Finite-time Ruin Probability of Aggregate Gaussian Processes

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Abstract: Let $\{\sum_{i=1}^n \lambda_i X_i(t), t \in [0,T]\}$ be an aggregate Gaussian risk process with $X_i, i \le n$ independent Gaussian processes satisfying Piterbarg conditions and λ_i's given positive weights. In this paper we derive exact asymptotics of the finite-time ruin probability given by

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left(\sum_{i=1}^n \lambda_i X_i(t) - g(t)\right) > u\right)$$

as $u \to \infty$ for some general trend function g. Further, we derive asymptotic results for the finite-time ruin probabilities of risk processes perturbed by an aggregate Gaussian process.

Key Words: ruin probability; Gaussian process; perturbed risk process; Lévy process; (sub- and bi-)fractional Brownian motion; risk aggregation; subexponential risks.

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1 Introduction

Numerous contributions have discussed the evaluation of the first-passage density of a random process $\{X(t), t \in [0,T]\}$ to a given deterministic boundary denoted by $u + g(t)$ with fixed $u \ge 0$. In a concrete insurance setup, let $X(t)$ model the surplus process of the whole company at time t, the decision to pay dividends can be objectively made once the surplus process crosses the boundary. Specifically, from the actuarial point of view, it is of interest to calculate the crossing probability

$$\mathbb{P}\left(\exists t \in [0,T],\ X(t) > u + g(t)\right) \tag{1.1}$$

for $u \ge 0$. However, an explicit formula for (1.1) is hard to obtain except for some very special cases, e.g., $\{X(t), t \in [0,T]\}$ is a Brownian motion (Bm) and $g(t)$ is a linear function. Therefore, usually the aim of the analysis is to find adequate approximations for it. From risk theory point of view Eq. (1.1) can also be seen as the finite-time ruin probability of an insurance company, i.e.,

$$\mathbb{P}\left(\exists t \in [0,T],\ X(t) > u + g(t)\right) = \mathbb{P}\left(\inf_{t\in[0,T]}(u + g(t) - X(t)) < 0\right),$$

where $u \ge 0$ is the initial capital, $g(t)$ is the premium amount received up to time t, and $X(t)$ represents the aggregate claim amount up to t. Recently, the study of surplus process with dependent risks becomes more and more popular since independent risks is not applicable to practice, see e.g., Denuit et al. (2005) and Constantinescu et al. (2011).

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In Michna (1998) it is shown that the finite-time ruin probability given by

$$\mathbb{P}\left(\inf_{t\in[0,T]}\Big(u+ct-B_H(t)\Big)<0\right) \tag{1.2}$$

is an adequate approximation of the finite-time ruin probability for a risk process with certain dependent risks, where $\{B_H(t), t\in[0,T]\}$ is a fractional Brownian motion (fBm) with Hurst index $H\in(0,1]$.
Nowadays, all insurance companies run diverse lines of business, with typically some lines of business (for non-life insurer) having very high premiums because of high risks. In order to reflect different portfolio variances, as well as different business volumes, it is adequate to consider a process which is a result of aggregation of the specific portfolios. A tractable choice here is the aggregate process

$$X(t) \quad = \quad \lambda_1 B_{H_1}(t)+\cdots+\lambda_n B_{H_n}(t), \quad t\in[0,T], \tag{1.3}$$

where $\lambda_i, i\le n$, are positive weights assigned to the processes $\{B_{H_i}(t), t\in[0,T]\}, i\le n$, being independent fBm's with Hurst indexes $H_i\in(0,1], i\le n$, respectively.
Clearly, $X(t), t\in[0,T]$ is not a fBm anymore; bounds and asymptotics of the finite-time ruin probability for $X(t), t\in[0,T]$ are given in Dębicki and Sikora (2011) for this multiplexed fBm's with a linear trend. The asymptotics of the infinite-time ruin probability of the multiplexed fBm's with a trend is discussed in Hüsler and Schmid (2006).
The perturbed risk model is an important extension of the classical risk model. Of course, instead of the Bm, general processes, including Lévy and Gaussian processes, can be considered as perturbations, see e.g., Schlegel (1998), Furrer (1998) and Frostig (2008). In fact, the Bm (and Lévy processes) can not be justified if the *perturbation terms* do not come from an i.i.d. framework, whereas some Gaussian processes can be. In practice, the surplus is influenced by various uncertainties such as premium adjustments, legislation changes, cost of repairs, and other related expanses. Therefore, in order to reflect different variances of the uncertainties, it is reasonable to consider an aggregate Gaussian process as the perturbation.

In this paper we present some extensions of Dębicki and Sikora (2011) and consider further the perturbed risk process. Specifically, instead of dealing with the aggregation of independent fBm's, we consider the aggregation of independent centered Gaussian processes $\{X_i(t), t\in[0,T]\}, i\le n$, with some positive weights $\lambda_i, i\le n$. Our analysis then focusses on the asymptotics of the finite-time ruin probability

$$\mathbb{P}\left(\sup_{t\in[0,T]}\Big(\sum_{i=1}^n\lambda_i X_i(t)-g(t)\Big)>u\right), \text{ as } u\to\infty,$$

with some bounded measurable trend function $g(t)$. It is worth noting that the aggregate Gaussian process $\sum_{i=1}^n \lambda_i X_i(t)$ is also a Gaussian process, but in order to see which of the components will contribute more to the asymptotics we would like to deal with the aggregate Gaussian process other than one single Gaussian process. This might also be necessary from practical point of view. Moreover, the finite-time ruin probability of a perturbed risk process with perturbation modeled by an aggregate Gaussian process defined by

$$\mathbb{P}\left(\sup_{t\in[0,T]}\Big(U(t)-c(t)+\sum_{i=1}^n\lambda_i X_i(t)\Big)>u\right), \quad u\ge0,$$

is also discussed, where $U(t)-c(t)$ is the claim surplus process, and $\sum_{i=1}^n \lambda_i X_i(t)$ is the aggregate Gaussian perturbation.
In the first result Theorem 3.1 we provide the asymptotic behaviour of the finite-time ruin probability for the aggregate Gaussian process, which indicates that the processes which have the smallest characteristic

constants will contribute more to the asymptotics. Furthermore, our second result Theorem 4.1 derives a novel asymptotic result for the finite-time ruin probabilities of some quite general perturbed risk processes including Gaussian perturbed risk process as a special case.

This paper is organized as follows. In Section 2 we introduce some notation. The main results are given in Section 3 and Section 4. Section 5 presents several examples. Proofs of all the results are relegated to Section 6.

2 Notation and Preliminaries

In this section we mention several abbreviations and notation needed in this paper and present the main assumptions. There are mainly two well known constants, namely Pickands constant and Piterbarg constant, which play important roles in the extreme theory of Gaussian processes. The former is defined by

$$\mathcal{H}_{\alpha/2}=\lim_{T\to\infty}T^{-1}\mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_{\alpha/2}(t)-t^{\alpha}\right)\right)\right),\quad \alpha\in(0,2],$$

and the latter is defined by

$$\mathcal{P}_{\alpha}^{R}:=\lim_{S\to\infty}\mathbb{E}\left(\exp\left(\sup_{t\in[0,S]}\left(\sqrt{2}B_{\alpha/2}(t)-(1+R)t^{\alpha}\right)\right)\right),\quad \alpha\in(0,2],\ R>0,$$

where $\{B_{\alpha/2}(t),t\in[0,\infty)\}$ is a fBm with Hurst index $\alpha/2$. See Pickands (1969) or Piterbarg (1996), for the main properties of Pickands and Piterbarg constants.

We shall impose two main common assumptions on the Gaussian processes of interest. Let $\{\xi(t),t\in[0,\infty)\}$ be a centered Gaussian process with variance function $\sigma_{\xi}^{2}(\cdot)$. Throughout this paper the process ξ with a bar represents a standardized process i.e., $\bar{\xi}(t):=\xi(t)/\sigma_{\xi}(t)$.

Assumption A1. The standard deviation function $\sigma_{\xi}(\cdot)$ of the Gaussian process $\xi(t)$ attains its maximum, denoted by $\widetilde{\sigma}$, over $[0,T]$ at the unique point $t=T$. Further, there exist some positive constants $\alpha\in(0,2],\beta,A,D$ such that

$$\sigma_{\xi}(t)\quad=\quad\widetilde{\sigma}-A(T-t)^{\beta}+o((T-t)^{\beta}),\quad t\to T, \tag{2.4}$$

and

$$\mathbb{C}ov\left(\bar{\xi}(s),\bar{\xi}(t)\right)=1-D|t-s|^{\alpha}+o(|t-s|^{\alpha}),\quad \min(t,s)\to T.$$

Assumption A2. There exist positive constants $\mathbb{C},\delta$ and γ such that, for all $s,t\in[\delta,T]$,

$$\mathbb{E}\left((\xi(t)-\xi(s))^{2}\right)\quad\le\quad\mathbb{C}|t-s|^{\gamma}. \tag{2.5}$$

Some recent studies in financial markets indicate that the class of H-self-similar (H-ss) Gaussian processes can adequately model the long-range dependence structure of the real financial data. Let us recall that a centered Gaussian process $\{X(t),t\in[0,\infty)\}$ with $X(0)=0$ is H-ss with an exponent $H\in(0,1]$ if the covariance function satisfies the condition

$$\mathbb{C}ov\left(X(at),X(as)\right)=a^{2H}\mathbb{C}ov\left(X(t),X(s)\right),\quad \forall a\in(0,\infty).$$

A prominent example of self-similar Gaussian processes is the bi-fractional Brownian motion (bi-fBm) $\{B_{K,H}(t),t\in[0,\infty)\}$ with covariance function given by

$$\mathbb{C}ov\left(B_{K,H}(t),B_{K,H}(s)\right)=\frac{1}{2^{K}}[(t^{2H}+s^{2H})^{K}-|s-t|^{2KH}],\quad K\in(0,1],\quad H\in(0,1).$$

Another interesting self-similar Gaussian process is the sub-fractional Brownian motion (sub-fBm) $\{S_H(t), t \in [0,\infty)\}$ with covariance function given by

$$\mathbb{C}ov\left(S_H(t), S_H(s)\right) = t^{2H} + s^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right], \quad H \in (0,1).$$

Important results for the bi-fBm and sub-fBm can be found in Houdré and Villa (2003) and Bojdecki et al. (2004).

3 Exact Asymptotics of the Finite-time Ruin Probability

Given n independent centered Gaussian processes $\{X_i(t), t \in [0,T]\}, i \le n$, with a.s. continuous sample paths and standard deviation functions $\sigma_i(\cdot), i \le n$, respectively, the extended Dębicki-Sikora Gaussian model consists in the specification of the aggregate Gaussian process

$$X(t) := \lambda_1 X_1(t) + \cdots + \lambda_n X_n(t), \quad t \in [0,T], \tag{3.6}$$

with $\lambda_i \ge 0, i \le n$. The finite-time ruin probability of this risk model is defined as

$$\mathbb{P}\left(\sup_{t\in[0,T]}\Big(X(t) - g(t)\Big) > u\right),$$

for the deterministic bounded measurable trend function $g(t)$ and $u \ge 0$.

In order to obtain the exact asymptotics of the finite-time ruin probability, some conditions on the Gaussian processes and the bounded measurable trend function $g(t)$ needed are fully described in Theorem 3.1. For our results below we need the following notation

$$\Lambda_{\alpha,\beta}(u) := \begin{cases} \left(\frac{u+g(T)}{\sqrt{\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}}}\right)^{2/\alpha - 2/\beta}, & \text{if } \alpha < \beta, \\ 1, & \text{if } \alpha \ge \beta, \end{cases} \quad \text{with} \quad \widetilde{\sigma}_i := \sigma_i(T).$$

Further, $\Gamma(\cdot)$ stands for the Euler Gamma function and $\mathbb{I}(\cdot)$ for the indicator function. Next we state our first result.

Theorem 3.1. *Let $\{X_i(t), t \in [0,T]\}, i \le n$, be independent centered Gaussian processes with a.s. continuous sample paths and standard deviation functions $\sigma_i(\cdot), i \le n$, and define $\{X(t), t \in [0,T]\}$ as in (3.6). If Assumptions A1 and A2 hold for each $\{X_i(t), t \in [0,T]\}, i \le n$, with constants $\alpha_i, \beta_i, A_i, D_i, \mathbb{C}, \delta, \gamma_i, i \le n$, respectively, then, for any bounded measurable trend function $g(t)$ satisfying*

$$\left|g(T) - g(t)\right| \le \mathcal{M}(T-t)^{\min_{i\le n}\beta_i}, \quad \forall t \in [\nu, T] \tag{3.7}$$

for some constant $\mathcal{M}$ and $\nu \in (0,T)$, we have

$$\mathbb{P}\left(\sup_{t\in[0,T]}(X(t)-g(t)) > u\right) \sim \mathcal{C}_{\alpha,\beta}\Lambda_{\alpha,\beta}(u)\Psi\left(\frac{u+g(T)}{\sqrt{\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}}}\right), \quad u \to \infty, \tag{3.8}$$

where

$$\mathcal{C}_{\alpha,\beta} = \begin{cases} \mathcal{H}_{\alpha/2}\Gamma(1/\beta+1)\widetilde{N}^{-1/\beta}\widetilde{G}^{1/\alpha}\left(\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}\right)^{1/\beta - 1/\alpha}, & \text{if } \alpha < \beta, \\ \mathcal{P}_\alpha^{\widetilde{N}/\widetilde{G}}, & \text{if } \alpha = \beta, \\ 1, & \text{if } \alpha > \beta, \end{cases}$$

with

$$\alpha = \min_{i\le n}\alpha_i, \ \beta = \min_{i\le n}\beta_i, \ \widetilde{N} = \sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i A_i \mathbb{I}(\beta_i = \beta), \ \widetilde{G} = \sum_{i=1}^n \lambda_i^2 D_i \widetilde{\sigma}_i^{\,2}\mathbb{I}(\alpha_i = \alpha).$$

Corollary 3.2. *Let $\{X_i(t), t \in [0,T]\}, i \le n$, $\{X(t), t \in [0,T]\}$ and $g(t)$ be as in Theorem 3.1.*
(i) If $\{X_i(t), t \in [0,T]\}, i \le n$, are bi-fBm's with parameters $K_i, H_i \in (0,1]$, $i \le n$, satisfying $0 < KH := K_1H_1 < K_2H_2 \le \cdots \le K_nH_n$, then we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} (X(t) - g(t)) > u\right) \sim \mathcal{C}_{2KH,1}\Lambda_{2KH,1}(u)\Psi\left(\frac{u+g(T)}{\sqrt{\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}}}\right), \quad u \to \infty, \tag{3.9}$$

where

$$\mathcal{C}_{2KH,1} = \begin{cases} \mathcal{H}_{KH}\left(\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}\right)^{\frac{2KH-1}{2KH}} \frac{(\frac{1}{2^K}\lambda_1^2)^{1/(2KH)}T}{\sum_{i=1}^n \lambda_i^2 K_i H_i \widetilde{\sigma}_i^{\,2}}, & \text{if } KH < 1/2, \\ 1 + \frac{\lambda_1^2 T}{2^K(\sum_{i=2}^n \lambda_i^2 K_i H_i \widetilde{\sigma}_i^{\,2} + \lambda_1^2 T/2)}, & \text{if } KH = 1/2, \\ 1, & \text{if } KH > 1/2, \end{cases} \quad \text{and} \quad \widetilde{\sigma}_i = T^{K_iH_i}.$$

(ii) If $\{X_i(t), t \in [0,T]\}, i \le n$, are sub-fBm's with parameters $H_i \in (0,1), i \le n$, satisfying $H := H_1 < H_2 \le \cdots \le H_n$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]} (X(t) - g(t)) > u\right) \sim \mathcal{C}_{2H,1}\Lambda_{2H,1}(u)\Psi\left(\frac{u+g(T)}{\sqrt{\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}}}\right), \quad u \to \infty, \tag{3.10}$$

where

$$\mathcal{C}_{2H,1} = \begin{cases} \mathcal{H}_{H}\left(\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma}_i^{\,2}\right)^{\frac{2H-1}{2H}} \frac{(\frac{1}{2}\lambda_1^2)^{1/(2H)}T}{\sum_{i=1}^n \lambda_i^2 H_i \widetilde{\sigma}_i^{\,2}}, & \text{if } H < 1/2, \\ 1 + \frac{\lambda_1^2 T}{2\sum_{i=2}^n \lambda_i^2 H_i \widetilde{\sigma}_i^{\,2} + \lambda_1^2 T}, & \text{if } H = 1/2, \\ 1 & \text{if } H > 1/2, \end{cases} \quad \text{and} \quad \widetilde{\sigma}_i^{\,2} = (2 - 2^{2H_i-1})T^{2H_i}.$$

4 Perturbed Risk Processes

This section is devoted to the analysis of finite-time ruin probabilities of some general perturbed risk models. In particular, we focus on perturbed risk processes, where the perturbation is an aggregate centered Gaussian process representing the aggregation of different types of perturbations. Consider the claim surplus process of an insurance company defined by

$$S(t) = U(t) - c(t), \quad t \ge 0, \tag{4.11}$$

where $\{U(t), t \in [0,\infty)\}$ is the aggregate claim process and $c(t)$ is a nonnegative increasing function modeling the premium income. Further, define the claim surplus process of the perturbed risk process as

$$\tilde{S}(t) = S(t) + X(t), \quad t \ge 0, \tag{4.12}$$

where the process $\{X(t), t \in [0,\infty)\}$ is a perturbation assumed to be independent of $\{S(t), t \in [0,\infty)\}$. For any $T \in (0,\infty)$, the finite-time ruin probability for the processes (4.11) and (4.12) are defined as

$$\psi(u,c,T) = \mathbb{P}\left(\sup_{t\in[0,T]} S(t) > u\right) \quad \text{and} \quad \tilde{\psi}(u,c,T) = \mathbb{P}\left(\sup_{t\in[0,T]} \tilde{S}(t) > u\right),$$

respectively, where $u \ge 0$ is the initial surplus. In general, the calculation of the finite-time ruin probability is more difficult than the infinite-time ruin probability. Therefore, often the aim of the analysis is to find good approximation for it. For notational simplicity set below

$$F_1(u) = \mathbb{P}\left(\sup_{t\in[0,T]} U(t) \le u\right), \quad F_2(u) = \mathbb{P}\left(\sup_{t\in[0,T]} X(t) \le u\right), \quad u \ge 0.$$

Let us first recall the class of long-tailed distributions and that of heavy-tailed distributions.
Heavy-tailed distribution class ($\mathcal{H}$): A distribution function F is said to be heavy-tailed if and only if

$$\int_{-\infty}^{\infty} e^{\lambda x} F(dx) = \infty \quad \text{for all } \lambda > 0.$$

Long-tailed distribution class ($\mathcal{L}$): A distribution function F is said to be long-tailed if and only if

$$\lim_{x\to\infty} \frac{1 - F(x+y)}{1 - F(x)} = 1 \quad \text{for all } y \in I\!R.$$

It is well-known that $\mathcal{L} \subset \mathcal{H}$, see e.g., Embrechts et al. (1997) and Foss et al. (2011) for the basic properties of heavy-tailed distributions. In addition, $F \in \mathcal{L}$ implies that there exists some function $d(u), u \geq 0$ such that

$$\lim_{u\to\infty} \frac{u}{d(u)} = \lim_{u\to\infty} d(u) = \infty$$

and

$$1 - F(u + d(u)) \ \sim \ 1 - F(u), \quad u \to \infty, \tag{4.13}$$

see e.g., Foss et al. (2011). Next, we present the main result of this section.

Theorem 4.1. *Assume that $F_1 \in \mathcal{L}$ and $1 - F_2(u) = o(1 - F_1(u))$ as $u \to \infty$, then*

$$\tilde{\psi}(u, c, T) \ \sim \ 1 - F_1(u) \ \sim \ \psi(u, c, T), \quad u \to \infty. \tag{4.14}$$

In the following, we consider Gaussian perturbed Lévy risk processes, where the perturbation is an aggregate Gaussian process $X(t) = \sum_{i=1}^{n} \lambda_i X_i(t), t \geq 0$, discussed in Section 3.

Corollary 4.2. *If $\{U(t), t \in [0, \infty)\}$ is a Lévy process such that*

$$U(T) \in \mathcal{L}, \tag{4.15}$$

and $\{X(t), t \in [0, T]\}$ is an aggregate Gaussian process satisfying the conditions of Theorem 3.1, then

$$\tilde{\psi}(u, c, T) \ \sim \ \mathbb{P}\left(U(T) > u\right) \ \text{as } u \to \infty.$$

Remark 4.3. *In the light of Albin and Sundén (2009), for a Lévy process $\{Y(t), t \in [0, \infty)\}$ with characteristic triple (d, σ^2, Π),*

$$\frac{\Pi([1, \infty) \cap \cdot)}{\Pi([1, \infty))} \in \mathcal{L}$$

implies that $Y(T) \in \mathcal{L}$.

5 Examples

In this section, we present several illustrating examples.
Example 1. Let $X(t) = B_H(t) + B_{1/2}(t^{2H}), t \in [0, T]$, with $H \in (0, 1/2)$. Assume that the trend function $g(t)$ satisfies (3.7) with some constant $\mathcal{M}$ and some $d \geq 1$. We have

$$\boldsymbol{P}\Big\{\sup_{t\in[0,T]} \Big(X(t) - g(t)\Big) > u\Big\} \sim \frac{\mathcal{H}_H}{4^{\frac{1}{2H}} H} \Lambda_u^{1/H-2} \Psi\Big(\Lambda_u\Big), \quad u \to \infty,$$

with

$$\Lambda_u = \frac{u + g(T)}{\sqrt{2} T^H}.$$

The following time average Gaussian process was discussed in Dębicki and Tabiś (2011).

Example 2. Let $\{B_{H_i}(t), t \in [0,T]\}, i \le n$, be independent fBm's with Hurst parameters $H_i \in (0,1]$, $i \le n$, satisfying $H_1 < H_2 < \cdots < H_n$. Set

$$X_i(t) = \begin{cases} \sqrt{2H_i+2}\frac{1}{t}\int_0^t B_{H_i}(s)ds, & t > 0, \\ 0, & t = 0. \end{cases}$$

Assume that the trend function $g(t)$ satisfies (3.7) with some constant $\mathcal{M}$ and some $d \ge 1$. It follows from Theorem 3.1 that

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left(\sum_{i=1}^n \lambda_i X_i(t) - g(t)\right) > u\right) \sim \Psi\left(\frac{u+g(T)}{\sqrt{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}}\right).$$

Example 3. Assume that $U(t) = \sum_{i=1}^{N(t)} Z_i, t \ge 0$, is a compound Poisson process, with i.i.d. claim inter-arrival times $\tau_i, i \in I\!N$, being exponentially distributed with parameter $\mu > 0$, and i.i.d claim sizes $Z_i, i \in I\!N$, having a Weibull distribution $F(y) = 1 - \exp(-y^\tau), y \ge 0$, with shape parameter $\tau \in (0,1)$. Furthermore, let $X(t) = \sum_{i=1}^n \lambda_i B_{1/2}(t^{2H_i})$ with $H_i \in (0,1], \lambda_i > 0, i \le n$. In view of Corollary 4.2, we conclude that

$$\tilde{\psi}(u,c,T) \sim \mu T e^{-u^\tau}, \quad \text{as} \quad u \to \infty.$$

Example 4. Consider a Gaussian perturbed α-stable risk process. Specifically, let $\{U(t), t \in [0,\infty)\}$ be an α-stable Lévy process with $\alpha \in (1,2)$, i.e. $U(t) \overset{d}{=} S_\alpha(t^{1/\alpha}, \beta, 0)$, where $S_\alpha(\sigma,\beta,d)$ denotes a stable random varible with index of stability α, scale parameter σ, skewness parameter β and drift parameter d (see e.g., Samorodnitsky and Taqqu (1994)). Moreover, let $X(t) = \sum_{i=1}^n \lambda_i B_{H_i}(t)$ with $B_{H_i}, i \le n$, being independent fBm's and $H_i \in (0,1], \lambda_i > 0, i \le n$. It is known that (4.15) is satisfied. Consequently, it follows from Corollary 4.2 and the tail behavior of stable distribution (e.g., Samorodnitsky and Taqqu (1994)) that

$$\tilde{\psi}(u,c,T) \sim \mathbb{P}(U(T) > u) \sim C_{\alpha,T^{1/\alpha}}\left(\frac{1+\beta}{2}\right)u^{-\alpha}, \quad \text{as} \quad u \to \infty,$$

where

$$C_{\alpha,T^{1/\alpha}} = \frac{T(1-\alpha)}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}.$$

6 Proofs

In this section we give detailed proofs of our previous results. Recall that $X(t) = \sum_{i=1}^n \lambda_i X_i(t)$ is the aggregate centered Gaussian process with variance function $\sigma_X^2(t) := \sum_{i=1}^n \lambda_i^2 \sigma_i^2(t)$ and $\bar{X}(t) := X(t)/\sigma_X(t)$.

Proof of Theorem 3.1 Define

$$m_u(t) := \frac{u+g(t)}{\sigma_X(t)} \quad \text{and} \quad \pi(u) := \mathbb{P}\left(\sup_{t\in[\delta,T]} \bar{X}(t)\frac{m_u(T)}{m_u(t)} > m_u(T)\right).$$

For any $u \ge 0$, as in Dębicki and Sikora (2011), we may further write

$$\pi(u) \le \mathbb{P}\left(\sup_{t\in[0,T]}(X(t)-g(t)) > u\right) \le \mathbb{P}\left(\sup_{t\in[0,\delta]}(X(t)-g(t)) > u\right) + \pi(u).$$

Obviously,

$$1-\frac{m_u(T)}{m_u(t)}=\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}+\frac{\sigma_X(t)[g(t)-g(T)]}{(u+g(t))\sigma_X(T)}.$$

Further, in view of (3.7), δ can be suitably chosen such that

$$\left|g(T)-g(t)\right|\le Const(\sigma_X(T)-\sigma_X(t))$$

for all $t\in[\delta,T]$. Therefore, for any $\varepsilon>0$, when u is sufficiently large, we have, uniformly in $[\delta,T]$,

$$1-(1+\varepsilon)\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}\le\frac{m_u(T)}{m_u(t)}\le 1-(1-\varepsilon)\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}. \tag{6.16}$$

Consequently, it follows from (6.16) that, for u sufficiently large,

$$\pi_{+\varepsilon}(u):=\mathbb{P}\left(\sup_{t\in[\delta,T]}Y_{+\varepsilon}(t)>m_u(T)\right)\le\pi(u)\le\pi_{-\varepsilon}(u):=\mathbb{P}\left(\sup_{t\in[\delta,T]}Y_{-\varepsilon}(t)>m_u(T)\right),$$

where

$$Y_{\pm\varepsilon}(t):=\bar{X}(t)\left(1-(1\pm\varepsilon)\frac{\sigma_X(T)-\sigma_X(t)}{\sigma_X(T)}\right),\quad t\ge0.$$

Next, we analyse $\pi_{-\varepsilon}(u)$ for fixed $\varepsilon\in(0,1)$, the asymptotics of $\pi_{+\varepsilon}(u)$ follows with the same arguments. Obviously, the standard deviation function $\sigma_{Y_{-\varepsilon}}(t)$ attains its unique maximum over $[\delta,T]$ at $t=T$, with $\sigma_{Y_{-\varepsilon}}(T)=1$. Further, by Assumption A1 (recall that $\tilde{\sigma}_i=\sigma_i(T)$),

$$\sigma_{Y_{-\varepsilon}}(t) \quad=\quad 1-(1-\varepsilon)\frac{\widetilde{N}}{\sum_{i=1}^n\lambda_i^2\widetilde{\sigma}_i^{\,2}}(T-t)^\beta+o((T-t)^\beta)$$

as $t\uparrow T$, with

$$\widetilde{N}=\lim_{t\to T}\sum_{i=1}^n\lambda_i^2\widetilde{\sigma}_iA_i(T-t)^{(\beta_i-\beta)}\in(0,\infty),$$

and

$$1-\mathbb{C}ov\left(\bar{Y}_{-\varepsilon}(s),\bar{Y}_{-\varepsilon}(t)\right)=\frac{\widetilde{G}}{\sum_{i=1}^n\lambda_i^2\widetilde{\sigma}_i^{\,2}}|t-s|^\alpha+o(|t-s|^\alpha)$$

as $\min(s,t)\to T$, with

$$\widetilde{G}=\lim_{t,s\to T}\sum_{i=1}^n\lambda_i^2D_i\widetilde{\sigma}_i^{\,2}|t-s|^{(\alpha_i-\alpha)}\in(0,\infty).$$

Moreover, in view of Assumption A2, we have, for $s,t\in[\delta,T]$ and some $\mathbb{C}>0$,

$$\begin{aligned}\mathbb{E}\left((Y_{-\varepsilon}(t)-Y_{-\varepsilon}(s))^2\right) &= \mathbb{E}\left(\left(\varepsilon(\bar{X}(t)-\bar{X}(s))+\frac{1-\varepsilon}{\sigma_X(T)}(X(t)-X(s))\right)^2\right)\\ &\le 2\varepsilon^2\mathbb{E}\left((\bar{X}(t)-\bar{X}(s))^2\right)+\frac{2(1-\varepsilon)^2}{\sigma_X^2(T)}\mathbb{E}\left((X(t)-X(s))^2\right)\\ &\le \left(\frac{2\varepsilon^2}{\sigma_X^2(\delta)}+\frac{2(1-\varepsilon)^2}{\sigma_X^2(T)}\right)\mathbb{E}\left((X(t)-X(s))^2\right)\\ &\le \mathbb{C}|s-t|^{\min_{1\le i\le n}\gamma_i}.\end{aligned}$$

Therefore, the Gaussian process $\{Y_{-\varepsilon}(t),t\in[0,T]\}$ satisfies the conditions of Theorem 8.2 of Piterbarg (1996) with

$$A=\left((1-\varepsilon)\frac{\widetilde{N}}{\sum_{i=1}^n\lambda_i^2\widetilde{\sigma}_i^{\,2}}\right)^{1/\beta},\quad C=\left(\frac{\widetilde{G}}{\sum_{i=1}^n\lambda_i^2\widetilde{\sigma}_i^{\,2}}\right)^{1/\alpha},$$

and thus, as $u\to\infty$,

$$\pi_{-\varepsilon}(u) \sim \begin{cases} \mathcal{H}_{\alpha/2}\beta^{-1}\Gamma(1/\beta)A^{-1}C\Lambda_{\alpha,\beta}(u)\widetilde{\Psi}(u), & \alpha<\beta, \\ \mathcal{P}_{\alpha}^{(1-\varepsilon)\widetilde{N}/\widetilde{G}}\widetilde{\Psi}(u), & \alpha=\beta, \\ \widetilde{\Psi}(u), & \alpha>\beta, \end{cases} \quad \text{with} \quad \widetilde{\Psi}(u) := \Psi\left(\frac{u+g(T)}{\sqrt{\sum_{i=1}^n \lambda_i^2 \widetilde{\sigma_i}^2}}\right).$$

Consequently, letting $\varepsilon \to 0$,

$$\pi(u) \sim \begin{cases} \mathcal{H}_{\alpha/2}\beta^{-1}\Gamma(1/\beta)C\left(\frac{\widetilde{N}}{\sum_{i=1}^n \lambda_i^2\widetilde{\sigma_i}^2}\right)^{-1/\beta}\Lambda_{\alpha,\beta}(u)\widetilde{\Psi}(u), & \alpha<\beta, \\ \mathcal{P}_{\alpha}^{\widetilde{N}/\widetilde{G}}\widetilde{\Psi}(u), & \alpha=\beta, \\ \widetilde{\Psi}(u), & \alpha>\beta, \end{cases}$$

as $u\to\infty$. Finally, using Borell-TIS inequality (e.g., Adler and Taylor (2007)) we conclude, as $u\to\infty$,

$$\begin{aligned} \mathbb{P}\left(\sup_{t\in[0,\delta]}(X(t)-g(t))>u\right) &\le \mathbb{P}\left(\sup_{t\in[0,\delta]}X(t)>u+\inf_{t\in[0,\delta]}g(t)\right) \\ &\le \exp\left(\frac{-\left(u+\inf_{t\in[0,\delta]}g(t)-\mathbb{E}\left(\sup_{t\in[0,\delta]}X(t)\right)\right)^2}{2\sigma_\delta^2}\right) = o(\pi(u)), \end{aligned}$$

since $\sigma_\delta^2 := \sup_{t\in[0,\delta]}(\sum_{i=1}^n \lambda_i^2\sigma_i^2(t)) < \sum_{i=1}^n \lambda_i^2\widetilde{\sigma_i}^2$. The proof is complete. □

The next lemma is crucial for the proof of Corollary 3.2. Details of its proof are omitted here since there are only some algebra calculations involved.

Lemma 6.1. *Under the conditions of Corollary 3.2, for any $i\le n$ and $T>0$, we have, as $s,t\to T$*
(i) if $\{X_i(t), t\in[0,T]\}$ is a bi-fBm , then

$$\begin{aligned} \sigma_i(t) &= T^{K_iH_i} - K_iH_iT^{K_iH_i-1}(T-t) + o((T-t)), \\ 1-\mathbb{C}ov\left(\bar{X}_i(t),\bar{X}_i(s)\right) &= \frac{1}{2^{K_i}T^{2K_iH_i}}|t-s|^{2K_iH_i} + o(|t-s|^{2K_iH_i}); \end{aligned}$$

(ii) if $\{X_i(t), t\in[0,T]\}$ is a sub-fBm, then

$$\begin{aligned} \sigma_i(t) &= \sqrt{2-2^{2H_i-1}}T^{H_i} - \sqrt{2-2^{2H_i-1}}H_iT^{H_i-1}(T-t) + o((T-t)), \\ 1-\mathbb{C}ov\left(\bar{X}_i(t),\bar{X}_i(s)\right) &= \frac{1}{2(2-2^{2H_i-1})T^{2H_i}}|t-s|^{2H_i} + o(|t-s|^{2H_i}); \end{aligned}$$

Additionally, the process $\{X_i(t), t\in[0,T]\}$ satisfies the condition of Assumption A2 for some positive $\delta,\mathbb{C}$, and $\gamma_i = 2K_iH_i$ and $H_i/2$ for bi-fBm and sub-fBm, respectively.

Proof of Corollary 3.2 The claim follows from Theorem 3.1 and Lemma 6.1, where $\beta := \beta_1 = \beta_2 = \cdots = \beta_n = 1$, $A_i = K_iH_iT^{K_iH_i-1}$, $\alpha_i = 2K_iH_i$ and $D_i = \frac{1}{2^{K_i}T^{2K_iH_i}}$ for the bi-fBm; $\beta := \beta_1 = \beta_2 = \cdots = \beta_n = 1$, $A_i = \sqrt{2-2^{2H_i-1}}H_iT^{H_i-1}$, $\alpha_i = 2H_i$ and $D_i = \frac{1}{2(2-2^{2H_i-1})T^{2H_i}}$ for the sub-fBm. □

Proof of Theorem 4.1 We first give the proof of the second tail equivalence of (4.14). It is easy to see that

$$\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t))>u\right) \le \mathbb{P}\left(\sup_{t\in[0,T]}U(t)+\sup_{t\in[0,T]}(-c(t))>u\right) \tag{6.17}$$

and thus

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t))>u\right)}{1-F_1(u)} \le 1. \tag{6.18}$$

Further we can write

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t))>u\right) &\geq \mathbb{P}\left(\sup_{t\in[0,T]}U(t)-\sup_{t\in[0,T]}c(t)>u\right)\\
&\geq \mathbb{P}\left(\sup_{t\in[0,T]}U(t)-\sup_{t\in[0,T]}c(t)>u,\ \sup_{t\in[0,T]}c(t)\leq d(u)\right)\\
&\geq \mathbb{P}\left(\sup_{t\in[0,T]}U(t)>u+d(u)\right)\mathbb{P}\left(c(T)\leq d(u)\right),
\end{aligned}$$

which together with (4.13) yields

$$\liminf_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t))>u\right)}{1-F_1(u)} \geq 1,$$

and thus the second tail equivalence of (4.14) is established. Since $F_1\in\mathcal{H}$, it follows using similar arguments and Theorem 2.13 in Foss et al. (2011) that

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t)+X(t))>u\right)}{1-F_1(u)} \leq \liminf_{u\to\infty}\frac{1-(1-\psi(\cdot,c,T))*F_2(u)}{\psi(u,c,T)}\frac{\psi(u,c,T)}{1-F_1(u)} = 1,$$

where $(1-\psi(\cdot,c,T))*F_2(u)$ denotes the convolution of distributions $1-\psi(u,c,T)$ and $F_2(u)$. Moreover

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t)+X(t))>u\right) &\geq \mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t))>u+d(u)\right)\mathbb{P}\left(\sup_{t\in[0,T]}(-X(t))\leq d(u)\right)\\
&\sim \mathbb{P}\left(\sup_{t\in[0,T]}U(t)>u+d(u)\right)\mathbb{P}\left(\sup_{t\in[0,T]}(-X(t))\leq d(u)\right)
\end{aligned}$$

as $u\to\infty$, which together with (4.13) yields

$$\liminf_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t)+X(t))>u\right)}{1-F_1(u)} \geq 1.$$

Consequently,

$$\mathbb{P}\left(\sup_{t\in[0,T]}(U(t)-c(t)+X(t))>u\right) \sim 1-F_1(u)$$

as $u\to\infty$, and thus the claim follows. □

Proof of Corollary 4.2 By Theorem 4.1 of Albin and Sundén (2009) $U(T)\in\mathcal{L}$ and $\sup_{t\in[0,T]}U(t)\in\mathcal{L}$ are equivalent. Consequently, the claim follows applying Theorem 3.1 and Theorem 4.1. □

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