partial_u^\alpha A_u f(g)|^q \, dg \, du \\
&\leq C \sum_{|\alpha|\leq l} \sup_{u\in G} \int_G |\partial_u^\alpha A_y f(g)|^p \, dg \\
&= C \sum_{|\alpha|\leq l} \sup_{u\in G} \|\partial_u^\alpha A_u f\|^q_{L^q(G)} \\
&\leq C \sum_{|\alpha|\leq l} \sup_{u\in G} \|f \mapsto \mathrm{Op}(\partial_u^\alpha \sigma_A) f\|^q_{\mathcal{L}(L^p(G)\to L^q(G))} \|f\|^q_{L^p(G)} \\
&\lesssim \left[\sum_{|\alpha|\leq l} \sup_{u\in G} \sup_{s>0} s \left(\sum_{\substack{\pi\in\widehat{G} \\ \|\partial_u^\alpha \sigma_A(u,\pi)\|_{\mathrm{op}}\geq s}} d_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}} \right]^q \|f\|^q_{L^p(G)},
\end{aligned}$$

where the last inequality holds due to Theorem 3.1. This completes the proof.

□

4. Proofs

In this section we prove results stated in the previous section. We start by proving the Paley inequality in Theorem 2.3 and then use it to deduce the Hardy-Littlewood Theorem 2.2.

4.1. Proof of Theorem 2.3

Proof of Theorem 2.3. Let ν give measure $\varphi^2(\pi) d_\pi k_\pi$ to the set consisting of the single point $\{\pi\}, \pi \in \widehat{G}_0$, i.e.

$$\nu(\{\pi\}) := \varphi^2(\pi) d_\pi k_\pi.$$

We define the corresponding space $L^p(\widehat{G}_0, \nu)$, $1 \leq p < \infty$, as the space of complex (or real) sequences $a = \{a_\pi\}_{\pi\in\widehat{G}_0}$ such that

$$\|a\|_{L^p(\widehat{G}_0,\nu)} := \left(\sum_{\pi\in\widehat{G}_0} |a_\pi|^p \varphi^2(\pi) d_\pi k_\pi \right)^{\frac{1}{p}} < \infty. \tag{55}$$

We will show that the sub-linear operator

$$A\colon L^p(G/K) \ni f \mapsto Af = \left\{ \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} \right\}_{\pi\in\widehat{G}_0} \in L^p(\widehat{G}_0, \nu)$$

is well-defined and bounded from $L^p(G/K)$ to $L^p(\widehat{G}_0,\nu)$ for $1 < p \leq 2$. In other words, we claim that we have the estimate

$$\|Af\|_{L^p(\widehat{G}_0,\nu)} = \left(\sum_{\pi\in\widehat{G}_0} \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} \right)^p \varphi^2(\pi) d_\pi k_\pi \right)^{\frac{1}{p}} \lesssim N_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G/K)}, \quad (56)$$

which would give (36) and where we set $N_\varphi := \sup_{t>0} t \sum\limits_{\substack{\pi\in\widehat{G}_0 \\ \varphi(\pi)\geq t}} d_\pi k_\pi$. We will show that A is of weak type (2,2) and of weak-type (1,1). For definition and discussions we refer to Section Appendix B where we give definitions of weak-type, formulate and prove Marcinkiewicz-type interpolation Theorem Appendix B.2 to be used in the present setting. More precisely, with the distribution function ν as in Theorem Appendix B.2, we show that

$$\nu_{\widehat{G}_0}(y; Af) \leq \left(\frac{M_2 \|f\|_{L^2(G/K)}}{y} \right)^2 \quad \text{with norm } M_2 = 1, \quad (57)$$

$$\nu_{\widehat{G}_0}(y; Af) \leq \frac{M_1 \|f\|_{L^1(G/K)}}{y} \quad \text{with norm } M_1 = M_\varphi, \quad (58)$$

where $\nu_{\widehat{G}_0}$ is defined in the Appendix in (B.2). Then (56) would follow by Marcinkiewicz interpolation theorem (Theorem Appendix B.2 from Section Appendix B) with $\Gamma = \widehat{G}_0$ and $\delta_\pi = d_\pi, \kappa_\pi = k_\pi$.

Now, to show (57), using Plancherel's identity (16), we get

$$\begin{aligned} y^2 \nu_{\widehat{G}_0}(y; Af) \leq \|Af\|^2_{L^p(\widehat{G}_0,\nu)} &= \sum_{\pi\in\widehat{G}_0} d_\pi k_\pi \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} \right)^2 \varphi^2(\pi) \\ &= \sum_{\pi\in\widehat{G}_0} d_\pi \|\widehat{f}(\pi)\|^2_{\mathtt{HS}} = \|\widehat{f}\|^2_{\ell^2(\widehat{G}_0)} = \|f\|^2_{L^2(G/K)}. \end{aligned}$$

Thus, A is of type (2,2) with norm $M_2 \leq 1$. Further, we show that A is of weak-type (1,1) with norm $M_1 = M_\varphi$; more precisely, we show that

$$\nu_{\widehat{G}_0}\{\pi \in \widehat{G}_0\colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\} \lesssim M_\varphi \frac{\|f\|_{L^1(G/K)}}{y}. \quad (59)$$

The left-hand side here is the weighted sum $\sum \varphi^2(\pi) d_\pi k_\pi$ taken over those $\pi \in \widehat{G}_0$ for which $\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y$. From the definition of the Fourier transform it follows that

$$\|\widehat{f}(\pi)\|_{\mathtt{HS}} \leq \sqrt{k_\pi}\|f\|_{L^1(G/K)}.$$

Therefore, we have

$$y < \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} \leq \frac{\|f\|_{L^1(G/K)}}{\varphi(\pi)}.$$

Using this, we get

$$\left\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \subset \left\{\pi \in \widehat{G}_0 \colon \frac{\|f\|_{L^1(G/K)}}{\varphi(\pi)} > y\right\}$$

for any $y > 0$. Consequently,

$$\nu\left\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \leq \nu\left\{\pi \in \widehat{G}_0 \colon \frac{\|f\|_{L^1(G/K)}}{\varphi(\pi)} > y\right\}.$$

Setting $v := \frac{\|f\|_{L^1(G/K)}}{y}$, we get

$$\nu\left\{\pi \in \widehat{G}_0 \colon \frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \leq \sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \leq v}} \varphi^2(\pi) d_\pi k_\pi. \tag{60}$$

We claim that

$$\sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \leq v}} \varphi^2(\pi) d_\pi k_\pi \lesssim M_\varphi v. \tag{61}$$

In fact, we have

$$\sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \leq v}} \varphi^2(\pi) d_\pi k_\pi = \sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \leq v}} d_\pi k_\pi \int_0^{\varphi^2(\pi)} d\tau.$$

We can interchange sum and integration to get

$$\sum_{\substack{\pi \in \widehat{G}_0 \\ \varphi(\pi) \leq v}} d_\pi k_\pi \int_0^{\varphi^2(\pi)} d\tau = \int_0^{v^2} d\tau \sum_{\substack{\pi \in \widehat{G}_0 \\ \tau^{\frac{1}{2}} \leq \varphi(\pi) \leq v}} d_\pi k_\pi.$$

Further, we make a substitution $\tau = t^2$, yielding

$$\int_0^{v^2} d\tau \sum_{\substack{\pi \in \widehat{G}_0 \\ \tau^{\frac{1}{2}} \leq \varphi(\pi) \leq v}} d_\pi k_\pi = 2 \int_0^{v} t\, dt \sum_{\substack{\pi \in \widehat{G}_0 \\ t \leq \varphi(\pi) \leq v}} d_\pi k_\pi \leq 2 \int_0^{v} t\, dt \sum_{\substack{\pi \in \widehat{G}_0 \\ t \leq \varphi(\pi)}} d_\pi k_\pi .$$

Since

$$t \sum_{\substack{\pi \in \widehat{G}_0 \\ t \leq \varphi(\pi)}} d_\pi k_\pi \leq \sup_{t>0} t \sum_{\substack{\pi \in \widehat{G}_0 \\ t \leq \varphi(\pi)}} d_\pi k_\pi = M_\varphi$$

is finite by the assumption that $M_\varphi < \infty$, we have

$$2 \int_0^{v} t\, dt \sum_{\substack{\pi \in \widehat{G}_0 \\ t \leq \varphi(\pi)}} d_\pi k_\pi \lesssim M_\varphi v.$$

This proves (70). Thus, we have proved inequalities (57), (58). Then by using the Marcinkiewicz interpolation theorem (Theorem Appendix B.2 from Section Appendix B) with $p_1 = 1, p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\left(\sum_{\pi \in \widehat{G}_0} \left(\frac{\|\widehat{f}(\pi)\|_{\mathtt{HS}}}{\varphi(\pi)} \right)^p \varphi^2(\pi) d_\pi k_\pi \right)^{\frac{1}{p}} = \|Af\|_{L^p(\widehat{G}_0, \mu)} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(G/K)}.$$

This completes the proof. □

We now prove the Hardy-Littlewood Theorem 2.2.

4.2. Proof of Theorem 2.2

Proof of Theorem 2.2. The second part of Theorem 2.2 follows from the first by duality, so we will concentrate on proving the first part.

Denote by $N(L)$ the eigenvalue counting function of eigenvalues (counted with multiplicities) of the first order elliptic pseudo-differential operator $(I - \Delta_{G/K})^{\frac{1}{2}}$ on the compact manifold G/K, i.e.

$$N(L) := \sum_{\substack{\pi \in \widehat{G}_0 \\ \langle \pi \rangle \leq L}} d_\pi k_\pi . \tag{62}$$

Using the eigenvalue counting function $N(L)$, we can reformulate condition (35) for $\varphi(\pi) = \langle\pi\rangle^{-n}$ in the following form

$$\sup_{0<u<+\infty} uN(u^{-\frac{1}{n}}) < \infty. \tag{63}$$

Since $N(L)$ is a right-continuous monotone function, the set of discontinuity points on $(0, +\infty)$ is at most countable. Therefore, without loss of generality, we can assume that $\psi(u) = uN\left(\left(\frac{1}{u}\right)^{\frac{1}{n}}\right)$ is a continuous function on $(0, +\infty)$. It is clear that $\lim_{u\to+\infty} \psi(u) = 0$. Further, we use the asymptotic of the Weyl eigenvalue counting function $N(L)$ for the first order elliptic pseudo-differential operator $(1-\Delta_{G/K})^{1/2}$ on the compact manifold G/K, to get that the eigenvalue counting function $N(L)$ (see e.g. Shubin [Shu87]) satisfies

$$N(L) = \sum_{\substack{\pi\in\widehat{G}_0 \\ \langle\pi\rangle\le L}} d_\pi k_\pi \cong L^n \quad \text{for large } L. \tag{64}$$

With $L = \left(\frac{1}{u}\right)^{\frac{1}{n}}$ and $n = \dim G/K$, this implies

$$\lim_{u\to 0} \psi(u) = \lim uN\left(\left(\frac{1}{u}\right)^{\frac{1}{n}}\right) = \lim_{u\to 0} u\left(\frac{1}{u^{\frac{1}{n}}}\right)^n = \lim_{u\to 0} 1 = 1.$$

Thus, we showed that $\psi(u)$ is a bounded function on $(0, +\infty)$, or equivalently, we established (63). Then, it is clear that $\varphi(\pi) = \langle\pi\rangle^{-n}$ satisfies condition (35). The application of the Paley inequality from Theorem 2.3 yields the Hardy-Littlewood inequality. This completes the proof. $\square$

4.3. Proof of Theorem 2.10

Proof of Theorem 2.10. In view of Theorem 2.2, it is sufficient to prove the converse inequality, i.e.

$$\|f\|^p_{L^p_*(\mathrm{SU}(2))} \lesssim \sum_{l\in\frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5p}{2}-4} \|\widehat{f}(l)\|^p_{\mathtt{HS}}. \tag{65}$$

We will first prove that there is $C > 0$ such that for any $\xi \in \frac{1}{2}\mathbb{N}_0$ we have

$$|f(u)| \le C\frac{1}{(2\xi+1)^2}\frac{1}{(\sin\pi\frac{t}{2})^2}\left|\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le\xi}} (2l+1)\,\mathrm{Tr}\,\widehat{f}(l)\right|, \tag{66}$$

where

$$u(t,\theta,\psi) = \begin{pmatrix} \cos(\frac{\theta}{2})e^{i(2\pi t+\psi)/2} & i\sin(\frac{\theta}{2})e^{i(2\pi t-\psi)/2} \\ i\sin(\frac{\theta}{2})e^{-i(2\pi t-\psi)/2} & \cos(\frac{\theta}{2})e^{-i(2\pi t+\psi)/2} \end{pmatrix} \tag{67}$$

is a parameterisation of SU(2), and the coordinates (t,θ,ψ) vary in the parameter ranges

$$0 \le t < 1, \quad 0 \le \theta \le \pi, \quad -2\pi \le \psi \le 2\pi. \tag{68}$$

We refer to [RT13] or [RT10] for the general discussion of the Euler angles in this setting. We also note that due to the assumption that the Fourier coefficients are monotone, they are nonnegative and decreasing, so the modulus on the right hand side of (66) can be actually dropped.

We fix an arbitrary half-integer $\xi \in \frac{1}{2}\mathbb{N}_0$ and let k be any half-integer greater than ξ, i.e. $k \ge \xi$, $k \in \frac{1}{2}\mathbb{N}_0$. Then we have

$$\left| \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le k}} (2l+1)\operatorname{Tr}[\widehat{f}(l)T^l(u)] \right| \le \left| \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le \xi}} (2l+1)\operatorname{Tr}[\widehat{f}(l)T^l(u)] \right| + \left| \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ \xi< l\le k}} (2l+1)\operatorname{Tr}[\widehat{f}(l)T^l(u)] \right|. \tag{69}$$

Since $\widehat{f}(k)$ is an almost scalar sequence of the Fourier coefficients, we have

$$\operatorname{Tr}[\widehat{f}(k)T^k(u)] \cong \widehat{f}_k \operatorname{Tr} T^k(u).$$

Thus

$$\left| \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le \xi}} (2l+1)\operatorname{Tr}[\widehat{f}(l)T^l(u)] \right| \le \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le \xi}} (2l+1)|\widehat{f}_l||\operatorname{Tr} T^l(u)|.$$

Since matrices $T^l(u)$ are unitary of size $(2l+1)\times(2l+1)$, we have

$$\left|\operatorname{Tr} T^l(u)\right| \le (2l+1).$$

Therefore

$$\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le \xi}} (2l+1)|\widehat{f}_l||\operatorname{Tr} T^l(u)| \le \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0 \\ l\le \xi}} (2l+1)^2|\widehat{f}_l|.$$

Applying the Abel transform to $\widehat{f}_l$ and $(2l+1)\,\mathrm{Tr}[T^l(u)]$ in the second term in the sum in (69), we get

$$\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k}} (2l+1)\widehat{f}_l\,\mathrm{Tr}[T^l(u)] = \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} (\widehat{f}_l - \widehat{f}_{l+\frac{1}{2}})D_l(t) + \widehat{f}_k D_k(t) - \widehat{f}_\xi D_{\xi-\frac{1}{2}}(t),$$

where $D_k(t) = \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le k}} (2l+1)\,\mathrm{Tr}\,T^l(u)$. We will now use the estimate (74) for the Dirichlet kernel from Proposition 4.2 that we postpone to be proved later. Thus, we first estimate

$$\left|\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k}} (2l+1)\,\mathrm{Tr}[\widehat{f}(l)T^l(u)]\right| \le \left|\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} (\widehat{f}_l - \widehat{f}_{l+\frac{1}{2}})D_l(t)\right| + \left|\widehat{f}_k D_k(t)\right|$$
$$+ \left|\widehat{f}_\xi D_{\xi-\frac{1}{2}}(t)\right| \le \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} \left|\widehat{f}_l - \widehat{f}_{l+\frac{1}{2}}\right| |D_l(t)| + \left|\widehat{f}_k\right| |D_k(t)| + \left|\widehat{f}_\xi\right| \left|D_{\xi-\frac{1}{2}}(t)\right|$$

Using estimate (74) for the Dirichlet kernel and monotonicity of $(2k+1)\widehat{f}_k$ we can estimate this as

$$\lesssim \frac{1}{t^2}\left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} [(2l+2)\widehat{f}_l - (2l+2)\widehat{f}_{l+\frac{1}{2}}] + (2k+1)\widehat{f}_k + 2\xi\widehat{f}_\xi\right)$$
$$= \frac{1}{t^2}\left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} [(2l+1)\widehat{f}_l - (2l+2)\widehat{f}_{l+\frac{1}{2}}] + \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} \widehat{f}_l + (2k+1)\widehat{f}_k + 2\xi\widehat{f}_\xi\right)$$
$$\lesssim \frac{1}{t^2}\left((2\xi+1)\widehat{f}_\xi - (2k+1)\widehat{f}_k + \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} \widehat{f}_l + (2k+1)\widehat{f}_k + 2\xi\widehat{f}_\xi\right)$$
$$\lesssim \frac{1}{t^2}(2\xi+1)\widehat{f}_\xi,$$

where the sum in the last line is finite even as $k\to\infty$ in view of the nonoscillating assumption (44), namely, since

$$\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ \xi\le l\le k-\frac{1}{2}}} \widehat{f}_l \le \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\ge\xi}} (d_l - d_{l+1})\widehat{f}_l < (2\xi+1)\widehat{f}_\xi.$$

Collecting these estimates, we get

$$\begin{aligned}\left|\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le k}} (2l+1)\,\mathrm{Tr}[\widehat{f}(l)T^l(u)]\right| &\le \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l + \frac{(2\xi+1)\widehat{f}_\xi}{t^2}\\ &= \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l + (2\xi+1)^3\widehat{f}_\xi\frac{(2\xi+1)}{(2\xi+1)^3}\frac{1}{t^2}.\end{aligned}$$

By Theorem Appendix A.2 the partial sums $\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le k}} (2l+1)\,\mathrm{Tr}[\widehat{f}(l)T^l(u)]$ converge to $f(x)$ for almost all $x\in G$. Then taking the limit as $k\to\infty$, we get

$$|f(u)| \lesssim \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l + (2\xi+1)^3\widehat{f}_\xi\frac{(2\xi+1)}{(2\xi+1)^3}\frac{1}{t^2}.$$

We assumed that $(2l+1)\widehat{f}_l$ is a monotone sequence. Then $\widehat{f}_k$ is also a monotone decreasing sequence. Therefore, we get

$$(2\xi+1)^3\widehat{f}_\xi \le \sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l.$$

Thus

$$\begin{aligned}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l+(2\xi+1)^3\widehat{f}_\xi\frac{2\xi+1}{(2\xi+1)^3}\frac{1}{t^2} &\le \left(1+\frac{1}{(2\xi+1)^2}\frac{1}{t^2}\right)\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l\\ &\lesssim \frac{1}{(2\xi+1)^2}\frac{1}{t^2}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)^2\widehat{f}_l.\end{aligned}$$

Since $\widehat{f}_l$ is *almost scalar*, by Definition 2.7, the last sum equals to

$$\frac{1}{(2\xi+1)^2}\frac{1}{t^2}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}} (2l+1)\,\mathrm{Tr}\,\widehat{f}(l).$$

Finally, we obtain

$$|f(u)| \lesssim \frac{1}{(2\xi+1)^2}\frac{1}{t^2}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l). \tag{70}$$

This proves (66). Using this inequality and applying Weyl's integral formula for class functions (cf. e.g. Hall [Hal03]), we immediately get

$$\begin{aligned}\|f\|^p_{L^p(\mathrm{SU}(2))} &= \int\limits_{[0,1]} |f(u)|^p \sin^2\frac{\pi t}{2}\,dt\\ &\lesssim \int\limits_{[0,1]}\left(\frac{1}{(2\xi+1)^2}\frac{1}{t^2}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p\sin^2\frac{\pi t}{2}\,dt.\end{aligned}$$

Here ξ is an arbitrary fixed half-integer. We split the interval $[0,1]$ as the union $[0,1]=\bigsqcup\limits_{\xi\in\frac{1}{2}\mathbb{N}_0}[(2\xi+1+1)^{-1},(2\xi+1)^{-1}]$. Using the estimate with the corresponding ξ in each interval of this decomposition, the last integral becomes

$$\begin{aligned}&\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}\int\limits_{\frac{1}{(2\xi+1+1)}}^{\frac{1}{(2\xi+1)}}\left(\frac{1}{(2\xi+1)^2}\frac{1}{t^2}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p\sin^2\frac{\pi t}{2}\,dt\\ &\qquad\cong\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}\int\limits_{\frac{1}{(2\xi+1+1)}}^{\frac{1}{(2\xi+1)}}\left(\frac{1}{(2\xi+1)^2}\frac{1}{t^2}\right)^p\left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p t^2\,dt.\end{aligned}$$

Now, we notice that the inner sum $\sum\limits_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ 2l+1\le2\xi+1}}(2l+1)\operatorname{Tr}\widehat{f}(l)$ does not depend on t. Therefore, we can interchange summation and integration to get

$$\begin{aligned}&\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}\int\limits_{\frac{1}{(2\xi+1+1)}}^{\frac{1}{(2\xi+1)}}\left(\frac{1}{(2\xi+1)^2}\frac{1}{t^2}\right)^p\left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p t^2\,dt\\ &\qquad=\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}\left(\frac{1}{(2\xi+1)^2}\right)^p\left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\le\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p\int\limits_{\frac{1}{(2\xi+1+1)}}^{\frac{1}{(2\xi+1)}}t^{2-2p}\,dt.\end{aligned}$$

The key observation now is the fact that

$$\left(\frac{1}{(2\xi+1)^2}\right)^p \int\limits_{\frac{1}{(2\xi+1+1)}}^{\frac{1}{(2\xi+1)}} t^{2-2p}\, dt \cong (2\xi+1)^2(2\xi+1)^{3(p-2)}\frac{1}{(2\xi+1)^{3p}}.$$

Thus, the last sum, up to constant, equals to

$$\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}(2\xi+1)^{-4}\left(\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ l\leq\xi}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p.$$

Thus, the last sum, up to constant, equals to

$$\sum_{\xi\in\frac{1}{2}\mathbb{N}_0}(2\xi+1)^2(2\xi+1)^{3(p-2)}\left(\frac{1}{(2\xi+1)^3}\sum_{\substack{l\in\frac{1}{2}\mathbb{N}_0\\ 2l+1\leq 2\xi+1}}(2l+1)\operatorname{Tr}\widehat{f}(l)\right)^p.$$

Now, we formulate and apply the following theorem proved by the authors in [ANR17]. Let G be a compact Lie group and $\widehat{G}$ its unitary dual. Let us denote by $\mathcal{M}_1$ the collection of all finite subsets $Q\subset\widehat{G}$ of $\widehat{G}$. Denote $\mu(Q)=\sum\limits_{\pi\in Q}d_\pi^2$ for $Q\in\mathcal{M}_1$.

Theorem 4.1 ([ANR17])**.** *Let $1<p\leq 2$. Then we have*

$$\sum_{\pi\in\widehat{G}}d_\pi^2\langle\pi\rangle^{n(p-2)}\left(\sup_{\substack{Q\in\mathcal{M}_1\\ \mu(Q)\geq\langle\pi\rangle^n}}\frac{1}{\mu(Q)}\left|\sum_{\xi\in Q}d_\xi\operatorname{Tr}\widehat{f}(\xi)\right|\right)^p =: \|\widehat{f}\|_{N_{p',p}(\widehat{G},\mathcal{M}_1)}\lesssim\|f\|_{L^p(G)}. \tag{71}$$

Here $N_{p',p}(\widehat{G},\mathcal{M}_1)$ is the net space on the lattice $\widehat{G}$ which has been discussed in [ANR17]. For an arbitrary collection of finite subsets M, in view of the embedding (cf. [ANR17])

$$N_{p',p}(\widehat{G},\mathcal{M})\hookrightarrow N_{p',p}(\widehat{G},\mathcal{M}_1) \tag{72}$$

and inequality (71), we get

$$\sum_{\pi\in\widehat{G}}d_\pi^2\langle\pi\rangle^{n(p-2)}\left(\sup_{\substack{Q\in\mathcal{M}\\ \mu(Q)\geq\langle\pi\rangle^n}}\frac{1}{\mu(Q)}\left|\sum_{\xi\in Q}d_\xi\operatorname{Tr}\widehat{f}(\xi)\right|\right)^p\leq\|f\|_{L^p(G)}. \tag{73}$$

In particular, for $G = \mathrm{SU}(2)$ and $\mathcal{M} = \{\{\xi \in \widehat{G} \colon \langle \xi \rangle \leq \langle \pi \rangle\} \colon \pi \in \widehat{G}\}$, we thus obtain from (73) that

$$\sum_{\xi \in \frac{1}{2}\mathbb{N}_0} (2\xi+1)^2 (2\xi+1)^{3(p-2)} \left(\frac{1}{(2\xi+1)^3} \sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ (2l+1)^3 \leq 2\xi+1}} (2l+1) \operatorname{Tr} \widehat{f}(l) \right)^p$$

$$\leq \sum_{\xi \in \frac{1}{2}\mathbb{N}_0} (2\xi{+}1)^2 (2\xi{+}1)^{3(p-2)} \left(\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ (2k+1)^3 \geq (2\xi+1)^3}} \frac{1}{(2k+1)^3} \sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ (2l+1)^3 \leq (2k+1)^3}} (2l+1) \operatorname{Tr} \widehat{f}(l) \right)^p$$

$$\leq \|f\|^p_{L^p(\mathrm{SU}(2))}.$$

This completes the proof. □

4.4. Dirichlet kernel on SU(2)

In the proof of Theorem 2.10 we made use of an estimate for the Dirichlet kernel on SU(2) which we now prove. We continue with the SU(2)-notation introduced in (67)–(68).

Proposition 4.2. *On* SU(2)*, the Dirichlet kernel*

$$D_l(t) := \sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \leq l}} (2k+1)\chi_k(t) = \sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \leq l}} (2k+1) \frac{\sin(2k+1)\pi t}{\sin \pi t}, \quad l \in \frac{1}{2}\mathbb{N}_0,$$

satisfies the estimate

$$|D_l(t)| \lesssim \frac{2l+1}{t^2}, \tag{74}$$

with a constant independent of t and l.

Proof. Since $\chi_k(t) = \operatorname{Tr} T^k(t) = \frac{\sin(2k+1)\pi t}{\sin \pi t}$, we have

$$D_l(t) = \sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \leq l}} (2k+1)\chi_k(t) = \sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \leq l}} (2k+1) \frac{\sin(2k+1)\pi t}{\sin \pi t}.$$

Using the fact that $\frac{d}{dt} \sin(2k+1)\pi t = (2k+1)\pi \cos(2k+1)\pi t$, we can represent the last sum as follows

$$\frac{1}{\sin \pi t} \sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \le l}} (2k+1) \sin(2k+1)\pi t = \left(\frac{-1}{\pi}\right) \frac{1}{\sin \pi t} \frac{d}{dt} \left(\sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \le l}} \cos(2k+1)\pi t \right)$$

$$= \left(\frac{-1}{\pi}\right) \frac{1}{\sin \pi t} \frac{d}{dt} \left(\frac{\sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \le l}} \cos(2k+1)\pi t \sin \pi t}{\sin \pi t} \right).$$

Using sine multiplication formula, we obtain

$$\left(\frac{-1}{\pi}\right) \frac{1}{\sin \pi t} \frac{d}{dt} \left(\frac{\sum_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \le l}} \sin(2k+1+1)\pi t - \sin(2k+1-1)\pi t}{\sin \pi t} \right)$$

$$= \left(\frac{-1}{\pi}\right) \frac{1}{\sin \pi t} \frac{d}{dt} \left(\frac{\sin(2l+1)\pi t + \sin(2l+2)\pi t}{\sin \pi t} \right)$$

$$= \frac{(\sin(2l+1)\pi t + \sin(2l+2)\pi t) \cos(\pi t)}{\sin^3 \pi t} - \frac{(2l+1)\cos(2l+1)\pi t + (2l+2)\cos(2l+2)\pi t}{\sin^2 \pi t}.$$

This proves (74). □

We can refer to Giulini and Travaglini [GT80] and to Travaglini [Tra93] for some other interesting properties of Fourier coefficients and Dirichlet kernels on SU(2).

Appendix A. Polyhedral summability on compact Lie groups

It has been shown by Stanton [Sta76] that for class functions on semisimple compact Lie groups the polyhedral Fourier partial sums $S_N f$ converge to f in L^p provided that $2 - \frac{1}{s+1} < p < 2 + \frac{1}{s}$. Here the number s depends on the root system $\mathcal{R}$ of the compact Lie group G, in the way we now describe. We also note that the range of indices p as above is sharp, see Stanton and Tomas [ST76, ST78] as well as Colzani, Giulini and Travaglini [CGT89].

Let G be a compact semisimple Lie group and let T be a maximal torus of G, with Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let $n = \dim G$ and $l = \dim T = \operatorname{rank} G$. We define a positive definite inner product on $\mathfrak{t}$ by putting $(\cdot,\cdot) = -B(\cdot,\cdot)$, where B is the Killing form. Let $\mathcal{R}$ be the set of roots of $\mathfrak{g}$. Choose in $\mathcal{R}$ a system $\mathcal{R}_+$ of positive roots (with cardinality r) and let $S = \{\alpha_1, \ldots, \alpha_l\}$ be the corresponding simple system. We define $\rho := \frac{1}{2} \sum\limits_{\alpha\in\mathcal{R}_+} \alpha$.

For every $\lambda \in it^*$ there exists a unique $H_\lambda \in \mathfrak{t}$ such that $\lambda(H) = i(H_\lambda, H)$ for every $H \in \mathfrak{t}$. The vectors $H_j = \frac{4\pi i H_{\alpha_j}}{\alpha_j(H_{\alpha_j})}$ generate the lattice sometimes denoted by Ker(exp). The elements of the set

$$\Lambda = \{\lambda \in it^* \colon \lambda(H) \in 2\pi i\mathbb{Z}, \quad \text{for any} \quad H \in \operatorname{Ker}(\exp)\}$$

are called the weights of G and the fundamental weights are defined by the relations $\lambda_j = 2\pi i\delta_{jk}$, $j, k = 1, \ldots, l$. The subset

$$\mathfrak{D} = \{\lambda \in \Lambda \colon \lambda = \sum_{j=1}^{l} m_j\lambda_j,\ m_j \in \mathbb{N}\}$$

of the set Λ with positive coordinates m_j is called the set of dominant weights. Here, the word 'dominant' means that with respect to a certain partial order on the set Λ every weight $\lambda = \sum_{j=1}^{l} m_j\lambda_j$ with $m_j > 0$ is maximal. There exists a bijection between $\widehat{G}$ and the semilattice $\mathfrak{D}$ of the dominant weights of G, i.e.

$$\mathfrak{D} \ni \lambda = (m_1, \ldots, m_l) \longleftrightarrow \pi \in \widehat{G}.$$

Therefore, we will not distinguish between π and the corresponding dominant weight λ and will write

$$\pi = (\pi_1, \ldots, \pi_l), \tag{A.1}$$

where we agree to set $\pi_i = m_i$. With $\rho = \frac{1}{2} \sum\limits_{\alpha\in\mathcal{R}_+} \alpha$, for a natural number $N \in \mathbb{N}$, we set

$$Q_N := \{\xi \in \widehat{G} \colon\ \xi_i \le N\rho_i,\ i = 1, \ldots, l\}. \tag{A.2}$$

We call Q_N a finite polyhedron of N^{th} order and denote by $\mathcal{M}_0$ the set of all finite polyhedrons in $\widehat{G}$ or in $\widehat{G}_0$.

Now, fix an arbitrary fundamental weight λ_j, $j = 1, \ldots, l$, and set $\mathcal{R}_{\lambda_j}^{\perp} := \{\alpha \in \mathcal{R}_+ \colon (\alpha, \lambda_j) = 0\}$, and $\mathcal{R}_+ = \mathcal{R}_{\lambda_j} \oplus \mathcal{R}_{\lambda_j}^{\perp}$. We will often use the number

$$s := \max_{j=1,\ldots,l} \operatorname{card} \mathcal{R}_{\lambda_j}. \tag{A.3}$$

We denote by $L^p_*(G/K)$ the Banach subspace of $L^p(G/K)$ of functions on G/K whose canonical liftings are central on G: if $\tilde{f}(g) = f(gK)$ is the canonical lifting of f from G/K to G, by definition

$$f \in L^p_*(G/K) \text{ if and only if } f \in L^p(G/K) \text{ and } \tilde{f}(gug^{-1}) = \tilde{f}(u) \quad \text{for all } u, g \in G.$$

We note that such functions have then K-invariance both on the right and on the left: $\tilde{f}(KuK) = \tilde{f}(u)$ for all $u \in G$. Consequently, for $\pi \in \widehat{G}_0$, with our choice of basis vectors for the invariant subspace of the representation space, the Fourier coefficient $\widehat{f}(\pi)$ vanishes outside the upper-left $k_\pi \times k_\pi$ block, i.e. $\widehat{f}(\pi)_{ij} = 0$ if $i > k_\pi$ or $j > k_\pi$.

Further, we formulate and apply a result on semisimple Lie groups by Robert Stanton [Sta76] for L^p-norm convergence of polyhedral Fourier partial sums. We also refer to Stanton and Tomas [ST76, ST78] and to Colzani, Giulini and Travaglini [CGT89] for the converse statement.

Let ρ denote the half-sum of positive roots of G. Recall also the notation $Q_N := \{\pi \in \widehat{G}_0 \colon \pi_i \leq N\rho_i, \; i = 1, \ldots, l\}$ and $D_N(u) := D_{Q_N}(u) = \sum_{\pi \in Q_N} \operatorname{Tr}[\pi(u)]$.

Theorem Appendix A.1 ([Sta76])**.** *Let G be a semisimple compact Lie group. Let $f \in L^p_*(G/K)$ and let $S_N f(x)$ be the associated polyhedral Fourier partial sum, i.e.*

$$S_N f(u) := T_{Q_N}(x).$$

Then $S_N f$ converges to f in $L^p(G/K)$ provided that $2 - \frac{1}{1+s} < p < 2 + \frac{1}{s}$, where s is defined by (A.3)*. If G is simply connected, this range of p is in general sharp.*

Consequently, one obtains

Theorem Appendix A.2. *Let $\frac{2n}{n+l} < p < +\infty$ and $f \in L^p_*(G)$. Then $S_N f(x)$ converges to $f(x)$ for almost all $x \in G$.*

Although Stanton's version of this theorem is on groups, by considering the canonical liftings from the homogeneous space we obtain the formulation above also for homogeneous spaces, at least for the sufficient condition. The only if part of 'in general sharp' follows from [CGT89], by for example taking $K = \{e\}$.

Appendix B. Marcinkiewicz interpolation theorem

In this section we formulate the Marcinkiewicz interpolation theorem on arbitrary σ-finite measure spaces. Then we show how to use this theorem for linear mappings between $C^\infty(G)$ and the space Σ of finite matrices on the discrete unitary dual $\widehat{G}$ or on the discrete set $\widehat{G}_0$ of class I representations with different measures on $\widehat{G}$ and $\widehat{G}_0$.

This approach will be instrumental in the proof of the Hardy-Littlewood Theorem 2.2 and of the Paley inequality in Theorem 2.3.

We now formulate the Marcinkiewicz theorem for linear mappings between functions on arbitrary σ-finite measure spaces (X, μ_X) and (Γ, ν_Γ).

Let $PC(X)$ denote the space of step functions on (X, μ_X). We say that a linear operator A is of strong type (p, q), if for every $f \in L^p(X, \mu_X) \cap PC(X)$, we have $Af \in L^q(\Gamma, \nu_\Gamma)$ and

$$\|Af\|_{L^q(\Gamma,\nu_\Gamma)} \leq C\|f\|_{L^p(X,\mu_X)},$$

where C is independent of f, and the space $\ell^q(\Gamma, \nu_\Gamma)$ defined by the norm

$$\|h\|_{L^q(\Gamma,\nu_\Gamma)} := \left(\int\limits_\Gamma |h(\pi)|^p \nu(\pi) \right)^{\frac{1}{q}}. \tag{B.1}$$

The least C for which this is satisfied is taken to be the strong (p, q)-norm of the operator A.

Denote the distribution functions of f and h by $\mu_X(x;f)$ and $\nu_\Gamma(y;h)$, respectively, i.e.

$$\mu_X(x;f) := \int\limits_{\substack{t\in X\\ |f(t)|\geq x}} d\mu(t), \quad x>0,$$
$$\nu_\Gamma(y;h) := \int\limits_{\substack{\pi\in\Gamma\\ |h(\pi)|\geq y}} d\nu(\pi), \quad y>0. \tag{B.2}$$

Then

$$\|f\|^p_{L^p(X,\mu_X)} = \int\limits_X |f(t)|^p\, d\mu(t) = p\int\limits_0^{+\infty} x^{p-1}\mu_X(x;f)\, dx,$$
$$\|h\|^q_{L^q(\Gamma,\nu_\Gamma)} = \int\limits_{\pi\in\Gamma} |h(\pi)|^q\nu(\pi) = q\int\limits_0^{+\infty} y^{q-1}\nu_\Gamma(y;h)\, dy.$$

A linear operator $A\colon \mathcal{P}C(X)\to L^q(\Gamma,\nu_\Gamma)$ satisfying

$$\nu_\Gamma(y;Af) \leq \left(\frac{M}{y}\|f\|_{L^p(X,\mu_X)}\right)^q, \quad \text{for any } y>0. \tag{B.3}$$

is said to be of *weak type* (p,q); the least value of M in (B.3) is called the weak (p,q) norm of A.

Every operation of strong type (p,q) is also of weak type (p,q), since

$$y\left(\nu_\Gamma(y;Af)\right)^{\frac{1}{q}} \leq \|Af\|_{L^q(\Gamma)} \leq M\|f\|_{L^p(X)}.$$

445 **Theorem Appendix B.1.** *Let $1\leq p_1<p<p_2<\infty$. Suppose that a linear operator A from $\mathcal{P}C(X)$ to $L^q(\Gamma,\nu_\Gamma)$ is simultaneously of weak types (p_1,p_1) and (p_2,p_2), with norms M_1 and M_2, respectively, i.e.*

$$\begin{aligned} \nu_\Gamma(y;Af) &\leq \left(\frac{M_1}{y}\|f\|_{L^{p_1}(X,\mu_X)}\right)^{p_1}, \\ \nu_\Gamma(y;Af) &\leq \left(\frac{M_2}{y}\|f\|_{L^{p_2}(X,\mu_X)}\right)^{p_2} \qquad \textit{hold for any } y>0. \end{aligned}$$

Then for any $p\in(p_1,p_2)$ the operator A is of strong type (p,p) and we have

$$\|Af\|_{L^p(\Gamma,\nu_\Gamma)} \lesssim M_1^{1-\theta}M_2^{\theta}\|f\|_{L^p(X,\mu_X)}, \quad 0<\theta<1,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

The proof is given in e.g. Folland [Fol99]. Now, we adapt this theorem to the setting of matrix-valued mappings.

Suppose Γ is a discrete set. Integral over Γ is defined as sum over Γ, i.e.

$$\int_{\Gamma} \nu_{\Gamma}(\pi) := \sum_{\pi\in\Gamma} \nu(\pi). \tag{B.4}$$

In this case, to define a measure on Γ means to define a real-valued positive sequence $\nu = \{\nu_\pi\}_{\pi\in\Gamma}$, i.e.

$$\Gamma \ni \pi \mapsto \nu_\pi \in \mathbb{R}_+.$$

We turn Γ into a σ-finite measure space by introducing a measure

$$\nu_{\Gamma}(Q) := \sum_{\pi\in Q} \nu_\pi,$$

where Q is arbitrary subset of Γ.

We consider two sequences $\delta = \{\delta_\pi\}_{\pi\in\Gamma}$ and $\kappa = \{\kappa_\pi\}_{\pi\in\Gamma}$, i.e.

$$\Gamma \ni \pi \mapsto \delta_\pi \in \mathbb{N},$$

$$\Gamma \ni \pi \mapsto \kappa_\pi \in \mathbb{N}.$$

We denote by Σ the space of matrix-valued sequences on Γ that will be realised via

$$\Sigma := \left\{ h = \{h(\pi)\}_{\pi\in\Gamma}, h(\pi) \in \mathbb{C}^{\kappa_\pi \times \delta_\pi} \right\}.$$

The ℓ^p spaces on Σ can be defined, for example, motivated by the Fourier analysis on compact homogeneous spaces, in the form

$$\|h\|_{\ell^p(\Gamma,\Sigma)} := \left(\sum_{\pi\in\Gamma} \left(\frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}} \right)^p \nu_\pi \right)^{\frac{1}{p}}, \quad h \in \Sigma.$$

If we put $X = G$, where G is a compact Lie group and let $\Gamma = \widehat{G}$, then Fourier transform can be regarded as an operator mapping a function $f \in L^p(G)$ to the matrix-valued sequence $\widehat{f} = \{\widehat{f}(\pi)\}_{\pi\in\widehat{G}}$ of the Fourier coefficients, with $\delta_\pi = \kappa_\pi = d_\pi$. For $\Gamma = \widehat{G}_0$ we put $\delta_\pi = d_\pi$ and $\kappa_\pi = k_\pi$, these spaces thus coincide with the $\ell^p(\widehat{G}_0)$ spaces introduced in [RT10]. In Section 4, choosing different measures $\{\nu_\pi\}_{\pi\in\Gamma}$ on the unitary dual $\widehat{G}$ or on the set $\widehat{G}_0$, we use this to prove the Paley inequality and Hausdorff-Young-Paley inequalitites. Let us denote by $|h|$ the sequence consisting of $\{\frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}}\}$, i.e.

$$|h| = \left\{\frac{\|h(\pi)\|_{\mathtt{HS}}}{\sqrt{k_\pi}}\right\}_{\pi\in\Gamma}.$$

Then, we have

$$\|h\|_{\ell^q(\Gamma,\Sigma)} = \||h|\|_{L^q(\Gamma,\nu_\Gamma)}.$$

Thus, we obtain

Theorem Appendix B.2. *Let $1 \leq p_1 < p < p_2 < \infty$. Suppose that a linear operator A from $\mathcal{P}C(X)$ to Σ is simultaneously of weak types (p_1, p_1) and (p_2, p_2), with norms M_1 and M_2, respectively, i.e.*

$$\nu_\Gamma(y; Af) \leq \left(\frac{M_1}{y}\|f\|_{L^{p_1}(X)}\right)^{p_1}, \tag{B.5}$$

$$\nu_\Gamma(y; Af) \leq \left(\frac{M_2}{y}\|f\|_{L^{p_2}(X)}\right)^{p_2} \quad \textit{hold for any } y > 0. \tag{B.6}$$

Then for any $p \in (p_1, p_2)$ the operator A is of strong type (p, p) and we have

$$\|Af\|_{\ell^p(\Gamma,\Sigma)} \leq M_1^{1-\theta} M_2^{\theta} \|f\|_{L^p(X)}, \quad 0 < \theta < 1, \tag{B.7}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

References

[ANR16] R. Akylzhanov, E. Nursultanov, and M. Ruzhansky. Hardy-Littlewood-Paley inequalities and Fourier multipliers on SU(2). *Studia Math.*, 234: 1–29, 2016. *arXiv:1403.1731*

[ANR17] R. Akylzhanov and M. Ruzhansky. Net spaces on lattices, Hardy-Littlewood type inequalities, and their converse. *Eurasian Math. J.*, 8: 10–27, 2017.

[BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[CdG71] R. R. Coifman and M. de Guzmán. Singular integrals and multipliers on homogeneous spaces. *Rev. Un. Mat. Argentina*, 25:137–143, 1970/71. Collection of articles dedicated to Alberto González Domínguez on his sixty-fifth birthday.

[CW71a] R. R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971. Étude de certaines intégrales singulières.

[CW71b] R. R. Coifman and G. Weiss. Multiplier transformations of functions on SU(2) and $\sum_2$. *Rev. Un. Mat. Argentina*, 25:145–166, 1971. Collection of articles dedicated to Alberto González Domínguez on his sixty-fifth birthday.

[CGT89] L. Colzani, S. Giulini, and G. Travaglini. Sharp results for the mean summability of Fourier series on compact Lie groups. *Math. Ann.*, 285(1):75–84, 1989.

[Cle71] J.-L. Clerc. Fonctions de Paley-Littlewood sur SU(2) attachées aux sommes de Riesz. *C. R. Acad. Sci. Paris Sér. A-B*, 272:A1697–A1699, 1971.

[Cle72] J.-L. Clerc. Les sommes partielles de la décomposition en harmoniques sphériques ne convergent pas dans $\mathbf{L}^p$ $(p \neq 2)$. *C. R. Acad. Sci. Paris Sér. A-B*, 274:A59–A61, 1972.

[Cle73] J. L. Clerc. *Thèse.* PhD thesis, Université Paris XI, 1973.

[DR14] A. Dasgupta and M. Ruzhansky. Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces. *Bull. Sci. Math.*, 138(6):756–782, 2014.

[Edw72] R. E. Edwards. *Integration and harmonic analysis on compact groups.* Cambridge Univ. Press, London, 1972. London Mathematical Society Lecture Note Series, No. 8.

[Fef71a] C. Fefferman. The multiplier problem for the ball. *Ann. of Math. (2)*, 94:330–336, 1971.

[Fef71b] C. Fefferman. On the divergence of multiple Fourier series. *Bull. Amer. Math. Soc.*, 77:191–195, 1971.

[Fol99] G. B. Folland. *Real analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[GT80] S. Giulini and G. Travaglini. L^p-estimates for matrix coefficients of irreducible representations of compact groups. *Proc. Amer. Math. Soc.*, 80(3):448–450, 1980.

[Hal03] B. C. Hall. *Lie groups, Lie algebras, and Representations. An elementary introduction.* Graduate Texts in Mathematics. Springer-Verlag, 2003.

[Her54] C. S. Herz. On the mean inversion of Fourier and Hankel transforms. *Proc. Nat. Acad. Sci. U. S. A.*, 40:996–999, 1954.

[HL27] G. H. Hardy and J. E. Littlewood. Some new properties of Fourier constants. *Math. Ann.*, 97(1):159–209, 1927.

[Hör60] L. Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960.

[HR70] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, New York, 1970.

[HR74] E. Hewitt and K. A. Ross. Rearrangements of L^r Fourier series on compact Abelian groups. *Proc. Lond. Math. Soc. (3)*, 29:317–330, 1974.

[Kun58] R. A. Kunze. L_p Fourier transforms on locally compact unimodular groups. *Trans. Amer. Math. Soc.*, 89:519–540, 1958.

[NRT14] E. Nursultanov, M. Ruzhansky, and S. Tikhonov. Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds. *http://arxiv.org/abs/1403.3430*, 2014.

[NT00] E. D. Nursultanov and N. T. Tleukhanova. Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces. *Funktsional. Anal. i Prilozhen.*, 34(2):86–88, 2000.

[Pes08] I. Pesenson. Bernstein-Nikolskii inequalities and Riesz interpolation formula on compact homogeneous manifolds. *J. Approx. Theory*, 150(2):175–198, 2008.

[RT10] M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries. Background analysis and advanced topics*, volume 2 of *Pseudo-Differential Operators. Theory and Applications.* Birkhäuser Verlag, Basel, 2010.

[RT13] M. Ruzhansky and V. Turunen. Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces. *Int. Math. Res. Not. IMRN*, (11):2439–2496, 2013.

[RW13] M. Ruzhansky and J. Wirth. On multipliers on compact Lie groups. *Funct. Anal. Appl.*, 47(1):87–91, 2013.

[RW15] M. Ruzhansky and J. Wirth. L^p Fourier multipliers on compact Lie groups. *Math. Z.*, 280: 621–642, 2015.

[Shu87] M. A. Shubin. *Pseudodifferential operators and spectral theory.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987. Translated from the Russian by Stig I. Andersson.

[ST76] R. J. Stanton and P. A. Tomas. Convergence of Fourier series on compact Lie groups. *Bull. Amer. Math. Soc.*, 82(1):61–62, 1976.

[ST78] R. J. Stanton and P. A. Tomas. Polyhedral summability of Fourier series on compact Lie groups. *Amer. J. Math.*, 100(3):477–493, 1978.

[Sta76] R. J. Stanton. Mean convergence of Fourier series on compact Lie groups. *Trans. Amer. Math. Soc.*, 218:61–87, 1976.

[Ste70] E. M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory.* Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.

[Tra93] G. Travaglini. Polyhedral summability of multiple Fourier series (and explicit formulas for Dirichlet kernels on $\mathbf{T}^n$ and on compact Lie groups). *Colloq. Math.*, 65(1):103–116, 1993.

[Vil68] N. J. Vilenkin. *Special functions and the theory of group representations.* Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, R. I., 1968.

[VK91] N. J. Vilenkin and A. U. Klimyk. *Representation of Lie groups and special functions. Vol. 1*, volume 72 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1991. Simplest Lie groups, special functions and integral transforms, Translated from the Russian by V. A. Groza and A. A. Groza.