Extremes of Aggregated Dirichlet Risks

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Abstract: The class of Dirichlet random vectors is central in numerous probabilistic and statistical applications. The main result of this paper derives the exact tail asymptotics of the aggregated risk of powers of Dirichlet random vectors when the radial component has df in the Gumbel or the Weibull max-domain of attraction. We present further results for the joint asymptotic independence and the max-sum equivalence.

Key words and phrases: Dirichlet distribution; Gumbel max-domain of attraction; Weibull max-domain of attraction; tail asymptoics; risk aggregation; Davis-Resnick tail property.

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1 Introduction and Main Result

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a d-dimensional Dirichlet random vector with parameter $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (0, \infty)^d$ and radial component $R > 0$ with some distribution function (df) F. By definition, $\boldsymbol{X}$ has the stochastic representation

$$\boldsymbol{X} \stackrel{\mathcal{D}}{=} \Big(R\frac{Y_1}{\sum_{i=1}^d Y_i}, \ldots, R\frac{Y_d}{\sum_{i=1}^d Y_i} \Big) =: (RU_1, \ldots, RU_d), \tag{1}$$

where $\stackrel{\mathcal{D}}{=}$ stands for equality of dfs and $Y_i, i \le d$ are independent random variables (rvs) such that Y_i has Gamma distribution with parameters α_i and 1 (in our notation the $Gamma(a, \lambda)$ distribution has probability density function (pdf) $\lambda^a x^{a-1} \exp(-\lambda x)/\Gamma(a)$ where $\Gamma(\cdot)$ is the Euler Gamma function). Further $R, Y_1, \ldots, Y_d$ and $\boldsymbol{U} = (U_1, \ldots, U_d)$ are mutually independent. Basic distributional and asymptotic properties of Dirichlet random vectors are discussed in numerous contributions; see e.g., [11, 21, 4, 22, 23, 27, 1, 2] and the references therein.

Clearly, for any $1 \le k < d$

$$\sum_{i=1}^k X_i \stackrel{\mathcal{D}}{=} R \sum_{i=1}^k U_i \stackrel{\mathcal{D}}{=} R \frac{\sum_{i=1}^k Y_i}{\sum_{i=1}^d Y_i} \stackrel{\mathcal{D}}{=} RB,$$

where R and B are independent, and B has the Beta distribution with parameters $\sum_{i=1}^k \alpha_i$ and $\sum_{k+1}^d \alpha_i$. Hence the df of the total risk $\sum_{i=1}^k X_i$ can be directly calculated if F is known. Clearly, when $k = 2$ the above holds with B almost surely equal to 1. Furthermore, if F is in the Gumbel or the Weibull max-domain of attraction (MDA), then the tail asymptotics of $\sum_{i=1}^k X_i$ follows immediately by Theorem 3.1 in [19].

In this paper we are concerned with the tail asymptotic behaviour of the aggregated risk $S_p := \sum_{i=1}^d \lambda_i X_i^p$ for some fixed constant $p > 0$ and for given non-negative weights $\lambda_i, i \le d$. We shall assume first that $\boldsymbol{X}$ with stochastic representation (1) has a radial component R such that its df F is in the Gumbel MDA, i.e., its survival function $\overline{F} = 1 - F$ satisfies for any $x \ge 0$

$$\overline{F}(u + x/w(u)) \quad \sim \quad \exp(-x)\overline{F}(u), \quad u \uparrow x_F \tag{2}$$

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for some positive scaling function w (here x_F is the upper endpoint of F and we abbreviate (2) as $F \in GMDA(w)$). We use in (2) the standard notation $\sim$ for the asymptotic equivalence of two real-valued functions. For the sake of simplicity we shall assume hereafter that $x_F = \infty$ or $x_F = 1$. See [26, 10] for basic results concerned with the Gumbel MDA. Throughout in the following

$$1 = \lambda_1 = \cdots = \lambda_m \geq \lambda_{m+1} \geq \cdots \geq \lambda_d \geq 0 \tag{3}$$

are given weights with $m \leq d$ the multiplicity of λ_1. For $p > 1$ and $m < d$, it turns out that $\lambda_{m+1}, \ldots, \lambda_d$ do not influence the tail asymptotics of $S_p = \sum_{i=1}^d \lambda_i X_i^p$, which is however not the case if $p \in (0, 1]$. Hereafter we set $\overline{\alpha} := \sum_{i=1}^d \alpha_i$ with α_i's being positive constants.

Our principal result below displays the exact asymptotics of the tail of S_p, for any $p > 0$.

Theorem 1.1 *Let $\boldsymbol{X}$ be a d-dimensional Dirichlet random vector with parameter $\boldsymbol{\alpha}$ and representation* (1). *Suppose that* (2) *holds with $x_F \in \{1, \infty\}$ and some positive scaling function w.*

a) If $p > 1$, then

$$\mathbb{P}\{S_p > u^p\} \quad \sim \quad \mathbb{P}\left\{\sum_{i=1}^m X_i^p > u^p\right\} \quad \sim \quad m^* \frac{\Gamma(\overline{\alpha})}{\Gamma(\widehat{\alpha})} (uw(u))^{\widehat{\alpha}-\overline{\alpha}} \overline{F}(u), \quad u \uparrow x_F, \tag{4}$$

where $\widehat{\alpha} = \max_{1 \leq i \leq m} \alpha_i$, and m^* is the number of elements of the index set $\{i \leq m : \alpha_i = \widehat{\alpha}\}$.

b) If $m < d$, then

$$\mathbb{P}\{S_1 > u\} \quad \sim \quad \left(\prod_{i=1}^{d-m} (1 - \lambda_{m+i})^{-\alpha_{m+i}}\right) \frac{\Gamma(\overline{\alpha})}{\Gamma(\sum_{i=1}^m \alpha_i)} (uw(u))^{-\sum_{i=1}^{d-m} \alpha_{m+i}} \overline{F}(u), \quad u \uparrow x_F. \tag{5}$$

c) If $\lambda_i > 0, i \leq d$, then for any $p \in (0, 1)$ we have

$$\mathbb{P}\left\{S_p > \widetilde{\lambda_d} u^p\right\} \quad \sim \quad C_{\boldsymbol{\alpha},d} (uw(u))^{-(d-1)/2} \overline{F}(u), \quad u \uparrow x_F, \tag{6}$$

with $C_{\boldsymbol{\alpha},d}$ some positive constant and $\widetilde{\lambda_d} = \left(\sum_{i=1}^d \lambda_i^{1/(1-p)}\right)^{1-p}$.

Remarks: a) An immediate consequence of Theorem 1.1 is that if F is as therein, then the aggregated risk S_p has df in the Gumbel MDA with scaling function $w_p(x) = x^{1/p-1} w(x^{1/p})/p$; see also Proposition 2.2 below. Consequently, in view of the properties of the scaling function w (see e.g., p.143 in [9]) we have assuming $x_F = \infty$

$$\mathbb{E}\{S_p | S_p > VaR_{S_p}(b)\} - VaR_{S_p}(b) \quad \sim \quad \frac{1}{w_p(VaR_{S_p}(b))}, \quad b \uparrow 1,$$

with $VaR_{S_p}(\tau)$ being the Value-at-Risk of S_p at $\tau \in (0, 1)$, implying thus

$$\mathbb{E}\{S_p | S_p > VaR_{S_p}(b)\} \quad \sim \quad VaR_{S_p}(b), \quad b \uparrow 1.$$

b) For any df $F \in GMDA(w)$ with upper endpoint $x_F = \infty$, the Davis-Resnick tail property is crucial, i.e., (see e.g., Proposition 1.1 in [6] and p. 113 in [15])

$$\lim_{u \to \infty} (uw(u))^\mu \frac{\overline{F}(cu)}{\overline{F}(u)} \quad = \quad 0 \tag{7}$$

holds for any $\mu \in \mathbb{R}$ and $c > 1$. Under the assumptions of statement c) in Theorem 1.1 we have $\widetilde{\lambda_d} > \lambda_i, i \leq d$. It follows further by (7) that for $x_F = \infty, i \leq d$ and $p \in (0, 1]$

$$\lim_{u \to \infty} \frac{\mathbb{P}\{\lambda_i X_i^p > u\}}{\mathbb{P}\{S_p > u\}} \quad = \quad 0. \tag{8}$$

Consequently, each risk $\lambda_i X_i^p$ has a different asymptotic behaviour compared to S_p.

c) The convergence in (8) reveals a key property of the Dirichlet dependence structure, namely the principle of a single big jump (see e.g., [12] for details) applies if $p > 1$. However, this principle does not apply when $p \in (0, 1]$, see (13) below. An example which demonstrates this is furnished by taking $\boldsymbol{X} = (X_1, \ldots, X_d)$ with independent components having unit exponential distribution, then $\boldsymbol{X}$ is a Dirichlet random vector with its radial component having $Gamma(d, 1)$ distribution. Hence since also $\sum_{i=1}^d X_i$ has $Gamma(d, 1)$ distribution if $p = 1$, then

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{\max_{1\le i\le d} X_i^p > u\right\}}{\mathbb{P}\left\{S_p > u\right\}} = 0,$$

which is valid also for any $p \in (0, 1)$.

d) The tail asymptotic behaviour of L_p type weighted norm $(\sum_{i=1}^d \lambda_i X_i^p)^{1/p}$ for various $\boldsymbol{X}$ has been considered by several authors; see e.g., [25, 14] and the references therein.

In the next section, we discuss our main result and present some important extensions. All the proof are relegated to Section 3 followed by an Appendix.

2 Discussions and Extensions

A canonical example of a d-dimensional Dirichlet random vector $\boldsymbol{X}$ is the so-called Kotz-Dirichlet random vector, with $X_i, i \le d$ independent such that X_i has $Gamma(\alpha_i, 1)$ distribution with $\alpha_i > 0, i \le d$; see e.g., [2]. Such a random vector has stochastic representation (1) with R having $Gamma(\overline{\alpha}, 1)$ distribution. Hence for this particular example Theorem 1.1 gives the tail asymptotics of the sum of powers of independent Gamma rvs.

Note that for any $p > 1$ the rv X_i^p is a subexponential one (see e.g., [9] for the definition and main properties), and therefore the statement a) in Theorem 1.1 for this case can be directly checked to hold. When $p = 1$, the claim of statement b) in Theorem 1.1 follows by Lemma 2.1 in [24], whereas for $p \in (0, 1)$ and $\alpha_i = \alpha > 0, i \le d$ the claim in statement *c*) of Theorem 1.1 is established by applying the result of [28], which also gives the explicit formula for the constant $C_{\boldsymbol{\alpha},d}$ with $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha) \in (0, \infty)^d$.

In the previous section we introduced the Dirichlet random vectors in the first quadrant. This restriction can be removed by introducing indicator rvs $I_1, \ldots, I_d$ independent of $\boldsymbol{X}$ with stochastic representation (1). If $\mathbb{P}\{I_i = 1\} = c_i = 1 - \mathbb{P}\{I_i = -1\}, i \le d$, then

$$\boldsymbol{Y} = (Y_1, \ldots, Y_d) \stackrel{\mathcal{D}}{=} (I_1 X_1^{1/p}, \ldots, I_d X_d^{1/p}), \quad p > 0$$

is referred to as a weighted L_p-Dirichlet random vector. For simplicity, we assume here that $X_i, i \le d$ has the $Gamma(\alpha_i, 1/p)$ distribution; the above extension allows us to include the Gaussian distribution in the class of L_p-Dirichlet random vectors. Indeed, if $p = 1/\alpha_1 = \cdots = 1/\alpha_d = 2$ and $I_1, \ldots, I_d$ are mutually independent with mean 0, then $\boldsymbol{Y}$ is a d-dimensional Gaussian random vector if additionally R^2 is chi-square distributed with d degrees of freedom. If the df of R is not specified in general, then $\boldsymbol{Y}$ is a spherical random vector (see the seminal contribution [3] for the main distributional properties). We have thus $\sum_{i=1}^d \lambda_i |Y_i|^2 = \sum_{i=1}^d \lambda_i X_i$, and hence for this particular case statement b) of Theorem 1.1 implies the claim of Theorem 3.1 in [14].

In the sequel $\mathbb{B}_{a,b}$ stands for the Beta distribution with positive parameters a and b, and $V \sim \mathbb{B}_{a,b}$ means that the rv V has the Beta distribution with parameters a and b.

Concerning the Gumbel MDA assumption imposed on F we first remark that under stronger assumptions on the scaling function w, namely w is regularly varying at infinity, then in view of [7], it follows that for any homogeneous function h of order p, i.e., $h(tx_1,\dots,tx_d) = t^p h(x_1,\dots,x_d)$ holds for any $t>0$ and $(x_1,\dots,x_d)\in\mathbb{R}^d$ we have that $h(\boldsymbol{X}) \overset{\mathcal{D}}{=} R^p h(\boldsymbol{U})$ has df in the Gumbel MDA. Using the terminology of [17] the rv $h(\boldsymbol{X})$ can be referred to as the Dirichlet chaos. In the light of the findings of the aforementioned contribution, the exact asymptotics of the Dirichlet chaos can be derived. In this paper we used a direct approach for the special case of aggregated risk.

As mentioned in the Introduction the Davis-Resnick property of F is crucial. In fact, if we assume that $\overline{F} = 1-F$ is rapidly varying at infinity, i.e., (7) holds for $\mu = 0$ and $c>1$, then for two Dirichlet random vectors $\boldsymbol{X}$ and $\boldsymbol{W}$ with corresponding radius R and R^* and parameter $\boldsymbol{\alpha}$, we obtain by applying Lemma 4.1 in Appendix

$$\mathbb{P}\left\{\sum_{i=1}^d \lambda_i X_i^p > u\right\} \quad\sim\quad L(u)\mathbb{P}\left\{\sum_{i=1}^d \lambda_i W_i^p > u\right\}, \quad u\to\infty, \tag{9}$$

provided that $\overline{F}$ is rapidly varying at infinity and $\mathbb{P}\{R>u\} \sim L(u)\mathbb{P}\{R^*>u\}$ where $L(u)$ is some slowly varying function at infinity.

2.1 Weibull MDA

Instead of the Gumbel MDA assumption in (2) we shall suppose that $\overline{F} = 1-F$ is regularly varying with index $\gamma\ge 0$ at the upper endpoint $x_F = 1$, i.e., for any $t>0$

$$\frac{\overline{F}(1-tu)}{\overline{F}(1-u)} \quad\sim\quad t^\gamma, \quad u\downarrow 0. \tag{10}$$

For $\gamma>0$, the above assumption means that F is in the MDA of the Weibull distribution $\Psi_\gamma(x) = \exp(-|x|^\gamma), x<0$. A canonical example of F in the Weibull MDA is the case of the Beta distribution $\mathbb{B}_{a,b}$ where $\gamma = b$. Under (10) we can derive similar results to those in Theorem 1.1. For simplicity we formulate only the claim of statement b) therein.

Theorem 2.1 *Under the assumptions of statement b) in Theorem 1.1, if further instead of* (2) *we suppose that the survival function* $\overline{F}$ *of* R *satisfies* (10) *for some* $\gamma\ge 0$*, then*

$$\mathbb{P}\{S_p > 1-u\} \quad\sim\quad \left(\prod_{i=1}^{d-m}(1-\lambda_{m+i})^{-\alpha_{m+i}}\right)\frac{\Gamma(\overline{\alpha})\Gamma(\gamma+1)}{\Gamma(\sum_{i=1}^m \alpha_i)\Gamma(\sum_{i=1}^{d-m}\alpha_{m+i}+\gamma+1)}u^{-\sum_{i=1}^{d-m}\alpha_{m+i}}\overline{F}(1-u) \tag{11}$$

holds as $u\downarrow 0$.

In the special case that $\alpha_1 = \dots = \alpha_d = 1/2 = 1/p$ the claim of Theorem 2.1 agrees with that of Theorem 3.6 in [14].

A specific of the Weibull MDA is that the upper endpoint x_F of F is necessarily finite. There is no possibility to convert x_F to be infinite such that the transformed $\boldsymbol{X}$ is still a Dirichlet random vector. Therefore, the result of this section cannot be retrieved by results available in the literature concerned with the aggregation of dependent unbounded risks dealt with for instance in [13, 16, 8].

2.2 Approximation by Max-Stable Distributions

Next, we present an application of Theorem 1.1; a similar application (omitted here) can be given using Theorem 2.1. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ be a random vector which is obtained by a linear transform of $(X_1^p, \ldots, X_d^p)$, i.e., for given constants $\lambda_{ij}, i, j \le d$

$$\boldsymbol{Y} \stackrel{\mathcal{D}}{=} (\sum_{i=1}^{d} \lambda_{i1} X_i^p, \ldots, \sum_{i=1}^{d} \lambda_{id} X_i^p).$$

We shall denote by G the df of $\boldsymbol{Y}$, and G_i is its ith marginal df. It is of interest to determine if G is in the max-domain of attraction of some multivariate max-stable df Q, i.e., if there are constants $a_{ni} > 0, b_{ni} \in \mathbb{R}, i \le d, n \ge 1$ such that

$$\lim_{n\to\infty} \sup_{x_i \in \mathbb{R}, 1 \le i \le d} \Big| G^n(a_{n1}x_1 + b_{n1}, \ldots, a_{nd}x_d + b_{nd}) - Q(x_1, \ldots, x_d) \Big| \quad = \quad 0. \tag{12}$$

Our next result shows that this is possible, if F is in the Gumbel MDA.

Proposition 2.2 *Let $\lambda_{ij}, i, j \le d$ be non-negative constants and denote by $A_j := \{i \le d : \lambda_{ij} = 1\}, j \le d$. Suppose for $p \ge 1$ that $A_j, j \le d$ is non-empty and $A_i \cap A_j$ has no elements for any pair (i, j) of different indices, and for $p \in (0, 1)$ that $\sum_{i=1}^d \lambda_{ij}^{1/(1-p)} = 1$ and $\lambda_{ij}, i, j \le d$ are non-negative such that for any i, j two different indices $\lambda_{ik} \ne \lambda_{jk}$ for some $k \le d$. Under the assumption of Theorem 1.1, then for $a_{ni} = 1/w_p(b_{ni}), i \le d$ with $b_{ni} = G_i^{-1}(1-1/n), n \ge 1$ and $w_p(x) = x^{1/p-1} w(x^{1/p})/p, x > 0$ we have that (12) holds with $Q(x_1, \ldots, x_d) = \exp\Big(-\sum_{i=1}^d \exp(-x_i)\Big)$.*

Clearly, the conditions in Proposition 2.2 on λ_{ij}'s are satisfied if $\lambda_{ii} = 1, i \le d$ and $\lambda_{ij} = 0$ for all i, j different indices. As in the proof of Proposition 2.2 we have

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{X_i^p > u, X_j^p > u\right\}}{\mathbb{P}\left\{X_i^p > u\right\}} \quad = \quad 0.$$

Consequently, for the case $p > 1$, by Bonferroni's inequality it follows that the sum and maximum of $\lambda_i X_i^p, i \le d$ are asymptotically equivalent, i.e., the principle of a single big jump holds. More precisely, if $x_F = \infty$ and $F \in GMDA(w)$, then for any $p > 1$

$$\mathbb{P}\left\{\sum_{i=1}^{d} \lambda_i X_i^p > u\right\} \quad \sim \quad \mathbb{P}\left\{\max_{i \le d} \lambda_i X_i^p > u\right\}, \quad u \to \infty. \tag{13}$$

2.3 Converse Results

So far we have assumed that the df of R is in the Gumbel or Weibull MDA and then we showed that the same holds for the aggregated risk. Recall that we do not consider the case that R has df in the Fréchet MDA since the answer follows immediately by Breiman's lemma.

At this point, the question on the validity of the converse results is natural. Namely, if for some $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ satisfying (3) the aggregated risk S_p has df $G_{p,\boldsymbol{\lambda}}$ in the Gumbel MDA, then does also F belong to the Gumbel MDA? Since in statistical applications, some observations might be missing, neither the radius R nor the total risk S_p can be observed, also of interest is if $G_{p,\boldsymbol{\lambda}}$ belongs to the Gumbel MDA for some $\boldsymbol{\lambda}$ implies that $G_{p,\boldsymbol{\lambda}}$ is in the Gumbel MDA for any $\boldsymbol{\lambda}$ that satisfies (3). Note that when F is in the Gumbel MDA, then $G_{p,\boldsymbol{\lambda}}$ is in the Gumbel MDA with scaling function $w_p(x) = x^{1/p-1} w(x^{1/p})/p$ for any $p \in (0, \infty)$.

We state next the converse of Theorem 1.1 omitting the corresponding result for the Weibull MDA which can be derived by utilising the same idea.

Theorem 2.3 *Let $F, x_F, \boldsymbol{X}$ be as in Theorem 1.1. If $\boldsymbol{\lambda}$ is a d-dimensional vector whose components satisfy* (3), *then $F \in GMDA(w)$ is equivalent with $G_{p,\boldsymbol{\lambda}}$ in the Gumbel MDA for some $\boldsymbol{\lambda}$ and some $p \in (0, \infty)$. Moreover, the latter assertion is equivalent with $G_{p,\boldsymbol{\lambda}}$ in the Gumbel MDA for any $\boldsymbol{\lambda}$ and any $p \in (0, \infty)$.*

Recent results concerning the asymptotics of products and converse results for the regularly varying case are derived in the deep contributions [20, 5]. Therefore, we omit the details for the case that R has a regularly varying survival function at infinity.

3 Proofs

We state first a lemma which is useful for the proof of Theorem 1.1. In particular, the following lemma shows that in the bivariate setup Theorem 1.1 can be extended to include some general bivariate random vectors which have similar dependence structure as the Dirichlet ones. In the sequel we say that Z is regularly varying at x_G with index $\tau \ge 0$ (we omit often the index τ) if this is the case for its survival function $\overline{G}$.

Lemma 3.1 *Let B, X, Y be three non-negative rvs with upper endpoints $\omega_B = \omega_X = 1, \omega_Y \le 1$.*
a) If $\omega_Y < 1$ and $B^p X$ is regularly varying at 1 for some $p > 1$, then for $\mathcal{S}_p := B^p X + (1-B)^p Y$

$$\mathbb{P}\left\{\mathcal{S}_p > 1-u\right\} \sim \mathbb{P}\left\{B^p X > 1-u\right\}, \quad u \downarrow 0. \tag{14}$$

b) Under the conditions of statement a), if further $\omega_Y = 1$ and $(1-B)^p Y$ is also regularly varying at 1, then

$$\mathbb{P}\left\{\mathcal{S}_p > 1-u\right\} \sim \mathbb{P}\left\{B^p X > 1-u\right\} + \mathbb{P}\left\{(1-B)^p Y > 1-u\right\}, \quad u \downarrow 0. \tag{15}$$

c) If B has a continuous pdf g, then for any c, λ positive and $p \in (0,1)$

$$\mathbb{P}\left\{B^p c + \lambda(1-B)^p > \widetilde{\theta} - u\right\} \quad \sim \quad 2^{3/2} \frac{g(\theta)}{\sqrt{h''(c,\theta)}} \sqrt{u}, \quad u \downarrow 0$$

holds with $h(c,\beta) = \beta^p c + \lambda(1-\beta)^p$ and $\theta = (\lambda/c)^{1/(p-1)}/(1+(\lambda/c)^{1/(p-1)}), \widetilde{\theta} = h(c,\theta) = (c^{1/(1-p)} + \lambda^{1/(1-p)})^{p-1}$.
d) Under the assumption and notation in statement c) if further X is regularly varying at $c := \omega_X > 0$ with index $\gamma > 0$, then for any $\lambda > 0$ and $p \in (0,1)$

$$\mathbb{P}\left\{B^p X + \lambda(1-B)^p > \widetilde{\theta} - u\right\} \quad \sim \quad \frac{\sqrt{2\pi} g(\theta)}{\sqrt{h''(c,\theta)}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)} \theta^{-\gamma p} \sqrt{u} \mathbb{P}\left\{X > c-u\right\}, \quad u \downarrow 0,$$

provided that B and X are independent.

Proof of Lemma 3.1 a) For some $u > 0$ sufficiently small, since $\omega_Y < 1$, the event $\{\mathcal{S}_p > 1-u\}$ is possible if $B^p > 1-u$ and $X > 1-u$ and thus in that case $(1-B)^p Y = O(u^p)$. Hence

$$\mathbb{P}\left\{\mathcal{S}_p > 1-u\right\} \quad \sim \quad \mathbb{P}\left\{B^p X > 1-u(1+o(1))\right\}, \quad u \downarrow 0.$$

Thus the claim follows by the uniform convergence theorem for regularly varying function, see e.g., [9].

b) As in the proof of a) the event $\{\mathcal{S}_p > 1-u\}$ is also possible if $B < u$ hence $B^p X \le u^p$. Consequently

$$\mathbb{P}\left\{\mathcal{S}_p > 1-u\right\} = \mathbb{P}\left\{B^p X > 1-u(1+o(1))\right\} + \mathbb{P}\left\{(1-B)^p Y > 1-u(1+o(1))\right\}, \quad u \downarrow 0$$

and again the claim follows by the uniform convergence theorem for regularly varying function.

c) First note that the unique maximum of the function $h(c,\beta)=\beta^p c+\lambda(1-\beta)^p$ for $\beta\in[0,1]$ is attained at

$$\theta=(\lambda/c)^{1/(p-1)}/(1+(\lambda/c)^{1/(p-1)}) \tag{16}$$

and we have thus $h'(c,\theta)=0$ and

$$\widetilde{\theta}=h(c,\theta)=\frac{\lambda}{(1+(\lambda/c)^{1/(p-1)})^{p-1}}=c\lambda(c^{-1/(1-p)}+\lambda^{-1/(1-p)})^{1-p}=(c^{1/(1-p)}+\lambda^{1/(1-p)})^{1-p}.$$

Consequently, since B has a continuous pdf g we get that for $\varepsilon_u=\sqrt{2u/h''(c,\theta)}$

$$\mathbb{P}\left\{B^p c+\lambda(1-B)^p>\widetilde{\theta}-u\right\}\quad\sim\quad\int_{\theta-\varepsilon_u}^{\theta+\varepsilon_u}g(s)\,ds\sim 2^{3/2}\frac{g(\theta)}{\sqrt{h''(c,\theta)}}\sqrt{u}$$

as $u\downarrow 0$, hence the claim follows.

d) Let Q denote the df of X and write $c>0$ for its upper endpoint. Since X is regularly varying at c with index $\gamma>0$, then for any $t>0$

$$\lim_{u\downarrow 0}\frac{\overline{Q}(c-tu)}{\overline{Q}(c-u)}=t^\gamma,\quad \overline{Q}=1-Q.$$

We proceed as above, but the choice of ε_u is different since we condition first on $X=c-tu$. Choosing $\varepsilon_u=\sqrt{\frac{2u(1-\theta^p t)}{h''(c,\theta)}}$ with θ as in (16), by the independence of X and B we may further write

$$\begin{aligned}
&\mathbb{P}\left\{B^pX+\lambda(1-B)^p>\widetilde{\theta}-u\right\}\\
&\sim\quad\int_{c-u/\theta^p}^{c}\int_{\theta-\varepsilon_u}^{\theta+\varepsilon_u}g(s)\,ds dQ(t)\\
&\sim\quad -2^{3/2}\frac{g(\theta)}{\sqrt{h''(c,\theta)}}\overline{Q}(c-u)\sqrt{u}\int_0^{1/\theta^p}\sqrt{1-\theta^p t}\,d\frac{\overline{Q}(c-tu)}{\overline{Q}(c-u)}\\
&\sim\quad 2^{3/2}\frac{g(\theta)}{\sqrt{h''(c,\theta)}}\overline{Q}(c-u)\sqrt{u}\gamma\int_0^{1/\theta^p}(1-\theta^p t)^{3/2-1}t^{\gamma-1}\,dt\\
&\sim\quad \sqrt{2\pi}\frac{g(\theta)}{\sqrt{h''(c,\theta)}}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)}\theta^{-\gamma p}\overline{Q}(c-u)\sqrt{u}
\end{aligned}$$

as $u\downarrow 0$, and thus the proof is complete. □

Proof of Theorem 1.1 In the sequel $\mathcal{B}_{\alpha,\beta}$ will denote a Beta rv with df $\mathbb{B}_{\alpha,\beta}$. Note that as $u\downarrow 0$

$$\mathbb{P}\left\{B^p>1-u\right\}\sim\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{1-u/p}^1(1-x)^{\beta-1}dx\sim\frac{\Gamma(\alpha+\beta)}{p^\beta\Gamma(\alpha)\Gamma(\beta+1)}u^\beta. \tag{17}$$

a) Assume next that $m=1$, i.e., $1=\lambda_1>\lambda_2\ge\cdots\ge\lambda_d\ge 0$. By the beta-independence splitting property of Dirichlet random vectors, we have the stochastic representation

$$(U_1,\ldots,U_d)\stackrel{\mathcal{D}}{=}\Big(\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1},(1-\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1})\tilde{U}_1,\ldots,(1-\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1})\tilde{U}_{d-1}\Big), \tag{18}$$

where $(\tilde{U}_1,\ldots,\tilde{U}_{d-1})$ is a standard $(d-1)$-dimensional Dirichlet random vector with parameter $(\alpha_1,\ldots,\alpha_{d-1})$ being independent of $\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1}$. Consequently

$$\sum_{i=1}^d\lambda_iU_i^p=\mathcal{B}^p_{\alpha_1,\overline{\alpha}-\alpha_1}+\lambda(1-\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1})^pW,$$

where $\lambda \in (0,1)$ is some constant, $\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1}$ and W are independent, and W has df with upper endpoint equal to 1. Applying statement a) of Lemma 3.1 we have as $u \downarrow 0$

$$\mathbb{P}\left\{\sum_{i=1}^d \lambda_i U_i^p > 1-u\right\} \sim \mathbb{P}\left\{U_1^p > 1-u\right\}.$$

Since further

$$\sum_{i=1}^d \lambda_i X_i^p \stackrel{\mathcal{D}}{=} R^p \sum_{i=1}^d \lambda_i U_i^p \tag{19}$$

and R^p has df in the Gumbel MDA with scaling function $w_p(x) = x^{1/p-1} w(x^{1/p})/p$, see e.g., Lemma 5.2 in [18], the claim follows by applying Theorem 4.2. Next, by repeating the above arguments, it follows that in the general case $1 \le m \le d$

$$\mathbb{P}\left\{\sum_{i=1}^d \lambda_i U_i^p > 1-u\right\} \sim \mathbb{P}\left\{\sum_{i=1}^m U_i^p > 1-u\right\}, \quad u \downarrow 0.$$

Since the case $m=1$ is shown above suppose that $m=2$. Again, by the beta-independence splitting property of Dirichlet random vectors

$$U_1^p + U_2^p \stackrel{\mathcal{D}}{=} \mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1}^p + (1-\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1})^p \tilde{U}_2, \tag{20}$$

with $\tilde{U}_2 \sim \mathbb{B}_{\alpha_2,\overline{\alpha}-\alpha_1-\alpha_2}$, provided that $d>2$. If $d=2$, then we simply have

$$U_1^p + U_2^p \stackrel{\mathcal{D}}{=} \mathcal{B}_{\alpha_1,\alpha_2}^p + (1-\mathcal{B}_{\alpha_1,\alpha_2})^p. \tag{21}$$

In both cases, applying statement $b)$ and $c)$ of Lemma 3.1 we obtain

$$\mathbb{P}\left\{U_1^p + U_2^p > 1-u\right\} \sim C_2 u^{\overline{\alpha}-\max(\alpha_1,\alpha_2)}, \quad u \downarrow 0,$$

with $C_m \in (0,\infty), m \le d$. By induction on m it follows that

$$\mathbb{P}\left\{\sum_{i=1}^m U_i^p > 1-u\right\} \sim C_m u^{\overline{\alpha}-\max_{i\le m}\alpha_i}, \quad u \downarrow 0$$

and further

$$\mathbb{P}\left\{\sum_{i=1}^m U_i^p > 1-u\right\} \sim \mathbb{P}\left\{\sum_{1\le i\le m:\alpha_i=\alpha_m^*} U_i^p > 1-u\right\}, \quad u \downarrow 0,$$

with $\alpha_m^* = \max_{i\le m}\alpha_i$. In order to simplify notation, assume that

$$\alpha_1 = \alpha_i, \quad 2 \le i \le m^* \le m,$$

where m^* denotes the number of elements in $\{1 \le i \le m : \alpha_i = \alpha_m^*\}$. Suppose for simplicity that $m = m^*$ and consider next the case $m=2$. Clearly, if $d=2$, then by (21) with $\alpha_1 = \alpha_2$ and Lemma 3.1 it follows that (recall (17))

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^{m^*} U_i^p > 1-u\right\} &\sim m^* \mathbb{P}\left\{\mathcal{B}_{\alpha_1,\alpha_1}^p > 1-u\right\} \\
&\sim m^* \frac{\Gamma(2\alpha_1)}{\Gamma(\alpha_1)\Gamma(\alpha_1+1)} (u/p)^{\alpha_1} \\
&\sim m^* \mathbb{P}\left\{U_1^p > 1-u\right\}, \quad u \downarrow 0.
\end{aligned}$$

For $d > 2$ we consider the representation (20), where $\tilde{U}_2$ has Beta df with parameters $\alpha_2 = \alpha_1, \overline{\alpha} - 2\alpha_1 > 0$. In view of statement $b)$ and $c)$ of Lemma 3.1 we have

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^{m^*} U_i^p > 1-u\right\} &\sim \mathbb{P}\left\{\mathcal{B}^p_{\alpha_1,\overline{\alpha}-\alpha_1} > 1-u\right\} + \mathbb{P}\left\{(1-\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1})^p \tilde{U}_2 > 1-u\right\}\\ &\sim m^* \frac{\Gamma(\overline{\alpha})}{\Gamma(\alpha_1)\Gamma(\overline{\alpha}-\alpha_1+1)}(u/p)^{\overline{\alpha}-\alpha_1}, \quad u \downarrow 0.\end{aligned}$$

Using induction and the above arguments, for any $m^* \geq 2$ we obtain

$$\begin{aligned}\mathbb{P}\left\{(\sum_{i=1}^{m^*} U_i^p)^{1/p} > 1-u\right\} &\sim m^*\mathbb{P}\left\{U_1^p > 1-pu\right\}\\ &\sim m^*\mathbb{P}\left\{\mathcal{B}_{\alpha_1,\overline{\alpha}-\alpha_1} > 1-u\right\}\\ &\sim m^* \frac{\Gamma(\overline{\alpha})}{\Gamma(\alpha_1)\Gamma(\overline{\alpha}-\alpha_1+1)} u^{\overline{\alpha}-\alpha_1}, \quad u \downarrow 0,\end{aligned}$$

hence the claim follows by Theorem 4.2.

$b)$ The case $m = d-1$ follows easily using the following representation

$$\sum_{i=1}^{m} U_i + \lambda_{m+1}U_{m+1} \stackrel{\mathcal{D}}{=} B(1-\lambda_{m+1}) + \lambda_{m+1},$$

where $B \stackrel{\mathcal{D}}{=} \mathcal{B}_{\sum_{i=1}^m \alpha_i, \alpha_{m+1}}$, and noting further that

$$\mathbb{P}\left\{B(1-\lambda_{m+1}) + \lambda_{m+1} > 1-u\right\} \sim (1-\lambda_{m+1})^{-\alpha_{m+1}} \frac{\Gamma(\overline{\alpha})}{\Gamma(\overline{\alpha}_m)\Gamma(\overline{\alpha}-\overline{\alpha}_m+1)} u^{\overline{\alpha}-\overline{\alpha}_m} \tag{22}$$

as $u \downarrow 0$. We consider next the case $m < d-1$. By the aggregation property of Dirichlet distributions and the beta-independence splitting property, we have

$$\sum_{i=1}^{m+1} \lambda_i U_i \stackrel{\mathcal{D}}{=} BX + \lambda_{m+1}(1-B),$$

where B and X are independent such that $B \stackrel{\mathcal{D}}{=} \mathcal{B}_{\overline{\alpha}-\alpha_{m+1},\alpha_{m+1}}$ and $X \stackrel{\mathcal{D}}{=} \mathcal{B}_{\overline{\alpha}_m, \overline{\alpha}-\sum_{i=1}^{m+1}\alpha_i}$. Consequently, (28) implies

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^{m+1} \lambda_i U_i > 1-u\right\} &\sim (1-\lambda_{m+1})^{-\alpha_{m+1}} \frac{\Gamma(\alpha_{m+1}+1)\Gamma(\overline{\alpha}-\sum_{i=1}^{m+1}\alpha_i+1)}{\Gamma(\overline{\alpha}-\overline{\alpha}_m+1)}\\ &\quad \times \mathbb{P}\left\{\mathcal{B}_{\overline{\alpha}-\alpha_{m+1},\alpha_{m+1}} > 1-u\right\}\mathbb{P}\left\{\mathcal{B}_{\overline{\alpha}_m,\overline{\alpha}-\sum_{i=1}^{m+1}\alpha_i} > 1-u\right\}\\ &\sim (1-\lambda_{m+1})^{-\alpha_{m+1}} \frac{\Gamma(\overline{\alpha})}{\Gamma(\overline{\alpha}_m)\Gamma(\overline{\alpha}-\overline{\alpha}_m+1)} u^{\overline{\alpha}-\overline{\alpha}_m}\end{aligned}$$

as $u \downarrow 0$. Since (22) holds also for $\lambda_{m+1} = 0$, repeating the above argument we have

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^{d} \lambda_i U_i > 1-u\right\} &\sim \left(\prod_{i=1}^{d-m}(1-\lambda_{m+i})^{-\alpha_{m+i}}\right) \frac{\Gamma(\overline{\alpha})}{\Gamma(\overline{\alpha}_m)\Gamma(\overline{\alpha}-\overline{\alpha}_m+1)} u^{\overline{\alpha}-\overline{\alpha}_m}\\ &\sim \left(\prod_{i=1}^{d-m}(1-\lambda_{m+i})^{-\alpha_{m+i}}\right)\mathbb{P}\left\{\sum_{i=1}^{m} U_i > 1-u\right\}, \quad u \downarrow 0\end{aligned}$$

and hence the proof follows by applying again Theorem 4.2.

$c)$ As above it suffices to determine the tail asymptotics of $Z_d = \sum_{i=1}^d \lambda_i U_i^p$ at $\widetilde{\lambda_d}$ the upper endpoint of the df of Z_d. In view of (18) and statement $d)$ in Lemma 3.1 we have with $X := \sum_{i=1}^{d-1} \lambda_i \tilde{U}_i$ being independent of $B \stackrel{\mathcal{D}}{=} \mathcal{B}_{\overline{\alpha}-\alpha_d,\alpha_d}$

$$\mathbb{P}\left\{Z_d > \widetilde{\lambda_d} - u\right\} = \mathbb{P}\left\{B^p X + \lambda_d(1-B)^p > \widetilde{\lambda_d} - u\right\}$$

$$\sim \sqrt{2\pi}\frac{g_{\overline{\alpha}-\alpha_d,\alpha_d}(\theta)}{\sqrt{h''(\widetilde{\lambda_{d-1}},\theta)}}\frac{\Gamma(\alpha_d+1)}{\Gamma(\alpha_d+3/2)}\theta^{-\alpha_d p}\sqrt{u}\mathbb{P}\left\{X>\widetilde{\lambda_{d-1}}-u\right\}, \quad u\downarrow 0,$$

where $\widetilde{\lambda_{d-1}}$ is the upper endpoint of the df of X, $g_{\overline{\alpha}-\alpha_d,\alpha_d}$ is the pdf of B and

$$\theta=\frac{\tau^{1/(p-1)}}{1+\tau^{1/(p-1)}}, \quad \tau=\frac{\lambda_d}{\widetilde{\lambda_{d-1}}}.$$

From the proof of Lemma 3.1 we see that

$$\widetilde{\lambda_d}=\frac{\lambda_d}{(1+(\lambda_d/\widetilde{\lambda_{d-1}})^{1/(p-1)})^{p-1}}=(\widetilde{\lambda_{d-1}}^{1/(1-p)}+\lambda_d^{1/(1-p)})^{1-p},$$

hence

$$\widetilde{\lambda_d}=\Big(\sum_{i=1}^d\lambda_i^{1/(1-p)}\Big)^{1-p}.$$

Note that above we used the fact that X has a regularly varying survival function at $\widetilde{\lambda_{d-1}}$, which follows by induction. We remark further that $\widetilde{\lambda_d}$ is the attained maximum of the function $h(\beta_1,\ldots,\beta_d)=\sum_{i=1}^d\lambda_i\beta_i^p$ for $\beta_i\in[0,1], i\le d$ satisfying $\sum_{i=1}^d\beta_i=1$. Continuing, we obtain that

$$\mathbb{P}\left\{Z_d>\widetilde{\lambda_d}-u\right\}\sim\mathbb{P}\left\{B^pX+\lambda_d(1-B)^p>\widetilde{\lambda_d}-u\right\}\sim\widetilde{C_d}u^{(d-1)/2}, \quad u\downarrow 0,$$

with $\widetilde{C_d}$ a positive constant which can be calculated explicitly, and hence by Theorem 4.2

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^d\lambda_iX_i^p>\widetilde{\lambda_d}u^p\right\} &= \mathbb{P}\left\{R(Z_d/\widetilde{\lambda_d})^{1/p}>u\right\}\\ &\sim \Gamma((d-1)/2+1)\mathbb{P}\left\{\frac{Z_d}{\widetilde{\lambda_d}}>1-\frac{p}{uw(u)}\right\}\mathbb{P}\left\{R>u\right\}\\ &\sim \Gamma((d+1)/2)\widetilde{C_d}\left(\frac{p\widetilde{\lambda_d}}{uw(u)}\right)^{(d-1)/2}\mathbb{P}\left\{R>u\right\}\end{aligned}$$

establishing the proof. ☐

Proof of Theorem 2.1 Applying Theorem 4.2 as in the proof of Theorem 1.1, we obtain

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^d\lambda_iX_i>1-u\right\} &= \mathbb{P}\left\{R\sum_{i=1}^d\lambda_iU_i>1-u\right\}\\ &\sim \frac{\Gamma(\sum_{i=1}^{d-m}\alpha_{m+i}+1)\Gamma(\gamma+1)}{\Gamma(\sum_{i=1}^{d-m}\alpha_{m+i}+\gamma+1)}\mathbb{P}\left\{\sum_{i=1}^d\lambda_iU_i>1-u\right\}\mathbb{P}\left\{R>1-u\right\}\\ &\sim \prod_{i=1}^{d-m}(1-\lambda_{m+i})^{-\alpha_{m+i}}\frac{\Gamma(\sum_{i=1}^{d-m}\alpha_{m+i}+1)\Gamma(\gamma+1)}{\Gamma(\sum_{i=1}^{d-m}\alpha_{m+i}+\gamma+1)}\\ &\quad\times\frac{\Gamma(\overline{\alpha})}{\Gamma(\sum_{i=1}^m\alpha_i)\Gamma(\sum_{i=1}^{d-m}\alpha_{m+i}+1)}u^{-\sum_{i=1}^{d-m}\alpha_{m+i}}\overline{F}(1-u)\end{aligned}$$

as $u\downarrow 0$, hence the proof follows. ☐

Proof of Proposition 2.2 In view of Theorem 3.1 and Lemma 5.2 in [18], it follows that $Y_j=\sum_{i=1}^d\lambda_{ij}X_i^p=R^p\sum_{i=1}^d\lambda_{ij}U_i^p$ has df in the Gumbel MDA with scaling function $w_p(x)=x^{1/p-1}w(x^{1/p})/p, x>0$, hence (see e.g., [10])

$$\lim_{n\to\infty}\sup_{x_i\in\mathbb{R}}\Big|G_i^n(a_{ni}x_i+b_{n1})-\exp(-\exp(-x_i))\Big|=0, \quad 1\le i\le d.$$

Now by [26], the claim follows if we show the pairwise asymptotic independence of Y_i, Y_j for two different indices i and j, i.e.,

$$\lim_{n\to\infty} \frac{\mathbb{P}\{Y_i > b_{ni}, Y_j > b_{ni}\}}{\mathbb{P}\{Y_i > b_{ni}\}} = 0.$$

By the result of Theorem 1.1, it follows that (see [15])

$$\lim_{n\to\infty} \frac{b_{ni}}{b_{n1}} = 1, \quad 2 \le i \le d.$$

Clearly,

$$\frac{\mathbb{P}\{Y_i > b_{ni}, Y_j > b_{ni}\}}{\mathbb{P}\{Y_i > b_{ni}\}} \le \frac{\mathbb{P}\{Y_i + Y_j > 2b_{ni}(1+o(1))\}}{\mathbb{P}\{Y_i > b_{ni}\}}$$

for all n large. For $p > 1$, since by assumption $Y_i + Y_j = \sum_{k=1}^d (\lambda_{ki} + \lambda_{kj}) X_k^p$ with $\delta_k := \lambda_{ki} + \lambda_{kj} < 2$. Applying Theorem 1.1 we obtain

$$\frac{\mathbb{P}\{Y_i + \lambda_j Y_j > (1+\lambda_j) b_{ni}(1+o(1))\}}{\mathbb{P}\{Y_i > b_{ni}\}} \quad \to \quad 0, \quad n \to \infty,$$

which follows by the Davis-Resnick property mentioned in (7). When $p = 1$, the claim follows by statement b) in Theorem 1.1 and (7). For $p \in (0,1)$, by the triangle inequality, and the assumption that $\left(\sum_{k=1}^d \lambda_{ki}^q\right)^{1/q} = \left(\sum_{k=1}^d \lambda_{kj}^q\right)^{1/q} = 1$ with $q := 1/(1-p)$, we have

$$\widetilde{\delta}_d = \left(\sum_{k=1}^d \delta_k^q\right)^{1/q} < \left(\sum_{k=1}^d \lambda_{ki}^q\right)^{1/q} + \left(\sum_{k=1}^d \lambda_{kj}^q\right)^{1/q} = 2.$$

Hence statement c) of Theorem 1.1 and (7) imply

$$\begin{aligned} \mathbb{P}\{Y_i + Y_j > 2b_{ni}(1+o(1))\} &= \mathbb{P}\left\{Y_i + Y_j > \widetilde{\delta}_d (2/\widetilde{\delta}_d) b_{ni}(1+o(1))\right\} \\ &= o(\mathbb{P}\{Y_i > b_{ni}\}), \quad n \to \infty \end{aligned}$$

and thus the claim follows. □

Proof of Theorem 2.3 In view of representation (19) and the tail behaviour of $\sum_{i=1}^d \lambda_i U_i^p$ found in the proof of Theorem 1.1, the claim follows by applying Theorem 4.2 in Appendix. □

4 Appendix

In Theorem 4.2 below we present results on the tail asymptotics of the products of two independent non-negative rvs. For its proof we need the next lemma, which is of some independent interest.

Lemma 4.1 *Let S, S^*, Y, Y^* be four independent positive rvs. Let further L be a slowly varying function at infinity and suppose that the dfs of S and S^* have upper endpoint equal to 1.*
i) Assume that $\mathbb{P}\{S > x\} \sim c\mathbb{P}\{S^* > x\}$ as $x \uparrow 1$ for some $c \in (0,\infty)$. If Y has a rapidly varying survival function satisfying further $\mathbb{P}\{Y > u\} \sim L(u)\mathbb{P}\{Y^* > u\}$ as $u \to \infty$, then for any $\delta \in (0,1)$

$$\mathbb{P}\{SY > u\} \sim c\mathbb{P}\{S^* Y > u\} \sim \mathbb{P}\{SY > u, S > \delta\} \sim L(u)\mathbb{P}\{SY^* > u\}, \quad u \to \infty. \tag{23}$$

ii) If Y and Y^* have dfs with upper endpoint equal to 1 and $\mathbb{P}\{Y > 1-1/u\} \sim c^*\mathbb{P}\{Y^* > 1-1/u\}, c^* \in (0,\infty)$ as $u \to \infty$, then we have

$$\mathbb{P}\{SY > 1-1/u\} \sim c^*\mathbb{P}\{SY^* > 1-1/u\}, \quad u \to \infty. \tag{24}$$

Proof of Lemma 4.1 *i*) Along the same lines of the proof of Lemma 1 in [8] for any $\delta \in (0,1)$ we have

$$\mathbb{P}\{SY > u\} \sim \int_\delta^1 \mathbb{P}\{Y > u/s\}\, dG(s) = \mathbb{P}\{Y > u/\delta\}\mathbb{P}\{S > \delta\} + \int_u^{u/\delta} \mathbb{P}\{S > u/y\}\, dF(y) \tag{25}$$

as $u \to \infty$, where F and G are the dfs of Y and S, respectively. Choosing δ close enough to 1 we obtain

$$\mathbb{P}\{SY > u\} \sim c\mathbb{P}\{Y > u/\delta\}\mathbb{P}\{S^* > \delta\} + c\int_u^{u/\delta} \mathbb{P}\{S^* > u/y\}\, dF(y) \sim c\mathbb{P}\{S^*Y > u\}$$

as $u \to \infty$. The other asymptotic equivalences are proved in [7], Lemma 4.1; the third claim is due to Lemma A.3 in [29].

ii) By the independence of S, Y, Y^* for all u and G the df of S we have

$$\begin{aligned}\mathbb{P}\{SY > 1-1/u\} &= \int_{1-1/u}^1 \mathbb{P}\{Y > (1-1/u)/s\}\, dG(s) \\ &\sim c^*\int_{1-1/u}^1 \mathbb{P}\{Y^* > (1-1/u)/s\}\, dG(s)\end{aligned}$$

as $u \to \infty$, hence the proof is complete. □

Remark: Let S, S^*, Y be three non-negative independent rvs. Let 1 be the upper endpoint of the dfs of S and S^* and suppose that the survival function of Y is rapidly varying at infinity. In view of Lemma 2 in [8]

$$\lim_{u\to\infty} \frac{\mathbb{P}\{SY > u\}}{\mathbb{P}\{Y > u\}} = \mathbb{P}\{S = 1\}, \quad \lim_{u\to\infty} \frac{\mathbb{P}\{S^*Y > u\}}{\mathbb{P}\{Y > u\}} = \mathbb{P}\{S^* = 1\},$$

hence if $c = \mathbb{P}\{S = 1\}/\mathbb{P}\{S^* = 1\} > 0$, then

$$\mathbb{P}\{SY > u\} \quad \sim \quad c\mathbb{P}\{S^*Y > u\}, \quad u \to \infty.$$

Theorem 4.2 *Let S, Y be two independent non-negative rvs. Let F and H denote the dfs of Y and SY, respectively. Suppose that for L some slowly varying function at infinity and some $\beta \ge 0$*

$$\mathbb{P}\{S > 1-1/u\} \quad \sim \quad L(u)u^{-\beta}, \quad u \to \infty. \tag{26}$$

Assume further that F has upper endpoint $x_F \in \{1, \infty\}$.

i) If $F \in GMDA(w)$, then

$$\mathbb{P}\{SY > u\} \quad \sim \quad \Gamma(\beta+1)\mathbb{P}\{S > 1-1/(uw(u))\}\mathbb{P}\{Y > u\}, \quad u \uparrow x_F. \tag{27}$$

Furthermore, if $\beta > 0$ and $L(x) = L > 0, \forall x > 0$, then $H \in GMDA(w)$ if and only if $F \in GMDA(w)$.

ii) If F with $x_F = 1$ satisfies (10) for some $\gamma \ge 0$, then for any $\lambda \in (-\infty, 1)$

$$\mathbb{P}\{S(Y-\lambda) > 1-1/u\} \quad \sim \quad (1-\lambda)^\gamma \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)}\mathbb{P}\{S > 1-1/u\}\mathbb{P}\{Y > 1-1/u\}, \quad u \to \infty. \tag{28}$$

Furthermore, if $\gamma > 0, L(x) = L > 0, \forall x > 0$, then F is in the Weibull MDA of Ψ_γ if and only if H is in the Weibull MDA of $\Psi_{\beta+\gamma}$.

Proof of Theorem 4.2 $i)$ Suppose that $x_F = \infty$. When S is beta distributed the claim follows from Theorem 4.1 in [18]. Let us consider some general S such that (26) holds. The claim in (27) follows by Theorem 3.1 in [19]. Next, we show that $H \in GMDA(w)$ implies $F \in GMDA(w)$. Since for any $\eta > 1, u > 0$

$$\mathbb{P}\{S > 1/\eta\}\,\mathbb{P}\{Y > \eta u\} = \mathbb{P}\{S > 1/\eta, Y > \eta u\} \le \mathbb{P}\{SY > u\} \le \mathbb{P}\{Y > u\}$$

and the fact that SY has df in the Gumbel MDA, we conclude that both SY and Y have a rapidly varying survival function. If $L(t) = L > 0, t > 0$ and $\beta > 0$, then for $\tilde{S} \stackrel{\mathcal{D}}{=} \mathcal{B}_{a,\beta}$ with $a > 0$ some arbitrary constant we find applying Theorem 3.1 in [19] that

$$\mathbb{P}\left\{\tilde{S}Y > u\right\} \sim \frac{\Gamma(a+\beta)}{L\Gamma(a)\Gamma(\beta+1)}\mathbb{P}\{SY > u\}, \quad u \to \infty,$$

provided that $\tilde{S}$ is independent of Y. Hence $\tilde{S}Y$ has df in the Gumbel MDA. It follows from Theorem 4.1 in [18] that Y has df in the Gumbel MDA with the same scaling function w as $\tilde{S}Y$. In view of (24) the case that $x_F = 1$ follows with similar arguments.

$ii)$ The idea of the proof is the same as that of the proof of the statement $i)$ making further use of $ii)$ in Lemma 4.1, Theorem 4.5 in [18] and Theorem 3.1 in [19]. □

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References

[1] R.B. Arellano-Valle and W.-D. Richter. On skewed continuous $l_{n,p}$-symmetric distributions. *Chilean J. Stat.*, 3(2):193–212, 2012.

[2] N. Balakrishnan and E. Hashorva. Scale mixtures of Kotz-Dirichlet distributions. *J. Multivariate Anal.*, 113:48–58, 2013.

[3] S. Cambanis, S. Huang, and G. Simons. On the theory of elliptically contoured distributions. *J. Multivariate Anal.*, 11(3):368–385, 1981.

[4] A. Charpentier and J. Segers. Tails of multivariate Archimedean copulas. *J. Multivariate Anal.*, 100(7):1521–1537, 2009.

[5] E. Damek, T. Mikosch, J. Rosiński, and G. Samorodnitsky. General inverse problems for regular variation. *Adv. Appl. Probab. to appear*, 2014.

[6] R. Davis and S. Resnick. Extremes of moving averages of random variables from the domain of attraction of the double exponential distribution. *Stochastic Process. Appl.*, 30(1):41–68, 1988.

[7] K. Dębicki, J. Farkas, and E. Hashorva. Random scaling of Gumbel risks. *http://arxiv.org/abs/1312.7132*, 2013.

[8] P. Embrechts, E. Hashorva, and T. Mikosch. Aggregation of log-linear risks. *J. Appl. Probab.*, 2014, in press.

[9] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling extremal events for insurance and finance*. Springer-Verlag, Berlin, 1997.

[10] M. Falk, J. Hüsler, and R.-D. Reiss. Laws of Small Numbers: Extremes and Rare Events. In *DMV Seminar*, volume 23. Birkhäuser, Basel, third edition, 2010.

[11] K.T. Fang and B.Q. Fang. Generalized symmetrized Dirichlet distributions. In *Statistical inference in elliptically contoured and related distributions*, pages 127–136. Allerton, New York, 1990.

[12] S. Foss, D. Korshunov, and S. Zachary. *An Introduction to Heavy-tailed and Subexponential Distributions.* Springer-Verlag, New York, 2011.

[13] S. Foss and A. Richards. On sums of conditionally independent subexponential random variables. *Math. Oper. Res.*, 35(1):102–119, 2010.

[14] E. Hashorva. Asymptotics of the norm of elliptical random vectors. *J. Multivariate Anal.*, 101(4):926–935, 2010.

[15] E. Hashorva. Exact tail asymptotics in bivariate scale mixture models. *Extremes*, 15(1):109–128, 2012.

[16] E. Hashorva. Exact tail asymptotics of aggregated parametrised risk. *J. Math. Anal. Appl.*, 400(1):187–199, 2013.

[17] E. Hashorva, D. Korshunov, and V.I. Piterbarg. Extremal behavior of Gaussian chaos. *arXiv:1307.5857v2*, 2013.

[18] E. Hashorva and A.G. Pakes. Tail asymptotics under beta random scaling. *J. Math. Anal. Appl.*, 372(2):496–514, 2010.

[19] E. Hashorva, A.G. Pakes, and Q. Tang. Asymptotics of random contractions. *Insurance Math. Econom.*, 47(3):405–414, 2010.

[20] M. Jacobsen, T. Mikosch, J. Rosiński, and G. Samorodnitsky. Inverse problems for regular variation of linear filters, a cancellation property for σ-finite measures and identification of stable laws. *Ann. Appl. Probab.*, 19(1):210–242, 2009.

[21] S. Kotz, N. Balakrishnan, and N. L. Johnson. *Continuous multivariate distributions. Vol. 1.* Wiley-Interscience, New York, second edition, 2000.

[22] A.J. McNeil and J. Nešlehová. Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. *Ann. Statist.*, 37(5B):3059–3097, 2009.

[23] A.J. McNeil and J. Nešlehová. From Archimedean to Liouville copulas. *J. Multivariate Anal.*, 101(8):1772–1790, 2010.

[24] A.G. Pakes. Convolution equivalence and infinite divisibility. *J. Appl. Prob.*, 41:407–424, 2004.

[25] V.I. Piterbarg and V.R. Fatalov. The Laplace method for probability measures in Banach spaces. *Uspekhi Mat. Nauk*, 50(6(306)):57–150, 1995.

[26] S.I. Resnick. *Extreme Values, Regular Variation, and Point Processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust.* Springer-Verlag, New York, 1987.

[27] W.-D. Richter. On skewed continuous $l_{n,p}$-symmetric distributions. *Lithuanian Mathematical Journal*, 51:440–449, 2011.

[28] H. Rootzén. A ratio limit theorem for the tails of weighted sums. *Ann. Proabb.*, 15:728–747, 1987.

[29] Q. Tang and G. Tsitsiashvili. Finite- and infinite-time ruin probabilities in the presence of stochastic returns on investments. *Adv. in Appl. Probab.*, 36(4):1278–1299, 2004.