University Polytechnic Catalunya

Department of Applied Mathematics

Ph.D. Thesis

Product of digraphs, (super) edge-magic valences and related problems

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Contents

Introduction

Discrete Mathematics, and in particular Graph Theory, has gained a lot of popularity during the last 7 decades. There are many branches of graph theory and a wide variety of beautiful and challenging problems have been originating from them. Among them, the area of graph labelings has experimented a fast development, in particular during the last decade of the twentieth century up to the present day. Over 2000 papers in the literature, a very complete dynamic survey by Joseph Gallian [17] and four books dedicated to this subject [4, 22, 31, 44] validate the fact that graph labelings are gaining more importance day after day.

The popularity of graph labelings is due to various factors. We enumerate some of them in what follows.

The beauty and challenge that graph labelings offer to researchers: Some of the problems that can be found in the literature have proven to be very challenging, since in spite of the efforts for decades and many important researchers working on them, they are still open. Among these problems we point out the graceful tree conjecture and the harmonious tree conjecture. A graph is *graceful* if there is an injective function from the vertex set of the graph to the numbers $\{0, 1, 2, ..., q\}$ where q is the size of the graph, such that the absolute value of the differences of the labels of adjacent vertices are all mutually different. Such a function is called a *graceful labeling* of the graph [19, 48]. The graceful tree conjecture states that all trees are graceful [48]. A graph is *harmonious* if there exists an injective function from the vertex set of the graph to the additive group $\mathbb{Z}_q$ such that all sums of the labels of adjacent vertices taken in $\mathbb{Z}_q$ are mutually different, where q denotes the size of the graph. If the graph is a tree then the injection condition is relaxed, and exactly two vertices are allowed to have the same label. Such a function

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is called a *harmonious labeling* [20]. The harmonious tree conjecture states that all trees are harmonious [20].

Applications of graph labelings to other branches of mathematics: In fact, graceful labelings defined above appeared as an alternative way to attack the Ringel-Kotzig conjecture which states that the complete graph on $2n+1$ vertices can be decomposed into $2n+1$ copies of a given tree of size n. Also harmonious labelings were originally introduced in relation to additive bases of integers and error correcting codes [20].

Applications of graph labelings to other branches of science: Gary Bloom and Solomon Golomb devoted many efforts to study applications of graph labelings to different parts of science. The following two papers by them [5, 6] are very popular and constitute a great source for those who are interested to study such applications. They can be found in X-ray, crystallography, coding theory, radar, astronomy, circuit designs and communications design.

We have already seen graceful and harmonious labelings. However there are many other types of labelings. Gallian's survey [17] constitutes a very complete source of information about the different types of labelings. However, due to their popularity and oldness, we feel that graceful and harmonious labelings are among the most important ones. Another very important type of labelings are super edge-magic labelings introduced in 1998 by Enomoto et al. [11] as a particular case of edge-magic labelings, introduced in 1970 by Kotzig and Rosa [26]. An *edge-magic labeling* of a graph G of order p and size q is a bijective function $f : V(G) \cup E(G) \to [1, p+q]$ such that the sum $f(x) + f(xy) + f(y)$ is constant for any $xy \in E(G)$. The constant is called the *valence* [26] or the *magic sum* [44] of the labeling f. We write $val(f)$ to denote the valence of f. If f has the extra property that $f(v) = [1, p]$, then G is called *super edge-magic* and f is a *super edge-magic labeling* of G. It is worthwhile mentioning that Acharya and Hegde had defined in 1991 the concept of strongly indexable graph [1], and the sets of strongly indexable graphs and super edge-magic graphs coincide. It turns out that super edge-magic labelings have become very important in the world of graph labelings since they constitute a very powerful link among labelings. The links have been deeply studied in [13, 23, 35, 38]. A key tool in the relation is the $\otimes_h$-product, a type of digraph product that was introduced by Figueroa-Centeno et al. in 2008 [16]. Furthermore, new relations among super edge-magic labelings and other well studied and important combinatorial

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problems, that constitute now days the main topic of many research papers, have been recently found. For instance, it has been shown recently in [30], that super edge-magic labelings can be used to create Skolem and Langford type sequences. Since such sequences have strong relations with Steiner triple systems and many other problems, it follows that these connections also exist between super edge magic labelings and all these problems. In summary, we can say that the importance and power of super edge-magic labelings lies in the fact that they are a connecting bridge among many different topics, some of them, seeming totally unrelated.

A problem that has caught the attention of many researchers since the beginning of (super) edge-magic labelings is to study the valences of such labelings. One of the first papers in which a problem of this type appeared was [18] and the authors studied the valences of edge magic labelings of small cycles. In the same paper it was asked to characterize the set of edge-magic valences for any cycle. Following this line of thinking, Figueroa-Centeno et al. [15] characterized the set of edge-magic and super edge magic valences for the family of stars. López et al. [37, 41] introduced the notion of perfect (super) edge-magic graphs to refer those graphs in which all theoretical (super) edge-magic valences are attained, and proved that a particular family of crowns are perfect (super) edge-magic. Results in the opposite direction can be found in [39] by López et al. Since then, not many papers have appeared in the literature focusing on this problem, and we will make this problem as one of the main focuses of the thesis.

It seems that the techniques used in order to study these valences can also be used in order to obtain new and surprising relations with graph decompositions. This will be the first time in which relations of this nature will be obtained and this will strength the relations existing between the fields of graph labelings and graph decompositions.

Now we will provide the main problems developed in the thesis.

Problem 1: The valence problem

One of our objectives is to study the problem of the valences for (super) edge-magic graphs and to enlarge the class of perfect edge-magic and perfect super edge-magic graphs. We will study the implications of the style: if a graph G is perfect (super) edge-magic then the graph H is also perfect (super) edge-magic, where H is obtained from G in some way. Also we will prove

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that certain graphs are not perfect (super) edge-magic. Finally, we would like to mention that a very important goal with our research is to better understand the problem of the valences of the cycles and develop techniques to obtain many edge-magic valences for cycles. We want to remark that due to the paper of Goldbold and Slater [18], this has become a very famous open problem in the area of graph labelings. In spite of the fact that a final solution of this problem seems to be quite far away, there are some interesting partial results obtained using different techniques. We will compare our new techniques with the ones that already exist as well as the results in [45].

Problem 2: The $\otimes_h$-product applied to labelings

We explore new labeling properties of the $\otimes_h$-product and how these properties allow us to enlarge the families of labeled graph. We are particularly interested in finding new relations among labelings. There is a big gap in the literature involving enumerative graph labeling results. That is to say, in almost all cases, the authors are happy enough when obtaining that a given family of graphs admits a labeling of one type or another. But they usually do not try to get bounds on the number of such labelings. The product constitutes a great tool to obtain lower bounds for the number of (super) edge-magic labelings, among other classes of labelings, that graphs admit. We intent to take advantage of this fact in the thesis to get enumerative results. We remark that such results are extremely rare in the literature.

Problem 3: The $\otimes_h$-product applied to decompositions

We explore new structural results obtained using the $\otimes_h$-product, edge-magic and super edge-magic labelings and how these properties allow us to obtain new results in terms of graph decompositions. Although we already have seen that relations between labelings and decompositions have been studied for a long time, there have never been an instance where such relations are studied in the context of (super)edge-magic labelings. The relations proposed in the thesis are absolutely novel.

The organisation of the thesis is as follows:

The first three sections of Chapter 1 explain the basic definitions and results that will help the reader to better understand the thesis. In the last two sections, we provide some of the most important definitions, examples and basic results related to graph labelings and digraph products that will

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be used in the later chapters.

In Chapter 2, we study the super edge-magic properties of some types of super edge-magic graphs of equal order and size. The negative results found in Section 2.3 are specially interesting since these kind of results are not common in the literature. Furthermore, the few results found in this direction usually meet one of the following reasons: too many vertices compared with the number of edges; too many edges compared with the number of vertices; or parity conditions. All previous reasons fail in our results.

In Chapter 3, we study the valences for (super) edge-magic labelings of crowns $C_m \odot \overline{K}_n$ and we prove that the crowns are perfect (super) edge-magic when $m = pq$ where p and q are different odd primes. We also provide a lower bound for the number of different valences of $C_m \odot \overline{K}_n$, in terms of the prime factors of m.

In Chapter 4, we introduce a new labeling construction by changing the role of the factors in the $\otimes_h$-product. Using this new construction the field of applications grows. In particular, we can improve the information about magic sums of cycles and crowns.

In Chapter 5, we establish a relationship existing between the (super) edge-magic valences of certain types of bipartite graphs (where labelings involving sums are used) to characterize the existence of a particular type of decompositions of bipartite graphs.

Chapter 1

Introduction to Graph Theory and Graph Labelings

1.1 Basic notation and terminology

In this first section, we provide some fundamental definitions and notation that will be used throughout in this thesis. For an overview on general graph theory, [7, 49] can be referred.

We begin this by introducing the definition of graph. A *graph* G is a finite nonempty set of objects called *vertices*, together with a set of unordered pairs of distinct vertices of G called *edges*. The vertex and edge sets of G are usually denoted by $V(G)$ and $E(G)$ respectively. The *order* is $|V(G)|$ and the *size* is $|E(G)|$. If an edge $e = \{u, v\}$, then we say that e joins the vertices u and v, and u and v are said to be *adjacent vertices* in G. From now on, for simplicity, we will denote $\{u, v\}$ by uv. We say that a graph G of order p and size q is a (p, q)-*graph*.

A graph can also be described using diagrams in which each element of the vertex set of the graph is represented by a dot and each edge $e = uv$ is represented by a curve joining the dots that represent the vertices u and v.

For example, if we consider the graph G with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_5v_6\}$, then a possible diagram is shown
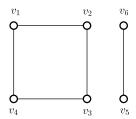
in Figure 1.1.

Figure 1.1: A graph G.

Another very common way is by means of *adjacency matrix*. Let G be a graph with $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$. Then define its $p \times p$ adjacency matrix $A = (a_{ij})$ to be

$$a_{ij} = \begin{cases} 1, & if \ v_i v_j \in E(G), \\ 0, & if \ v_i v_j \notin E(G). \end{cases}$$

For example, the adjacency matrix for the graph of Figure 1.1 is shown in Figure 1.2.

Two adjacent vertices are referred to as neighbors of each other. If uv and vw are distinct edges in G, then uv and vw are *adjacent edges*. The vertex u and the edge uv are said to be *incident* with each other. Similarly, v and uv are incident. Edges with identical end-vertices are called *loops* and repeated edges are called *multiple edges*.

For the graph G in Figure 1.1, the vertices v_1 and v_2 are therefore adjacent in G, while the vertices v_1 and v_3 are not adjacent. The edges $v_1 v_2$ and $v_2 v_3$ are adjacent in G, while the edges $v_1 v_2$ and $v_3 v_4$ are not adjacent.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

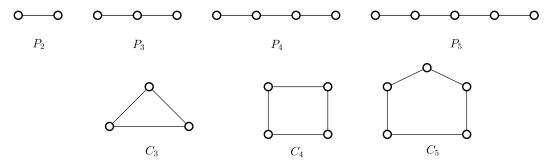Figure 1.2: The adjacency matrix for the graph in Figure 1.1.

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Figure 1.3: Paths and cycles of order 5 or less.

Two other classes of graphs that are often referred in this thesis are paths and cycles. For an integer $n \geq 1$, the *path* P_n is a graph of order n and size $n - 1$ whose vertices can be labeled by $v_1, v_2, v_3, \cdots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, 3, \cdots, n - 1$. For an integer $n \geq 3$, the *cycle* C_n is a graph of order and size n whose vertices can be labeled by $v_1, v_2, v_3, \cdots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, 3, \cdots, n - 1$.

A parameter that appears often when studying graphs is the degree of a vertex. The *degree of a vertex* v in a graph G is the number of the vertices in G that are adjacent to v, denoted by $deg_G(v)$ or simply $deg(v)$. A vertex v of a graph G is called *even* if its degree is even and *odd* if its degree is odd. A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end-vertex*. An edge incident with an end-vertex is called a *pendant edge*.

Next, we will define what it means to say two graphs are equal. This is the concept of isomorphism. Two graphs G and H are *isomorphic* if there exists a bijective function $\phi : V(G) \rightarrow V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is called an *isomorphism* from G to H. If G and H are isomorphic, we write $G \cong H$.

The graphs G and H of Figure 1.4 are isomorphic and the function $\phi : V(G) \rightarrow V(H)$ defined by $\phi(u_1) = v_5, \phi(u_2) = v_6, \phi(u_3) = v_1, \phi(u_4) = v_7, \phi(u_5) = v_2, \phi(u_6) = v_4, \phi(u_7) = v_3$ is an isomorphism.

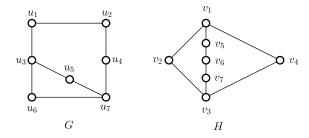A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

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Figure 1.4: Two isomorphic graphs.

If H is a subgraph of G, then we will write $H \subseteq G$. For a vertex v in a nonempty graph $G = (V, E)$, the *subgraph* $G - v$ is the graph obtained by deleting v from G, that is with the vertex set $V(G - v) = V(G) - v$ and $E(G - v) = E(G) - \{e \in E(G) : v \text{ is incident with } e\}$. For a nonempty subset S of $V(G)$, the *subgraph induced by* S, denoted by $G[S]$ of G, has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. For any edge e in a nonempty graph $G = (V, E)$, the *subgraph* $G - e$ is defined as $V(G - e) = V(G)$ and $E(G - e) = E(G) - \{e\}$.

Next, we introduce the concept of connectedness. Two vertices u and v in a graph G are *connected* if there exists a path between the vertices u and v. The graph G is *connected* if every pair of vertices of G are connected. A graph G that is not connected is a *disconnected graph*.

A connected subgraph H of a graph G is a *component* of G if H is not a proper subgraph of any connected subgraph of G. Thus every component of G is an induced subgraph of G.

The graph G in Figure 1.5 is connected since there is a path between every pair of vertices in G. On the other hand, graph H is disconnected since, for example, H contains no path between u_3 and u_5.

A *directed graph* or *digraph* D is a finite nonempty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of D called *arcs* or *directed edges*. As with graphs, the vertex set of D is denoted by $V(D)$ and the arc set of D is denoted by $E(D)$. The order and the size are defined to be $|V(D)|$ and $|E(D)|$, respectively. When a digraph is represented as a diagram, the direction of each arc is indicated
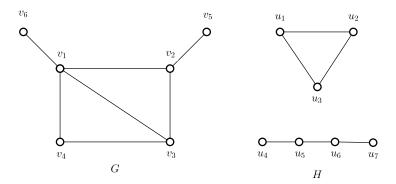
Figure 1.5: A connected graph G and a disconnected graph H.

by an arrowhead.

Another way of representing digraphs is by means of *adjacency matrix*. Let D be a digraph with $V(D) = \{v_1, v_2, v_3, \ldots, v_p\}$. Then $p \times p$ adjacency matrix $A = (a_{ij})$ is defined by

$$a_{ij} = \left\{ \begin{array}{ll} 1, & if \ (v_i v_j) \in E(D), \\ 0, & if \ (v_i v_j) \notin E(D). \end{array} \right.$$

Much of the terminology used for digraphs is quite similar to that used for graphs. However, the concept of degree of a vertex for graphs is substituted by *indegree* and *outdegree* of a vertex in the case of digraphs. For a vertex v in a digraph D, the *outdegree* $o(v)$ of v is defined as

$$o(v) = |\{u : (v, u) \in E(D)\}|.$$

The *indegree* $i(v)$ of v is defined as

$$i(v) = |\{u : (u, v) \in E(D)\}|.$$

Figure 1.6 shows a digraph D with vertex set $V = \{u, v, w, x\}$ and arc set $E = \{(u, v), (u, w), (v, u), (v, w), (w, x), (x, u), (x, w)\}$.

A digraph is called an *oriented graph* if whenever (u, v) is an arc of D, then (v, u) is not an arc of D. An oriented graph D can be obtained from a

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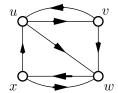
Figure 1.6: A digraph.
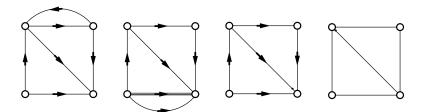
Figure 1.7: Digraphs with the same underlying graph.

graph G by assigning a direction to each edge of G and hence transforming every edge of G into an arc. The digraph D is also called an orientation of G.

The *underlying graph* of a digraph D is that graph obtained by replacing each arc (u, v) or symmetric pair $(u, v), (v, u)$ of arcs by the edge uv. Figure 1.7 shows the digraphs with the same underlying graph.

For integers $m \leq n$, we use $[m, n]$ to denote $\{m, m + 1, \ldots, n\}$. Next, we introduce concept of decomposition of graphs. A *decomposition* of a simple graph G is a collection $\{H_i : i \in [1, m]\}$ of subgraphs of G such that $\cup_{i \in [1,m]} E(H_i)$ is a partition of the edge set of G. If the set $\{H_i : i \in [1, m]\}$ is a decomposition of G, then we denote it by $G \cong H_1 \oplus H_2 \oplus \ldots \oplus H_m = \oplus_{i=1}^{n} H_i$. An example of graph decomposition is shown in Figure 1.8.

K_7

$3C_7$
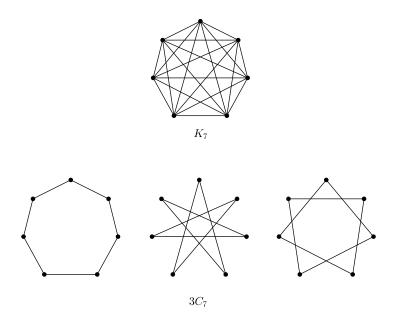
Figure 1.8: A decomposition of K_7 into $3C_7$.

1.2 Some special classes of graphs

Throughout this thesis, there are certain classes of graphs that occur very often. We will describe some of them now.

A graph G is *r*-regular if $deg(v) = r$ for all the vertices $v \in G$. All the cycles are 2-regular graphs. Figure 1.9 shows a 3-regular graph.

A graph G is *bipartite* if $V(G)$ can be partitioned into two sets X and
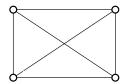
Figure 1.9: A 3-regular graph.

$K_{2,3}$ $K_{1,3}$
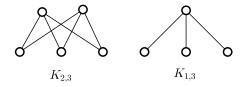
Figure 1.10: Complete bipartite graphs.
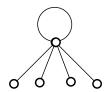
Figure 1.11: The graph $K_{1,4}^l$.

Y so that every edge of G joins a vertex of X and a vertex of Y. A graph G is a *complete bipartite graph* if $V(G)$ can be partitioned into two sets X and Y such that xy is an edge of G if and only if $x \in X$ and $y \in Y$ and is denoted by $K_{|X|,|Y|}$. The complete bipartite graph $K_{1,|y|}$ is called a *star*. The complete bipartite graphs $K_{2,3}$ and $K_{1,3}$ are shown in Figure 1.10. $K_{1,n}^l$ is the graph obtained from the star $K_{1,n}$ with a loop attached to its central vertex. Figure 1.11 shows $K_{1,4}^l$.

1.3   Operations on graphs

There are many ways of producing a new graph from one or more given graphs. We begin this section with the complement of a graph. The *complement* $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. A graph G is *self-complementary* if $G \cong \overline{G}$. A graph G and its complement are shown in Figure 1.12.

The *union* $G = G_1 \cup G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of $k(\geq 2)$
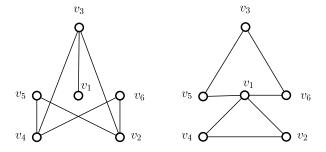
Figure 1.12: A graph and its complement.

disjoint copies of a graph H, then we write $G = kH$.

The *Cartesian product* of graphs G_1 and G_2 is the graph $G = G_1 \times G_2$ obtained in the following way.

$$V(G) = V(G_1) \times V(G_2)$$

and

$$(x_1, x_2)(y_1, y_2) \in E(G) \Leftrightarrow x_1 = y_1 \text{ and } x_2 y_2 \in E(G_2)$$

or

$$x_2 = y_2 \text{ and } x_1 y_1 \in E(G_1).$$

An example is given in Figure 1.13.
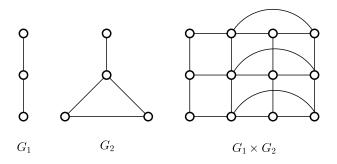
Figure 1.13: The Cartesian product of two graphs.
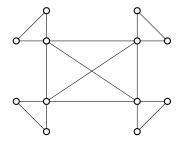
Figure 1.14: The corona product $K_4 \odot K_2$.

The next operation is the corona product of two graphs. The *corona product* of two graphs G and H is the graph $G \odot H$ obtained by placing a copy of G and $|V(G)|$ copies of H and then joining each vertex of G with all vertices in one copy of H in such a way that all vertices in the same copy of H are joined exactly to one vertex of G. If we denote by $\overline{K}_n$, the complementary graph of the complete graph K_n, $n \in \mathbb{N}$, then the corona product of C_m and $\overline{K}_n$ is the family $C_m \odot \overline{K}_n$ that has been deeply studied. An example for corona product is shown in Figure 1.14.

1.4 (Super) edge-magic labelings

In this section, we will provide definitions and results related to edge-magic and super edge-magic labelings. We provide some proofs as a matter of completeness. We will also provide examples, in order to familiarize the reader with these concepts.

In 1970, Kotzig and Rosa [26] introduced the concepts of edge-magic graphs and edge-magic labelings as follows: Let G be a (p, q)-graph. Then G is called *edge-magic* if there is a bijective function $f : V(G) \cup E(G) \to [1, p+q]$ such that the sum $f(x)+f(xy)+f(y) = k$ for any $xy \in E(G)$. Such a function is called an *edge-magic labeling* of G and k is called the *valence* [26] or the *magic sum* [44] of the labeling f. We denote the valence of f by $val(f)$.

Motivated by the concept of edge-magic labelings, Enomoto et al. [11] introduced in 1998 the concepts of super edge-magic graphs and labelings
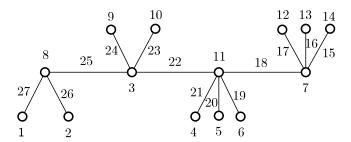
Figure 1.15: An edge-magic labeling of a caterpillar with valence 36.

as follows: Let $f : V(G) \cup E(G) \rightarrow [1, p + q]$ be an edge-magic labeling of a (p, q)-graph G with the extra property that $f(V(G)) = [1, p]$. Then G is called *super edge-magic* and f is a *super edge-magic labeling* of G. It is worthwhile mentioning that Acharya and Hegde had already defined in [1] the concept of *strongly indexable graph* that turns out to be equivalent to the concept of super edge-magic graph. We take this opportunity to mention that although the original definitions of (super) edge-magic graphs and labelings were originally provided for simple graphs (that is to say, graphs with no loops nor multiple edges), we understand these definitions for any graph. Therefore, unless otherwise specified, the graphs considered in this thesis are not necessarily simple. In [13], Figueroa-Centeno et al. provided the following useful characterization of super edge-magic simple graphs, that works in exactly the same way for graphs in general.

Lemma 1.4.1. [13] Let G be a (p, q)-graph. Then G is super edge-magic if and only if there is a bijective function $g : V(G) \longrightarrow [1, p]$ such that the set $S = \{g(u) + g(v) : uv \in E(G)\}$ is a set of q consecutive integers. In this case, g can be extended to a super edge-magic labeling f with valence $p + q + \min S$.

Proof. Assume that there exists such a function g and let $xy \in E(G)$ such that $g(x) + g(y) = \min S$. Then for any edge $uv \in E(G)$, $g(uv) = p + q + \min S - g(u) - g(v)$. Then $g(E(G)) = \{p + 1, p + 2, \cdots, p + q\}$.

Conversely, if G is a super edge-magic graph with a super edge-magic

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labeling g with valence k, then $S = \{k - g(uv) : uv \in E(G)\} = \{k - (p + 1), k - (p + 2), \cdots, k - (p + q)\}$. □

The next result is a necessary condition for a regular graph to be super edge-magic.

Lemma 1.4.2. [13] If G is an r-regular super edge-magic (p, q)-graph, where $r > 0$, then q is odd and the valence of any super edge-magic labeling of G is $(4p + q + 3)/2$.

Enomoto et al. [11] were the first ones to observe the following result for which we provide the proof also.

Lemma 1.4.3. [11] A cycle of order n is super edge-magic if and only if n is odd.

Proof. Assume first that n is odd. Let $V(C_n) = \{v_i\}_{i=1}^{n}$ and $E(C_n) = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_n v_1\}$. The function $f : V(C_n) \to \{i\}_{i=1}^{n}$ defined by the rule

$$f(v_i) = \begin{cases} (i+1)/2, & \text{if } i \text{ is odd,} \\ (i+1+n)/2, & \text{if } i \text{ is even.} \end{cases}$$

is a super edge-magic labeling of C_n.

Conversely, let C_n is a super edge-magic cycle of order n. By Lemma 1.4.2, the valence of C_n is $(5n+3)/2$. Since valence is an integer, this implies that n is odd. □

In particular, according to Lemma 1.4.1, the minimum induced sum of a cycle with n vertices is

$$\frac{n+3}{2}. \tag{1.4.1}$$

1.5 The $\otimes_h$-product

Next, we introduce the definition of the $\otimes_h$-product. In [16], Figueroa et al. defined the following product: Let D be a digraph and let Γ be a family of

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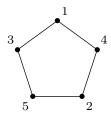
Figure 1.16: A super edge-magic labeling of C_5.

digraphs with the same set V of vertices. Assume that $h : E(D) \rightarrow \Gamma$ is any function that assigns elements of Γ to the arcs of D. Then the digraph $D \otimes_h \Gamma$ is defined by (i) $V(D \otimes_h \Gamma) = V(D) \times V$ and (ii) $((a, i), (b, j)) \in E(D \otimes_h \Gamma) \Leftrightarrow (a, b) \in E(D)$ and $(i, j) \in E(h(a, b))$. Note that when h is constant, $D \otimes_h \Gamma$ is the Kronecker product.

Let D be the digraph with vertex set $\{1, 2, 3\}$ and arc set $\{(1, 1), (1, 2), (1, 3)\}$. Let F_1 and F_2 be the digraphs on $V = \{1, 2, 3\}$, such that $E(F_1) = \{(1, 2), (2, 3), (3, 1)\}$ and $E(F_2) = \{(1, 3), (3, 2), (2, 1)\}$. Let $h : E(D) \longrightarrow \{F_1, F_2\}$ be the function defined by $h((1, 1)) = F_1$, $h((1, 2)) = F_2$ and $h((1, 3)) = F_2$. Then $D \otimes_h \Gamma$ is the digraph that appears in Figure 1.17.
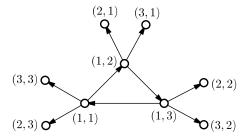
Figure 1.17: An example of the $\otimes_h$-product.

There is a different way to represent this digraph product. It is by means of adjacency matrices of the digraphs involved and the adjacency matrix of the product itself. Let $A(D)$ and $A(F)$ be the adjacency matrices of D and $F \in \Gamma$ respectively. Let $\{a_1, a_2, \cdots, a_m\}$ be the vertices of D and

$\{b_1, b_2, \cdots, b_n\}$ be the vertices of F. The vertices of $D \otimes_h \Gamma$ be labeled as $\{(a_1, x_1), (a_1, x_2), \cdots, (a_1, x_n)(a_2, x_1), \cdots, (a_m, x_n)\}$. Then the adjacency matrix of the product, denoted by $A(D \otimes_h \Gamma)$ is obtained by multiplying every 0 entry of $A(D)$ by the $n \times n$ null matrix and every 1 entry of $A(D)$ by $A(h(a, b))$, where (a, b) is the arc related to the corresponding 1 entry. When h is constant, the adjacency matrix corresponding to the product is nothing but the Kronecker product $A(D) \otimes A(h(a, b))$.

The adjacency matrix of the resulting digraph in Figure 1.17 with the vertices $\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ is given in Figure 1.18. It has been divided into nine parts in order to have a better understanding of all the adjacency matrices involved in the product. Let S_p

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Figure 1.18: The adjacency matrix for the digraph in Figure 1.17.

be the set of all the super edge-magic 1-regular digraphs of order p where each vertex is identified by the label assigned to it. The next result was proved in [16].

Theorem 1.5.1. *[16] Let D be a (super) edge-magic digraph and let $h : E(D) \to S_p$ be any function. Then $und(D \otimes_h S_p)$ is (super) edge-magic.*

The key point in the proof of Theorem 1.5.1 is to assign label $p(a-1)+i$ to the vertex (a, i). By doing this, we obtain a super edge-magic labeling of $D \otimes_h S_p$. The super edge-magic labeling for the digraph obtained in Figure 1.17 is shown in Figure 1.19.

In order to bring this introduction to its end, it is worth mentioning that the product of digraphs introduced in [16] by Figueroa-Centeno et al. has
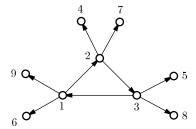
Figure 1.19: The super edge-magic labeling corresponding to the digraph product.

become a very powerful technique to study graph labelings. Many of the results obtained in the thesis will be using this product, however there are also other results that will be obtained using other techniques.

Chapter 2

On super edge-magicness of graphs of equal order and size

2.1   Introduction

The super edge-magicness of graphs of equal order and size has been shown
to be important since such graphs can be used as seeds to answer many
questions related to (super) edge-magic labelings and other types of well
studied labelings, as for instance harmonious labelings. Also other questions
related to the area of combinatorics can be attacked and understood from
the point of view of super edge-magic graphs of equal order and size. For
instance, the design of Steiner triple systems, the study of the set of dual
shuffle primes and the Jacobsthal numbers [40]. In this chapter, we study
the super edge-magic properties of some types of super edge-magic graphs
of equal order and size, with the hope that they can be used later in the
study of other related questions. The negative results found in Section 2.3
are specially interesting since these kind of results are not common in the
literature. Furthermore, the few results found in this direction usually meet
one of the following reasons: too many vertices compared with the number
of edges; too many edges compared with the number of vertices; or parity
conditions. In this case, all previous reasons fail. All the results in this
chapter are proved in [32] unless otherwise mentioned.

All graphs contained in this chapter may contain loops, however multiple edges are not allowed. Recall that a (p, q)-graph we mean a graph of order p and size q and for integers $m \leq n$, we use $[m, n]$ to denote $\{m, m+1, \ldots, n\}$.

Next, we provide some results that will be proven to be useful. The next result is a direct consequence of Lemma 1.4.1.

Lemma 2.1.1. Let D be a super edge-magic digraph in which each vertex is identified by the labels assigned by a super edge-magic labeling. Then the adjacency matrix $A(D)$ has the following properties.

(i) Each counterdiagonal contains all 0's or all 0's except one 1.

(ii) The set of counterdiagonals containing 1's in $A(D)$ is a set of consecutive diagonals.

In Lemma 2.1.1, (i) is equivalent to the fact that each induced sum is unique and (ii) that the set of induced sums is a set of consecutive integers.
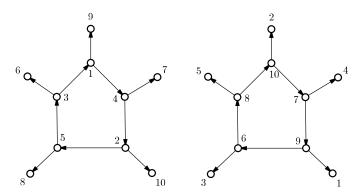

Figure 2.1: A super edge-magic labeling and its complementary.

It is easy to check the properties of Lemma 2.1.1 with the adjacency matrix given in Figure 2.2 corresponding to the digraph (on the left) in Figure 2.1.

Although the definitions of (super)edge-magic graphs and the original Lemma 1.4.1 in [13] were established for simple graphs (that is to say, graphs

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Figure 2.2: The adjacency matrix of Figure 2.1(left).

without loops or multiple edges), it works exactly the same for graphs with loops. From now on, whenever we talk about super edge-magic labelings, we will refer to labelings with the property provided in Lemma 1.4.1, unless otherwise specified.

Let f be a super edge-magic labeling f of a graph G. The *super edge-magic complementary labeling*, f^c is the labeling defined by the rule, $f^c(x) = p + 1 - f(x)$, for all $x \in V(G)$. Notice that, the labeling f^c is also super edge-magic. Figure 2.1 shows a super edge-magic labeling of $C_5 \odot \overline{K_1}$ and its complementary. The next lemma is an easy observation.

Lemma 2.1.2. Let D be a digraph and f be a super edge-magic labeling of D. Let $A(D_f)$ denote the adjacency matrix of D where each vertex takes the name of their labels in f. Then the matrix $A(D_{f^c})$ is a π radians clockwise rotation of $A(D_f)$.

The adjacency matrix of the super edge-magic labeled digraphs in Figure 2.1 are shown in Figure 2.2 and Figure 2.3. An easy check shows that the two vertices are related by a rotation of π radians clockwise.

We conclude this introduction by stating the following theorem which allows us to use super edge-magic labeled (di)graphs of equal order and size as seeds in order to get new families of super edge-magic labeled (di)graphs.

Theorem 2.1.1. *[38] Let D be a (super) edge-magic digraph and let S_n^k be the set of all super edge-magic labeled digraphs of order and size n with minimum*

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

Figure 2.3: The adjacency matrix of Figure 2.1(right).

induced sum k. Assume that $h : E(D) \to S_n^k$. Then the graph $und(D \otimes_h S_n^k)$ is (super) edge-magic.

2.2 Families of super edge-magic graphs of equal order and size

We begin this section by providing some families of super edge-magic graphs of equal order and size. Then we will use these families in order to get other families of super edge-magic graphs. Recall that, $K_{1,n}^l$ is the graph formed by a star $K_{1,n}$ with a loop attached at its central vertex.

Theorem 2.2.1. *The graph $2K_{1,1}^l \cup K_{1,n}^l$ is super edge-magic for all $n \in \mathbb{N}$.*

Proof. Label the central vertex of one component $K_{1,1}^l$ with 3 and the other vertex of this component with 6. Then label the central vertex of the other component isomorphic to $K_{1,1}^l$ with 5 and the other vertex of this component with 2. This gives us the edge induced sums $\{6, 7, 9, 10\}$. Finally, label the central vertex of the component isomorphic to $K_{1,n}^l$ with 4, and the vertices of degree 1 in $K_{1,n}^l$ with the remaining labels in $[1, n + 5]$. Then the edge induced sums $\{5, 8, 11, 12, \cdots, n + 9\}$ together with $\{6, 7, 9, 10\}$ gives a set of $n + 5$ consecutive integers. Hence, by Lemma 1.4.1, the labeling is super edge-magic . $\square$

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We illustrate the labeling of Theorem 2.2.1 in Figure 2.4. Theorem 2.2.1 can be generalized as follows.
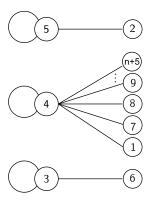
Figure 2.4: A super edge-magic labeling of $2K_{1,1}^l \cup K_{1,n}^l$.

Theorem 2.2.2. *The graph $2K_{1,m}^l \cup K_{1,n}^l$ is super edge-magic for all $m, n \in \mathbb{N}$.*

Proof. In order to label $2K_{1,m}^l \cup K_{1,n}^l$, consider the labeling of $2K_{1,1}^l \cup K_{1,n}^l$ obtained in the proof of Theorem 2.2.1. Add $m-1$ new edges with their corresponding vertices of degree 1 attached to the central vertex of each of the two components that are originally isomorphic to $K_{1,1}^l$. Label the new vertices of the component that has the central vertex labeled 5 with the numbers $-1, -3, -5, \ldots, -(2m-3)$. Label the remaining vertices (that is to say the new vertices in the component with the central vertex labeled 3 with the numbers $0, -2, -4, -6, \ldots, -(2m-4)$. Then by adding $(2m-2)$ to each of the original labels, we obtain the edge induced sums $[2m+3, 4m+6]$ from the two components isomorphic to $K_{1,m}^l$ and $[4m+1, 4m+n+5]$ from $K_{1,n}^l$. This results in a set of $2m+n+3$ consecutive integers and hence by Lemma 1.4.1, it is a super edge-magic labeling of the graph $2K_{1,m}^l \cup K_{1,n}^l$. $\square$

We illustrate the procedure of the above proof in Figure 2.5. Notice that, when $n = m$, we get that $3K_{1,n}^l$ is super edge-magic for all $n \in \mathbb{N}$. This fact can be generalized as follows.

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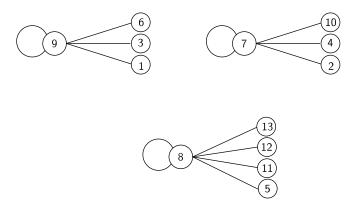
Figure 2.5: A super edge-magic labeling of $2K_{1,3}^l \cup K_{1,4}^l$.

Theorem 2.2.3. *The graph $(2s + 1)K_{1,n}^l$ is super edge-magic for all $n \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$.*

Proof. For $s = 0$, it is easy to check that any bijective function $f : V(K_{1,n}^l) \to [1, n + 1]$ is super edge-magic, for all $n \in \mathbb{N}$ (see for instance, in Lemma 3.1.1). Hence, we can assume that $s \in \mathbb{N}$. Let us define the graph $(2s + 1)K_{1,n}^l$ as follows: $V((2s + 1)K_{1,n}^l) = \{v_i\}_{i=1}^{2s+1} \cup \{v_i^j\}_{i=1}^{2s+1}, j = 1, 2, \ldots, n$ and $E((2s+1)K_{1,n}^l) = \{v_i v_i\}_{i=1}^{2s+1} \cup \{v_i v_i^j\}_{i=1}^{2s+1}, j = 1, 2, \ldots, n$. Next, we show that $(2s+1)K_{1,n}^l$ is super edge-magic. Consider the labeling $f : V((2s+1)K_{1,n}^l) \to [1, (n + 1)(2s + 1)]$ defined by

$$f(v) = \begin{cases} s + i, & v = v_i, i \in [1, 2s + 1], \\ i - s - 1, & v = v_i^1, i \in [s + 2, 2s + 1], \\ f(v_i) + (2s + 1), & v = v_i^1, i \in [1, s + 1], \\ f(v_i^1) + (2s + 1)(j - 1), & v = v_i^j, i \in [1, s + 1] \text{ and } j \neq 1, \\ f(v_i) + (2s + 1)(j - 1), & v = v_i^j, i \in [s + 2, 2s + 1] \text{ and } j \neq 1. \end{cases}$$

Now, we show that this labeling produces a set of $(n + 1)(2s + 1)$ consecutive integers. The loop attached to the central vertex v_i together with the edges $v_i v_i^1$ for $i = 1, 2, \ldots, 2s + 1$ gives the edge induced sums $[2s + 2, 6s + 3]$ and for a fixed k, the edges $v_i v_i^k$ and $v_{s+i} v_{s+i}^k$, $i = 1, 2, \ldots, s + 1$ gives the

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induced sums $[6s+4+(k-2)(2s+1), 6s+3+(k-1)(2s+1)]$. For $k = 2, \ldots, n$, these two sets together is a set of $(n + 1)(2s + 1)$ consecutive integers and hence by Lemma 1.4.1, f is a super edge-magic labeling of $(2s + 1)K_{1,n}^l$. $\quad\square$
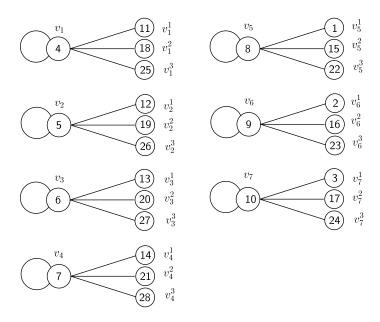
Figure 2.6: A super edge-magic labeling of $7K_{1,3}^l$.

It is worth to mention that the labeling provided in the previous proof has been motivated by the technique introduced in [37] in order to prove that the crowns of certain cycles are perfect super edge-magic(see Section 3.1). In fact, when using this technique, there is one step in which super edge-magic labelings of $(2s + 1)K_{1,n}^l$ are obtained. The next result that we want to consider is the following one.

Theorem 2.2.4. *The graph* $K_{1,m}^l \cup 2K_{1,n}^l \cup (2s)K_{1,1}^l, \ s \in \mathbb{N}$ *is super edge-magic.*

Proof. Let us define the graph $G = K_{1,m}^l \cup 2K_{1,n}^l \cup (2s)K_{1,1}^l, \ s \in \mathbb{N}$ as follows: $V(G) = \{v_i\}_{i=1}^{2s+3} \cup \{v_1^j\}_{j=1}^n \cup \{v_2^k\}_{k=1}^m \cup \{v_3^j\}_{j=1}^n \cup \{v_i^1\}_{i=4}^{2s+3}$ and $E(G) =$

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$\{v_i v_i\}_{i=1}^{2s+3} \cup \{v_1 v_1^j\}_{j=1}^{n} \cup \{v_2 v_2^k\}_{k=1}^{m} \cup \{v_3 v_3^j\}_{j=1}^{n} \cup \{v_i v_i^1\}_{i=4}^{2s+3}$. Consider the labeling $f : V(G) \to [-2n + 3, 4s + m + 5]$ defined by

$$
f(v) = \begin{cases}
2s + 1 + i, & v = v_i, i \in [1,3], \\
s - 2 + i, & v = v_i, i \in [4, s+3], \\
s + 1 + i, & v = v_i, i \in [s+4, 2s+3], \\
f(v_i) + (2s+3), & v = v_i^1, i \in [1,2], \\
f(v_3) - (2s+3), & v = v_3^1, \\
-2j + 4, & v = v_1^j, j \in [2,n], \\
f(v_2^1) + k - 1, & v = v_2^k, k \in [2,m], \\
-2j + 3, & v = v_3^j, j \in [2,n], \\
f(v_i) + (2s+3), & v = v_i^1, i \in [4, s+3], \\
f(v_i) - (2s+3), & v = v_i^1, i \in [s+4, 2s+3].
\end{cases}
$$

By adding $(2n - 2)$ to each of the original labels, we obtain a super edge-magic labeling of the graph $K_{1,m}^l \cup 2K_{1,n}^l \cup (2s)K_{1,1}^l$, $s \in \mathbb{N}$. The labeling pattern of the graph $K_{1,m}^l \cup 2K_{1,n}^l \cup (2s)K_{1,1}^l$, $s \in \mathbb{N}$ is shown in Figure 2.7. □

Next, we introduce the concept of deer graph. Consider any caterpillar with an odd spine whose edges can be embedded in a horizontal line and the degree sequence of the vertices of the spine read the same from left to right than from right to left. If we attach a loop to the central vertex of the spine, we get a *deer graph*. An example of a deer graph appears in Figure 2.8.

Theorem 2.2.5. *All deer graphs are super edge-magic.*

Proof. It suffices to label the vertices of the caterpillar in a traditional super edge-magic way. Figure 2.8 shows the labeling pattern to get a super edge-magic labeling. □

Let $\overline{K}_n$ be the complementary graph of the complete graph K_n, $n \in \mathbb{N}$. The *corona product* $C_k \odot \overline{K}_n$ is the graph obtained from a cycle of size k by attaching n-pendant edges to each vertex of the cycle. The next lemma is an easy exercise.

Lemma 2.2.1. *If C_k is a cycle with k vertices, then* $\text{und}(\overrightarrow{C_k} \otimes_h \overrightarrow{K}_{1,n}^l) \cong C_k \odot \overline{K}_n$.
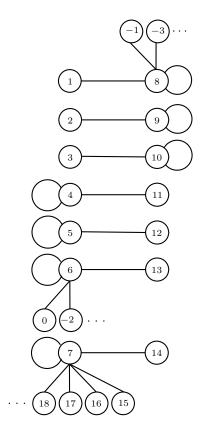
Figure 2.7: A labeling pattern for $K_{1,m}^l \cup 2K_{1,n}^l \cup (2s)K_{1,1}^l$.

Combining Lemma 2.2.1 with Theorems 2.2.1, 2.2.2, 2.2.3 and 2.2.4, we get the following result.

Corollary 2.2.1. The following graphs are edge-magic.

(i) $(2C_k \odot \overline{K}_1) \cup (C_k \odot \overline{K}_n)$, for all $k, n \in \mathbb{N}$,

(ii) $(2C_k \odot \overline{K}_m) \cup (C_k \odot \overline{K}_n)$, for all $k, m, n \in \mathbb{N}$

(iii) $(2s+1)C_k \odot \overline{K}_n$, for all $k, n \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$,

(iv) $(C_k \odot \overline{K}_m) \cup (2C_k \odot \overline{K}_n) \cup ((2s)C_k \odot \overline{K}_1)$, for all $k, s \in \mathbb{N}$.

In particular, if k is odd, all the graphs above are super edge-magic.

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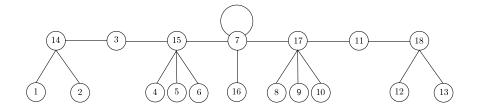
Figure 2.8: A super edge-magic labeling of a deer graph with a spine of order 7.

Proof. (i) By Lemma 2.2.1, $C_k \otimes_h (2K_{1,1}^l \cup K_{1,n}^l) \cong (2C_k \odot \overline{K_1}) \cup (C_k \odot \overline{K_n})$. Combining this with Theorem 2.2.1 and Theorem 2.1.1, $(2C_k \odot \overline{K_1}) \cup (C_k \odot \overline{K_n})$ is edge-magic. In particular, if k is odd, C_k is super edge-magic and hence the graph $(2C_k \odot \overline{K_1}) \cup (C_k \odot \overline{K_n})$ is super edge-magic.

(ii) By Lemma 2.2.1, $C_k \otimes_h (2K_{1,m}^l \cup K_{1,n}^l) \cong (2C_k \odot \overline{K_m}) \cup (C_k \odot \overline{K_n})$. Combining this with Theorem 2.2.2 and Theorem 2.1.1, $(2C_k \odot \overline{K_m}) \cup (C_k \odot \overline{K_n})$ is edge-magic. In particular, if k is odd, C_k is super edge-magic and hence the graph $(2C_k \odot \overline{K_m}) \cup (C_k \odot \overline{K_n})$ is super edge-magic.

(iii) By Lemma 2.2.1, $C_k \otimes_h (2s+1)K_{1,n}^l \cong (2s+1)C_k \odot \overline{K_n}$. Combining this with Theorem 2.2.3 and Theorem 2.1.1, $(2s+1)C_k \odot \overline{K_n}$ is edge-magic. In particular, if k is odd, C_k is super edge-magic and hence the graph $(2s+1)C_k \odot \overline{K_n}$ is super edge-magic.

(iv) By Lemma 2.2.1, $C_k \otimes_h ((K_{1,m}^l) \cup (2K_{1,n}^l) \cup ((2s)K_{1,1}^l)) \cong (C_k \odot \overline{K_m}) \cup (2C_k \odot \overline{K_n}) \cup ((2s)C_k \odot \overline{K_1})$. Combining this with Theorem 2.2.4 and Theorem 2.1.1, $(C_k \odot \overline{K_m}) \cup (2C_k \odot \overline{K_n}) \cup ((2s)C_k \odot \overline{K_1})$ is edge-magic. In particular, if k is odd, C_k is super edge-magic and hence the graph $(C_k \odot \overline{K_m}) \cup (2C_k \odot \overline{K_n}) \cup ((2s)C_k \odot \overline{K_1})$ is super edge-magic. $\square$

2.3 Families of graphs of equal order and size which are not super edge-magic

Gallian identifies in [17] three usual reasons why graphs fail to admit labelings of certain types. The reasons are enumerated next.

(a) Divisibility conditions.

(b) Too many edges when we compare this number with the number of vertices.

(c) Too many vertices when we compare this number with the number of edges.

Our immediate goal is to prove a result about an infinite family of graphs that fails to be super edge-magic for different reasons than the ones enumerated above. We will prove that the all graphs of the family $K_{1,m}^l \cup K_{1,n}^l$ are not super edge-magic, for all positive integers m and n. However, what we will really end up showing is that the digraph $D \cong \overrightarrow{K}_{1,m}^l \cup \overrightarrow{K}_{1,n}^l$ where D is obtained by orienting the edges of $K_{1,m}^l \cup K_{1,n}^l$ in such a way that all vertices of degree 1 in $K_{1,m}^l \cup K_{1,n}^l$ have outdegree 0 in D is not super edge-magic. We will do this using a contradiction argument.

Next, we describe some properties that the adjacency matrix of D has.

Lemma 2.3.1. Let D be the digraph obtained from $K_{1,m}^l \cup K_{1,n}^l$, $m, n \in \mathbb{N}$, by orienting its edges in such a way that all vertices have indegree 1 in D. Then the adjacency matrix of D, denoted by $A(D)$ satisfies the following conditions:

(i) Each column of $A(D)$ contains exactly one 1.

(ii) All entries of $A(D)$ are either 0 or 1 and the entries 1 are located in exactly two rows of $A(D)$.

(iii) Since each component of D contains a loop, it follows that the two rows contain exactly one 1 in the main diagonal of $A(D)$.

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Keeping all the above information in mind, we are now ready to state and prove the next result.

Theorem 2.3.1. *The graph $K_{1,m}^l \cup K_{1,n}^l$ is not super edge-magic, for all positive integers m and n.*

Proof. Assume to the contrary that $K_{1,m}^l \cup K_{1,n}^l$ is a super edge-magic graph and let D be the digraph obtained from it by orienting its edges in such a way that all vertices in D have indegree 1. By definition, D is also super edge-magic. Let f be a super edge-magic labeling of D, and let $A(D) = (a_{ij})$ be the adjacency matrix induced by f where the vertices take the name of their labels in the super edge-magic labeling. By Lemma 2.3.1, all the entries 1 are located in exactly two rows of $A(D)$. Let these two row be row i and row j and assume that $i < j$. If $a_{i1} = a_{i2} = \ldots = a_{il} = 1$ and $a_{il+1} = 0$, for some $l \geq 1$, then there is no 1 in the diagonal with induced sum $i + l + 1$, contradicting Lemma 2.1.1. Thus, we only have two possible generic forms for the adjacency matrix, either row i is of the form $(0 \ldots 01 \ldots 10 \ldots \ldots 10 \ldots 0)$, with one more block of zeros than blocks of ones, or, $(0 \ldots 01 \ldots 10 \ldots \ldots 01 \ldots 1)$, with exactly the same number of blocks of zeros and ones. Notice that, the first possibility is forbidden by Lemma 2.1.2, since otherwise, the adjacency matrix induced by the complementary labeling of f^c would have exactly two rows with 1 entries, namely k, l with $k < l$, and row k of the form $(1 \ldots 10 \ldots)$, which is a contradiction, as we have shown above. Hence, in what follows assume that row i and row j are of the form

$$(\overbrace{0 \ldots 0}^{|B_1|} T_1 \overbrace{0 \ldots 0}^{|B_2|} T_2 \ldots \ldots 0 T_k) \text{ and } (B_1 \overbrace{0 \ldots 0}^{|T_1|} B_2 \overbrace{0 \ldots 0}^{|T_2|} \ldots \ldots 1 \overbrace{0 \ldots 0}^{|T_k|}),$$

respectively, where the T_l and B_l are blocks of 1's. By Lemma 2.1.2, we can assume that $\sum_{i=1}^{k} |T_i| = m + 1$ and $\sum_{i=1}^{k} |B_i| = n + 1$.

Notice that the block B_1 must be used to cover the zeros between T_1 and T_2, B_2 must be used to cover the zeros between T_2 and T_3 and so on. Since the length of B_1 must be equal to the number of zeros between T_1 and T_2 which is equal to the length of B_2 and so on, we get $\mid B_i \mid = (n+1)/k$ for $i = 1, 2, \ldots, k$. A similar argument shows that $\mid T_i \mid = (m + 1)/k$ for $i = 1, 2, \ldots, k$. More over, since the first zero in the top row after T_1 appears in column $((m+n+2)/k) + 1$, this can only be covered if the bottom row is $((m+n+2)/k) + i)$. This implies that

$$j - i = (m + n + 2)/k \tag{2.3.1}$$

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On the other hand, the possible positions of 1's in the top row and bottom row are $(l'(m+n+2)-(m+1))/k+y$, $l' = 1, 2, \ldots, k-1$, $y \in [1, (m+1)/k]$ and $(l-1)(m+n+2)/k+w$, $l = 1, 2, \ldots, k-1$, $w \in [1, (n+1)/k]$, respectively. Since we assume that the graph is super edge-magic, each of the two rows contains a 1 in the main diagonal. Thus, there exist l, l', y and w such that, $i = (l'(m+n+2)-(m+1))/k+y$ and $j = (l-1)(m+n+2)/k+w$. Hence, by (2.3.1) we obtain that $w = (2-l+l')(m+n+2)/k-(m+1)/k+y$. Since $l' \leq l$, w is either negative or greater than $(n+1)/k$, which contradicts that $w \in [1, (n+1)/k]$. $\square$

Let L be the loop graph. Using a similar reasoning to the one used above, we are able to prove the following result.

Theorem 2.3.2. *The graph $L \cup K_{1,n}^l$ is not super edge-magic for any $n \in \mathbb{N}$.*

Proof. Suppose to the contrary that there exists $n \in \mathbb{N}$ such that $L \cup K_{1,n}^l$ is super edge-magic and assume that each vertex is identified with the label assigned by a super edge-magic labeling. By definition, the digraph $D \cong \overrightarrow{L} \cup \overrightarrow{K}_{1,n}^l$ obtained from $L \cup K_{1,n}^l$ by orienting its edges in such a way that all vertices in D have indegree 1 in D is also super edge-magic. The adjacency matrix of D contains exactly two rows with 1 entries, namely row i and row j, $1 \leq i < j \leq n+2$. If $a_{i1} = a_{i2} = \ldots = a_{il} = 1$ and $a_{il+1} = 0$, for some $l \geq 1$, then there is no 1 in the diagonal with induced sum $i+l+1$, contradicting Lemma 2.1.1.

Thus, we only have three possible generic forms for the adjacency matrix, either row i, $1 \leq i < n+2$, is of the form a) $(0\ldots01)$ or, b) $(01\ldots1)$ or, c) $(0\ldots010\ldots0)$. Notice that, a) and b) are not possible since each of the two rows should contain a 1 in the main diagonal, which is not possible in any of the two configurations. Finally, (c) is forbidden by Lemma 2.1.2, since otherwise, the adjacency matrix induced by the complementary labeling of f^c would have exactly two rows with 1 entries, namely k, l with $k < l$, and row k of the form $(1\ldots101\ldots1)$, which is a contradiction, as we have shown above. $\square$

Out next immediate goal is to study the super edge-magicness of the graph $2L \cup K_{1,n}^l$.

Theorem 2.3.3. *The graph* $2L \cup K_{1,n}^l$ *is not super edge-magic for any* $n \in \mathbb{N}$.

Proof. Assume to the contrary that there exists a $n \in \mathbb{N}$ such that $2L \cup K_{1,n}^l$ is super edge-magic. Following the same idea that we used in the two previous results, we consider the digraph $D \cong 2\overrightarrow{L} \cup \overrightarrow{K}_{1,n}^l$ where the star is oriented as in the previous two proofs. By definition, $2L \cup K_{1,n}^l$ is super edge-magic if and only if the digraph $D \cong 2\overrightarrow{L} \cup \overrightarrow{K}_{1,n}^l$ is super edge-magic. Let f be a super edge-magic labeling of D, and let $A(D) = (a_{ij})$ be the adjacency matrix induced by f where the vertices take the name of their labels in the super edge-magic labeling. Since all vertices of D have indegree 1, we can represent the adjacency matrix of D by a vector $(v_1, \ldots, v_n)$ such that $v_k = i$ if and only if $a_{ik} = 1$. There are three possible generic forms.

Case 1: Let $v_1 = v_2 = \ldots = v_k = 1$, $1 \leq k \leq n+1$ and $v_{k+1} > 1$. Then there is no 1 in the diagonal with induced sum $k + 2$, contradicting Lemma 2.1.1.

Case 2: Let i be such that $a_{1i} = 1$. By case 1, $i > 1$. Let $v_1 = v_2 = \ldots = v_k = i$ and $v_{k+1} \neq i$, $2 \leq k \leq n+1$ and $2 \leq i \leq n+2$. Since each row with entries being 1 has a 1 in the main diagonal and $i > 1$, the other two rows of the adjacency matrix represent the two loops. Let v_{k+1} and $v_l, k+2 \leq l \leq n+3$ be the components that represent the two loops. Then, $\min\{v_{k+1}, v_l\} > v_k$. If $v_l > v_{k+1}$, then there is no 1 in the diagonal with induced sum $i + k + 1$. If $v_l < v_{k+1}, v_l \neq i$, then $v_k < v_l < v_{k+1}, k+2 \leq l \leq n+2$. This implies that one of the two rows representing the loop components cannot have a 1 in the main diagonal. Hence we get a contradiction in each of the above possible scenarios.

Case 3: Let $v_1 = v_2 = \ldots = v_k = n+3$, $1 \leq k \leq n+1$. Notice that this must be a part of the component $LK_{1,n}$ otherwise there is no 1 in the main diagonal. In particular, this implies that $v_{n+3} = 1$, which in view of Lemma 2.1.2 and case 1, is also a contradiction. $\square$

2.4 Open questions

In this chapter, we have proved the families $2K_{1,1}^l \cup K_{1,n}^l$, $2K_{1,m}^l \cup K_{1,n}^l$, $(2s+1)K_{1,n}^l$, $K_{1,m}^l \cup 2K_{1,n}^l \cup (2s)K_{1,1}^l$, for all $m, n, s \in \mathbb{N}$ and deer graphs are

super edge-magic. We have also proved the families $K_{1,m}^l \cup K_{1,n}^l$, $L \cup K_{1,n}^l$, $2L \cup K_{1,n}^l$ are not super edge-magic. From what we have seen so far, the following open questions raise naturally.

Open question 2.4.1. Characterize the set of super edge-magic graphs of the form $mL \cup nK_{1,s}^l$.

Next, we propose the most general open question, although we think that it may be a really hard question to solve.

Open question 2.4.2. Characterize the set of super edge-magic graphs whose components are isomorphic to loops and graphs that are stars with a loop attached at their central vertices.

Open question 2.4.3. For which values of $s \in \mathbb{N}$ the graph $(2s)K_{1,n}^l$ is super edge-magic?

In general, the study of graphs of equal order and size is very interesting. According to Theorem 2.1.1, the $\otimes_h$-product can be applied to generate further families of (super) edge-magic graphs using these graphs. Thus we will bring this section to its end with the following general question that is probably very hard to answer.

Open question 2.4.4. Which graphs of equal order and size are super edge-magic?

Chapter 3

Perfect (super) edge-magic crowns

3.1 Introduction

Intuitively speaking, a (super) edge-magic graph is perfect (super) edge-magic if all possible theoretical valences occur. A famous conjecture of Godbold and Slater [18] states that, for $n = 2t + 1 \geq 7$ and $5t + 4 \leq j \leq 7t + 5$ and for $n = 2t \geq 4$ and $5t + 2 \leq j \leq 7t + 1$ there is an edge-magic labeling of C_n, with valence $k = j$. In other words, for odd $n \geq 7$ and for even $n \geq 4$ the cycle C_n is perfect edge-magic. Notice that this conjecture explicitly excludes $n = 5$, since as it was proved in [3], there are two possible edge-magic valences that are not attained.

The formal definition of perfect (super) edge-magic graph was introduced by López et al. in [37, 41].

Let $G = (V, E)$ be a (p, q)-graph and let $g : V \cup E \to [1, p + q]$ be a bijective function. Then the set, denoted by T_G is

$$T_G = \left\{ \frac{\sum_{u \in V} \deg(u) g(u) + \sum_{e \in E} g(e)}{q} \right\}. \tag{3.1.1}$$

If $\lceil \min T_G \rceil \leq \lfloor \max T_G \rfloor$ then the *magic interval* of G, denoted by J_G, is

defined to be the set $J_G = [\lceil \min T_G \rceil, \lfloor \max T_G \rfloor] \cap \mathbb{Z}$ and the *magic set* of G, denoted by τ_G, is the set

$$\tau_G = \{n \in J_G : \ n \text{ is the valence of some edge-magic labeling of } G\}.$$

It is clear that $\tau_G \subseteq J_G$. A graph G is called *perfect edge-magic* [41] if $\tau_G = J_G$.

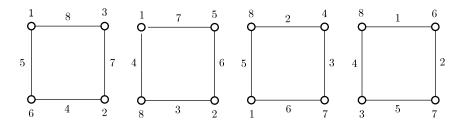Figure 3.5 shows edge-magic labelings of C_4 with all the possible valences in the magic interval $[12, 15]$.

Figure 3.1: Edge-magic labelings of C_4.

Let $G = (V, E)$ be a (p, q)-graph. Then the set S_G is defined as $S_G = \{1/q(\Sigma_{u \in V} deg(u)g(u) + \Sigma_{i=p+1}^{p+q} i) : \text{the function } g : V \to \{i\}_{i=1}^{p} \text{ is bijective}\}$. If $\lceil \min S_G \rceil \leq \lfloor \max S_G \rfloor$ then the *super edge-magic interval* of G, denoted by I_G, is defined to be the set $I_G = [\lceil \min S_G \rceil, \lfloor \max S_G \rfloor] \cap \mathbb{Z}$ and the *super edge-magic set* of G, denoted by σ_G, is the set formed by all integers $k \in I_G$ such that k is the valence of some super edge-magic labeling of G. A graph G is called *perfect super edge-magic graph* [37] if $\sigma_G = I_G$.

We would like to mention that the concepts of perfect edge-magic and perfect super edge-magic are not equivalent, since we can find examples of graph that are perfect super edge-magic but they are not perfect edge-magic. One of the smallest ones is the cycle C_5. Let us see some of them.

Proposition 3.1.1. [37] An r-regular graph G is perfect super edge-magic if and only if G is super edge-magic.

The result is true since any r-regular graph is super edge-magic with only one valence (see Lemma 1.4.2) and hence it is perfect super edge-magic.

Theorem 3.1.1. *[14] The star $K_{1,n}$ is edge-magic. Furthermore, there are only three possible valences for edge-magic labelings of $K_{1,n}$. These valences are $2n+4, 3n+3$ and $4n+2$. Moreover, the first two valences correspond to super edge-magic labelings of $K_{1,n}$.*
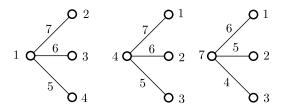
Figure 3.2: Three possible valences for edge-magic labelings of $K_{1,3}$.

Theorem 3.1.2. *[37] The path P_n is perfect super edge-magic for every $n \in \mathbb{N}$.*

Though the path is perfect super edge-magic for all $n \in \mathbb{N}$, there is only one possible valence $(5n+3)/2$ when n is even and two possible valences, $(5n+1)/2$ and $(5n+3)/2$ when n is odd.

Figure 3.3: All possible super edge-magic valences of P_3 and P_4.

Lemma 3.1.1. The graph formed by a star $K_{1,n}$ and a loop attached to its central vertex, denoted by $K_{1,n}^l$, is perfect super edge-magic for all positive integers n. Furthermore, $|I_{K_{1,n}^l}| = |\sigma_{K_{1,n}^l}| = n+1$.

Proof. By Lemma 1.4.1, it is a very easy observation that any bijection $f : V(K_{1,n}^l) \to [1, n+1]$ is a super edge-magic labeling of $K_{1,n}^l$. Further more, the valence of any super edge-magic labeling of $K_{1,n}^l$ depends only on the

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label assigned to the central vertex of $K_{1,n}^l$ (that is, the vertex of $K_{1,n}^l$ with degree different from 1). If two labelings of $K_{1,n}^l$ assign consecutive labels to the central vertex of $K_{1,n}^l$, then the resulting valences are also consecutive. Since there are exactly $(n+1)$ possible consecutive labels to assign to the central vertex, it follows that $|I_{K_{1,n}^l}| = |\sigma_{K_{1,n}^l}| = n+1$. $\qquad\square$
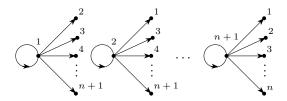

Figure 3.4: All possible super edge-magic labelings of an orientation of $K_{1,n}^l$.

Let $\overline{K}_n$ be the complementary graph of the complete graph K_n, $n \in \mathbb{N}$. The following theorem was proved by López et al. in [37, 41].

Theorem 3.1.3. *[37, 41] Let C_m be a cycle of order $m = p^k$, where $p > 2$ is a prime number. Then the graph $G \cong C_m \odot \overline{K}_n$ is perfect (super) edge-magic.*

In this chapter, we study the (super) edge-magic valences of crowns $C_m \odot \overline{K}_n$ and we extend the result to $m = pq$, where p and q are different odd primes. The chapter is organized as follows: in Section 3.2, we provide all the necessary results needed. In Section 3.3, we prove that each element in the family $C_m \odot \overline{K}_n$ where $m = pq$, with p and q being different odd primes is a perfect (super) edge-magic graph. In Section 3.4, we provide a lower bound for the number of valences of general crowns $C_m \odot \overline{K}_n$. All the results in this chapter are proved in [34] unless otherwise mentioned. In the next section, we will give all the necessary results needed to reach our goals.

3.2 The tools

We start by considering the labeling of the cycle C_n introduced in the proof of Lemma 1.4.3.
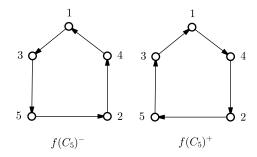
Figure 3.5: Two strong orientations of a super edge-magic labeled C_5.

Let $V(C_n) = \{v_i\}_{i=1}^n$ and $E(C_n) = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_n v_1\}$. The function $f : V(C_n) \to \{i\}_{i=1}^n$ defined by the rule

$$f(v_i) = \begin{cases} (i+1)/2, & \text{if } i \text{ is odd} \\ (i+1+n)/2, & \text{if } i \text{ is even} \end{cases}$$
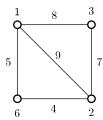
We will refer this labeling as the *canonical* labeling of the cycle. When we say that a digraph has a labeling we mean that its underlying graph has such labeling, see [16]. We denote the underlying graph of a digraph D by $und(D)$.

Remark 3.2.1. Let $\{f(C_n)^+, f(C_n)^-\}$ be the strong orientations of the super edge-magic labeled cycle C_n introduced in the proof of Lemma 1.4.3. Then

$$(a,b) \in E(f(C_n)^+) \Leftrightarrow b - a \equiv \frac{n+1}{2} \ (mod \ n) \qquad (3.2.1)$$

$$(a,b) \in E(f(C_n)^-) \Leftrightarrow b - a \equiv \frac{n-1}{2} \ (mod \ n) \qquad (3.2.2)$$

Let f be an edge-magic labeling of a (p,q)-graph G. The *complementary labeling* of f, denoted by $\overline{f}$, is the labeling defined by the rule: $\overline{f}(x) = p + q + 1 - f(x)$, for all $x \in V(G) \cup E(G)$. Notice that, if f is an edge-magic labeling of G, we have that $\overline{f}$ is also an edge-magic labeling of G with valence $\mathrm{val}(\overline{f}) = 3(p+q+1) - \mathrm{val}(f)$. An example is shown in Figure 2.1. In the case of a super edge-magic labeling f of a graph G, there is also the corresponding

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Figure 3.6: An edge-magic labeling of $K_4 - e$ and its complementary.

super edge-magic complementary labeling, f_c, which is also super edge-magic. In this case f_c is defined by the rule $f_c(x) = p + 1 - f(x)$, for all $x \in V(G)$ and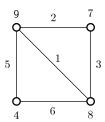 $f_c(ab)$ is obtained as described in Lemma 1.4.1, for all $ab \in E(G)$. An example is shown in Figure 2.1. Let $\mathrm{val}(f) = k + p + q$ where k is the minimum induced sum of the labeling f. The corresponding maximum induced sum is $k + q - 1$. Then, the valence of f_c is given by

$$\mathrm{val}(f_c) = p + q + \text{minimum induced sum of } f_c$$
$$= p + q + 2(p + 1) - \text{maximum induced sum of } f$$
$$= p + q + 2(p + 1) - k - q + 1$$
$$= 2(p + 1) + p + 1 - \mathrm{val}(f) + p + q$$

and hence the valence of f_c can be expressed in terms of the valence of f as follows:

$$\mathrm{val}(f_c) = 4p + q + 3 - \mathrm{val}(f). \tag{3.2.3}$$

The complementary labeling of an edge-magic labeling is a powerful tool that allows us to increase the number of valences of certain families of graphs dramatically. Using the complementary labeling we may even prove the perfect edge-magicness of many graphs. The following proposition can serve as an illustration of this fact.

Proposition 3.2.1. The graph $K_{1,n}^l$ is perfect edge-magic for all positive integers. Furthermore, $|J_{K_{1,n}^l}| = |\tau_{K_{1,n}^l}| = 2n + 2$.

Proof. We claim that $J_{K_{1,n}^l} = [2n + 4, 4n + 5] \cap \mathbb{Z}$. The edge-magic interval defined in (3.1.1) attains its maximum when the biggest label is assigned to

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the vertex with maximum degree and in our case, the label $2n+2$ assigned to the central vertex with degree $n+2$. This implies, $((n+2)(2n+2) + (1+2+\cdots+2n+1))/(n+1) = 4n+5$. In the same way, the minimum is achieved by assigning the label 1 to the central vertex and this leads to $((n+2)(1)+(2+\cdots+2n+2))/(n+1) = 2n+4$. Hence, $J_{K_{1,n}^l} = [2n+4, 4n+5] \cap \mathbb{Z}$. In Lemma 3.1.1 it is shown that all numbers in the set $\{2n+4, 2n+5, \ldots, 3n+4\}$ are in $\tau_{K_{1,n}^l}$. Now, using the fact that if f is a super edge-magic labeling of $K_{1,n}^l$ then $\mathrm{val}(\overline{f}) = 3(2n+3) - \mathrm{val}(f)$, we obtain that $\{3n+5, 3n+6, \ldots, 4n+5\} \subseteq \tau_{K_{1,n}^l}$, showing the result. $\qquad\square$
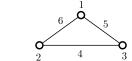
Also, in the case that G is a graph of equal order and size, new edge-magic labelings can be obtained from known super edge-magic labelings of G. The *odd labeling* and the *even labeling* [41] obtained from f, denoted respectively by $o(f)$ and $e(f)$, are the labelings $o(f), e(f) : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$ defined as follows: (i) on the vertices: $o(f)(x) = 2f(x) - 1$ and $e(f)(x) = 2f(x)$, for all $x \in V(G)$, (ii) on the edges: $o(f)(xy) = 2\mathrm{val}(f) - 2p - 2 - o(f)(x) - o(f)(y)$ and $e(f)(xy) = 2\mathrm{val}(f) - 2p - 1 - e(f)(x) - e(f)(y)$, for all $xy \in E(G)$. Figure 3.7 shows an example of these constructions.

Lemma 3.2.1. [41] Let G be a (p,q)-graph with $p = q$ and let $f : V(G) \cup E(G) \to [1, p+q]$ be a super edge-magic labeling of G. Then, the odd labeling $o(f)$ and the even labeling $e(f)$ obtained from f are edge-magic labelings of G with valences $\mathrm{val}(o(f)) = 2\mathrm{val}(f) - 2p - 2$ and $\mathrm{val}(e(f)) = 2\mathrm{val}(f) - 2p - 1$ respectively.
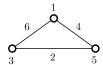
Proof. Since f is super edge-magic, the set $S_{odd} = \{o(f)(x) + o(f)(y) : xy \in E(G)\} = \{2(f(x) + f(y)) - 2 : xy \in E(G)\}$ is an arithmetic progression of difference 2 and the minimum value of the set is $2(\mathrm{val}(f) - 2p) - 2$. Thus, an edge-magic labeling with valence $\mathrm{val}(o(f)) = 2\mathrm{val}(f) - 2p - 2$ can be obtained by assigning the even labels to the edges.
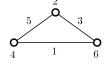
Similarly, the set $S_{even} = \{e(f)(x) + e(f)(y) : xy \in E(G)\} = \{2(f(x) + f(y)) : xy \in E(G)\}$ is an arithmetic progression of difference 2, starting at $2(\mathrm{val}(f) - 2p)$. Hence, by assigning the odd labels to the edges, we obtain an edge-magic labeling of valence $\mathrm{val}(o(f)) = 2\mathrm{val}(f) - 2p - 1$. $\qquad\square$

At this point, we want to observe that Proposition 3.2.1 can also be proved using the labelings provided in the proof of Lemma 3.1.1 and the odd

A Super edge − magic labeling with valence 9

The induced odd labeling with valence 10   *The induced even labeling with valence* 11

Figure 3.7: The odd and the even labeling from a super edge-magic labeling of C_3.

and even labelings just defined above. Clearly, the set of minimum induced sum (see Figure 3.4) is $\{2, \cdots, n+2\}$. Thus, the set of valences is obtained by adding $2n+2$. That is, $\sigma_{K_{1,n}^l} = \{2n+4, \cdots, 3n+4\}$. Using "the odd labeling construction", we get all the even numbers between $[2n+4, 4n+5]$ and "the even labeling construction" produces all the odd numbers between $[2n+4, 4n+5]$ and hence we get $|J_{K_{1,n}^l}| = |\tau_{K_{1,n}^l}| = 2n+2$.

Many relations among labelings have been established using the $\otimes_h$-product introduced in Section 1.4 and some particular families of graphs, namely $\mathcal{S}_p$ and $\mathcal{S}_p^k$ (see for instance, [23, 35, 38, 42]). The family $\mathcal{S}_p$ contains all super edge-magic 1-regular labeled digraphs of order p where each vertex takes the name of the label that has been assigned to it. A super edge-magic digraph F is in $\mathcal{S}_p^k$ if $|V(F)| = |E(F)| = p$ and the minimum sum of the labels of the adjacent vertices is equal to k (see Lemma 1.4.1). Notice that, since each 1-regular digraph of order p has minimum edge induced sum equal to $(p+3)/2$, it follows that $\mathcal{S}_p \subset \mathcal{S}_p^{(p+3)/2}$. The following result was introduced in [38], generalizing Theorem 1.5.1 found in [16] :

Theorem 3.2.1. *[38] Let D be a (super) edge-magic digraph and let $h : E(D) \to \mathcal{S}_p^k$ be any function. Then $D \otimes_h \mathcal{S}_p^k$ is (super) edge-magic.*

Remark 3.2.2. The key point in the proof of Theorem 3.2.1 is to rename the vertices of D and each element of $\mathcal{S}_p^k$ after the labels of their corre-

sponding (super) edge-magic labeling f and their super edge-magic labelings respectively. Then the labels of the product are defined as follows: (i) the vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$ receives the label: $p(a - 1) + i$ and (ii) the arc $((a, i), (b, j)) \in E(D \otimes_h \mathcal{S}_p^k)$ receives the label: $p(e-1)+(k+p)-(i+j)$, where e is the label of (a, b) in D. Thus, for each arc $((a, i), (b, j)) \in E(D \otimes_h \mathcal{S}_p^k)$, coming from an arc $e = (a, b) \in E(D)$ and an arc $(i, j) \in E(h(a, b))$, the sum of labels is constant and equal to $p(a + b + e - 3) + (k + p)$. That is, $p(\mathrm{val}(f) - 3) + k + p$. Thus, the next result is obtained.

Lemma 3.2.2. [38] Let $\hat{f}$ be the (super) edge-magic labeling of the graph $D \otimes_h \mathcal{S}_p^k$ induced by a (super) edge-magic labeling f of D (see Remark 3.2.2). Then the valence of $\hat{f}$ is given by the formula

$$\mathrm{val}(\hat{f}) \;=\; p(\mathrm{val}(f) - 3) + k + p. \tag{3.2.4}$$

To prove the main result, we need some technical lemmas. The next lemma was proved in [37].

Lemma 3.2.3. [37] Let p and q be odd coprime numbers. Then there exist integers α and β with $1 = \alpha p + \beta q$ and $\max\{|\alpha p|, |\beta q|\} \le (pq + 1)/2$.

Proof. Bézout's identity states that there exist integers α, β such that $1 = \alpha p + \beta q$, with $\alpha p > |\beta q|$. Thus, for any $k \in \mathbb{R}$, the identity $1 = (\alpha - kq)p + (\beta + kp)q$ holds.

If $\alpha p \le (m + 1)/2$, then $max\{|\alpha p|, |\beta q|\} \le (pq + 1)/2$ where $m = pq$.

Assume that $\alpha p > (m+1)/2$. Let k be an integer such that $|\alpha - kq| < q/2$ (it exists since q is odd). Hence with such a choice of k, we have $|\alpha - kq|p \le pq/2$ and $|\beta + kp|q = |1 - (\alpha - kq)p| \le 1 + |\alpha - kq|p \le 1 + pq/2$. Thus, since $2 + pq$ is odd, we have that $|\beta + kp|q \le (m + 1)/2$. $\qquad\square$

The following lemma was partially proved in [37].

Lemma 3.2.4. Let p and q be different odd primes. Then, there exists an integer x with $1 < x < pq$ such that $\gcd(x, pq) \ne 1$, $\gcd(x-1, pq) \ne 1$. Moreover, if there exists a different x' with $1 < x' < pq$ such that $\gcd(x', pq) \ne 1$, $\gcd(x' - 1, pq) \ne 1$, then $x' = pq - x + 1$.

Proof. By Lemma 3.2.3, there exist two integers α and β such that $\alpha p + \beta q = 1$ and $\max\{|\alpha p|, |\beta q|\} \leq (pq + 1)/2$. Assume, without loss of restriction that, $\alpha p > 0$. Let $x = \alpha p$. Then we have that $x - 1 = -\beta q$. Thus, $\gcd(x, pq) = p$ and $\gcd(x - 1, pq) = q$. Let $x' = pq - x + 1$. Then we have that $\gcd(x', pq) = q$, $\gcd(x' - 1, pq) = p$. Now, we show that $1 < x, x' < pq$. Since α and p are positive integers, $x = \alpha p > 1$. Using Lemma 3.2.3, we have $x = \alpha p \leq (pq + 1)/2 < pq$. Thus, we obtain that $1 < x' = pq - \alpha p + 1 < pq$. Hence, $1 < x, x' < pq$.

Finally, we prove that x and x' are unique. Suppose that there exists another y such that $1 < y < pq$ with $\gcd(y, pq) \neq 1$ and $\gcd(y - 1, pq) \neq 1$. By considering $y' = pq - y + 1$, we can assume that $\gcd(y, pq) = p$ and $\gcd(y - 1, pq) = q$. Let α' and β' be such that $y = \alpha' p$ and $y - 1 = \beta' q$. Then, $1 \leq \alpha' < q$ and $1 \leq \beta' < p$. Hence, $|y - x| = |\alpha' - \alpha| p < pq$ and $|y - x| = |y - 1 - (x - 1)| = |\beta' + \beta| q$. However, $|\beta' + \beta| q = |\alpha' - \alpha| p$, a contradiction since p and q are different primes. Therefore, $x = y$. $\square$

Corollary 3.2.1. Let p and q be different odd primes. Then, there exist exactly $2p^{k-1}$ integers y with $1 < y < p^k q$ such that $\gcd(y, p^k q) \neq 1$, $\gcd(y - 1, p^k q) \neq 1$. Moreover, these integers are of the form $x + \lambda pq$, $x' + \lambda pq \in [1, p^k q]$, where λ is an integer in $[0, p^{k-1} - 1]$ and x, x' are the numbers described in Lemma 3.2.4.

Proof. Let x and x' be the integers described in Lemma 3.2.4. Then, for every integer $\lambda \in [0, p^{k-1} - 1]$, we get $x + \lambda pq$, $x' + \lambda pq \in [1, p^k q]$ and $\gcd(x, p^k q) \neq 1$, $\gcd(x - 1, p^k q) \neq 1$. Similarly, for every $y \in [1, p^k q]$ with $\gcd(y, p^k q) \neq 1$, $\gcd(y - 1, p^k q) \neq 1$, there exists a positive integer $\lambda \in [0, p^{k-1} - 1]$ such that $\lambda pq < y < (\lambda + 1)pq$. Thus, $y - \lambda pq \in [1, pq]$, with $\gcd(y, pq) \neq 1$, $\gcd(y - 1, pq) \neq 1$, since p and q are different primes. Hence, $y - \lambda pq$ is one of the two possible integers described in Lemma 3.2.4. $\square$

3.3 A family of perfect edge-magic graphs of the form $C_m \odot \overline{K}_n$

Let L be the set of vertices of degree 1 of $G = C_m \odot \overline{K}_n$ and $C = V(G) \setminus L$. Assume that $C = \{v_0, v_1, \ldots, v_{m-1}\}$, $L = \{v_i^j\}_{i=0,1,2,\ldots,m-1}^{j=1,2,\ldots,n}$ and $E(G) =$

$\{v_i v_{i+_m 1}, v_i v_i^j\}_{i=0,1,2,\dots,m-1}^{j=1,2,\dots,n}$ where $+_m$ denotes the sum modulo m. Let $\overrightarrow{G}$ be an orientation of G such that, the subdigraph induced by C is strongly connected and all vertices of degree 1 have indegree 1(see, for instance, Figure 3.8). Note that, $\overrightarrow{G} \cong \overrightarrow{K}_{1,n}^l \otimes C_m^+$, where $\overrightarrow{K}_{1,n}^l$ is the digraph obtained by orienting $K_{1,n}^l$ in such a way that all vertices of degree 1 have indegree 1 and C_m^+ is a strong orientation of C_m. Thus by Theorem 3.2.1, $\overrightarrow{G}$ is super edge-magic when m is an odd positive integer.

The following construction and lemmas are inspired by the construction introduced by López et al. in [37]. Let $\mathcal{M}_m$ be the set of all matrices of order $m \times m$ and let g_1 be the labeling of $\overrightarrow{G}$ induced by the product $\overrightarrow{K}_{1,n}^l \otimes C_m^+$, when considering the super edge-magic labeling of $\overrightarrow{K}_{1,n}^l$ that assigns label 1 to the central vertex and a super edge-magic labeling g of C_m. By identifying each vertex of $\overrightarrow{G}$ with the label assigned to it by g_1, we can construct the adjacency matrix of the digraph $\overrightarrow{G}$, which is of the form: $\mathbf{A}_g^1 = (\mathbf{A}_{ij}^1)$, where each $\mathbf{A}_{ij}^1 \in \mathcal{M}_m, \mathbf{A}_{ij}^1 = 0$ for $i > 1$ and $\mathbf{A}_{1j}^1$ has the structure of the adjacency matrix of $g(C_m)^+$, in which each vertex of C_m^+ is identified with the label assigned to it by g. For instance, in case g is the canonical labeling, the structure of the adjacency matrix is given by

$$\begin{pmatrix} M & Id_{(m-1)/2} \\ Id_{(m+1)/2} & N \end{pmatrix},$$

where M and N are two null matrices of size respectively, $(m-1)/2 \times (m+1)/2$ and $(m+1)/2 \times (m-1)/2$, and $Id_k = diag(\overbrace{1,\dots,1}^{k})$. An example of this structure can be observed in the first 5 rows of the adjacency matrix, $\mathbf{A}_g^1$, of the digraph (on the left) that appears in Figure 3.8.

We can also consider the opposite strong orientation of the labeled cycle denoted by $g(C_m)^-$. If we identify each vertex of $\overrightarrow{G} \cong \overrightarrow{K}_{1,n}^l \otimes C_m^-$ with the labels induced by the product, we obtain an adjacency matrix of $\overrightarrow{G}$ with the same structure as $\mathbf{A}_g^1$. Let us denote this matrix by $\mathbf{B}_g^1$. Then $\mathbf{B}_g^1 = (\mathbf{B}_{ij}^1)$, where each $\mathbf{B}_{ij}^1 \in \mathcal{M}_m, \mathbf{B}_{ij}^1 = 0$ for $i > 1$ and $\mathbf{B}_{1j}^1$ has the structure of the adjacency matrix of $g(C_m)^-$, in which each vertex of C_m^- is identified with the label assigned to it by g and is given by
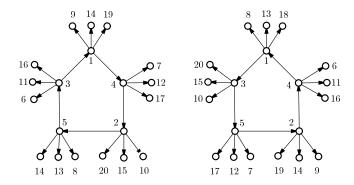
Figure 3.8: Two orientations of $C_5 \odot \overline{K_3}$ with the super edge-magic labelings induced by the product.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Figure 3.9: The first five rows of the matrix $\mathbf{A}_g^1$ of the Figure 3.8 (left).

$$\begin{pmatrix} N & Id_{(m+1)/2} \\ Id_{(m-1)/2} & M \end{pmatrix}.$$

The next matrix corresponds to the first 5 rows of the adjacency matrix $\mathbf{B}_g^1$ of the digraph (on the right) that appears in Figure 3.8.

Let $\mathbf{A}_g^r$ and $\mathbf{B}_g^r$ be the matrices obtained from $\mathbf{A}_g^1$ and $\mathbf{B}_g^1$ respectively by translating each row $r - 1$ units, for $1 \leq r \leq mn + 1$. Thus, if $\mathbf{A}_g^r = (a_{ij}^r)$, then

$$a_{ij}^r = \begin{cases} a_{(i-r+1)j}^1, & i \geq r \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Let $G(\mathbf{A}_g^r)$ and $G(\mathbf{B}_g^r)$ be the digraphs with adjacency matrices $\mathbf{A}_g^r$ and $\mathbf{B}_g^r$ respectively. We also denote by $S(\mathbf{A}_g^r)$ and $S(\mathbf{B}_g^r)$ the subdigraphs of $G(\mathbf{A}_g^r)$

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

Figure 3.10: The first five rows of the matrix $\mathbf{B}_g^1$ of the Figure 3.8 (right).

and $G(\mathbf{B}_g^r)$ induced by the set of vertices $\{r, \ldots, r-1+m\}$, respectively. From the adjacency matrices $\mathbf{A}_g^r$ and $\mathbf{B}_g^r$, it is easy to check the following lemma.

Lemma 3.3.1. Let g be a super edge-magic labeling of C_m. The vertices of $G(\mathbf{A}_g^r)$ and $G(\mathbf{B}_g^r)$ define a super edge-magic labeling g_r^+ and g_r^-, respectively, with valence $\mathrm{val}(g_r^+) = \mathrm{val}(g_r^-) = \mathrm{val}(g_1) + r - 1$, $1 \leq r \leq mn + 1$.

The digraphs $S(\mathbf{A}_g^r)$ and $S(\mathbf{B}_g^r)$ are 1-regular and the graphs $\mathrm{und}(G(\mathbf{A}_g^r))$ and $\mathrm{und}(G(\mathbf{B}_g^r))$ are of the form $H_g^r \odot \overline{K}_n$ where H_g^r is a 2-regular graph. Moreover, $H_g^r \cong H_g^{r+\lambda m}$, for every positive integer λ with $r + \lambda m \leq (m+1)n$.

Proof. The first part of the lemma comes from Lemma 1.4.3, since the minimum induced sum of two adjacent vertices increases by one unit at every step of the translation, and for $r = 1$ this minimum sum is the minimum sum of adjacent vertices of a super edge-magic labeled cycle. The second part is due to the structure of the adjacency matrices. $\square$

Lemma 3.3.2. Let f be the canonical super edge-magic labeling of C_m. If neither $\mathrm{und}(S(\mathbf{A}_f^r))$ nor $\mathrm{und}(S(\mathbf{B}_f^r))$ is isomorphic to a cycle, then $\gcd((m+1)/2 - (r-1), m) \neq 1$ and $\gcd((m-1)/2 - (r-1), m) \neq 1$.

Proof. By (3.3.1), it is clear that $(a, b) \in E(S(\mathbf{A}_f^r))$ if and only if $(a - (r-1), b) \in E(G(\mathbf{A}_f^1))$. That is, if and only if $(b-a) \equiv (m+1)/2 - (r-1) \pmod{m}$, by (3.2.1) in Remark 3.2.1. Similarly, $(a, b) \in E(G(\mathbf{B}_f^r))$, if and only if $(b - a) \equiv (m-1)/2 - (r-1) \pmod{m}$, by (3.2.2). $\square$

Now, we present one of the main contributions of this chapter.

Theorem 3.3.1. *Let $m = pq$ where p and q are different odd primes. Let n be a positive integer. Then, the graph $G = C_m \odot \overline{K}_n$ is perfect super edge-magic.*

Proof. Let us first determine the super edge-magic interval I_G of G. Let $g : V \to \{i\}_{i=1}^{m+mn}$ be a bijective function. Then, the corresponding element in I_G is given by

$$\frac{\sum_{u \in V} \deg(u)g(u) + \sum_{i=m+mn+1}^{2(m+mn)} i}{m+mn}.$$

That is,

$$\frac{\sum_{u \in C}(2+n)g(u) + \sum_{u \in L} g(u) + \sum_{i=m+mn+1}^{2(m+mn)} i}{m+mn},$$

The maximum of I_G occurs when $\{g(u) : u \in L\} = \{1, 2, 3, \ldots, mn\}$ and the minimum when $\{g(u) : u \in L\} = \{m+1, m+2, \ldots, m+mn\}$ where L denotes the set of vertices of degree 1 of G and $g : V(G) \to \{i\}_{i=1}^{m+mn}$ is any bijective function. Thus, $I_G = [(3+5m)/2 + 2mn, (3+5m)/2 + 3mn] \cap \mathbb{Z}$.

Let f be the canonical labeling of the cycle. By Lemmas 3.3.1 and 3.3.2, we obtain that for all r with $1 \le r \le mn+1$, with either $\gcd((m+1)/2 - (r-1), m) = 1$ or $\gcd((m-1)/2 - (r-1), m) = 1$, either $\mathbf{A}_f^r$ or $\mathbf{B}_f^r$ is the adjacency matrix of a super edge-magic labeled digraph, whose underlying graph is G. Moreover, if f_r is the induced super edge-magic labeling of G, then $\mathrm{val}(f_r) = \mathrm{val}(f_1) + r - 1$. Notice that, by Lemma 3.2.2, $\mathrm{val}(f_1) = (5m+3)/2 + 2mn$.

Now, we provide a construction to cover the missing valences of G. That is, $\mathrm{val}(f_1) + r - 1$, with $\gcd((m+1)/2 - (r-1), m) \ne 1$ and $\gcd((m-1)/2 - (r-1), m) \ne 1$. What happens for this values is that, by Lemma 3.3.2, we can not guarantee that H_f^r is a cycle. Also, by Lemma 3.3.2, we have that $H_f^r \cong H_f^{r+\lambda m}$, for every positive integer λ with $r + \lambda m \le (m+1)n$. Thus, in what follows, we will assume that $n = 1$.

Let $\alpha p + \beta q = 1$ be the Bézout identity where $\alpha p > 0$ and $\max\{\alpha p, |\beta q|\} \le (pq+1)/2$ (such α and β exist by Lemma 3.2.3). Then, $x = \alpha p$ is one of the integers of Lemma 3.2.4. The other one is $x' = pq - \alpha p + 1$. Thus, one of the missing valences is $\mathrm{val}(f_1) + r - 1$, where $r - 1 = (pq+1)/2 - \alpha p$. That is,

$$r - 1 = p(\frac{q+1}{2} - \alpha - 1) + \frac{p+1}{2}.$$

Let $\alpha' = (q+1)/2 - \alpha$ and $\beta' = (p+3)/2$. Then, $r - 1 = p(\alpha' - 1) + \beta' - 1$. Notice that, if we prove the existence of a super edge-magic labeling g_r of G with valence $\mathrm{val}(f_1) + r - 1$, then the other missing valence, namely, $\mathrm{val}(f_1) + (pq - 1)/2 + \alpha p$ will be realized by the super edge-magic complementary labeling of g_r, namely g_r^c (see (3.2.3)).

Let g be the labeling of C_m^+ induced by the product $f(C_q)^+ \otimes f(C_p)^-$, when considering the canonical super edge-magic labeling of C_q and C_p, respectively. We will prove that $H_g^r \cong \mathrm{und}(S(\mathbf{A}_g^r))$ is a cycle of length pq.

Let $(a', b') \in E(S(\mathbf{A}_g^r))$, that is $r \leq a', b' \leq r - 1 + m$. Thus, $(a' - (r-1), b') \in E(G(\mathbf{A}_g^1))$. In particular, there exists a nonnegative integer $\lambda_0 = \lambda_0(b')$ such that $(a' - (r-1), b' - \lambda_0 m) \in E(S(\mathbf{A}_g^1))$. Let $(a, i), (b, j)$ be such that $a' = p(a-1)+i$, $b' - \lambda_0 m = p(b-1)+j$ where $1 \leq a \leq 2q$, $1 \leq b \leq q$ and $1 \leq i, j \leq p$. This implies, $(p(a-\alpha')+(i-\beta'+1), p(b-1)+j) \in E(S(\mathbf{A}_g^1))$. That is, $(p(a - \alpha') + (i - \beta' + 1), p(b - 1) + j) \in E(f(C_q)^+ \otimes f(C_p)^-)$.

We have two types of adjacencies:

Type i: $1 \leq i - \beta' + 1 \leq p$. By definition of $\otimes$-product and the labeling induced (see Remark 3.2.2), we obtain that $(a - \alpha' + 1, b) \in E(f(C_q)^+)$ and $(i-\beta'+1, j) \in E(f(C_p)^-)$. That is, using (3.2.1) and (3.2.2) in Remark 3.2.1, $b - (a - \alpha' + 1) \equiv (q+1)/2 \pmod{q}$ and $j - (i - \beta' + 1) \equiv (p-1)/2 \pmod{p}$. Equivalently, $b - a \equiv \alpha + 1 \pmod{q}$ and $j - i \equiv -1 \pmod{p}$.

Type ii: $-p + 2 \leq i - \beta' + 1 \leq 0$. Again by definition of $\otimes$-product and the labeling induced, we obtain $(a-\alpha, b) \in E(f(C_q)^+)$ and $(p+i-\beta'+1, j) \in E(f(C_p)^-)$. Thus, using (3.2.1) and (3.2.2), $b - (a - \alpha') \equiv (q+1)/2 \pmod{q}$ and $j - (p+i-\beta'+1) \equiv (p-1)/2 \pmod{p}$. Equivalently, $b - a \equiv \alpha \pmod{q}$ and $j - i \equiv -1 \pmod{p}$.

Assume that r is contained in a cycle C_l^+, $l < pq$, with I edges of type i. Then, $l = kp$, $I = k(p-1)/2$, for some positive integer k, and

$$kp\alpha + k(p-1)/2 = sq, \qquad (3.3.2)$$

for some integer s. Using that $\alpha p = 1 - \beta q$ and (3.3.2), we obtain that

q is a divisor of $(p + 1)/2$. Note that, this implies that $p + 1 = \lambda q$, for some positive λ, and hence, $p(\alpha + 1) = (\lambda - \beta)q$. Therefore, q divides $\alpha + 1$ contradicting that $\alpha p < (pq + 1)/2$. $\square$

The magic interval of crowns of the form $C_m \odot \overline{K}_n$ was obtained in [41].

Lemma 3.3.3. [41] Let m and n be positive integers with $m \geq 3$. Then, the magic interval of $C_m \odot \overline{K}_n$ is given by

$$J_{C_m \odot \overline{K}_n} = \left[\frac{3 + 5m}{2} + 2mn, \frac{3 + 7m}{2} + 4mn \right] \cap \mathbb{Z}.$$

Theorem 3.3.1 implies that for every element k included in the super edge-magic interval, there exists a super edge-magic labeling with valence k. Taking the complementary labeling of these labelings, we get that all natural numbers from $3mn + (3 + 7m)/2$ up to $4mn + (3 + 7m)/2$ appear as valences of edge-magic labelings of $C_m \odot \overline{K}_n$. Therefore, in order to prove that $C_m \odot \overline{K}_n$ is perfect edge-magic, we only need to show that for each $k \in \mathbb{N}$, with $3mn + (3 + 5m)/2 < k < 3mn + (3 + 7m)/2$, there exists an edge-magic labeling with valence k. We do this using the odd and even labelings of the labelings f_r and g_r introduced in the proof of Theorem 3.3.1. The details are contained in the proof of the following lemma.

Lemma 3.3.4. Let m be the product of two different odd primes and let n be any positive integer. Then, for each k with $2mn + 3m + 1 \leq k \leq 4mn + 3m + 2$ there exists an edge-magic labeling of $C_m \odot \overline{K}_n$ with valence k.

Proof. Let $m = pq$, where p and q are different odd primes. Let x and x' be the integers introduced in Lemma 3.2.4. Consider the super edge-magic labelings f_r and g_r of $C_m \odot \overline{K}_n$, introduced in the proof of Theorem 3.3.1. Then, the set $\{\text{val}(f_r); 1 \leq r \leq mn + 1, r - 1 \notin \{(pq + 1) - x, (pq + 1) - x'\}\} \cup \{\text{val}(g_r), \text{val}(g_r^c) : r - 1 = (pq + 1) - x\}$ is a set of consecutive integers. Thus, Lemma 3.2.1 implies that the set $\{\text{val}(o(f_r)), \text{val}(e(f_r)); 1 \leq r \leq mn + 1, r - 1 \notin \{(pq + 1) - x, (pq + 1) - x'\}\} \cup \{\text{val}(o(g_r)), \text{val}(o(g_r^c)), \text{val}(e(g_r)), \text{val}(e(g_r^c)) : r - 1 = (pq + 1) - x\}$ contains all integers from $\text{val}(o(f_1))$ up to $\text{val}(e(f_{mn+1}))$. That is, all integers from $2mn + 3m + 1$ up to $4mn + 3m + 2$. $\square$

Since $2mn + 3m + 1 \leq 3mn + (3 + 5m)/2$ and $3mn + (3 + 7m)/2 \leq 4mn + 3m + 2$ for $n \geq 1$, we obtain the next theorem which is the other main contribution of the chapter.

Theorem 3.3.2. *Let $m = pq$ where p and q are different odd primes. Let n be a positive integer. Then, the graph $G = C_m \odot \overline{K}_n$ is perfect edge-magic.*

3.4 Edge-magic labelings of crowns

The fact that even cycles admit edge-magic labelings has been known for several decades already. See the next theorem.

Theorem 3.4.1. *[44] Every even cycle C_n has an edge-magic labeling with magic sum $(5n + 4)/2$.*

In fact this result has been improved recently as shown in the next theorem. It is also worth to mention that McQuillian [45] has made important contributions in this direction.

Theorem 3.4.2. *[42] Let $m = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the unique prime factorization (up to ordering) of an even number m. Then C_m admits at least $\Sigma_{i=1}^k \alpha_i$ edge-magic labelings with at least $\Sigma_{i=1}^k \alpha_i$ mutually different magic sums. If $\alpha \geq 2$, this lower bound can be improved to $1 + \Sigma_{i=1}^k \alpha_i$.*

Similarly, the next result was established in [42] for cycles of odd order.

Theorem 3.4.3. *[42] Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the unique prime factorization (up to ordering) of an odd number m. Then C_m admits at least $1 + \Sigma_{i=1}^k \alpha_i$ edge-magic labelings with at least $1 + \Sigma_{i=1}^k \alpha_i$ mutually different magic sums.*

The next lemma will be useful in improving the lower bounds of crowns as in Theorem 3.4.4 and Theorem 3.4.5.

Lemma 3.4.1. Let $g : V(C_m^+) \cup E(C_m^+) \to \{1, 2, \ldots, 2m\}$ be an edge-magic labeling of C_m^+, and let $\gamma_r : V(\overrightarrow{K}_{1,n}^l) \to \{1, 2, \ldots, n+1\}$ be a super edge-magic labeling of $\overrightarrow{K}_{1,n}^l$ that assigns label r to the central vertex with $\text{val}(\gamma_r) = r + 2n + 3$, $1 \leq r \leq n + 1$. Then the induced edge-magic labeling $\widehat{g}_r$ of $C_m^+ \otimes_h \overrightarrow{K}_{1,n}^l$ has valence $(n+1)(\text{val}(g) - 2) + r + 1$. Let g' be a different edge-magic labeling of C_m^+ with $\text{val}(g) < \text{val}(g')$, then $\text{val}(\widehat{g}_{n+1}) < \text{val}(\widehat{g}'_1)$, where $\widehat{g}'_r$ is the induced edge-magic labeling of $C_m^+ \otimes \overrightarrow{K}_{1,n}^l$ when $\overrightarrow{K}_{1,n}^l$ is labeled with γ_r and C_m^+ with g'.

Proof. By Lemma 3.2.2, $\text{val}(\widehat{g}_r) = (n+1)[\text{val}(g) - 3] + r + 1 + n + 1$, that is, $\text{val}(\widehat{g}_r) = (n+1)[\text{val}(g) - 2] + r + 1$. Let g' be a different edge-magic labeling of C_m^+ with $\text{val}(g) < \text{val}(g')$, then $\text{val}(\widehat{g}_{n+1}) = (n+1)[\text{val}(g) - 2] + n + 2 \leq (n+1)[\text{val}(g') - 1 - 2] + n + 2 < \text{val}(\widehat{g}'_1)$. Hence the result follows. □

Now using Theorem 3.4.2 and Lemmas 3.1.1 and 3.4.1, we can prove the next theorem.

Theorem 3.4.4. *Let $m = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the unique prime factorization (up to ordering) of an even number m. Then $G = C_m \odot \overline{K}_n$ admits at least $(\Sigma_{i=1}^k \alpha_i)(n+1)$ mutually different magic sums. If $\alpha \geq 2$, this lower bound can be improved to $(1 + \Sigma_{i=1}^k \alpha_i)(n+1)$.*

Proof. Note that $G \cong \text{und}(C_m^+ \otimes \overrightarrow{K}_{1,n}^l)$. Let $\widehat{g}$ and g be edge-magic labelings of $C_m^+ \otimes \overrightarrow{K}_{1,n}^l$ and C_m^+ respectively and let γ_r be a super edge-magic labeling of $\overrightarrow{K}_{1,n}^l$ that assigns label r to the central vertex with $\text{val}(\gamma_r) = r + 2n + 3$, $1 \leq r \leq n+1$. By Lemma 3.4.1, we get $\text{val}(\widehat{g}_r) = (n+1)[\text{val}(g) - 2] + r + 1$. Thus, $\text{val}(\widehat{g})$ depends on the valences of g and r. We know that by Lemma 3.1.1, $\overrightarrow{K}_{1,n}^l$ has $n+1$ valences and by Theorem 3.4.2, C_m has at least $\Sigma_{i=1}^k \alpha_i$ mutually different valences. Thus, using Lemma 3.4.1, $G = C_m \odot \overline{K}_n$ admits at least $(\Sigma_{i=1}^k \alpha_i)(n+1)$ mutually different magic sums. If $\alpha \geq 2$, this lower bound can be improved to $(1 + \Sigma_{i=1}^k \alpha_i)(n+1)$. □

Similarly, using Theorem 3.4.3 and Lemmas 3.1.1 and 3.4.1, we can prove the next theorem.

Theorem 3.4.5. *Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the unique prime factorization (up to ordering) of an odd number m. Then $G = C_m \odot \overline{K}_n$ admits at least $(1 + \Sigma_{i=1}^k \alpha_i)(n+1)$ mutually different magic sums.*

Let f be the canonical labeling of the cycle $C_{p^k q}$, where p and q are different odd primes and k is a positive integer. The construction provided in Section 3.3 guarantees the existence of a super edge-magic labeling of the crown $C_{p^k q} \odot \overline{K}_n$, with valence $\text{val}(f_1) + r - 1$, for many values of r. The possible exceptions can be obtained from Corollary 3.2.1.

3.5 Open questions

In this chapter, we have proved that $C_m \odot \bar{K}_n$ is perfect (super) edge-magic for $m = pq$ where p and q are different odd primes. We have also improved a lower bound for the number of edge-magic valences of $C_m \odot \bar{K}_n$. From what we have done in this chapter so far, we propose the following open questions.

Open question 3.5.1. Prove or disprove that $C_{p^k q} \odot \overline{K}_n$, where p and q are different odd primes and k is a positive integer is perfect (super) edge-magic.

Open question 3.5.2. Prove or disprove that $C_{p^k q^l} \odot \overline{K}_n$ is perfect (super) edge-magic, where p and q are different odd primes and k, l are positive integers.

Open question 3.5.3. Prove or disprove that $C_{pqr} \odot \overline{K}_n$ is perfect (super) edge-magic, where p, q and r are primes.

The previous open questions are the initial steps to solve the following more general open question.

Open question 3.5.4. Characterize the set of perfect (super) edge-magic graphs of the form $C_m \odot \overline{K}_n$.

The *valence density* of G, $\delta(G)$, was introduced in [39] as the quotient

$$\delta(G) = \frac{|\tau(G)|}{|J(G)|}.$$

Similarly, the *super valence density* of G, $\delta_s(G)$, introduced in [39], is the quotient

$$\delta_s(G) = \frac{|\sigma(G)|}{|I(G)|}.$$

We end this section with the following open question on the lower bound of valence density and super valence density.

Open question 3.5.5. Is there any $\alpha \in (0, 1)$ that is possible to ensure that all $C_m \odot \overline{K}_n$ have $\delta(G) \geq \alpha$ or $\delta_s(G) \geq \alpha$?

Chapter 4

A new labeling construction from the $\otimes_h$-product

4.1 Introduction

Since the $\otimes_h$-product was first introduced in 2008 [16], it has been proven to be an excellent technique to better understand many different types of labelings, as for instance (super) edge-magic labelings and harmonious labelings. The lack of enumerative results involving graph labelings constitutes a big gap in the literature of graph labelings that this product has helped to fill enormously. Also further applications outside the world of graph labeling have been found for the $\otimes_h$-product, as for instance it introduces new ways to construct Skolem and Langford type sequences [30]. In summary, the $\otimes_h$-product constitutes a big breakthrough into the world of graph labeling that allows to have a better and deeper understanding of the subject. In all the results involving the $\otimes_h$-product, since the very beginning, it seems to be constant to use super edge-magic labeled graphs as the second factor of the product, or at least graphs that in a way or another come from super edge-magic graphs [23, 35, 38].

In this chapter, we characterize some relations among the induced labelings obtained from the $\otimes_h$-product, when we combine the odd and the even labelings of a particular super edge-magic labeling f, together with the com-

plementary and the super edge-magic complementary constructions of the labelings involved. This is the content of Section 4.2. The main result of the chapter is Theorem 4.3.1, where, in some sense, we exchange the role of the factors established in Theorem 3.2.1. Thus, we can enlarge the family of labeled graphs that we can obtain from the product. We conclude this chapter with an application of this fact in Section 4.4. All the results in this chapter are proved in [33] unless otherwise mentioned.

Recall that, if f is a super edge-magic labeling of a (p, q)-graph G, with $p = q$, then the odd labeling $o(f)$ and the even labeling $e(f)$ are the labelings defined on the vertices as follows:

$$o(f)(x) = 2f(x) - 1, \text{ and } e(f)(x) = 2f(x).$$

We start with an easy lemma that establishes a relation with the odd and the even labelings introduced in Section 3.2, with the complementary and super edge-magic complementary labeling of a super edge-magic labeling f (see Section 1.4 and Section 2.1 respectively).

Lemma 4.1.1. Let f be a super edge-magic labeling of a (p, q)-graph G with $p = q$. Then,

$$\overline{e(f)} \simeq o(f_c) \quad \text{and} \quad \overline{o(f)} \simeq e(f_c).$$

Proof. Let f_c be the super edge-magic complementary labeling of f. In the case that G is a graph of equal order and size, $o(f)$ and $e(f)$ denote the odd and the even labelings of f [see Section 3.2]. Then, by definition, for all $x \in V(G)$,

$$\begin{aligned}
\overline{e(f)}(x) &= 2p + 1 - 2f(x) \\
&= 2(p + 1 - f(x)) - 1 \\
&= 2f_c(x) - 1 \\
&= o(f_c)(x).
\end{aligned}$$

Similarly, for all $x \in V(G)$,

$$\begin{aligned}
\overline{o(f)}(x) &= 2p + 2 - 2f(x) \\
&= 2(p + 1 - f(x)) \\
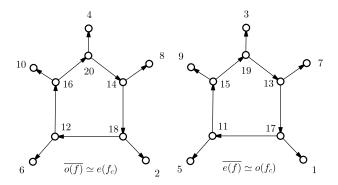&= 2f_c(x) \\
&= e(f_c)(x).
\end{aligned}$$

Figure 4.1: An example for $\overline{e(f)} \simeq o(f_c)$ and $\overline{o(f)} \simeq e(f_c)$.

Hence, $\overline{e(f)} \simeq o(f_c)$ and $\overline{o(f)} \simeq e(f_c)$.   □

Figure 4.1 is an example for the Lemma 4.1.1 based on the super edge-magic labeling of $C_5 \odot \overline{K_1}$ introduced in Figure 2.1 (right).

4.2   Some labeling properties obtained from the $\otimes_h$-product

One of the research lines when we deal with edge-magic labelings of a particular graph G is the study of the theoretical valences that are realizable. This problem has been completely solved for crowns of the form $C_m \odot \overline{K}_n$, where $m = p^k$ and $m = pq$, where p and q are primes (see [37, 41] and Chapter 3, respectively). In both cases, the proof is based on the construction of all theoretical super edge-magic valences and then, with the help of the odd and the even labelings, to complete the remaining valences. In this section we show some labeling properties in which we combine these labelings together with other labeling constructions.

The following proposition shows a relation among complementary labelings and the induced labelings obtained from the $\otimes_h$-product.

Proposition 4.2.1. Let f be an edge-magic labeling of a digraph D. Consider any function $h : E(D) \to \mathcal{S}_p^k$. Then, there exists $\bar{h} : E(D) \to \mathcal{S}_p^{p+3-k}$ such that

$$D \otimes_h \mathcal{S}_p^k \cong D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k} \quad \text{and} \quad \overline{\hat{f}} \simeq \hat{\bar{f}},$$

where $\overline{\hat{f}}$ is the complementary labeling of the induced labeling of f of $D \otimes_h \mathcal{S}_p^k$ and $\hat{\bar{f}}$ is the labeling of $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ induced by the complementary labeling of f.

Proof. We will prove that the induced edge-magic labeled digraphs are isomorphic. Let $\phi : \mathcal{S}_p^k \to \mathcal{S}_p^{p+3-k}$ be the function defined by $\phi(F) = F^c$, where $(\bar{i}, \bar{j}) \in F^c$ if and only if $(p + 1 - \bar{i}, p + 1 - \bar{j}) \in F$. Notice that the minimum induced edge sum of F^c is $2p + 2 - (i + j)$, where $i + j$ is the maximum induced edge sum of F, that is, $i + j = k + (p - 1)$. Thus, the minimum induced edge sum of F^c is $(p + 3) - k$.

Assume that D is a (n, m)-digraph in which each vertex is identified with the label assigned to it by f. Then, the induced labeling $\hat{f}$ of the product $D \otimes_h \mathcal{S}_p^k$ is defined by $\hat{f}(a, i) = p(a - 1) + i$, for any vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$ and $\hat{f}((a, i), (b, j)) = p(e - 1) + k + p - (i + j)$, where e is the label of (a, b) assigned by f. Then, since $|V(D \otimes_h \mathcal{S}_p^k)| = pn$ and $|E(D \otimes_h \mathcal{S}_p^k)| = pm$, the complementary labeling of $\hat{f}$ is defined by

- $\overline{\hat{f}}(a, i) = p(m + n) + 1 - p(a - 1) - i$, for any vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$ and

- $\overline{\hat{f}}((a, i), (b, j)) = p(m + n) + 1 - p(e - 1) - k - p + (i + j)$, where e is the label of (a, b) assigned by f.

Let $\bar{h} = \phi \circ h : E(D) \to \mathcal{S}_p^{p+3-k}$ and consider the labeling $\bar{f}$ of D. Then the induced labeling $\hat{\bar{f}}$ of the product $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ is defined by $\hat{\bar{f}}(\bar{a}, \bar{i}) = p(m + n + 1 - a - 1) + p + 1 - i$, that is,

- $\hat{\bar{f}}(\bar{a}, \bar{i}) = p(m + n) + 1 - p(a - 1) - i$, for any vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$ and

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- $\hat{\overline{f}}((\bar{a}, \bar{i}), (\bar{b}, \bar{j})) = p(m + n + 1 - e - 1) + p + 3 - k + p - (\bar{i} + \bar{j})$,
  where e is the label of (a, b) assigned by f. That is, $\hat{\overline{f}}((\bar{a}, \bar{i}), (\bar{b}, \bar{j})) = p(m + n) + 1 - p(e - 1) - k - p + (i + j)$.

This proves the result. □

An example for the above proposition is shown in Figure 4.2. An edge-magic labeling f and its complementary $\overline{f}$ are shown in Figure 4.2(a). Let $\overrightarrow{K}_{1,2}^{l}$ be the super edge-magic labeled digraph from the family $\mathcal{S}_3^2$ with a set of vertices $\{1, 2, 3\}$ and $E(\overrightarrow{K}_{1,2}^{l}) = \{(1, 1), (1, 2), (1, 3)\}$. Figure 4.2(b) shows the induced labeling of f and hence $\overline{\hat{f}} \simeq \hat{\overline{f}}$. Notice that, by using the missing labels, there is only one way to complete the edge-magic labelings obtained.
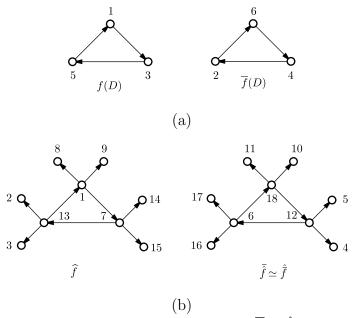

Figure 4.2: An example for $\overline{\hat{f}} \simeq \hat{\overline{f}}$.

Corollary 4.2.1. Let D be a (super) edge-magic digraph. Let f and $\overline{f}$ be a (super) edge-magic labeling and its complement of D respectively. Assume that $k = (p + 3)/2$ and let $\hat{f}$ and $\hat{\overline{f}}$ be the edge-magic labeling and its

complementary labeling of the graph $und(D \otimes_h \mathcal{S}_p^{p+3})$ obtained from the labeling f of D. Then,

$$\text{val}(\overline{\hat{f}}) = \text{val}(\hat{\overline{f}}).$$

Proof. It sufficies to observe that if $k = (p+3)/2$ then $p+3-k = (p+3)/2$. $\square$

For digraphs D with the same order and size, we obtain the next two results.

Corollary 4.2.2. Let f be a super edge-magic labeling of a (n, m)-digraph D with $m = n$. Consider any function $h : E(D) \to \mathcal{S}_p^k$. Then, there exists $\bar{h} : E(D) \to \mathcal{S}_p^{p+3-k}$ such that

$$\overline{\widehat{o(f)}} \simeq \widehat{e(f_c)},$$

where $\overline{\widehat{o(f)}}$ is the complementary labeling of the induced labeling of $o(f)$ of $D \otimes_h \mathcal{S}_p^k$ and $\widehat{e(f_c)}$ is labeling of $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ induced by the even labeling of f_c.

Proof. Let f be a super edge-magic labeling of D, by Proposition 4.2.1 applied to $o(f)$, there exists $\bar{h} : E(D) \to \mathcal{S}_p^{p+3-k}$ such that

$$\overline{\widehat{o(f)}} \simeq \widehat{\overline{o(f)}},$$

where $\overline{\widehat{o(f)}}$ is the complementary labeling of the induced labeling of $o(f)$ of $D \otimes_h \mathcal{S}_p^k$ and $\widehat{\overline{o(f)}}$ is labeling of $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ induced by the complementary of the odd labeling of f. By Lemma 4.1.1, $\overline{o(f)} \simeq e(f_c)$. Thus, we obtain the result. $\square$
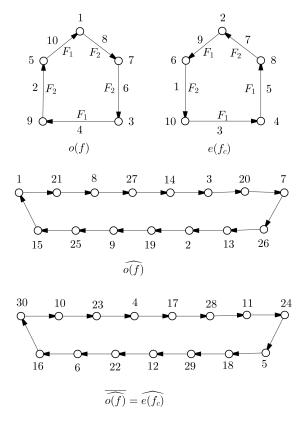
Example 4.2.1. Let D be a digraph with a super edge-magic labeling f defined by $V(D) = [1, 5]$ and $E(D) = \{(1, 4), (4, 2), (2, 5), (5, 3), (3, 1)\}$ and $\mathcal{S}_3^3 = \{F_1, F_2\}$, where F_1 is the super edge-magic labeled digraph defined by $V(F_1) = \{1, 2, 3\}$ and $E(F_1) = \{(1, 2), (2, 3), (3, 1)\}$ and F_2 is the digraph obtained from F_1 by reversing all its arcs. Let f_c be the super edge-magic complementary labeling of D with $V(D) = [1, 5]$ and $E(D) = \{(5, 2), (2, 4), (4, 1), (1, 3), (3, 5)\}$. Figure 4.3 shows the mapping h and $\bar{h}$ together with $\overline{\widehat{o(f)}} \simeq \widehat{e(f_c)}$.

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Figure 4.3: An example for $\overline{\widehat{o(f)}} \simeq \widehat{e(f_c)}$.

With a similar proof of Corollary 4.2.2, we obtain the next result.

Corollary 4.2.3. Let f be a super edge-magic labeling of (n, m)-digraph D with $m = n$. Consider any function $h : E(D) \to \mathcal{S}_p^k$. Then, there exists $\bar{h} : E(D) \to \mathcal{S}_p^{p+3-k}$ such that

$$\overline{\widehat{e(f)}} \simeq \widehat{o(f_c)},$$

where $\overline{\widehat{e(f)}}$ is the complementary labeling of the induced labeling of $e(f)$ of $D \otimes_h \mathcal{S}_p^k$ and $\widehat{o(f_c)}$ is labeling of $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ induced by the odd labeling of f_c.

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Proof. Let f be a super edge-magic labeling of D, by Proposition 4.2.1 applied to $e(f)$, there exists $\bar{h} : E(D) \to \mathcal{S}_p^{p+3-k}$ such that

$$\overline{\widehat{e(f)}} \simeq \widehat{\overline{e(f)}},$$

where $\overline{\widehat{e(f)}}$ is the complementary labeling of the induced labeling of $e(f)$ of $D \otimes_h \mathcal{S}_p^k$ and $\widehat{\overline{e(f)}}$ is labeling of $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ induced by the complementary of the even labeling of f. By Lemma 4.1.1, $\overline{e(f)} \simeq o(f_c)$. Thus, we obtain the result. $\qquad\square$

The next result is similar to Proposition 4.2.1.

Proposition 4.2.2. Let f be a super edge-magic labeling of digraph D. Consider any function $h : E(D) \to \mathcal{S}_p^k$. Then, there exists $\bar{h} : E(D) \to \mathcal{S}_p^{p+3-k}$ such that

$$(\hat{f})_c \simeq \widehat{f_c},$$

where $(\hat{f})_c$ is the super edge-magic complementary labeling of the induced labeling of f of $D\otimes_h\mathcal{S}_p^k$ and $\widehat{f_c}$ is the labeling of $D\otimes_{\bar{h}}\mathcal{S}_p^{p+3-k}$ induced by the super edge-magic complementary labeling of f. Moreover, val $((\hat{f})_c) = val(\widehat{f_c})$.

Proof. Let $\phi : \mathcal{S}_p^k \to \mathcal{S}_p^{p+3-k}$ be the function defined by $\phi(F) = F^c$, where $(\bar{i}, \bar{j}) \in F^c$ if and only if $(p + 1 - \bar{i}, p + 1 - \bar{j}) \in F$. Notice that the minimum induced edge sum of F^c is $2p + 2 - (i + j)$, where $i + j$ is the maximum induced edge sum of F, that is, $i + j = k + (p - 1)$. Thus, the minimum induced edge sum of F^c is $(p + 3) - k$.

Assume that D is a (n, m)-digraph in which each vertex is identified with the label assigned to it by f. Then, the induced (super edge-magic) labeling $\hat{f}$ of the product $D \otimes_h \mathcal{S}_p^k$ is defined by $\hat{f}(a, i) = p(a - 1) + i$, for any vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$. Since $|V(D \otimes_h \mathcal{S}_p^k)| = pn$, the super edge-magic complementary labeling of $\hat{f}$ is defined by

- $(\hat{f})_c(a, i) = pn + 1 - (p(a - 1) + i)$, for any vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$.

Let $\bar{h} = \phi \circ h : E(D) \to \mathcal{S}_p^{p+3-k}$ and consider the labeling f_c of D. Then the induced labeling $\widehat{f_c}$ of the product $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ is defined by $\widehat{f_c}(\bar{a}, \bar{i}) = p(\bar{a} - 1) + \bar{i}$, that is,

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- $\widehat{f}_c(\bar{a}, \bar{i}) = p(n-a) + p + 1 - i$, for any vertex $(a, i) \in V(D \otimes_h \mathcal{S}_p^k)$.

This proves the result. $\qquad\square$

Example 4.2.2. Let D be a super edge-magic labeled digraph with $V(D) = [1, 3]$ and $E(D) = \{(1, 2), (2, 3), (3, 1)\}$. Let F be a member of $\mathcal{S}_3^2$ defined by $V(F) = \{1, 2, 3\}$ and $E(F) = \{(1, 1), (1, 2), (1, 3)\}$. Let f_c be the super edge-magic complementary labeling of D with $V(D) = [1, 3]$ and $E(D) = \{(1, 3), (3, 2), (2, 1)\}$. Consider $h : E(D) \longrightarrow \mathcal{S}_3^2$ defined by: $h(1, 2) = h(2, 3) = h(3, 1) = F$. Let H be a member of $\mathcal{S}_3^4$ with $V(H) = \{1, 2, 3\}$ and $E(H) = \{(3, 1), (3, 2), (3, 3)\}$. Consider the mapping $\bar{h} : E(D) \longrightarrow \mathcal{S}_3^4$ defined by: $\bar{h}(1, 3) = \bar{h}(3, 2) = \bar{h}(2, 1) = H$. Then, Figure 4.4 shows the induced labeling of f of $D \otimes_h \mathcal{S}_p^k$ and $\widehat{f}_c$, the labeling of $D \otimes_{\bar{h}} \mathcal{S}_p^{p+3-k}$ induced by the super edge-magic complementary labeling of f.
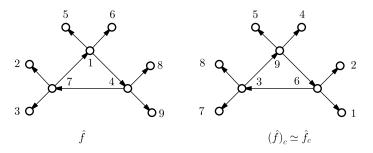
Figure 4.4: An example for $(\hat{f})_c \simeq \widehat{f}_c$.

4.3   The main result

The goal of this section lies in the fact that it allows us to use other types of labeled graphs as a second factor of the product and this allows to refresh the ways of attacking old famous problems in the subject of graph labelings as we will see in the next lines. We now introduce a new family $\mathcal{T}_\sigma^q$ of edge-magic labeled graphs. An edge-magic labeled digraph F is in $\mathcal{T}_\sigma^q$ if $V(F) = V$, $|E(F)| = q$ and the magic sum of the edge-magic labeling is equal to σ.

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Theorem 4.3.1. *Let $D \in \mathcal{S}_n^k$ and let h be any function $h : E(D) \to \mathcal{T}_\sigma^q$. Then $D \otimes_h \mathcal{T}_\sigma^q$ is edge-magic.*

Proof. Let $p = |V|$. We identify the vertices of D and each element of $\mathcal{T}_\sigma^q$ after the labels of their corresponding super edge-magic labeling and edge-magic labeling, respectively. Consider the following labeling of $D \otimes_h \mathcal{T}_\sigma^q$:

1. If $(i, a) \in V(D \otimes_h \mathcal{T}_\sigma^q)$ we assign to the vertex the label:

$$(p + q)(i - 1) + a.$$

2. If $((i, a), (j, b)) \in E(D \otimes_h \mathcal{T}_\sigma^q)$ we assign to the arc the label:

$$(p + q)(k + n - (i + j) - 1) + (\sigma - (a + b)).$$

Notice that, since $D \in \mathcal{S}_n^k$ is labeled with a super edge-magic labeling with minimum sum of the adjacent vertices equal to k, we have

$$\{(k + n) - (i + j) : \ (i, j) \in E(D)\} = [1, n].$$

Moreover, since each element $F \in \mathcal{T}_\sigma^q$, it follows that

$$\{\sigma - (a + b) : \ (a, b) \in E(F)\} = [1, p + q] \setminus V.$$

Thus, the set of labels in $D \otimes_h \mathcal{T}_\sigma^q$ covers all elements in $[1, n(p+q)]$. Moreover, for each arc $((i, a)(j, b)) \in E(D \otimes_h \mathcal{T}_\sigma^q)$ the sum of the labels is constant and is equal to: $(p + q)(k + n - 3) + \sigma$. □

From the previous proof, we also conclude the next result.

Lemma 4.3.1. *Let $D \in \mathcal{S}_n^k$ and $\widetilde{h}$ be the edge-magic labeling of the digraph $D \otimes_h \mathcal{T}_\sigma^q$, induced by the super edge-magic labeling of D and the function $h : E(D) \to \mathcal{T}_\sigma^q$. Then the valence of $\widetilde{h}$ is given by the formula*

$$\mathrm{val}(\widetilde{h}) \ = \ (p + q)(k + n - 3) + \sigma, \tag{4.3.1}$$

where $p = |V(F)|$, for every $F \in \mathcal{T}_\sigma^q$.

Example 4.3.1. Let D be the edge-magic labeled digraph defined by $V(D) = [1,3]$ and $E(D) = \{(1,2),(2,3),(3,1)\}$ and $\mathcal{T}_{36}^4 = \{F_1, F_2\}$, where F_1 is the edge-magic labeled digraph defined by $V(F_1) = \{1,2,3,6\}$ and $E(D) = \{(1,3),(3,2),(2,6),(6,1)\}$ and F_2 is the digraph obtained from F_1 by reversing all its arcs. Notice that, by using the missing labels, there is only one way to complete an edge-magic labeling of D, F_1 and F_2. Consider $h : E(D) \longrightarrow \mathcal{T}_{36}^4$ defined by: $h(1,2) = F_2$ and $h(2,3) = h(3,1) = F_1$. Then, the digraph $D \otimes_h \mathcal{T}_{36}^4$ appears in Figure 4.5(a). The edge-magic labeling induced by the product (where only the labels of the vertices are showed) appears in Figure 4.5(b).
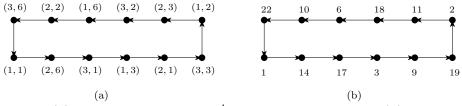
Figure 4.5: (a) The product $D \otimes_h \mathcal{T}_{36}^4$ in Example 4.3.1 and (b) the induced labeling on vertices.

4.3.1 More labeling properties obtained from the $\otimes_h$-product

Recall that, for every labeled digraph $D \in \mathcal{S}_n^k$ we can consider $D^c \in \mathcal{S}_n^{n+3-k}$, such that $D \cong D^c$, just by taking the super edge-magic complementary labeling that defines D.

Proposition 4.3.1. Let $D \in \mathcal{S}_n^k$ and let $h : E(D) \to \mathcal{T}_\sigma^q$ be any function. Then, there exists $h^c : E(D^c) \to \mathcal{T}_{3(p+q+1)-\sigma}^q$ such that

$$D \otimes_h \mathcal{T}_\sigma^q \simeq D^c \otimes_{h^c} \mathcal{T}_{3(p+q+1)-\sigma}^q, \text{ and } \widetilde{h^c} \simeq \widetilde{\overline{h}},$$

where $\widetilde{\overline{h}}$ is the edge-magic complementary labeling of the induced labeling of $D \otimes_h \mathcal{T}_\sigma^q$ and $\widetilde{h^c}$ is the induced labeling of $D^c \otimes_{h^c} \mathcal{T}_{3(p+q+1)-\sigma}^q$.

Proof. Let $\phi : \mathcal{S}_n^k \to \mathcal{S}_n^{n+3-k}$ be the function defined by $\phi(D) = D^c$, where $(\bar{i}, \bar{j}) \in D^c$ if and only if $(p+1-\bar{i}, p+1-\bar{j}) \in D$ and $\psi : \mathcal{T}_\sigma^q \to \mathcal{T}_{3(p+q+1)-\sigma}^q$

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be the function defined by $\psi(F) = \bar{F}$, where $(\bar{i}, \bar{j}) \in \bar{F}$ if and only if $(p + q + 1 - \bar{i}, p + q + 1 - \bar{j}) \in F$.

Let $h : E(D) \to \mathcal{T}_\sigma^q$ be any function. Then, the induced edge-magic labeling $\widetilde{h}$ of the product $D \otimes_h \mathcal{T}_\sigma^q$ is defined by $\widetilde{h}(i, a) = (p + q)(i - 1) + a$, for any vertex $(i, a) \in V(D \otimes_h \mathcal{T}_\sigma^q)$ and by $\widetilde{h}((i, a), (j, b)) = (p + q)(k + n - (i + j) - 1) + (\sigma - a - b)$, for any arc $((i, a), (j, b)) \in E(D \otimes_h \mathcal{T}_\sigma^q)$. Then, since $|V(D \otimes_h \mathcal{T}_\sigma^q)| = pn$ and $|E(D \otimes_h \mathcal{T}_\sigma^q)| = qn$, the complementary labeling $\widetilde{\bar{h}}$ of $D \otimes_h \mathcal{T}_\sigma^q$ is defined by

$$
\begin{aligned}
\widetilde{\bar{h}}(i, a) &= (p + q)n + 1 - (p + q)(i - 1) - a \\
&= (p + q)(n + 1 - i - 1) + (p + q + 1 - a),
\end{aligned}
$$

for any vertex $(i, a) \in V(D \otimes_h \mathcal{T}_\sigma^q)$ and

$$
\begin{aligned}
\widetilde{\bar{h}}((i, a), (j, b)) &= (p + q)n + 1 - (p + q)(k + n - (i + j) - 1) - (\sigma - a - b) \\
&= (p + q)(n + 3 - k + n - (n + 1 - i) - (n + 1 - j) - 1) \\
&\quad + 3(p + q + 1) - \sigma - (p + q + 1 - a) - (p + q + 1 - b).
\end{aligned}
$$

Thus, the function $h^c : E(D^c) \to \mathcal{T}_{3(p+q+1)-\sigma}^q$ defined by

$$
h^c(i, j) = \psi(h(n + 1 - i, n + 1 - j)),
$$

induces a labeling $\widetilde{h^c}$ of $D^c \otimes_{h^c} \mathcal{T}_{3(p+q+1)-\sigma}^q$, which is isomorphic to the labeling $\widetilde{\bar{h}}$ of $D \otimes_h \mathcal{T}_\sigma^q$. Therefore, the result follows. $\qquad\square$

4.4 Magic sums of cycles

Let G be a (p, q)-graph and $f : V(G) \cup E(G) \to [1, p + q]$ be a bijective function. The f-weight of a vertex $v \in V(G)$, $w_f(v)$, is defined to be $w_f(v) = f(v) + \sum f(e)$, where the sum is taken over all edges e incident to v. The function f is said to be a *vertex-magic total labeling* [43], if the vertex weight $w_f(v)$ does not depend on v. It turns out, that for 2-regular graphs the notions of edge-magic labeling and vertex-magic total labeling coincide, since

we can easily obtain a vertex-magic total labeling from an edge-magic labeling and viceversa, just by translating one unit clockwise the labels: the label of each edge is assigned to one of its adjacent vertices, and the label of the other one is assigned to the edge.

Dan McQuillan proved in [45] the next result that was originally stated in terms of vertex-magic total labelings.

Proposition 4.4.1. [45] Let p be odd. Assume that C_m has an edge-magic labeling f. Then,

(i) C_{pm} has an edge-magic labeling with valence $p(\mathrm{val}(f)) - 3(p-1)/2$, and

(ii) C_{pm} has an edge-magic labeling with valence $3(p-1)m + \mathrm{val}(f)$.

The following structural results will be useful to prove that Proposition 4.4.1 can also be obtained by means of the $\otimes_h$-product. We denote by $\overrightarrow{C_n}$ and by $\overleftarrow{C_n}$ the two possible strong orientations of the cycle C_n, where the vertices of C_n are the elements of the set $\{i\}_{i=1}^n$. It is well known that

$$\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} = \gcd(m, n)\overrightarrow{C}_{\mathrm{lcm}[m,n]}.$$

Theorem 4.4.1. [2] Let $m, n \in \mathbb{N}$ and consider the product $\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$ where $h : E(\overrightarrow{C}_m) \longrightarrow \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$. Let g be a generator of a cyclic subgroup of $\mathbb{Z}_n$, namely $\langle g \rangle$, such that $|\langle g \rangle| = k$. Also let $N_g(h^-) < m$ be a natural number that satisfies the congruence relation $m - 2N_g(h^-) \equiv g \pmod{n}$.

If the function h assigns $\overleftarrow{C}_n$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C}_m$ then the product

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$$

consists of exactly n/k disjoint copies of a strongly oriented cycle $\overrightarrow{C}_{mk}$. In particular if $\gcd(g, n) = 1$, then $\langle g \rangle = \mathbb{Z}_n$ and if the function h assigns $\overleftarrow{C}_n$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C}_m$ then

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\} \cong \overrightarrow{C}_{mn}.$$

Corollary 4.4.1. [41] Let $n \geq 3$ be an odd integer and suppose that $m \geq 3$ is an integer such that either m is odd or $m \geq n$. Then there exists a function $h : E(\overrightarrow{C_m}) \rightarrow \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ such that

$$\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} \cong \overrightarrow{C_{mn}}.$$

Now, by combining the previous two results and Lemmas 3.2.2 and 4.3.1, we obtain the next result, which, except for the technical condition in (i), is the same result that McQuilian obtained in [45] (see Proposition 4.4.1).

Proposition 4.4.2. Let p be odd. Assume that C_m has an edge-magic labeling f. Then,

(i) C_{pm} has an edge-magic labeling with valence $p(\text{val}(f)) - 3(p-1)/2$, when m is odd or $m \geq p$.

(ii) C_{pm} has an edge-magic labeling with valence $3(p-1)m + \text{val}(f)$.

Proof. (i) By Corollary 4.4.1, there exists a function $h : E(\overrightarrow{C_m}) \rightarrow \{\overrightarrow{C_p}, \overleftarrow{C_p}\}$ such that $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_p}, \overleftarrow{C_p}\} \cong \overrightarrow{C_{pm}}$. Assume that each vertex of C_p is identified by the label assigned to it by a super edge-magic labeling. Then, by Lemma 3.2.2, the induced labeling of the product $\overrightarrow{C_{pm}}$ has valence: $\text{val}(\hat{f}) = p(\text{val}(f) - 3) + (p+3)/2 + p$, that is, $p(\text{val}(f)) - 3(p-1)/2$.

(ii) Similarly, By Theorem 4.4.1, there exists a function $h : E(\overrightarrow{C_p}) \rightarrow \{\overrightarrow{C_m}, \overleftarrow{C_m}\}$ such that $\overrightarrow{C_p} \otimes_h \{\overrightarrow{C_m}, \overleftarrow{C_m}\} \cong \overrightarrow{C_{pm}}$.

Assume that each vertex of C_p is identified by the label assigned to it by a super edge-magic labeling and each vertex of C_m is identified by the label assigned to it by f. Then, by Lemma 4.3.1, the induced labeling of the product $\overrightarrow{C_{pm}}$ has valence: $\text{val}(\tilde{f}) = 2m((p+3)/2 + p - 3) + \text{val}(f)$, that is, $3(p-1)m + \text{val}(f)$. Thus, the result holds. $\square$

Example 4.4.1 and Example 4.4.2 shows the two possible cases when $m \geq p$ and m is odd in Proposition 4.4.2(i) respectively.

Example 4.4.1. Consider the edge-magic labeled cycle $\overrightarrow{C_4}$ in which the vertices are identified by its labels, defined by $V(\overrightarrow{C_4}) = \{1, 2, 3, 6\}$ and

$E(\overrightarrow{C_4}) = \{(1,3),(3,2),(2,6),(6,1)\}$ and the super edge-magic labeled cycle $\overrightarrow{C_3}$ defined by $V(\overrightarrow{C_3}) = \{1,2,3\}$ and $E(\overrightarrow{C_3}) = \{(1,2),(2,3),(3,1)\}$. $\overleftarrow{C_3}$ and $\overleftarrow{C_4}$ are obtained by reversing all the arcs. Then, by Proposition 4.4.2 (i)(for $m \geq p$), C_{12} has an edge-magic labeling with valence 33 and by Proposition 4.4.2 (ii), it has another edge-magic labeling with valence 36 as shown in Figure 4.6. Notice that, by using the missing the labels, there is only one way to complete the edge-magic labelings defined in this example.
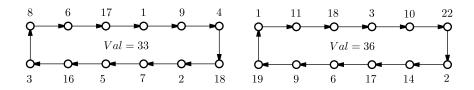
Figure 4.6: An edge-magic labeled C_{12} with valences 33 and 36.

Example 4.4.2. Consider the edge-magic labeled cycle $\overrightarrow{C_3}$ in which the vertices are identified by its labels, defined by $V(\overrightarrow{C_3}) = \{2,4,6\}$ and $E(\overrightarrow{C_3}) = \{(2,4),(4,6),(6,2)\}$ and the super edge-magic labeled cycle $\overrightarrow{C_5}$ defined by $V(\overrightarrow{C_5}) = \{1,2,3,4,5\}$ and $E(\overrightarrow{C_5}) = \{(1,4),(4,2),(2,5),(5,3),(3,1)\}$ by Proposition 4.4.2(when m is odd), C_{15} has two edge-magic labelings with valences 47 and 49 as shown in Figure 4.7.
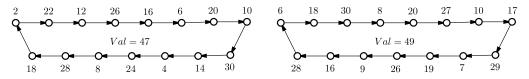
Figure 4.7: C_{15} with edge-magic valences 47 and 49.

4.5 Open questions

We started the chapter by providing some properties of odd and even labeling construction related to the (super) edge-magic complementary labeling construction and also with respect to the $\otimes_h$-product. Then we introduced a new labeling construction of $\otimes_h$-product by changing the role of the factors. Finally, with the help of this construction, we attacked the problem of magic valences of cycles. We obtained the same result of McQuilian in [45] (with an additional condition) showing that both constructions can be complementary.

Open question 4.5.1. Can the additional condition given in Proposition 4.4.2(i) be removed?

But the original conjecture of Godbold and slater still remains open. Motivated by Theorem 4.3.1 and the existence of the family $\mathcal{T}_\sigma^q$, we propose the following open question.

Open question 4.5.2. Prove or disprove that more additional conditions can be placed to a family $\mathcal{T}$ such that we get other type of labelings such as bimagic labeling and harmonious labeling. If so, what are the additional conditions that should be placed?

Everything is known for perfect super edge-magic paths. We propose the following open questions related to perfect edge-magic paths.

Open question 4.5.3. For which values of $n \in \mathbb{N}$, P_n is perfect edge-magic?

Open question 4.5.4. Prove or disprove that union of paths are perfect (super) edge-magic?

Open question 4.5.5. What about the caterpillars obtained from $C_m \odot \overline{K_n}$ by removing an edge of the cycle? Are they perfect (super) edge-magic? Can we find an upper bound and lower bound for the number of non-isomorphic (super) edge-magic labelings of these types of caterpillars?

Chapter 5

(Di)graph decompositions and magic type labelings: A dual relation

5.1 Introduction

The origin of graph labeling is strongly connected to graph decompositions. In fact, one of the first motivations in order to study graph labelings was due to Ringel's conjecture [47] which states that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Rosa introduced β-valuation, α-labeling and ρ-valuation in [48] as a means of attacking the Ringel's conjecture and most of the labelings originated from that. Rosa also proved that a graph G with n edges cyclically decomposes the edge set of the complete graph K_{2n+1} if and only if it admits a ρ-valuation. In 1989, Graham and Häggvist [21] generalized Ringel's conjecture and stated that every tree with m edges decomposes every $2m$-regular graph and every bipartite m-regular graph. As it happens in the non-bipartite graphs, the natural way to approach this problem is to look for cyclic decompositions using graph labelings. In [9], El-Zanaty et al. introduced the concept of near α-labeling of a bipartite graph and proved that if a graph G with n edges has a near α-labeling, then there is a cyclic G-decomposition of both $K_{n,n}$ and K_{2nx+1} for all positive integers x. In [29] (see also in [27]), bigraceful labelings

were introduced to study cyclic decompositions of bipartite graphs. Another approach to the problem was given in [25] by Kézdy and Snevily. They proved that a tree with n edges and radius r decomposes K_{2h+1} for some $h \leq (r + 4)n^2$. Lladó and López in [27] proved that when a tree T of size n is not known to be bigraceful it is shown, using similar techniques to the ones by Kézdy Snevily in[25], that T decomposes $K_{2hn,2hn}$ for some $h \leq \lceil r/4 \rceil$, where r is the radius of T. Another attempt in this direction was given in [28] in which Lladó et al. proved that every tree with m edges is contained in tree that decomposes $K_{n,n}$, for a bounded n using bigraceful labelings. In [46], Pasotti introduced d-divisible graceful labeling, a generalization of graceful labeling, and proved that this can be used to obtain certain cyclic decompositions of complete bipartite graphs. Using labeling methods, Dufour [8] and Eldergill [10] proved some results on the decomposition of complete graphs. Inayah et al. [24] showed that K_{2m+1} admits T-magic compositions by any graceful tree with m edges using the result on the sumset partition problem. In [36], López et al. introduced the concept of $\{H_i\}_{i \in I}-$ super edge-magic decomposable graphs and proved several results on graph decompositions.

The main goal of this chapter is to show a new application of labeled super edge-magic digraphs to graph decompositions. What we believe that it is new and surprising in the relation established in this paper is that, as far as we know, there are no relations between labelings involving sums and graph decompositions. In fact, we believe that this is the first relation found in this direction and we believe that to explore this relationship is a very interesting line for future research.

The study of the (super) edge-magic properties of the graph $C_m \odot \overline{K_n}$ has been of interest during the years and some papers on the topic can be found in the literature. See for instance [34, 37, 41]. Due to this, many things are known on the (super) edge-magic properties of these graphs. However, many other things remain a mystery, and we believe that it is worth the while to work in this direction. In fact, a big hole in the literature, appears when considering graphs of the form $C_m \odot \overline{K_n}$ for m even. In this chapter, we will devote one of the sections to these type of graphs. This study leads us to consider other classes of graphs and to study the relation existing between the valences of edge-magic and super edge-magic labelings and the well known problem of graph decompositions.

The organization of the chapter is as follows: in Section 2, we provide the

lower bound of for the number of edge-magic valences of $C_m \odot \overline{K}_n$ when m is even. In Section 3, we establish a relationship existing between the (super) edge-magic labelings and graph decompositions where the labelings involving sums used and end this chapter with some open questions in Section 4.

5.2 More about valences

As we have already mentioned in the introduction, not too much is known about the valences of (super) edge-magic labelings for the graph $C_m \odot \overline{K}_n$ when m is even. In fact, as far as we know, the only papers that deal with (super) edge-magic labelings of $C_m \odot \overline{K}_n$ for m even are [12, 34]. Hence almost all such results involve only odd cycles. Next, we study the edge-magic valences of $C_m \odot \overline{K}_n$ when m is even. Unless otherwise specified, $\overrightarrow{G}$ denotes any orientation of G. In the next lemma, we provide a well known result that gives a lower bound and an upper bound for edge-magic valences. We add the proof as a matter of completeness. Recall that the complementary labeling of an edge-magic labeling f is the labeling $\overline{f}(x) = p + q + 1 - f(x)$, for all $x \in V(G) \cup E(G)$, and that $\mathrm{val}(\overline{f}) = 3(p + q + 1) - \mathrm{val}(f)$.

Lemma 5.2.1. Let G be a (p, q)-graph with an edge-magic labeling f. Then $p + q + 3 \le \mathrm{val}(f) \le 2(p + q)$.

Proof. Let $f : V(G) \cup E(G) \to [1, p+q]$ be an edge-magic labeling of G. The two lowest possible integers in $[1, p + q - 1]$ that can be added to $p + q$ are 1 and 2. Thus, $\mathrm{val}(f) \ge p + q + 3$. By using the complementary labeling, the maximum possible valence has the form $3(p+q+1) - \mathrm{val}(g)$ where $\mathrm{val}(g)$ is the minimum possible valence. Thus, $\mathrm{val}(f) \le 3(p+q+1) - \mathrm{val}(g) \le 2(p+q)$. $\square$

The next lemma is an generalization of Lemma 3.4.1.

Lemma 5.2.2. Let g be a (super) edge-magic labeling of a graph G, and let f_r be the super edge-magic labeling of $K_{1,n}^l$ that assigns label r to the central vertex, $1 \le r \le n + 1$. Then,

(i) the induced (super) edge-magic labeling $\widehat{g}_r$ of $\overrightarrow{G} \otimes \overrightarrow{K}_{1,n}^l$ has valence $(n + 1)(\mathrm{val}(g) - 2) + r + 1$.

(ii) Let g' be a different (super) edge-magic labeling of G with $\mathrm{val}(g) < \mathrm{val}(g')$, then $\mathrm{val}(\widehat{g}_{n+1}) < \mathrm{val}(\widehat{g}'_1)$, where $\widehat{g}'_r$ is the induced (super) edge-magic labeling of $\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}$ when $K^l_{1,n}$ is labeled with f_r and G with g'.

Proof. The labeling f_r of $\overrightarrow{K}^l_{1,n}$ has minimum induced sum $r+1$. Thus, $\overrightarrow{K}^l_{1,n} \in \mathcal{S}^{r+1}_{n+1}$. By Lemma 3.2.2,

$$\mathrm{val}(\widehat{g}_r) = (n+1)[\mathrm{val}(g) - 3] + r + 1 + n + 1$$
$$= (n+1)[\mathrm{val}(g) - 2] + r + 1.$$

Let g' be a different (super) edge-magic labeling of G with $\mathrm{val}(g) < \mathrm{val}(g')$, then

$$\mathrm{val}(\widehat{g}_{n+1}) = (n+1)[\mathrm{val}(g) - 2] + n + 2$$
$$\leq (n+1)[\mathrm{val}(g') - 1 - 2] + n + 2$$
$$\leq (n+1)[\mathrm{val}(g') - 2] + 1$$
$$< \mathrm{val}(\widehat{g}'_1).$$

Hence the result follows. $\qquad\square$

Theorem 5.2.1. *Let G be an edge-magic (p,q)-graph. Then $|\tau_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| \geq (n+1)|\tau_{\overrightarrow{G}}| + 2$.*

Proof. Let f_r be the super edge-magic labeling of $K^l_{1,n}$ that assigns the label r to the central vertex, $1 \leq r \leq n+1$. Let $g : V(G) \cup E(G) \to [1, p+q]$ be an edge-magic labeling of G. By Lemma 5.2.2, $\mathrm{val}(\widehat{g}_r) = (n+1)[\mathrm{val}(g) - 2] + r + 1$ and if $\mathrm{val}(g) < \mathrm{val}(g')$, then $\mathrm{val}(\widehat{g}_{n+1}) < \mathrm{val}(\widehat{g}'_1)$ where $\widehat{g}_r$ is the induced edge-magic labeling of $\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}$. Therefore, $|\tau_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| \geq (n+1)|\tau_{\overrightarrow{G}}|$.

Consider $\overrightarrow{K}^l_{1,n} \otimes \overrightarrow{G}$. By Theorem 4.3.1, $\mathrm{val}(\tilde{g}_r) = (p+q)[n+r-1] + \mathrm{val}(g), 1 \leq r \leq n+1$ where $\tilde{g}_r$ is the induced labeling of $\overrightarrow{K}^l_{1,n} \otimes \overrightarrow{G}$ when $\overrightarrow{K}^l_{1,n}$ is labeled with f_r and $\overrightarrow{G}$ with g'. We claim that $\mathrm{val}(\tilde{g}_1) < \mathrm{val}(\widehat{g}_1)$ and $\mathrm{val}(\widehat{g}_{n+1}) < \mathrm{val}(\tilde{g}_{n+1})$. Assume to the contrary that $\mathrm{val}(\tilde{g}_1) \geq \mathrm{val}(\widehat{g}_1)$, we get $\mathrm{val}(g) \leq p + q + 2$ which is a contradiction to Lemma 5.2.1. Similarly, if $\mathrm{val}(\widehat{g}_{n+1}) \geq \mathrm{val}(\tilde{g}_{n+1})$, we get $\mathrm{val}(g) \geq 2(p+q) + 1$ which again is a contradiction to Lemma 5.2.1. Hence $|\tau_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| \geq (n+1)|\tau_{\overrightarrow{G}}| + 2$. $\qquad\square$

By adding an extra condition on the smallest and the biggest valence, we can improve the lower bound given in the previous result.

Theorem 5.2.2. *Let G be an edge-magic (p, q)-graph. If α and β are the smallest and the biggest valences of G, respectively, and $\beta - \alpha < (\alpha - (p + q + 2))n$ then $|\tau_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| \geq (n + 3)|\tau_{\overrightarrow{G}}|$.*

Proof. The previous proof guarantees that, using Lemma 5.2.2, we get

$$|\tau_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| \geq (n + 1)|\tau_{\overrightarrow{G}}|.$$

Next we will use Theorem 4.3.1 to complete the remaining valences. Consider now, the reverse order $\overrightarrow{K}^l_{1,n} \otimes \overrightarrow{G}$. By Theorem 4.3.1, $\mathrm{val}(\widetilde{g}_r) = (p + q)[n + r - 1] + \mathrm{val}(g), 1 \leq r \leq n + 1$ where $\widetilde{g}_r$ is the induced labeling of $\overrightarrow{K}^l_{1,n} \otimes \overrightarrow{G}$ when $\overrightarrow{K}^l_{1,n}$ is labeled with f_r and $\overrightarrow{G}$ with g. Let g be an edge-magic labeling of G with valence α and g' an edge-magic labeling with valence β. We claim that $\mathrm{val}(\widetilde{g'}_1) < \mathrm{val}(\widehat{g}_1)$ and $\mathrm{val}(\widehat{g'}_{n+1}) < \mathrm{val}(\widetilde{g}_{n+1})$.

Assume to the contrary that $\mathrm{val}(\widetilde{g'}_1) \geq \mathrm{val}(\widehat{g}_1)$, then we get $\beta - \alpha \geq (\alpha - (p + q + 2))n$ which is a contradiction to the statement. Similarly, if $\mathrm{val}(\widehat{g'}_{n+1}) \geq \mathrm{val}(\widetilde{g}_{n+1})$, we get $\beta - \alpha \geq (1 + 2(p + q) - \beta)n$. Notice that, since α and β correspond to the valences of two complementary labelings of G, $\beta = 3(p + q + 1) - \alpha$ and this inequality is equivalent to $\beta - \alpha \geq (\alpha - (p + q + 2))n$ which is again a contradiction. Since by construction of the induced labeling, if $\mathrm{val}(g) < \mathrm{val}(g')$, then $\mathrm{val}(\widetilde{g}_r) < \mathrm{val}(\widetilde{g'}_r)$, we obtain $\mathrm{val}(\widetilde{g}_1) < \ldots < \mathrm{val}(\widetilde{g'}_1) < \mathrm{val}(\widehat{g}_1) < \ldots < \mathrm{val}(\widehat{g'}_{n+1}) < \mathrm{val}(\widetilde{g}_{n+1}) < \ldots < \mathrm{val}(\widetilde{g'}_{n+1})$. Hence $|\tau_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| \geq (n + 3)|\tau_{\overrightarrow{G}}|$. $\qquad\square$

Corollary 5.2.1. Let G be any edge-magic (bipartite) 2-regular graph. Then $|\tau_{G \odot \overline{K}_n}| \geq (n + 1)|\tau_G| + 2$.

Proof. Let $G = C_{m_1} \oplus C_{m_2} \oplus \cdots \oplus C_{m_k}$ and let $\overrightarrow{G} = C^+_{m_1} \oplus C^+_{m_2} \oplus \cdots \oplus C^+_{m_k}$ be an orientation of G in which each cycle is strongly oriented. Then $\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n} = (C^+_{m_1} \otimes \overrightarrow{K}^l_{1,n}) \oplus (C^+_{m_2} \otimes \overrightarrow{K}^l_{1,n}) \oplus \cdots \oplus (C^+_{m_k} \otimes \overrightarrow{K}^l_{1,n})$. Note that since G is bipartite, all cycles should be of even length and by definition of $\otimes$-product, $G \odot \overline{K}_n \cong und(\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n})$. Thus by Theorem 5.2.1, $|\tau_{G \odot \overline{K}_n}| \geq (n + 1)|\tau_G| + 2$. $\qquad\square$
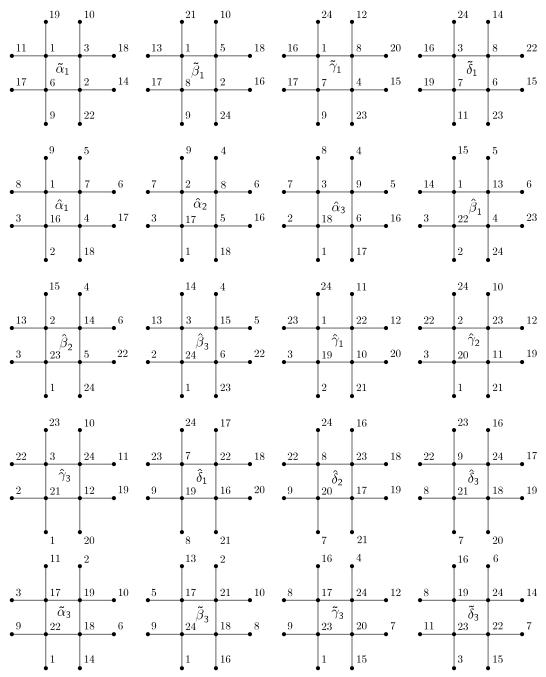
b

Figure 5.1: All theoretical valences are realizable for $C_4 \odot \overline{K}_2$.

Example 5.2.1. Let g be an edge-magic labeling of $\overrightarrow{C_4}$ and f_r be the super edge-magic labeling of $\overrightarrow{K}_{1,2}^l$ that assigns the label r to the central vertex, $1 \le r \le 3$. Then the valence of the induced labeling $\widehat{g}_r$ is $\mathrm{val}(\widehat{g}_r) = 3(\mathrm{val}(g)-2) + r+1 \in [3(\mathrm{val}(g)-2)+2, 3(\mathrm{val}(g)-2)+4]$. Let $\alpha : 1\bar{5}6\bar{4}2\bar{7}3\bar{8}1$, $\beta = 1\bar{7}5\bar{6}2\bar{3}8\bar{4}1$, $\gamma = 1\bar{5}8\bar{2}4\bar{3}7\bar{6}1$ and $\delta = 8\bar{4}3\bar{5}7\bar{2}6\bar{1}8$, where $i\bar{m}j$ indicates that m is the label assigned to the edge ij. Since $\tau_{C_4} = [12, 15] = [\mathrm{val}(\alpha), \mathrm{val}(\beta)]$ we get different 12 edge-magic valences $[32, 43]$ for the induced labeling of $C_4 \odot \overline{K}_2 \cong und(\overrightarrow{C_4} \otimes \overrightarrow{K}_{1,2})$. Moreover, since the condition $\mathrm{val}(\beta) - \mathrm{val}(\alpha) < (\mathrm{val}(\alpha) - (p+q+2))n$, is satisfied for $n \ge 2$, by using Theorem 4.3.1, $\mathrm{val}(\widetilde{g}_r) = 8(1+r) + \mathrm{val}(g)$ which gives, associated to a labeling g two new valences, namely $\mathrm{val}(\widetilde{g}_1)$ and $\mathrm{val}(\widetilde{g}_3)$ which gives in total 20 valences. The induced labelings and they are shown in Figure 5.1, according to the notation introduced above (for clarity reasons, only the labels of the vertices are shown). Notice that, by using the missing labels, there is only one way to complete the edge-magic labelings obtained in Figure 5.1. The minimum induced sum together with the maximum unused label provides the valence of the labeling.

Remark 5.2.1. For a given even m, the magic interval for crowns of the form $C_m \odot \overline{K}_n$ is $[mn + 2 + 5m/2, 2mn + 1 + 7m/2]$ (see Section 2, in [41]). Thus, for $m = 4$, the magic interval is $[28, 47]$. Hence, the crown $C_4 \odot \overline{K}_2$ is perfect edge-magic.

It is well known that all cycles are edge-magic [18]. Thus, the following corollary follows:

Corollary 5.2.2. Fix $m \in \mathcal{N}$. Then $\lim_{n \to \infty} |\tau_{C_m \odot \overline{K}_n}| = \infty$.

A similar argument to that of the first part in Theorem 5.2.1 can be used to prove the following theorem.

Theorem 5.2.3. *Let G be a super edge-magic graph. Then* $|\sigma_{\overrightarrow{G} \otimes \overrightarrow{K}_{1,n}^l}| \ge (n+1)|\sigma_{\overrightarrow{G}}|$.

5.3 A relation between (super) edge-magic labelings and graph decompositions

Let G be a bipartite graph with stable sets $X = \{x_i\}_{i=1}^s$ and $Y = \{y_j\}_{j=1}^t$. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then we denote by $S_2(G; H_1, H_2)$ the graph with vertex and edge sets defined as follows:

$$V(S_2(G; H_1, H_2)) = X \cup Y \cup X' \cup Y',$$
$$E(S_2(G; H_1, H_2)) = E(G) \cup \{x_i y_j' : x_i y_j \in E(H_1)\} \cup \{x_i' y_j : x_i y_j \in E(H_2)\},$$

where $X' = \{x_i'\}_{i=1}^s$ and $Y' = \{y_j'\}_{j=1}^t$.

We are ready to state and prove the next theorem.

Theorem 5.3.1. *Let G be a bipartite (super) edge-magic simple graph with stable sets X and Y. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then, the graph $S_2(G; H_1, H_2)$ is (super) edge-magic.*

Proof. Let f be a (super) edge-magic labeling of G, and assume that the edges of H_1 are directed from X to Y and the edges of H_2 are directed from Y to X in G, obtaining the digraph $\overrightarrow{G}$. Let $\overrightarrow{K}_{1,1}^l$ be the super edge-magic labeled digraph with $V(\overrightarrow{K}_{1,1}^l) = \{1, 2\}$ and $E(\overrightarrow{K}_{1,1}^l) = \{(1,1), (1,2)\}$. By Theorem 3.2.1, we have that the graph $\text{und}(\overrightarrow{G} \otimes \overrightarrow{K}_{1,1}^l)$ is (super) edge-magic. Moreover, an easy check shows that the bijective function $\phi : V(\overrightarrow{G} \otimes \overrightarrow{K}_{1,1}^l) \to V(S_2(G; H_1, H_2))$ defined by $\phi(v, 1) = v$ and $\phi(v, 2) = v'$ is an isomorphism between $\text{und}(\overrightarrow{G} \otimes \overrightarrow{K}_{1,1}^l)$ and $S_2(G; H_1, H_2)$. Therefore, the graph $S_2(G; H_1, H_2)$ is (super) edge-magic. $\square$

Next, we show an example.

Example 5.3.1. Consider the edge-magic labeling of $K_{3,3}$ shown in Figure 5.2. The same figure shows a partition of the edges and a possible orientation of them when $X = \{1, 2, 3\}$ and $Y = \{4, 8, 12\}$. The construction given in the proof of Theorem 5.3.1 when each vertex (a, i) is labeled $2(a-1)+i$ and each edge $(a, i)(b, j)$ is labeled $2(e-1) + 4 - (i+j)$ (where e is the label of (a, b) in D) results into the graph in Figure 5.3.
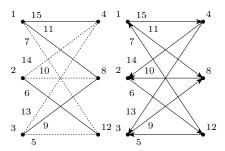
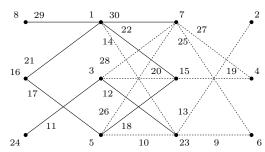Figure 5.2: A decomposition of $K_{3,3}$ and the induced orientation.

Figure 5.3: An edge-magic labeling of $S_2(K_{3,3}; H_1, H_2)$.

Kotzig and Rosa [26] proved that every complete bipartite graph is edge-magic. It is clear that Theorem 5.3.1 works very nicely when the graph G under consideration is a complete bipartite graph and many new edge-magic graphs can be obtained. Theorem 5.3.1 can be easily extended. Let us do so next.

Let G be a bipartite graph with stable sets $X = \{x_i\}_{i=1}^s$ and $Y = \{y_j\}_{j=1}^t$. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then we define $S_{2n}(G; H_1, H_2)$ to be the graph with vertex and edge sets as follows:

$$
\begin{aligned}
V(S_{2n}(G; H_1, H_2)) &= X \cup Y \cup (\cup_{k=1}^n X_k) \cup (\cup_{k=1}^n Y_k), \\
E(S_{2n}(G; H_1, H_2)) &= E(G) \cup \{x_i y_j^k : x_i y_j \in E(H_1)\} \cup \{x_i^k y_j : x_i y_j \in E(H_2)\},
\end{aligned}
$$

where $X_k = \{x_i^k\}_{i=1}^s$ and $Y_k = \{y_j^k\}_{j=1}^t$.

Lemma 5.3.1. Let G be a bipartite simple graph with stable sets X and Y. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then, there exists an orientation of G and $K_{1,n}^l$, namely $\overrightarrow{G}$ and $\overrightarrow{K}_{1,n}^l$ respectively, such that $S_{2n}(G; H_1, H_2) \cong und(\overrightarrow{G} \otimes \overrightarrow{K}_{1,n}^l)$.

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Proof. Assume that the digraph $\overrightarrow{G}$ is obtained from G by orienting the edges of H_1 from X to Y and the edges of H_2 from Y to X in G. Let $\overrightarrow{K}^l_{1,n}$ be the digraph with $V(\overrightarrow{K}^l_{1,n}) = [1, n+1]$ and $E(\overrightarrow{K}^l_{1,n}) = \{(1, k) : k \in [1, n+1]\}$. An easy check shows that the bijective function $\phi : V(\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}) \to V(S_{2n}(G; H_1, H_2))$ defined by $\phi(v, 1) = v$ and $\phi(v, k+1) = v^k$, $k \in [1, n]$ is an isomorphism between $und(\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n})$ and $S_{2n}(G; H_1, H_2)$. $\square$

We are ready to state and prove the next theorem.

Theorem 5.3.2. *Let G be a bipartite (super) edge-magic simple graph with stable sets X and Y. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then, the graph $S_{2n}(G; H_1, H_2)$ is (super) edge-magic.*

Proof. Let f be a (super) edge-magic labeling of G, and assume that the edges of H_1 are directed from X to Y and the edges of H_2 are directed from Y to X in G, obtaining the digraph $\overrightarrow{G}$. Let $\overrightarrow{K}^l_{1,n}$ be the super edge-magic labeled digraph with $V(\overrightarrow{K}^l_{1,n}) = [1, n+1]$ and $E(\overrightarrow{K}^l_{1,n}) = \{(1, k) : k \in [1, n+1]\}$. By Theorem 3.2.1, we have that the graph $und(\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n})$ is (super) edge-magic. By Lemma 5.3.1, $S_{2n}(G; H_1, H_2) \cong und(\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n})$. Therefore, the graph $S_{2n}(G; H_1, H_2)$ is (super) edge-magic. $\square$

With the help of Lemma 3.1.1, we can generalize Theorem 5.3.2 very easily. We do it in the following two results.

Theorem 5.3.3. *Let G be a bipartite super edge-magic simple graph with stable sets X and Y. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then $|\sigma_{S_{2n}(G; H_1, H_2)}| \geq (n+1)|\sigma_G|$.*

Proof. Let h be a super edge-magic labeling of G, and assume that the edges of H_1 are directed from X to Y and the edges of H_2 are directed from Y to X in G, obtaining the digraph $\overrightarrow{G}$. Let f_r be the super edge-magic labeling of $\overrightarrow{K}^l_{1,n}$ that assigns the label r to the central vertex with $\mathrm{val}(f_r) = 2n + 3 + r$, $1 \leq r \leq n + 1$. Then by Lemma 5.3.1, $S_{2n}(G; H_1, H_2) \cong und(\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n})$ and by Theorem 5.3.2, it is super edge-magic. By Theorem 5.2.3, $|\sigma_{S_{2n}(G; H_1, H_2)}| \geq (n+1)|\sigma_G|$. $\square$

A similar argument to the one of Theorem 5.3.3, but now using Theorem 5.2.1, allows us to prove the following theorem.

Theorem 5.3.4. *Let G be a bipartite edge-magic simple graph with stable sets X and Y. Assume that G admits a decomposition $G \cong H_1 \oplus H_2$. Then $|\tau_{S_{2n}(G;H_1,H_2)}| \geq (n+1)|\tau_G| + 2$.*

Once again, we have the following two easy corollaries.

Corollary 5.3.1. Let G be a bipartite super edge-magic simple graph with stable sets X and Y. If G admits a decomposition $G \cong H_1 \oplus H_2$, then $\lim_{n \to \infty} |\sigma_{S_{2n}(G;H_1,H_2)}| = \infty$.

Corollary 5.3.2. Let G be a bipartite edge-magic simple graph with stable sets X and Y. If G admits a decomposition $G \cong H_1 \oplus H_2$, then $\lim_{n \to \infty} |\tau_{S_{2n}(G;H_1,H_2)}| = \infty$.

At this point, consider any graph G^* whose vertex set admits a partition of the form $V(G^*) = X \cup Y \cup_{k=1}^n X_k \cup_{k=1}^n Y_k$ and that decomposes as a union of three bipartite graphs $G^* \cong G \oplus H_1 \oplus H_2$, where $G^*[X \cup Y] \cong G$, $G^*[X \cup Y_k] \cong H_1$ and $G^*[X_k \cup Y] \cong H_2$ for all $k \in [1,n]$. By Theorem 5.3.1, we have the following remarks.

Remark 5.3.1. If G is a (super) edge-magic graph and G^* is not, then H_1 and H_2 do not decompose G.

Remark 5.3.2. If $|\sigma_{S_{2n}(G;H_1,H_2)}| < (n+1)|\sigma_G|$ provided that G is a bipartite super edge-magic graph, then $G \not\cong H_1 \oplus H_2$.

Remark 5.3.3. If $|\tau_{S_{2n}(G;H_1,H_2)}| < (n+1)|\tau_G| + 2$ provided that G is a bipartite edge-magic graph, then $G \not\cong H_1 \oplus H_2$.

We will bring this section to its end, by mentioning that, although some labelings involving differences as for instance, graceful labelings and α-valuat--ions have a strong relationship with graph decompositions, the results mentioned in this section are the only ones known relating the subject of decompositions with addition type labelings. This is why we consider these results interesting.

5.4 Open questions

As we have already mentioned, the popularity of graph labelings come from the relation existing between graph labelings and decompositions. There are two important links between labelings and decompositions. The first one that we mention is the relation between graceful labelings and decompositions of complete graphs into isomorphic subgraphs [48]. The second one was established by Lladó et al. in [29] and relates the concept of bigraceful labelings with decompositions of complete bipartite graphs into trees. We believe that the relations existing between graceful labelings and decompositions and bigraceful labelings and decompositions share a similar flavour. Also, in [9], El-Zanati et al. established a relationship between near α-labelings and cyclic G-decompositions. However, the relations established in this thesis is of a different nature than the previous ones and relates (super) edge-magic labelings, decompositions and the $\otimes_h$-product. Hence, we believe that a further study of this relation is important and we would like to take this opportunity to introduce the following open problem, that we feel more than an open problem and it constitutes a line of research by itself.

Open question 5.4.1. Find new bridges between graph decompositions and graph labelings.

In this chapter, we also provided a lower bound for the magic-sums of the graphs $G \odot \overline{K}_n$ when G is an edge-magic (p,q)-graph and in particular, when G is a (bipartite) 2-regular graph. Then, we established a relationship with (super) edge-magic labelings and graph decompositions and provided a lower bound for the number of (super) edge-magic valences of a new family of graphs $S_{2n}(G; H_1, H_2)$. Especially interesting are the results established in Remarks 5.3.1, 5.3.2 and 5.3.3. We propose the following open problems in finding the families where these lower bounds are tight.

Open question 5.4.2. Prove of disprove that the lower bound obtained in Theorem 5.2.1 is tight. If the lower bound is tight, provide the families of edge-magic graphs G for which $|\tau_{G \odot \overline{K}_n}| = (n+1)|\tau_G| + 2$. Otherwise, can the lower bound be improved?

Open question 5.4.3. Prove of disprove that the lower bound obtained in Theorem 5.2.3 is tight. If the lower bound is tight, provide the families of

super edge-magic graphs G for which $|\sigma_{\overrightarrow{G} \otimes \overrightarrow{K}^l_{1,n}}| = (n+1)|\sigma_{\overrightarrow{G}}|$. Otherwise, can the lower bound be improved?

The results we get start from a partition of a complete bipartite graph. The following open question is proposed in the context of non-bipartite graphs.

Open question 5.4.4. When G is a non-bipartite graph, can we obtain results similar to Theorem 5.3.2 , Theorem 5.3.3 and Theorem 5.3.4?

List of Figures

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List of Publications

1. S.C. LÓPEZ, F. A. MUNTANER-BATLE AND M. PRABU, Perfect (super) edge-magic crowns, *Results in Math.* **71** (2017), 1459–1471.

2. S.C. LÓPEZ, F. A. MUNTANER-BATLE AND M. PRABU, A new labeling construction from the $\otimes_h$-product, *Discrete Math.* **340** (2017), 1903–1909.

3. S.C. LÓPEZ, F. A. MUNTANER-BATLE AND M. PRABU, On the super edge-magicness of graphs of equal order and size, *Ars. Combin.* To appear.

4. S.C. LÓPEZ, F. A. MUNTANER-BATLE AND M. PRABU, (Di)graph decompositions and magic type labelings: a dual relation, Submitted.

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