Extremal behavior of squared Bessel processes attracted by the Brown-Resnick process

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Abstract

The convergence of properly time-scaled and normalized maxima of independent standard Brownian motions to the Brown-Resnick process is well-known in the literature. In this paper, we study the extremal functional behavior of non-Gaussian processes, namely squared Bessel processes and scalar products of Brownian motions. It is shown that maxima of independent samples of those processes converge weakly on the space of continuous functions to the Brown-Resnick process.

Keywords: Bessel process, Brown-Resnick process, extreme value theory, functional convergence

1. Introduction

The study of Gaussian processes, their suprema and sojourns has been of interest to researchers for quite some time; see the excellent monographs by Leadbetter et al. [25], Berman [4], Lifshits [26], Piterbarg [28], Adler and Taylor [1], Azaïs and Wschebor [3] and Yakir [32] for a detailed overview. These studies involve investigations of the asymptotic behavior of the maximum of a Gaussian (and sometimes non-Gaussian) process over a specific set under time and space scalings. On the other hand, in spatial extreme value theory, the main focus is on *pointwise maxima* of independent processes representing regular measurements of an environmental quantity, for instance.

Suppose a large number, n, of particles start at the origin and move along the trajectories of independent Brownian motions in an m-dimensional Euclidean space. Denote by $M_n(t)$, $t \geq 0$, the maximal squared displacement from the origin of those n particles at time t. It is well-known that for a fixed $t > 0$ and suitable normalizing sequences $a_n > 0$, $b_n \in \mathbb{R}$, we have the weak convergence

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_n(t) - b_n t}{a_n t} \leq x\right) = \exp(-\exp(-x)), \qquad x \in \mathbb{R}, \tag{1}$$

see e.g., [11, p.156]. In this paper we are interested in the functional convergence of the quantity in (1) on the space of continuous functions. In the one-dimensional case, Brown and Resnick [6] showed that the

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functional limit is given by a stationary, max-stable process. This Brown-Resnick process and its generalizations in Kabluchko et al. [24] and Kabluchko [23] are now well-known in extreme value theory and have recently found importance as models for spatial extreme weather events; see Davis et al. [7], Davison et al. [8], Engelke et al. [14].

The finite-dimensional distributions of a Brown-Resnick process can be naturally identified as the so-called Hüsler-Reiss distributions introduced in Hüsler and Reiss [22] which appear as the limits of maxima of a triangular array of Gaussian random vectors. Those distributions arise even in more general, non-Gaussian settings, as shown in Hashorva [16] and Hashorva et al. [17]. In fact the latter paper provides conditions for the weak convergence of maxima of independent, multivariate chi-square random vectors to the Hüsler-Reiss distribution. Such an observation naturally points us towards the question whether there are some non-Gaussian processes whose maxima are attracted by the Brown-Resnick process under appropriate linear scaling.

This is the principal focus of our paper which is organized as follows. In Section 2 we introduce necessary notation, recall the definition of Brown-Resnick processes and provide the two main theorems (Subsection 2.1). They state the functional convergence of maxima of independent (weighted) squared Bessel processes and, furthermore, it is shown that the Brown-Resnick process also appears as the limit of maxima processes obtained by the scalar product of two independent, m-dimensional Brownian motions. Subsection 2.2 gives a sketch of the proof of these results. The main lemma, which might be of some independent interest, and the rigorous proofs of the theorems are relegated to Section 3. Section 4 concludes the paper.

2. Extremal behavior of squared Bessel processes and Brownian scalar product processes

In the sequel, for $T > 0$ we denote by $C[0, T]$ and $C[0, \infty)$ the space of real-valued continuous functions on $[0, T]$ and $[0, \infty)$, respectively, equipped with the topology of uniform convergence (on bounded intervals).

Let $\{X_i, i \in \mathbb{N}\}$ be the points of a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}dx$, $x \in \mathbb{R}$, and let $\{B_i, i \in \mathbb{N}\}$ be independent standard Brownian motions on $[0, \infty)$ which are also independent of $\{X_i, i \in \mathbb{N}\}$. The original Brown-Resnick process presented in [6] is denoted by M_B and defined as

$$M_B(t) = \max_{i\in\mathbb{N}}\Big(X_i + B_i(t) - t/2\Big), \qquad t \geq 0. \tag{2}$$

More generally, for a centered Gaussian process $\{\eta(t), t \in \mathbb{R}\}$ with stationary increments and variance function $\sigma^2(t)$ the corresponding max-stable, stationary Brown-Resnick process M_η is defined by

$$M_\eta(t) = \max_{i\in\mathbb{N}}\Big(X_i + \eta_i(t) - \sigma^2(t)/2\Big), \qquad t \geq 0, \tag{3}$$

where $\eta_i, i \in \mathbb{N}$, are independent and identically distributed (i.i.d.) copies of η, see Kabluchko et al. [24], Kabluchko [23], Dombry and Eyi-Minko [10].

Originally, the standard Brown-Resnick process was derived as the limit of the maximum of i.i.d. Gaussian processes, namely Brownian motions and Ornstein-Uhlenbeck processes. Motivated by the recent findings of Hashorva et al. [17], in this section we show that two other classes of non-Gaussian processes lead to the same limit process M_B. More precisely, we investigate (weighted chi-square, or squared Bessel processes, and scalar-product processes related to standard Brownian motions.

2.1. Main results

We first state the two main results of this paper. To this end, let $\{B_{i,j}, i \in \mathbb{N}, 1 \leq j \leq m\}$ be independent standard Brownian motions on $[0, \infty)$ and denote by $\mathbf{B}_i = (B_{i,1}, \ldots, B_{i,m})$ the vector process. Furthermore, let Σ be a positive-definite $m \times m$ matrix with eigenvalues $1 = \lambda_1 \geq \cdots \geq \lambda_m$ in decreasing order, and let p be the multiplicity of the maximum eigenvalue $\lambda_1 = 1$, that is, $p = \#\{j : \lambda_j = 1, 1 \leq j \leq m\}$. By the eigen decomposition of Σ we can write $\Sigma = U^T \Lambda U$, where Λ is the diagonal matrix with the eigenvalues of Σ and U is the orthogonal matrix of its eigenvectors. Define for $i \in \mathbb{N}$ the (weighted) squared Bessel process of dimension $m \geq 1$ as

$$\xi_i(t) = \mathbf{B}_i^T(t)\Sigma\mathbf{B}_i(t) = \|\Lambda^{1/2}U\mathbf{B}_i(t)\|_2^2, \qquad t \geq 0, \tag{4}$$

where $\|x\|_2$ denotes the Euclidean norm of $x \in \mathbb{R}^m$. Set $C_{p,m} := \prod_{i=p+1}^m \frac{1}{\sqrt{1-\lambda_i}}$ and $C_{m,m} := 1$. It follows easily from Lemma 4.1 in Appendix that for constants a_n, b_n defined by

$$a_n = 2, \quad b_n = 2\ln n + (m-2)\ln(\ln n) - 2\ln(\Gamma(m/2)/C_{p,m}), \quad n \geq 1, \tag{5}$$

the maximum $M_{n,\xi}(t) = \max\{\xi_1(t), \ldots, \xi_n(t)\}$ for any fixed $t > 0$ satisfies

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_{n,\xi}(t) - b_n t}{a_n t} \leq x\right) = \exp(-\exp(-x)), \qquad x \in \mathbb{R}. \tag{6}$$

In their paper, Hashorva et al. [17] prove that the normalized maxima of independent chi-square random vectors converge to the Hüsler-Reiss distribution [22] which are the finite dimensional distributions of the Brown-Resnick processes M_η defined above. On the other hand, Brown and Resnick [6] showed that the rescaled maxima of an independent sequence of rescaled Brownian motions tend to the Brown-Resnick process. Thus a Brown-Resnick limit for the maxima of squared Bessel processes is quite intuitive. The sequence of processes $M_{n,\xi}, n \geq 1$, is defined on $C[0,\infty)$, but weak convergence of $M_{n,\xi}$ holding on $C[0,T]$ for all $T > 0$ implies convergence on $C[0,\infty)$ and proving convergence on $C[0,T]$ is similar to proving it for $C[0,1]$; see Brown and Resnick [6]. For the sake of simplicity, we thus show weak convergence only on $C[0,1]$.
For $1 \leq i \leq n$, $n \in \mathbb{N}$, define the local, or rescaled, processes

$$\xi_{i,n}(t) = \frac{\xi_i(1 + t/b_n) - b_n(1 + t/b_n)}{2}, \qquad t \geq 0. \tag{7}$$

Our first result below shows the functional convergence of the maximum process $\max_{1\leq i\leq n} \xi_{i,n}$ to the standard Brown-Resnick process M_B.

Theorem 2.1. *We have the weak convergence, as $n \to \infty$,*

$$\max_{i=1,\ldots,n} \xi_{i,n}(t) \xrightarrow{d} M_B(t), \quad t \in [0,1]$$

on the space of continuous functions $C[0,1]$.

Remark 2.2. *a) The process $\left\{\Lambda^{1/2}U\mathbf{B}_i(t), t \geq 0\right\}$ has the same distribution as $\left\{\Lambda^{1/2}\mathbf{B}_i(t), t \geq 0\right\}$, since U is an orthogonal matrix and the law of the m-dimensional Brownian motion $\mathbf{B}_i$ is invariant with respect to orthogonal transformations. Consequently, we have the equality in distribution*

$$\left\{\|\Lambda^{1/2}U\mathbf{B}_i(t)\|_2^2, t \geq 0\right\} \stackrel{d}{=} \left\{\mathbf{B}_i^T(t)\Lambda\mathbf{B}_i(t), t \geq 0\right\} = \left\{\lambda_1 B_{i,1}^2(t) + \cdots + \lambda_m B_{i,m}^2(t), t \geq 0\right\}.$$

b) If $\Sigma = I$, the identity matrix, then the above theorem implies that the maximum over squared Bessel processes converges to the Brown-Resnick process, that is,

$$\max_{i=1,\ldots,n} \frac{B_{i,1}^2(1 + t/b_n) + \ldots + B_{i,m}^2(1 + t/b_n) - b_n(1 + t/b_n)}{2} \xrightarrow{d} M_B(t), \quad t \in [0,1].$$

Since Bessel processes are defined as the norm of multivariate Brownian motions, we shall investigate further the extremal behavior of the scalar product of two independent Brownian motion vector processes. Let therefore $\{B_{i,j}, \widetilde{B}_{i,j}, i \in \mathbb{N}, 1 \le j \le m\}$ be independent standard Brownian motions on $[0, \infty)$ and define for $i \in \mathbb{N}$

$$\gamma_i(t) = B_{i,1}(t)\widetilde{B}_{i,1}(t) + \ldots + B_{i,m}(t)\widetilde{B}_{i,m}(t), \quad t \in [0, \infty). \tag{8}$$

By Lemma 4.1 in the Appendix it follows that for constants a_n^*, b_n^* defined by

$$a_n^* = 1, \quad b_n^* = \ln n + (m/2 - 1)\ln(\ln n) - (m/2 - 1)\ln 2 - \ln\Gamma(m/2), \quad n \ge 1 \tag{9}$$

the maximum process $M_{n,\gamma}(t) = \max\{\gamma_1(t), \ldots, \gamma_n(t)\}$ for a fixed $t > 0$ satisfies

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{M_{n,\gamma}(t) - b_n^* t}{a_n^* t} \le x\right) = \exp\left(-\exp(-x)\right), \qquad x \in \mathbb{R}. \tag{10}$$

Note in passing that a_n^*, b_n^* are however different than in the case of squared Bessel processes. Similarly as above we define for $1 \le i \le n$ the local processes

$$\gamma_{i,n}(t) = \gamma_i\left(1 + t/(2b_n^*)\right) - b_n^*\left(1 + t/(2b_n^*)\right), \qquad t \ge 0. \tag{11}$$

We have the following result for the convergence of $\max_{1\le i\le n} \gamma_{i,n}$, as $n \to \infty$.

Theorem 2.3. *For $n \to \infty$, we have the weak convergence*

$$\max_{i=1,\ldots,n} \gamma_{i,n}(t) \overset{d}{\to} M_B(t), \quad t \in [0, 1]$$

on the space of continuous functions $C[0, 1]$.

2.2. Sketch of the proof

In this subsection we outline the main steps for proving Theorem 2.1 with Σ being the identity matrix. The proof for general Σ and of Theorem 2.3 are similar. The details can be found in the next section.
Let us first remark that the space $C[0, 1]$ of continuous functions is not locally compact. This fact prevents us from applying the standard theory for Poisson point processes in extreme value theory. In particular, [30, Theorem 5.3] is not applicable for Poisson point processes on the space $C[0, 1]$. We thus rely on a similar technique as in the proof of Theorem 17 in Kabluchko et al. [24] in order to show negligibility of lower order statistics.

The key idea is to represent the process (7) in the following way. For $i \in \mathbb{N}$ and $1 \le j \le m$, write

$$B_{i,j}\left(1 + t/b_n\right) \overset{d}{=} B_{i,j}(1) + \frac{1}{\sqrt{b_n}} B_{i,j}^*(t), \quad t \ge 0, \tag{12}$$

where $\{B_{i,j}^*(t), i \in \mathbb{N}, 1 \le j \le m\}$ are independent standard Brownian motions being further independent of $\{B_{i,j}(1), i \in \mathbb{N}, 1 \le j \le m\}$. Naturally we denote $\mathbf{B}_i^* = (B_{i,1}^*, \ldots, B_{i,m}^*)$. Plugging (12) into the definition of $\xi_{i,n}$ (with $\Sigma = I$), we thus have

$$\begin{aligned}\xi_{i,n}(t) &\overset{d}{=} \frac{\sum_{j=1}^m (B_{i,j}(1))^2 - b_n}{2} + \left(\frac{1}{\sqrt{b_n}} \sum_{j=1}^m B_{i,j}(1) B_{i,j}^*(t) - \frac{t}{2}\right) + \frac{1}{2b_n} \sum_{j=1}^m (B_{i,j}^*(t))^2 \\ &=: X_{i,n} + R_{i,n}(t) - t/2 + \delta_{i,n}(t), \quad t \in [0, 1]. \end{aligned} \tag{13}$$

In order to prove that the pointwise maximum of n copies of $\xi_{i,n}$ converges to the Brown-Resnick process M_B, as $n \to \infty$, the following facts will either be used or established.

- The error term $\{\delta_{i,n}(t), t \in [0,1]\}$ becomes uniformly small, as $n \to \infty$. Therefore it does not affect the maximum process (cf., Corollary 3.4 below).
- The collection of random variables $\{X_{i,n}, 1 \leq i \leq n\}$ converges to a Poisson point process $\{X_i, i \in \mathbb{N}\}$ on $\mathbb{R}$ with intensity measure $e^{-x}dx$ which is used in the definition of M_B in (2). This fact is well-known from univariate extreme value theory [11].
- Among the processes $\{\xi_{i,n}, 1 \leq i \leq n\}$ for computing $\max_{i=1,\dots,n} \xi_{i,n}$, only the ones where $X_{i,n}$ is in a compact interval contribute to the maximum on the space $C[0,1]$. In fact, we show that conditional on $X_{i,n}$ being outside this compact set, the contribution of $\{R_{i,n}(t) - t/2, t \in [0,1]\}$ is asymptotically negligible (cf. Lemma 3.3 below).
- On the other hand, $\{R_{i,n}(t) - t/2, t \in [0,1]\}$ conditional on $X_{i,n}$ being bounded in the compact interval, converges weakly to the drifted Brownian motion $\{B(t) - t/2, t \in [0,1]\}$ as in the definition of M_B in (2) (cf. Lemma 3.1 below).

In summary, these points show that the maximum over n processes in (13) converges (as $n \to \infty$) to the maximum of the Poisson point process of the sum of the $\{X_i, i \in \mathbb{N}\}$ and the drifted Brownian motions. This is nothing else than the definition of the Brown-Resnick process (2).

3. Proofs

3.1. Preliminary lemmas

We first prove the following main lemma, which is of some independent interest as a tool for showing weak convergence to the Brown-Resnick process. It gives explicit conditions under which the reasoning of Subsection 2.2 can be made rigorous. For instance, the conditions apply to the framework in Brown and Resnick [6] and thus our lemma implies their weak convergence results. In the following, for a continuous function $f \in C[0,1]$ we denote $\|f\| = \sup_{t\in[0,1]} |f(t)|$.

Lemma 3.1. *For $n \in \mathbb{N}$, $1 \leq i \leq n$, let the following triangular arrays be given, where the elements within the rows of each array are i.i.d.:*

1. *Identically distributed random variables $Y_{i,n}$ satisfying*

$$\mathbb{P}(Y_{1,1} > u) = (1 + o(1))Ku^{\beta}e^{-cu}du, \quad u \to \infty, \tag{14}$$

with constants $K, c > 0$, $\beta \in \mathbb{R}$. By Theorem 3.3.26 in Embrechts et al. [11], this implies that

$$\lim_{n\to\infty} n\mathbb{P}(X_{1,n} > s) = e^{-s}, \quad \forall s \in \mathbb{R}, \tag{15}$$

where $X_{i,n} = a_n^{-1}(Y_{i,n} - b_n)$ and

$$a_n = c^{-1}, \quad b_n = c^{-1}\left(\ln n + \beta \ln(c^{-1}\ln n) + \ln K\right), \quad n \geq 1. \tag{16}$$

Assume further that for all large r and any $p > 0$

$$\limsup_{n\to\infty} n \int_{-b_n/(2a_n)}^{-r} e^{-px^2}\mathbb{P}(X_{i,n} \in dx) < \infty. \tag{17}$$

2. *Stochastic processes* $\{R_{i,n}(t), t \in [0,1]\}$*, such that the vector* $(X_{i,n}, R_{i,n}(\cdot))$ *has the same distribution as* $(X_{i,n}, \phi_{i,n} W_{i,n}(\cdot))$*, where* $W_{i,n} \sim \{W(t), t \in [0,1]\}$ *are standard Brownian motions independent of the* $X_{i,n}$*, and* $\phi_{i,n}$ *are positive random variables, independent of* $W_{i,n}$ *such that for some* $q > 0$

$$\lim_{n\to\infty} n\mathbb{P}(\phi_{1,n} > q) = 0 \tag{18}$$

and for any compact set $K \subset \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}(|1-\phi_{1,n}| > \epsilon \,|\, X_{1,n} \in K) = 0, \quad \forall \epsilon > 0. \tag{19}$$

3. *Stochastic processes* $\{\delta_{i,n}(t), t \in [0,1]\}$*, independent of* $X_{i,n}$*, such that*

$$\lim_{n\to\infty} \mathbb{P}(\|\delta_{1,n}\| > \epsilon) \quad = 0, \quad \forall \epsilon > 0, \tag{20}$$

$$\lim_{n\to\infty} n\mathbb{P}(\|\delta_{1,n}\| > C) \quad = 0, \; \textit{for some } C > 0. \tag{21}$$

Then, we have the weak convergence

$$\eta_n(t) := \max_{i=1,\dots,n} \zeta_{i,n}(t) := \max_{i=1,\dots,n} \Big(X_{i,n} + R_{i,n}(t) - t/2 + \delta_{i,n}(t) \Big) \xrightarrow{d} M_B(t), \quad t \in [0,1] \tag{22}$$

on the function space $C[0,1]$*, where* $\{M_B(t), t \in [0,1]\}$ *is the original Brown-Resnick process given by* (2)*.*

Remark 3.2. *If* (14) *holds, then condition* (17) *is satisfied if* $Y_{1,1}$ *possesses a density* h *such that for some* $c > 0$

$$\mathbb{P}(Y_{1,1} > u) = (1 + o(1))h(u)/c, \quad u \to \infty.$$

We will prove first the following useful result.

Lemma 3.3. *With the notation and under the assumptions of Lemma 3.1, for any* $\epsilon > 0$ *we can find constants* $R, N > 0$ *such that for any* $r > R$ *and* $n > N$*, we have*

$$\mathbb{P}(A_n) := \mathbb{P}\left(\exists t \in [0,1] : \eta_n(t) \neq \max_{i \in \{1,\dots,n\}, |X_{i,n}| < r} \zeta_{i,n}(t) \right) \leq \epsilon. \tag{23}$$

Proof. We apply a similar technique as in the proof of Theorem 17 in Kabluchko et al. [24]. First note that

$$A_n \subset C_n \cup D_n \cup (A_n \setminus [C_n \cap D_n]),$$

where for some $r, r_1 > 0$

$$C_n = \left\{ \inf_{t\in[0,1]} \eta_n(t) < -r_1 \right\}, \qquad D_n = \bigcup_{i=1}^n \{X_{i,n} > r\}.$$

Clearly by (15), for N and R large enough it holds that $\mathbb{P}(D_n) = n\mathbb{P}(X_{1,n} > r) < \epsilon$, for any $n > N$, $r > R$. Moreover, note that $C_n \subset \bigcap_{i=1}^n F_{i,n}^c$, where

$$F_{i,n} = \left\{ X_{i,n} \in [-r, r], \inf_{t\in[0,1]} \Big(R_{i,n}(t) - t/2 + \delta_{i,n}(t) \Big) \geq r - r_1 \right\}.$$

In view of assumption (19), for

$$\Delta_{i,n}(t) := W_{i,n}(t)(\phi_{i,n} - 1), \qquad t \in [0,1] \tag{24}$$

we obtain for any $\delta > 0$

$$\begin{aligned}\mathbb{P}(\|\Delta_{i,n}\| > \delta | X_{i,n} \in [-r, r]) &\leq \mathbb{P}(\|W_{i,n}\| > \delta/\tau) + \mathbb{P}(|(\phi_{i,n} - 1)| > \tau | X_{i,n} \in [-r, r]) \\ &\leq \overline{\Phi}(\delta/\tau) + \epsilon/2 \\ &\leq \epsilon \end{aligned} \tag{25}$$

for sufficiently small $\tau > 0$ with $\overline{\Phi}$ the survival function of an $N(0, 1)$ random variable. Further, using assumption 2 of Lemma 3.1, (20) and (25), we obtain for any $\delta > 0$

$$\begin{aligned}&\mathbb{P}\left(\inf_{t\in[0,1]}\Big(R_{i,n}(t) - t/2 + \delta_{i,n}(t)\Big) < r - r_1 \Big| X_{i,n} \in [-r, r]\right) \\ &\quad\leq \mathbb{P}(\|\delta_{i,n}\| > \delta) + \mathbb{P}\left(\inf_{t\in[0,1]}\Big(W_{i,n}(t) - t/2 + \Delta_{i,n}(t)\Big) < r - r_1 + \delta \Big| X_{i,n} \in [-r, r]\right) \\ &\quad\leq \mathbb{P}(\|\delta_{i,n}\| > \delta) + \mathbb{P}(\|\Delta_{i,n}\| > \delta | X_{i,n} \in [-r, r]) + \mathbb{P}\left(\inf_{t\in[0,1]} W_{i,n}(t) < r - r_1 + 2\delta + 1/2\right) \\ &\quad\leq \frac{1}{2},\end{aligned}$$

for n and r_1 sufficiently large. Thus, by (15)

$$\mathbb{P}(F_{i,n}) \geq \frac{1}{2}\,\mathbb{P}(X_{i,n} \in [-r, r]) \geq \frac{r}{n} + o(1/n), \quad n \to \infty$$

for r large enough and uniformly in $i \in \mathbb{N}$, and consequently

$$\mathbb{P}(C_n) \; \leq (1 - \mathbb{P}(F_{1,n}))^n \leq \left(1 - \frac{r}{n} + o(1/n)\right)^n \leq 2e^{-r} < \epsilon$$

for r and n large. It remains to show that $\mathbb{P}(A_n \setminus (C_n \cap D_n))$ becomes small. To this end, define events

$$E_{i,n} = \Big\{X_{i,n} < -r, \sup_{t\in[0,1]} \zeta_{i,n}(t) > -r_1\Big\}$$

and note that $A_n \setminus (C_n \cap D_n)$ is a subset of the union $\bigcup_{i=1}^{n} E_{i,n}$. Let $C > 0$ be the constant in (21) and recall the stochastic representation of $R_{i,n}$ from assumption 2. Then

$$\mathbb{P}(E_{i,n}) \leq \mathbb{P}(\|\delta_{i,n}\| > C) + \mathbb{P}\left(X_{i,n} < -r, \sup_{t\in[0,1]}\Big(X_{i,n} + \phi_{i,n} W_{i,n}(t) - t/2\Big) > -r_1 - C\right). \tag{26}$$

For n large enough, (21) implies that the first summand is bounded by ϵ/n. A coupling argument yields that the second summand can be bounded from above by

$$\mathbb{P}(\phi_{i,n} > q) + \mathbb{P}\left(X_{i,n} < -r, \sup_{t\in[0,1]}\Big(X_{i,n} + qW_{i,n}(t)\Big) > -r_1 - C\right),$$

where again the first summand is bounded by ϵ/n by (18). We can write

$$\begin{aligned}\mathbb{P}\left(X_{i,n} < -r, \sup_{t\in[0,1]}\Big(X_{i,n} + qW_{i,n}(t)\Big) > -r_1 - C\right) &\leq \int_{-\infty}^{-b_n/(2a_n)} \mathbb{P}\left(\sup_{t\in[0,1]} W_{i,n}(t) > \frac{-r_1 - x - C}{q} \;\Big|\; X_{i,n} = x\right)\mathbb{P}(X_{i,n} \in dx) \\ &\quad+ \int_{-b_n/(2a_n)}^{-r} \mathbb{P}\left(\sup_{t\in[0,1]} W_{i,n}(t) > \frac{-r_1 - x - C}{q} \;\Big|\; X_{i,n} = x\right)\mathbb{P}(X_{i,n} \in dx).\end{aligned} \tag{27}$$

Recall the estimate

$$\mathbb{P}\left(\sup_{t\in[0,1]} W_{i,n}(t) > u\right) \leq 2\overline{\Phi}(u) \leq e^{-u^2/2}$$

for large $u > 0$. Choosing $r > 2(r_1 + C)$ implies $(-r_1 - x - C)/q > -x/(2q)$ for all $x < -r$ and thus the first term in (27) is bounded from above by

$$\mathbb{P}\left(\sup_{t\in[0,1]} W_{i,n}(t) > \frac{-r_1 - C + b_n/(2a_n)}{q}\right) \leq e^{-(b_n/(4a_n q))^2/2} < \epsilon/n,$$

with a_n and b_n defined in (16). Similarly, the second term in (27) is bounded from above by

$$\int_{-b_n/(2a_n)}^{-r} e^{-x^2/(8q)}\mathbb{P}(X_{i,n} \in dx) \leq \epsilon/n,$$

as a consequence of (17). Collecting all parts together yields

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_{i,n}\right) \leq nP\left(E_{i,n}\right) \leq \epsilon$$

and thus $\mathbb{P}(A_n) \leq 3\epsilon$ for all $n > N$ and $r > R$ with N, R large enough. □

Corollary 3.4. *With the notation and under the assumptions of Lemma 3.1, for any $\epsilon > 0$ we can find an $N \in \mathbb{N}$, such that for all $n > N$ we have*

$$\mathbb{P}\left(\sup_{t\in[0,1]} |\eta_n(t) - \tilde{\eta}_n(t)| > \epsilon\right) \leq \epsilon,$$

where

$$\tilde{\eta}_n(t) = \max_{i=1,\dots,n}\Big(X_{i,n} + R_{i,n}(t) - t/2\Big), \quad t \in [0,1]. \tag{28}$$

Proof. For any $\epsilon > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,1]} |\eta_n(t) - \tilde{\eta}_n(t)| > \epsilon\right)\\
&\leq \mathbb{P}\left(\exists t \in [0,1] : \eta_n(t) \neq \max_{i\in\{1,\dots,n\},|X_{i,n}|<r}\Big(X_{i,n} + R_{i,n}(t) - t/2 + \delta_{i,n}(t)\Big)\right)\\
&\quad +\mathbb{P}\left(\exists t \in [0,1] : \tilde{\eta}_n(t) \neq \max_{i\in\{1,\dots,n\},|X_{i,n}|<r}\Big(X_{i,n} + R_{i,n}(t) - t/2\Big)\right)\\
&\quad +\mathbb{P}\left(\exists i \in \{1,\dots,n\} : |X_{i,n}| < r, \|\delta_{i,n}\| > \epsilon\right)\\
&\leq \epsilon/3 + \epsilon/3 + n\mathbb{P}\left(|X_{i,n}| < r\right)\mathbb{P}(\|\delta_{i,n}\| > \epsilon)\\
&\leq \epsilon,
\end{aligned}$$

where for the first and second summand r and N can be chosen according to Lemma 3.3. The last inequality then follows from assumptions (15) and (20). □

Proof of Lemma 3.1. The proof will consist of two steps. First, we establish convergence of the finite dimensional margins in (22), and, second, we show that the sequence of probability measures $\{\eta_n\}_{n\in\mathbb{N}}$ on $C[0,1]$ is tight. In fact, by Corollary 3.4, $\{\eta_n\}_{n\in\mathbb{N}}$ converges weakly on $C[0,1]$ if and only if the sequence of probability measures $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$ in (28) converges weakly on $C[0,1]$, and, in this case, the limits are equal. In the sequel we will therefore consider $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$ instead of $\{\eta_n\}_{n\in\mathbb{N}}$.
For the first part, let $\mathbf{t} = (t_1,\dots,t_m) \in [0,1]^m$ and $(y_1,\dots,y_m) \in \mathbb{R}^m$ be fixed. It follows from Lemma 4.1.3 in Falk et al. [15] that it suffices to proof the convergence

$$\lim_{n\to\infty} n\mathbb{P}(\forall j : X_{1,n} + R_{1,n}(t_j) - t_j/2 > y_j) = \int_{\mathbb{R}} e^{-y}\mathbb{P}\left(\forall j : W(t_j) - t_j/2 > y_j - y\right) dy, \tag{29}$$

where $\{W(t), t \in [0,1]\}$ is a standard Brownian motion. To this end, we recall the definition of $\Delta_{1,n}$ in (24) and, for clarity, denote by $\{\overline{W}_{1,n}(t) = W_{1,n}(t) - t/2, t \in [0,1]\}$ the drifted process. For arbitrary $\delta, r > 0$ we obtain the estimate

$$\begin{aligned}\mathbb{P}(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) + \Delta_{1,n}(t_j) > y_j) \quad \le \quad & \mathbb{P}(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) > y_j - \delta, |X_{1,n}| < r) \\ & + \mathbb{P}(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) + \Delta_{1,n}(t_j) > y_j, |X_{1,n}| > r) \\ & + \mathbb{P}(\|\Delta_{1,n}\| > \delta, |X_{1,n}| < r). \end{aligned} \tag{30}$$

Furthermore, as $n \to \infty$, the first summand fulfills

$$\begin{aligned} n\mathbb{P}(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) > y_j - \delta, |X_{1,n}| < r) &= \int_{-r}^{r} \mathbb{P}(\forall j : \overline{W}_{1,n}(t_j) > y_j - y - \delta) n\mathbb{P}(X_{1,n} \in dy) \\ &\to \int_{-r}^{r} e^{-y}\mathbb{P}(\forall j : \overline{W}_{1,1}(t_j) > y_j - y - \delta)\, dy, \end{aligned}$$

since by (15), $n\mathbb{P}(X_{1,n} \in dy)$ converges weakly to $e^{-y}dy$, as $n \to \infty$. Now, in view of the calculations following (26) for the second summand in (30), and (25) and (15) for the third summand in (30), we have

$$\begin{aligned} &\limsup_{n\to\infty} nP(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) + \Delta_{1,n}(t_j) > y_j) \\ &\qquad \le \lim_{r\to\infty} \int_{-r}^{r} e^{-y}\mathbb{P}(\forall j : \overline{W}_{1,1}(t_j) > y_j - y - \delta)\, dy \\ &\qquad\quad + \lim_{r\to\infty} \limsup_{n\to\infty} n\mathbb{P}(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) + \Delta_{1,n}(t_j) > y_j, |X_{1,n}| > r) \\ &\qquad\quad + \lim_{r\to\infty} \limsup_{n\to\infty} n\mathbb{P}(\|\Delta_{1,n}\| > \delta, |X_{1,n}| < r) \\ &\qquad = \int_{\mathbb{R}} e^{-y}\mathbb{P}(\forall j : \overline{W}_{1,1}(t_j) > y_j - y - \delta)\, dy. \end{aligned} \tag{31}$$

Similarly, we can show that

$$\liminf_{n\to\infty} nP(\forall j : X_{1,n} + \overline{W}_{1,n}(t_j) + \Delta_{1,n}(t_j) > y_j) \ge \int_{\mathbb{R}} e^{-y}\mathbb{P}(\forall j : \overline{W}_{1,1}(t_j) > y_j - y + \delta)\, dy. \tag{32}$$

Since $\delta > 0$ was arbitrary, (29) follows from (31) and (32) as $\delta \searrow 0$, and thus the convergence of finite dimensional margins.

In order to show the tightness of the sequence $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$ we note that the sequence $\{\tilde{\eta}_n(0)\}_{n\in\mathbb{N}}$ is tight since it equals $\{\max_{i=1,\dots,n} X_{i,n}\}_{n\in\mathbb{N}}$ which converges to the Gumbel distribution by (15). For a function $g \in C[0,1]$

and any $\kappa > 0$, we define the modulus of continuity $\omega_\kappa(g)$

$$\omega_\kappa(g) = \sup_{s,t\in[0,1],|s-t|\le\kappa} |g(s) - g(t)|.$$

By Theorem 7.3 in Billingsley [5] it suffices to find for any $\epsilon, \alpha > 0$ a $\kappa > 0$ and $N \in \mathbb{N}$ such that

$$\mathbb{P}(\omega_\kappa(\tilde{\eta}_n) > \alpha) < \epsilon, \quad n > N.$$

By choosing $\kappa > 0$ small enough, we get for any $r > 0$

$$\begin{aligned} \mathbb{P}(\omega_\kappa(X_{1,n} + \overline{W}_{1,n} + \Delta_{i,n}) > \alpha \,|\, X_{i,n} \in [-r, r]) &\le \mathbb{P}(\omega_\kappa(\overline{W}_{1,n}) > \alpha/2) + \mathbb{P}(\|\Delta_{1,n}\| > \alpha/2 \,|\, X_{i,n} \in [-r, r]) \\ &\le \epsilon/2 \end{aligned} \tag{33}$$

for all $n > N$ with N large enough, because of the fact that $\overline{W}_{1,n}$ is independent of $X_{1,n}$ and its distribution does not depend on n, and condition (25). We proceed by noting that for any n, we have

$$\{\omega_\kappa(\tilde{\eta}_n) > \alpha\} \subset \left(\{\omega_\kappa(\tilde{\eta}_n) > \alpha\} \cap \tilde{A}_n^C\right) \cup \tilde{A}_n \subset \left(\bigcup_{i=1}^n G_{i,n},\right) \cup \tilde{A}_n, \tag{34}$$

where

$$\tilde{A}_n = \left\{\exists t \in [0,1] : \tilde{\eta}_n(t) \neq \max_{i\in\{1,\dots,n\},|X_{i,n}|<r} \left(X_{i,n} + R_{i,n}(t) - t/2\right)\right\}$$

and

$$G_{i,n} = \left\{X_{i,n} \in [-r, r], \omega_\kappa(X_{i,n} + \overline{W}_{1,n} + \Delta_{i,n}) > \alpha\right\}.$$

Conditioning we obtain for any $\epsilon' > 0$

$$\mathbb{P}(G_{i,n}) = \mathbb{P}(\omega_\kappa(X_{i,n} + \overline{W}_{1,n} + \Delta_{i,n}) > \alpha \,|\, X_{i,n} \in [-r, r])\mathbb{P}(X_{i,n} \in [-r, r]) \le \epsilon' \, \mathbb{P}(X_{i,n} \in [-r, r])$$

by (33) and $\kappa > 0$ small enough, for any $n > N$. Thus, since by (15), $\mathbb{P}(X_{i,n} \in [-r, r])$ is of order $1/n$, we have for any $n > N$ with N large enough

$$\mathbb{P}\left(\bigcup_{i=1}^n G_{i,n}\right) \le n\mathbb{P}(G_{i,n}) < \epsilon/2. \tag{35}$$

Consequently, (34) together with (23) and (35) implies $\mathbb{P}(\omega_\kappa(\tilde{\eta}_n) > \alpha) < \epsilon$, for $n > N$, and hence the tightness of $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$. □

3.2. Proofs of Theorems 2.1 and 2.3

Proof of Theorem 2.1. For simplicity, we drop the index $1 \le i \le n$ in the proof, since all random objects are i.i.d. in this index. With Remark 2.2 and (12) in Subsection 2.2 we obtain the stochastic representation

$$\begin{aligned} \xi_n(t) &\overset{d}{=} \frac{\left(\mathbf{B}(1) + \sqrt{\frac{1}{b_n}}\mathbf{B}^*(t)\right)^T \Lambda \left(\mathbf{B}(1) + \sqrt{\frac{1}{b_n}}\mathbf{B}^*(t)\right) - b_n\,(1 + t/b_n)}{2} \\ &= \frac{\mathbf{B}(1)^T\Lambda\mathbf{B}(1) - b_n}{2} + \left(\frac{1}{\sqrt{b_n}}\mathbf{B}(1)^T\Lambda\mathbf{B}^*(t) - \frac{t}{2}\right) + \frac{1}{2b_n}\mathbf{B}^*(t)^T\Lambda\mathbf{B}^*(t) && (36) \\ &=: X_n + R_n(t) - t/2 + \delta_n(t), \quad t \in [0,1]. && (37) \end{aligned}$$

We check the assumptions of Lemma 3.1. By Lemma 4.1 in the Appendix, $Y_n := 2X_n + b_n$ satisfies for $u \to \infty$ (recall $C_{m,p} = \prod_{i=p+1}^{m} \frac{1}{\sqrt{1-\lambda_i}}$ for $p < m$ and $C_{m,m} = 1$)

$$\mathbb{P}(Y_1 > u) = \frac{C_{m,p}}{2^{p/2-1}\Gamma(p/2)} u^{p/2-1} \exp(-u/2)\Big(1 + O(1/u)\Big) = 2\mathbb{P}(Y_1 \in du)\Big(1 + O(1/u)\Big)$$

and hence assumption 1 of Lemma 3.1 holds (recall Remark 3.2).

A simple calculation with characteristic functions yields

$$\begin{aligned}\Big(\mathbf{B}(1)^T \Lambda \mathbf{B}(1), \mathbf{B}(1)^T \Lambda \mathbf{B}^*(\cdot)\Big) &\stackrel{d}{=} \Big(\mathbf{B}(1)^T \Lambda \mathbf{B}(1),\ \sqrt{\mathbf{B}(1)^T \Lambda^2 \mathbf{B}(1)}\, W_n(\cdot)\Big) \\ &\stackrel{d}{=} \left(\sum_{j=1}^{m} \lambda_j B_j^2(1),\ \sqrt{\sum_{j=1}^{m} \lambda_j^2 B_j^2(1)}\, W_n(t)\right),\end{aligned}$$

where $\{W_n(t) : t \in [0, 1]\}$ are i.i.d. standard Brownian motions, independent of $\mathbf{B}(1)$. Thus, for X_n and R_n in (37) we have the joint stochastic representation

$$(X_n, R_n) \stackrel{d}{=} (X_n, \phi_n W_n(\cdot)), \qquad \phi_n := \sqrt{\frac{\sum_{j=1}^{m} \lambda_j^2 B_j^2(1)}{b_n}}.$$

Assume next for simplicity that $p < m$, the case $p = m$ is easy. By Lemma 4.1 the tail asymptotics of $\sum_{j=1}^{m} \lambda_j^2 B_j^2(1)$ and $\sum_{j=1}^{m} \lambda_j B_j^2(1)$ are the same up to a positive constant. Hence, using (15) for any $q > 1$

$$\lim_{n\to\infty} n\mathbb{P}(\phi_{1,n} > q) = \lim_{n\to\infty} n\mathbb{P}\left(\sum_{j=1}^{m} \lambda_j^2 B_j^2(1) > b_n q^2\right) = 0.$$

For arbitrary $\epsilon, r > 0$ observe

$$\begin{aligned}\mathbb{P}(|1-\phi_n| > \epsilon | X_n \in [-r, r]) &= \mathbb{P}\left(\sum_{j=1}^{m} \lambda_j^2 B_j^2(1) \notin [b_n(1-\epsilon)^2, b_n(1+\epsilon)^2] \,\Big|\, \sum_{j=1}^{m} \lambda_j B_j^2(1) \in [b_n - 2r, b_n + 2r]\right) \\ &\le \mathbb{P}\left(\sum_{j=1}^{p} B_j^2(1) < b_n(1-\epsilon/2)^2 \Big| \sum_{j=1}^{m} \lambda_j B_j^2(1) \in [b_n - 2r, b_n + 2r]\right), \qquad (38)\end{aligned}$$

since

$$\sum_{j=1}^{p} B_j^2(1) \le \sum_{j=1}^{m} \lambda_j^2 B_j^2(1) \le \sum_{j=1}^{m} \lambda_j B_j^2(1).$$

Using the same arguments as in the proof of Theorem 1 in Hüsler et al. [21], we can show that (38) converges to 0, as $n \to \infty$. Thus, assumption 2 of Lemma 3.1 is fulfilled.

We note that δ_n in (37) is independent of X_n and for any $\epsilon > 0$

$$\mathbb{P}(\|\delta_n\| > \epsilon) = \mathbb{P}\left(\sup_{t\in[0,1]} \left(\mathbf{B}^*(t)^T \Sigma \mathbf{B}^*(t)\right) > 2b_n\epsilon\right) \to 0, \quad n \to \infty.$$

Moreover, for a $C > 1$, in view of the Piterbarg inequality given in Proposition 3.2 in Tan and Hashorva [31] (see also Theorem 8.1 in Piterbarg [28], or in Piterbarg [29]), we have for some positive constant λ

$$\begin{aligned} n\mathbb{P}(\|\delta_n\| > C) &= n\mathbb{P}\left(\sup_{t\in[0,1]}\left(\mathbf{B}^*(t)^T\Sigma\mathbf{B}^*(t)\right) > 2b_nC\right) \\ &\le n\mathbb{P}\left(\sup_{t\in[0,1]}\left(\mathbf{B}^*(t)^T I\mathbf{B}^*(t)\right) > 2b_nC\right) \\ &\le nb_n^{\lambda}e^{-b_nC} \\ &\to 0, \quad n\to\infty \end{aligned}$$

and thus assumption 3 of Lemma 3.1 holds, and the assertion of the theorem follows. □

Proof of Theorem 2.3. Again, for simplicity, we drop the index $1 \le i \le n$ in the proof. By the independent increment property of Brownian motion, we write for $1 \le j \le m$

$$B_j\left(1 + t/(2b_n^*)\right) \stackrel{d}{=} B_j(1) + \frac{1}{\sqrt{2b_n^*}}B_j^*(t), \quad \widetilde{B}_j\left(1 + t/(2b_n^*)\right) \stackrel{d}{=} \widetilde{B}_j(1) + \frac{1}{\sqrt{2b_n^*}}\widetilde{B}_j^*(t) \quad t \ge 0, \tag{39}$$

where $\{B_j^*(t), \widetilde{B}_j^*, 1 \le j \le m\}$ are independent standard Brownian motions being further independent of $\{B_j(1), \widetilde{B}_j, 1 \le j \le m\}$. Plugging (39) into the definition of γ_n in (11), we get

$$\begin{aligned} \gamma_n(t) &\stackrel{d}{=} \left(\sum_{j=1}^m B_j(1)\widetilde{B}_j(1) - b_n^*\right) + \left(\frac{1}{\sqrt{2b_n^*}}\sum_{j=1}^m B_j(1)\widetilde{B}_j^*(t) + \frac{1}{\sqrt{2b_n^*}}\sum_{j=1}^m \widetilde{B}_j(1)B_j^*(t) - \frac{t}{2}\right) + \frac{1}{2b_n^*}\sum_{j=1}^m B_j^*(t)\widetilde{B}_j^*(t) \\ &=: X_n + R_n(t) - t/2 + \delta_n(t), \quad t\in[0,1]. \end{aligned} \tag{40}$$

As above, we only have to check the assumptions of Lemma 3.1. By Lemma 4.1 in the Appendix, $Y_n := X_n + b_n^*$ satisfies for $u \to \infty$

$$\mathbb{P}(Y_1 > u) = (1 + o(1))\frac{1}{2^{m/2}\Gamma(m/2)}u^{m/2-1}\exp(-u) = (1 + o(1))\mathbb{P}(Y_1 \in du)$$

and hence assumption 1 of Lemma 3.1 holds (recall again Remark 3.2).

A simple calculation with characteristic functions yields for X_n and R_n in (40) the joint stochastic representation

$$(X_n, R_n) \stackrel{d}{=} (X_n, \phi_n W_n(\cdot)), \qquad \phi_n := \sqrt{\frac{\Psi_n}{2b_n}}, \qquad \Psi_n := \sum_{j=1}^m \left(B_j^2(1) + \widetilde{B}_j^2(1)\right),$$

where $\{W_n(t), t \in [0, 1]\}$ are i.i.d. standard Brownian motions, independent of the X_n. Clearly, since Ψ_n is chi-square distributed with $2m$ degrees of freedom, it holds for any $q > 1$ that

$$\begin{aligned} \lim_{n\to\infty} n\mathbb{P}(\phi_n > q) &= \lim_{n\to\infty} n\mathbb{P}\left(\Psi_n > 2b_nq^2\right) \\ &\le \lim_{n\to\infty} nK\exp(-b_nq^2) = 0, \end{aligned}$$

where $K > 0$ is a constant. Furthermore, for arbitrary $\epsilon, r > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(|1-\phi_n| > \epsilon | X_n \in [-r,r]\right) \qquad (41)\\
&= \mathbb{P}\left(\Psi_n \notin [2b_n(1-\epsilon)^2, 2b_n(1+\epsilon)^2] \,\Big|\, \sum_{j=1}^m B_j(1)\widetilde{B}_j(1) \in [b_n - r, b_n + r]\right)\\
&= \frac{\mathbb{P}\left(\Psi_n \notin [2b_n(1-\epsilon)^2, 2b_n(1+\epsilon)^2]\,,\, \sum_{j=1}^m B_j(1)\widetilde{B}_j(1) \in [b_n - r, b_n + r]\right)}{\mathbb{P}\left(\sum_{j=1}^m B_j(1)\widetilde{B}_j(1) \in [b_n - r, b_n + r]\right)}.
\end{aligned}$$

By Lemma 4.1, for large $n \in \mathbb{N}$ the denominator can be bounded by

$$\mathbb{P}\left(\sum_{j=1}^m B_j(1)\widetilde{B}_j(1) \in [b_n - r, b_n + r]\right) \geq K'\left((b_n - r)^{m/2-1}e^r - (b_n + r)^{m/2-1}e^{-r}\right)e^{-b_n} \qquad (42)$$

for some constant $K' > 0$. For the numerator we first note that

$$\Psi_n = \sum_{j=1}^m \left(B_j^2(1) + \widetilde{B}_j^2(1)\right) \geq 2\sum_{j=1}^m B_j(1)\widetilde{B}_j(1)$$

and thus for n large enough it suffices to consider

$$\begin{aligned}
&\mathbb{P}\left(\Psi_n > 2b_n(1+\epsilon)^2\,,\, 2\sum_{j=1}^m B_j(1)\widetilde{B}_j(1) \in [2b_n - 2r, 2b_n + 2r]\right)\\
&\leq \mathbb{P}\left(\sum_{j=1}^m \left(B_j(1) + \widetilde{B}_j(1)\right)^2 > 2b_n(1+\epsilon)^2 + 2b_n - 2r\right)\\
&\leq \mathbb{P}\left(2\chi_m^2 > 4b_n(1+\epsilon) - 2r\right)\\
&\leq K''(2b_n(1+\epsilon) - r)^{m/2-1}e^{-b_n(1+\epsilon)}, \qquad (43)
\end{aligned}$$

where χ_m^2 is a chi-square distribution with m degrees of freedom and $K'' > 0$ is a constant. From (42) and (43) it is now obvious, that the probability in (41) turns to 0, as $n \to \infty$. Thus, assumption 2 of Lemma 3.1 is fulfilled.

Note that δ_n in (40) is independent of X_n. For any $\epsilon > 0$

$$\begin{aligned}
\mathbb{P}(\|\delta_n\| > \epsilon) &= \mathbb{P}\left(\sup_{t\in[0,1]}\left|\sum_{j=1}^m B_j^*(t)\widetilde{B}_j^*(t)\right| > 2b_n\epsilon\right)\\
&\leq m\mathbb{P}\left(\sup_{t\in[0,1]}\left|B_1^*(t)\right| \sup_{t\in[0,1]}\left|\widetilde{B}_1^*(t)\right| > 2b_n\epsilon/m\right)\\
&\leq M\exp\left(-\frac{b_n\epsilon}{m}\right)\\
&= M' n^{-\epsilon/m}(\ln(n))^{-(m/2-1)\epsilon/m)} \to 0, \quad n \to \infty,
\end{aligned}$$

where the second inequality follows from Lemma 2.1 in [2] (see also Corollary 2.2 in [20]) and M, M' are positive constants. Clearly, for $C > m$ we further have

$$n\mathbb{P}(\|\delta_n\| > C) \to 0, \quad n \to \infty.$$

Thus assumption 3 of Lemma 3.1 holds, and the proof is complete. □

4. Conclusion and further work

Brown-Resnick processes have gained a lot of attention recently both because of their theoretical intricacies as well as their potential applicability, especially in space-time modeling of extreme events; see Davison et al. [8]. To this end, it is an important fact that this class of max-stable processes naturally appears as max-limits of Gaussian processes (cf. Kabluchko et al. [24], Kabluchko [23]). We have shown that these processes appear more generally as limits of maxima of not only Gaussian, but also squared Bessel processes and Brownian scalar product processes. Further generalizations are under investigation. A recent work by Engelke et al. [13] shows that Hüsler-Reiss type limit distributions are also obtained for non-identically distributed independent Gaussian random vectors. A natural extension could be thus to consider maxima of non-identically distributed independent Gaussian processes and their functional limits. Furthermore, the independence assumption can be eventually relaxed as in Hashorva and Weng [19], so that the limit process still remains Brown-Resnick. With regard to applications, there have been some developments in simulating Brown-Resnick processes [9, 12, 27]. An alternative formulation as the limit of other processes as described in this paper can potentially lead to further techniques for simulation.

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Appendix

Lemma 4.1. *If $X_i, Y_i, i \geq 1$, are independent $N(0,1)$random variables, then for any two integers $p > 1, k \geq 0$ and $1 = \lambda_1 = \cdots = \lambda_p > \lambda_{p+1} \geq \cdots \geq \lambda_{p+k} \geq 0$, as $x \to \infty$ we have*

$$\mathbb{P}\left(\sum_{i=1}^{p+k} \lambda_i X_i Y_i > x\right) = \frac{C_k}{2^{p/2}\Gamma(p/2)} x^{p/2-1} \exp(-x)\Big(1 + O(1/x)\Big) = f(x)\Big(1 + O(1/x)\Big), \tag{44}$$

$$\mathbb{P}\left(\sum_{i=1}^{p+k} \lambda_i X_i^2 > x\right) = \frac{C_k^*}{2^{p/2-1}\Gamma(p/2)} x^{p/2-1} \exp(-x/2)\Big(1 + O(1/x)\Big) = 2g(x)\Big(1 + O(1/x)\Big), \tag{45}$$

where f and g are the densities of $\sum_{i=1}^{p+k} \lambda_i X_i Y_i$ and $\sum_{i=1}^{p+k} \lambda_i X_i^2$, respectively. Here $C_0 = C_0^* := 1$ and

$$C_k := \prod_{i=1}^{k} \frac{1}{\sqrt{1 - \lambda_{p+i}}}, \quad C_k^* := \prod_{i=1}^{k} \frac{1}{\sqrt{1 - \lambda_{p+i}^2}}.$$

Furthermore $\sum_{i=1}^{p+k} \lambda_i X_i Y_i$ is in the Gumbel max-domain of attraction with norming constants

$$a_n^* = 1, \quad b_n^* = \ln n + (p/2 - 1)\ln(\ln n) - (p/2 - 1)\ln 2 - \ln(\Gamma(p/2)/C_p).$$

Proof. The proof follows from Example 5 and 6 in Hashorva et al. [18]. The norming constants can be easily found; see e.g., [11, p.155]. □

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