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Bohr's equivalence relation in the space of Besicovitch almost periodic functions

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Abstract Based on Bohr's equivalence relation which was established for general Dirichlet series, in this paper we introduce a new equivalence relation on the space of almost periodic functions in the sense of Besicovitch, $B(\mathbb{R}, \mathbb{C})$, defined in terms of polynomial approximations. From this, we show that in an important subspace $B^2(\mathbb{R}, \mathbb{C}) \subset B(\mathbb{R}, \mathbb{C})$, where Parseval's equality and Riesz-Fischer theorem holds, its equivalence classes are sequentially compact and the family of translates of a function belonging to this subspace is dense in its own class.

Keywords Almost periodic functions · Besicovitch almost periodic functions · Bochner's theorem · Exponential sums · Fourier series

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1 Introduction

The class of almost periodic functions, whose theory was created and developed in its main features by H. Bohr during the 1920's, is the class of continuous functions possessing a certain structural property, which is a generalization of pure periodicity. This theory opened a way to study a wide class of trigonometric series of the general type and even exponential series. In this context, we can cite, among others, the papers [3,4,5,6,8,10].

Let $f(t)$ be a real or complex function of an unrestricted real variable t. The notion of almost periodicity given by Bohr involves the fact that $f(t)$ must be continuous, and for every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such

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that each interval of length l contains a number τ satisfying $|f(t+\tau)-f(t)| \leq \varepsilon$ for all t. As in [8], we will denote as $AP(\mathbb{R},\mathbb{C})$ the space of almost periodic functions in the sense of this definition (Bohr's condition). A very important result of this theory is the approximation theorem according to which the class of almost periodic functions $AP(\mathbb{R},\mathbb{C})$ coincides with the class of limit functions of uniformly convergent sequences of trigonometric polynomials of the type

$$a_1e^{i\lambda_1 t} + \ldots + a_ne^{i\lambda_n t} \tag{1}$$

with arbitrary real exponents λ_j and arbitrary complex coefficients a_j. Moreover, S. Bochner observed that Bohr's notion of almost periodicity of a function f is equivalent to the relative compactness, in the sense of uniform convergence, of the family of its translates $\{f(t+h)\}$, $h \in \mathbb{R}$.

In the course of time, some variants and extensions of Bohr's concept have been introduced, most notably by A. S. Besicovitch, W. Stepanov and H. Weyl. We refer the reader to the papers by Besicovitch [3, Chapter II], Bohr and Fœlner [7], Corduneanu [8], and by Andres, Bersani and Grande [1] and the references therein for a comprehensive treatment of this subject.

In particular, A.S. Besicovitch enlarged the class of almost periodic functions by considering the convergence of sequences of functions in a more general sense than uniform convergence. In this way, the Besicovitch spaces $B^p(\mathbb{R},\mathbb{C})$, $1 \leq p < \infty$, are obtained by the completion of the trigonometric polynomials of the form (1) with respect to the seminorms

$$\left(\limsup_{l\to\infty}(2l)^{-1}\int_{-l}^{l}|f(t)|^p dt\right)^{1/p}.$$

This topology is certainly weaker than that of the uniform convergence. In particular, the space $B^1(\mathbb{R},\mathbb{C})$ is denoted by $B(\mathbb{R},\mathbb{C})$ and contains $AP(\mathbb{R},\mathbb{C})$, $B^2(\mathbb{R},\mathbb{C})$ and all variants of almost periodic functions which were mentioned above.

Moreover, for any function $f \in B(\mathbb{R},\mathbb{C})$ there exists the mean value

$$M(f) = \lim_{l\to\infty}(2l)^{-1}\int_{-l}^{l} f(t)dt \tag{2}$$

and, at most, a countable set of values of $\lambda_k \in \mathbb{R}$ such that $a_k = a(f,\lambda_k) = M(f(t)e^{-\lambda_k t}) \neq 0$. Thus the series $\sum_{k\geq 1} a_ke^{i\lambda_k t}$ is called the Fourier series of f [8, Section 4.2]. Also, λ_k and a_k are called the Fourier exponents and coefficients of the function f, respectively.

In the case that $f \in B^2(\mathbb{R},\mathbb{C})$, with $\sum_{k\geq 1} a_ke^{i\lambda_k t}$ the Fourier series of f, it is accomplished the Parseval's equality [3, p. 109]

$$\sum_{k\geq 1}|a_k|^2 = M(|f(t)|^2) < \infty.$$

In this respect, if $f_1, f_2 \in B^2(\mathbb{R},\mathbb{C})$, then f_1 and f_2 have the same Fourier series if and only if $M(|f_1(t) - f_2(t)|^2) = 0$. That is, two functions satisfying this condition belong to the same class of equivalence defined in terms of the Fourier series. This equivalence relation is inherent to the classes $B^p(\mathbb{R},\mathbb{C})$ and it is different from the generalization of Bohr's equivalence of Definition 2 which is the main tool of this paper.

Besicovitch's generalization is interesting because, for this extension, the analogue of the Riesz-Fischer theorem is also valid, that is to say, any trigonometric series $\sum_{n\geq 1} a_n e^{i\lambda_n t}$ with $\sum_{n\geq 1} |a_n|^2$ finite is the Fourier series of a $B^2(\mathbb{R},\mathbb{C})$ almost periodic function [3, p. 110] (in this sense, $B^2(\mathbb{R},\mathbb{C})$ is also called $AP_2(\mathbb{R},\mathbb{C})$ in [8]). This is not the case for some Stepanov or Weyl functions [10]. As a consequence of the above, a Fourier series $\sum_{n\geq 1} a_n e^{i\lambda_n t}$ so that $\sum_{n\geq 1} |a_n|^2 < \infty$ represents an equivalence class of functions in $B^2(\mathbb{R},\mathbb{C})$.

In this paper we extend Bohr's equivalence relation to the Fourier series associated with the Besicovitch almost periodic functions, and hence to the Besicovitch almost periodic functions too. In this way, in view of the analogue of the Riesz-Fischer theorem and with respect to the topology of $B^2(\mathbb{R},\mathbb{C})$, the main result of our paper shows that, fixed an almost periodic function in $B^2(\mathbb{R},\mathbb{C})$, the limit points of the set of its translates are precisely the functions which are equivalent to it (see Theorem 2 in this paper). This means that the Bochner-type property, which is satisfied for the Besicovitch classes of almost periodic functions defined as above in terms of polynomials approximations (see [1, Definition 5.5, Definition 5.17 and Theorem 5.34] or [8, Section 3.4, p. 65]), is now refined for $B^2(\mathbb{R},\mathbb{C})$ in the sense that we show that the condition of almost periodicity in the Besicovitch sense implies that every sequence of translates has a subsequence that converges in $B^2(\mathbb{R},\mathbb{C})$ to an equivalent function.

2 Preliminary definitions and results on exponential sums

We shall refer to the expressions of the type

$$P_1(p)e^{\lambda_1 p} + \ldots + P_j(p)e^{\lambda_j p} + \ldots$$

as exponential sums, where the frequencies λ_j are complex numbers and the $P_j(p)$ are polynomials in p. In this paper we are going to consider some functions which are associated with a concrete subclass of these exponential sums, where the parameter p will be changed by t in the real case. In this way, as in [11], we take the following definition.

Definition 1 Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ be an arbitrary countable set of distinct real numbers, which we will call a set of exponents or frequencies. We will say that an exponential sum is in the class $\mathcal{S}_\Lambda$ if it is a formal series of type

$$\sum_{j\geq 1} a_j e^{\lambda_j p},\ a_j \in \mathbb{C},\ \lambda_j \in \Lambda. \tag{3}$$

We next introduce an equivalence relation on the classes $\mathcal{S}_\Lambda$.

Definition 2 Given an arbitrary countable set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ of distinct real numbers, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. We will say that A_1 is equivalent to A_2 (in that case, we will write $A_1 \overset{*}{\sim} A_2$) if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$, where V_n is the $\mathbb{Q}$-vector space generated by $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, such that

$$b_j = a_j e^{i\psi_n(\lambda_j)},\ j = 1, \ldots, n.$$

It is plain that the relation $\overset{*}{\sim}$ considered in the foregoing definition is an equivalence relation.

Let $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ be a basis of the vector space over the rationals generated by a set Λ of exponents, which implies that G_Λ is linearly independent over the rationals and each λ_j is expressible as a finite linear combination of terms of G_Λ, say

$$\lambda_j = \sum_{k=1}^{q_j} r_{j,k} g_k,\ \text{for some } r_{j,k} \in \mathbb{Q}. \tag{4}$$

By abuse of notation, we will say that G_Λ is a basis for Λ. Moreover, we will say that G_Λ is an integral basis for Λ when $r_{j,k} \in \mathbb{Z}$ for any j, k. By taking this into account, the equivalence relation introduced in Definition 2 can be characterized in terms of a basis for Λ (see [11, Proposition 1']).

Proposition 1 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. Fixed a basis G_Λ for Λ, for each $j \geq 1$ let $\mathbf{r}_j$ be the vector of rational components satisfying (4). Then $A_1 \overset{*}{\sim} A_2$ if and only if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a vector $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n>i}$ for $j = 1, 2, \ldots, n$. Furthermore, if G_Λ is an integral basis for Λ then $A_1 \overset{*}{\sim} A_2$ if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0>i}$ for every $j \geq 1$.*

Proof For each integer value $n \geq 1$, let V_n be the $\mathbb{Q}$-vector space generated by $\{\lambda_1, \ldots, \lambda_n\}$, V the $\mathbb{Q}$-vector space generated by Λ, and $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ a basis of V. If $A_1 \overset{*}{\sim} A_2$, by Definition 2 for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$ such that $b_j = a_j e^{i\psi_n(\lambda_j)}$, $j = 1, 2 \ldots, n$. Hence $b_j = a_j e^{i\sum_{k=1}^{i_j} r_{j,k}\psi_n(g_k)}$, $j = 1, 2 \ldots, n$ or, equivalently, $b_j = a_j e^{i<\mathbf{r}_j, \mathbf{x}_n>}$, $j = 1, 2 \ldots, n$, with $\mathbf{x}_n := (\psi_n(g_1), \psi_n(g_2), \ldots)$. Conversely, suppose the existence, for each integer value $n \geq 1$, of a vector of the form $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n>i}$, $j = 1, 2 \ldots, n$. Thus a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$ can be defined from $\psi_n(g_k) := x_{n,k}$, $k \geq 1$. Therefore $\psi_n(\lambda_j) = \sum_{k=1}^{i_j} r_{j,k}\psi(g_k) = < \mathbf{r}_j, \mathbf{x}_n >$, $j = 1, 2 \ldots, n$, and the result follows.

Now, suppose that G_Λ is an integral basis for Λ and $A_1 \overset{*}{\sim} A_2$. Thus, by above, for each fixed integer value $n \geq 1$, let $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ be a vector such that $b_j = a_j e^{i<\mathbf{r}_j, \mathbf{x}_n>}$, $j = 1, 2 \ldots, n$. Since each component of $\mathbf{r}_j$ is an integer number, without loss of generality, we can take $\mathbf{x}_n \in [0, 2\pi)^{\sharp G_\Lambda}$ as the unique vector in $[0, 2\pi)^{\sharp G_\Lambda}$ satisfying the above equalities, where we assume $x_{n,k} = 0$ for any k such that $r_{j,k} = 0$ for $j = 1, \ldots, n$. Therefore, under this assumption, if $m > n$ then $x_{m,k} = x_{n,k}$ for any k so that $x_{n,k} \neq 0$. In this way, we can construct a vector $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in [0, 2\pi)^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0>i}$ for every $j \geq 1$. Indeed, if $r_{1,k} \neq 0$ then the component $x_{0,k}$ is chosen as $x_{1,k}$, and if $r_{1,k} = 0$ then each component $x_{0,k}$ is defined as $x_{n+1,k}$ where $r_{j,k} = 0$ for $j = 1, \ldots, n$ and $r_{n+1,k} \neq 0$. Conversely, if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0>i}$ for every $j \geq 1$, then it is clear that $A_1 \overset{*}{\sim} A_2$ under Definition 2.

On the other hand, we will say that G_Λ is the *natural basis* for Λ, and we will denote it as G_Λ^*, when it is constituted by elements in Λ. That is, firstly if $\lambda_1 \neq 0$ then $g_1 := \lambda_1 \in G_\Lambda^*$. Secondly, if $\{\lambda_1, \lambda_2\}$ are $\mathbb{Q}$-rationally independent, then $g_2 := \lambda_2 \in G_\Lambda^*$. Otherwise, if $\{\lambda_1, \lambda_3\}$ are $\mathbb{Q}$-rationally independent, then $g_2 := \lambda_3 \in G_\Lambda^*$, and so on. In this way, if $\lambda_j \in G_\Lambda^*$ then $r_{j,m_j} = 1$ and $r_{j,k} = 0$ for $k \neq m_j$, where m_j is such that $g_{m_j} = \lambda_j$. In fact, each element in G_Λ^* is of the form g_{m_j} for j such that λ_j is $\mathbb{Q}$-linear independent of the previous elements in the basis. Furthermore, if $\lambda_j \notin G_\Lambda^*$ then $\lambda_j = \sum_{k=1}^{i_j} r_{j,k} g_k$, where $\{g_1, g_2, \ldots, g_{i_j}\} \subset \{\lambda_1, \lambda_2, \ldots, \lambda_{j-1}\}$.

In terms of the natural basis, we next prove another characterization which will be used later.

Proposition 2 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j \geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j \geq 1} b_j e^{\lambda_j p}$. Fixed the natural basis $G_\Lambda^* = \{g_1, g_2, \ldots, g_k, \ldots\}$ for Λ, for each $j \geq 1$ let $\mathbf{r}_j \in \mathbb{R}^{\sharp G_\Lambda^*}$ be the vector of rational components verifying (4). Then $A_1 \overset{*}{\sim} A_2$ if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in [0, 2\pi)^{\sharp G_\Lambda^*}$ such that for each $j = 1, 2, \ldots$ it is satisfied $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j>i}$ for some $\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$, with $n_{j,k} \in \mathbb{Z}$.*

Proof Suppose that $A_1 \overset{*}{\sim} A_2$. Consider $I = \{1, 2, \ldots, k, \ldots : \lambda_k \in G_\Lambda^*\}$ and $I_n = \{1, 2, \ldots, k, \ldots, n : \lambda_k \in G_\Lambda^*\}$. Let $j \in I$, then $r_{j,m_j} = 1$ and $r_{j,k} = 0$ for $k \neq m_j$, where m_j is such that $g_{m_j} = \lambda_j$. Thus, by Proposition 1, let $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$ be a vector such that

$$b_j = a_j e^{i<\mathbf{r}_j, \mathbf{x}_j>} = a_j e^{i \sum_{k=1}^{i_j} r_{j,k} x_{j,k}} = a_j e^{i r_{j,m_j} x_{j,m_j}} = a_j e^{i x_{j,m_j}}.$$

Define $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*} = \mathbb{R}^{\sharp I}$ as $x_{0,m_j} := x_{j,m_j}$ for $j \in I$. Thus, by taking $\mathbf{p}_j = (0, 0, \ldots)$, the result trivially holds for those j's such that $\lambda_j \in G_\Lambda^*$, i.e. for $j \in I$. Now, let j be such that $\lambda_j \notin G_\Lambda^*$, i.e. $j \notin I$. By Proposition 1, let $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$ be a vector such that

$$b_p = a_p e^{i<\mathbf{r}_p, \mathbf{x}_j>} = a_p e^{i \sum_{k=1}^{i_j} r_{p,k} x_{j,k}}, \ p = 1, 2, \ldots, j.$$

Note that if $p = 1, 2, \ldots, j$ is such that $\lambda_p \in G_\Lambda^*$, then

$$b_p = a_p e^{i r_{p,m_p} x_{j,m_p}},$$

which necessarily implies that $r_{p,m_p} x_{j,m_p} = r_{p,m_p} x_{p,m_p} + 2\pi n_p$, i.e. $x_{j,m_p} = x_{p,m_p} + 2\pi n_{j,p}$ for some $n_{j,p} \in \mathbb{Z}$. Hence

$$b_j = a_j e^{i<\mathbf{r}_j, \mathbf{x}_j>} = a_j e^{i \sum_{k=1}^{i_j} r_{j,k} x_{j,k}} = a_j e^{i \sum_{p \in I_{j-1}} r_{j,m_p} x_{j,m_p}} =$$

$$a_j e^{i \sum_{p \in I_{j-1}} r_{j,m_p} (x_{p,m_p} + 2\pi n_{j,p})} = a_j e^{i<\mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j>},$$

where $\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots, 0, 0, \ldots)$. Moreover, by changing conveniently the vectors $\mathbf{p}_j$, we can take $\mathbf{x}_0 \in [0, 2\pi)^{\sharp G_\Lambda^*}$ without loss of generality.

Conversely, suppose the existence of $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$ satisfying $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0 + \mathbf{p}_j> i}$ for some $\mathbf{p}_j = (2\pi n_{j,1}, 2\pi n_{j,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda^*}$, with $n_{j,k} \in \mathbb{Z}$. Let $r_{j,k} = \frac{p_{j,k}}{q_{j,k}}$ with $p_{j,k}$ and $q_{j,k}$ coprime integer numbers, and define $q_{n,k} := \mathrm{lcm}(q_{1,k}, q_{2,k}, \ldots, q_{n,k})$ for each $k = 1, 2, \ldots$. Thus, for any integer number $n \geq 1$, take $\mathbf{x}_n = \mathbf{x}_0 + \mathbf{m}_n$, where $m_{n,k} = 2\pi p_{1,k} p_{2,k} \cdots p_{n,k} q_{n,k}$, $k = 1, 2, \ldots$. Therefore, it is satisfied $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n> i}$ for each $j = 1, 2, \ldots, n$, which implies that $A_1 \overset{*}{\sim} A_2$.

As corollary, we can formulate the following result.

Corollary 1 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j \geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j \geq 1} b_j e^{\lambda_j p}$. Fixed a basis $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ for Λ, for each $j \geq 1$ let $\mathbf{r}_j \in \mathbb{R}^{\sharp G_\Lambda}$ be the vector of rational components verifying (4). Then $A_1 \overset{*}{\sim} A_2$ if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in [0, 2\pi)^{\sharp G_\Lambda}$ such that for each $j = 1, 2, \ldots$ it is satisfied $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_0 + \mathbf{q}_j> i}$ for some $\mathbf{q}_j \in \mathbb{R}^{\sharp G_\Lambda}$ which are of the form $\mathbf{q}_j = T \cdot \mathbf{p}_j^t$, where T is the change of basis matrix, with respect to the natural basis, and $\mathbf{p}_j$ is of the form $(2\pi n_{j,1}, 2\pi n_{j,2}, \ldots, 2\pi n_{j,k}, \ldots)$, $n_{j,k} \in \mathbb{Z}$.*

In particular, note that the coefficients of equivalent exponential sums have the same modulus.

3 The finite exponential sums of the classes $\mathcal{P}_{\mathbb{R},\Lambda}$

This section is focused on the following classes of finite exponential sums.

Definition 3 Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a set of $n \geq 1$ distinct real numbers. We will say that a function $f : \mathbb{R} \mapsto \mathbb{C}$ is in the class $\mathcal{P}_{\mathbb{R},\Lambda}$ if it is of the form

$$f(t) = a_1 e^{i\lambda_1 t} + \ldots + a_n e^{i\lambda_n t}, \ a_j \in \mathbb{C}, \ \lambda_j \in \Lambda, \ j = 1, \ldots, n. \tag{5}$$

The functions $f(t)$ of type (5) are also called trigonometric polynomials.

Note that Definition 2 can be particularized to the classes $\mathcal{P}_{\mathbb{R},\Lambda}$. Furthermore, if Λ is finite it is clear that it is always possible to find an integral basis for Λ. In this context, we next prove the following important result.

Theorem 1 *Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ a set of $n \geq 1$ exponents, let $a_1e^{i\lambda_1 t} + \ldots + a_ne^{i\lambda_n t}$ and $b_1e^{i\lambda_1 t} + \ldots + b_ne^{i\lambda_n t}$ be two equivalent functions in the class $\mathcal{P}_{\mathbb{R},\Lambda}$. Fixed $d > 0$ and $\varepsilon > 0$, there exists $\tau > d$ such that*

$$\sum_{j=1}^{n} |a_je^{i\lambda_j\tau} - b_j| < \varepsilon.$$

Proof Let $G_\Lambda = \{g_1, \ldots, g_m\}$, for a certain $m \geq 1$, be linearly independent over the rationals so that each $\lambda_j \in \Lambda$ is expressible as a linear combination of its terms, say

$$\lambda_j = \sum_{k=1}^{m} r_{j,k}g_k, \text{ for some } r_{j,k} = \frac{p_{j,k}}{q_{j,k}} \in \mathbb{Q},\ j = 1, 2, \ldots, n. \tag{6}$$

Consider $\varepsilon > 0$, $q := \operatorname{lcm}(q_{j,k} : j = 1, \ldots, n, k = 1, \ldots, m)$, $r := \max\{|r_{j,k}| : j = 1, \ldots, n, k = 1, \ldots, m\} > 0$ and $a := \max\{|a_j| : j = 1, 2, \ldots, n\} > 0$. Since $a_1e^{i\lambda_1 t} + \ldots + a_ne^{i\lambda_n t}$ and $b_1e^{i\lambda_1 t} + \ldots + b_ne^{i\lambda_n t}$ are equivalent, Proposition 1 assures the existence of a vector of real numbers $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,m})$ such that

$$b_j = a_je^{<\mathbf{r}_j,\mathbf{x}_0>i} = a_je^{i\sum_{k=1}^{m} r_{j,k}x_{0,k}},\ j = 1, 2, \ldots, n. \tag{7}$$

Now, as the numbers $c_k = \frac{g_k}{2\pi q}$, $k = 1, 2, \ldots, m$, are rationally independent, we next apply Kronecker's theorem [9, p.382] with the following choice: c_k, $\varepsilon_1 = \frac{\varepsilon}{a\cdot m\cdot n\cdot r\cdot E} > 0$ and $d_k = \frac{x_{0,k}}{2\pi q}$, $k = 1, 2, \ldots, m$. In this manner we assure the existence of a real number $\tau > d > 0$ and integer numbers $e_1, e_2, \ldots, e_m$ such that

$$|\tau c_k - e_k - d_k| = \left|\frac{\tau g_k}{2\pi q} - e_k - \frac{x_{0,k}}{2\pi q}\right| < \varepsilon_1,$$

that is

$$\tau g_k = 2\pi q e_k + x_{0,k} + \eta_k, \text{with } |\eta_k| < \varepsilon_1. \tag{8}$$

Therefore, from (6) and (7), with $t \in \mathbb{R}$, we have

$$\sum_{j=1}^{n} |a_je^{i\lambda_j\tau} - b_j| = \sum_{j=1}^{n} \left|a_je^{i\lambda_j\tau} - a_je^{i\sum_{k=1}^{m} r_{j,k}x_{0,k}}\right| \leq$$

$$\sum_{j=1}^{n} |a_j| \left|e^{i\tau\lambda_j} - e^{i\sum_{k=1}^{m} r_{j,k}x_{0,k}}\right| \leq a\sum_{j=1}^{n} \left|e^{i\tau\lambda_j} - e^{i\sum_{k=1}^{m} r_{j,k}x_{0,k}}\right| =$$

$$a\sum_{j=1}^{n} \left|e^{i\tau\sum_{k=1}^{m} r_{j,k}g_k} - e^{i\sum_{k=1}^{m} r_{j,k}x_{0,k}}\right|,$$

which, from (8), is equal to

$$a\sum_{j=1}^{n} \left|e^{i\sum_{k=1}^{m}(r_{j,k}2\pi q e_k + r_{j,k}x_{0,k} + r_{j,k}\eta_k)} - e^{i\sum_{k=1}^{m} r_{j,k}x_{0,k}}\right| =$$

$$a\sum_{j=1}^{n}\left|e^{i\sum_{k=1}^{m}r_{j,k}\eta_k}-1\right|\leq a\sum_{j=1}^{n}\left|\sum_{k=1}^{m}r_{j,k}\eta_k\right|\leq$$

$$anr\sum_{k=1}^{m}|\eta_k|<anr\sum_{k=1}^{m}\frac{\varepsilon}{a\cdot m\cdot n\cdot r}=\varepsilon.$$

As an immediate consequence of Theorem 1, we obtain the following corollary (compare with [11, Corollary 3]).

Corollary 2 *Given $\Lambda=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ a finite set of exponents, let $f_1(t)=\sum_{j=1}^{n}a_je^{i\lambda_jt}$ and $f_2(t)=\sum_{j=1}^{n}b_je^{i\lambda_jt}$ be two equivalent functions in the class $\mathcal{P}_{\mathbb{R},\Lambda}$. Fixed $\varepsilon>0$, there exists a relatively dense set of real numbers τ such that*

$$|f_1(t+\tau)-f_2(t)|<\varepsilon\quad\forall t\in\mathbb{R}.$$

Proof Fixed $\tau>0$, note that for any $t\in\mathbb{R}$ it is accomplished that

$$|f_1(t+\tau)-f_2(t)|\leq\sum_{j=1}^{n}|a_je^{i\lambda_j(t+\tau)}-b_je^{i\lambda_jt}|=\sum_{j=1}^{n}|a_je^{i\lambda_j\tau}-b_j|.$$

Thus, by Theorem 1 and given $d>0$, there exists $\tau_1>d$ such that

$$|f_1(t+\tau_1)-f_2(t)|<\varepsilon/2\ \forall t\in\mathbb{R}.\tag{9}$$

Moreover, since $f_1(t)$ is almost periodic, there exists a real number $l=l(\varepsilon)$ such that every interval of length l contains at least one translation number τ, associated with ε, satisfying

$$|f_1(t+\tau)-f_1(t)|\leq\varepsilon/2\text{ for all }t\in\mathbb{R}.\tag{10}$$

Consequently, from (9) and (10) we deduce the existence of a relatively dense set of real numbers τ such that any $t\in\mathbb{R}$ satisfies

$$|f_1(t+\tau+\tau_1)-f_2(t)|\leq|f_1(t+\tau_1+\tau)-f_1(t+\tau_1)|+|f_1(t+\tau_1)-f_2(t)|<\varepsilon.$$

This proves the result.

It was proved in [11, Proposition 2] that, with respect to the topology of uniform convergence, the equivalence classes in $\mathcal{P}_{\mathbb{R},\Lambda}/\overset{*}{\sim}$ are sequentially compact. We can analogously prove that this property is also true with respect to the topology of $B^2(\mathbb{R},\mathbb{C})$ (see the proof in [11, Proposition 2]).

Proposition 3 *Let Λ be a finite set of exponents and $\mathcal{G}$ an equivalence class in $\mathcal{P}_{\mathbb{R},\Lambda}/\overset{*}{\sim}$. Thus $\mathcal{G}$ is sequentially compact.*

4 Besicovitch almost periodic functions in terms of an equivalence relation

For our purposes, we next focus our attention on the Besicovitch space $B(\mathbb{R},\mathbb{C})$, whose functions are obtained by the completion of the trigonometric polynomials with respect to the seminorm $\limsup_{l\to\infty}\left(\frac{1}{2l}\int_{-l}^{l}|f(t)|dt\right)$ (see for example [8, Section 3.4]). In particular, the space of functions $B(\mathbb{R},\mathbb{C})$ contains those of the space of the almost periodic functions $AP(\mathbb{R},\mathbb{C})$ and those functions of $B^2(\mathbb{R},\mathbb{C})$. We recall that every function in $B(\mathbb{R},\mathbb{C})$ is associated with a real exponential sum with real frequencies of the form $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, which is called its Fourier series.

Definition 4 Let $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_j,\ldots\}$ be an arbitrary countable set of distinct real numbers. We will say that a function $f:\mathbb{R}\to\mathbb{C}$ is in the class $\mathcal{F}_{B^2,\Lambda}$ if it is an almost periodic function in $B^2(\mathbb{R},\mathbb{C})$ whose associated Fourier series is of the form

$$\sum_{j\geq 1} a_j e^{i\lambda_j t},\ a_j\in\mathbb{C},\ \lambda_j\in\Lambda. \tag{11}$$

It is worth noting that, in general, when we write that a function f is in $B(\mathbb{R},\mathbb{C})$ we do not have in mind the function f itself, it does represent a whole class of equivalent functions according to the relation $f_1\simeq f_2$ if and only if

$$\limsup_{l\to\infty}\left(\frac{1}{2l}\int_{-l}^{l}|f(t)-g(t)|^2dt\right)=0.$$

In terms of Definition 2, we can define an equivalence relation on the functions in $B(\mathbb{R},\mathbb{C})$, in particular on the classes $\mathcal{F}_{B^2,\Lambda}$. More specifically, we establish the following definition.

Definition 5 Given $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_j,\ldots\}$ a set of exponents, let f_1 and f_2 denote two equivalence classes of $B(\mathbb{R},\mathbb{C})/\simeq$ whose associated Fourier series are given by

$$\sum_{j\geq 1} a_j e^{i\lambda_j t} \text{ and } \sum_{j\geq 1} b_j e^{i\lambda_j t},\ a_j,b_j\in\mathbb{C},\ \lambda_j\in\Lambda.$$

We will say that f_1 is equivalent to f_2 if for each integer value $n\geq 1$ there exists a $\mathbb{Q}$-linear map $\psi_n:V_n\to\mathbb{R}$, where V_n is the $\mathbb{Q}$-vector space generated by $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$, such that

$$b_j=a_j e^{i\psi_n(\lambda_j)},\ j=1,\ldots,n.$$

In that case, we will write $f_1\overset{*}{\sim}f_2$.

The next important lemma allows us to prove that if a function f_2 is equivalent (in the sense of Definition 5) to a function f_1 belonging to the space $B^2(\mathbb{R},\mathbb{C})$, then f_2 also belongs to $B^2(\mathbb{R},\mathbb{C})$. This is clearly a consequence of Riesz-Fischer theorem [3, p. 110].

Lemma 1 *Let $f_1(t) \in B^2(\mathbb{R},\mathbb{C})$ be an almost periodic function whose Fourier series is given by $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, $a_j \in \mathbb{C}$, where $\{\lambda_1,\ldots,\lambda_j,\ldots\}$ is a set of distinct exponents. Consider $b_j \in \mathbb{C}$ such that $\sum_{j\geq 1} b_j e^{i\lambda_j t}$ and $\sum_{j\geq 1} a_j e^{i\lambda_j t}$ are equivalent. Then $\sum_{j\geq 1} b_j e^{i\lambda_j t}$ is the Fourier series associated with an almost periodic function $f_2(t) \in B^2(\mathbb{R},\mathbb{C})$ so that $f_1 \overset{*}{\sim} f_2$.*

Proof Take $\Lambda = \{\lambda_1,\ldots,\lambda_j,\ldots\}$. By the hypothesis, $f_1 \in \mathcal{F}_{B^2,\Lambda} \subset B^2(\mathbb{R},\mathbb{C})$ is determined by the series $\sum_{j\geq 1} a_j e^{i\lambda_j t}$, $a_j \in \mathbb{C}$, $\lambda_j \in \Lambda$. Moreover, since $\sum_{j\geq 1} a_j e^{i\lambda_j t} \overset{*}{\sim} \sum_{j\geq 1} b_j e^{i\lambda_j t}$, we deduce from Corollary 1 that $|b_j| = |a_j|$ for $j \geq 1$ and hence

$$\sum_{j\geq 1} |b_j|^2 = \sum_{j\geq 1} |a_j|^2 < \infty.$$

By Riesz-Fischer theorem [3, p. 110], there exists a function $f_2 \in B^2(\mathbb{R},\mathbb{C})$ such that the values b_n are the Fourier coefficients of f_2.

As it was said before, it is worth noting that a Fourier series $\sum_{n\geq 1} a_n e^{i\lambda_n t}$, such that $\sum_{n\geq 1} |a_n|^2 < \infty$, represents an equivalence class (according to the relation $\simeq$) of functions in $B^2(\mathbb{R},\mathbb{C})$ (not a single function). In fact, as we pointed out in introduction, since two almost periodic functions in the Besicovitch sense are connected in $B^2(\mathbb{R},\mathbb{C})$ when they have the same Fourier series ([3, p. 148] or [8, Section 4.2]), we immediately deduce from the results above the following corollary.

Corollary 3 *Let $f_1(t)$ and $f_2(t)$ be two equivalent functions in $B(\mathbb{R},\mathbb{C})$. If $f_1(t) \in B^2(\mathbb{R},\mathbb{C})$, then $f_2(t) \in B^2(\mathbb{R},\mathbb{C})$.*

The following result is concerned with the concept of convergence in $B^2(\mathbb{R},\mathbb{C})$ which is certainly weaker than the uniform convergence. Under this topology, we next show that the equivalence classes of $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$ are closed. In fact, more specifically, they are sequentially compact.

Proposition 4 *Let Λ be a set of exponents and $\mathcal{G}$ an equivalence class in $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$. Thus $\mathcal{G}$ is sequentially compact.*

Proof Let $\{f_l\}_{l\geq 1}$ be a sequence in an equivalence class $\mathcal{G}$ in $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$. For each $l = 1,2,\ldots$, suppose that the Fourier series which is associated with $f_l(t)$ is given by

$$\sum_{j\geq 1} a_{l,j} e^{i\lambda_j t} \text{ with } a_{l,j} \in \mathbb{C},\ \lambda_j \in \Lambda.$$

Fixed a basis $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ for Λ, let $\mathbf{r}_j = (r_{j,1}, r_{j,2}, \ldots)$ be the vector satisfying $< \mathbf{r}_j, \mathbf{g} >= \lambda_j$ for each $j \geq 1$, where $\mathbf{g} = (g_1, g_2, \ldots, g_k, \ldots)$. Since $f_1 \overset{*}{\sim} f_l$ for each $l = 1,2,\ldots$, we deduce from Proposition 1 that for each integer value $n \geq 1$ there exists $\mathbf{x}_{l,n} = (x_{l,n,1}, x_{l,n,2}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that

$$a_{l,j} = a_{1,j} e^{i<\mathbf{r}_j,\mathbf{x}_{l,n}>},\ j = 1,2\ldots,n. \qquad (12)$$

Given $l \geq 1$, let $P_{l,k}(t) = \sum_{j\geq 1} p_{j,k} a_{l,j} e^{i\lambda_j t}$, $k = 1, 2, \ldots$, be the Bochner-Fejér polynomials which converge to f_l with respect to the topology of $B^2(\mathbb{R}, \mathbb{C})$ (and converge formally to its Fourier series on $\mathbb{R}$) [3, p. 105, Theorem II]. It is worth noting that for each k only a finite number of the factors $p_{j,k}$ differ from zero, and these factors $p_{j,k}$ do not depend on l [3, p. 48]. Thus, by taking into account (12), it is clear that $\{P_{l,1}(t)\}_{l\geq 1}$ is a sequence of equivalent trigonometric polynomials and, by Proposition 3, there exists a subsequence $\{P_{l_{m,1},1}(t)\}_{m\geq 1} \subset \{P_{l,1}(t)\}_{l\geq 1}$ convergent to a certain $P_1(t) = \sum_{j\geq 1} p_{j,1} a_j e^{i\lambda_j t} \in \mathcal{P}_{\mathbb{R},\Lambda_1}$, where $\Lambda_1 = \{\lambda_j \in \Lambda : p_{j,1} \neq 0\}$, which is in the same equivalence class as $P_{1,1}(t)$. Furthermore, by Proposition 1, this means that there exists $\mathbf{x}_0^{(1)} = (x_{0,1}^{(1)}, x_{0,2}^{(1)}, \ldots) \in \mathbb{R}^{m_1}$ such that

$$p_{j,1} a_j = p_{j,1} a_{1,j} e^{i<\mathbf{r}_j, \mathbf{x}_0^{(1)}>},\ j = 1, 2 \ldots,\ \text{with } \lambda_j \in \Lambda_1,$$

where m_1 is the number of elements of any basis for Λ_1. Equivalently

$$a_j = a_{1,j} e^{i<\mathbf{r}_j, \mathbf{x}_0^{(1)}>},\ j = 1, 2 \ldots,\ \text{with } \lambda_j \in \Lambda_1.$$

Analogously, from the sequence $\{P_{l_{m,1},2}(t)\}_{m\geq 1}$, we can draw a subsequence $\{P_{l_{m,2},2}(t)\}_{m\geq 1} \subset \{P_{l_{m,1},2}(t)\}_{m\geq 1}$ convergent to a certain

$$P_2(t) = \sum_{j\geq 1} p_{j,2} a_j e^{i\lambda_j t} \in \mathcal{P}_{\mathbb{R},\Lambda_2},$$

where $\Lambda_2 = \{\lambda_j \in \Lambda : p_{j,2} \neq 0\} \cup \Lambda_1$, which is in the same equivalence class as $P_{1,2}(t)$. This implies that there exists $\mathbf{x}_0^{(2)} = (x_{0,1}^{(2)}, x_{0,2}^{(2)}, \ldots) \in \mathbb{R}^{m_2}$ such that

$$a_j = a_{1,j} e^{i<\mathbf{r}_j, \mathbf{x}_0^{(2)}>},\ j = 1, 2 \ldots,\ \text{with } \lambda_j \in \Lambda_2,$$

where m_2 is the number of elements of any basis for Λ_2. In general, for each $k = 2, 3, \ldots$, we can extract a subsequence $\{P_{l_{m,k},k}(t)\}_{m\geq 1} \subset \{P_{l_{m,k-1},k}(t)\}_{m\geq 1}$ convergent to a certain

$$P_k(t) = \sum_{j\geq 1} p_{j,k} a_j e^{i\lambda_j t} \in \mathcal{P}_{\mathbb{R},\Lambda_k},$$

where $\Lambda_k = \{\lambda_j \in \Lambda : p_{j,k} \neq 0\} \cup \Lambda_{k-1}$, which is in the same equivalence class as $P_{1,k}(t)$ and hence there exists $\mathbf{x}_0^{(k)} = (x_{0,1}^{(k)}, x_{0,2}^{(k)}, \ldots) \in \mathbb{R}^{m_k}$ (m_k is the number of elements of any basis for Λ_k) such that

$$a_j = a_{1,j} e^{i<\mathbf{r}_j, \mathbf{x}_0^{(k)}>},\ j = 1, 2 \ldots,\ \text{with } \lambda_j \in \Lambda_k. \tag{13}$$

So we get by induction a sequence $\{P_k(t)\}_{k\geq 1}$ of trigonometric polynomials which converges formally to the series

$$\sum_{j\geq 1} a_j e^{i\lambda_j t},\ \lambda_j \in \Lambda, \tag{14}$$

and, since (13) is satisfied for any $k = 1, 2, \ldots$, we can construct, for each integer value $n \geq 1$, a vector $\mathbf{x}_{0,n} \in \mathbb{R}^{\sharp G_\Lambda}$ such that

$$a_j = a_{1,j} e^{i<\mathbf{r}_j, \mathbf{x}_{0,n}>},\ j = 1, 2 \ldots, n \text{ with } \lambda_j \in \Lambda.$$

Hence the series (14) is equivalent to $\sum_{j\geq 1} a_{1,j} e^{i\lambda_j t}$ and, by Lemma 1, it is the Fourier series associated with an almost periodic function $h(t) \in B^2(\mathbb{R}, \mathbb{C})$ such that $h \overset{*}{\sim} f_1$. Consequently, $\{P_k(t)\}_{k\geq 1}$ converges with respect to the topology of $B^2(\mathbb{R}, \mathbb{C})$ to $h(t) \in \mathcal{G}$ and we can extract a subsequence of $\{f_l(t)\}_{l\geq 1}$ which also converges in $B^2(\mathbb{R}, \mathbb{C})$ to $h(t)$.

As a consequence of Proposition 4, in the topology of $B^2(\mathbb{R}, \mathbb{C})$, we next show that the family of translates of a function $f \in \mathcal{F}_{B^2,\Lambda}$ is closed on its equivalence class of $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$.

Corollary 4 *Let Λ be a set of exponents and $f \in \mathcal{F}_{B^2,\Lambda}$. Thus the limit points of the set of functions $\mathcal{T}_f = \{f_\tau(t) := f(t+\tau) : \tau \in \mathbb{R}\}$ are functions which are equivalent to f.*

Proof Since it is plain that the functions included in $\mathcal{T}_f = \{f_\tau(p) := f(t+\tau) : \tau \in \mathbb{R}\}$ are in the same equivalence class of f (see in [8, Section 4.2] the Fourier series of the translates of a function in the Besicovitch spaces), the result follows easily from Proposition 4.

Now Corollary 4 can be improved with the following result. Indeed, we next prove that, fixed a function $f \in \mathcal{F}_{B^2,\Lambda}$, the limit points of the set of the translates $\mathcal{T}_f = \{f(t+\tau) : \tau \in \mathbb{R}\}$ of f are precisely the almost periodic functions which are equivalent to f.

Theorem 2 *Let Λ be a set of exponents, $\mathcal{G}$ an equivalence class in $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$ and $f \in \mathcal{G}$. Thus the set of functions $\mathcal{T}_f = \{f_\tau(t) := f(t+\tau) : \tau \in \mathbb{R}\}$ is dense in $\mathcal{G}$.*

Proof Let $f(t)$ be a function in the class $\mathcal{F}_{B^2,\Lambda}$. We know by Corollary 4 that the limit points of the set of functions $\mathcal{T}_f = \{f_\tau(t) := f(t+\tau) : \tau \in \mathbb{R}\}$ are functions in $B^2(\mathbb{R}, \mathbb{C})$ which are equivalent to f. We next demonstrate that any function $h(t)$ which is equivalent to $f(t)$ is also a limit point of $\mathcal{T}_f$. If $\sharp\Lambda < \infty$, given $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, Corollary 2 assures the existence of an increasing sequence $\{\tau_n\}_{n\geq 1}$ of positive real numbers such that any $n \in \mathbb{N}$ verifies

$$|f(t+\tau_n) - h(t)|^2 < \varepsilon_n \ \forall t \in \mathbb{R}.$$

Hence $M(|f_{\tau_n}(t) - h(t)|^2) \to 0$ as n goes to ∞ (see (2) for the definition of the mean value $M(f)$), and the result holds for the case $\sharp\Lambda < \infty$. Consider $\sharp\Lambda = \infty$ and let $\sum_{j\geq 1} a_j e^{i\lambda_j t}$ and $\sum_{j\geq 1} b_j e^{i\lambda_j t}$ be the Fourier series of $f \in \mathcal{F}_{B^2,\Lambda}$ and $h \overset{*}{\sim} f$, respectively. Take $\varepsilon_1 = \sum_{j>1} |a_j|^2 > 0$, then Theorem 1 assures the existence of $\tau_1 > 0$ such that

$$\left| a_1 e^{i\lambda_1(t+\tau_1)} - b_1 e^{i\lambda_1 t} \right| < \sqrt{\varepsilon_1} \ \forall t \in \mathbb{R},$$

which implies

$$\left|a_1 e^{i\lambda_1 \tau_1} - b_1\right|^2 < \varepsilon_1. \tag{15}$$

Thus, from (15) and $|a_j| = |b_j|$ for any $j \geq 1$ (Corollary 1), we have that

$$\sum_{j\geq 1} |a_j e^{i\lambda_1 \tau_1} - b_j|^2 < \varepsilon_1 + \sum_{j>1} |a_j e^{i\lambda_j \tau_1} - b_j|^2 \leq \varepsilon_1 + \sum_{j>1} (|a_j| + |b_j|)^2 =$$

$$\varepsilon_1 + 4\sum_{j>1} |a_j|^2 = 5\varepsilon_1.$$

Consequently,

$$M(|f_{\tau_1}(t) - h(t)|^2) < 5\varepsilon_1.$$

Similarly, take $\varepsilon_2 = \sum_{j>2} |a_j|^2 > 0$, then Theorem 1 assures the existence of $\tau_2 > \tau_1$ such that

$$\sum_{j=1}^{2} \left|a_j e^{i\lambda_j (t+\tau_2)} - b_j e^{i\lambda_j t}\right| < \sqrt{\varepsilon_2},$$

which implies

$$\left(\sum_{j=1}^{2} \left|a_j e^{i\lambda_j \tau_2} - b_j\right|\right)^2 < \varepsilon_2. \tag{16}$$

Therefore, from (16) and $|a_j| = |b_j|$ for any $j \geq 1$, we have

$$\sum_{j\geq 1} |a_j e^{i\lambda_1 \tau_2} - b_j|^2 = |a_1 e^{i\lambda_1 \tau_2} - b_1|^2 + |a_2 e^{i\lambda_1 \tau_2} - b_2|^2 + \sum_{j>2} |a_j e^{i\lambda_j \tau_2} - b_j|^2 \leq$$

$$(|a_1 e^{i\lambda_1 \tau_2} - b_1| + |a_2 e^{i\lambda_1 \tau_2} - b_2|)^2 + \sum_{j>2} |a_j e^{i\lambda_j \tau_2} - b_j|^2 \leq$$

$$\leq \varepsilon_2 + \sum_{j>2} (|a_j| + |b_j|)^2 = \varepsilon_2 + 4\sum_{j>2} |a_j|^2 = 5\varepsilon_2.$$

Consequently,

$$M(|f_{\tau_2}(t) - h(t)|^2) < 5\varepsilon_2.$$

In general, by repeating this process, we can construct an increasing sequence $\{\tau_n\}_{n\geq 1}$ such that each τ_n satisfies that

$$\sum_{j=1}^{n} \left|a_j e^{i\lambda_j (t+\tau_n)} - b_j e^{i\lambda_j t}\right| < \sqrt{\varepsilon_n},$$

which implies

$$\left(\sum_{j=1}^{n} \left|a_j e^{i\lambda_j \tau_n} - b_j\right|\right)^2 < \varepsilon_n. \tag{17}$$

with $\varepsilon_n = \sum_{j>n} |a_j|^2$. Thus, from (17) we have

$$M(|f_{\tau_n}(t) - h(t)|^2) = \sum_{j\geq 1} |a_j e^{i\lambda_1\tau_n} - b_j|^2 =$$

$$\sum_{j=1}^{n} |a_j e^{i\lambda_j\tau_n} - b_j|^2 + \sum_{j>n} |a_j e^{i\lambda_j\tau_n} - b_j|^2 \leq$$

$$\left(\sum_{j=1}^{n} |a_j e^{i\lambda_j\tau_n} - b_j|\right)^2 + \sum_{j>n} |a_j e^{i\lambda_j\tau_n} - b_j|^2 \leq$$

$$\leq \varepsilon_n + \sum_{j>n} (|a_j| + |b_j|)^2 = \varepsilon_n + 4\sum_{j>n} |a_j|^2 = 5\varepsilon_n.$$

Note that $\sum_{j\geq 1} |a_j|^2 < \infty$, then $\sum_{j>n} |a_j|^2$ tends to 0 when n goes to ∞. Consequently, the sequence of functions $\{f(t+\tau_n)\}_{n\geq 1}$ converges in $B^2(\mathbb{R},\mathbb{C})$ to $h(t)$, and the result holds.

Corollary 5 *Let $f \in B^2(\mathbb{R},\mathbb{C})$ and $f_1 \overset{*}{\sim} f$. There exists an increasing unbounded sequence $\{\tau_n\}_{n\geq 1}$ of positive numbers such that the sequence of functions $\{f(t+\tau_n)\}_{n\geq 1}$ converges in $B^2(\mathbb{R},\mathbb{C})$ to $f_1(t)$. In fact, given $\varepsilon > 0$ there exists a satisfactorily uniform set of positive numbers τ such that*

$$M(|f(t+\tau) - f_1(t)|^2) < \varepsilon.$$

Proof Let $f \in B^2(\mathbb{R},\mathbb{C})$, then $f \in \mathcal{F}_{B^2,\Lambda}$ for some set Λ of exponents. Let $\mathcal{G}$ be the equivalence class in $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$ so that $f \in \mathcal{G}$ and let $f_1 \overset{*}{\sim} f$. Thus, by Theorem 2 (see also its proof), there exists an increasing unbounded sequence $\{\tau_n\}_{n\geq 1}$ of positive numbers such that the sequence of functions $\{f(t+\tau_n)\}_{n\geq 1}$ converges in $B^2(\mathbb{R},\mathbb{C})$ to $f_1(t)$. Equivalently, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$M(|f(t+\delta_n) - f_1(t)|^2) < \varepsilon/2 \ \forall n \geq n_0.$$

Moreover, since $f(t)$ is almost periodic in the sense of Besicovitch, there exist a set $S = \{\tau_k\} \subset \mathbb{R}$ and $l = l(\varepsilon) > 0$ such that the ratio of the maximum number of elements of S included in an interval $(a, a+l)$ to the minimum number is less than 2 and satisfy

$$M(|f(t+\tau_k) - f(t)|^2) < \varepsilon/2.$$

Hence any τ_k satisfies

$$M(|f(t+\delta_n+\tau_k) - f_1(t)|^2) \leq M(|f(t+\delta_n+\tau_k) - f(t+\delta_n)|^2) + \\ + M(|f(t+\delta_n) - f_1(t)|^2) < \varepsilon \quad \forall n \geq n_0,$$

which proves the result.

It is known that the almost periodic functions in the Besicovitch spaces $B^p(\mathbb{R},\mathbb{C})$, $1 \leq p < \infty$, satisfy the Bochner-type property consisting of the relative compactness of the set $\{f(t+\tau)\}$, $\tau \in \mathbb{R}$, associated with an arbitrary function $f \in B^p(\mathbb{R},\mathbb{C})$ (see [1, Theorem 5.34] or [8, Section 3.4]). As an important consequence of Theorem 2, we next refine this property for the case of $B^2(\mathbb{R},\mathbb{C})$ in the sense that we show that the condition of almost periodicity, in the sense of Besicovitch, of a function $f(t)$ implies that every sequence $\{f(t+\tau_n)\}$, $\tau_n \in \mathbb{R}$, of translates of f has a subsequence that converges with the topology of $B^2(\mathbb{R},\mathbb{C})$ to a function which is equivalent to f.

Corollary 6 *If $f \in B^2(\mathbb{R},\mathbb{C})$, then the compact closure of its set of translates coincides with its equivalence class.*

Proof First of all, we recall that any function $f \in B(\mathbb{R},\mathbb{C})$ has an associated Fourier series. Let $f \in B^2(\mathbb{R},\mathbb{C})$, then $f \in \mathcal{F}_{B^2,\Lambda}$ for some set Λ of exponents. Now, let $\mathcal{G}$ be the equivalence class in $\mathcal{F}_{B^2,\Lambda}/\overset{*}{\sim}$ so that $f \in \mathcal{G}$. By Theorem 2, all the limit points of the translates of f are exponential sums which are included in $\mathcal{G}$ and, in fact, the compact closure of the set of the translates of f coincides with $\mathcal{G}$.

Remark 1 Given $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ a set of exponents, consider $A_1(p)$ and $A_2(p)$ two exponential sums in the class $\mathcal{S}_\Lambda$, say $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$. Let V be the $\mathbb{Q}$-vector space generated by Λ. We will say that A_1 is B-equivalent to A_2 if there exists a $\mathbb{Q}$-linear map $\psi_n : V \to \mathbb{R}$ such that

$$b_j = a_j e^{i\psi_n(\lambda_j)},\ j = 1, 2, \ldots.$$

It is easy to prove that, fixed a basis G_Λ for Λ, A_1 is B-equivalent to A_2 if and only if there exists $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j,\mathbf{x}_0>i}$ for every $j \geq 1$, where the $\mathbf{r}_j$'s are the vectors of rational components verifying (4).

From this and Proposition 1, it is worth noting that Definition 2 and definition of B-equivalence are equivalent in the case that it is possible to obtain an integral basis for the set of exponents Λ. Consequently, all the results of this paper which can be formulated in terms of an integral basis are also valid under the B-equivalence. (in particular, those related to the finite exponential sums in Section 3).

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