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Stable solutions of nonlinear fractional elliptic problems

by

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To Alba,
for her unconditional support.

"This thesis was written using 100% recycled words."[1]

[1] The quote is also recycled, from Terry Pratchett's "Wyrd sisters".*
* The original quote uses *book* instead of *thesis*.

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Summary

This thesis is devoted to study regularity and qualitative properties of solutions to semilinear integro-differential equations of the form

$$L_K u = f(u), \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where L_K is defined by

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(x+z)\} K(z)\, \mathrm{d}z\,.$$

The most canonical example of such operators is the fractional Laplacian $(-\Delta)^s$, which corresponds to the choice $K(z) = |z|^{-n-2s}$, with $s \in (0,1)$. Stable solutions are those at which the linearized operator is nonnegative. That is, u is a stable solution to the previous equation if

$$\int_\Omega f'(u)\xi^2\, \mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x) - \xi(x+z)|^2 K(z)\, \mathrm{d}x\, \mathrm{d}z \quad \text{for all } \xi \in C_c^\infty(\Omega).$$

The study of integro-differential equations is nowadays a very active field of research which has important applications in modeling real-life phenomena where nonlocal interactions appear. For instance, the Peierls-Nabarro equation, as well as the Benjamin-Ono and the fractional Allen-Cahn equations are semilinear integro-differential problems as above. They constitute fundamental models for describing crystal dislocation, water waves, and phase-transitions respectively.

In the first part of the thesis we study the boundedness of stable solutions to

$$\begin{cases} (-\Delta)^s u &= f(u) & \text{in} & \Omega \subset \mathbb{R}^n\,, \\ u &= 0 & \text{in} & \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with Ω a bounded domain. Our regularity results depend on the dimension n and the power $s \in (0,1)$, and one of the main open problems (still under our investigation) is to find the optimal threshold of these parameters for stable solutions to be bounded. Before our work, there were only two articles available concerning the boundedness of stable solutions in the fractional setting.

Our first result concerns the case when $\Omega = B_1$, the unit ball of $\mathbb{R}^n$. We establish an L^∞ bound for stable solutions whenever the dimension satisfies $2 \le n < 2(s + 2 + \sqrt{2(s+1)})$. We also prove some symmetry and monotonicity properties for bounded stable solutions to the problem in a ball.

To establish these results we use the extension problem for the fractional Laplacian. This is an important technique that relates a fractional problem in $\mathbb{R}^n$ with a local one in the half-space $\mathbb{R}^{n+1}_+$.

In the second part of the thesis we are focused on the study of saddle-shaped solutions to the integro-differential Allen-Cahn equation

$$L_K u = f(u), \quad \text{in } \mathbb{R}^{2m},$$

where f is of bistable type, the model case being $f(u) = u - u^3$. Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only on this set. The importance of studying this type of solution is due to its relation with the theory of nonlocal minimal surfaces and a fractional version of a conjecture by De Giorgi. Saddle-shaped solutions are the simplest non 1D candidates to be global minimizers in high dimensions, a property not yet established in any dimension.

The first chapter of this second part is devoted to study saddle-shaped solutions to the fractional Allen-Cahn equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{2m}$ using the extension problem. Our results establish the uniqueness of the saddle-shaped solution, and its stability in dimensions $2m \geq 14$. Before this work, it was known that these solutions exist in all even dimensions and are unstable in dimensions 2, 4, and 6. Thus, after our result, the stability remains an open problem only in dimensions 8, 10, and 12.

In the same dimensions $2m \geq 14$, and for $s < 1/2$, our result leads to the stability of the Simons cone as a nonlocal $(2s)$-minimal surface. This is the first analytical proof of a stability result for the Simons cone in the nonlocal setting.

In the last two chapters of this dissertation, we study for first time in the literature saddle-shaped solutions to general integro-differential equations $L_K u = f(u)$ in $\mathbb{R}^{2m}$, with L_K a rotation invariant and uniformly elliptic operator. Since the extension problem is no longer available, some new nonlocal techniques are developed in this thesis.

Our first accomplishment is to establish an appropriate setting for which a theory of existence and uniqueness for saddle-shaped solutions can be developed. More precisely, we characterize the kernels K for which we can carry out such a theory, and it turns out that a necessary and sufficient condition for this is that K is radially symmetric and $K(\sqrt{\cdot})$ is strictly convex. These results are achieved by writing the operator L_K acting on a doubly radial odd function as a new integro-differential operator acting on functions defined only at one side of the cone, $\{|x'| > |x''|\}$.

Under the previous assumption on the kernel K, we establish existence, uniqueness, and asymptotic behavior of the saddle-shaped solution to $L_K u = f(u)$ in $\mathbb{R}^{2m}$. For this, we prove, among others, an energy estimate for doubly radial minimizers, a Liouville type result, the one-dimensional symmetry of positive solutions to semilinear problems in a half-space, and maximum principles in "narrow" sets.

The thesis is made up of the following articles:

- [131] T. Sanz-Perela, *Regularity of radial stable solutions to semilinear elliptic equations for the fractional Laplacian*, Commun. Pure Appl. Anal. **17** (2018), 2547–2575.
- [88] J.C. Felipe-Navarro and T. Sanz-Perela, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, preprint available at arXiv (2018).
- [86] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone*, preprint available at arXiv (2019).
- [87] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation*, preprint (2019).

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Introduction

This thesis is devoted to study regularity and qualitative properties of solutions to semilinear integro-differential equations of the form

$$L_K u = f(u), \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where L_K is defined as in (1) below. The model example of such operator is the fractional Laplacian.

The study of integro-differential equations is nowadays a very active field of research, due to the multiple applications in modeling real-life phenomena where nonlocal interactions appear. Important examples are the Peierls-Nabarro, the Benjamin-Ono, and the fractional Allen-Cahn equations.

In the following, we introduce integro-differential equations and some of their main applications. We also present the fractional Laplacian and its main features. Finally, we recall the definition of stability in this nonlocal context, and we describe very briefly the topics studied in this thesis, which will be more extensively presented in the subsequent introductions to Parts I and II.

From PDEs to integro-differential equations

A Partial Differential Equation (PDE) is a relation between a function and its derivatives of different orders. This type of equations has been widely used in the last centuries to model a great variety of phenomena. They are useful tools, for instance, to describe the vibration of a string, the transfer of heat, or the motion of fluids. The reason why PDEs arise in modeling is that there is often a physical law governing the rate of change of some quantity, which is described through derivatives —a prime example is Newton's Second Law, which relates force and mass with acceleration, the second derivative of position with respect to time.

Despite the success of PDEs in many applications, there are some behaviors appearing in nature that cannot be modeled with precision through a PDE, due to the local character of such equations. Let us explain this briefly. To verify if a function satisfies a PDE at a point, it suffices to know the values of such function in an arbitrarily small neighborhood —to be able to compute the derivatives. As a consequence, PDEs can be a good tool to describe situations in which the state at a point depends on the nearby states but it is not influenced by changes in distant points. Nevertheless, there exist some natural phenomena which depend strongly on long range interactions and for which a PDE can not be a satisfactory model. An example would be the spreading of a disease. If one person is infected by a contagious disease, people around him or her are more likely to become sick. This is a local interaction that could be modeled through a

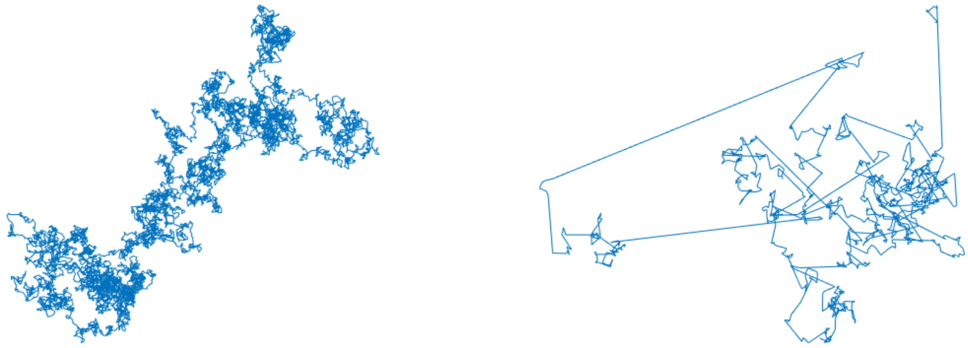

Figure 1: Graphical representation of a Brownian motion (left) and a Lévy flight (right).

PDE. However, imagine that someone who is infected takes a flight and travels to a distant country. In a short period, people which were a priori unlikely to become infected are now exposed to the disease. This is an example of a nonlocal interaction.

Situations as in the previous example may be modeled with *nonlocal* equations that involve integral operators. An example would be the following:

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(x+z)\} K(z) \, dz \,, \tag{1}$$

where $K(z)$ is a nonnegative function called the *kernel* of the operator. Note that $L_K u(x)$ is a weighted average of how the function u differs from its value at x. Obviously, to compute $L_K u(x)$ we need to know the values of u in the whole $\mathbb{R}^n$ and not only in a neighborhood of x.

Quite often the kernel K has a nonintegrable singularity at the origin and the previous integral must be understood in principal value (that is, integrating in $\mathbb{R}^n \setminus B_\varepsilon$ and taking the limit $\varepsilon \to 0$). In this case L_K is said to be an *integro-differential* operator, since, on account of the singularity of the kernel, L_K differentiates (in some fractional sense, as we will see) the function u. Indeed, in order to L_K be well defined, the singularity of the kernel forces u to be regular in such a way that there is a cancellation in the term $u(x) - u(x+z)$, which will compensate for the singularity of the kernel. This will make the integral in (1) to be finite in principal value sense.

Important models involving integro-differential equations

Integro-differential operators such L_K appear naturally in many situations, since they are the infinitesimal generators of Lévy processes —stochastic processes with independent and stationary increments. Informally speaking, a Lévy process (sometimes called Lévy flight) represents the motion of a particle which moves randomly and is allowed to jump large distances with certain probability (see Figure 1), in such a way that successive displacements are independent and statistically identical over different intervals of time with equal length.

These processes generalize the concept of Brownian motion and appear when one relaxes the assumption of continuity of paths by assuming only stochastic continuity. In the same way that second order elliptic operators are associated to Brownian motion, integro-differential operators arise naturally when one considers Lévy processes. Thus, they appear as models for many phenomena. Let us illustrate this with an example.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider a Lévy process X_t, $t \geq 0$, starting at $x \in \Omega$ —we should imagine a particle which is allowed to move randomly, but not necessarily in a continuous way, i.e., it may jump. Assume that when the particle exits Ω for the first time τ, we get a payoff $g(X_\tau)$, and define u to be the expected payoff obtained starting at x, that is, $u(x) := \mathbb{E}[g(X_\tau)]$. If the probability to jump from one point x to another $x+z$ is given by $K(z)$, appropriately normalized, then u solves the problem

$$\begin{cases} L_K u &= 0 \quad \text{in}\, \Omega\,, \\ \quad u &= g \quad \text{in}\, \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Note that, in contrast with local equations, here we have a Dirichlet condition in $\mathbb{R}^n \setminus \Omega$ and not only on $\partial\Omega$.

As the previous example shows, many situations where there is diffusion of nonlocal nature can be modeled with an integro-differential equation. This type of diffusion (sometimes called "anomalous diffusion") appears in many real-life phenomena: besides the situations described next in more detail, we could add also transport in turbulent plasma, bacterial motion, quantum optics, transport in heterogeneous rocks, or the flight of an albatross; see [117] and the references therein.

(a) Fluid Mechanics: An important nonlocal equation is the Benjamin-Ono equation (see [10, 5]), which describes one-dimensional internal waves in deep water. In this context, the irrotationality of the fluid leads to consider a stream function, which is harmonic in a half-space and whose flux at the boundary satisfies an equation (see for instance [22, 148]). This situation is strongly related to the extension problem for the fractional Laplacian, that we will describe in the next section. An important result regarding this equation is the uniqueness of the ground state solutions to the Benjamin-Ono equation, proved in $\mathbb{R}$ by Frank and Lenzmann [92], and in $\mathbb{R}^n$ by Frank, Lenzmann, and Silvestre [93].

Within Fluid Mechanics, another remarkable example where nonlocal equations arise is in oceanography. In this field, the surface quasi-geostrophic equation is used to model the dynamics associated with surface buoyancy conservation, see for instance [62, 110]. An important problem concerning the regularity of solutions to this equation was solved in the celebrated paper of Caffarelli and Vasseur [53], using the methods developed in the last decade for solving fractional problems using the extension technique (see also [52] and the lecture notes [54]).

(b) Crystal dislocations: Other nonlocal equations appear in Elasticity and Material Science. For instance, in a crystal, which is a material whose atoms are distributed in a regular way, certain dislocations of atoms can lead to macroscopic changes in the material. To model this situation it is of great importance the Peierls-Nabarro equation, see [147, 114, 77]. After the seminal contributions of Toland [147] in dimension $n = 1$,

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Usually one considers $h\mathbb{Z}^n$, with h small, as a discretization of $\mathbb{R}^n$. If the probability to jump from x to $x + hj$, with $j \in \mathbb{Z}^n$, is given by $K_h(j) = K(jh)/\sum_{j\in\mathbb{Z}^n\setminus\{0\}} K(jh)$, it is easy to deduce that the expected payoff for the step h, u^h, satisfies

$$\sum_{j\in\mathbb{Z}^n\setminus\{0\}} \{u^h(x) - u^h(x+hj)\}K_h(j) = 0\,.$$

Under suitable assumptions for K on its integrability at infinity and its growth at the origin, multiplying the previous equation by the right power of h and letting $h \to 0$ one obtains $L_K u(x) = 0$.

there has been a lot of activity concerning this equation, which is in fact of fractional Allen-Cahn type. The second part of the thesis treats this equation extensively.

(c) Front propagation in Ecology: In optimal search theory, it has been observed that depending on the distribution of prey, predators adopt search strategies that can be of nonlocal nature. Indeed, in contrast with the case when prey is abundant (in which Brownian motion is a good search strategy), if the prey distribution is sparse, strategies based in long jumps are more effective (see [150, 104, 115]). In addition to this, in 2013 Berestycki, Roquejoffre, and Rossi [17] introduced an important model to describe biological invasions in the plane when a strong diffusion takes place on a line (see also [13, 18]). Intuitively speaking, their model describe the dynamics of a population that moves in a field where there is a road that allows fast displacements through it. This is deeply connected to the extension problem for the fractional Laplacian, described in the next section.

(d) Mathematical Finance: Apart from modeling phenomena in nature, integro-differential equations are of great importance in other fields, such as Mathematical Finance. Since the prices of assets can have sudden changes, they are frequently modeled following a Lévy process and therefore a good understanding of integro-differential equations is of crucial importance. See for instance [116, 63, 111, 47].

(e) Image Processing: This is another important field in which integro-differential equations appear. In 2005, Buades, Coll, and Morel [26, 27] introduced an algorithm for image denoising based on a nonlocal average of all pixels in an image. They showed that this is better than PDE based models in detecting patterns and contours of images with noise. The algorithm of Buades, Coll, and Morel has been widely used since its introduction and has been the basis for other nonlocal techniques in Image Processing (see also [28, 99, 108] and the references therein).

The fractional Laplacian and the extension problem

One of the most important —and most widely studied— integro-differential operators is the fractional Laplacian $(-\Delta)^s$, defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, \mathrm{d}z \,,$$

where $c_{n,s}$ is a normalizing constant. This operator is translation and rotation invariant, and it is homogeneous of order $2s$. Thus, it the canonical model for integro-differential operators in the same way as the Laplacian is for second-order elliptic PDEs.

Alternatively, one can define $(-\Delta)^s$ through the Fourier transform:

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi) \,.$$

In this way the fractional Laplacian can be seen as a pseudo-differential operator of symbol $|\xi|^{2s}$ and from this it follows that $(-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}$ and that the classical Laplacian is recovered when $s = 1$. These two definitions are equivalent for an appropriate choice of the constant $c_{n,s}$ (see for instance [30]).

A particularly interesting (and useful) feature of the fractional Laplacian is the so-called extension problem. In a few words, by adding another variable $y > 0$, we can consider a local problem in $\mathbb{R}^{n+1}_+ = \{(x,y) \,:\, x \in \mathbb{R}^n,\, y > 0\}$ and recover the fractional

Laplacian in $\mathbb{R}^n$ through a weighted flux on $\{y = 0\}$. More precisely, if v solves the problem

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty), \\ v = u & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{2}$$

with $a = 1 - 2s$, then

$$-\lim_{y \downarrow 0} y^a v_y = \frac{1}{d_s} (-\Delta)^s u, \tag{3}$$

for a positive constant d_s which depends only on s. Through the thesis we will call v, the solution of (2), the *s-harmonic extension* of u.

Let us show an heuristic argument which illustrates the relation (3) in the simplest case $s = 1/2$, i.e., $a = 0$. We will see that $u \mapsto \Psi_{1/2}(u) := -\lim_{y\downarrow 0} \partial_y v$ applied two times is the same as applying $-\Delta_x$. To show this, we first extend u in $\mathbb{R}^{n+1}_+$ by considering its harmonic extension v, and we have $\Psi_{1/2}(u) = -\lim_{y\downarrow 0} v_y$. Since $-v_y$ is also harmonic in $\mathbb{R}^{n+1}_+$, it is in fact the harmonic extension of $\Psi_{1/2}(u)$ and thus $\Psi_{1/2}(\Psi_{1/2}(u)) = \lim_{y\downarrow 0} v_{yy} = \lim_{y\downarrow 0}(-\Delta_x v) = -\Delta_x u$.

This previous relation for the half-Laplacian had already been noticed in the nineties by Amick and Toland (see [5]) from an observation made by Benjamin in [10] in the context of the Benjamin-Ono equation. In 2006, Caffarelli and Silvestre [51] studied the extension problem for all powers $s \in (0, 1)$ of the fractional Laplacian. This development has led to a huge amount of new discoveries in nonlinear equations for the fractional Laplacian (just to cite some of them, see [50, 48, 49, 92, 93, 40, 41, 60, 37]).

The extension problem is a quite powerful tool since it allows to use local techniques in a nonlocal setting. As a consequence, a lot of known results for the Laplacian have been generalized to $(-\Delta)^s$ using the extension technique (some of them quite straightforward and some others with a lot of effort). However, the use of the extension may not be the best strategy to deal with certain problems involving the fractional Laplacian. Indeed, in this thesis we will encounter two examples of this. The first one concerns a trace inequality that relates quantities in $\mathbb{R}^n$ and in $\mathbb{R}^{n+1}_+$. It will be essential to study stability of solutions, although some information is lost when using this trace inequality (see Remark I.4). The second example concerns a maximum principle in "narrow" sets. As we will see, working directly in $\mathbb{R}^n$ provide better and simpler proofs than using the extension problem (see the comments right after Propositions II.7 and II.13).

More importantly, the extension technique is only available for the fractional Laplacian and thus, to obtain results for more general integro-differential operators, one needs to develop an intrinsically nonlocal approach (without the "help" of the extension). Regarding this last issue, let us point out some results that are only available in the literature, at current time, for the fractional Laplacian and not for other integro-differential equations, since they rely very strongly on the local characterization through the s-harmonic extension. A first example consists on the results in [37, 76, 90] regarding the nonlocal analogue of a conjecture by De Giorgi on the Allen-Cahn equation (see more details in the introduction to Part II). Another result only available for $(-\Delta)^s$ is the uniqueness of a ground state solution to Benjamin-Ono or fractional Schrödinger type equations (see [92, 93]). In the proof of this result, it is crucial to use a Hamiltonian that

In other very particular cases there is also a local formulation through an extension problem, such as for the fractional harmonic oscillator $(-\Delta + |x|^2)^s$ or for the sum of fractional Laplacians, see [144, 46].

was only available through the extension problem (see [40]), together with a topological lemma that concerns the level sets in $\mathbb{R}^{n+1}_+$ of the extension. Finally, in the theory of nonlocal minimal surfaces (see the introduction to Part II) the available monotonicity formula relies on the extension problem in $\mathbb{R}^{n+1}_+$ (see [48]), and a proof without using it is not known.

It would be of great interest to find proofs of the above results which do not use the extension problem. This perhaps would open the way to establish similar results for other kernels.

To conclude this section, we make a short comment on the class of kernels K that we consider along the thesis. Recall that, when studying second order PDEs of the form $\sum_{i,j} \partial_i(a_{ij}(x)\partial_j u) = h$, a crucial assumption in many problems is the uniform ellipticity of the coefficients a_{ij}. In the integro-differential setting, this assumption is translated into the following bounds on the kernel:

$$\frac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{n+2s}}, \tag{4}$$

for two positive constants $0 < \lambda \leq \Lambda$. In this thesis we will always assume that (4) holds. This condition is usually assumed in the literature, and it is known to yield Hölder regularity of solutions. Essentially (and speaking very informally), if u solves $L_K u = h \in C^\alpha$ in B_1, say, with K satisfying (4) and $\alpha \geq 0$, it follows that $u \in C^{\alpha+2s}$ in $B_{1/2}$; see [125, 139] for the precise statement. In addition to the uniform ellipticity, along this thesis we will also assume that K is symmetric, i.e., $K(z) = K(-z)$.

For a more detailed introduction to the fractional Laplacian, we refer to [30] and the references therein. For a general exposition on integro-differential equations and the regularity of solutions, see the survey [125].

Stable solutions

A system is said to be in a stable state if it can recover from small perturbations. Think for instance in a membrane of a drum that is hit with a stick. After a certain amount of time, in which the membrane is vibrating, it will eventually stop and return to a state of equilibrium.

We can understand stability by looking at the variations of the energy of the system, as we illustrate in the next simple example. Assume that the state of a (physical) system is described by a parameter $t \in \mathbb{R}$ and consider a smooth function $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ denoting the potential energy of a system. The most common situation in physics is that steady states of the system are those which minimize the energy (at least locally). Thus, if t_0 is a point of local minimum of $\mathcal{E}$, it follows that $\mathcal{E}'(t_0) = 0$ and $\mathcal{E}''(t_0) \geq 0$. We say that this point is *stable*. A simple and intuitive way to imagine this is by considering a round object placed at the center of bowl. If we push it slightly it will move for some time, but eventually it will return to its stable position.

In the previous simple example we considered a system whose state is described by a single parameter $t \in \mathbb{R}$. However, in most situations the state of a system is described a function and therefore we need to consider an energy $\mathcal{E}$ defined in a suitable functional space, and the corresponding definition of stability will concern the second variation of

More recently, in [8], a Hamiltonian for a related problem has been found without the use of the extension, but still only for the fractional Laplacian.

the energy. We next define properly stable solutions in the context of semilinear integro-differential equations.

This thesis is devoted to semilinear equations of the form

$$L_K u = f(u) \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{5}$$

with L_K a linear integro-differential operator with a positive and symmetric kernel satisfying the ellipticity assumption (4). These equations have a variational structure (that is, solutions are critical points of an associated energy functional). Indeed, if we define $F(t) := \int_0^t f(\tau)\,\mathrm{d}\tau$, then (5) is the Euler-Lagrange equation of the functional

$$\mathcal{E}(w,\Omega) := \frac{1}{4}\left\{ \int_\Omega \int_\Omega |w(x) - w(z)|^2 K(x-z)\,\mathrm{d}x\,\mathrm{d}z \right. \\ \left. + 2\int_\Omega \int_{\mathbb{R}^n\setminus\Omega} |w(x) - w(z)|^2 K(x-z)\,\mathrm{d}x\,\mathrm{d}z \right\} - \int_\Omega F(w)\,\mathrm{d}x\,.$$

In other words,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(u+\varepsilon\xi,\Omega) = 0 \quad \text{for all } \xi \in C_c^\infty(\Omega).$$

If $f \in C^1$ one can consider the second variation of $\mathcal{E}$, namely

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Big|_{\varepsilon=0} \mathcal{E}(u+\varepsilon\xi,\Omega) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x)-\xi(z)|^2 K(x-z)\,\mathrm{d}x\,\mathrm{d}z - \int_\Omega f'(u)\xi^2\,\mathrm{d}x.$$

Then, one says that u is a *stable solution* in Ω if the second variation of the energy at u is nonnegative, that is, if

$$\int_\Omega f'(u)\xi^2\,\mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x)-\xi(z)|^2 K(x-z)\,\mathrm{d}x\,\mathrm{d}z \quad \text{for all } \xi \in C_c^\infty(\Omega).$$

These solutions represent steady states of a system that are stable under small perturbations. Obviously, local minimizers of the energy are stable solutions.

Topics and structure of the thesis

The thesis is divided into two parts. In these parts, each chapter corresponds to a research paper.

In Part I we study the boundedness of stable solutions to (5) in bounded domains, with $L_K = (-\Delta)^s$, and assuming zero Dirichlet exterior data. We study the problem in a ball, and we establish that stable solutions are bounded if $2 \le n < 2(s+2+\sqrt{2(s+1)})$. We believe to know which is the optimal range of parameters n and s for stable solutions to be bounded, which is slightly better than the one in our results. However, establishing this is a delicate open problem still under our investigation.

Part I consists on the paper

- [131] T. Sanz-Perela, *Regularity of radial stable solutions to semilinear elliptic equations for the fractional Laplacian*, Commun. Pure Appl. Anal. **17** (2018), 2547–2575.

In Part II, we study saddle-shaped solutions to (5) when $\Omega = \mathbb{R}^{2m}$ and f is of bistable type (the basic model is the Allen-Cahn nonlinearity $f(u) = u - u^3$). One of the main questions is to determine in which dimensions the saddle-shaped solution is stable. This issue is strongly related with a nonlocal version of a conjecture by De Giorgi and with the regularity theory of nonlocal minimal surfaces.

In the case $L_K = (-\Delta)^s$, we prove the uniqueness of the saddle-shaped solution and, in dimensions $2m \geq 14$, its stability. We also study for first time in the literature saddle-shaped solutions to integro-differential equations with more general kernels. We establish a necessary and sufficient condition on the kernel K to be able to develop a theory of existence and uniqueness on the saddle-shaped solution. We also prove an energy estimate, symmetry and Liouville type results, and maximum principles in "narrow" sets.

Part II consists on the papers

- [88] J.C. Felipe-Navarro and T. Sanz-Perela, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, preprint available at arXiv (2018).
- [86] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone*, preprint available at arXiv (2019).
- [87] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation*, preprint (2019).

Part I

Regularity of stable solutions to semilinear fractional equations in bounded domains

Introduction to Part I

This first part of the thesis is devoted to study the regularity of stable solutions to the semilinear equation $(-\Delta)^s u = f(u)$ in $\Omega \subset \mathbb{R}^n$, as well as its associated Dirichlet problem, whenever Ω is a bounded domain. We obtain several estimates, depending on the dimension n and the power $s \in (0,1)$, that yield the regularity of stable solutions in low dimensions.

The structure of this part consists in a chapter which corresponds to the article:

- [131] T. Sanz-Perela, *Regularity of radial stable solutions to semilinear elliptic equations for the fractional Laplacian*, Commun. Pure Appl. Anal. **17** (2018), 2547–2575.

Regularity of stable solutions to reaction-diffusion equations

Reaction-diffusion equations play an important role in PDE theory, as well as in its applications to other sciences. They are used to model several phenomena in different fields, such as Biology, Physics or Chemistry (for instance in the study of population evolution, epidemiology, fluids, combustion, chemical reactions, pattern formation, etc.). They also appear in many geometric problems, for instance in the conformal classification of varieties, in the problem of prescribing a curvature on a manifold, or in the study of parabolic flows on manifolds.

The regularity of minimizers to reaction-diffusion equations —or, more generally, minimizers to nonlinear elliptic equations— constitutes a classical problem in the Calculus of Variations. It appears, for instance, in Hilbert's 19th problem. A fundamental example in Geometry is the regularity of minimizing minimal surfaces in $\mathbb{R}^n$. These surfaces are not only stationary points of the area functional, but also minimizers. Several important works from the 1970s established that these surfaces are smooth in dimensions $n \leq 7$, while if $n = 8$, there exists a minimizing minimal hypersurface which has a singularity at the origin. It is the so-called Simons cone, defined in $\mathbb{R}^8$ by

$$\mathscr{C} = \left\{ (x', x'') \in \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8 \ : \ |x'| = |x''| \right\}.$$

For more details on this, see the introduction of Part II and the references therein.

A natural question to ask is if the same holds for stable minimal surfaces (a more general class of stationary points than minimizers). By the previous comments, stable minimal surfaces in $\mathbb{R}^n$ are not smooth in general if $n \geq 7$. On the other hand, in dimension $n = 3$ all stable minimal surfaces are smooth (see [91, 80]). However, nothing is known in dimensions $4 \leq n \leq 7$, and solving this is a very relevant open problem at the current time.

A PDE analogue to the previous question is to ask whether stable solutions to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{I.1}$$

are smooth. Recall that u is a stable solution to (I.1) if

$$\int_\Omega f'(u)\xi^2\, dx \le \int_\Omega |\nabla \xi|^2\, dx$$

for all $\xi \in C_c^\infty(\Omega)$. The seminal papers in the 1970s of Crandall and Rabinowitz [68], and Joseph and Lundgren [107], established a certain number of results on this problem —described next. However, as we will see, many central questions are still open.

A similar phenomenon as in the theory of minimal surfaces occurs with problem (I.1) too: stable solutions are regular in low dimensions, while singularities may appear if the dimension is big. To illustrate this, consider the function $u(x) = -2\log|x|$. An easy computations shows that $u \in H_0^1(B_1)$ is a solution to $-\Delta u = 2(n-2)e^u$ in B_1 if $n \ge 3$. Now, the linearized operator at u is given by

$$-\Delta - \frac{2(n-2)}{|x|^2}.$$

If this operator is nonnegative, then u is stable. Using the Hardy inequality

$$\frac{(n-2)^2}{4}\int_{B_1} \frac{\xi^2}{|x|^2}\, dx \le \int_{B_1} |\nabla\xi|^2\, dx \quad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n),$$

and the fact that $2(n-2) \le (n-2)^2/4$ if $n \ge 10$, we readily obtain that $u(x) = -2\log|x|$ is a singular $H_0^1(B_1)$ stable solution to (I.1) with $\Omega = B_1$ whenever $n \ge 10$.

Instead, if $f(u) = e^u$ or $f(u) = (1+u)^p$ with $p > 1$, for all bounded domains Ω it holds that in dimensions $n \le 9$ all stable $H_0^1(\Omega)$ solutions are bounded —and thus, by standard elliptic regularity theory, smooth. The same is true if $f \in C^2$ is positive, increasing, convex, and $\lim_{t\to+\infty} f(t)f''(t)/f'(t)^2$ exists in $[0,+\infty]$. See the papers of Crandall and Rabinowitz [68], and Joseph and Lundgren [107].

From these two previous results, a natural conjecture arises.

Conjecture I.1. *Let $u \in H_0^1(\Omega)$ be a stable solution to* (I.1) *and assume that f is positive, nondecreasing, superlinear at infinity, and convex. Then, if $n \le 9$, u is bounded.*

In the last decades there have been several attempts to prove the regularity of stable solutions . In 2000, Nedev [121] proved the boundedness of stable solutions in dimensions $n \le 3$. Later, Cabré and Capella [35] obtained an L^∞ bound when $\Omega = B_1$, whenever $n \le 9$. The best known result at the moment for general f is due to Cabré [33], and it states that stable solutions are regular in dimensions $n \le 4$ for every convex domain Ω. This result was extended by Villegas [149] to nonconvex domains. Nevertheless, the problem is still open in dimensions $5 \le n \le 9$.

The problem of the extremal solution

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary, and let $\lambda > 0$ be a positive parameter. We consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{I.2}$$

where f is a positive and increasing function such that it is superlinear at infinity. In some cases, f is also assumed to be convex.

Problem (I.2) appears in several models, for instance in combustion. The first extensive study of (I.2) appeared in [95] and, after some questions raised by Brezis and Vázquez in [25], it aroused the interest of many experts in PDEs. Intuitively, one may think of (I.2) as the steady state of a substance where there is a competition between diffusion and reaction, and this is controlled by the parameter $\lambda > 0$. If λ is small, the diffusion term dominates and this suggests that such steady state may exist. Instead, if λ is big, the reaction term is so powerful that there cannot be a steady state. This heuristic argument can be formalized (see the excellent monograph [81]) and, nowadays, it is well known that, under the previous assumptions on f, there exists a finite extremal parameter λ^* such that, if $0 < \lambda < \lambda^*$, then problem (I.2) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Here, *minimal* means that u_λ is smaller than any other solution or supersolution and *classical* means that $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

It is also well known that the family $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ and that its pointwise limit when $\lambda \nearrow \lambda^*$ is a weak solution to (I.2) with $\lambda = \lambda^*$. Such solution is called the *extremal solution* of (I.2) and it is denoted by u^*. Here by weak solution we mean that $u^* \in L^1(\Omega)$, that $f(u^*)\,\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$, and that

$$\int_\Omega u^*(-\Delta)\zeta \, dx = \lambda^* \int_\Omega f(u^*)\zeta \, dx \quad \text{for all } \zeta \in C^2(\Omega) \text{ with } \zeta_{|\partial\Omega} = 0.$$

In the nineties, Brezis and Vázquez [25] raised the question of determining the regularity of u^*, depending on the dimension n (see also the problems raised by Brezis in [23]). This is equivalent to determine whether u^* is bounded or unbounded. The relevancy of this question lies in the fact that the existence of other nonminimal solutions for $\lambda < \lambda^*$ depends strongly on the regularity of the extremal solution u^* (see [25, 81]). Moreover, every L^1 stable solution to $-\Delta u = f(u)$ can be expressed as a limit of regular solutions. Indeed, for every $\varepsilon \in (0, 1)$ there exists a bounded solution u_ε to $-\Delta u_\varepsilon = (1-\varepsilon)f(u_\varepsilon)$, see [24, 81], and $u_\varepsilon \to u$ as $\varepsilon \to 1$. As a consequence, the problem of the regularity of the extremal solution is deeply related with the previous conjecture for stable solutions to $-\Delta u = f(u)$.

The usual strategy to prove the boundedness of u^* is to find universal a priori estimates for bounded solutions to (I.1). Here by universal we mean that they depend, essentially, on the L^1 norm of the solutions. Then, we apply these estimates to the classical solutions u_λ with $\lambda < \lambda^*$ and since $u^* \in L^1(\Omega)$, by monotone convergence the estimates also hold for u^*.

Regularity of stable solutions for semilinear problems with the fractional Laplacian: available results

In this thesis we consider the fractional analogues of the previous questions concerning stable solutions to semilinear problems, replacing $-\Delta$ by the fractional Laplacian. We

It is not necessary to assume f convex to have existence of the family $\{u_\lambda : 0 < \lambda < \lambda^*\}$. However, the convexity assumption on f is necessary to guarantee that it is a continuous family in λ (see the comments in [35]).

are interested in finding suitable a priori estimates for stable solutions to

$$(-\Delta)^s u = f(u) \quad \text{in } \Omega\,, \tag{I.3}$$

as well as its associated Dirichlet problem

$$\begin{cases} (-\Delta)^s u &= f(u) & \text{in } \Omega\,, \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{I.4}$$

We say that a solution u of (I.3) is *stable* in Ω if

$$\int_\Omega f'(u)\xi^2 \,\mathrm{d}x \le \frac{c_{n,s}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\xi(x)-\xi(z)|^2}{|x-z|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}z \tag{I.5}$$

for all $\xi \in H^s(\mathbb{R}^n)$ such that $\xi \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Recall that

$$H^s(\mathbb{R}^n) = \left\{ w \in L^2(\mathbb{R}^n) \;:\; [w]_{H^s(\mathbb{R}^n)} < +\infty \right\},$$

where

$$[w]^2_{H^s(\mathbb{R}^n)} = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(x)-w(z)|^2}{|x-z|^{n+2s}},$$

and that this is the natural Hilbert space where equation (I.3) is posed.

Regarding the extension problem, it is well known that the space $H^s(\mathbb{R}^n)$ coincides with the trace of $H^1(\mathbb{R}^{n+1}_+, y^a)$ on $\partial\mathbb{R}^{n+1}_+$ (see for instance [92, 93]). In particular, every function $\xi : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ such that $\xi \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^a)$ and $\nabla\xi \in L^2(\mathbb{R}^{n+1}_+, y^a)$ has a trace on $H^s(\mathbb{R}^n)$, $\operatorname{tr}\xi$, and it satisfies the following inequality (see Proposition 3.6 in [92]):

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\operatorname{tr}\xi(x) - \operatorname{tr}\xi(z)|^2}{|x-z|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}z \le d_s \int_{\mathbb{R}^{n+1}_+} y^a |\nabla\xi|^2 \,\mathrm{d}x\,\mathrm{d}y\,, \tag{I.6}$$

where d_s is the constant appearing in the relation $-d_s \lim_{y\downarrow 0} y^a v_y = (-\Delta)^s u$. In addition, d_s is the optimal constant in (I.6) and the equality is achieved only when ξ is the s-harmonic extension of its trace on $\mathbb{R}^n$, as explained in Chapter 1 (see also [92, 93]). We will write simply ξ and not $\operatorname{tr}\xi$ when no confusion is possible.

As a consequence of (I.6), if u is stable solution to (I.4), it follows that

$$\int_\Omega f'(u)\xi^2 \,\mathrm{d}x \le d_s \int_{\mathbb{R}^{n+1}_+} y^a |\nabla\xi|^2 \,\mathrm{d}x\,\mathrm{d}y\,, \tag{I.7}$$

for every $\xi \in H^1(\mathbb{R}^{n+1}_+, y^a)$ such that its trace has compact support in Ω. This form of the stability condition will be used in Chapter 1 to study stable solutions to (I.4) using the extension problem.

Obviously, the same type of questions as in the previous section can be asked about problem (I.4): are stable solutions to (I.4) bounded, at least in low dimensions? In the fractional setting, the answer to this question depends not only on the dimension n, but also on the power $s \in (0,1)$. Let us illustrate this with an example.

First, by using the Fourier transform, it is not difficult to see that

$$(-\Delta)^s \log\frac{1}{|x|^{2s}} = \lambda_0 |x|^{-2s}, \qquad \text{where } \lambda_0 = 2^{2s}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)}.$$

As a consequence, $u(x) = -2s \log |x|$ solves $(-\Delta)^s u = \lambda_0 e^u$ in $\mathbb{R}^n$. To check if u is stable we have to consider the linearization of the equation at u, which leads to the Hardy type operator

$$\mathscr{L}_u := (-\Delta)^s - \frac{\lambda_0}{|x|^{2s}}.$$

Thus, u is stable in $\mathbb{R}^n$ if $\mathscr{L}_u$ is a nonnegative operator, i.e., if $\langle \xi, \mathscr{L}_u \xi \rangle_{L^2(\mathbb{R}^n)} \geq 0$ for all $\xi \in C_c^\infty(\mathbb{R}^n)$. This condition can be rewritten as

$$\lambda_0 \int_{\mathbb{R}^n} \frac{\xi^2}{|x|^{2s}} \, dx \leq \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\xi(x) - \xi(z)|^2}{|x-z|^{n+2s}} \, dx \, dz, \qquad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n).$$

To check this, recall that the fractional Hardy inequality ([94]) states that for every $\xi \in C_c^\infty(\mathbb{R}^n)$ it holds

$$H_{n,s} \int_{\mathbb{R}^n} \frac{\xi^2}{|x|^{2s}} \, dx \leq \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\xi(x) - \xi(z)|^2}{|x-z|^{n+2s}} \, dx \, dz, \qquad \text{where } H_{n,s} = 2^{2s} \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}.$$

Moreover, $H_{n,s}$ is the optimal constant in the previous inequality. As a consequence, u is stable whenever $\lambda_0 \leq H_{n,s}$, and unstable if $\lambda_0 > H_{n,s}$, that is, if

$$\frac{\Gamma(\frac{n}{2})\Gamma(1+s)}{\Gamma(\frac{n-2s}{2})} > \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}. \tag{I.8}$$

As it will be mentioned below, this condition is believed to be the optimal threshold of the parameters n and $s \in (0,1)$ for stable solutions to be bounded, but establishing it is still an open problem under our investigation.

Before stating the results of the thesis, let us describe briefly what is known regarding the regularity of stable solutions to (I.4). Actually, prior to our results there were only two articles in the literature concerning boundedness of stable solutions to (I.4): [127] and [124].

The first paper where the boundedness of stable solutions for fractional problems is addressed is the one by Ros-Oton and Serra [127]. In that article, they study the problem of the extremal solution for the fractional Laplacian, that is,

$$\begin{cases} (-\Delta)^s u &= \lambda f(u) \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{I.9}$$

The first result in [127] is the existence of the extremal parameter λ^* and the extremal solution u^* for (I.9). More precisely, what they showed is that there exists a family of stable solutions $\{u_\lambda \ : \ 0 < \lambda < \lambda^*\}$ which is increasing in λ and whose pointwise limit u^*, when $\lambda \nearrow \lambda^*$, is a stable weak solution to (I.9). In the fractional context, this means that $u^* \in L^1(\Omega)$, that $f(u^*) \operatorname{dist}(\cdot, \partial\Omega)^s \in L^1(\Omega)$, and that

$$\int_\Omega u^* (-\Delta)^s \zeta \, dx = \lambda^* \int_\Omega f(u^*) \zeta \, dx$$

for all ζ such that ζ and $(-\Delta)^s \zeta$ is bounded and $\zeta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

Apart from this, Ros-Oton and Serra extended the techniques of Crandall and Rabinowitz [68] and Nedev [121, 122] to the fractional setting, establishing the following:

- If f is C^2 and the limit $\lim_{t\to+\infty} f(t)f''(t)/f'(t)^2$ exists, then u^* is bounded whenever $n < 10s$ (this extends the results of Crandall and Rabinowitz [68]).

- If f is convex, then u^* is bounded whenever $n < 4s$ (this extends the results of Nedev [121]).

- If Ω is convex, then $u^* \in H^s(\mathbb{R}^n)$ for all $n \geq 1$ and all $s \in (0,1)$ (this extends the result of Nedev in [122]).

These statements give, as $s \to 1$, the results of [68, 121, 122] for the classical problem. In both local and nonlocal scenarios, the proof of the first two statements is based in the following strategy. First, using the stability condition one shows that $f(u^*) \in L^p$ for certain exponents p. Then, by estimates of Calderón-Zygmund type —see [98] for $s = 1$ and [127] for $s \in (0,1)$—, it follows that $u^* \in W^{2s,p}$. Finally, if $n < 2sp$ the space $W^{2s,p}$ is embedded in a Hölder space, and thus u^* is bounded. Note, however, that for s small this argument does not provide any boundedness result. The third statement above is proved using the Pohozaev identity for the fractional Laplacian (see [128]) together with a moving planes argument that provide some estimates near the boundary $\partial\Omega$; see the details in [127]. It should be mentioned that no proof in [127] uses the extension problem for the fractional Laplacian.

The second paper in the literature addressing problem (I.9) is the one by Ros-Oton [124], concerning the case $f(u) = e^u$. In that paper, the following is established:

- If $f(u) = e^u$, Ω is a domain which is symmetric and convex in each x_i-direction, and either $n \leq 2s$, or $n > 2s$ and (I.8) holds, then u^* is bounded. In particular, the extremal solution u^* is bounded for all $s \in (0,1)$ whenever $n \leq 7$.

To prove this, one assumes by contradiction that u^* is singular. Then, it is possible to deduce a lower bound for u^* near its singular point, which is the origin due to the convexity and symmetry of the domain. By taking an explicit function $\xi(x) \sim |x|^{-\beta}$ in the stability condition and choosing β appropriately, if (I.8) holds one arrives at a contradiction with the previously established lower bound for u^*. As in [127], the proof in [124] does not use the extension problem.

It should be mentioned that the previous argument does not depend at all on the behavior of the solution outside Ω, and thus it also holds for a more general class of problems with nonhomogeneous Dirichlet data in $\mathbb{R}^n \setminus \Omega$. Nevertheless, the proof depends strongly on the fact that $f(u) = e^u$, in particular in the relation $f'(u) = f(u)$, which allows to combine the stability condition with the weak form of the equation. Therefore, it seems difficult to extend the proof to include more general nonlinearities.

This last result by Ros-Oton in [124], together with the above comments on the stability of $-2s\log|x|$ when (I.8) does not hold, both seem to suggest that (I.8) could be the optimal range of parameters n and s for which stable solutions are bounded (as already pointed out in [127]). However, this is still an open problem.

Results for the spectral fractional Laplacian

Both results appearing in [127] and [124] are obtained without the use of the extension problem. Apart from these two papers, before the results of this thesis there was only one more work on boundedness of stable solutions for semilinear problems involving fractional operators. This is the article of Capella, Dávila, Dupaigne, and Sire [55] concerning problem (I.9) but with $(-\Delta)^s$ replaced by the spectral fractional Laplacian A^s. This operator, defined via the Dirichlet eigenvalues of the Laplacian in Ω (see below), has an associated extension problem, which it is used in [55] to study the regularity of the extremal solution for the spectral fractional Laplacian when $\Omega = B_1$. This suggested that similar techniques to the ones introduced in [55] could be carried out to study (I.9) with the extension problem for the fractional Laplacian, as we indeed do in Chapter 1.

Next we explain some of the main features of [55]. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let λ_k and e_k, with $k \in \mathbb{N}$, be the eigenvalues and the corresponding eigenfunctions of the Laplace operator $-\Delta$ in Ω with homogeneous Dirichlet boundary data on $\partial\Omega$, i.e.,

$$\begin{cases} -\Delta e_k & = & \lambda_k e_k & \text{in } \Omega\,, \\ \quad e_k & = & 0 & \text{on } \partial\Omega\,. \end{cases}$$

The eigenfunctions e_k are normalized, that is, $||e_k||_{L^2(\Omega)} = 1$. For every $s \in (0,1)$ and every $u \in H_0^1(\Omega)$ with

$$u(x) = \sum_{k\in\mathbb{N}} a_k e_k(x)\,,$$

the operator A^s is defined as

$$A^s u = \sum_{k\in\mathbb{N}} a_k \lambda_k^s e_k(x)\,.$$

The operators A^s and $(-\Delta)^s$ are related but not equal. In particular, solutions of the Dirichlet problem for A^s are smooth up to the boundary if Ω is smooth enough, while in the case of $(-\Delta)^s$ they are $C^s(\overline{\Omega})$ and not better (see [126]). For a more detailed discussion about the differences and similarities between both operators we refer to [141].

The operator A^s has also an associated extension problem. Indeed, one can see that the spectral fractional Laplacian can be realized as the boundary Neumann operator of a suitable extension in the half-cylinder $\Omega \times (0, +\infty)$. More precisely, one considers the extension problem

$$\begin{cases} \operatorname{div}(y^a \nabla w) & = & 0 & \text{in } \Omega \times (0,+\infty)\,, \\ w & = & 0 & \text{on } \partial\Omega \times [0,+\infty)\,, \\ w & = & u & \text{on } \Omega \times \{0\}\,, \end{cases}$$

with $a = 1 - 2s$. Then, it can be proven that $-\lim_{y\downarrow 0} y^a w_y$ agrees with $A^s u$ up to a multiplicative constant. Notice that the solution w (extended by 0 to all $\overline{\mathbb{R}^{n+1}_+}$) is a subsolution of

$$\begin{cases} \operatorname{div}(y^a \nabla v) & = & 0 & \text{in } \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,+\infty)\,, \\ v & = & u & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n\,, \end{cases}$$

and thus, thanks to the maximum principle, one can use the Poisson formula for this last problem to obtain estimates for w. This is what is done in [55] and suggested that similar arguments could be carried out for the fractional Laplacian, as we do in Chapter 1.

The main result in [55] concerns the extremal solution for the operator A^s in a ball. The authors establish that if $2 \le n < 2(s + 2 + \sqrt{2(s+1)})$, then $u^* \in L^\infty(B_1)$. In particular, u^* is bounded in dimensions $2 \le n \le 6$ for all $s \in (0,1)$. In the first chapter of this thesis, we use similar ideas to the ones in [55] to study the same problem in B_1, but now with A^s replaced by the fractional Laplacian. We obtain the same condition on n and s guaranteeing regularity of the extremal solution to (I.9). Moreover, in the arguments of [55] there are two points where an auxiliary estimate is missing. We establish it in Chapter 1 both for the operators $(-\Delta)^s$ and A^s.

Results of the thesis: Part I

In Chapter 1 we study the regularity of stable solutions to (I.4) when $\Omega = B_1$, the unit ball of $\mathbb{R}^n$. We consider

$$\begin{cases} (-\Delta)^s u &= f(u) \quad \text{in} \quad B_1\,, \\ \qquad u &= 0 \qquad\ \ \text{in} \quad \mathbb{R}^n \setminus B_1\,, \end{cases} \tag{I.10}$$

with f a C^2 nonlinearity. The main result that we establish is the following.

Theorem I.2. *Let $n \geq 2$, $s \in (0,1)$, and f be a C^2 nondecreasing function. Let $u \in H^s(\mathbb{R}^n)$ be a stable radially decreasing weak solution to (I.10). We have that:*

(i) If $n < 2\left(s+2+\sqrt{2(s+1)}\right)$, then $u \in L^\infty(B_1)$. Moreover,

$$||u||_{L^\infty(B_1)} \leq C$$

for some constant C that depends only on n, s, f, and $||u||_{L^1(\mathbb{R}^n)}$.

(ii) If $n \geq 2\left(s+2+\sqrt{2(s+1)}\right)$, then for every $\mu > n/2 - s - 1 - \sqrt{n-1}$, it holds

$$u(x) \leq C|x|^{-\mu} \quad \text{in } B_1\,,$$

for some constant C that depends only on n, s, μ, f, and $||u||_{L^1(\mathbb{R}^n)}$.

Note that this result yields the boundedness of the extremal solution in dimensions $n < 2(s+2+\sqrt{2(s+1)})$. Indeed, the classical stable solutions u_λ with $\lambda < \lambda^*$ are radially symmetric and decreasing (see Proposition I.5 below), and therefore the same holds for u^*. Then, since $u^* \in H^s(\mathbb{R}^n)$ by the results of [127], we can apply Theorem I.2 to deduce that u^* is bounded in dimensions $2 \leq n < 2(s+2+\sqrt{2(s+1)})$. In particular, u^* is bounded in dimensions $2 \leq n \leq 6$ for all $s \in (0,1)$.

The principal ingredient to establish Theorem I.2 is the following result.

Proposition I.3. *Let $n \geq 2$, $s \in (0,1)$, and f be a nondecreasing C^2 function. Let $u \in H^s(\mathbb{R}^n)$ be a stable radially decreasing solution of (I.10) and let v be its s-harmonic extension. Assume that α is any real number satisfying*

$$1 \leq \alpha < 1 + \sqrt{n-1}\,. \tag{I.11}$$

Then, denoting by $\rho = |x|$ the horizontal radius and $v_\rho = (x/\rho)\cdot\nabla_x v$, we have

$$\int_0^\infty \int_{B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\, \mathrm{d}x\, \mathrm{d}y \leq C\,, \tag{I.12}$$

for a positive constant C depending only on n, s, α, $||u||_{L^1(B_1)}$, $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$, and $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$.

The key point to establish Proposition I.3 —as well as its analogue in [55]— is the particular choice of the test function ξ in the stability condition (I.7). We take

$$\xi = \rho^{1-\alpha} v_\rho \zeta = |x|^{-\alpha}(x \cdot \nabla_x v)\zeta\,, \tag{I.13}$$

where α satisfies (I.11), v_ρ is the horizontal radial derivative of v, and ζ is a cut-off function. This choice, after controlling a number of integrals, leads to (I.12). A similar idea was already used by Cabré and Capella in [35] to prove the boundedness of u^* in the radial case for the classical Laplacian, and later by Capella, Dávila, Dupaigne, and Sire in [55] for A^s.

Remark I.4. The dimensions $n < 2(s+2+\sqrt{2(s+1)})$ obtained in Theorem I.2 are not optimal (in the sense that for the exponential nonlinearity, the result of Ros-Oton in [124] involving (I.8) is better). The reason of this is explained next.

In the stability condition (I.7), it is not necessary to take ξ as the s-harmonic extension of its trace (that is, test functions ξ need not solve $\operatorname{div}(y^a\nabla\xi) = 0$ in $\mathbb{R}^{n+1}_+$). This gives us more flexibility for the choice of functions in the stability condition. However, if we want an inequality completely equivalent to (I.5) —in the sense that we do not lose information when going to $\mathbb{R}^{n+1}_+$—, one would need to consider always test functions solving $\operatorname{div}(y^a\nabla\xi) = 0$ in $\mathbb{R}^{n+1}_+$.

Typically, in order to obtain the best possible dimensions from the stability condition, one has to try to get estimates that are sharp. Thus, it seems that the extension problem is not the best tool in this setting, since we do not have any control on the reminder that appears when using the trace inequality (I.6). Indeed, the test function chosen in (I.13) does not solve $\operatorname{div}(y^a\nabla\xi) = 0$ in $\mathbb{R}^{n+1}_+$, and this could be reason why we obtain $n < 2(s+2+\sqrt{2(s+1)})$ and not (I.8) —though condition $n < 2(s+2+\sqrt{2(s+1)})$ is not far from (I.8).

We believe that a more direct nonlocal approach (without the extension problem) could lead to the optimal dimensions (I.8) for the regularity of stable solutions in the fractional case. This is ongoing research.

Note that the radially decreasing assumption in Theorem I.2 is automatically satisfied for stable solutions. Indeed, in Chapter 1 we prove the following.

Proposition I.5. *Let $n \geq 2$ and let u be a bounded stable solution of* (I.10) *with $f \in C^2$. Then, u is radially symmetric. Moreover, if u is not identically zero then u is either increasing or decreasing in $B_1 \setminus \{0\}$.*

As it is well known, when $u \geq 0$ is a bounded solution of (I.10), then u is radially symmetric and decreasing. This was proved in [19] using the moving planes method. Furthermore, by the Poisson formula, the s-harmonic extension of u is also radially symmetric in the horizontal direction, that is, it only depends on ρ and y. Moreover, $v_\rho < 0$ for $\rho > 0$.

In the moving planes argument, the hypothesis of $u \geq 0$ cannot be omitted, since there can be changing-sign solutions to (I.10) that are not radially symmetric. Nevertheless, this is not the case for stable solutions, as we prove in Chapter 1.

Chapter 1

Regularity of radial stable solutions to semilinear elliptic equations for the fractional Laplacian

We study the regularity of stable solutions to the problem

$$\left\{ \begin{array}{rcll} (-\Delta)^s u & = & f(u) & \text{in} \quad B_1\,, \\ u & \equiv & 0 & \text{in} \quad \mathbb{R}^n \setminus B_1\,, \end{array} \right.$$

where $s \in (0,1)$. Our main result establishes an L^∞ bound for stable and radially decreasing H^s solutions to this problem in dimensions $2 \le n < 2(s+2+\sqrt{2(s+1)})$. In particular, this estimate holds for all $s \in (0,1)$ in dimensions $2 \le n \le 6$. It applies to all nonlinearities $f \in C^2$.

For such parameters s and n, our result leads to the regularity of the extremal solution when f is replaced by λf with $\lambda > 0$. This is a widely studied question for $s = 1$, which is still largely open in the nonradial case both for $s = 1$ and $s < 1$.

1.1 Introduction

This paper is devoted to the study of the regularity of stable solutions to the semilinear problem

$$\left\{ \begin{array}{rcll} (-\Delta)^s u & = & f(u) & \text{in} \quad B_1\,, \\ u & = & 0 & \text{in} \quad \mathbb{R}^n \setminus B_1\,, \end{array} \right. \tag{1.1.1}$$

where B_1 is the unit ball in $\mathbb{R}^n$ and f is a C^2 function. The operator $(-\Delta)^s$ is the fractional Laplacian, defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) := c_{n,s}\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z\,,$$

where $c_{n,s} > 0$ is a normalizing constant depending only on n and s and P. V. stands for principal value.

Our results are motivated by the following problem, a variation of (1.1.1):

$$\left\{ \begin{array}{rcll} (-\Delta)^s u & = & \lambda f(u) & \text{in} \quad \Omega\,, \\ u & = & 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega\,, \end{array} \right. \tag{1.1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $\lambda > 0$ is a real parameter and the function $f : [0, \infty) \longrightarrow \mathbb{R}$ satisfies

$$f \in C^1([0,\infty))\,,\ f \text{ is nondecreasing}, f(0) > 0, \text{ and } \lim_{t\to+\infty} \frac{f(t)}{t} = +\infty\,. \tag{1.1.3}$$

In this article we study (1.1.2) when $\Omega = B_1$.

It is well known (see [127]) that, for f satisfying (1.1.3), there exists a finite extremal parameter λ^* such that, if $0 < \lambda < \lambda^*$ then problem (1.1.2) admits a minimal classical solution u_λ which is stable —see (1.1.6) below—, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. The family $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ and its pointwise limit when $\lambda \nearrow \lambda^*$ is a weak solution of (1.1.2) with $\lambda = \lambda^*$. Such solution, denoted by u^*, is called *extremal solution* of (1.1.2). As in [127], we say that u is a weak solution of (1.1.2) when $u \in L^1(\Omega)$, $f(u)\delta^s \in L^1(\Omega)$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, and

$$\int_\Omega u(-\Delta)^s \zeta \, dx = \int_\Omega \lambda f(u)\zeta \, dx \tag{1.1.4}$$

for all ζ such that ζ and $(-\Delta)^s\zeta$ are bounded in Ω and $\zeta \equiv 0$ on $\partial\Omega$.

In the nineties, H. Brezis and J.L. Vázquez [25] raised the question of determining the regularity of u^* depending on the dimension n for the local version ($s = 1$) of (1.1.2) —see also the open problems raised by H. Brezis in [23]. This is equivalent to determine whether u^* is bounded or unbounded. There are several results in this direction for the classical problem (see Remark 1.1.4 for more details and also the monograph [81]).

Regarding the problem for the fractional Laplacian, there are fewer results concerning the regularity of stable solutions and in particular of the extremal solution of (1.1.2). This problem was first studied for the fractional Laplacian by X. Ros-Oton and J. Serra in [127]. There, the authors proved the existence of the family of minimal and stable solutions u_λ, as well as the existence of the extremal solution u^*. They also showed that if f is convex then u^* is bounded whenever $n < 4s$, and that if f is C^2 and $ff''/(f')^2$ has a limit at infinity, the same happens if $n < 10s$ (see Remark 1.1.4 for more comments on this). Later, X. Ros-Oton [124] improved this result in the case of the exponential nonlinearity $f(u) = e^u$, showing that u^* is bounded whenever $n \le 7$ for all $s \in (0,1)$. More precisely, the condition involving n and s that he found is the following:

$$\frac{\Gamma(\frac{n}{2})\Gamma(1+s)}{\Gamma(\frac{n-2s}{2})} > \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}\,. \tag{1.1.5}$$

In particular, for $s \gtrsim 0.63237\ldots$, u^* is bounded up to dimension $n = 9$. As explained in Remark 2.2 of [124], condition (1.1.5) is expected to be optimal, since if (1.1.5) does not hold, then $\log |x|^{-2s}$ is a singular extremal solution of the problem $(-\Delta)^s u = \lambda e^u$ in all $\mathbb{R}^n$. Nevertheless, this is still an open problem, since this last example is not our Dirichlet problem in a bounded domain.

To our knowledge, [127, 124] are the only papers where problem (1.1.2) is studied. However, the article by A. Capella, J. Dávila, L. Dupaigne, and Y. Sire [55] deals with a similar problem to (1.1.2) but for a different operator, the spectral fractional Laplacian A^s defined via the Dirichlet eigenvalues and eigenfunctions of the Laplace operator. It studies the problem of the extremal solution for the operator A^s in the unit ball and it establishes that, if $2 \le n < 2(s + 2 + \sqrt{2(s+1)})$, then $u^* \in L^\infty(B_1)$. In particular, u^* is bounded in dimensions $2 \le n \le 6$ for all $s \in (0,1)$. In the present work, we use similar ideas to the ones in [55] to study the same problem in B_1, but now with A^s replaced by the fractional Laplacian. We obtain the same condition on n and s guaranteeing regularity of the extremal solution to (1.1.2). Moreover, in the arguments of [55] there are two points where an estimate is missing and hence the result is not completely proved. In this paper we establish such estimate (given in Proposition 1.3.4) which is valid for the fractional Laplacian and also for the spectral fractional Laplacian. Hence, we complete the proofs of [55] (see the comment before Remark 1.1.4 and also Remarks 1.6.2 and 1.5.2).

The following is our main result, concerning the boundedness of the extremal solution.

Theorem 1.1.1. *Let $n \geq 2$, $s \in (0,1)$, and f be a C^2 function satisfying (1.1.3). Let u^* be the extremal solution of (1.1.2) with $\Omega = B_1$, the unit ball of $\mathbb{R}^n$. Then, u^* is radially symmetric and decreasing in $B_1 \setminus \{0\}$ and we have that:*

(i) If $n < 2\left(s+2+\sqrt{2(s+1)}\right)$, then $u^* \in L^\infty(B_1)$.

(ii) If $n \geq 2\left(s+2+\sqrt{2(s+1)}\right)$, then for every $\mu > n/2 - s - 1 - \sqrt{n-1}$, it holds $u^*(x) \leq C|x|^{-\mu}$ in B_1 for some constant $C > 0$.

As a consequence, u^* is bounded for all $s \in (0,1)$ whenever $2 \leq n \leq 6$. The same holds if $n = 7$ and $s \gtrsim 0.050510\ldots$, if $n = 8$ and $s \gtrsim 0.354248\ldots$, and if $n = 9$ and $s \gtrsim 0.671572\ldots$. Note that the assumption in (i) never holds for $n \geq 10$. In the limit $s \uparrow 1$, the condition on n in statement (i) corresponds to the optimal one for the local problem in the ball, that is $n < 10$ —see [35]. Instead, for powers $s < 1$, the hypothesis in (i) is not optimal: for the exponential nonlinearity $f(u) = e^u$ a better assumption is (1.1.5) —see [124].

Theorem 1.1.1 is a consequence of the stability of u^*. We say that a weak solution $u \in L^1(\Omega)$ of (1.1.2) is *stable* if

$$\lambda \int_\Omega f'(u)\xi^2 \, dx \leq [\xi]^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\xi|^2 \, dx \tag{1.1.6}$$

for all $\xi \in H^s(\mathbb{R}^n)$ such that $\xi \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Note that the integral in the left-hand side of (1.1.6) is well defined if f is nondecreasing, an assumption that we make throughout all the paper.

In case of problem (1.1.2), all the solutions u_λ with $\lambda < \lambda^*$, as well as the extremal solution, are stable. This property follows from their minimality. When $u \in H^s(\mathbb{R}^n)$, stability is equivalent to the nonnegativeness of the second variation of the energy associated to (1.1.2) at u.

The proof of Theorem 1.1.1 is based only on the stability of solutions. First, we show that bounded stable solutions are radially symmetric and monotone (see Section 1.4). Then, we use this, the stability condition and the equation to prove our estimates. This procedure is first applied to u_λ, with $\lambda < \lambda^*$, which are bounded stable solutions and thus regular enough, and we establish some estimates that are uniform in $\lambda < \lambda^*$. More precisely, they depend essentially on $||u_\lambda||_{L^1(\mathbb{R}^n)}$, a quantity that can be bounded independently of λ —see Remark 1.3.1 for more details about this fact. Once we have these uniform estimates, we can pass to the limit $\lambda \to \lambda^*$ and use monotone convergence to prove the result for u^*.

This result, Theorem 1.1.1, is a consequence of the following more general statement, which applies to the class of stable and radially decreasing H^s weak solutions —not necessarily bounded— to (1.1.1). Recall that our notion of weak solution is given in (1.1.4). Recall also (see Section 1.4) that positive bounded stable solutions to (1.1.1) will be shown to be radially decreasing in B_1.

Theorem 1.1.2. *Let $n \geq 2$, $s \in (0,1)$, and f be a C^2 nondecreasing function. Let $u \in H^s(\mathbb{R}^n)$ be a stable radially decreasing weak solution to (1.1.1). We have that:*

(i) If $n < 2\left(s+2+\sqrt{2(s+1)}\right)$, then $u \in L^\infty(B_1)$. Moreover,

$$||u||_{L^\infty(B_1)} \leq C$$

for some constant C that depends only on n, s, f and $||u||_{L^1(\mathbb{R}^n)}$.

(ii) If $n \geq 2\left(s+2+\sqrt{2(s+1)}\right)$, then for every $\mu > n/2 - s - 1 - \sqrt{n-1}$, it holds

$$u(x) \leq C|x|^{-\mu} \quad \text{in } B_1\,,$$

for some constant C that depends only on n, s, μ, f and $||u||_{L^1(\mathbb{R}^n)}$.

A main tool used in the present article is the extension problem for the fractional Laplacian, due to L. Caffarelli and L. Silvestre [51]. Namely, for $s \in (0,1)$ and given a function $u : \mathbb{R}^n \to \mathbb{R}$, consider v the solution of

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ v = u & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n\,, \end{cases} \tag{1.1.7}$$

where $a = 1-2s$ and $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} \,:\, x \in \mathbb{R}^n,\ y \in (0,+\infty)\}$. As it is well known (see [51]), the limit $-\lim_{y\downarrow 0} y^a \partial_y v$ agrees with $(-\Delta)^s u$ up to a positive multiplicative constant. We will refer to the solution of (1.1.7), v, as the *s-harmonic extension* of u. This terminology is motivated by the fact that, when $s = 1/2$, then $a = 0$ and v is the harmonic extension of u.

Throughout the paper, (x,y) denote points in $\mathbb{R}^n \times (0,+\infty) = \mathbb{R}^{n+1}_+$. We also use the notation

$$\frac{\partial v}{\partial \nu^a} = -\lim_{y\downarrow 0} y^a v_y$$

for the conormal exterior derivative and we will always assume the relation

$$a = 1-2s \in (-1,1)\,.$$

Moreover, we denote by

$$\rho = |x| \quad \text{and } r = \sqrt{\rho^2 + y^2}$$

the modulus in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_+$, respectively. Therefore, v_ρ will denote the derivative of v in the horizontal radial direction, that is

$$v_\rho(x,y) = \frac{x}{\rho} \cdot \nabla_x v(x,y) \quad \text{with } \rho = |x|.$$

We will always use the letter u to denote a function defined in $\mathbb{R}^n$ and the letter v for its s-harmonic extension in $\mathbb{R}^{n+1}_+$.

In [55], the authors use also an extension problem for the spectral operator A^s. Indeed, one can see that the spectral fractional Laplacian can be realized as the boundary Neumann operator of a suitable extension in the half-cylinder $\Omega \times (0, +\infty)$. More precisely, one considers the extension problem

$$\begin{cases} \operatorname{div}(y^a \nabla w) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\ w &= 0 \quad \text{on } \partial\Omega \times [0, +\infty), \\ w &= u \quad \text{on } \Omega \times \{0\}, \end{cases}$$

with $a = 1 - 2s$. Then, it can be proven that $-\lim_{y \downarrow 0} y^a w_y$ agrees with $A^s u$ up to a multiplicative constant. Notice that the solution w (extended by 0 to all $\overline{\mathbb{R}^{n+1}_+}$) is a subsolution of (1.1.7) and thus, thanks to the maximum principle, one can use the Poisson formula for (1.1.7) to obtain estimates for w. This is what is done in [55] and suggested that similar arguments could be carried out for the fractional Laplacian, as we indeed do.

The proof of Theorem 1.1.2 is mostly based on two ideas. First, by the representation formula for the fractional Laplacian, we see that the L^∞ norm of a solution u can be bounded by the integral over B_1 of $f(u)/|x|^{n-2s}$ (see Lemma 1.2.2). Thus, it remains to estimate this integral. We bound it in $B_1 \setminus B_{1/2}$ using that the solution is radially decreasing (see Section 1.4). Regarding the integral in $B_{1/2}$, we can relate it with

$$\int_{B_{1/2} \times (0,1)} y^a r^{-(n+2-2s)} \rho v_\rho \, dx \, dy + \int_{B_{1/2} \times (0,1)} y^a r^{-(n+2-2s)} y v_y \, dx \, dy,$$

after an integration by parts in $B_{1/2} \times (0,1) \subset \mathbb{R}^{n+1}_+$ and seeing $f(u)$ as the flux $d_s \partial_{\nu^a} v$ —the other boundary terms are estimated using the results of Section 1.3. On the one hand, the integral involving v_y can be absorbed in the left-hand side of the estimates by using the identity given in Lemma 1.6.1 (see Section 1.6 for the details). On the other hand, the integral involving v_ρ can be estimated, after using the Cauchy-Schwarz inequality, thanks to the next key proposition. It provides an estimate for a weighted Dirichlet integral involving the s-harmonic extension of stable solutions to (1.1.1).

Proposition 1.1.3. *Let $n \geq 2$, $s \in (0,1)$, and f be a nondecreasing C^2 function. Let $u \in H^s(\mathbb{R}^n)$ be a stable radially decreasing solution of* (1.1.1) *and v be its s-harmonic extension as in* (1.1.7)*. Assume that α is any real number satisfying*

$$1 \leq \alpha < 1 + \sqrt{n-1}. \tag{1.1.8}$$

Then,

$$\int_0^\infty \int_{B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha} \, dx \, dy \leq C, \tag{1.1.9}$$

for a positive constant C depending only on n, s, α, $||u||_{L^1(B_1)}$, $||f'(u)||_{L^\infty(B_{7/8} \setminus B_{3/8})}$, and $||u||_{L^\infty(B_{7/8} \setminus B_{3/8})}$.

The key point to establish Proposition 1.1.3 —as well as its analogous in [55]— is the particular choice of the test function ξ in the stability condition (1.2.5), which is equivalent to (1.1.6) when considering the extension to $\mathbb{R}^{n+1}_+$ of functions defined in $\mathbb{R}^n$. We take

$$\xi = \rho^{1-\alpha} v_\rho \zeta\,, \tag{1.1.10}$$

where α satisfies (1.1.8), v_ρ is the horizontal radial derivative of v, and ζ is a cut-off function. This choice, after controlling a number of integrals, will lead to (1.1.9). A similar idea was already used by X. Cabré and A. Capella in [35] to prove the boundedness of u^* in the radial case for the classical Laplacian, and later by A. Capella, J. Dávila, L. Dupaigne and Y. Sire in [55] for A^s.

Furthermore, another important ingredient in order to establish Theorem 1.1.2 and Proposition 1.1.3 is a crucial estimate for the s-harmonic extension of solutions to (1.1.1). In Proposition 1.3.4 we establish such estimate, whose proof was missing in [55], as mentioned before. It controls pointwise the horizontal gradient of v, where v is the s-harmonic extension of u, in a cylindrical annulus about the origin.

Remark 1.1.4. The local version of problem (1.1.2) was first studied in the seventies and eighties, essentially for the exponential and power nonlinearities. When $f(u) = e^u$, it is known that $u^* \in L^\infty(\Omega)$ if $n \le 9$ (see [68]), while $u^*(x) = \log|x|^{-2}$ when $\Omega = B_1$ and $n \ge 10$ (see [107]). Similar results hold for $f(u) = (1+u)^p$, and also for functions f satisfying a limit condition at infinity (see [130]). This is extended to the nonlocal case in [127], where the condition $n \le 9$ becomes now $n < 10s$.

For the local case and general nonlinearities, the first result concerning the boundedness of the extremal solution was obtained by G. Nedev [121], who proved that u^* is bounded in dimensions $n \le 3$ whenever f is convex. The result in [127] for $n < 4s$ also extends this to the nonlocal setting.

Later, X. Cabré and A. Capella [35] obtained an L^∞ bound for u^*, when $s = 1$ and $\Omega = B_1$, whenever $n \le 9$. The best known result at the moment for general f and $s = 1$ is due to X. Cabré [33], and states that in dimensions $n \le 4$ the extremal solution is bounded for every convex domain Ω. This result was extended by S. Villegas [149] to nonconvex domains. Nevertheless, the problem is still open in dimensions $5 \le n \le 9$.

As mentioned before, to our knowledge the only articles dealing with problem (1.1.2) are [127] and [124]. There, the authors work in $\mathbb{R}^n$ and do not use the extension problem for the fractional Laplacian. For this reason, we include in the appendix of this article an alternative proof —which uses the extension problem— of the result of X. Ros-Oton and J. Serra [127] that establishes the boundedness of the extremal solution in dimensions $n < 10s$ in any domain when $f(u) = e^u$. This is Proposition 1.7.1 below.

The paper is organized as follows. Section 1.2 is devoted to recall some results concerning the extension problem for the fractional Laplacian, as well as to express the stability condition using the extension problem. In Section 1.3, we establish some preliminary results which are used in the following sections. Section 1.4 focuses on the symmetry and monotonicity of bounded stable solutions. Proposition 1.1.3 is proved in Section 1.5, and Theorem 1.1.2 in Section 1.6. Finally, in Appendix 1.7 we give an alternative proof of the result of [127] concerning the exponential nonlinearity.

1.2 The extension problem for the fractional Laplacian

In this section we recall briefly some results concerning the extension problem for the fractional Laplacian. The main feature is the following well known relation: if v is the solution of the extension problem (1.1.7), then

$$(-\Delta)^s u = (-\Delta)^s\{v(\cdot,0)\} = d_s \frac{\partial v}{\partial \nu^a}\,, \tag{1.2.1}$$

for a positive constant d_s which only depends on s.

Hence, given $s \in (0,1)$, a function u defined in $\mathbb{R}^n$ is a solution of $(-\Delta)^s u = h$ in $\mathbb{R}^n$ if, and only if, its s-harmonic extension in $\mathbb{R}^{n+1}_+$ solves the problem

$$\begin{cases} \operatorname{div}(y^a \nabla v) &= 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ \dfrac{\partial v}{\partial \nu^a} &= \dfrac{h}{d_s} & \text{on } \mathbb{R}^n\,. \end{cases} \tag{1.2.2}$$

Recall that for problem (1.1.7) we have an explicit Poisson formula:

$$v(x,y) = P * u = \int_{\mathbb{R}^n} P(x-z,y)u(z)\,\mathrm{d}z\,, \quad \text{where} \quad P(x,y) = P_{n,a}\frac{y^{1-a}}{(|x|^2+y^2)^{\frac{n+1-a}{2}}}$$

and the constant $P_{n,s}$ is such that, for every $y > 0$, $\int_{\mathbb{R}^n} P(x,y)\,\mathrm{d}x = 1\,.$

The relation between v and $-y^a v_y$ via a conjugate equation gives a useful formula for the y-derivative of the solution of (1.2.2).

Lemma 1.2.1 (see [51])**.** *Let $s \in (0,1)$, $a = 1-2s$, $h : \mathbb{R}^n \to \mathbb{R}$ and v be the solution of* (1.2.2)*. Then,*

$$-v_y(x,y) = \Gamma * \frac{h}{d_s} = \frac{1}{d_s}\int_{\mathbb{R}^n} \Gamma(x-z,y)h(z)\,\mathrm{d}z\,.$$

where

$$\Gamma(x,y) = \Gamma_{n,s}\frac{y}{(|x|^2+y^2)^{\frac{n+1+a}{2}}} = \Gamma_{n,s}\frac{y}{(|x|^2+y^2)^{\frac{n+2-2s}{2}}}\,,$$

with a constant $\Gamma_{n,s}$ depending only on n and s.

This is proved by considering the function $w = -y^a v_y$. A simple computation shows that w solves the conjugate problem

$$\begin{cases} \operatorname{div}(y^{-a} \nabla w) &= 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ w &= h/d_s & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n\,. \end{cases}$$

Then, we use the Poisson formula for this problem to obtain

$$-y^a v_y = w = \frac{y^{1+a}}{d_s}\int_{\mathbb{R}^n} P_{n,-a}\frac{h(z)}{(|x-z|^2+y^2)^{\frac{n+1+a}{2}}}\,\mathrm{d}z\,.$$

Recall also that the fundamental solution of the fractional Laplacian is well known. Namely, given $h : \mathbb{R}^n \to \mathbb{R}$ regular enough (for instance h continuous with compact support), the unique continuous and bounded solution of $(-\Delta)^s u = h$ in $\mathbb{R}^n$ is given by

$$u(x) = C\int_{\mathbb{R}^n} \frac{h(z)}{|x-z|^{n-2s}}\,\mathrm{d}z\,,$$

for a constant C which depends only on n and s (see [51, 40]). Using this last formula and the maximum principle, we easily deduce a useful pointwise bound for solutions of the Dirichlet problem for the fractional Laplacian. It is given by the following lemma.

Lemma 1.2.2. *Let $\Omega \subset \mathbb{R}^n$ a bounded smooth domain, $s \in (0,1)$ and $h : \Omega \to \mathbb{R}$ a nonnegative bounded function. Let $u \in H^s(\mathbb{R}^n)$ be a weak solution of the Dirichlet problem*

$$\begin{cases} (-\Delta)^s u &= h \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Then, for every $x \in \mathbb{R}^n$,

$$u(x) \le C \int_\Omega \frac{h(z)}{|x-z|^{n-2s}}\, \mathrm{d}z\,, \tag{1.2.3}$$

for a constant C which depends only on n and s.

This result is the analogous of Lemma 6.1 in [55] and is the first step in order to prove Theorem 1.1.2. Indeed, we will estimate the L^∞ norm of a solution by controlling the right-hand side of (1.2.3), which can be related to the Dirichlet integral in (1.1.9) through an integration by parts (see Section 1.6).

As mentioned in the introduction, the main property in which are based our estimates is stability. Recall that a solution of (1.1.2) is stable if it satisfies (1.1.6). Since we want to work with the s-harmonic extension of such solutions, we need to rewrite the stability condition (1.1.6) in terms of the extension of functions in $\mathbb{R}^{n+1}_+$.

It is well known that the space $H^s(\mathbb{R}^n)$ coincides with the trace of $H^1(\mathbb{R}^{n+1}_+, y^a)$ on $\partial\mathbb{R}^{n+1}_+$ (see for instance [92]). In particular, every function $\xi : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ such that $\xi \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^a)$ and $\nabla\xi \in L^2(\mathbb{R}^{n+1}_+, y^a)$ has a trace on $H^s(\mathbb{R}^n)$ and satisfies the following inequality (see Proposition 3.6 in [92]):

$$[\mathrm{tr}\,\xi]_{H^s(\mathbb{R}^n)} \le d_s\, [\xi]_{H^1(\mathbb{R}^{n+1}_+, y^a)}\,, \tag{1.2.4}$$

where we use the notation

$$[\varphi]_{H^1(\mathbb{R}^{n+1}_+, y^a)} = \int_{\mathbb{R}^{n+1}_+} y^a |\nabla\varphi|^2\, \mathrm{d}x\, \mathrm{d}y$$

and d_s is the constant appearing in (1.2.1). In addition, d_s is the optimal constant in (1.2.4), as seen next.

To show why d_s is the optimal constant, we find a case where the equality is attained. Consider $w \in H^s(\mathbb{R}^n)$ and let W denote the solution of

$$\begin{cases} \mathrm{div}(y^a \nabla W) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+\,, \\ W &= w \quad \text{on } \mathbb{R}^n\,. \end{cases}$$

Notice that W minimizes the seminorm $[\cdot]_{H^1(\mathbb{R}^{n+1}_+, y^a)}$ among all functions whose trace on $\mathbb{R}^n$ is w, because it solves the Euler-Lagrange equation of the functional

$$E(\varphi) = \int_{\mathbb{R}^{n+1}_+} y^a |\nabla\varphi|^2\, \mathrm{d}x\, \mathrm{d}y\,.$$

Therefore, integrating by parts and using that $d_s \dfrac{\partial W}{\partial \nu^a} = (-\Delta)^s w$ at $\mathbb{R}^n$, we have

$$\begin{aligned} d_s\,[W]_{H^1(\mathbb{R}^{n+1}_+,y^a)} &= d_s \int_{\mathbb{R}^{n+1}_+} y^a |\nabla W|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= d_s \int_{\mathbb{R}^n} w \frac{\partial W}{\partial \nu^a}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} w(-\Delta)^s w \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} (-\Delta)^{s/2} w (-\Delta)^{s/2} w \,\mathrm{d}x \\ &= [w]_{H^s(\mathbb{R}^n)}\,. \end{aligned}$$

This shows that the optimal constant in (1.2.4) is d_s and that the equality is achieved when one takes the s-harmonic extension of a function defined in $\mathbb{R}^n$.

Using (1.2.4), we say that u is a stable solution to

$$\left\{ \begin{array}{rcll} (-\Delta)^s u &=& f(u) & \text{in} \quad \Omega\,, \\ u &=& 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega\,. \end{array} \right.$$

if

$$\int_\Omega f'(u)\xi^2 \,\mathrm{d}x \le d_s \int_{\mathbb{R}^{n+1}_+} y^a |\nabla \xi|^2 \,\mathrm{d}x\,\mathrm{d}y\,, \tag{1.2.5}$$

for every $\xi \in H^1(\mathbb{R}^{n+1}_+, y^a)$ such that its trace has compact support in Ω. Notice that it is not necessary to take ξ as the s-harmonic extension of its trace (that is, ξ need not solve $\operatorname{div}(y^a \nabla \xi) = 0$ in $\mathbb{R}^{n+1}_+$). This gives us more flexibility for the choice of functions in the stability condition. However, if we want an inequality completely equivalent to (1.1.6) —in the sense that we do not lose anything when going to $\mathbb{R}^{n+1}_+$—, we need to consider always test functions solving $\operatorname{div}(y^a \nabla \xi) = 0$ in $\mathbb{R}^{n+1}_+$.

1.3 Preliminary results: estimates for solutions of (1.1.1)

The purpose of this section is to provide some estimates for solutions of (1.1.1) that will be used in the subsequent sections. In particular, we give estimates for the derivatives of the s-harmonic extension of solutions to (1.1.1).

The three main estimates of this section are stated below. The first two results, Lemma 1.3.2 and Proposition 1.3.3, concern the decay at infinity of ∇v, where v solves $\operatorname{div}(y^a \nabla v) = 0$ in $\mathbb{R}^{n+1}_+$. We control the decay at infinity since we deal with integrals in $\mathbb{R}^{n+1}_+$ weighted by y^a, with $a \in (-1,1)$, and y^a is not integrable at infinity. In [55], the authors use that the extension of solutions for the spectral fractional Laplacian, as well as their derivatives, have exponential decay as $y \to +\infty$. This allows them to overcome the problem of integrability at infinity. Instead, in the case of the fractional Laplacian, such exponential decay does not hold. Nevertheless, we establish a power decay in Lemma 1.3.2 and in Proposition 1.3.3, and this will be enough for our purposes. The estimates we deduce in these two results are in terms of u, the trace of v on $\mathbb{R}^n$, but we do not assume that u solves any equation in $\mathbb{R}^n$. On the contrary, the third result of this section, Proposition 1.3.4, is an estimate up to $\{y = 0\}$ and in this case we assume that u is a solution to (1.1.1).

Before presenting the three results of this section, let us make a comment on the right-hand sides of the estimates that we establish. We point out that the constants appearing in the statement of Lemma 1.3.2 depend on $||u||_{L^\infty(B_1)}$ instead of $||u||_{L^1(B_1)}$, in contrast with the other two main estimates of this section (Propositions 1.3.3 and 1.3.4). This will cause no problem since the lemma will be used only in Section 1.4, where we will assume that $u \in L^\infty(B_1)$, to show that certain boundary terms go to zero as $r \to \infty$. Therefore, the specific dependence of the constants is not relevant as long as they are finite. Instead, for the terms that remain through the estimates, it is important to have dependency only on the L^1 norm of the solution —since weak solutions are only assumed to be in $L^1(B_1)$, and since for problem (1.1.2) the L^1 norm of u_λ, with $\lambda < \lambda^*$, is bounded uniformly in λ, as explained next.

Remark 1.3.1. When one considers stable solutions u_λ of (1.1.2) in general domains Ω, the only available estimate that is uniform in λ is the following:

$$||u_\lambda||_{L^1(\Omega)} \le ||u^*||_{L^1(\Omega)} \quad \text{for all } \lambda < \lambda^* .$$

Indeed, a simple argument shows that $||u_\lambda||_{L^1(\Omega)}$ is uniformly bounded as $\lambda \uparrow \lambda^*$. Then, it follows that u^* is a weak solution of (1.1.2), i.e., belonging to $L^1(\Omega)$ (see [127] for the details). In the case $\Omega = B_1$, the solutions u_λ are radially decreasing (see Section 1.4). Hence, the L^∞ norm of u_λ in sets that are away from the origin is also bounded independently of λ, since in those sets it can be controlled by the L^1 norm of u^*. We have indeed

$$||u_\lambda||_{L^\infty(B_1\setminus\overline{B_R})} \le \frac{C}{R^n}\,||u_\lambda||_{L^1(B_1)} \le \frac{C}{R^n}\,||u^*||_{L^1(B_1)} \quad \text{for every } R \in (0,1) \text{ and } \lambda < \lambda^* .$$

In fact, if $u \in L^1(B_1)$ is a weak solution of (1.1.1) that is radially decreasing, automatically $u \in L^\infty_{\rm loc}(B_1 \setminus \{0\})$. Then, by interior estimates for the fractional Laplacian (see Corollaries 2.3 and 2.5 in [126]), u is, in $B_1 \setminus \{0\}$, at least as regular as the nonlinearity f. Since in this paper we assume $f \in C^2$, then we have $u \in C^{2,\alpha}_{\rm loc}(B_1 \setminus \{0\})$ for some $\alpha > 0$. The hypothesis on f can be slightly weakened depending on s, as it is explained in Remark 1.5.4.

The following is the first result of this section (recall that we use the notation $r = |(x,y)|$).

Lemma 1.3.2. *Let $u \in C^1(B_1) \cap L^\infty(B_1)$ be such that $u \equiv 0$ in $\mathbb{R}^n \setminus B_1$, and let v be its s-harmonic extension. Then, we have the following estimates:*

$$|v_{x_i}(x,y)| \le C\frac{y^{2s}}{r^{n+1+2s}} \quad \textit{for } |x| > 2,\ y > 0\,, \tag{1.3.1}$$

for $i = 1, \ldots, n$, and

$$|v_y(x,y)| \le C\frac{y^{2s-1}}{r^{n+2s}} \quad \textit{for } |x| > 2,\ y > 0\,, \tag{1.3.2}$$

for some constants C depending only on n, s and $||u||_{L^\infty(B_1)}$.

The second result of this section also deals with the decay of ∇v as $y \to +\infty$. The main difference with the previous one is that we establish an estimate that does not depend on the L^∞ norm of the solution, only on its L^1 norm. Therefore, it holds not only for bounded solutions but also for weak solutions —recall (1.1.4). As we will see, the result follows from an argument in the proof of Proposition 4.6 in [40], and is the following.

Proposition 1.3.3. *Let $s \in (0,1)$ and $a = 1-2s$. Let $v \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^a)$ satisfy $\nabla v \in L^2(\mathbb{R}^{n+1}_+, y^a)$ and $\operatorname{div}(y^a \nabla v) = 0$ in $\mathbb{R}^{n+1}_+$. Let u be its trace on $\mathbb{R}^n$. Then,*

$$|\nabla v(x,y)| \leq \frac{C}{y^{n+1}} ||u||_{L^1(\mathbb{R}^n)} \quad \textit{for every } y > 0 \tag{1.3.3}$$

and for a constant C depending only on n and s.

The third result we present is new and important. It provides an estimate for the horizontal gradient in the set $(B_{3/4} \setminus \overline{B_{1/2}}) \times (0,1)$. As it is commented in Remark 1.3.8, this gradient estimate is also valid for the problem studied in [55] for the operator A^s. Therefore, it can be used in the arguments of [55] in order to complete their proofs at the points where an estimate of this kind is missing (see Remarks 1.5.2 and 1.6.2).

Proposition 1.3.4. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$ be a radially decreasing weak solution of (1.1.1), with $f \in C^2$. Let v be the s-harmonic extension of u given by (1.1.7) and*

$$A := (B_{3/4} \setminus \overline{B_{1/2}}) \times (0,1) \subset \mathbb{R}^{n+1}_+ .$$

Then,

$$||\nabla_x v||_{L^\infty(A)} \leq C \tag{1.3.4}$$

for some constant C depending only on n, s, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8} \setminus B_{3/8})}$, and $||f'(u)||_{L^\infty(B_{7/8} \setminus B_{3/8})}$.

The rest of this section is devoted to prove Lemma 1.3.2, Proposition 1.3.3 and Proposition 1.3.4. We start with the proof of the first lemma, which only relies on the Poisson formula for the s-harmonic extension of u.

Proof of Lemma 1.3.2. Since u has compact support in $\overline{B_1}$, by the Poisson formula we have

$$v(x,y) = P * u = P_{n,s} \int_{B_1} \frac{y^{2s}}{(|x-z|^2 + y^2)^{\frac{n+2s}{2}}} u(z)\, \mathrm{d}z .$$

If we differentiate the previous expression with respect to x_i, $i = 1, \ldots, n$, we get

$$|v_{x_i}| \leq C \, ||u||_{L^\infty(B_1)} \, y^{2s} \int_{B_1} \frac{|x_i - z_i|}{(|x-z|^2 + y^2)^{\frac{n+2+2s}{2}}} \, \mathrm{d}z .$$

Now, on the one hand we use that $|x| > 2$ to see that

$$|x_i - z_i| \leq |x| + 1 \leq 2|x| \leq 2r .$$

On the other hand,

$$\frac{1}{(|x-z|^2 + y^2)^{\frac{n+2+2s}{2}}} \leq \frac{1}{((|x|-1)^2 + y^2)^{\frac{n+2+2s}{2}}} \leq \frac{4^{\frac{n+2+2s}{2}}}{(|x|^2 + y^2)^{\frac{n+2+2s}{2}}} = \frac{C}{r^{n+2+2s}} ,$$

where in the first inequality we have used that $|x - z| \geq |x| - 1$ and in the second one, that $4|x-1|^2 \geq |x|^2$ if $|x| > 2$. Combining all this we get the estimate (1.3.1).

The proof for v_y is completely analogous. □

We deal now with estimates for weak solutions. We start with the proof of Proposition 1.3.3, establishing a gradient estimate for v (the s-harmonic extension of u) in sets which are far from $y = 0$. To establish it we follow the ideas of Proposition 4.6 of [40], but with a careful look on the right-hand side of the estimates.

Proof of Proposition 1.3.3. Let $(x_0, y_0) \in \mathbb{R}^{n+1}_+$ with $y_0 > 0$ and note that v satisfies the equation $\operatorname{div}(y^a \nabla v) = 0$ in $B_{y_0/2}(x_0, y_0)$. We perform the scaling $\overline{v}(\overline{x}, \overline{y}) = v(x_0 + y_0\overline{x}, y_0\overline{y})$ and then $\overline{v}$ satisfies $\operatorname{div}(\overline{y}^a \nabla \overline{v}) = 0$ in $B_{1/2}(0, 1)$. Since $\overline{y} \in (1/2, 3/2)$ in this ball, $\overline{v}$ satisfies a uniformly elliptic equation and we can use classical interior estimates for the gradient (see [98], Corollary 6.3) to obtain

$$||\nabla \overline{v}||_{L^\infty(B_{1/4}(0,1))} \leq C ||\overline{v}||_{L^\infty(B_{1/2}(0,1))} ,$$

for a constant C depending only on n. Undoing the scaling we have

$$|\nabla v(x_0, y_0)| \leq \frac{1}{y_0} ||\nabla \overline{v}||_{L^\infty(B_{1/4}(0,1))} \leq \frac{C}{y_0} ||\overline{v}||_{L^\infty(B_{1/2}(0,1))} = \frac{C}{y_0} ||v||_{L^\infty(B_{y_0/2}(x_0,y_0))} .$$

Finally, we estimate $||v||_{L^\infty(B_{y_0/2}(x_0,y_0))}$. Recall that $v = P * u$ and we can bound $P(x, y)$ by $P_{n,s}/y^n$ for every $y > 0$. Then,

$$|v(x,y)| \leq \int_{\mathbb{R}^n} P(x - z, y) |u(z)| \, dz \leq \frac{P_{n,s}}{y^n} \int_{\mathbb{R}^n} |u(z)| \, dz = \frac{P_{n,s}}{y^n} ||u||_{L^1(\mathbb{R}^n)} \quad \text{for every } y > 0 .$$

Combining this with the previous estimate, we get (1.3.3). □

The estimate given by Proposition 1.3.3 is useful to bound quantities far from $\{y = 0\}$. However, in the proofs of Proposition 1.1.3 and Theorem 1.1.2 we also need to bound quantities up to $\{y = 0\}$. This is done thanks to Proposition 1.3.4. To prove it we need two preliminary results, which are estimates in half-balls of $\mathbb{R}^{n+1}_+$. Regarding such sets, we use the notation

$$B_R^+ = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ \ : \ |(x,y)| < R \right\},$$

$$\Gamma_R^0 = \left\{ (x,0) \in \partial\mathbb{R}^{n+1}_+ \ : \ |x| < R \right\}.$$

We also write $B_R^+(x_0)$ and $\Gamma_R^0(x_0)$ in order to denote that the center of the balls is $(x_0, 0)$ and not the origin.

The first lemma we need is the following. It is used to bound the L^∞ norm of v in a half-ball B_R^+ by some quantities that only refer to the trace of v on $\mathbb{R}^n$, u.

Lemma 1.3.5. *Let $s \in (0, 1)$ and $u \in L^1(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Let v be the s-harmonic extension of u given by (1.1.7). Then,*

$$||v||_{L^\infty(B_R^+)} \leq C \left(||u||_{L^\infty(\Gamma_{2R}^0)} + ||u||_{L^1(\mathbb{R}^n)} \right) , \tag{1.3.5}$$

where C is a constant depending only on n, s and R.

Proof. Let $(x, y) \in B_R^+$. By the Poisson formula,

$$v(x,y) = \int_{\mathbb{R}^n} P(x - z, y) u(z) \, dz , \quad \text{where} \quad P(x,y) = P_{n,s} \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{n+2s}{2}}} .$$

Now, we split the integral into two parts:

$$\int_{\mathbb{R}^n} P(x-z,y)u(z)\,dz = \int_{\Gamma^0_{2R}} P(x-z,y)u(z)\,dz + \int_{\mathbb{R}^n\setminus\Gamma^0_{2R}} P(x-z,y)u(z)\,dz\,.$$

For the first term we find the estimate

$$\int_{\Gamma^0_{2R}} P(x-z,y)u(z)\,dz \le ||u||_{L^\infty(\Gamma^0_{2R})} \int_{\mathbb{R}^n} P(x-z,y)\,dz = ||u||_{L^\infty(\Gamma^0_{2R})}\,,$$

where we have used that $P(x,y)$ is positive and for all $y>0$ it integrates 1 in $\mathbb{R}^n$. For the second term, note that since $|x|<R$ and $|z|\ge 2R$, $|x-z|\ge R$ and therefore

$$\frac{y^{2s}}{(|x-z|^2+y^2)^{\frac{n+2s}{2}}} \le \frac{y^{2s}}{(R^2+y^2)^{\frac{n+2s}{2}}}\,.$$

Hence, for $(x,y)\in B_R^+$,

$$\int_{\mathbb{R}^n\setminus\Gamma^0_{2R}} P(x-z,y)u(z)\,dz \le \int_{\mathbb{R}^n\setminus\Gamma^0_{2R}} P_{n,s}\frac{y^{2s}}{(R^2+y^2)^{\frac{n+2s}{2}}}u(z)\,dz \le C\,||u||_{L^1(\mathbb{R}^n)}\,,$$

where C is a constant depending only on n, s and R. □

The second lemma we need in order to prove Proposition 1.3.4 is a Harnack inequality:

Lemma 1.3.6 (Lemma 4.9 of [40]). *Let $a\in(-1,1)$ and $\varphi\in H^1(B^+_{4R},y^a)$ be a nonnegative weak solution of*

$$\begin{cases} \operatorname{div}(y^a\nabla\varphi) &= 0 \quad \text{in } B^+_{4R}\,,\\ \dfrac{\partial\varphi}{\partial\nu^a}+d(x)\varphi &= 0 \quad \text{in } \Gamma^0_{4R}\,,\end{cases}$$

where d is a bounded function in Γ^0_{4R}. Then,

$$\sup_{B_R^+}\varphi \le C\inf_{B_R^+}\varphi\,, \tag{1.3.6}$$

for some constant C depending only on n, a and $R^{1-a}\,||d||_{L^\infty(\Gamma^0_{4R})}$.

Remark 1.3.7. Since the operator $\operatorname{div}(y^a\nabla\cdot)$ is invariant under translations in the x variable, the two previous results also hold for half-balls not necessarily centered at the origin.

Once we have the two previous lemmas, we can establish Proposition 1.3.4:

Proof of Proposition 1.3.4. We first claim that, for $x_0\in\{x\in\mathbb{R}^n\,:\,1/2\le|x|\le 3/4\}$, we have

$$||\nabla_x v||_{L^\infty(B^+_{1/32}(x_0))} \le C\,, \tag{1.3.7}$$

with C depending only on n, s, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(\Gamma^0_{1/8}(x_0))}$ and $||f'(u)||_{L^\infty(\Gamma^0_{1/8}(x_0))}$. Assuming that the claim is true we complete the proof. First, we use a standard covering argument to deduce

$$||\nabla_x v||_{L^\infty((\Gamma^0_{3/4}\setminus\overline{\Gamma^0_{1/2}})\times(0,1/32))} \le C\,,$$

with a constant C depending on the same quantities as the previous one. Then, we use Proposition 1.3.3 to bound $\nabla_x v$ in $(\Gamma^0_{3/4} \setminus \overline{\Gamma^0_{1/2}}) \times (1/32, 1)$. Combining these last two estimates we deduce (1.3.4).

Let us show (1.3.7). By the radial symmetry of the domain, it is enough to prove the estimate for a point x_0 of the form $x_0 = (c, c, \ldots, c)$ with c such that $1/2 \le |x_0| \le 3/4$. Under these assumptions, the ball $\Gamma^0_{1/8}(x_0)$ is inside the first orthant of $\mathbb{R}^n$, i.e., $\Gamma^0_{1/8}(x_0) \subset \{x_i \ge 0,\ i = 1, \ldots, n\}$, and there we have $u_{x_i} < 0$ for all $i = 1, \ldots, n$ (and the same happens for v_{x_i}). Since the equation that v satisfies is invariant under translations in the x variable, we can assume from now on that the ball is centered at the origin, so we write just $B^+_{1/8}$.

Then, differentiating the equation that v satisfies in $B^+_{1/8}$, for all $i = 1, \ldots, n$ we get

$$\begin{cases} \operatorname{div}(y^a \nabla v_{x_i}) &= 0 \quad \text{in } B^+_{1/8}\,, \\ \dfrac{\partial v_{x_i}}{\partial \nu^a} - f'(u) v_{x_i} &= 0 \quad \text{in } \Gamma^0_{1/8}\,. \end{cases}$$

At this point we use Lemma 1.3.6 with $\varphi = -v_{x_i} \ge 0$ and $d = -f'(u)$, obtaining

$$\sup_{B^+_{1/32}} -v_{x_i} \le C \inf_{B^+_{1/32}} -v_{x_i}\,,$$

with a constant C depending only on n, s and $||f'(u)||_{L^\infty(\Gamma^0_{1/8})}$.

We bound $\inf_{B^+_{1/32}} -v_{x_i}$ by $C\,||v||_{L^\infty(B^+_{1/32})}$ with a constant C depending only on n. To see this, we use integration by parts:

$$\inf_{B^+_{1/32}} -v_{x_i} \le ||v_{x_i}||_{L^1(B^+_{1/32})} = \left| \int_{B^+_{1/32}} v_{x_i}\,dx \right| = \left| \int_{\partial(B^+_{1/32})} v \nu_i \,d\sigma \right| \le C\,||v||_{L^\infty(B^+_{1/32})}\,.$$

Finally, (1.3.7) is obtained by estimating $||v||_{L^\infty(B^+_{1/32})}$ in terms of u, the trace of v on $\mathbb{R}^n$, using Lemma 1.3.5. □

Obviously, in the definition of the set A of Proposition 1.3.4 we can replace $B_{3/4} \setminus \overline{B_{1/2}}$ by every other annuli $B_L \setminus \overline{B_l}$ with $0 < l < L < 1$, and we can also replace $(0, 1)$ by any other open interval. Then, estimate (1.3.4) also holds for $A = (B_L \setminus \overline{B_l}) \times (0, T)$ with a different constant C which depends on l and L.

Remark 1.3.8. Let $u \ge 0$ be a bounded solution of (1.1.1) and let $w \ge 0$ be a bounded solution of the problem

$$\begin{cases} A^s w &= f(w) \quad \text{in} \quad B_1\,, \\ w &= 0 \qquad \text{on} \quad \partial B_1\,. \end{cases}$$

Then, in the half-ball $B^+_{1/8}$ (or in every half-ball with base strictly contained in B_1), both u and w satisfy the same degenerate elliptic problem. Therefore, the proof of Proposition 1.3.4 can be applied without any change to w. Thus, we obtain an estimate for $\nabla_x w$ that can be used in the arguments of [55] in order to complete the proof of their main theorem (see Remarks 1.5.2 and 1.6.2).

1.4 Radial symmetry and monotonicity of stable solutions

In this section we establish the radial symmetry of bounded stable solutions and that, when they are not identically zero, they are either increasing or decreasing.

As it is well known, when $u \geq 0$ is a bounded solution of (1.1.1), then u is radially symmetric and decreasing ($u_\rho < 0$ for $1 > \rho > 0$). This was proved in [19] using the celebrated moving planes method. Furthermore, by the Poisson formula, the s-harmonic extension of u is also radially symmetric in the horizontal direction, that is, it only depends on ρ and y. Moreover, $v_\rho < 0$ for $\rho > 0$.

In the moving planes argument, the hypothesis of $u \geq 0$ cannot be omitted, since there can be changing-sign solutions of (1.1.1) that are not radially symmetric. Nevertheless, this is not the case for stable solutions, as the next result states:

Proposition 1.4.1. *Let $n \geq 2$ and let u be a bounded stable solution of (1.1.1) with $f \in C^2$. Then, u is radially symmetric. Moreover, if u is not identically zero then u is either increasing or decreasing in $B_1 \setminus \{0\}$.*

The first part of this result is already well known (see for instance Remark 5.3 of [127]), but we will present here the proof for completeness. Instead, to our knowledge, the second part of the proposition about the monotonicity has not been established in the nonlocal setting. In order to prove it, we follow the main ideas in the classical proof of the analogous result for the Laplacian ($s = 1$), which can be found for instance in [35, 81]. The argument in the local case is quite simple: one must show that if u_ρ is not identically zero in B_1, then it cannot vanish in $B_1 \setminus \{0\}$. As a consequence, either $u_\rho > 0$ or $u_\rho < 0$ in $B_1 \setminus \{0\}$. Hence, to complete the proof, we assume that there exists $\rho_\star \in (0, 1)$ for which $u_\rho(\rho_\star) = 0$ and $u_\rho \not\equiv 0$ in $\omega := B_{\rho_\star}$. Therefore, $u_\rho \chi_\omega \in H_0^1(B_1)$ and we can take it as a test function in the stability condition. Finally, we get the contradiction after an integration by parts in ω.

Adapting the previous argument to the nonlocal case using the extension problem is not a straightforward task. To do it, we choose $v_\rho \chi_\Omega$ as a test function in (1.2.5) to arrive at a contradiction. Here, v is the s-harmonic extension of u and $\Omega \subseteq \mathbb{R}^{n+1}_+$ is a certain connected component of the set $\{v_\rho \neq 0\}$ that must be chosen appropriately to satisfy the following condition. We need that $\overline{\partial\Omega \cap B_1} \cap \partial B_1 = \emptyset$, since this condition guarantees that $u \in C^2(\overline{\partial\Omega \cap B_1})$, a property that will be used in our arguments. Note that u is not C^2 in a neighborhood of ∂B_1. Recall —see [126]— that $u \sim \delta^s$ near ∂B_1, where $\delta = \operatorname{dist}(\cdot, \partial B_1)$. In particular, $u_\rho \notin L^2(B_1)$ for $s \leq 1/2$. As a consequence of this, $\partial\Omega \cap B_1$ may differ from $B_{\rho_\star}$ (where $u_\rho(\rho_\star) = 0$) in contrast with the local case.

In addition, Ω may turn to be unbounded. For this reason we need Lemma 1.3.2 and Proposition 1.3.3 to control the decay at infinity of ∇v. This is necessary in order to perform correctly an integration by parts in Ω.

We proceed now with the detailed proof.

Proof of Proposition 1.4.1. We first show the symmetry of u, following [127]. For $i \neq j$ and $i, j = 1, \ldots, n$, consider $w = x_i u_{x_j} - x_j u_{x_i}$, which is a function defined in $\mathbb{R}^n$. Define its extension in $\mathbb{R}^{n+1}_+$ as $W = x_i v_{x_j} - x_j v_{x_i}$, where v is the s-harmonic extension of u.

Then,

$$\begin{aligned}
\operatorname{div}(y^a \nabla W) &= y^a \Delta_x W + \partial_y (y^a W_y) \\
&= y^a (x_i \Delta_x v_{x_j} - x_j \Delta_x v_{x_i}) + \partial_y (y^a x_i (v_y)_{x_j} - y^a x_j (v_y)_{x_i}) \\
&= x_i (\operatorname{div}(y^a \nabla v_{x_j})) - x_j (\operatorname{div}(y^a \nabla v_{x_i})) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
d_s \frac{\partial W}{\partial \nu^a} &= x_i d_s \frac{\partial v_{x_j}}{\partial \nu^a} - x_j d_s \frac{\partial v_{x_i}}{\partial \nu^a} \\
&= x_i f'(u) u_{x_j} - x_j f'(u) u_{x_i} \\
&= f'(u) w \,.
\end{aligned}$$

This means that W is a solution of the linearized problem

$$\begin{cases} \operatorname{div}(y^a \nabla W) = 0 & \text{in } \mathbb{R}^{n+1}_+ \,, \\ d_s \dfrac{\partial W}{\partial \nu^a} = f'(u) w & \text{in } B_1 \subset \mathbb{R}^n \,. \end{cases}$$

Equivalently, the trace of W on $\mathbb{R}^n$, w, solves

$$\begin{cases} (-\Delta)^s w = f'(u) w & \text{in } B_1 \,, \\ w = 0 & \text{in } \mathbb{R}^n \setminus B_1 \,. \end{cases}$$

Let us prove that $w \equiv 0$ for every $i \neq j$, $i, j = 1, \ldots, n$. This leads to the radial symmetry of u since all its tangential derivatives are zero.

Due to the stability of u, we have that $\lambda_1((-\Delta)^s - f'(u); B_1) \geq 0$, that is, the first eigenvalue of the operator $(-\Delta)^s - f'(u)$ in B_1 with zero Dirichlet data outside B_1 is nonnegative. Here we have to consider two cases. If $\lambda_1((-\Delta)^s - f'(u); B_1) > 0$, then $w \equiv 0$. On the contrary, if $\lambda_1((-\Delta)^s - f'(u); B_1) = 0$ then $w = K\phi_1$, that is, w is a multiple of the first eigenfunction ϕ_1, which is positive. But since w is a tangential derivative, it cannot have constant sign along a sphere $\{|x| = R\}$ for $R \in (0, 1)$. Hence, $K = 0$, which leads to $w \equiv 0$. Thus, u is radially symmetric.

We prove now the second part of the result. In order to establish the monotonicity of u, it is enough to see that if $u_\rho \not\equiv 0$ in B_1, then u_ρ does not vanish in $B_1 \setminus \{0\}$. If this is shown to be true, then either $u_\rho > 0$ or $u_\rho < 0$ in $B_1 \setminus \{0\}$.

Arguing by contradiction, we assume that there exists $\rho_\star \in (0, 1)$ such that $u_\rho(\rho_\star) = 0$. Let

$$A^+ = \{v_\rho > 0\} \quad \text{and } A^- = \{v_\rho < 0\} \,.$$

Assume first that one of these two open sets is empty, for instance $A^- = \varnothing$ (the other case is analogous). Then, we find a contradiction with Hopf's lemma. Indeed, since $A^- = \varnothing$, v_ρ satisfies

$$\begin{cases} \operatorname{div}(y^a \nabla v_\rho) = y^a \dfrac{n-1}{\rho^2} v_\rho & \text{in } \mathbb{R}^{n+1}_+ \,, \\ v_\rho \geq 0 & \text{in } \mathbb{R}^{n+1}_+ \,, \\ \dfrac{\partial v_\rho}{\partial \nu^a} = f'(u) u_\rho & \text{in } B_1 \,. \end{cases}$$

At the same time, $v_\rho(\rho_\star, 0) = u_\rho(\rho_\star) = 0$ and thus

$$\frac{\partial v_\rho}{\partial \nu^a}(\rho_\star, 0) = f'(u(\rho_\star))u_\rho(\rho_\star) = 0\,.$$

This contradicts the Hopf's lemma for the operator $\tilde{L}_a w := \operatorname{div}(y^a \nabla w) - y^a c(x) w$, with $c = (n-1)/\rho^2$, which can be proved with the same arguments as in Proposition 4.11 of [40].

Assume now that $A^+ \neq \emptyset$ and $A^- \neq \emptyset$. Our goal is to get a contradiction with the stability of u. For this, we need to define a set $\Omega \subset \mathbb{R}^{n+1}_+$ for which $v_\rho \chi_\Omega \in H^1(\mathbb{R}^{n+1}_+, y^a)$ —note that this forces $v_\rho \equiv 0$ on $\partial\Omega \cap \{y > 0\}$— and, thus, $v_\rho \chi_\Omega$ is a valid test function in the stability condition. The resulting relation must then be integrated by parts in Ω. This will require the integral

$$\int_{\partial\Omega \cap B_1} f'(u) u_\rho^2 \zeta_\varepsilon \, \mathrm{d}x\,,$$

to be finite, where ζ_ε is a smooth function. Now, since $u_\rho \notin L^2(B_1)$ for $s \leq 1/2$, we need to choose Ω such that $\overline{\partial\Omega \cap B_1} \cap \partial B_1$ is empty and, therefore, $u \in C^2(\overline{\partial\Omega \cap B_1})$.

To accomplish this, we first make the following

Claim 1: There exists a set $\Omega \subset \mathbb{R}^{n+1}_+$ (perhaps unbounded) such that v_ρ does not vanish in Ω, $v_\rho = 0$ on $\partial\Omega \cap \{y > 0\}$ and such that

$$\overline{\partial\Omega \cap B_1} \cap \partial B_1 = \emptyset\,.$$

To show this, we define

$$A_0^+ = \partial A^+ \cap B_1 \quad \text{ and } A_0^- = \partial A^- \cap B_1\,.$$

Note that if $u_\rho \leq 0$ in B_1, the Poisson formula yields $v_\rho \leq 0$ in $\mathbb{R}^{n+1}_+$. Similarly, $u_\rho \geq 0$ in B_1 ensures that $v_\rho \geq 0$ in $\mathbb{R}^{n+1}_+$. Therefore, since $A^+ \neq \emptyset$ and $A^- \neq \emptyset$, we also have $A_0^+ \neq \emptyset$ and $A_0^- \neq \emptyset$.

Since v is radially symmetric in the horizontal variables, we can identify the sets A^+, A^-, A_0^+ and A_0^- with their projections into $\mathbb{R}^2_{++} := \{(\rho, y) \in \mathbb{R}^2 : \rho, y \geq 0\}$ and recover the original sets by a revolution about the y-axis. With this identification in mind, let $(\rho_-, 0) \in A_0^-$ and $(\rho_+, 0) \in A_0^+$. Without loss of generality, we can assume that $\rho_- < \rho_+$ —the argument in the other case is analogous. Let Ω_- be the connected component of A^- whose closure contains $(\rho_-, 0)$, and let Ω_+ be the connected component of A^+ whose closure contains $(\rho_+, 0)$. Now, we distinguish two cases.

Case 1: $\overline{\partial\Omega_- \cap B_1} \cap \partial B_1 = \emptyset$**.**

In this case we define

$$\Omega := \Omega_-\,.$$

Case 2: $\overline{\partial\Omega_- \cap B_1} \cap \partial B_1 \neq \emptyset$**.**

In this case, a simple topological argument yields that $\overline{\partial\Omega_+ \cap B_1} \cap \partial B_1 = \emptyset$. Indeed, under the assumption of Case 2, there exists $(\rho_-', 0) \in \partial\Omega_- \cap B_1$ as close as we want to ∂B_1 and such that $\rho_- < \rho_+ < \rho_-' < 1$. Since Ω_- is arc-connected, we can join $(\rho_-, 0)$ and $(\rho_-', 0)$ by a curve in $\Omega_- \cap \{y > 0\}$. By the Jordan curve theorem, the connected component Ω_+, whose closure contains $(\rho_+, 0)$, is bounded and satisfies $\overline{\partial\Omega_+ \cap B_1} \cap \partial B_1 = \emptyset$.

Thus, in Case 2 we define

$$\Omega := \Omega_+$$

and Claim 1 is proved.

To proceed, me make the following

Claim 2: $v_\rho \chi_\Omega \in H^1(\mathbb{R}^{n+1}_+, y^a)$ and the following formula holds:

$$(n-1)d_s \int_\Omega y^a \frac{v_\rho^2}{\rho^2}\,\mathrm{d}x\,\mathrm{d}y = -d_s \int_\Omega y^a |\nabla v_\rho|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{B_1\cap\partial\Omega} f'(u)u_\rho^2\,\mathrm{d}x\,. \tag{1.4.1}$$

To prove Claim 2, note first that v_ρ satisfies the equation

$$\operatorname{div}(y^a \nabla v_\rho) = y^a \frac{n-1}{\rho^2} v_\rho \quad \text{in } \mathbb{R}^{n+1}_+\,.$$

Take $\zeta_\varepsilon = \zeta_\varepsilon(\rho)$ a smooth cut-off function such that $\zeta_\varepsilon = 0$ in B_ε and $\zeta_\varepsilon = 1$ outside $B_{2\varepsilon}$. Multiply the above equation by $d_s v_\rho(\rho, y)\zeta_\varepsilon(\rho)\chi_\Omega(\rho, y)$ and integrate in $\mathbb{R}^{n+1}_+$. Using integration by parts and the fact that $u_\rho = 0$ in $\mathbb{R}^n \setminus B_1$, we get

$$\begin{aligned}(n-1)d_s \int_\Omega y^a \frac{v_\rho^2}{\rho^2}\zeta_\varepsilon\,\mathrm{d}x\,\mathrm{d}y &= d_s \int_\Omega \operatorname{div}(y^a\nabla v_\rho)v_\rho\zeta_\varepsilon\,\mathrm{d}x\,\mathrm{d}y \\ &= -d_s \int_\Omega y^a \nabla v_\rho \cdot \nabla(v_\rho\zeta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y + \int_{B_1\cap\partial\Omega} f'(u)u_\rho^2\zeta_\varepsilon\,\mathrm{d}x\,.\end{aligned} \tag{1.4.2}$$

At this point, we need to justify this integration by parts. On the one hand, we know that $v_\rho = 0$ on $\partial\Omega \cap \{y > 0\}$, and therefore there are no boundary terms except for the one in $\partial\Omega \cap B_1$. Note that, since $u \in C^2(\overline{B_1 \cap \partial\Omega})$, we have

$$\int_{B_1\cap\partial\Omega} f'(u)u_\rho^2\zeta_\varepsilon\,\mathrm{d}x < +\infty\,. \tag{1.4.3}$$

On the other hand, since Ω may be unbounded, the right way to do the computation in (1.4.2) is the following: we first integrate by parts in half-balls B_R^+ and then we make $R \to \infty$. We need to ensure that the boundary terms in $\{y > 0\}$ go to zero, i.e.,

$$\int_{\partial B_R^+\cap\{y>0\}\cap\Omega} y^a v_\rho \frac{\partial v_\rho}{\partial \nu}\zeta_\varepsilon\,\mathrm{d}\sigma \to 0 \quad \text{as } R \to +\infty\,. \tag{1.4.4}$$

This can be easily seen by using the estimate of Lemma 1.3.2 at the points with $|x| > 2$. For the other points, by Proposition 1.3.3 we have

$$|\nabla v(x,y)| \le \frac{C}{y^{n+1}} = \frac{C}{(R^2 - |x|^2)^{\frac{n+1}{2}}}\,,$$

for a constant C depending only on n, s and $||u||_{L^1(B_1)}$. Here we have used that $|x|^2 + y^2 = R^2$. Then, we take into account that

$$\frac{1}{R^2 - |x|^2} \le \frac{2}{R^2} \quad \text{if } R > 2\sqrt{2} \text{ and } |x| < 2$$

to deduce

$$|\nabla v(x,y)| \le \frac{C}{R^{n+1}} \quad \text{if } R > 2\sqrt{2} \text{ and } |x| < 2. \tag{1.4.5}$$

Combining this estimate with the ones in Lemma 1.3.2, we deduce (1.4.4).

In addition, by Lemma 1.3.2 and (1.4.5), the left-hand side of (1.4.2) is finite for all $\varepsilon \in [0, 1/2]$. Hence all the quantities appearing in (1.4.2) are finite —recall (1.4.3)— and, letting $\varepsilon \to 0$, we deduce (1.4.1). Furthermore,

$$\int_\Omega y^a |\nabla v_\rho|^2 \, dx \, dy < +\infty .$$

This and the fact that $v_\rho = 0$ on $\partial\Omega \cap \{y > 0\}$ yield that $v_\rho \chi_\Omega \in H^1(\mathbb{R}^{n+1}_+, y^a)$. Therefore, Claim 2 is proved.

We conclude now the proof. Since $v_\rho \chi_\Omega \in H^1(\mathbb{R}^{n+1}_+, y^a)$, we can take it in the stability condition (1.2.5) to obtain

$$0 \le d_s \int_\Omega y^a |\nabla v_\rho|^2 \, dx \, dy - \int_{B_1 \cap \partial\Omega} f'(u) u_\rho^2 \, dx . \tag{1.4.6}$$

Combining this with (1.4.1) and using that $n \ge 2$ and $d_s > 0$, we get

$$0 \le d_s \int_\Omega y^a |\nabla v_\rho|^2 \, dx \, dy - \int_{B_1 \cap \partial\Omega} f'(u) u_\rho^2 \, dx = -(n-1) d_s \int_\Omega y^a \frac{v_\rho^2}{\rho^2} \, dx \, dy < 0 ,$$

a contradiction. $\square$

1.5 Weighted integrability. Proof of Proposition 1.1.3

This section is devoted to establish Proposition 1.1.3, which is the key ingredient in the proof of Theorem 1.1.2. To do so, we first need the following lemma, which is an expression of the stability condition when the test function ξ is taken as $\xi = c\eta$, with c to be chosen freely and η with compact support.

Lemma 1.5.1. *Let $s \in (0,1)$ and $a = 1 - 2s$. Let f be a nondecreasing C^1 function and u a stable weak solution of*

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega . \end{cases}$$

Let v be the s-harmonic extension of u.

Then, for all $c \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^a)$ and $\eta \in C^1(\overline{\mathbb{R}^{n+1}_+})$ with compact support and such that its trace has support in $\Omega \times \{0\}$,

$$\int_\Omega \left\{ f'(u)c - d_s \frac{\partial c}{\partial \nu^a} \right\} c\eta^2 \, dx \le d_s \int_{\mathbb{R}^{n+1}_+} y^a c^2 |\nabla \eta|^2 \, dx \, dy - d_s \int_{\mathbb{R}^{n+1}_+} \operatorname{div}(y^a \nabla c) c\eta^2 \, dx \, dy,$$

where d_s is the best constant of the trace inequality (1.2.4).

Proof. Simply take $\xi = c\eta$ in the stability condition (1.2.5) and integrate by parts:

$$\begin{aligned}\int_\Omega f'(u)c^2\eta^2\,dx &\le d_s \int_{\mathbb{R}^{n+1}_+} y^a\left\{c^2|\nabla\eta|^2 + \eta^2|\nabla c|^2 + c\nabla c\cdot\nabla\eta^2\right\}\,dx\,dy\\ &= d_s\int_\Omega \frac{\partial c}{\partial \nu^a}c\eta^2\,dx + d_s\int_{\mathbb{R}^{n+1}_+} y^a c^2|\nabla\eta|^2\,dx\,dy\\ &\quad - d_s\int_{\mathbb{R}^{n+1}_+}\operatorname{div}(y^a\nabla c)c\eta^2\,dx\,dy\,.\end{aligned}$$

□

Thanks to this lemma we can now prove Proposition 1.1.3:

Proof of Proposition 1.1.3. We first note that we can replace the conditions on c and η in Lemma 1.5.1 by the following: $c \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\setminus\{0\}, y^a)$ and $\eta \in C^1(\mathbb{R}^{n+1}_+)$ with $\operatorname{tr}\eta \in C^1_0(B_1\setminus\{0\})$, where tr denotes the trace on $\mathbb{R}^n$. Therefore, we can take $c = v_\rho$, which belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\setminus\{0\}, y^a)$. To see this, recall that $u \in C^2_{\mathrm{loc}}(B_1\setminus\{0\})$ (see Remark 1.3.1). Hence, using the estimates given by Proposition 1.3.3 and Proposition 1.3.4, we deduce that $\nabla_x v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\setminus\{0\})$, which yields $v_\rho \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\setminus\{0\}, y^a)$.

Differentiating with respect to ρ the equation $\operatorname{div}(y^a\nabla v) = 0$ and the boundary condition $d_s\partial_{\nu^a}v = f(u)$, we have the following equations for $c = v_\rho$:

$$\operatorname{div}(y^a\nabla c) = \operatorname{div}(y^a\nabla v_\rho) = y^a\frac{n-1}{\rho^2}v_\rho \quad \text{in } \mathbb{R}^{n+1}_+$$

and

$$d_s\frac{\partial c}{\partial\nu^a} = d_s\frac{\partial v_\rho}{\partial \nu^a} = f'(u)u_\rho = f'(u)c \quad \text{in } B_1\,.$$

Therefore, we take $c = v_\rho$ in Lemma 1.5.1 to get

$$(n-1)\int_{\mathbb{R}^{n+1}_+} y^a\frac{(v_\rho\eta)^2}{\rho^2}\,dx\,dy \le \int_{\mathbb{R}^{n+1}_+} y^a v_\rho^2|\nabla\eta|^2\,dx\,dy\,,$$

for every $\eta \in C^1(\mathbb{R}^{n+1}_+)$ with compact support and such that $\operatorname{tr}\eta \in C^1_0(B_1\setminus\{0\})$. For the purpose of our computations, it is convenient to replace η by $\rho\eta$, thus obtaining

$$(n-1)\int_{\mathbb{R}^{n+1}_+} y^a v_\rho^2\eta^2\,dx\,dy \le \int_{\mathbb{R}^{n+1}_+} y^a v_\rho^2|\nabla(\rho\eta)|^2\,dx\,dy\,. \tag{1.5.1}$$

Now, we proceed with some cut-off arguments. Let ζ_δ and ψ_T be two functions in $C^\infty(\mathbb{R})$ such that

$$\zeta_\delta(\rho) = \begin{cases}0 & \text{if } \rho\le\delta\,,\\ 1 & \text{if } \rho \ge 2\delta\,,\end{cases}\quad \zeta_\delta'(\rho) \le \frac{C}{\delta} \text{ if } \rho\in(\delta,2\delta) \quad\text{and}\quad \psi_T(y) = \begin{cases}1 & \text{if } y\le T\,,\\ 0 & \text{if } y\ge T+1\,.\end{cases}$$

Then, we take

$$\eta(\rho,y) = \eta_\varepsilon(\rho)\psi_T(y)\zeta_\delta(\rho)$$

in (1.5.1), where η_ε is a C^1 function with compact support in B_1 to be choosen later. We assume that η_ε and $|\nabla(\rho\eta_\varepsilon)|$ are bounded in B_1. Therefore, we obtain

$$\begin{aligned}(n-1)\int_0^T\int_{B_1} y^a v_\rho^2\eta_\varepsilon^2\zeta_\delta^2\,\mathrm{d}x\,\mathrm{d}y &\le \int_0^{T+1}\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon\psi_T\zeta_\delta)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\le \int_0^{T+1}\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon\zeta_\delta)|^2\psi_T^2\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_T^{T+1}\int_{B_1} y^a v_\rho^2\eta_\varepsilon^2|\nabla(\psi_T)|^2\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}\tag{1.5.2}$$

Now, we see that

$$\begin{aligned}&\int_0^{T+1}\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon\zeta_\delta)|^2\psi_T^2\,\mathrm{d}x\,\mathrm{d}y=\\ &=\int_0^{T+1}\int_{B_1} y^a v_\rho^2\psi_T^2\left\{|\nabla(\rho\eta_\varepsilon)|^2\zeta_\delta^2+\rho^2\eta_\varepsilon^2|\nabla\zeta_\delta|^2+2\rho\eta_\varepsilon\zeta_\delta\nabla\zeta_\delta\cdot\nabla(\rho\eta_\varepsilon)\right\}\,\mathrm{d}x\,\mathrm{d}y\\ &\le\int_0^{T+1}\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad+C\int_0^{T+1}\int_{B_{2\delta}\setminus B_\delta} y^a v_\rho^2|\eta_\varepsilon|\left\{\frac{\rho^2}{\delta^2}|\eta_\varepsilon|+\frac{\rho}{\delta}\zeta_\delta|\nabla(\rho\eta_\varepsilon)|\right\}\,\mathrm{d}x\,\mathrm{d}y\\ &\le\int_0^{T+1}\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}y+C\int_0^{T+1}\int_{B_{2\delta}\setminus B_\delta} y^a v_\rho^2\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}\tag{1.5.3}$$

Note that in the last inequality we have used that η_ε and $|\nabla(\rho\eta_\varepsilon)|$ are bounded. Since $u\in H^s(\mathbb{R}^n)$, we have that its s-harmonic extension, v, is in $H^1(\mathbb{R}^{n+1}_+,y^a)$ (see the comments in Section 1.2). Therefore, the last term in the previous inequalities tends to zero as $\delta\to 0$. Exactly as in the local case (see the proof of Lemma 2.3 in [35]), this point is the only one where we use that $u\in H^s(\mathbb{R}^n)$. Hence, combining (1.5.2) and (1.5.3), and letting $\delta\to 0$, by monotone convergence we have

$$\begin{aligned}(n-1)\int_0^T\int_{B_1} y^a v_\rho^2\eta_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}y &\le \int_0^T\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_T^{T+1}\int_{B_1} y^a v_\rho^2\{|\nabla(\rho\eta_\varepsilon)|^2+\eta_\varepsilon^2|\nabla(\psi_T)|^2\}\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Now we want to make $T\to\infty$. We claim that the last term in the previous inequality goes to zero as $T\to\infty$. Indeed, to see this we use the power decay of v_ρ as $y\to\infty$ given by Proposition 1.3.3, and the bounds for $|\nabla(\psi_T)|$, η_ε and $|\nabla(\rho\eta_\varepsilon)|$. Hence, letting $T\to\infty$ in the previous expression, we obtain

$$(n-1)\int_0^\infty\int_{B_1} y^a v_\rho^2\eta_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}y\le\int_0^\infty\int_{B_1} y^a v_\rho^2|\nabla(\rho\eta_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}y\,,\tag{1.5.4}$$

for every $\eta_\varepsilon(\rho)\in C^1(B_1)$ with compact support and such that $|\nabla(\rho\eta_\varepsilon)|$ is bounded. By approximmation, η_ε can be taken to be Lipschitz instead of C^1.

Now, for $\varepsilon \in (0,1/2)$ and α satisfying (1.1.8), we define

$$\eta_\varepsilon(\rho) = \begin{cases} \varepsilon^{-\alpha} & \text{if } 0 \le \rho \le \varepsilon\,, \\ \rho^{-\alpha}\varphi(\rho) & \text{if } \varepsilon \le \rho\,, \end{cases}$$

where $\varphi \ge 0$ is a smooth cut-off function such that $\varphi(\rho) \equiv 1$ if $\rho \le 1/2$ and $\varphi(\rho) \equiv 0$ if $\rho \ge 3/4$. Taking η_ε in (1.5.4) and using that $\varphi \ge 0$, we get

$$\begin{aligned}(n-1)\int_0^\infty \int_{B_{1/2}\setminus B_\varepsilon} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy + (n-1)\varepsilon^{-2\alpha}\int_0^\infty\int_{B_\varepsilon} y^a v_\rho^2\,dx\,dy \\ \le (1-\alpha)^2 \int_0^\infty \int_{B_{1/2}\setminus B_\varepsilon} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy + \varepsilon^{-2\alpha}\int_0^\infty\int_{B_\varepsilon} y^a v_\rho^2\,dx\,dy \\ + C\int_0^\infty \int_{B_{3/4}\setminus B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy\,, \end{aligned} \tag{1.5.5}$$

for a constant C depending only on α and n. Since $n \ge 2$ and α satisfies (1.1.8), we obtain

$$\int_0^\infty \int_{B_{1/2}\setminus B_\varepsilon} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy \le C\int_0^\infty \int_{B_{3/4}\setminus B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy\,, \tag{1.5.6}$$

for another constant depending only on n and α. Finally, we estimate the right hand side of this last inequality using the estimates developed in Section 1.3. To do this, we split the integral into two parts:

$$\int_0^1 \int_{B_{3/4}\setminus B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy + \int_1^\infty \int_{B_{3/4}\setminus B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy\,.$$

We bound the first term using Proposition 1.3.4, obtaining:

$$\int_0^1 \int_{B_{3/4}\setminus B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy \le C\left(\int_0^1 y^a\,dy\right)\left(\int_{B_{3/4}\setminus B_{1/2}} \rho^{-2\alpha}\,dx\right) \le C,$$

where the constant C above depends only on n, s, α, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$ and $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$. For the second term, we use the estimate $|\nabla_x v| \le C\,||u||_{L^1(B_1)}/y^{n+1}$, given by Proposition 1.3.3, to get

$$\int_1^\infty \int_{B_{3/4}\setminus B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,dx\,dy \le C\,||u||^2_{L^1(B_1)} \int_1^\infty \int_{B_{3/4}\setminus B_{1/2}} y^{a-2n-2} \rho^{-2\alpha}\,dx\,dy \le C,$$

for a constant C depending only on n, s, α and $||u||_{L^1(B_1)}$.

Finally, using these estimates in (1.5.6) and letting $\varepsilon \to 0$, we conclude the proof. □

Remark 1.5.2. As mentioned in the introduction, in the proof of Proposition 5.1 of [55] —which is similar to the previous one—, there is a missing term which remains to be estimated. This is the one appearing in (1.5.5), but with a different power of ρ. In the case of the spectral fractional Laplacian, the estimate we need is given by Proposition 1.3.4, which is valid for both operators A^s and $(-\Delta)^s$ (see Remark 1.3.8). Therefore, the proof of Proposition 5.1 of [55] is now complete.

With a small modification of the previous proof, we can replace the constant on the right-hand side of (1.1.9) by $C\,||u||_{H^s(\mathbb{R}^n)}$ with C depending only on n, s and α.

Proposition 1.5.3. *Under the same hypotheses of Proposition 1.1.3, we have*

$$\int_0^\infty \int_{B_{1/2}} y^a v_\rho^2 \rho^{-2\alpha}\,\mathrm{d}x\,\mathrm{d}y \le C\,[u]_{H^s(\mathbb{R}^n)}\,, \tag{1.5.7}$$

where C is a constant which depends only on n, s and α.

Proof. We follow the previous proof up to (1.5.6) and then we use that

$$[v]_{H^1(\mathbb{R}^{n+1}_+,y^a)} = \frac{1}{d_s}\,[u]_{H^s(\mathbb{R}^n)}\,.$$

This follows from the fact that v solves $\operatorname{div}(y^a\nabla v) = 0$ in $\mathbb{R}^{n+1}_+$ (see Section 1.2). □

Remark 1.5.4. The hypotheses for f in Proposition 1.1.3 —and also in Theorem 1.1.2— can be slightly weakened. Indeed, the statements remain true if, instead of f being C^2 we assume that $f\in C^{2-2s+\varepsilon}([0,+\infty))$ for $\varepsilon>0$. In particular, for $s>1/2$, it is enough to assume $f\in C^1$. This regularity is needed in order to have $u\in C^2_{\text{loc}}(B_1\setminus\{0\})$, a fact that is used in the previous proofs.

1.6 Proof of the main theorem

In this section we prove Theorem 1.1.2. As explained before the statement of Proposition 1.1.3, to get an L^∞ bound for u we still need a crucial identity and a precise bound on a universal constant. This is the content of Lemma 6.2 in [55]. We include it here with a slightly different statement and proof that probably make the result and proof more transparent.

Lemma 1.6.1. *Let $w:\mathbb{R}^n\to\mathbb{R}$ be a bounded function with compact support and such that $(-\Delta)^s w\in L^\infty_{\text{loc}}(\mathbb{R}^n)$. Let W be its s-harmonic extension and let β be a real number such that $0<\beta<n+2-2s$. Then,*

$$-d_s\beta\int_{\mathbb{R}^{n+1}_+} y^a r^{-\beta-2} yW_y\,\mathrm{d}x\,\mathrm{d}y = A_{n,s,\beta}\int_{\mathbb{R}^n}\rho^{-\beta}(-\Delta)^s w\,\mathrm{d}x\,,$$

for a constant $A_{n,s,\beta}$ depending only on n, s, and β and satisfying

$$0<A_{n,s,\beta}<1\,.$$

Proof. Consider the following two operators:

$$\mathcal{F}^\varepsilon_\beta(w) := -d_s\beta\int_{\mathbb{R}^{n+1}_+} y^a(|x|^2+y^2+\varepsilon)^{-(\beta+2)/2} yW_y\,\mathrm{d}x\,\mathrm{d}y\,,$$

$$\mathcal{G}_\beta(w) = \int_{\mathbb{R}^n}\rho^{-\beta}(-\Delta)^s w\,\mathrm{d}x\,.$$

First, we will show that $\lim_{\varepsilon\to 0}\mathcal{F}^\varepsilon_\beta(w) = A_{n,s,\beta}\mathcal{G}_\beta(w)$ and later we will see that we have $0<A_{n,s,\beta}<1$.

Using the Poisson formula for W_y (Lemma 1.2.1), we find that

$$-d_s W_y(x,y) = \Gamma_{n,s} y \int_{\mathbb{R}^n} \frac{(-\Delta)^s w(z)}{(|x-z|^2+y^2)^{\frac{n+2-2s}{2}}}\, \mathrm{d}z\,.$$

Now, multiply the previous equation by $\beta y^{a+1}(|x|^2+y^2+\varepsilon)^{-(\beta+2)/2}$ and integrate in the whole $\mathbb{R}^{n+1}_+$ to obtain

$$\begin{aligned}&\mathcal{F}^{\varepsilon}_{\beta}(w) = \\ &= \int_{\mathbb{R}^n} (-\Delta)^s w(z) \left(\beta\Gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} \frac{y^{a+2}}{(|x-z|^2+y^2)^{\frac{n+2-2s}{2}}(|x|^2+y^2+\varepsilon)^{\beta/2+1}}\, \mathrm{d}x\, \mathrm{d}y \right) \mathrm{d}z\,.\end{aligned}$$

After the change of variables $x = |z|x'$, $y = |z|y'$, we get

$$\mathcal{F}^{\varepsilon}_{\beta}(w) = \int_{\mathbb{R}^n} (-\Delta)^s w(z) |z|^{-\beta} A_{n,s,\beta}\left(\frac{\varepsilon}{|z|^2}\right) \mathrm{d}z\,,$$

where

$$A_{n,s,\beta}(t) = \beta\Gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} \frac{y^{a+2}}{(\rho^2+y^2+t)^{\frac{\beta+2}{2}}(|x-\frac{z}{|z|}|^2+y^2)^{\frac{n+2-2s}{2}}}\, \mathrm{d}x\, \mathrm{d}y\,.$$

Notice that $A_{n,s,\beta}(t)$ does not depend on z and that

$$A_{n,s,\beta}\left(\frac{\varepsilon}{|z|^2}\right) \to A_{n,s,\beta} := A_{n,s,\beta}(0) \quad \text{for all } z \in \mathbb{R}^n$$

as $\varepsilon \to 0$. Moreover, this limit is finite for $0 < \beta < n+2-2s$. Hence, we have proved that

$$\mathcal{F}_{\beta}(w) := \lim_{\varepsilon\to 0} \mathcal{F}^{\varepsilon}_{\beta}(w) = A_{n,s,\beta}\mathcal{G}_{\beta}(w)\,,$$

with a nonnegative constant $A_{n,s,\beta}$ given by

$$A_{n,s,\beta} = \beta\Gamma_{n,s} \int_{\mathbb{R}^{n+1}_+} \frac{y^{a+2}}{(\rho^2+y^2)^{\frac{\beta+2}{2}}(|x-e|^2+y^2)^{\frac{n+2-2s}{2}}}\, \mathrm{d}x\, \mathrm{d}y$$

for an arbitrary unitary vector e.

Now, let us prove that the constant $A_{n,s,\beta}$ is smaller than one. Take $h \in C^\infty(\mathbb{R}^n)$, $h \not\equiv 0$, a smooth nonnegative radially decreasing function with compact support. Let $w \geq 0$ be the solution of $(-\Delta)^s w = h$ in $\mathbb{R}^n$ and let W be its s-harmonic extension. Note that, by the moving planes argument, w is radially decreasing and so it is W in the horizontal direction by the Poisson formula.

Take the equation that W satisfies, that is, $\operatorname{div}(y^a\nabla W) = 0$ and multiply it by $d_s r^{-\beta} = d_s(|x|^2+y^2)^{-\beta/2}$. After integration by parts we find that

$$\begin{aligned}0 &= d_s \int_{\mathbb{R}^{n+1}_+} \operatorname{div}(y^a\nabla W) r^{-\beta}\, \mathrm{d}x\, \mathrm{d}y = \\ &= \beta d_s \int_{\mathbb{R}^{n+1}_+} y^a(\rho W_\rho + yW_y) r^{-\beta-2}\, \mathrm{d}x\, \mathrm{d}y + \int_{\mathbb{R}^n} \rho^{-\beta}(-\Delta)^s w\, \mathrm{d}x\,.\end{aligned}$$

Therefore, we have

$$\frac{1}{A_{n,s,\beta}}\mathcal{F}_\beta(w) = \mathcal{G}_\beta(w) = -\beta d_s \int_{\mathbb{R}^{n+1}_+} y^a \rho W_\rho r^{-\beta-2}\,\mathrm{d}x\,\mathrm{d}y + \mathcal{F}_\beta(w)\,,$$

which leads to

$$\left(\frac{1}{A_{n,s,\beta}} - 1\right)\mathcal{F}_\beta(w) = -\beta d_s \int_{\mathbb{R}^{n+1}_+} y^a \rho W_\rho r^{-\beta-2}\,\mathrm{d}x\,\mathrm{d}y > 0\,,$$

since W is radially decreasing, i.e., $W_\rho < 0$. This leads to $0 < A_{n,s,\beta} < 1$. □

Once this lemma is established, we have all the ingredients to present the proof of our main result:

Proof of Theorem 1.1.2. We divide our proof into two steps.

Step 1. We claim that, for α satisfying (1.1.8) and $\beta > 0$ a real number such that $2(\beta + s - \alpha) < n$,

$$\int_{B_1} f(u)\rho^{-\beta}\,\mathrm{d}x \le C \tag{1.6.1}$$

with a positive constant C that depends only on n, s, α, β, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$, $||f(u)||_{L^\infty(B_1\setminus B_{1/2})}$, and $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$.

To prove the claim, we first multiply $\operatorname{div}(y^a\nabla v) = 0$ by $d_s(\rho^2 + y^2 + \varepsilon)^{-\beta/2}$ and integrate it in the cylinder $B_{1/2}\times(0,1)$. We get

$$\begin{aligned}
0 &= d_s \int_{B_{1/2}\times(0,1)} \operatorname{div}(y^a\nabla v)(\rho^2+y^2+\varepsilon)^{-\beta/2}\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{B_{1/2}} f(u)(\rho^2+\varepsilon)^{-\beta/2}\,\mathrm{d}x + d_s\int_{B_{1/2}} v_y(\rho,1)(\rho^2+1+\varepsilon)^{-\beta/2}\,\mathrm{d}x\\
&\quad + d_s\int_0^1 y^a v_\rho(1/2,y)(1/4+y^2+\varepsilon)^{-\beta/2}\,\mathrm{d}y\\
&\quad + d_s\beta\int_{B_{1/2}\times(0,1)} y^a(\rho^2+y^2+\varepsilon)^{-\beta/2-1}(\rho v_\rho + y v_y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

We rewrite this as

$$\int_{B_{1/2}} f(u)(\rho^2+\varepsilon)^{-\beta/2}\,\mathrm{d}x = -I_1 - I_2 + I_3\,, \tag{1.6.2}$$

where

$$I_1 = d_s\int_{B_{1/2}} v_y(\rho,1)(\rho^2+1+\varepsilon)^{-\beta/2}\,\mathrm{d}x\,,$$

$$I_2 = d_s\int_0^1 y^a v_\rho(1/2,y)(1/4+y^2+\varepsilon)^{-\beta/2}\,\mathrm{d}y\,,$$

and

$$I_3 = -d_s\beta\int_0^1\int_{B_{1/2}} y^a(\rho^2+y^2+\varepsilon)^{-\beta/2-1}(\rho v_\rho + y v_y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

We decompose $I_3 = I_\rho + I_y$, where

$$I_\rho = -d_s\beta \int_0^1 \int_{B_{1/2}} y^a(\rho^2+y^2+\varepsilon)^{-\beta/2-1}\rho v_\rho \, \mathrm{d}x \, \mathrm{d}y$$

and

$$I_y = -d_s\beta \int_0^1 \int_{B_{1/2}} y^a(\rho^2+y^2+\varepsilon)^{-\beta/2-1} y v_y \, \mathrm{d}x \, \mathrm{d}y \, .$$

We can estimate $\lim_{\varepsilon\to 0} I_y$ following the arguments of Lemma 1.6.1 to obtain

$$\lim_{\varepsilon\to 0} I_y \leq A_{n,s,\beta} \int_{B_1} f(u)|x|^{-\beta} \, \mathrm{d}x \, , \tag{1.6.3}$$

where $A_{n,s,\beta}$ is the constant appearing in Lemma 1.6.1. Recall that by this lemma, $0 < A_{n,s,\beta} < 1$. Indeed, we have that

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_y &= \int_{\mathbb{R}^n} (-\Delta)^s u(z) \left(\beta\Gamma_{n,s} \int_0^1 \int_{B_{1/2}} \frac{y^{a+2}}{(|x-z|^2+y^2)^{\frac{n+2-2s}{2}} (|x|^2+y^2)^{\beta/2+1}} \, \mathrm{d}x \, \mathrm{d}y \right) \mathrm{d}z \\
&\leq \int_{B_1} f(u) \left(\beta\Gamma_{n,s} \int_0^1 \int_{B_{1/2}} \frac{y^{a+2}}{(|x-z|^2+y^2)^{\frac{n+2-2s}{2}} (|x|^2+y^2)^{\beta/2+1}} \, \mathrm{d}x \, \mathrm{d}y \right) \, \mathrm{d}z \\
&\leq A_{n,s,\beta} \int_{B_1} f(u)|x|^{-\beta} \, \mathrm{d}x \, .
\end{aligned}$$

Here we have used the Poisson formula for v_y in the first equality. Then, we have used that $(-\Delta)^s u < 0$ in $\mathbb{R}^n \setminus B_1$ and also the equation $(-\Delta)^s u = f(u)$ in B_1. The last inequality is easily deduced using exactly the same arguments that are described in the proof of Lemma 1.6.1.

From (1.6.2) and (1.6.3), we deduce that

$$\begin{aligned}
\int_{B_1} f(u)\rho^{-\beta} \, \mathrm{d}x &= \int_{B_1\setminus B_{1/2}} f(u)\rho^{-\beta} \, \mathrm{d}x + \int_{B_{1/2}} f(u)\rho^{-\beta} \, \mathrm{d}x \\
&\leq C_{n,\beta} \, ||f(u)||_{L^\infty(B_1\setminus B_{1/2})} + \limsup_{\varepsilon\to 0} \left(|I_1| + |I_2| + |I_\rho| \right) \\
&\quad + A_{n,s,\beta} \int_{B_1} f(u)\rho^{-\beta} \, \mathrm{d}x \, .
\end{aligned}$$

Since u is radially decreasing, $f(u)\rho^{-\beta}$ is bounded in $B_1 \setminus B_{1/2}$. Thus, we obtain

$$(1 - A_{n,s,\beta}) \int_{B_1} f(u)\rho^{-\beta} \, \mathrm{d}x \leq C + \limsup_{\varepsilon\to 0} \left(|I_1| + |I_2| + |I_\rho| \right) ,$$

for a constant C depending only on n, β and $||f(u)||_{L^\infty(B_1\setminus B_{1/2})}$. Moreover, thanks to Lemma 1.6.1, $1 - A_{n,s,\beta} > 0$ and therefore

$$\int_{B_1} f(u)\rho^{-\beta} \, \mathrm{d}x \leq C(1 + \limsup_{\varepsilon\to 0} \left(|I_1| + |I_2| + |I_\rho| \right)) \, ,$$

with a constant C depending only on n, s, β and $||f(u)||_{L^\infty(B_1\setminus B_{1/2})}$.

Hence, in order to prove our claim, we only need to obtain suitable bounds for $\limsup_{\varepsilon\to 0}\left(|I_1|+|I_2|+|I_\rho|\right)$. This is done using some previous results, as follows.

We first bound $|I_1|$. Since this integral is computed over $B_{1/2}\times\{1\}$, we can use the gradient estimate $|\nabla v|\leq C$ (see Proposition 1.3.3) with a constant C depending only on n, s and $||u||_{L^1(\mathbb{R}^n)}$.

For $|I_2|$, we just use Proposition 1.3.4 to bound $|v_\rho|$ in $\{\rho = 1/2\}\times(0,1)$ by a constant depending only on n, s, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$ and $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$.

Finally, for $|I_\rho|$, using the Cauchy-Schwarz inequality we get

$$|I_\rho| \leq d_s\beta\left(\int_{B_{1/2}\times(0,1)} y^a\rho^{-2\alpha}v_\rho^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\left(\int_{B_{1/2}\times(0,1)} \frac{y^a\rho^{2+2\alpha}}{(\rho^2+y^2+\varepsilon)^{\beta+2}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

The first of these integrals is bounded by a constant which depends only on α and on the same quantities as the previous one, thanks to Proposition 1.1.3. To see that the second integral is finite, we notice that

$$\begin{aligned}\int_{B_{1/2}\times(0,1)} \frac{y^a\rho^{2+2\alpha}}{(\rho^2+y^2+\varepsilon)^{\beta+2}}\,\mathrm{d}x\,\mathrm{d}y &\leq \int_0^\infty\int_{B_{1/2}} \frac{y^a\rho^{2\alpha}}{(\rho^2+y^2)^{\beta+1}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \left(\int_{B_{1/2}} \rho^{a+2\alpha-2\beta-1}\,\mathrm{d}x\right)\left(\int_0^\infty \frac{t^a}{(1+t)^{\beta+1}}\,\mathrm{d}t\right),\end{aligned}$$

where we have made the change $y=\rho t$. These integrals are finite if $\beta>0$ and $n>2(\beta+s-\alpha)$ —recall that $a=1-2s$. Therefore, the claim (1.6.1) is proved.

Step 2. We prove point (i) of the statement of the theorem. Thanks to the representation formula for the fractional Laplacian and the fact that u is radially decreasing, it is easy to see that

$$||u||_{L^\infty(B_1)} = u(0) \leq C\int_{B_1}\frac{f(u(x))}{|x|^{n-2s}}\,\mathrm{d}x\,, \tag{1.6.4}$$

where C is a constant depending only on n and s. Indeed, we just use Lemma 1.2.2 with a truncation of $f(u)$ (recall that in such lemma h is assumed to be bounded) and then use monotone convergence to deduce (1.6.4). In order to use the claim of Step 1, we take $\beta=n-2s$ and we must choose α satisfying $2(\beta+s-\alpha)<n$ and $1\leq\alpha<1+\sqrt{n-1}$. Therefore, we require that $n/2-s<\alpha$ (thus $1\leq\alpha$ provided that $n\geq 2$) and $\alpha<1+\sqrt{n-1}$. Hence, such α exists if and only if $n/2-s<1+\sqrt{n-1}$, which is equivalent to

$$2\left(s+2-\sqrt{2(s+1)}\right) < n < 2\left(s+2+\sqrt{2(s+1)}\right). \tag{1.6.5}$$

Notice that the lower bound for n is automatically satisfied for $n\geq 2$ and $s\in(0,1)$. Thus, if n satisfies (1.6.5), we can take α such that (1.6.1) holds for $\beta=n-2s$. Therefore, by (1.6.4) and Step 1, we obtain

$$||u||_{L^\infty(B_1)} \leq C$$

with a constant C depending only on n, s, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$, $||f(u)||_{L^\infty(B_1\setminus B_{1/2})}$ and $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$. Next, we replace this C by another constant depending only on n, s, f and $||u||_{L^1(B_1)}$. To do this, we control the L^∞ norm of u in sets away from the origin by the L^1 norm of u. Indeed, since u is radially decreasing, we have that

$$||u||_{L^\infty(B_1\setminus\overline{B_R})} \leq \frac{C}{R^n}\,||u||_{L^1(B_1)} \quad \text{for every } R \in (0,1)\,.$$

Finally, we prove (ii). Assume that α and β satisfy the hypotheses of Step 1. Then, using that f is nondecreasing, that u is radially decreasing, and (1.6.1), we have

$$c_n\rho^{n-\beta} f(u(\rho)) = f(u(\rho)) \int_{B_{2\rho}\setminus B_\rho} |x|^{-\beta}\,dx \leq \int_{B_1} f(u)|x|^{-\beta}\,dx \leq C \quad \text{for } \rho \leq 1/2.$$

Therefore,

$$f(u(\rho)) \leq C\rho^{\beta-n} \quad \text{for } 0<\rho\leq 1\,, \tag{1.6.6}$$

with a constant C depending only on n, s, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$, $||f(u)||_{L^\infty(B_1\setminus B_{1/2})}$ and $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$.

Assume additionally that $\beta < n-2s$. Using Lemma 1.2.2 and (1.6.6), we obtain

$$u(x) \leq \frac{C}{|x|^{n-\beta-2s}} \quad \text{for all } x \in B_1\,.$$

From the restrictions on α and β that we assumed, we conclude that for every μ with $\mu > n/2 - s - 1 - \sqrt{n-1}$, we have

$$u(x) \leq \frac{C}{|x|^{\mu}} \quad \text{for all } x \in B_1\,,$$

for a constant C depending only on n, s, μ, $||u||_{L^1(B_1)}$, $||u||_{L^\infty(B_{7/8}\setminus B_{3/8})}$, $||f(u)||_{L^\infty(B_1\setminus B_{1/2})}$ and $||f'(u)||_{L^\infty(B_{7/8}\setminus B_{3/8})}$. As before, using that u is radially decreasing we can deduce the same estimate but with a constant C depending only on n, s, μ, f and $||u||_{L^1(B_1)}$. $\square$

Remark 1.6.2. In [55] there is a mistake in the proof of their analogous theorem (Theorem 1.6 there). The authors state that the integral I_2 can be controlled using an estimate that only holds for y away from $\{y=0\}$. Since I_2 is an integral up to $\{y=0\}$, a bound for I_2 requires an additional argument. As we show in our proof, the proper way to bound it is by using Proposition 1.3.4, which is valid also for the spectral fractional Laplacian (see Remark 1.3.8).

We conclude by applying the previous result to show the boundedness of the extremal solution u^*.

Proof of Theorem 1.1.1. First, note that the estimate given in point (i) of Theorem 1.1.2 is valid for the classical stable solutions u_λ for $\lambda < \lambda^*$. This is because, obviously, $u_\lambda \in H^s(\mathbb{R}^n)$ and, since u_λ are bounded and positive, they are also radially decreasing (see Proposition 1.4.1). Therefore, by Theorem 1.1.2, we have

$$||u_\lambda||_{L^\infty(B_1)} \leq C$$

for some constant C depending only on n, s, f and $||u_\lambda||_{L^1(\mathbb{R}^n)}$. Note that all these quantities are uniform in $\lambda < \lambda^*$ (see Remark 1.3.1). Hence, by letting $\lambda \to \lambda^*$ we conclude

$$||u^*||_{L^\infty(B_1)} \leq C$$

for some constant C depending only on n, s, f and $||u^*||_{L^1(\mathbb{R}^n)}$.

The way to deduce point (ii) from Theorem 1.1.2 is completely analogous. $\square$

1.7 Appendix: An alternative proof of the result of Ros-Oton and Serra for the exponential nonlinearity

In this appendix, we present an alternative proof of the following result of X. Ros-Oton and J. Serra. In contrast with theirs, our proof uses the extension problem.

Proposition 1.7.1 (Proposition 3.1 in [127]). *Let Ω be a smooth and bounded domain in $\mathbb{R}^n$, and let u^* be the extremal solution of (1.1.2). Assume that $f(u) = e^u$ and $n < 10s$. Then, u^* is bounded.*

The procedure used to prove the boundedness of the extremal solution is, as usual, to deduce an L^∞ estimate for u_λ uniform in $\lambda < \lambda^*$. Then, the result follows from monotone convergence. To prove the uniform bound for u_λ, we proceed as in the classical proof of Crandall-Rabinowitz [68]: we take $\xi = e^{\alpha u_\lambda} - 1$ in the stability condition to obtain a uniform L^p bound for e^{u_λ} for certain values of p. This, combined with the following result, will lead to the desired L^∞ estimate.

Lemma 1.7.2 ([127]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $s \in (0,1)$, $n > 2s$, $h \in C(\overline{\Omega})$, and u be the solution of*

$$\begin{cases} (-\Delta)^s u &= h \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $\frac{n}{2s} < p < \infty$. Then, there exists a constant C, depending only on n, s, p and $|\Omega|$, such that

$$||u||_{C^\beta(\mathbb{R}^n)} \le C\, ||h||_{L^p(\Omega)}\,, \qquad \text{where } \beta = \min\left\{ s, 2s - \frac{n}{p} \right\}.$$

With this bound in hand, we can proceed with the alternative proof of the result on the boundedness of u^* in the case $f(u) = e^u$.

Proof of Proposition 1.7.1. Let α be a positive real number that will be chosen later. Let u_λ be the minimal stable solution of (1.1.2) for $\lambda < \lambda^*$. Take $\xi = e^{\alpha u_\lambda} - 1$, which is 0 in $\mathbb{R}^n \setminus \Omega$, in the stability condition (1.2.5) to obtain

$$\lambda \int_\Omega e^{u_\lambda}(e^{\alpha u_\lambda} - 1)^2\, dx \le d_s \int_{\mathbb{R}^{n+1}_+} y^a \alpha^2 e^{2\alpha v_\lambda} |\nabla v_\lambda|^2 \, dx\, dy\,,$$

where v_λ denotes the s-harmonic extension of u_λ. Note that we have taken $e^{\alpha v_\lambda} - 1$ as the extension of ξ in $\mathbb{R}^{n+1}_+$. Then, integrating by parts we compute

$$\begin{aligned} d_s \int_{\mathbb{R}^{n+1}_+} y^a \alpha^2 e^{2\alpha v_\lambda} |\nabla v_\lambda|^2 \, dx\, dy &= d_s \frac{\alpha}{2} \int_{\mathbb{R}^{n+1}_+} y^a \nabla v_\lambda \cdot \nabla (e^{2\alpha v_\lambda} - 1)\, dx\, dy \\ &= \frac{\alpha}{2} \int_\Omega \lambda e^{u_\lambda}(e^{2\alpha u_\lambda} - 1)\, dx \end{aligned}$$

(recall that $\operatorname{div}(y^a \nabla v_\lambda) = 0$ in $\mathbb{R}^{n+1}_+$) and hence

$$\int_\Omega e^{u_\lambda}(e^{2\alpha u_\lambda} - 2e^{\alpha u_\lambda} + 1)\, dx \le \frac{\alpha}{2} \int_\Omega e^{u_\lambda}(e^{2\alpha u_\lambda} - 1)\, dx\,.$$

This leads to

$$\left(1 - \frac{\alpha}{2}\right) \int_\Omega e^{(2\alpha+1)u_\lambda}\, dx - 2 \int_\Omega e^{(\alpha+1)u_\lambda}\, dx + \left(1 + \frac{\alpha}{2}\right) \int_\Omega e^{u_\lambda}\, dx \le 0\,. \tag{1.7.1}$$

Now, using Hölder inequality we have

$$\int_\Omega e^{(\alpha+1)u_\lambda}\,\mathrm{d}x \le C\left(\int_\Omega e^{(2\alpha+1)u_\lambda}\,\mathrm{d}x\right)^{\frac{\alpha+1}{2\alpha+1}}$$

for a constant C depending only on α and $|\Omega|$. Therefore, from (1.7.1) we see that for each $\alpha < 2$, we have

$$||e^{u_\lambda}||_{L^{2\alpha+1}(\Omega)} \le C \tag{1.7.2}$$

for a constant C depending only on α and $|\Omega|$.

Finally, if $n < 10s$, we can choose $\alpha < 2$ such that $\frac{n}{2s} < 2\alpha + 1 < 5$. Then, taking $p = 2\alpha + 1$ in Lemma 1.7.2 and using (1.7.2) we obtain

$$||u_\lambda||_{L^\infty(\Omega)} \le C\lambda\, ||e^{u_\lambda}||_{L^{2\alpha+1}(\Omega)} \le C\,,$$

for a constant C depending only on n, s and Ω (and hence independent of λ). By monotone convergence, letting $\lambda \to \lambda^*$ we conclude that u^* is bounded. □

Remark 1.7.3. In the previous proof, we have taken $\xi = e^{\alpha v_\lambda} - 1$ in the stability condition, where v_λ is the s-harmonic extension of u_λ. Nevertheless, the inequality obtained with this choice of the extension is not sharp, since $e^{\alpha v_\lambda} - 1$ is not the s-harmonic extension of $\xi = e^{\alpha u_\lambda} - 1$. This choice simplifies a lot the computations but makes us wonder if there could be a smarter choice of the extension leading to a better result.

Part II

Integro-differential Allen-Cahn equations: the saddle-shaped solution

Introduction to Part II

This part of the thesis is devoted to study a certain type of solutions to the problem $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{2m}$ with f of bistable type. These are the so-called saddle-shaped solutions.

The structure of this part consists in three chapters, each of them corresponding respectively to the articles

- [88] J.C. Felipe-Navarro and T. Sanz-Perela, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, preprint (2018).
- [86] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone*, preprint (2019).
- [87] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation*, preprint (2019).

The Allen-Cahn equation and its connection with minimal surfaces

In the last 50 years, there has been a great interest in the equation

$$-\Delta u = u - u^3 \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{II.1}$$

This PDE is known as the *Allen-Cahn equation*, after the works of the chemists Allen and Cahn in the 1970s on the study of interfaces (see for instance [2, 3]). In the last decades it has attracted the interest of many mathematicians in Analysis, PDEs, and Geometry due to its deep connection with the theory of minimal surfaces. In this section we present a simple model from which this equation arises, and we describe very briefly its connection with minimal surfaces.

The Allen-Cahn equation (II.1) arises in the study of interfaces, in the theory of superconductors and superfluids, or in cosmology (see [3, 109, 56, 97]). Let us discuss a simple physical model that leads to (II.1). Assume that in a container $\Omega \subset \mathbb{R}^n$ we have confined a substance that may present two pure phases. We denote by $u(x)$ the proportion, between -1 and 1, of these two phases at a point $x \in \Omega$. That is, $u(x) = \pm 1$ represents that the whole substance at a point x is in one of the two phases, while $u(x) = 0$ denotes a point where the two phases coexist in equal proportion. Our goal is to describe mathematically the pattern and the separation of the two phases at a stationary configuration.

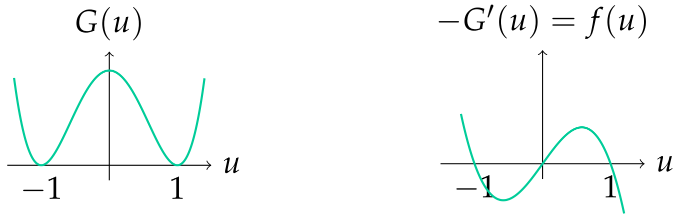

Figure 1.1: (a) A double-well potential G. (b) A bistable nonlinearity f.

As in many physical situations, it is natural to assume that the steady state pattern minimizes an energy. A first naive guess would be to consider the potential energy

$$\mathcal{P}(w,\Omega) = \int_\Omega G(w(x))\,\mathrm{d}x,$$

where G is a *double-well potential* —see Figure 1.1 (a)—, that is, $G \in C^2([-1,1])$ and satisfying

$$G(\pm 1) = 0, \quad G > 0 \text{ in } (-1,1), \quad G'(\pm 1) = 0, \quad \text{and} \quad G'(\pm 1) > 0. \tag{II.2}$$

The most classical example, appearing in the Allen-Cahn equation, is $G(u) = (1-u^2)^2/4$. The minimization of $\mathcal{P}$ favors states u that attain the pure phases -1 and $+1$ in big regions of Ω, but note that any function that takes only the values -1 and 1 is a minimizer for the energy. Therefore, the separation of the phases in a minimizing configuration could be arbitrary, and thus very irregular. However, in any reasonable physical situation this cannot happen, since small forces such as friction would not allow it.

To overcome this issue, in our model we can add to $\mathcal{P}$ an energy term that penalizes the formation of unnecessary interfaces. Typically, the energy that one considers is

$$\mathcal{E}_\varepsilon(w,\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega G(w)\,\mathrm{d}x, \tag{II.3}$$

with ε small. This is called in the literature the Ginzburg-Landau energy functional (see [100]).

To understand minimizers of (II.3), we study the equation that they satisfy. First, by rescaling in space we may assume that $\varepsilon = 1$. Then, setting $f = -G'$, it follows that the Euler-Lagrange equation of the energy functional $\mathcal{E}_1$ is

$$-\Delta u = f(u) \quad \text{in } \Omega\,. \tag{II.4}$$

If $G(u) = (1-u^2)^2/4$, then $f(u) = u - u^3$ and we have precisely (II.1). We will call Allen-Cahn any equation of the form (II.4) with $f = -G'$ and G satisfying (II.2). When f satisfies this, we say that f is of *bistable* type —see Figure 1.1 (b). Quite usually, G is assumed to be even (thus f is odd) as in the model example (II.1).

From equation (II.4) it follows that minimizers of the energy will be smooth functions taking values between -1 and $+1$, and the pure phases ± 1 will not be attained in general (by the strong maximum principle). Moreover, since $|\nabla u|$ will be globally bounded, the transitions between states close to 1 and -1 will occur in regions of universal width of order 1. Thus, if we rescale back and consider again minimizers of $\mathcal{E}_\varepsilon$, one can observe "sharp" transitions between pure states happening in regions of width of order ε.

The crucial fact concerning these transitions is that, as ε goes to zero, the minimization of the energy forces the interfaces to approach hypersurfaces with the least possible area. Let us give an heuristic argument explaining this. By noticing that

$$\mathcal{E}_\varepsilon(w,\Omega) \geq \int_\Omega \sqrt{2G(w)}|\nabla w|\,dx,$$

we can use the coarea formula to obtain

$$\mathcal{E}_\varepsilon(w,\Omega) \geq \int_{-1}^{1} \sqrt{2G(t)}\mathcal{H}^{n-1}(\{w=t\}\cap\Omega)\,dt\,.$$

This leads to think that the energy $\mathcal{E}_\varepsilon$ would be minimized by a function w if every level set $\{w=t\}\cap\Omega$ was a minimizing minimal surface and, in addition, we had an equality in the previous inequality. This last statement is equivalent to have $\sqrt{2G(w)}=\varepsilon|\nabla w|$, which yields that

$$w(x) = w_0\left(\frac{d_0(x)}{\varepsilon}\right),$$

where d_0 is the signed distance to the level set $\{w=0\}$ and w_0 is the unique solution to the ODE

$$-\ddot{w}_0 = f(w_0), \qquad w_0(0)=0. \tag{II.5}$$

Note that, in general, w defined as above does not solve the Allen-Cahn equation (see the computations in [132]) and thus it cannot be a minimizer. In addition, the level sets of w (which are the level sets of d_0) cannot be all minimizing minimal surfaces in general. Despite all of this, the function w provides intuition about the behavior of minimizers. If, for instance, the zero level set of w minimizes the area, then the level sets $\{w=t\}$ are almost minimizers whenever t is not too close to ±1 and provided that ε is small. Note, on the other hand, that if t is close to ±1, the level sets $\{w=t\}$ are negligible when computing the energy due to the term involving $G(w)$. These rough arguments suggest that w is close to be a minimizer, and that the level sets of an actual minimizer converge to a minimizing minimal surface as $\varepsilon\to0$. This is indeed true, as explained next (see also the classical papers [118, 119]).

To state the convergence to minimizing minimal surfaces, we consider the bow-down sequence $u_\varepsilon = u(\cdot/\varepsilon)$. Note that if u is a minimizer of $\mathcal{E}_1(\cdot,\mathbb{R}^n)$, then $u_\varepsilon=u(\cdot/\varepsilon)$ is a minimizer of $\mathcal{E}_\varepsilon(\cdot,\mathbb{R}^n)$. The classical convergence result by Modica in [118] states that there exists a subsequence $\varepsilon_k\to0$ such that

$$u_{\varepsilon_k} \to \chi_E - \chi_{\mathbb{R}^n\setminus E} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

and E is a set of minimal perimeter, i.e., ∂E is a minimizing minimal surface. Informally speaking, this result states that when we look at the minimizer u "from very far" (that is, considering the blow-down sequence u_ε), the level sets $\{u_\varepsilon=t\}$ with t between, say,

First, we consider $h(t)$ to be the primitive of $1/\sqrt{2G(t)}$ that vanish at the origin. Then, by multiplying the ODE $-\ddot{w}_0=-G'(w_0)$ by $\dot{w}_0$ and integrating it is not difficult to see that $h=w_0^{-1}$, that is, $h(w_0(z))=z$. Now, if w is such that $\sqrt{2G(w)}=\varepsilon|\nabla w|$, then the function $\varepsilon h(w)$ satisfies $\varepsilon|\nabla h(w)|=1$ and thus $\varepsilon h(w)$ must coincide with d_0, the signed distance to the zero level set $\{w=0\}$. Thus, $h(w)=d_0/\varepsilon$ and since $h=w_0^{-1}$, the result follows.

-0.9 and 0.9, get closer and closer as $\varepsilon \to 0$. In the limit, we only see two pure phases -1 and $+1$ occupying all the space, and an interface (being the limit of the zero level set) which is a minimizing minimal surface.

One of the main results concerning the regularity of minimal surfaces can be summarized as follows:

- If an $(n-1)$-dimensional manifold is a minimizing minimal surface in $\mathbb{R}^n$ and $n \le 7$, then it is a hyperplane. On the other hand, the Simons cone

$$\mathscr{C} = \left\{ (x', x'') \in \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8 \;:\; |x'| = |x''| \right\}$$

  is a minimizing minimal surface in $\mathbb{R}^8$ which is not a hyperplane.

Let us recall also that the same dichotomy holds in one dimension higher if we restrict to minimal graphs:

- If the graph $\{x_n = u(x_1, \dots, x_{n-1})\}$ is a minimal surface (in the sense that u solves the minimal surface equation in $\mathbb{R}^{n-1}$), and $n \le 8$, then u is affine. This is often called Bernstein theorem in the literature. On the other hand, there exists a non-affine entire minimal graph in dimension $n = 9$.

The following are the original references where these results were proved: [71, 72] by De Giorgi, [142] by Simons, and [21] by Bombieri, De Giorgi, and Giusti. See also the book [101] by Giusti.

Due to the previous connection between the Allen-Cahn equation and minimal surfaces, similar rigidity results as above are expected to hold for level sets of solutions to the Allen-Cahn equation, as we explain next.

A conjecture by De Giorgi

Let u be a global solution to the Allen-Cahn equation which is monotone in one direction. That is, satisfying

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n, \quad \text{with } u_{x_n} > 0.$$

Consider again the rescaled functions $u_\varepsilon = u(\cdot/\varepsilon)$ —which solve $-\varepsilon^2 \Delta u_\varepsilon = f(u_\varepsilon)$. The reasonings of the previous section suggest that each level set $\{u_\varepsilon = t\}$ with $t \in (-1, 1)$ is close to a surface of minimal area if ε is small enough. Actually, the monotonicity assumption $u_{x_n} > 0$ would entail that $\{u_\varepsilon = t\}$ is close to a minimal graph. If it was the graph of a function defined in all of $\mathbb{R}^{n-1}$ solving the minimal surface equation, then we can expect a similar result to that of Bernstein mentioned above. Thus, it would follow that if $n \le 8$, the level set $\{u_\varepsilon = t\}$ would be close to a flat hyperplane (provided that ε is small). At this point, one is led to think that if $\{u_\varepsilon = t\}$ is close enough to a hyperplane, it is indeed a hyperplane —since elliptic problems enjoy some kind of rigidity. Finally, if we scale back, the previous reasoning would yield that the level sets of u would be hyperplanes, and thus u would be one-dimensional.

This heuristic argument is the motivation for the following conjecture, raised by De Giorgi:

Conjecture II.1 (De Giorgi, 1978 [73])**.** *Let u be a bounded solution of the Allen-Cahn equation*

$$-\Delta u = u - u^3 \quad \textit{in } \mathbb{R}^n$$

such that it is monotone in one direction, say $\partial_{x_n} u > 0$. Then, if $n \le 8$, u is one-dimensional, i.e., u depends only on one Euclidean variable.

Obviously, there are several gaps in the previous argument. First of all, we did not assume minimality on u, but just that it was a stationary point of the energy (i.e., that it solved the Allen-Cahn equation). Therefore, one should prove the convergence of level sets to minimal surfaces. Moreover, the monotonicity of a function in the x_n-direction does not guarantee that its level sets are entire graphs in general. In addition, we should have on hand some type of rigidity argument for level sets close to hyperplanes. As a consequence of all this, establishing the validity of Conjecture II.1 is not an easy task and, in fact, the problem is not completely closed yet.

About twenty years after it was raised, the conjecture by De Giorgi was established in dimension $n = 2$ by Ghoussoub and Gui [96] and in dimension $n = 3$ by Ambrosio and Cabré [4]. In both cases, the one-dimensional symmetry of the solutions is true for any semilinear equation with a smooth nonlinearity, not necessarily coming from a double-well potential. Later, in the celebrated paper [132], Savin proved the validity of the conjecture in dimensions $4 \le n \le 8$, but under the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1} . \tag{II.6}$$

The proof of Savin is in the spirit of the argument described at the beginning of the section (indeed, an important result in [132] is the "improvement of flatness" for the level sets of u). Note that the assumption (II.6) is only used to guarantee that u is a minimizer in bounded sets of the energy. After the paper of Savin [132], del Pino, Kowalczyk, and Wei [74] built a counterexample to the conjecture in dimensions $n \ge 9$ by using the gluing method. Summarizing, Conjecture II.1 without the assumption (II.6) is nowadays open in dimensions $4 \le n \le 8$.

As it is well known, there is a connection between monotone and stable solutions. First, we have that monotone solutions (in the x_n-direction, say) are stable. This is because u_{x_n} is a positive supersolution to the linearized operator $-\Delta - f'(u)$. In addition, if u is monotone one can define the following two functions in $\mathbb{R}^{n-1}$

$$u^{\pm} := \lim_{x_n \to \pm\infty} u.$$

One can prove that $u^{\pm}$ are stable solutions to (II.4) $\mathbb{R}^{n-1}$. Then, if one can show that $u^{\pm}$ are one-dimensional, the results of Savin in [132] would yield that u is also one-dimensional.

Rephrasing this last statement, if one proves that stable solutions to the Allen-Cahn equation are one-dimensional in $\mathbb{R}^{n-1}$, this would automatically yield that the conjecture by De Giorgi is true in $\mathbb{R}^n$. Therefore, a natural associated conjecture arises.

Conjecture II.2 ("Stable De Giorgi Conjecture")**.** *Let u be a bounded stable solution to* (II.4)*. Then, if $n \le 7$, u is one-dimensional.*

Consider for instance $w(x_1, \dots, x_n) = h(x_1 + e^{x_n})$ with h a monotone bounded function.

This conjecture is only solved in dimension $n = 2$ (see [96, 11]) and it is deeply connected with the one mentioned in the introduction of Part I concerning the regularity of stable minimal surfaces. A weaker statement than Conjecture II.2 is to ask the same but for minimizers. In this setting, Savin already proved in [132] that it is true in dimensions $n \leq 7$. More recently, Liu, Wang, and Wei provided a counterexample for this conjecture on minimizers in dimensions $n \geq 8$ (see [113]). As we will see, this counterexample is related to the saddle-shaped solution, that we describe next.

The saddle-shaped solution

Recall that the Simons cone is the simplest example of nonplanar minimizing minimal surface. It can be defined in each even dimension $n = 2m$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \; : \; |x'| = |x''| \right\} .$$

In what follows it will be of importance the set

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \; : \; |x'| > |x''| \right\},$$

which is one of the parts in which the Simons cone divides the space $\mathbb{R}^{2m}$.

The comments in the previous sections can make one wonder whether there exists an analogous object to $\mathscr{C}$ for the Allen-Cahn equation, that is, a simple example of a solution which is not one-dimensional and which is stable in dimensions $n \geq 8$. A candidate for this could be a function whose zero level set is the Simons cone. This property leads to the following definition (recall that we use the notation $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m$ for points in $\mathbb{R}^{2m}$).

Definition II.3 (Saddle-shaped solution)**.** We say that a bounded solution u to $-\Delta u = f(u)$ in $\mathbb{R}^{2m}$ is a *saddle-shaped solution* (or simply *saddle solution*) if

(i) u is a doubly radial function, that is, $u = u(|x'|, |x''|)$.

(ii) u is odd with respect to the Simons cone, that is, $u(|x'|, |x''|) = -u(|x''|, |x'|)$.

(iii) $u > 0$ in $\mathcal{O} = \{|x'| > |x''|\}$.

Saddle-shaped solutions to the Allen-Cahn equation were first studied by Dang, Fife, and Peletier in [69] in dimension $2m = 2$. They established the existence and uniqueness of this type of solutions, as well as some monotonicity properties and asymptotic behavior. In [138], Schatzman studied the instability property of saddle solutions in $\mathbb{R}^2$. Later, Cabré and Terra proved the existence of a saddle solution in every dimension $2m \geq 2$, and they established some qualitative properties such as asymptotic behavior, monotonicity properties, as well as instability in dimensions $2m = 4$ and $2m = 6$ (see [43, 44]). The uniqueness in dimensions higher than 2 was established by Cabré in [34], where he also proved that the saddle solution is stable in dimensions $2m \geq 14$. The possible stability in dimensions 8, 10, and 12 is still an open problem, as well as the possible minimality of this solution in dimensions $2m \geq 8$.

For instance, the construction of the counterexample to the conjecture of De Giorgi in dimensions $n \geq 9$ is quite technical and relies on an infinite-dimensional form of Lyapunov–Schmidt reduction (sometimes also called gluing method), see [74].

Open problem II.4. Is the saddle-shaped solution to (II.4) stable in dimensions 8, 10, and 12? It is a minimizer in dimensions $2m \geq 8$?

Note that thanks to their uniqueness, saddle-shaped solutions are canonical objects associated to the Simons cone $\mathscr{C}$ and the Allen-Cahn equation. Thus, they seem to be the natural counterexamples to Conjecture II.2 on the one-dimensional symmetry of stable solutions. In addition to this, they are also natural objects from which one can build a counterexample for the conjecture by De Giorgi, in an alternative way to that of [74]. This is due to a result by Jerison and Monneau [106], where they show that a counterexample to the original conjecture of De Giorgi in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate. The saddle-shaped solution is of special interest in relation with the Jerison-Monneau program since it is even with respect to all the coordinate axis and it is expected to be a minimizer in high dimensions.

Let us make a last remark on the recent result of Liu, Wang, and Wei [113] concerning the existence of minimizers in $\mathbb{R}^8$ which are not one-dimensional. The authors proved that there exists an ordered family of solutions W_λ with their zero level set being asymptotic to the cone $\mathscr{C}$. From this ordering, they can establish that each solution W_λ is a minimizer of the Allen-Cahn equation. However, their construction only gives solutions W_λ for which $\{W_\lambda = 0\}$ is far from the origin of $\mathbb{R}^8$ (even if this set is asymptotic to the Simons cone at infinity). Therefore, this family does not include the saddle-shaped solution.

The integro-differential setting

In this thesis we are concerned with the integro-differential version of the Allen-Cahn equation, namely,

$$L_K u = f(u) \quad \text{in } \mathbb{R}^n,$$

with f of bistable type and L_K a linear uniformly elliptic integro-differential operator of the form

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(z)\} K(x-z)\,\mathrm{d}y.$$

In particular, we are interested in studying saddle-shaped solutions to this problem in $\mathbb{R}^{2m}$. Before stating our results, let us describe briefly the state of the art concerning the fractional and integro-differential versions of the Allen-Cahn equation.

Let us focus first on the simplest scenario, that is, when $L_K = (-\Delta)^s$. A similar derivation of a model for phase transitions can be done as for the local case, but now replacing the gradient term $[u]^2_{H^1}$ by $[u]^2_{H^s}$, where

$$[u]^2_{H^s} := \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z\,.$$

This choice tries to mimic a long-range particle interaction. That is, the energy still depends on the variation of the phase u but, in this case, far away changes in phase may influence each other (although such influence is weaker and weaker as the distance of the interaction increases). By considering the Euler-Lagrange equation of the nonlocal energy functional, we obtain the fractional Allen-Cahn equation $(-\Delta)^s u = f(u)$.

Obviously, different nonlocal seminorms involving general kernels K lead to integro-differential equations of the form $L_K u = f(u)$.

Recall that the blow-down sequence of minimizers of the Allen-Cahn energy converge (up to subsequence) to the characteristic function of a set of minimal perimeter. A similar fact holds for the equation with the fractional Laplacian, though we have two different scenarios depending on the parameter $s \in (0,1)$. If $s \geq 1/2$, the rescaled energy functionals associated to the equation Γ-converge to the classical perimeter (see [1, 102]), while in the case $s \in (0,1/2)$ they Γ-converge to the *fractional perimeter* (see [135]). Indeed, if $s \in (0,1/2)$, it can be proven that

$$u_{\varepsilon_k} \to \chi_E - \chi_{\mathbb{R}^n \setminus E} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n),$$

and E is a set which locally minimizes the fractional perimeter, defined by

$$\text{Per}_{2s}(E) = \int_E \int_{\mathbb{R}^n \setminus E} \frac{\mathrm{d}x \, \mathrm{d}z}{|x-z|^{n+2s}} \,.$$

Let us make some comments about this functional. The fractional perimeter as presented above was first introduced by Caffarelli, Roquejoffre, and Savin in the seminal paper [48]. It can be proved that, as $s \uparrow 1/2$, it converges to the classical perimeter up to a multiplicative constant . Moreover, the fractional perimeter has several important applications, for instance in Image Processing (see [79] and the references therein). In [48], it is proven that $\text{Per}_{2s}(\cdot)$ has good variational properties (lower semicontinuity and compactness) and thus one can prove existence of minimizers. In analogy with the classical theory of minimal surfaces, if E is a stationary point of $\text{Per}_{2s}(\cdot)$, then ∂E is said to be a *nonlocal minimal surface* (or $(2s)$-minimal surface). If, in addition, E minimizes $\text{Per}_{2s}(\cdot)$, we say that ∂E is a *minimizing nonlocal minimal surface*.

In the recent years there has been an increasing interest in developing a regularity theory for nonlocal minimal surfaces, although very few results are known for the moment. It is beyond the scope of this dissertation to describe all of them in detail, and we refer the interested reader to [66, 30] and the references therein. Let us just make some comments on the scarce available results concerning the possible minimality of the Simons cone as a nonlocal minimal surface, since this is connected to our work on saddle-shaped solutions. Note first that, by all its symmetries, it is easy to check that the Simons cone $\mathscr{C}$ is stationary for the fractional perimeter. If $2m = 2$, it cannot be a minimizer since in [136] Savin and Valdinoci proved that all minimizing nonlocal minimal cones in $\mathbb{R}^2$ are flat. In higher dimensions, the only available results regarding the possible minimality of $\mathscr{C}$ appear in [70] and in our paper [88] (corresponding to Chapter 2 of this thesis) but they concern stability, a weaker property than minimality.

A very interesting characterization of the stability of Lawson cones —a more general class of cones that includes $\mathscr{C}$— has been found by Dávila, del Pino, and Wei [70]. It consists of an inequality involving two hypergeometric constants which depend only on s and the dimension n. This inequality is checked numerically in [70], finding that, in dimensions $n \leq 6$ and for s close to zero, no Lawson cone with zero nonlocal mean

That is, E minimizes the fractional perimeter in bounded sets $\Omega \Subset \mathbb{R}^n$ against all competitors which coincide with the set E in $\mathbb{R}^n \setminus \Omega$. To treat this minimization problem correctly, one should consider a localized fractional perimeter $\text{Per}_{2s}(E, \Omega)$ where contributions that are the same for all competitors are removed in order to have finite integrals. For the details see [48].

Dimension $n = 2$ is the only one where a complete classification of minimizing nonlocal minimal cones is available. In higher dimensions the problem is still open.

curvature is stable. Numerics also show that all Lawson cones in dimension 7 are stable if s is close to zero. These two results for small s fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $s \in (0, 1/2)$, though this is still an open problem.

In contrast with the numeric computations in [70], our proof of Corollary II.9 establishing the stability of $\mathscr{C}$ in dimensions $2m \geq 14$ (see Chapter 2) is the first analytical proof of a stability result for the Simons cone in any dimension (in the nonlocal setting). This shows that the saddle-shaped solution does not only have its interest in the context of the Allen-Cahn equation, but it can also provide strategies to prove stability and minimality results in the theory of nonlocal minimal surfaces.

By the connection between the fractional Allen-Cahn equation and both local and nonlocal minimal surfaces, it is natural to ask the same type of questions as in Conjectures II.1 and II.2 but for equation (II.9). More interestingly, a reasonable question is whether the optimal dimensions 8 and 7 in the previous conjectures are also the right threshold in the nonlocal setting and if they depend on $s \in (0, 1)$. The numerical results obtained by Dávila, del Pino, and Wei in [70] suggest that for small s one could lower by one the previous optimal dimensions, although this is still an open problem. We describe next the available results in this direction.

Conjecture II.1 with $-\Delta$ replaced by $(-\Delta)^s$ has been proven to be true in dimension $n = 2$ by Cabré and Solà-Morales in [42] for $s = 1/2$, and extended to every power $0 < s < 1$ by Cabré and Sire in [41] and also by Sire and Valdinoci in [143]. In dimension $n = 3$, the conjecture has been proved by Cabré and Cinti for $1/2 \leq s < 1$ in [36, 37] and by Dipierro, Farina, and Valdinoci for $0 < s < 1/2$ in [76]. Recently, in [133, 134] Savin has established the validity of the conjecture in dimensions $4 \leq n \leq 8$ and for $1/2 \leq s < 1$, but assuming the additional hypothesis (II.6) on the limits as $x_n \to \pm\infty$. Under such extra assumption, the conjecture is true in the same dimensions $4 \leq n \leq 8$ for $0 < s < 1/2$ and s close to $1/2$, as proved by Dipierro, Serra, and Valdinoci in [78]. The most recent result concerning the proof of the conjecture is the one by Figalli and Serra in [90], where they have established the conjecture in dimension $n = 4$ and $s = 1/2$ without requiring the additional limiting assumption (II.6). Note that, without (II.6), the analogous result for the Laplacian in dimension $n = 4$ is not known. In the forthcoming paper [39], Cabré, Cinti, and Serra prove the conjecture in dimension $n = 4$ for $0 < s < 1/2$ and s sufficiently close to $1/2$. A counterexample to the De Giorgi conjecture for fractional Allen-Cahn equation in dimensions $n \geq 9$ for $s \in (1/2, 1)$ has been recently announced in [57].

Regarding the fractional analogue of Conjecture II.2 on the one-dimensional symmetry of stable solutions, the problem is only solved in dimension $n = 2$ (see [42, 41]). Regarding the same issue but for minimizers, Savin [133, 134] proved that if $s \in [1/2, 1)$ and $n \leq 7$, minimizers to the equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ are one-dimensional. In the case $s \in (0, 1/2)$, Dipierro, Serra, and Valdinoci [78] proved that minimizers are one-dimensional provided that their level sets are asymptotically flat. Therefore, if one could prove a classification result for nonlocal minimal cones in some dimension n — for $(s \in (0, 1/2)$—, this would entail the one-dimensional symmetry of minimizers to $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{n-1}$. As mentioned above, the classification of nonlocal minimal cones is still a fundamental open problem in dimensions $n \geq 3$.

Concerning saddle-shaped solutions to

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{II.7}$$

there are only two works in the literature where they are studied. In [60, 61], first for $s = 1/2$ and then for $s \in (0, 1)$, Cinti proved the existence of a saddle-shaped solution to (II.4) as well as some qualitative properties such as asymptotic behavior, monotonicity in some directions, and instability in dimensions $2m = 4$ and $2m = 6$ (this is also true in dimension $2m = 2$: by a result of Cabré and Sire [41] all stable solutions to (II.7) in $\mathbb{R}^2$ are one-dimensional). As explained in the next section, in Chapter 2 we continue the study initiated by Cinti by establishing the uniqueness of the saddle-shaped solution and, in dimensions $2m \geq 14$, its stability. After our result, the same questions as in the local case remain open:

Open problem II.5. Is the saddle-shaped solution to $(-\Delta)^s u = f(u)$ in $\mathbb{R}^{2m}$ stable in dimensions 8, 10, and 12? It is a minimizer in dimensions $2m \geq 8$?

To conclude this section, we make some quick comments on the very few known results concerning the general integro-differential Allen-Cahn equation $L_K u = f(u)$ when L_K is not the fractional Laplacian. As mentioned above, this equation arises when one replaces the gradient term in the Ginzburg-Landau energy by a seminorm of the type

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(z)|^2 K(x - z) \, dx \, dz \, .$$

This choice takes into account more general nonlocal interactions such as nonhomogeneous and anisotropic ones. Since the extension problem is not available for these operators, it is not a straightforward task to generalize some of the results known for the fractional Laplacian (recall for instance that the proofs in [36, 37, 90] rely very strongly on the local extension technique). Hence, to attack this problem one needs to consider a nonlocal approach. We next present some of the available results concerning the integro-differential Allen-Cahn equation $L_K u = f(u)$ in $\mathbb{R}^n$.

Concerning the integro-differential analogue of the conjecture by De Giorgi, with general operators like L_K, very few results are known. In dimension $n = 2$ the conjecture is proved in [103, 29, 85], under different assumptions on the kernel K and even for more general nonlinear operators. In addition, the above results of Dipierro, Serra, and Valdinoci in [78] also hold for a particular class of uniformly elliptic kernels.

In the same context, we shall mention the work of Cozzi and Passalacqua [67], where they study layer solutions to $L_K u = f(u)$ in $\mathbb{R}^n$. Layer solutions are monotone in one direction, say $e \in \mathbb{S}^{n-1}$, and have limit ± 1 as $x \cdot e \to \pm\infty$. In [67], the authors construct layer solutions in $\mathbb{R}$ for an associated equation involving a kernel $K_1 : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$. The kernel K_1 is related to K in such a way that the solutions in $\mathbb{R}$ are one-dimensional solutions to the associated problem in $\mathbb{R}^n$ (see Chapter 4 for the details). In particular, Cozzi and Passalacqua prove in [67] that under the assumptions (II.10) on f, for every uniformly elliptic kernel K_1 there exist a solution to $L_{K_1} w = f(w)$ in $\mathbb{R}$ which is increasing, $w(x) \to \pm 1$ as $x \to \pm\infty$ and it is odd with respect to some point. This layer solution is unique up to translations, and the one vanishing at the origin is usually denoted in the literature by u_0 —it is the nonlocal analogue to the solution of (II.5). To summarize, u_0 is the unique solution to

$$\begin{cases} L_{K_1} u_0 &= f(u_0) \quad \text{in } \mathbb{R}\,, \\ \dot{u}_0 &> 0 \quad \text{in } \mathbb{R}\,, \\ u_0(x) &= -u_0(-x) \quad \text{in } \mathbb{R}\,, \\ \lim_{x \to \pm\infty} u_0(x) &= \pm 1. \end{cases} \tag{II.8}$$

The importance of the layer solution u_0 in relation with saddle-shaped solutions is that the associated function

$$U(x) := u_0\left(\frac{|x'| - |x''|}{\sqrt{2}}\right)$$

will describe the asymptotic behavior of saddle solutions at infinity (as we prove in this thesis, see Theorem II.11). Note that $(|x'| - |x''|)/\sqrt{2}$ is the signed distance to the Simons cone (see Lemma 4.2 in [44]). Therefore, the function U consists of "copies" of the layer solution u_0 centered at each point of the Simons cone and oriented in the normal direction to the cone.

Prior to this thesis, there were no available results concerning saddle-shaped solutions to the Allen-Cahn equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$. In this thesis, and more precisely in Chapters 3 and 4, we initiate this study, which is not a straightforward extension of the techniques used for $(-\Delta)^s$ due to the lack of the associated local problem in $\mathbb{R}^{2m+1}_+$. As a consequence, some purely nonlocal techniques are developed in these chapters. For instance, we find a necessary and sufficient condition on the kernel K to be able to carry out a theory on existence and uniqueness for saddle-shaped solutions. As we will see, this will involve a new convexity assumption on K.

Results of the thesis: Part II

Chapter 2 (corresponding to [88]) is devoted to the saddle-shaped solution to the fractional Allen-Cahn equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{II.9}$$

where f satisfies

$$f(0) = f(1) = 0,\ f \text{ is odd},\ \text{and } f'' < 0 \text{ in } (0,1). \tag{II.10}$$

One of our main results is the following.

Theorem II.6. *Let $s \in (0,1)$ and let f be a function satisfying (II.10). Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution to problem (II.9).*

To prove it, we follow the ideas of Cabré [34] for the local problem with $s = 1$, using the extension technique in our case. The arguments rely on the asymptotic behavior of the saddle solution (proved by Cinti in [60, 61]) and a maximum principle in $\mathcal{O} = \{|x'| > |x''|\}$ for the linearized operator at a saddle-shaped solution, presented next. In the following statement we use the notation

$$\partial_L \Omega := \overline{\partial\Omega \cap \{y > 0\}} \quad \text{and} \quad \partial_0\Omega := \partial\Omega \setminus \partial_L\Omega \subset \{y = 0\}$$

for a set $\Omega \subset \mathbb{R}^{2m+1}_+$.

Proposition II.7. *Let u be a saddle-shaped solution of (II.9). Let $\Omega \subset \mathcal{O} \times (0, +\infty) \subset \mathbb{R}^{2m+1}_+$ be an open set such that $\partial_0\Omega$ is nonempty. Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$ be bounded from above and such that $y^a w_y \in C(\overline{\Omega})$.*

Consider the operator $\mathscr{L}_u$ defined by

$$\mathscr{L}_u w := -d_s \lim_{y \downarrow 0} y^a w_y - f'(u) w \quad \text{on } \partial_0\Omega \subset \mathbb{R}^{2m} \times \{0\},$$

and assume that

$$
\begin{cases}
-\operatorname{div}(y^a \nabla w) & \leq & b(x,y)w & \text{in } \Omega \subset \mathcal{O} \times (0,+\infty)\,, \\
\mathscr{L}_u w & \leq & 0 & \text{on } \partial_0\Omega \subset \mathcal{O}\,, \\
w & \leq & 0 & \text{on } \partial_L \Omega\,, \\
\limsup\limits_{x\in\partial_0\Omega,\ |x|\to+\infty} w(x,0) & \leq & 0\,, &
\end{cases}
\tag{II.11}
$$

with $b \leq 0$. *Then,* $w \leq 0$ *in* Ω.

The proof of an analogous result in [34] uses a maximum principle in "narrow" sets which is not straightforward to extend to the fractional case. The reason for this is that we need to combine the property of being "narrow" in $\mathbb{R}^{2m}$ with the fact that we are considering sets in $\mathbb{R}^{2m+1}_+$, and this requires to define a new notion of "narrowness" (see Chapter 2 for a detailed explanation). We should remark that in Chapter 4 we prove an analogous maximum principle in "narrow" sets for general integro-differential operators L_K and the proof does not require such technicalities (once the operator L_K is written in a suitable way, see (II.14) below). This is another example illustrating that working directly in $\mathbb{R}^n$ is more natural than using the extension problem (which here seems a rather artificial technique), and it leads to better and simpler proofs.

The second important result in Chapter 2 is the following stability theorem for the saddle-shaped solution. Its proof relies on finding a positive supersolution to the linearized problem (II.11), see the details in Chapter 2.

Theorem II.8. *Assume that* f *satisfies* (II.10). *If* $2m \geq 14$, *then the saddle-shaped solution* u *of* (II.9) *is stable in* $\mathbb{R}^{2m}$.

An important consequence of this result is Corollary II.9, stated next, on the stability of the Simons cone as a $(2s)$-minimal surface in dimensions $2m \geq 14$. This is the first analytical proof of its stability for some s and m (as explained before). It follows directly from the convergence results proved in [39] which state that the convergence $u_\varepsilon \to \chi_E - \chi_{\mathbb{R}^n \setminus E}$ of the blow-down sequence u_ε does not only hold for minimizers, but also for stable solutions; thus if u_ε are stable, the boundary of the limit set E is a stable $(2s)$-minimal surface.

Corollary II.9. *Let* $s \in (0, 1/2)$ *and* $2m \geq 14$. *Then, the Simons cone* $\mathscr{C} \subset \mathbb{R}^{2m}$ *is a stable* $(2s)$*-minimal surface.*

In Chapters 3 and 4 (corresponding to [86] and [87], respectively) we initiate the study of saddle-shaped solutions to

$$
L_K u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{II.12}
$$

where L_K is a linear uniformly elliptic integro-differential operator and f is of bistable type, that is, it satisfies (II.10). We establish an appropriate setting to study saddle solutions and we characterize the kernels for which one can develop a theory on saddle-shaped solutions. For these kernels, we prove existence, uniqueness, and asymptotic behavior of the saddle-shaped solution. To accomplish this, we prove, among others, an energy estimate, a Liouville type result, the one-dimensional symmetry of positive solutions to semilinear problems in a half-space, and maximum principles in "narrow" sets. This is explained in more detail next.

Chapter 3 is mainly devoted to settle the framework to work with doubly radial odd solutions (like saddle-shaped solutions). Recall that we say that a function $w : \mathbb{R}^{2m} \to \mathbb{R}$ is doubly radial if it depends only on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, $w(Rx) = w(x)$ for every $R \in O(m)^2$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$.

To state our results, we need first to introduce an isometry that plays a significant role in both Chapters 3 and 4. It is defined by

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \,. \end{array}$$

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathbb{R}^{2m} \setminus \overline{\mathcal{O}}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant —although not all points in $\mathscr{C}$ are fixed points of $(\cdot)^\star$. Taking into account this transformation, we say that a doubly radial function w is odd with respect to the Simons cone if $w(x) = -w(x^\star)$.

Regarding the doubly radial symmetry we define the following variables

$$\sigma := |x'| \quad \text{and} \quad \tau := |x''| \,.$$

They are specially useful when dealing with the Laplacian in these coordinates, since

$$\Delta w = w_{\sigma\sigma} + w_{\tau\tau} + \frac{m-1}{\sigma} w_\sigma + \frac{m-1}{\tau} w_\tau$$

becomes an expression suitable to work with. A similar formula appears in the case of the fractional Laplacian thanks to the local extension problem. Having a PDE in the two variables $(\sigma, \tau) \in \mathbb{R}^2$ is useful to perform certain computations (see [43, 44, 34, 45] for the local case and [60, 61, 88] for the fractional framework). Indeed, these variables are used in Chapter 2 when considering the extension problem.

The situation changes if we consider a general integro-differential operator L_K. Indeed, if we try to follow the same strategy by writing a rotation invariant operator L_K in (σ, τ) variables, the expression of the new operator is quite complex. Namely, if $w : \mathbb{R}^{2m} \to \mathbb{R}$ is doubly radial and we define $\widetilde{w}(\sigma, \tau) := w(\sigma, 0, ..., 0, \tau, 0, ..., 0)$, it holds

$$L_K w(x) = \widetilde{L}_K \widetilde{w}(|x'|, |x''|)$$

with

$$\widetilde{L}_K \widetilde{w}(\sigma, \tau) := \int_0^{+\infty} \int_0^{+\infty} \widetilde{\sigma}^{m-1} \widetilde{\tau}^{m-1} \big(\widetilde{w}(\sigma, \tau) - \widetilde{w}(\widetilde{\sigma}, \widetilde{\tau}) \big) J(\sigma, \tau, \widetilde{\sigma}, \widetilde{\tau}) \, d\widetilde{\sigma} \, d\widetilde{\tau}$$

and

$$J(\sigma, \tau, \widetilde{\sigma}, \widetilde{\tau}) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{\sigma^2 + \widetilde{\sigma}^2 - 2\sigma\widetilde{\sigma}\omega_1 + \tau^2 + \widetilde{\tau}^2 - 2\tau\widetilde{\tau}\widetilde{\omega}_1} \Big) \, d\omega \, d\widetilde{\omega} \,.$$

Note that $\widetilde{L}_K$ is an integro-differential operator in $(0, +\infty) \times (0, +\infty)$, but the expression of its kernel is quite involved. Indeed, such an expression does not become simpler even when L_K is the fractional Laplacian. In this case, the kernel J involves hypergeometric functions of two variables, the so-called Appell functions (see Chapter 3 for more details on it), but this does not simplify computations.

Instead of working with the (σ, τ) variables, we follow another approach that we find more clear and concise. It consists on rewriting the operator L_K with a different kernel $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ that is doubly radial with respect to its both arguments, but in such a way that it still acts on functions defined in $\mathbb{R}^{2m}$ —and not in $(0, +\infty)^2$. As it is explained in detail in Chapter 3, if K is a radially symmetric kernel, then we can write L_K acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(z)\} \overline{K}(x, z) \, \mathrm{d}z \,,$$

where $\overline{K}$ is doubly radial in both variables and is defined by

$$\overline{K}(x, z) := \fint_{O(m)^2} K(|Rx - z|) \, \mathrm{d}R \,. \tag{II.13}$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$ (see Chapter 3 for the details). It is important to notice that, in contrast with $K = K(x - z)$, $\overline{K}$ is no longer translation invariant (i.e., it is a function of x and z but not of the difference $x - z$).

If we consider doubly radial functions that are, in addition, odd with respect to the Simons cone, we can use the involution $(\cdot)^\star$ to find that

$$L_K w(x) = \int_{\mathcal{O}} \{w(x) - w(z)\} \{\overline{K}(x, z) - \overline{K}(x, z^\star)\} \, \mathrm{d}z + 2w(x) \int_{\mathcal{O}} \overline{K}(x, z^\star) \, \mathrm{d}z \,. \tag{II.14}$$

Furthermore,

$$\frac{1}{C} \operatorname{dist}(x, \mathscr{C})^{-2s} \leq \int_{\mathcal{O}} \overline{K}(x, z^\star) \, \mathrm{d}z \leq C \operatorname{dist}(x, \mathscr{C})^{-2s}, \tag{II.15}$$

with $C > 0$ depending only on m, s, and the ellipticity constants of K (see the details in Chapter 3).

Note that the expression (II.14) has an integro-differential part plus a term of order zero with a positive coefficient. Thus, the most natural assumption to make in order to have an elliptic operator (when acting on doubly radial odd functions) is the positiveness of the kernel in the integro-differential term. That is, $\overline{K}(x, z) - \overline{K}(x, z^\star) > 0$. One of the main results in Chapter 3, stated next, establishes a necessary and sufficient condition on the original kernel K for L_K to have a positive kernel when acting on doubly radial odd functions.

Theorem II.10. *Let $K : (0, +\infty) \to (0, +\infty)$ and consider the radially symmetric kernel $K(|x - z|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (II.13).*

If

$$K(\sqrt{t}) \text{ is a strictly convex function of } t \,, \tag{II.16}$$

then L_K has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x, z) > \overline{K}(x, z^\star) \quad \text{for every } x, z \in \mathcal{O} \,. \tag{II.17}$$

In addition, if $K \in C^2((0, +\infty))$, then (II.16) is not only a sufficient condition for (II.17) to hold, but also a necessary one.

The assumption (II.17) is crucial to prove most of the results of Chapter 4 concerning saddle-shaped solutions to (II.12). The main one is the following.

Theorem II.11. *Let f satisfy (II.10). Let K be a radially symmetric kernel satisfying the positivity condition (II.17) and such that L_K is uniformly elliptic.*

(i) For every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution u to (II.12). In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.

(ii) Let U be the function defined by

$$U(x) := u_0\left(\frac{|x'| - |x''|}{\sqrt{2}}\right),$$

where u_0 is the layer solution to (II.8).

Then,

$$||u - U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + \left|\left|D^2 u - D^2 U\right|\right|_{L^\infty(\mathbb{R}^n \setminus B_R)} \to 0$$

as $R \to +\infty$.

The existence of the saddle-shaped solution can be proved following two strategies, both using crucially the hypothesis (II.17) on the kernel $\overline{K}$. The first one consists on using the monotone iteration method (adapted to odd functions), and it is presented in Chapter 4. The second strategy uses variational techniques and appears in Chapter 3. It relies on energy estimates for odd minimizers (odd with respect to $\mathscr{C}$) of the energy

$$\begin{aligned}\mathcal{E}(w,\Omega) := \frac{1}{4}\bigg\{ &\int_\Omega \int_\Omega |w(x) - w(z)|^2 K(x-z)\,\mathrm{d}x\,\mathrm{d}z \\ &+ 2\int_\Omega \int_{\mathbb{R}^{2m}\setminus\Omega} |w(x) - w(y)|^2 K(x-z)\,\mathrm{d}x\,\mathrm{d}z \bigg\} + \int_\Omega G(w)\,\mathrm{d}x\,,\end{aligned}$$

a topic that had never been studied in the literature for general kernels. To define such minimizers properly, we denote by $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ the space of doubly radial odd functions that vanish outside B_R and for which the energy $\mathcal{E}$ is well defined (see Chapter 3 for the precise definition). We say that $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ is a doubly radial odd minimizer of $\mathcal{E}$ in B_R if $\mathcal{E}(u, B_R) \leq \mathcal{E}(w, B_R)$ for every $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. We then have:

Theorem II.12. *Let K be a radially symmetric kernel satisfying the positivity condition (II.17) and such that L_K is uniformly elliptic. Assume that G is a potential satisfying (II.2). Let $S \geq 2$ and let $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ be a doubly radial odd minimizer of $\mathcal{E}$ in B_R, with $R > S + 4$.*

Then

$$\mathcal{E}(u, B_S) \leq \begin{cases} C\,S^{2m-2s} & \text{if } s \in (0, 1/2), \\ C\,S^{2m-1}\log S & \text{if } s = 1/2, \\ C\,S^{2m-1} & \text{if } s \in (1/2, 1), \end{cases}$$

where C is a positive constant depending only on m, s, $||G||_{C^2([-1,1])}$, and the ellipticity constants of K.

Note that Theorem II.12 does not follow from the energy estimate for general minimizers stated in [137] by Savin and Valdinoci. The minimizers that they consider do not have any type of symmetry. In our case, the function u in the previous statement minimizes the energy in a smaller class of functions and the result in [137] cannot be applied. Nevertheless, we are able to adapt the arguments of Savin and Valdinoci to our setting (see Chapter 3). In our proof it is fundamental, again, to assume (II.17).

To prove the uniqueness of saddle-shaped solutions we need two ingredients. The first one is the asymptotic behavior of saddle solutions given in statement (ii) of Theorem II.11 above (we will make some comments on this result later on). The second ingredient is the following maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$.

Proposition II.13. *Let $\Omega \subset \mathcal{O}$ be an open set (not necessarily bounded) and let K be a radially symmetric kernel satisfying the positivity condition (II.17) and such that L_K is uniformly elliptic. Let u be a saddle-shaped solution to (II.12), and let $w \in C^s(\overline{\Omega}) \cap C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, for some $\alpha > 2s$, be a doubly radial function satisfying*

$$\begin{cases} L_K w - f'(u)w - c(x)w & \leq & 0 & \text{in } \Omega\,, \\ w & \leq & 0 & \text{in } \mathcal{O}\setminus\Omega\,, \\ -w(x^\star) & = & w(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in\Omega,\ |x|\to\infty} w(x) & \leq & 0\,, & \end{cases}$$

with $c \leq 0$ in Ω.

Then, $w \leq 0$ in Ω.

To establish it, the key tool is to use a maximum principle in "narrow" sets, also proved in Chapter 4. The proof of this result is much simpler than analogue maximum principles for the Laplacian. Indeed, this is an example of how the nonlocality of the operator makes the arguments easier and less technical (informally speaking, L_K "sees more", or 'further", than the Laplacian). Needless to mention, the proof of Proposition II.13 is by far simpler than the one for Proposition II.7 using the extension problem. In the proof, it is crucial again the positivity condition (II.17) together with the bounds (II.15).

Let us now make some comments on the proof of point (ii) of Theorem II.11, on the asymptotic behavior of saddle-shaped solutions. To establish it we use a compactness argument as in [44, 60, 61], together with two important results proved Chapter 4. The first one, Theorem II.14, is the following Liouville type result for nonnegative solutions to a semilinear equation in the whole space.

Theorem II.14. *Let L_K be a uniformly elliptic integro-differential operator and let w be a bounded solution to*

$$\begin{cases} L_K w & = & f(w) & \text{in } \mathbb{R}^n\,, \\ w & \geq & 0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

with a nonlinearity $f \in C^1$ satisfying

- $f(0) = f(1) = 0$,
- $f'(0) > 0$,
- $f > 0$ *in* $(0,1)$, *and* $f < 0$ *in* $(1,+\infty)$.

Then, $w \equiv 0$ or $w \equiv 1$.

Similar classification results have been proved for the fractional Laplacian in [59, 112] (either using the extension problem or not) with the method of moving spheres, which uses crucially the scale invariance of the operator $(-\Delta)^s$. To the best of our knowledge, there is no similar result available in the literature for general uniformly elliptic integro-differential operators (which are not necessarily scale invariant). Our proof is based on the techniques introduced by Berestycki, Hamel, and Nadirashvili [15] for the local equation with the classical Laplacian. It relies on the maximum principle, the translation invariance of the operator, a Harnack inequality, and a stability argument.

The second ingredient needed to prove the asymptotic behavior of saddle-shaped solutions is a symmetry result for equations in a half-space, stated next. Here, we use the notation $\mathbb{R}^n_+ = \{(x_H, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, x_n > 0\}$.

Theorem II.15. *Let L_K be a uniformly elliptic integro-differential operator and let w be a bounded solution to one of the following two problems: either to*

$$\begin{cases} L_K w &= f(w) & \text{in } \mathbb{R}^n_+, \\ w &> 0 & \text{in } \mathbb{R}^n_+, \\ w(x_H, x_n) &= -w(x_H, -x_n) & \text{in } \mathbb{R}^n, \end{cases} \tag{P1}$$

or to

$$\begin{cases} L_K w &= f(w) & \text{in } \mathbb{R}^n_+, \\ w &> 0 & \text{in } \mathbb{R}^n_+, \\ w &= 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases} \tag{P2}$$

Assume that, in $\mathbb{R}^n_+$, the kernel K of the operator L_K is decreasing in the direction of x_n, i.e., it satisfies

$$K(x_H - z_H, x_n - z_n) \geq K(x_H - z_H, x_n + z_n) \quad \text{for all} \quad x, z \in \mathbb{R}^n_+.$$

Suppose that $f \in C^1$ and

- $f(0) = f(1) = 0$,
- *$f'(0) > 0$, and $f'(t) \leq 0$ for all $t \in [1-\delta, 1]$ for some $\delta > 0$,*
- *$f > 0$ in $(0, 1)$, and*
- *f is odd in the case of (P1).*

Then, w depends only on x_n and it is increasing in this direction.

The result for (P2) has been proved for the fractional Laplacian under some assumptions on f (weaker than the ones in Theorem II.15) in [123, 7, 8, 83]. Instead, no result was available for general integro-differential operators. To the best of our knowledge, problem (P1) on odd solutions with respect to a hyperplane has not been treated even for the fractional Laplacian. In our case, the fact that f is of Allen-Cahn type allows us to use rather simple arguments that work for both problems (P1) and (P2) —moving planes and sliding methods. Moreover, the fact that the kernel of the operator is $|\cdot|^{-n-2s}$ or a general K satisfying uniform ellipticity bounds does not affect significantly the proof.

Note: In the following chapters there is a change of notation. First, the power of the fractional Laplacian will be denoted by γ instead of s. The reason is that s, together with t, will denote the doubly radial variables

$$s := |x'| \quad \text{and} \quad t := |x''|.$$

In addition, the variable for the extension problem will be $\lambda > 0$ instead of y, this last one denoting other type of variables depending on the chapter.

Chapter 2

Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation

In this paper we prove the uniqueness of the saddle-shaped solution to the semilinear nonlocal elliptic equation $(-\Delta)^\gamma u = f(u)$ in $\mathbb{R}^{2m}$, where $\gamma \in (0,1)$ and f is of Allen-Cahn type. Moreover, we prove that this solution is stable if $2m \geq 14$. As a consequence of this result and the connection of the problem with nonlocal minimal surfaces, we show that the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \,:\, |x'| = |x''|\}$ is a stable nonlocal (2γ)-minimal surface in dimensions $2m \geq 14$.

Saddle-shaped solutions of the fractional Allen-Cahn equation are doubly radial, odd with respect to the Simons cone, and vanish only in this set. It was known that these solutions exist in all even dimensions and are unstable in dimensions 2, 4 and 6. Thus, after our result, the stability remains an open problem only in dimensions 8, 10 and 12.

The importance of studying this type of solution is due to its relation with the fractional version of a conjecture by De Giorgi. Saddle-shaped solutions are the simplest non 1D candidates to be global minimizers in high dimensions, a property not yet established in any dimension.

2.1 Introduction

This paper is devoted to the study of saddle-shaped solutions to the fractional Allen-Cahn equation

$$(-\Delta)^\gamma u = f(u) \quad \text{in } \mathbb{R}^n\,, \tag{2.1.1}$$

where $n = 2m$ is an even integer, f is of bistable type (see (2.1.2) below), and $(-\Delta)^\gamma$ is the fractional Laplacian, defined for $\gamma \in (0,1)$ by

$$(-\Delta)^\gamma u(x) := c_{n,\gamma}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x) - u(\tilde{x})}{|x - \tilde{x}|^{n+2\gamma}}\,\mathrm{d}\tilde{x}\,.$$

Here $c_{n,\gamma} > 0$ is a normalizing constant depending only on n and γ, and P. V. stands for principal value. This problem is motivated by the fractional De Giorgi conjecture and it is closely related to the theory of nonlocal minimal surfaces, as we will explain later in this introduction.

Throughout the paper we assume that $f \in C^{2,\alpha}((-1,1))$, for some $\alpha \in (0,1)$, and that is of bistable type, i.e.,

$$f \text{ is odd}, \ f(0) = f(1) = 0, \text{ and } f'' < 0 \text{ in } (0,1). \tag{2.1.2}$$

Note that as a consequence we have $f > 0$ in $(0,1)$. A typical example of this kind of nonlinearity is $f(u) = u - u^3$.

An important role in this paper is played by the Simons cone, which is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x'| = |x''|\} \ .$$

It is well known that the Simons cone has zero mean curvature at every point $x \in \mathscr{C} \setminus \{0\}$, in every dimension $2m \geq 2$. However, it is only in dimensions $2m \geq 8$ that $\mathscr{C}$ is a minimizer of the area functional, as established by Bombieri, De Giorgi, and Giusti in [21]. Regarding the fractional setting, for every $\gamma \in (0, 1/2)$, $\mathscr{C}$ has zero *nonlocal* mean curvature in every even dimension but it is not known if, in addition, it is a minimizer of the fractional perimeter in dimensions $2m \geq 8$. We recall that it is only in dimension $2m = 2$ where we have a complete classification of minimizing nonlocal minimal cones, establishing that they must be straight lines (see [136]). The only other result concerning the possible minimality of the Simons cone refers to its stability, a weaker property than minimality, and it is proved in [70] by Dávila, del Pino, and Wei (recall that by stability we understand that the second variation of the energy functional is nonnegative). In that paper, the authors characterize the stability of Lawson cones through an inequality involving only two hypergeometric constants which depend only on γ and the dimension n. It is a hard task to verify the criterion analytically, and this has not been accomplished. It seems also delicate to check it numerically, but some cases are treated in [70]. With a numerical computation, [70] finds that, in dimensions $n \leq 6$ and for γ close to zero, no Lawson cone with zero nonlocal mean curvature is stable. The Simons cone is a particular case of Lawson cone corresponding to $C_m^m(2\gamma)$ in the notation of [70]. Numerics also shows that all Lawson cones in dimension 7 are stable if γ is close to zero. These results for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem.

In the present paper, we make a first contribution to the previous question by showing that the Simons cone is a stable (2γ)-minimal cone in dimensions $2m \geq 14$. Our proof uses the so-called saddle-shaped solution to the Allen-Cahn equation. As we will see in more detail, by the fractional Modica-Mortola type Γ-convergence result, the remarks above on the stability of the Simons cone are expected to hold also for saddle-shaped solutions. Indeed, our proof proceeds by establishing the stability of such solution to the fractional Allen-Cahn equation in dimensions $2m \geq 14$ (see Theorem 2.1.6 below). Then, as a consequence of this and a recent result by Cabré, Cinti, and Serra in [39] (see also the comments in [38]) concerning the preservation of stability along a blow-down procedure for the fractional Allen-Cahn equation, we deduce the stability of the Simons cone as a nonlocal minimal surface in these dimensions (see Corollary 2.1.7).

To introduce saddle-shaped solutions, we define the following variables:

$$s := \sqrt{x_1^2 + \ldots + x_m^2} \quad \text{and} \quad t := \sqrt{x_{m+1}^2 + \ldots + x_{2m}^2}\,,$$

for which the Simons cone becomes $\mathscr{C} = \{s = t\}$. Through the paper we will also use the letter $\mathcal{O}$ to denote one of the sets in which the cone divides the space:

$$\mathcal{O} := \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x'| > |x''|\} = \{s > t\}.$$

We define saddle-shaped solutions as follows.

Definition 2.1.1. We say that a bounded solution u to (2.1.1) is a *saddle-shaped solution* (or simply *saddle solution*) if

(i) u is a doubly radial function, that is, $u = u(s,t)$.

(ii) u is odd with respect to the Simons cone, that is, $u(s,t) = -u(t,s)$.

(iii) $u > 0$ in $\mathcal{O} = \{s > t\}$.

Saddle-shaped solutions for the classical Allen-Cahn equation involving the Laplacian were first studied by Dang, Fife, and Peletier in [69] in dimension $2m = 2$. They established the existence and uniqueness of this type of solutions, as well as some monotonicity properties and asymptotic behavior. In [138], Schatzman studied the instability property of saddle solutions in $\mathbb{R}^2$. Later, Cabré and Terra proved the existence of a saddle solution in every dimension $2m \geq 2$, and they established some qualitative properties such as asymptotic behavior, monotonicity properties, as well as instability in dimensions $2m = 4$ and $2m = 6$ (see [43, 44]). The uniqueness in dimensions higher than 2 was established by Cabré in [34], where he also proved that the saddle solution is stable in dimensions $2m \geq 14$.

In the nonlocal framework, there are only two works concerning saddle-shaped solutions to (2.1.1). In [60, 61], first for $\gamma = 1/2$ and then for $\gamma \in (0,1)$, Cinti proved the existence of a saddle-shaped solution to (2.1.1) as well as some qualitative properties such as asymptotic behavior, monotonicity properties, and instability in low dimensions (see Theorem 2.1.2 below).

In the present paper, we prove further properties of these solutions, the main ones being uniqueness and, when $2m \geq 14$, stability. Uniqueness is important since then the saddle-shaped solution becomes a canonical object associated to the Allen-Cahn equation and the Simons cone.

In [60, 61], the main tool used is the extension problem for the fractional Laplacian, due to Caffarelli and Silvestre [51] (see (2.1.3) below). This is also the approach of the present paper. It should be remarked that the extension technique has the limitation that it only works for the fractional Laplacian, and therefore the same arguments cannot be carried out for more general integro-differential operators of the form

$$L_K u(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} \{u(x) - u(\tilde{x})\} K(x - \tilde{x})\, d\tilde{x}.$$

In two forthcoming papers [86, 87] we address this problem by studying saddle-shaped solutions to equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is an elliptic integro-differential operator of the previous form with a radially symmetric kernel K. One of the most basic tools that we need is a maximum principle for the operator acting on functions which are odd with respect to the Simons cone. In [86] we find a sufficient condition to have such a maximum principle and, as we will see there, this will require a certain convexity property of the kernel K.

Let us now introduce the extension problem for the fractional Laplacian, which is the main tool used in this paper. First we should settle the notation. We call $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ and denote points by $(x, \lambda) \in \mathbb{R}^{n+1}_+$ with $x \in \mathbb{R}^n$ and $\lambda > 0$. As it is well known, see [51], if $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ solves $\operatorname{div}(\lambda^a \nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ with $a = 1 - 2\gamma$, then

$$\frac{\partial u}{\partial \nu^a}(x) := -\lim_{\lambda \downarrow 0} \lambda^a u_\lambda(x, \lambda) = \frac{(-\Delta)^\gamma u(x,0)}{d_\gamma},$$

where d_γ is a positive constant depending only on γ. Therefore, problem (2.1.1) is equivalent to

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ d_\gamma \dfrac{\partial u}{\partial \nu^a} &= f(u) \quad \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n. \end{cases} \tag{2.1.3}$$

We will always consider functions defined in $\mathbb{R}^{n+1}_+$ and not only in $\mathbb{R}^n$, and we will use the same letter to denote both the function and its trace on $\mathbb{R}^n$. Regarding sets in $\mathbb{R}^{n+1}_+$, we use the following notation. If $\Omega \subset \mathbb{R}^{n+1}_+$, we define

$$\partial_L \Omega := \overline{\partial\Omega \cap \{\lambda > 0\}} \quad \text{and} \quad \partial_0\Omega := \partial\Omega \setminus \partial_L\Omega \subset \{\lambda = 0\}.$$

We write

$$B_R^+ := \Big\{ (x,\lambda) \in \mathbb{R}^{n+1}_+ \ : \ |(x,\lambda)| < R \Big\},$$

for half-balls in $\mathbb{R}^{n+1}_+$. If $x_0 \in \mathbb{R}^n$, $B_R^+(x_0) = (x_0, 0) + B_R^+$.

A certain solution of problem (2.1.1) in dimension 1, the so-called *layer solution*, plays a crucial role through this paper. It is the unique solution of the following problem:

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_0) &= 0 \quad \text{in } \mathbb{R}^2_+ = \mathbb{R}\times(0,+\infty), \\ d_\gamma \dfrac{\partial u_0}{\partial \nu^a} &= f(u_0) \quad \text{on } \partial\mathbb{R}^2_+ = \mathbb{R}, \\ \partial_x u_0 &> 0 \quad \text{on } \partial\mathbb{R}^2_+ = \mathbb{R}, \\ u_0(0,0) &= 0, \\ \lim\limits_{x\to\pm\infty} u_0(x,0) &= \pm 1. \end{cases} \tag{2.1.4}$$

Under the assumptions on f in (2.1.2), the existence and uniqueness of such solution are well known (see [40]).

The importance of the layer solution comes from the fact that the associated function

$$U(x,\lambda) := u_0\left(\frac{s-t}{\sqrt{2}}, \lambda\right) \quad \text{for } x \in \mathbb{R}^{2m} \text{ and } \lambda > 0, \tag{2.1.5}$$

which is odd with respect to the Simons cone and positive in $\mathcal{O} \times [0,+\infty)$, describes the asymptotic behavior of saddle-shaped solutions at infinity (as shown in [60, 61]; see Theorem 2.1.2 below). Note that from Lemma 4.2 in [43], we know that $|s-t|/\sqrt{2}$ is the distance to the Simons cone. Therefore, we can understand the function U as the layer solution centered at each point of the Simons cone and oriented in the normal direction to the cone. Moreover, in this paper we show (see Proposition 2.1.5) that the saddle-shaped solution lies below U in $\mathcal{O}$, as it occurs in the local case (see Proposition 1.5 in [43]).

It is sometimes useful to consider also the following variables:

$$y := \frac{s+t}{\sqrt{2}} \quad \text{and} \quad z := \frac{s-t}{\sqrt{2}},$$

which satisfy $y \geq 0$ and $-y \leq z \leq y$. In these variables, $\mathscr{C} = \{z = 0\}$ and $\mathcal{O} = \{z > 0\}$. Therefore, we can write $U(x,\lambda) = u_0(z,\lambda)$.

To study the minimality and stability of the saddle-shaped solution, we recall the energy functional associated to equation (2.1.3):

$$\mathcal{E}(w,\Omega) := \frac{d_\gamma}{2}\int_\Omega \lambda^a |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}\lambda + \int_{\partial_0\Omega} G(w)\,\mathrm{d}x, \quad \text{where } G' = -f.$$

We say that u is a minimizer for problem (2.1.3) in $\Omega \subset \mathbb{R}^{2m+1}_+$ if

$$\mathcal{E}(u,\Omega) \leq \mathcal{E}(w,\Omega)$$

for every w such that $w = u$ on $\partial_L \Omega$. Observe that the admissible competitors do not have the boundary condition prescribed on $\partial_0 \Omega$. This is in correspondence with the Neumann condition in (2.1.3). We say that u is a global minimizer if it is a minimizer in every bounded domain Ω of $\mathbb{R}^{2m+1}_+$.

A bounded solution to (2.1.3) is said to be *stable* if the second variation of the energy with respect to perturbations ξ which have compact support in $\overline{\mathbb{R}^{2m+1}_+}$ is nonnegative. That is, if

$$\int_{\mathbb{R}^{2m}} f'(u)\,\xi^2 \,\mathrm{d}x \leq d_\gamma \int_0^\infty \int_{\mathbb{R}^{2m}} \lambda^a \,|\nabla \xi|^2 \,\mathrm{d}x \,\mathrm{d}\lambda \tag{2.1.6}$$

for every $\xi \in C^\infty_c(\overline{\mathbb{R}^{2m+1}_+})$.

In the following theorem we collect the known results concerning saddle-shaped solutions to (2.1.1).

Theorem 2.1.2 ([60, 61, 42, 41]). *Let $\gamma \in (0,1)$ and let $f \in C^{2,\alpha}\big((-1,1)\big)$ be a function satisfying* (2.1.2).

(i) For every even dimension $2m \geq 2$, there exists a saddle-shaped solution to problem (2.1.1) *with $|u| < 1$.*

(ii) For every even dimension $2m \geq 2$, every saddle-shaped solution to problem (2.1.1) *satisfies*

$$\big|\big|\,|u-U| + |\nabla_x(u-U)|\,\big|\big|_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} \to 0, \quad \text{as } R \to +\infty,$$

where U is defined in (2.1.5).

(iii) In dimension $2m$ with $2 \leq 2m \leq 6$, every saddle-shaped solution is unstable.

Here ∇_x denotes the gradient only in the horizontal variables $x \in \mathbb{R}^{2m}$, not to be confused with the gradient $\nabla = \nabla_{(x,\lambda)}$ in (2.1.3) or (2.1.6), for instance.

Points (i) and (ii) of Theorem 2.1.2 were proved by Cinti, first for $\gamma = 1/2$ in [60] and then extended to all powers $\gamma \in (0,1)$ in [61]. Instability in dimension $2m = 2$ follows from a general result on stable solutions established in [41] (previously proved for $\gamma = 1/2$ in [42]). Instead, instability in dimensions $2m = 4$ and $2m = 6$ was proved in [60, 61].

Our first main result is the uniqueness of the saddle-shaped solution. As a consequence, such solution to the fractional Allen-Cahn equation becomes a canonical object associated to the cone $\mathscr{C}$.

Theorem 2.1.3. *Let $\gamma \in (0,1)$ and let f be a function satisfying* (2.1.2). *Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution to problem* (2.1.3).

As in the paper of Cabré [34] for the classical case, the proof of the uniqueness result follows from the asymptotic behavior of the saddle solution (point (ii) in Theorem 2.1.2) and a maximum principle in $\mathcal{O}$ for the linearized operator at a saddle-shaped solution. The maximum principle is the following.

Proposition 2.1.4. *Let u be a saddle-shaped solution of* (2.1.3)*. Let $\Omega \subset \mathcal{O} \times (0,+\infty) \subset \mathbb{R}^{2m+1}_+$ be an open set such that $\partial_0\Omega$ is nonempty. Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be bounded from above and such that $\lambda^a v_\lambda \in C(\overline{\Omega})$.*

Consider the operator $\mathscr{L}_u$ defined by

$$\mathscr{L}_u v := d_\gamma \frac{\partial v}{\partial \nu^a} - f'(u)v \quad \text{on } \partial_0\Omega \subset \mathbb{R}^{2m} \times \{0\}, \tag{2.1.7}$$

and assume that

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \leq & b(x,\lambda)v & \text{in } \Omega \subset \mathcal{O} \times (0,+\infty)\,, \\ \mathscr{L}_u v & \leq & 0 & \text{on } \partial_0\Omega \subset \mathcal{O}\,, \\ v & \leq & 0 & \text{on } \partial_L\Omega\,, \\ \limsup\limits_{x\in\partial_0\Omega,\ |x|\to+\infty} \quad v(x,0) & \leq & 0\,, & \end{cases}$$

with $b \leq 0$. Then, $v \leq 0$ in Ω.

To establish the previous maximum principle we follow the proof of the analogous result for the local case ($\gamma = 1$) in [34]. It involves a maximum principle in "narrow" sets (see also [32, 16]). The main difference between our proof and the one in [34] is that, since we are using the extension problem, a new notion of narrowness is needed to carry out the same type of arguments (see Section 2.2 for the details).

The second main result of this paper is the following pointwise estimate for the saddle-shaped solution. We prove that the function $U(s,t,\lambda) := u_0((s-t)/\sqrt{2},\lambda)$ is a barrier for the saddle-shaped solution. This result was established in the local setting ($\gamma = 1$) in [43], but in such case the proof is quite simple by using the so-called Modica estimate (see [43] for the details). In the fractional framework, this estimate is only available (in a nonlocal form) in dimension 1 (see [42, 40]) and therefore we need another type of argument. Our strategy is to use a maximum principle for the linearized operator at U, similar to the one in Proposition 2.1.4. The pointwise estimate we establish is the following.

Proposition 2.1.5. *Let u be the saddle-shaped solution of* (2.1.3)*, let u_0 be the layer solution given by* (2.1.4) *and let U be defined by* (2.1.5)*. Then,*

$$|u(x,\lambda)| \leq |U(x,\lambda)| = |u_0(\operatorname{dist}(x,\mathscr{C}),\lambda)| \quad \text{for every } (x,\lambda) \in \overline{\mathbb{R}^{2m+1}_+}. \tag{2.1.8}$$

The third main result of the present paper establishes the stability of the saddle solution in high dimensions. This is an extension of Theorem 1.4 in [34] to the nonlocal case. For its proof, it is crucial to use the extension problem.

Theorem 2.1.6. *Assume that f satisfies* (2.1.2)*. If $2m \geq 14$, then the saddle-shaped solution u of* (2.1.3) *is stable in $\mathbb{R}^{2m+1}_+$, i.e.,* (2.1.6) *holds.*

Its stability is a consequence of the following fact. For every constant $b > 0$ satisfying $b(b-m+2) \leq -(m-1)$, the function

$$\varphi := t^{-b}\, u_s - s^{-b} u_t,$$

defined in $\mathbb{R}^{2m+1}_+ \setminus \{st = 0\}$, is even with respect to the Simons cone and is a positive supersolution of the linearized operator. More precisely, $-\operatorname{div}(\lambda^a \nabla \varphi) \geq 0$ in $\mathbb{R}^{2m+1}_+ \setminus \{st = 0\}$ and $\mathscr{L}_u \varphi \geq 0$ in $\mathbb{R}^{2m} \setminus \{st = 0\}$, where $\mathscr{L}_u$ is defined in (2.1.7)*.*

An important consequence of this result is Corollary 2.1.7, stated next, on the stability of the Simons cone as a (2γ)-minimal surface in dimensions $2m \geq 14$. This is the first analytical proof of its stability for some γ and m. It follows directly from the convergence results proved in [39] for stable solutions to the Allen-Cahn equation after a blow-down, together with the preservation of the stability along this procedure (see also the comments at the end of this introduction).

Corollary 2.1.7. *Let $\gamma \in (0, 1/2)$ and $2m \geq 14$. Then, the Simons cone $\mathscr{C} \subset \mathbb{R}^{2m}$ is a stable (2γ)-minimal surface.*

The key ingredients to prove Theorem 2.1.6 are some monotonicity and second derivative properties for the saddle-shaped solution. In fact, φ being a positive supersolution will follow from such properties. More precisely, our arguments will use the following.

Proposition 2.1.8. *Let u be the saddle-shaped solution to (2.1.3). Then,*

(i) $u_y > 0$ *in* $\mathcal{O} \times [0, +\infty)$.

(ii) $-u_t > 0$ *in* $(\mathcal{O} \setminus \{t = 0\}) \times [0, +\infty)$.

(iii) $u_{st} > 0$ *in* $(\mathcal{O} \setminus \{t = 0\}) \times [0, +\infty)$.

As a consequence, for every direction $\partial_\eta = \alpha\partial_y - \beta\partial_t$, where α and β are nonnegative constants, $\partial_\eta u > 0$ in $\{s > t > 0,\ \lambda \geq 0\}$.

The monotonicity properties (i) and (ii) were proved in the papers of Cinti [60, 61] for the so-called maximal saddle solution —note that in those papers the uniqueness of the saddle-shaped solution was not known yet. From her result and our uniqueness theorem, (i) and (ii) in Proposition 2.1.8 follow. Nevertheless, we present here a new proof of them by applying the maximum principle for the linearized operator to certain equations satisfied by u_s and u_t. A similar argument will establish the new property (iii) for the crossed second derivative u_{st}.

To conclude this introduction, let us comment briefly on the importance of problem (2.1.1) and its relation with a conjecture of De Giorgi and the theory of minimal surfaces.

The interest on problem (2.1.1) originates from a famous conjecture of De Giorgi for the classical Allen-Cahn equation. It reads as follows. Let u be a bounded solution to $-\Delta u = u - u^3$ in $\mathbb{R}^n$ which is monotone in one direction, say $\partial_{x_n} u > 0$. Then, if $n \leq 8$, u is one dimensional, i.e., u depends only on one Euclidean variable. This conjecture was proved to be true in dimension $n = 2$ by Ghoussoub and Gui in [96], and in dimension $n = 3$ by Ambrosio and Cabré in [4]. For dimensions $4 \leq n \leq 8$, it was established by Savin in [132] but with the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{2.1.9}$$

A counterexample to the conjecture in dimensions $n \geq 9$ was given by del Pino, Kowalczyk and Wei in [74].

The corresponding conjecture in the nonlocal setting, where one replaces the operator $-\Delta$ by $(-\Delta)^\gamma$, has been widely studied in the last years. In this framework, the conjecture has been proven to be true in dimension $n = 2$ by Cabré and Solà-Morales in [42] for $\gamma = 1/2$, and extended to every power $0 < \gamma < 1$ by Cabré and Sire in [41] and also by Sire and Valdinoci in [143]. In dimension $n = 3$, the conjecture has been proved by Cabré and Cinti for $1/2 \le \gamma < 1$ in [36, 37] and by Dipierro, Farina, and Valdinoci for $0 < \gamma < 1/2$ in [76]. Recently, in [133, 134] Savin has established the validity of the conjecture in dimensions $4 \le n \le 8$ and for $1/2 \le \gamma < 1$, but assuming the additional hypothesis (2.1.9). Under the same extra assumption, the conjecture is true in the same dimensions for $0 < \gamma < 1/2$ and γ close to $1/2$, as proved by Dipierro, Serra, and Valdinoci in [78]. The most recent result concerning the proof of the conjecture is the one by Figalli and Serra in [90], where they have established the conjecture in dimension $n = 4$ and $\gamma = 1/2$ without requiring the additional limiting assumption (2.1.9). Note that, without (2.1.9), the analogous result for the Laplacian in dimension $n = 4$ is not known. In the forthcoming paper [39], Cabré, Cinti, and Serra prove the conjecture in dimension $n = 4$ for $0 < \gamma < 1/2$ and γ sufficiently close to $1/2$. A counterexample to the De Giorgi conjecture for fractional Allen-Cahn equation in dimensions $n \ge 9$ for $\gamma \in (1/2, 1)$ has been very recently announced in [57].

Coming back to the local Allen-Cahn equation, while studying this conjecture by De Giorgi, another question arose naturally: do global minimizers in $\mathbb{R}^n$ of the Allen-Cahn energy have one-dimensional symmetry? A deep result from Savin [132] states that in dimension $n \le 7$ this is indeed true. On the other hand, it is conjectured that this is false for $n \ge 8$ and that the saddle-shaped solution is a counterexample (since the Simons cone is a global minimizer of the perimeter functional in these dimensions). The answer to this question would provide an alternative construction of a counterexample to the original conjecture of De Giorgi, different from the one of [74]. This is due to a result by Jerison and Monneau [106], where they show that a counterexample to the original conjecture of De Giorgi in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate. The saddle-shaped solution is of special interest in relation with the Jerison-Monneau program since it is even with respect to all the coordinate axis and it is expected to be a minimizer in high dimensions.

Let us explain why the Allen-Cahn equation has a very strong connection with the theory of minimal surfaces. A deep result from the seventies by Modica and Mortola (see [118, 119]) states that considering an appropriately rescaled version of the Allen-Cahn equation, the corresponding energy functionals Γ-converge to the perimeter functional. Thus, the minimizers of the equation converge to the characteristic function of a set of minimal perimeter. This same fact holds for the equation with the fractional Laplacian, though we have two different scenarios depending on the parameter $\gamma \in (0, 1)$. If $\gamma \ge 1/2$, the rescaled energy functionals associated to (2.1.1) Γ-converge to the classical perimeter (see [**?**, 102]), while in the case $\gamma \in (0, 1/2)$ they Γ-converge to the fractional perimeter (see [135]). As a consequence, if the saddle-shaped solution was proved to be a minimizer in a certain dimension for some $\gamma \in (0, 1/2)$, it would follow that the Simons cone $\mathscr{C}$ would be a minimizing nonlocal (2γ)-minimal surface in such dimensions. As mentioned before, this last statement is an open problem in any dimension. Our Corollary 2.1.7 on stability is related to this question, but for a weaker property than minimality.

By a result of Cabré, Cinti, and Serra in [39], also the stability is preserved in the blow-down limit when $\gamma \in (0, 1/2)$. Therefore, a limit of stable solutions to (2.1.1) with $\gamma \in (0, 1/2)$ will be a stable set for the (2γ)-perimeter. Thus, as a consequence of Theorem 2.1.6 we deduce Corollary 2.1.7.

The paper is organized as follows. In section 2.2 we prove the maximum principle for the linearized operator in $\mathcal{O}$, Proposition 2.1.4. Section 2.3 is devoted to show Theorem 2.1.3 concerning the uniqueness of the saddle-shaped solution. In Section 2.4 we establish some monotonicity properties of the layer solution u_0, as well as the pointwise estimate for the saddle solution in terms of the layer u_0, stated in Proposition 2.1.5. In Section 2.5 we prove the monotonicity and second derivative properties of the saddle solution stated in Proposition 2.1.8. Finally, Section 2.6 concerns the proof of the stability results, Theorem 2.1.6 and Corollary 2.1.7.

2.2 Maximum principles for the linearized operator

In this section we establish Proposition 2.1.4, a maximum principle for the linearized operator. To prove it, we follow the ideas appearing in [34], where an analogous maximum principle is proved for the local case $\gamma = 1$. The proof for the Laplacian uses a maximum principle in "narrow" sets (see for instance [32, 16]). In our case, the use of the extension problem requires a similar maximum principle but in pairs of sets that we will call "extension-narrow", defined next.

Definition 2.2.1 ("Extension-narrow" pair of sets). Let $\Omega \subset \mathbb{R}^{n+1}_+$ be an open set, not necessarily bounded, and let $\Gamma \subset \partial_0\Omega$ be nonempty. Given $\theta \in (0,1)$ and $a \in (-1,1)$, we define $R_a(\Omega,\Gamma,\theta) \in (0,+\infty]$ to be the smallest positive constant R for which

$$\frac{|B_R^+(x) \setminus \Omega|_a}{|B_R^+(x)|_a} \geq \theta \quad \text{for every } x \in \Gamma\,, \tag{2.2.1}$$

where

$$|E|_a := \int_E \lambda^a \,\mathrm{d}x\,\mathrm{d}\lambda\,.$$

We say that $R_a(\Omega,\Gamma,\theta) = +\infty$ if no such radius exists.

From this definition, we will say that a pair (Ω,Γ) is "extension-narrow" if $R_a(\Omega,\Gamma,\theta)$ is small enough depending on certain quantities.

Note that if in (2.2.1) we consider $a = 0$ and full balls centered at every point $x \in \Omega$, we recover the usual definition of "narrow" set. Here, instead, we only consider half-balls centered at points $x \in \Gamma \subset \partial_0\Omega$.

Once the quantity $R_a(\Omega,\Gamma,\theta)$ is defined, we can state precisely the maximum principle in "extension-narrow" pairs.

Proposition 2.2.2 (Maximum principle in "extension-narrow" pairs). *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be an open set and let $\Gamma \subset \partial_0\Omega$ be nonempty. Assume that there exists a nonempty open cone $E \subset \partial_0\mathbb{R}^{n+1}_+ = \mathbb{R}^n$ such that $(E \times (0,+\infty)) \cap \Omega = \varnothing$.*

Let $a \in (-1,1)$ and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a function bounded from above such that $\lambda^a v_\lambda \in C(\overline{\Omega})$, and assume that it satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \leq & b(x,\lambda)v & \text{in } \Omega\,, \\ \dfrac{\partial v}{\partial \nu^a} + c(x)v & \leq & 0 & \text{on } \Gamma\,, \\ v & \leq & 0 & \text{on } \partial\Omega \setminus \Gamma\,, \end{cases} \tag{2.2.2}$$

where $b \leq 0$ in Ω and c is bounded from below on Γ.

Then, for every $\theta \in (0,1)$ there exists a constant R^*, depending only on n, a, θ, and $||c_-||_{L^\infty(\Gamma)}$, such that $v \leq 0$ in Ω whenever $R_a(\Omega,\Gamma,\theta) \leq R^*$.

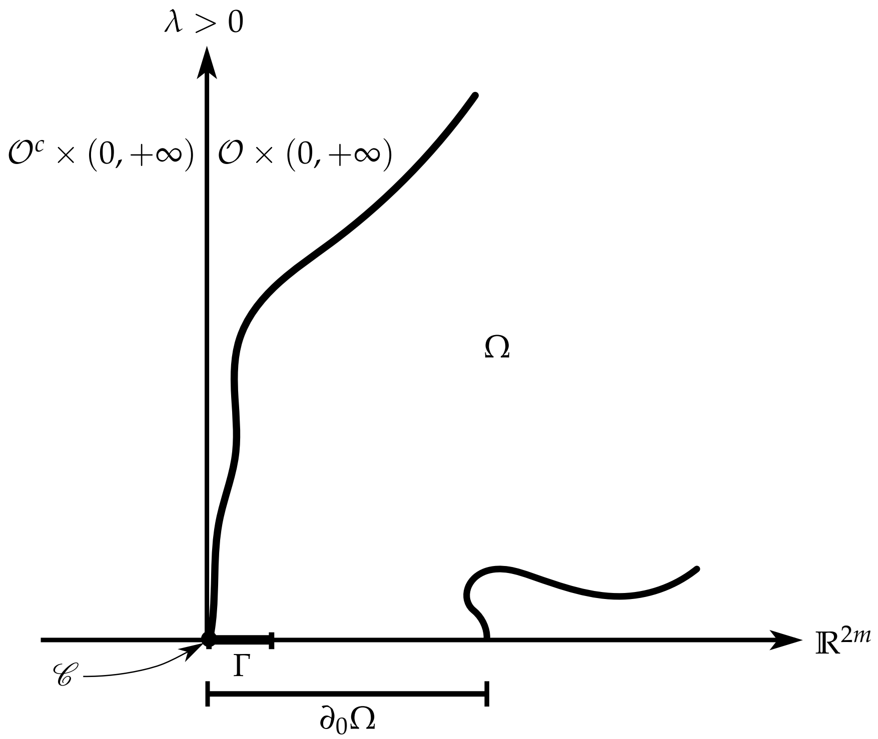

Figure 2.1: An example of a pair (Ω, Γ) which is "extension-narrow".

Before proving this result, let us explain why we need to introduce the notion of "extension-narrowness". We will use this maximum principle in a pair (Ω, Γ) with $\Omega \subset \mathcal{O} \times (0,+\infty)$ and $\Gamma \subset \partial_0\Omega$ in an ε-neighborhood in $\mathcal{O}$ of the cone $\mathscr{C}$. In this case, Ω could be very big (and not "narrow" in the usual sense) in $\mathbb{R}^{2m+1}_+$, as in Figure 2.1. However, $\mathcal{O}^c \times (0,+\infty)$ is contained in the complement of Ω — even if Ω filled all $\mathcal{O} \times (0,+\infty)$. Thus, it follows readily that (Ω, Γ) is "extension-narrow" by using that balls in this notion are centered in Γ (see Corollary 2.2.5 below for the details).

To prove Proposition 2.2.2 we need the following weak Harnack inequality.

Proposition 2.2.3 (Proposition 3.2 of [146]). *Let $v \in H^1(B_R^+, \lambda^a)$ be a nonnegative function that weakly satisfies*

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \geq & 0 & \text{in } B_R^+\,, \\ \dfrac{\partial v}{\partial \nu^a} & \geq & 0 & \text{on } \partial_0 B_R^+\,. \end{cases}$$

Then, there exists a constant $p_0 > 0$, depending only on n and a, such that for all $p \leq p_0$,

$$\left(\int_{B^+_{R/2}} \lambda^a v^p \, \mathrm{d}x \, \mathrm{d}\lambda \right)^{1/p} \leq C_h R^{\frac{n+1+a}{p}} \inf_{B^+_{R/4}} v\,, \tag{2.2.3}$$

for a positive constant C_h depending only on n and a.

With this result available, we can now present the proof of the maximum principle in "extension-narrow" pairs.

Proof of Proposition 2.2.2. Define the sets

$$\Omega_+ := \{(x,\lambda) \in \Omega \,:\, v(x,\lambda) > 0\} \quad \text{and} \quad \Gamma_+ := \partial\Omega_+ \cap \Gamma\,,$$

and by contradiction assume that Ω_+ is nonempty. Then, since $b \le 0$, v satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \le & 0 & \text{in } \Omega_+\,, \\ \dfrac{\partial v}{\partial \nu^a} + c(x)v & \le & 0 & \text{on } \Gamma_+ \text{ (if this set is nonempty)}\,, \\ v & \le & 0 & \text{on } \partial\Omega_+ \setminus \Gamma_+\,. \end{cases}$$

Now, we proceed in two steps in order to arrive at a contradiction.

Step 1. First, we claim that if Γ_+ is nonempty then $\sup_{\Omega_+} v = \sup_{\Gamma_+} v$. That is, if we call

$$\overline{v} := v - \sup_{\Gamma_+} v,$$

we then have $\overline{v} \le 0$ in Ω_+. To prove this, we use a classical Phragmen-Lindelöf-type argument, as follows. Similar methods appear, among many others, in the proof of Theorem 1.2 of [12], or Section 2.4 of [42].

We now claim that, since the cone E is open, there exists a nonempty open cone $F \subset E$ satisfying

$$|x-y| \ge c_0 > 0 \quad \text{for every } x \in E^c \text{ and } y \in \overline{F}\,, \tag{2.2.4}$$

for some positive constant c_0.

Indeed, since E is an open cone (with vertex, say, $z \in \partial E$), there exists a circular cone $E' \subset E$ with the same vertex z. Then, by sliding this circular cone in the direction of its axis, which can be assumed to be $e_n = (0,\ldots,0,1)$, we obtain a new open cone $F \subset E$. Let us now show (2.2.4). Since $F \subset E' \subset E$, it is enough to prove (2.2.4) for $x \in \partial E'$ and $y \in \partial F$. Hence, we have

$$x_n - z_n = \omega|x'-z'| \quad \text{and} \quad y_n - z_n = \tau + \omega|y'-z'|,$$

for some positive constants ω and τ. Here, we are using the notation $z = (z', z_n)$. Now, if we call $\sigma = |x'-z'| - |y'-z'|$, we have $|x'-y'| \ge |\sigma|$ and thus

$$\begin{aligned} |x-y|^2 &= |x'-y'|^2 + |x_n-y_n|^2 \ge \sigma^2 + (\omega\sigma - \tau)^2 \\ &= \left(\sqrt{1+\omega^2}\sigma - \frac{\omega\tau}{\sqrt{1+\omega^2}}\right)^2 + \frac{\tau^2}{1+\omega^2} \ge \frac{\tau^2}{1+\omega^2}, \end{aligned}$$

where the last constant is in fact the minimum distance between points on $\partial E'$ and ∂F.

Now, without loss of generality, we may assume that the vertex of F is the origin. Let F' be an open cone with the same vertex as F, and such that $\overline{F' \cap \mathbb{S}^{n-1}} \subset F \cap \mathbb{S}^{n-1}$. Let ϕ be the first eigenfunction of the Laplace-Beltrami operator in $\mathbb{S}^{n-1} \setminus \overline{F'} \subset \mathbb{R}^n$ with zero Dirichlet boundary conditions on $\partial F' \cap \mathbb{S}^{n-1}$, and let $\mu > 0$ be its associated eigenvalue. Since $\partial F' \cap \mathbb{S}^{n-1}$ is contained in F, there exists a positive constant δ such that $\phi \ge \delta > 0$ in $\mathbb{S}^{n-1} \setminus \overline{F}$. Now, define the auxiliary function

$$\psi(x,\lambda) = (1+\lambda^{2\gamma})|x|^\beta \phi(x/|x|),$$

where β is a positive real number and $\gamma = (1-a)/2 \in (0,1)$. Then, $\phi(x/|x|) \ge \delta$ for each $(x,\lambda) \in \Omega_+$, since $x/|x| \in \mathbb{S}^{n-1} \setminus \overline{F}$. Moreover, by (2.2.4) with $y = 0$, we deduce that

$$\psi(x,\lambda) \ge \delta(1+\lambda^{2\gamma})|x|^\beta \ge \delta c_0^\beta > 0 \quad \text{in } \Omega_+,$$

since 0 is the vertex of F. On the other hand, note that if we choose $\beta > 0$ solving $\beta(\beta + n - 2) = \mu$, we have that ψ satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla \psi) &= 0 \quad \text{in } \Omega_+, \\ \lim\limits_{(x,\lambda)\in\Omega_+,\ |(x,\lambda)|\to+\infty} \psi &= +\infty. \end{cases}$$

Thus, if we define

$$\overline{w} := \frac{\overline{v}}{\psi} = \frac{v - \sup_{\Gamma_+} v}{\psi},$$

proving that $\overline{v} \leq 0$ in Ω_+ is equivalent to showing that $\overline{w} \leq 0$ in Ω_+, since ψ is positive. Now, since $\sup_{\Gamma_+} v \geq 0$, it is easy to show that $\overline{w}$ satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla \overline{w}) - 2\lambda^a \dfrac{\nabla \psi}{\psi} \cdot \nabla \overline{w} &\leq 0 \quad \text{in } \Omega_+, \\ \overline{w} &\leq 0 \quad \text{on } \partial\Omega_+, \\ \lim\limits_{(x,\lambda)\in\Omega_+,\ |(x,\lambda)|\to+\infty} \overline{w} &\leq 0. \end{cases} \tag{2.2.5}$$

Then, by the classical maximum principle we deduce that $\overline{w} \leq 0$ in Ω_+, which yields $\overline{v} \leq 0$ in Ω_+.

Note that if Γ_+ is empty, the same argument applied to v instead of $\overline{v}$ yields a contradiction with the assumption that Ω_+ is nonempty. From now on in this proof, we will assume that $\Gamma_+ \neq \varnothing$.

Step 2. By Step 1 and the definition of Ω_+, we have that

$$M := \sup_{\Gamma_+} v > 0. \tag{2.2.6}$$

Therefore, since $v \leq 0$ on $\partial\Omega_+ \setminus \Gamma_+$, there exists a sequence $(x_k, 0) \in \Gamma_+$ such that

$$v(x_k) = v(x_k, 0) \geq M\left(1 - \frac{1}{k}\right),$$

where we are identifying v with its trace on $\mathbb{R}^n$ to simplify the notation.

Now, given any $R > 0$, let $\bar{c}_{n,\gamma}$ be the constant such that

$$(-\Delta)^\gamma \{\bar{c}_{n,\gamma}(R^2 - |x - x_k|^2)_+^\gamma\} = 1 \quad \text{in } B_R(x_k),$$

(see [20] for its explicit value) and take $\phi = \phi(x, \lambda)$ to be the γ-harmonic extension of

$$\phi(x, 0) = c_1 M \bar{c}_{n,\gamma}(R^2 - |x - x_k|^2)_+^\gamma,$$

where c_1 is a positive constant to be chosen later. Thus, ϕ solves

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla \phi) &= 0 \quad \text{in } B_R^+(x_k), \\ \dfrac{\partial \phi}{\partial \nu^a} &= \dfrac{c_1 M}{d_\gamma} \quad \text{on } \partial_0 B_R^+(x_k). \end{cases}$$

Moreover, on $\partial_0 B_R^+(x_k) \cap \Gamma_+$ we have

$$\frac{\partial v}{\partial \nu^a} \leq -cv \leq ||c_-||_{L^\infty(\Gamma)}\, v \leq ||c_-||_{L^\infty(\Gamma)}\, M \leq \frac{\partial \phi}{\partial \nu^a}$$

if we choose $c_1 > d_\gamma \, ||c_-||_{L^\infty(\Gamma)}$.

Thus, $v - \phi$ is γ-subharmonic in $B_R^+(x_k) \cap \Omega_+$ and has a nonpositive flux on the set $\partial_0 B_R^+(x_k) \cap \Gamma_+$. In addition, $v - \phi \le v \le 0$ in $B_R^+(x_k) \cap (\partial\Omega_+ \setminus \Gamma_+)$. Therefore, its positive part $(v-\phi)_+$ extended to be zero in $B_R^+(x_k) \setminus \Omega_+$ is a continuous function which is γ-subharmonic in $B_R^+(x_k)$ and has a nonpositive flux on $\partial_0 B_R^+(x_k)$, both properties in a weak sense.

We define $w := M - (v-\phi)_+$, which is a continuous nonnegative function and satisfies in a weak sense

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla w) & \ge & 0 & \text{in } B_R^+(x_k)\,, \\ \dfrac{\partial w}{\partial \nu^a} & \ge & 0 & \text{on } \partial_0 B_R^+(x_k)\,. \end{cases}$$

Hence, w fulfills the hypotheses of Proposition 2.2.3 and thus (2.2.3) holds. As a consequence, if we take $R = 2\,R_a(\Omega,\Gamma,\theta)$ and p as in (2.2.3), we have

$$\begin{aligned}
\theta^{1/p} M &\le \left(\frac{|B_{R/2}^+(x_k) \setminus \Omega|_a}{|B_{R/2}^+(x_k)|_a} M^p \right)^{1/p} \\
&\le \left(\frac{|B_{R/2}^+(x_k) \setminus \Omega_+|_a}{|B_{R/2}^+(x_k)|_a} M^p \right)^{1/p} \\
&= \frac{1}{|B_{R/2}^+(x_k)|_a^{1/p}} \left(\int_{B_{R/2}^+(x_k)\setminus\Omega_+} \lambda^a M^p \, \mathrm{d}x \, \mathrm{d}\lambda \right)^{1/p} \\
&\le |B_1^+|_a^{-1/p} (R/2)^{-\frac{n+1+a}{p}} \left(\int_{B_{R/2}^+(x_k)} \lambda^a w^p \, \mathrm{d}x \, \mathrm{d}\lambda \right)^{1/p} \\
&\le 2^{\frac{n+1+a}{p}} |B_1^+|_a^{-1/p} C_h \inf_{B_{R/4}^+(x_k)} w \\
&\le 2^{\frac{n+1+a}{p}} |B_1^+|_a^{-1/p} C_h \, w(x_k).
\end{aligned}$$

Here we have used the definition of $R_a(\Omega,\Gamma,\theta)$, the fact that $w \equiv M$ in $B_R^+(x_k) \setminus \Omega_+$, the scaling properties of $|\cdot|_a$ and the weak Harnack inequality (2.2.3).

Now, if $c_1 \bar{c}_{n,\gamma} R^{2\gamma} \le 1/2$, then $w(x_k) = M - v(x_k) + \phi(x_k)$ for k large enough. Therefore, for such indices k we conclude

$$\begin{aligned}
\theta^{1/p} M &\le 2^{\frac{n+1+a}{p}} |B_1^+|_a^{-1/p} C_h \{ M - v(x_k) + \phi(x_k) \} \\
&\le 2^{\frac{n+1+a}{p}} |B_1^+|_a^{-1/p} C_h \{ 1/k + c_1 \bar{c}_{n,\gamma} R^{2\gamma} \} M\,.
\end{aligned}$$

Hence, if we take $R_a(\Omega,\Gamma,\theta)$ small enough such that $c_1 \bar{c}_{n,\gamma} (2R_a(\Omega,\Gamma,\theta))^{2\gamma} < 1$ and $2^{\frac{n+1+a}{p}} |B_1^+|_a^{-1/p} C_h c_1 \bar{c}_{n,\gamma} (2R_a(\Omega,\Gamma,\theta))^{2\gamma} < \theta^{1/p}$, we get that

$$M\left(1 - \frac{C}{k}\right) \le 0$$

for some positive constant C independent of k. Letting $k \to +\infty$, this leads to $M \le 0$, which contradicts (2.2.6).

Therefore, our initial assumption stating $\Omega_+ \neq \varnothing$ is false. This means that $v \le 0$ in Ω. $\square$

Remark 2.2.4. It will be useful later to note that the previous result (and as a consequence, Proposition 2.1.4) is also valid not requiring v to be C^2 in the whole Ω. Indeed, we only need to assume that $v \in C(\Omega)$, that the equation $\operatorname{div}(\lambda^a \nabla v) \leq b(x,\lambda)v$ holds pointwise where v is regular, and that v cannot have a local maximum at a nonregular point.

This will be used in the proof of Proposition 2.1.5 with $v = u - CU$ in $\Omega = \mathcal{O} \times (0, +\infty)$, where u is a saddle-shaped solution, U is defined by (2.1.5), and C is a positive constant. Note that U is Lipschitz but not C^2 across $\{t = 0, \lambda \geq 0\}$. Therefore, as we will see in Section 2.4, U is only γ-superharmonic (pointwise) in $\Omega \setminus \{t = 0, \lambda \geq 0\}$. Nevertheless, by this remark, Proposition 2.2.2 will hold in this case thanks to the fact that the graph of $v = u - CU$ in its nonregular points makes the "good angle" for the maximum principle to hold (see the proof of Proposition 2.1.5 for the details).

As a consequence of Proposition 2.2.2, next we establish that the maximum principle holds in pairs (Ω, Γ) with $\Omega \subset \mathcal{O} \times (0, +\infty) \subset \mathbb{R}^{2m+1}_+$ and $\Gamma \subset \partial_0 \Omega$ lying in an ε-neighborhood of the Simons cone.

Corollary 2.2.5. *Let $\Omega \subset \mathcal{O} \times (0, +\infty) \subset \mathbb{R}^{2m+1}_+$ and let $\Gamma \subset \partial_0 \Omega$ be nonempty. Assume that $\Gamma \subset \mathcal{N}_\varepsilon := \{t < s < t + \varepsilon,\ \lambda = 0\}$.*

Then, if ε is small enough, depending only on n, γ, and $||c_-||_{L^\infty(\Gamma)}$, the maximum principle holds in Ω in the sense of Proposition 2.2.2. That is, if $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is bounded from above, $\lambda^a v_\lambda \in C(\overline{\Omega})$, and v satisfies (2.2.2), then $v \leq 0$ in Ω.

To prove it, it is enough to realize that the Simons cone separates every ball centered at a point in the cone into two regions with comparable measure. In fact, it is interesting to note that these two regions have exactly the same measure, as stated next.

Lemma 2.2.6. *Let $x_0 \in \mathscr{C} \subset \mathbb{R}^{2m}$. Then,*

$$|B_r(x_0) \cap \mathcal{O}| = |B_r(x_0) \setminus \mathcal{O}| = \frac{1}{2}|B_r(x_0)| \quad \textit{for all} \ \ r > 0\,.$$

This result was stated in [34], but without a proof. For the sake of completeness, we include here a simple one.

Proof of Lemma 2.2.6. First, let us call $\mathcal{I} := \mathbb{R}^{2m} \setminus \overline{\mathcal{O}}$. Since $x_0 \in \mathscr{C}$, we have that $x_0 = (x_0', x_0'') \in \mathbb{R}^m \times \mathbb{R}^m$ satisfies that $|x_0'| = |x_0''|$. Therefore, there exists an orthogonal transformation $R \in O(m)$ such that $Rx_0' = x_0''$. Let us define $\overline{R} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by $\overline{R}(x', x'') = (Rx', x'')$, which is a linear isometry that keeps invariant $\mathcal{O}$ and $\mathcal{I}$. With these properties it is easy to check that for every $y \in \mathbb{R}^{2m}$ it holds

$$|B_r(y) \cap \mathcal{I}| = |\overline{R}\left(B_r(y) \cap \mathcal{I}\right)| = |B_r(\overline{R}y) \cap \mathcal{I}|\,, \tag{2.2.7}$$

and the same replacing $\mathcal{I}$ with $\mathcal{O}$.

On the other hand, let us define $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by $S(x', x'') = (x'', x')$, which is also a linear isometry and transforms $\mathcal{O}$ into $\mathcal{I}$ and vice versa. Therefore, for every $y \in \mathbb{R}^{2m}$ we have

$$|B_r(y) \cap \mathcal{I}| = |S\left(B_r(y) \cap \mathcal{I}\right)| = |B_r(Sy) \cap \mathcal{O}|\,. \tag{2.2.8}$$

Finally, note that by the definition of $\overline{R}$, it is satisfied $S\overline{R}x_0 = \overline{R}x_0$. By combining this with (2.2.7) and (2.2.8) applied to $y = x_0$ and $y = \overline{R}x_0$ respectively, we obtain

$$|B_r(x_0) \cap \mathcal{I}| = |B_r(\overline{R}x_0) \cap \mathcal{I}| = |B_r(S\overline{R}x_0) \cap \mathcal{O}| = |B_r(\overline{R}x_0) \cap \mathcal{O}| = |B_r(x_0) \cap \mathcal{O}|\,.$$

□

With this lemma available we proceed with the proof of Corollary 2.2.5.

Proof of Corollary 2.2.5. Note that $\mathbb{R}^{2m} \setminus \overline{\mathcal{O}}$ is an open cone outside $\mathcal{O}$, and thus $\{(\mathbb{R}^{2m} \setminus \overline{\mathcal{O}}) \times (0,+\infty)\} \cap \Omega$ is empty. Hence, we can use Proposition 2.2.2 by noticing that, if we take $\theta = 2^{-4m-3-2a}$, then $R_a(\Omega, \Gamma, \theta) \leq \varepsilon$. Indeed, recall first that by Lemma 4.2 in [43], $|s-t|/\sqrt{2}$ is the distance to the cone. Then, let $x \in \Gamma$ and let $\overline{x} \in \mathscr{C}$ a point realizing this distance. Since $x \in \Gamma \subset \mathcal{N}_\varepsilon$, we have that $|x - \overline{x}| \leq \varepsilon/\sqrt{2} < 3\varepsilon/4$ and therefore

$$B^+_{\varepsilon/4}(\overline{x}) \setminus \big(\mathcal{O} \times (0,+\infty)\big) \subset B^+_{\varepsilon/4}(\overline{x}) \setminus \Omega \subset B^+_{\varepsilon}(x) \setminus \Omega \,.$$

Hence, by the scaling properties of $|\cdot|_a$ and Lemma 2.2.6 —used at each level $\{\lambda = \lambda_0\}$, with $\lambda_0 \in (0, \varepsilon/4)$—, we have

$$2^{-4m-3-2a} |B^+_\varepsilon(x)|_a = \frac{1}{2} |B^+_{\varepsilon/4}(\overline{x})|_a = |B^+_{\varepsilon/4}(\overline{x}) \setminus \big(\mathcal{O} \times (0,+\infty)\big)|_a \leq |B^+_\varepsilon(x) \setminus \Omega|_a \,.$$

□

With this result at hand we can now establish the maximum principle for the linearized operator in $\mathcal{O} \times (0,+\infty)$ at a saddle-shaped solution.

Proof of Proposition 2.1.4. Let u be a saddle-shaped solution. A key point in the proof is that u is a positive supersolution in $\mathcal{O} \times (0,+\infty)$ of the linearized problem at u. Indeed, since $u > 0$ in $\partial_0 \Omega \subset \mathcal{O}$,

$$\mathscr{L}_u u = d_\gamma \frac{\partial u}{\partial \nu^a} - f'(u)u = f(u) - f'(u)u > 0 \quad \text{on } \partial_0 \Omega \,. \tag{2.2.9}$$

We have used that since $f'' < 0$ in $(0,1)$ and $f(0) = 0$, it satisfies $f'(\tau)\tau < f(\tau)$ for all $\tau \in (0,1)$.

Now, we define

$$w := \frac{v}{u} \,.$$

Note that w is well defined in Ω, since u is positive in such set. The usual strategy (see [16]) in some proofs of the maximum principle is to assume that the supremum of w in Ω is positive and then arrive at a contradiction. Nevertheless, a priori we do not know that $\sup_\Omega w < +\infty$, since u vanishes on $\mathscr{C} \times [0,+\infty)$ and $\partial\Omega$ could intersect this set. Thus, in the following arguments we will consider the supremum of w in a subset of $\partial_0\Omega$ that is at a positive distance to the zero level set of u. Then, using the maximum principle in "extension-narrow" pairs we will see that, assuming this supremum to be positive, it will indeed agree with the supremum in the whole set Ω (see the details below). After some arguments, we will arrive at a contradiction. A similar strategy was used by Cabré in [34], to prove an analogous maximum principle in the local case $\gamma = 1$.

Les us proceed with the details. For $\varepsilon > 0$, set

$$\mathcal{O}_\varepsilon := \{t + \varepsilon < s,\ \lambda = 0\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \{t < s < t + \varepsilon,\ \lambda = 0\} \,,$$

and take ε small enough such that for each set $\Gamma \subset \partial_0\Omega$ satisfying $\Gamma \subset \mathcal{N}_\varepsilon$, the pair (Ω, Γ) is "extension-narrow". Hence, the maximum principle, as in Corollary 2.2.5, holds for the pair (Ω, Γ).

Next, we claim that

$$u \geq \delta > 0 \quad \text{in } \mathcal{O}_\varepsilon \tag{2.2.10}$$

for some positive constant δ. Indeed, thanks to the asymptotic behavior of u (see part (ii) of Theorem 2.1.2), and since $U(x) \geq u_0(\varepsilon/\sqrt{2})$ for $x \in \mathcal{O}_\varepsilon$, there exists a radius $R > 0$ such that $u(x) \geq u_0(\varepsilon/\sqrt{2})/2$ if $|x| > R$ and $x \in \mathcal{O}_\varepsilon$. Since u is positive in the compact set $\overline{\mathcal{O}_\varepsilon} \cap \overline{B_R}$, we conclude the claim.

We define

$$\Gamma := \partial_0\Omega \cap \mathcal{N}_\varepsilon\,,$$

and let

$$S := \sup_{\partial_0\Omega\cap\mathcal{O}_\varepsilon} w\,,$$

which is finite by the fact that u is bounded from below by $\delta > 0$ in $\mathcal{O}_\varepsilon$ and v is bounded from above. Assume by contradiction that $S > 0$.

First, we claim that $S = \sup_\Omega w$. To see this, we only need to show that $w \leq S$ in Ω. Define $\varphi := v - Su$ and note that since $S \geq 0$, φ satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla\varphi) & \leq & b(x,\lambda)\varphi & \text{in } \Omega\,, \\ \dfrac{\partial\varphi}{\partial\nu^a} & \leq & c(x)\varphi & \text{on } \Gamma\,, \\ \varphi & \leq & 0 & \text{on } \partial\Omega\setminus\Gamma\,, \end{cases}$$

with $c(x) = f'(u)/d_\gamma$. By the maximum principle in the "extension-narrow" pair (Ω,Γ), we have $\varphi \leq 0$ in Ω, which yields $w = v/u \leq S$ in Ω. Thus, the claim is proved.

Now, by the hypothesis on $\partial_L\Omega$ and at infinity on v, and the fact that $u > \delta$ in $\mathcal{O}_\varepsilon$, we have that S is attained at some point $(x_0,0) \in \partial_0\Omega \subset \mathcal{O}$. At this point we have

$$\frac{\partial w}{\partial\nu^a}(x_0) = -\lim_{\lambda\downarrow 0} \lambda^a w_\lambda(x_0,\lambda) = \lim_{\lambda\downarrow 0} \frac{w(x_0,0) - w(x_0,\lambda)}{\lambda^{2\gamma}} \geq 0\,, \tag{2.2.11}$$

since $w(x_0,0)$ is the maximum.

On the other hand, observe that

$$d_\gamma u^2 \frac{\partial w}{\partial\nu^a} = d_\gamma \frac{\partial v}{\partial\nu^a} u - d_\gamma \frac{\partial u}{\partial\nu^a} v = u\mathscr{L}_u v - v\mathscr{L}_u u \leq -v\mathscr{L}_u u \quad \text{on } \partial_0\Omega \subset \mathcal{O}\,,$$

since $u > 0$ in $\mathcal{O}$ and $\mathscr{L}_u v \leq 0$ in $\partial_0\Omega$. Therefore, at the point x_0 we have, using also (2.2.9),

$$\frac{\partial w}{\partial\nu^a}(x_0) \leq -\frac{S}{d_\gamma u(x_0)} \mathscr{L}_u u(x_0) < 0\,,$$

which contradicts (2.2.11). Note that in this last argument is crucial the fact that $x_0 \in \partial_0\Omega \subset \mathcal{O}$ and thus $u(x_0) > 0$ and $\mathscr{L}_u u(x_0) > 0$.

Hence, the assumption $S > 0$ is false and therefore $w \leq 0$ in $\partial_0\Omega \cap \mathcal{O}_\varepsilon$. Since $u > 0$ in $\mathcal{O}$, this yields that $v \leq 0$ in $\partial_0\Omega \cap \mathcal{O}_\varepsilon$. Finally, by the maximum principle in the "extension-narrow" pair (Ω,Γ) applied to v, it follows that $v \leq 0$ in Ω. □

2.3 Uniqueness of the saddle-shaped solution

Thanks to the maximum principle in $\mathcal{O} \times (0,+\infty)$ for the linearized operator we can now establish the uniqueness of the saddle-shaped solution.

Proof of Theorem 2.1.3. Let u_1 and u_2 be two saddle-shaped solutions. Define $v := u_1 - u_2$, a function that depends only on s and t and that is odd with respect to $\mathscr{C}$. Then, $\operatorname{div}(\lambda^a \nabla v) = 0$ in $\mathcal{O} \times (0, +\infty)$, $v = 0$ on $\partial_L (\mathcal{O} \times (0, +\infty)) = \mathscr{C} \times [0, +\infty)$ and

$$d_\gamma \frac{\partial v}{\partial \nu^a} = f(u_1) - f(u_2) \leq f'(u_2)(u_1 - u_2) = f'(u_2)v \quad \text{on } \mathcal{O} \times \{0\}\,,$$

since f is concave in $(0,1)$. Moreover, by the asymptotic result (see Theorem 2.1.2), we have

$$\limsup_{x \in \mathcal{O},\ |x| \to +\infty} v(x, 0) = 0\,.$$

Finally, by the maximum principle for the linearized operator in $\mathcal{O} \times (0, +\infty)$, see Proposition 2.1.4, we deduce that $v \leq 0$ in $\mathcal{O} \times [0, +\infty)$, which yields $u_1 \leq u_2$ in $\mathcal{O} \times [0, +\infty)$. Interchanging u_1 and u_2, we obtain $u_1 \geq u_2$ in $\mathcal{O} \times [0, +\infty)$. Therefore, $u_1 = u_2$ in $\mathbb{R}^{2m+1}_+$. □

2.4 The layer solution and a pointwise estimate for the saddle-shaped solution

This section is devoted to establish some monotonicity properties of the layer solution u_0 and a pointwise estimate for the saddle-shaped solution (Proposition 2.1.5). We start with a maximum principle similar to Proposition 2.1.4, but for the linearized operator at u_0 in the set $\{u_0 > 0\}$, which plays the role that $\mathcal{O} \times (0, +\infty)$ had for the saddle-shaped solution.

Proposition 2.4.1. *Let $u_0 : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ be the layer solution of (2.1.4) and let $\mathscr{L}_{u_0}$ be defined by*

$$\mathscr{L}_{u_0} v := d_\gamma \frac{\partial v}{\partial \nu^a} - f'(u_0) v \quad \text{on } \mathbb{R} = \partial_0 \mathbb{R}^2_+\,.$$

Let $\Omega \subset (0, +\infty) \times (0, +\infty)$ be an open set such that $\partial_0 \Omega$ is nonempty.

Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be bounded from above and satisfying $\lambda^a v_\lambda \in C(\overline{\Omega})$. Assume that

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \leq & b(x,\lambda) v & \text{in } \Omega \subset (0,+\infty) \times (0,+\infty)\,, \\ \mathscr{L}_{u_0} v & \leq & 0 & \text{on } \partial_0 \Omega \subset (0, +\infty)\,, \\ v & \leq & 0 & \text{on } \partial_L \Omega\,, \\ \limsup\limits_{x \in \partial_0 \Omega,\ |x| \to +\infty} v(x,0) & \leq & 0\,, & \end{cases}$$

with $b \leq 0$. Then, $v \leq 0$ in Ω.

Proof. Since it is analogous (and simpler) to the proof of Proposition 2.1.4, we just sketch it here pointing out what needs to be adapted. The key fact is that u_0 is a positive supersolution to the linearized problem. This is an analogous situation to that of Proposition 2.1.4 . That is, u_0 is γ-harmonic in $(0, +\infty) \times (0, +\infty)$, positive in $(0, +\infty) \times [0, +\infty)$, and

$$d_\gamma \frac{\partial u_0}{\partial \nu^a} = f(u_0) > f'(u_0) u_0 \quad \text{on } (0, +\infty) \times \{0\}\,, \tag{2.4.1}$$

where we have used that $f'' < 0$ in $(0,1)$ and $f(0) = 0$.

Then, one defines $w := v/u_0$ and proceeds exactly as in the proof of Proposition 2.1.4, replacing u by u_0 in the whole argument, and also replacing $\mathcal{O}_\varepsilon$ and $\mathcal{N}_\varepsilon$ by $(\varepsilon, +\infty)$ and $(0, \varepsilon)$ respectively. In addition, (2.2.10) follows immediately from the fact that $u_0(x, 0)$ is increasing. The rest of the proof is completely analogous by using (2.4.1). $\square$

With this maximum principle we can now prove the following monotonicity and concavity properties of the layer solution.

Lemma 2.4.2. *Let u_0 be the layer solution of* (2.1.4). *Then,*

$$\frac{\partial}{\partial x} u_0(x, \lambda) > 0 \quad \text{in} \quad \mathbb{R} \times [0, +\infty)$$

and

$$\frac{\partial^2}{\partial x^2} u_0(x, \lambda) < 0 \quad \text{in} \quad (0, +\infty) \times [0, +\infty)\,.$$

Proof. First of all, let us remark that u_0 has the required regularity to apply the following arguments by the results of [40] (see Section 2.5 for more details in the more involved setting of the saddle-shaped solution).

The monotonicity of the first derivative was already stated in Remark 4.7 of [40], but we include here the short proof for completeness. By differentiating (2.1.4) with respect to x, we obtain that $\operatorname{div}(\lambda^a \nabla(\partial_x u_0)) = 0$ in $\mathbb{R} \times (0, +\infty)$. Moreover, $\partial_x u_0(x, 0) > 0$ for $x \in \mathbb{R}$; see (2.1.4). Then, the result follows directly from the Poisson formula.

Next, we show the second statement. If we call

$$v(x, \lambda) := \partial_{xx} u_0(x, \lambda)\,,$$

by differentiating (2.1.4) twice with respect to x, we get

$$\left\{ \begin{array}{rcll} \operatorname{div}(\lambda^a \nabla v) & = & 0 & \text{in } (0, +\infty) \times (0, +\infty)\,, \\ d_a \dfrac{\partial v}{\partial \nu^a} - f'(u_0) v = f''(u_0)(\partial_x u_0)^2 & \leq & 0 & \text{on } (0, +\infty) \times \{0\}\,, \\ v & = & 0 & \text{on } \{0\} \times (0, +\infty)\,. \end{array} \right.$$

Notice that $v = 0$ on $\{0\} \times (0, +\infty)$ since v is an odd function with respect to the first variable (recall that u_0 is odd in x).

Moreover, by repeating the argument of Lemma 4.8 in [40] for $\partial_{xx} u_0$, it is easy to see that $\partial_{xx} u_0(x, 0) \to 0$ as $|x| \to +\infty$. Therefore, by Proposition 2.4.1 we deduce that $v \leq 0$ in $[0, +\infty) \times [0, +\infty)$. Finally, we get that it is in fact negative in $(0, +\infty) \times [0, +\infty)$ by applying the strong maximum principle. $\square$

Now we prove that the function

$$U(s, t, \lambda) := u_0 \left(\frac{s - t}{\sqrt{2}}, \lambda \right)$$

is a barrier for the saddle-shaped solution. To do it, we will use a maximum principle in $\mathcal{O} \times (0, +\infty)$ for the linearized problem at U.

Proof of Proposition 2.1.5. The idea is to repeat the arguments in the proof of Proposition 2.1.4, but using U instead of u as the positive supersolution to the linearized problem involving the operator

$$\mathscr{L}_U w := d_\gamma \frac{\partial w}{\partial \nu^a} - f'(U)w.$$

In order to do it, we need to point out several facts.

First, note that

$$U \in C^2\big((\mathcal{O} \times (0,+\infty)) \setminus \{t=0,\ \lambda>0\}\big) \cap \operatorname{Lip}\big(\overline{\mathbb{R}^{2m+1}_+}\big)\,,$$

and U cannot have a local minimum at $\{t=0,\ \lambda \geq 0\}$. Indeed, for every $\lambda \geq 0$,

$$\lim_{\tau \to 0^-} \partial_{x_{m+1}} U(x_1,...x_m,\tau,0,...,0,\lambda) = \frac{1}{\sqrt{2}} \partial_x u_0 \left(\frac{s}{\sqrt{2}}, \lambda \right) > 0\,,$$

and

$$\lim_{\tau \to 0^+} \partial_{x_{m+1}} U(x_1,...x_m,\tau,0,...,0,\lambda) = -\frac{1}{\sqrt{2}} \partial_x u_0 \left(\frac{s}{\sqrt{2}}, \lambda \right) < 0\,.$$

Note that the same property concerning a local minimum at $\{t=0,\ \lambda \geq 0\}$ holds if we add to U a regular function.

Next, we claim that U is a positive supersolution in $\mathcal{O}$ to the linearized problem for $\mathscr{L}_U$. Indeed, by the concavity of f, we have that

$$\mathscr{L}_U U = f(U) - f'(U)U > 0 \quad \text{in } \mathcal{O}\,.$$

Moreover, a simple computation in the (s,t,λ) variables shows that

$$\operatorname{div}(\lambda^a \nabla U) = \lambda^a \frac{m-1}{\sqrt{2}} \frac{t-s}{st} \partial_x u_0 \left(\frac{s-t}{\sqrt{2}}, \lambda \right) \qquad \text{in } \mathbb{R}^{2m+1}_+ \setminus \{st=0,\ \lambda>0\}\,. \tag{2.4.2}$$

Therefore, U is γ-superharmonic in $(\mathcal{O} \times (0,+\infty)) \setminus \{t=0,\ \lambda>0\}$ —recall that $\partial_x u_0 > 0$ by Lemma 2.4.2.

Now, we define

$$v := u - U \quad \text{and} \quad \Omega := \mathcal{O} \times (0,+\infty)\,,$$

and we want to see that $v \leq 0$ in Ω. First, since u is γ-harmonic, we have that

$$-\operatorname{div}(\lambda^a \nabla v) \leq 0 \quad \text{in } \Omega \setminus \{t=0,\ \lambda>0\}$$

and that v cannot have a local maximum at $\{t=0,\ \lambda \geq 0\}$. In addition, both u and U vanish at $\mathscr{C} \times [0,+\infty)$ and by the asymptotic behavior of u (see Theorem 2.1.2), we have $\lim_{x \in \mathcal{O},\, |x| \to +\infty} v(x,0) = 0$. On the other hand, since f is concave in $(0,1)$, we get

$$d_\gamma \frac{\partial v}{\partial \nu^a} = f(u) - f(U) \leq f'(U)v \quad \text{on } \partial_0 \Omega\,.$$

Collecting all these facts, we can repeat the proof of Proposition 2.1.4, using U instead of u as the positive supersolution to the linearized problem for $\mathscr{L}_U$ to see that $v \leq 0$ in Ω. All the arguments are analogous, taking into account Remark 2.2.4 when using the maximum principles in "extension-narrow" pairs. Therefore, we conclude that $v \leq 0$ in Ω and, by the odd symmetry of u and U, we get (2.1.8). □

2.5 Monotonicity properties

In this section we establish the monotonicity properties of u stated in Proposition 2.1.8. For this, we will apply the maximum principle of Proposition 2.1.4 to some derivatives of u. Therefore, we need some regularity results that we collect next.

Recall that we assume that $f \in C^{2,\alpha}$ for some $\alpha \in (0,1)$. Since u is a bounded solution to the first equation in (2.1.3), then $u \in C^\infty(\mathbb{R}^{2m+1}_+)$. Regarding the regularity on $\{\lambda = 0\}$, $u(\cdot, 0) \in C^{2,\alpha}(\mathbb{R}^{2m})$ by applying Lemma 4.4 from [40]. Moreover, [40] also gives the following uniform bound:

$$||u||_{C^\alpha\left(\overline{\mathbb{R}^{2m+1}_+}\right)} + ||\nabla_x u||_{C^\alpha\left(\overline{\mathbb{R}^{2m+1}_+}\right)} + ||D^2_x u||_{C^\alpha\left(\overline{\mathbb{R}^{2m+1}_+}\right)} \le C,$$

for some $C > 0$ depending only on m, γ , $||f||_{C^{2,\alpha}}$, and $||u||_{L^\infty(\mathbb{R}^{2m+1}_+)}$.

Next, since the horizontal first derivatives of u satisfy $\operatorname{div}(\lambda^a \nabla u_{x_i}) = 0$ and also $d_\gamma \partial_{\nu^a} u_{x_i} = f'(u)\, u_{x_i} \in C^\alpha(\mathbb{R}^{2m})$, and the horizontal second derivatives of u satisfy $\operatorname{div}(\lambda^a \nabla u_{x_i x_j}) = 0$ and also $d_\gamma \partial_{\nu^a} u_{x_i x_j} = f''(u)\, u_{x_i} u_{x_j} + f'(u)\, u_{x_i x_j} \in C^\alpha(\mathbb{R}^{2m})$ for all indices i and j from 1 to $2m$, we can apply Lemma 4.5 from [40] to obtain that

$$||\lambda^a\, u_\lambda||_{C^\beta(\mathbb{R}^{2m}\times[0,1])} + ||\lambda^a\, (u_{x_i})_\lambda||_{C^\beta(\mathbb{R}^{2m}\times[0,1])} + ||\lambda^a\, (u_{x_i x_j})_\lambda||_{C^\beta(\mathbb{R}^{2m}\times[0,1])} \le C,$$

for some $C > 0$ and $\beta \in (0,1)$ depending only on m, γ, $||f||_{C^{2,\alpha}}$, and $||u||_{L^\infty(\mathbb{R}^{2m+1}_+)}$.

Now, since u depends only on s, t and λ, from the previous results we obtain

$$\begin{aligned}
u_s \in C^{2,\alpha}(\mathbb{R}^{2m+1}_+ \setminus \{s = 0,\ \lambda \ge 0\}) \ \text{ and } \ \lambda^a\,(u_s)_\lambda \in C^\alpha(\overline{\mathbb{R}^{2m+1}_+} \setminus \{s = 0,\ \lambda \ge 0\}),\\
u_t \in C^{2,\alpha}(\mathbb{R}^{2m+1}_+ \setminus \{t = 0,\ \lambda \ge 0\}) \ \text{ and } \ \lambda^a\,(u_t)_\lambda \in C^\alpha(\overline{\mathbb{R}^{2m+1}_+} \setminus \{t = 0,\ \lambda \ge 0\}),\\
u_{st} \in C^{2,\alpha}(\mathbb{R}^{2m+1}_+ \setminus \{st = 0,\ \lambda \ge 0\}) \ \text{ and } \ \lambda^a\,(u_{st})_\lambda \in C^\alpha(\overline{\mathbb{R}^{2m+1}_+} \setminus \{st = 0,\ \lambda \ge 0\}).
\end{aligned}$$

Furthermore, as it is explained in Section 4 of [34], the regularity and the symmetry of u, in s and t, yield

$$u_s = 0 \text{ in } \{s = 0,\ \lambda \ge 0\}, \quad u_t = 0 \text{ in } \{t = 0,\ \lambda \ge 0\}, \quad u_{st} = 0 \text{ in } \{st = 0,\ \lambda \ge 0\},$$

and

$$u_s,\ u_t,\ u_{st} \in C(\overline{\mathbb{R}^{2m+1}_+}).$$

Before proceeding to the proof of Proposition 2.1.8, we first need the following asymptotic result for the second derivatives in x of u. This derivative was not included in the asymptotic theorem of [60, 61]. We will use it to show that $u_{st} > 0$ in $\{s > t > 0\} \times [0, +\infty)$.

Lemma 2.5.1. *Let f satisfy conditions (2.1.2), and let u be the saddle-shaped solution of (2.1.3). Then, denoting $U(x,\lambda) := u_0((s-t)/\sqrt{2}, \lambda) = u_0(z,\lambda)$, we have*

$$||D^2_x u(\cdot,\lambda) - D^2_x U(\cdot,\lambda)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} \to 0, \quad \text{as } R \to +\infty,$$

for every $\lambda \in [0, +\infty)$.

Proof. The proof follows the ones of the analogous results in [61, 34, 44], where a compactness argument is used. Therefore, we only give here the main ideas, since the details can be found in those papers. Arguing by contradiction, we suppose that the asymptotic result does not hold. Hence, there exists an $\varepsilon > 0$ and a sequence $\{x_k\} \subset \mathcal{O}$ such that

$$|D_x^2 u(x_k,\lambda) - D_x^2 U(x_k,\lambda)| \geq \varepsilon \quad \text{and} \quad |x_k| \to +\infty. \tag{2.5.1}$$

Now we distinguish two cases, depending on whether the sequence $\{\operatorname{dist}(x_k,\mathscr{C})\}$ is unbounded or bounded. In the first case, we show that, up to a subsequence, the function $u_k(x,\lambda) := u(x+x_k,\lambda)$ converges to a solution u_∞ of the semilinear Neumann problem in the half-space $\mathbb{R}^{2m+1}_+$ appearing in the statement of Theorem 5.3 in [61] (see [112] for the proof). Using this result and the stability of u_∞ we get that $u_\infty \equiv 1$. Thus, $|D_x^2 u(x_k,\lambda)| \to 0$, and since $|D_x^2 U(x_k,\lambda)| \to 0$, we arrive at a contradiction with (2.5.1).

In the second case, we have $\operatorname{dist}(x_k,\mathscr{C}) = |x_k - x_k^0|$ bounded, where $x_k^0 \in \mathscr{C}$. Since the Simons cone converges to a hyperplane at infinity (see the details in [44]), it can be proved that, up to a subsequence and a rotation, the function $u_k(x,\lambda) := u(x + x_k^0,\lambda)$ converges to a positive solution u_∞ of an equation in the quarter-space $\mathbb{R}^{2m+1}_{++} = \mathbb{R}^{2m+1}_+ \cap \{x_{2m} > 0\}$ with zero Dirichlet boundary conditions, as in the statement of Theorem 5.5 in [61] (see [145] for the proof). Applying this last theorem and the stability again, we conclude that u_∞ must be the 2D solution u_0 depending only on x_{2m} and λ. Hence, $D_x^2(u-U)(x_k,\lambda)$ converges to zero, and we arrive at a contradiction with (2.5.1). □

With the help of the maximum principle of Proposition 2.1.4, the asymptotic result for the saddle-shaped solution, and the monotonicity properties of the layer solution, we can prove Proposition 2.1.8.

Proof of Proposition 2.1.8. We write (2.1.3) in (s,t,λ) variables:

$$\begin{cases} u_{ss} + u_{tt} + u_{\lambda\lambda} + (m-1)\left(\dfrac{u_s}{s} + \dfrac{u_t}{t}\right) + \dfrac{a}{\lambda}u_\lambda &= 0 \qquad \text{in } \{st > 0,\ \lambda > 0\}, \\ u_s &= 0 \qquad \text{on } \{s = 0,\ \lambda \geq 0\}, \\ u_t &= 0 \qquad \text{on } \{t = 0,\ \lambda \geq 0\}, \\ d_\gamma \dfrac{\partial u}{\partial \nu^a} &= f(u) \quad \text{on } \{\lambda = 0\}. \end{cases} \tag{2.5.2}$$

Differentiating the previous equation with respect to s we find that

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_s) &= (m-1)\dfrac{\lambda^a}{s^2}u_s \quad \text{in } \{s > t,\ \lambda > 0\}, \\ d_\gamma \dfrac{\partial u_s}{\partial \nu^a} &= f'(u)u_s \qquad \text{on } \{s > t,\ \lambda = 0\}. \end{cases}$$

Since $u = 0$ on $\{s = t,\ \lambda \geq 0\}$ and $u > 0$ in $\{s > t,\ \lambda \geq 0\}$, we have that $u_s \geq 0$ on $\partial_L\{s > t,\ \lambda > 0\} = \{s = t,\ \lambda \geq 0\}$. Moreover, by the asymptotic result (point (ii) of Theorem 2.1.2) and the monotonicity properties of the layer solution u_0 (Lemma 2.4.2), we have

$$\liminf_{\{s>t\},\ |(s,t)|\to+\infty} u_s(s,t,0) \geq 0\,.$$

Indeed, if u_0 is the layer solution,

$$\partial_s U(x,0) = \frac{1}{\sqrt{2}}\partial_x u_0\left(\frac{s-t}{\sqrt{2}},0\right) \geq 0 \quad \text{and} \quad \lim_{R\to+\infty} ||(u_s - \partial_s U)(\cdot,0)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} = 0\,.$$

Thus, by the maximum principle for the linearized operator (Proposition 2.1.4) applied to $v = -u_s$, with $b(x,\lambda) = -(m-1)\lambda^a/s^2 \le 0$, we conclude that $u_s \ge 0$ in $\{s \ge t,\ \lambda \ge 0\}$.

Similarly, if we differentiate (2.5.2) with respect to t, we obtain

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_t) &= (m-1)\dfrac{\lambda^a}{t^2} u_t \quad \text{in } \{s > t > 0,\ \lambda > 0\}\,, \\ d_\gamma \dfrac{\partial u_t}{\partial \nu^a} &= f'(u) u_t \qquad\qquad \text{on } \{s > t > 0,\ \lambda = 0\}\,. \end{cases}$$

In the lateral boundary $\partial_L\{s > t > 0,\ \lambda > 0\} = \{s = t,\ \lambda \ge 0\} \cup \{t = 0,\ \lambda \ge 0\}$ we have $-u_t \ge 0$. Indeed, $u_t = 0$ on $\{t = 0,\ \lambda \ge 0\}$, and since $u = 0$ on $\{s = t,\ \lambda \ge 0\}$ and $u > 0$ in $\{s > t,\ \lambda \ge 0\}$, it holds $-u_t \ge 0$ on $\{s = t,\ \lambda \ge 0\}$. Furthermore, the asymptotic behavior of u and the monotonicity properties of the layer solution u_0 yield

$$\limsup_{\{s>t>0\},\ |(s,t)|\to+\infty} u_t(s,t,0) \le 0\,.$$

Indeed,

$$\partial_t U(x,0) = -\frac{1}{\sqrt{2}} \partial_1 u_0 \left(\frac{s-t}{\sqrt{2}}, 0 \right) \le 0 \quad \text{and} \quad \lim_{R\to+\infty} ||(u_t - \partial_t U)(\cdot,0)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} = 0\,.$$

Thus, using again the maximum principle for the linearized operator we find that $-u_t \ge 0$ in $\{s \ge t,\ \lambda \ge 0\}$.

By the odd symmetry of u, i.e., $u(s,t) = -u(t,s)$, we conclude that $u_s \ge 0$ and $u_t \le 0$ in $\mathbb{R}^{2m} \times [0,+\infty)$. This fact and the strong maximum principle give that $u_s > 0$ in $(\mathbb{R}^{2m} \setminus \{s = 0\}) \times [0,+\infty)$ and $-u_t > 0$ in $(\mathbb{R}^{2m} \setminus \{t = 0\}) \times [0,+\infty)$.

Now we check the sign of the y-derivative. We use that $\partial_y = (\partial_s + \partial_t)/\sqrt{2}$ to see that

$$\operatorname{div}(\lambda^a \nabla u_y) = (m-1)\frac{\lambda^a}{\sqrt{2}} \left(\frac{u_s}{s^2} + \frac{u_t}{t^2} \right) = (m-1)\frac{\lambda^a}{s^2} u_y + (m-1)\frac{\lambda^a}{\sqrt{2}} \frac{s^2 - t^2}{s^2 t^2} u_t\,.$$

Hence, using that $u_t \le 0$ in $\{s > t > 0,\ \lambda > 0\}$ we get

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_y) &\le (m-1)\dfrac{\lambda^a}{s^2} u_y \quad \text{in } \{s > t > 0,\ \lambda > 0\}\,, \\ d_\gamma \dfrac{\partial u_y}{\partial \nu^a} &= f'(u) u_y \qquad\qquad \text{on } \{s > t > 0,\ \lambda = 0\}\,. \end{cases}$$

Note that, since u vanishes at $\mathscr{C} \times [0,+\infty)$, $u_y = 0$ on $\{s = t,\ \lambda \ge 0\}$. Moreover, $u_s \ge 0$ and $u_t = 0$ on $\{t = 0,\ \lambda \ge 0\}$. Therefore, $u_y \ge 0$ on $\partial_L\{s > t > 0,\ \lambda > 0\} = \{s = t,\ \lambda \ge 0\} \cup \{t = 0,\ \lambda \ge 0\}$. Furthermore, by the asymptotic behavior of u and the monotonicity properties of the layer solution u_0 we have

$$\liminf_{\{s>t>0\},\ |(s,t)|\to+\infty} u_y(s,t,0) = 0\,,$$

since

$$\partial_y U(x,0) = \partial_y u_0(z,0) = 0 \quad \text{and} \quad \lim_{R\to+\infty} ||(u_y - \partial_y U)(\cdot,0)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} = 0\,.$$

Again, by using the maximum principle of Proposition 2.1.4, we deduce that $u_y \geq 0$ in $\{s \geq t,\ \lambda \geq 0\}$, and the strong maximum principle yields $u_y > 0$ on $\{s > t,\ \lambda \geq 0\}$.

Finally, we prove the last statement concerning the crossed derivatives. By differentiating (2.5.2), first with respect to s and then with respect to t, we find

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_{st}) &= (m-1)\lambda^a \left(\dfrac{1}{s^2} + \dfrac{1}{t^2}\right) u_{st} & \text{in } \{s > t > 0,\ \lambda > 0\}\,, \\ d_\gamma \dfrac{\partial u_{st}}{\partial \nu^a} &= f'(u) u_{st} + f''(u) u_s u_t \geq f'(u) u_{st} & \text{on } \{s > t > 0,\ \lambda = 0\}\,. \end{cases}$$

Here we have used that $f''(\tau) \leq 0$ if $\tau \in (0,1)$ and that $u_s u_t \leq 0$ in $\{s > t > 0,\ \lambda = 0\}$. Note that, by symmetry, $u_{st} = 0$ on $\{s = t,\ \lambda \geq 0\}$. Moreover, since $u_t(s,0,\lambda) = 0$ for every $s > 0$ and $\lambda \geq 0$, $u_{st} = 0$ on $\{t = 0,\ \lambda \geq 0\}$. Therefore, $u_{st} = 0$ on $\partial_L\{s > t > 0,\ \lambda > 0\}$. In addition, by the asymptotic result of Lemma 2.5.1 and the monotonicity properties of the layer solution u_0 (Lemma 2.4.2), we have

$$\liminf_{\{s>t>0\},\ |(s,t)| \to +\infty} u_{st}(s,t,0) \geq 0\,,$$

since

$$U_{st}(x,0) = -\frac{1}{2}\partial_1^2 u_0\left(\frac{s-t}{\sqrt{2}}, 0\right) \geq 0 \quad \text{and} \quad \lim_{R \to +\infty} ||(u_{st} - U_{st})(\cdot, 0)||_{L^\infty(\mathbb{R}^{2m} \setminus B_R)} = 0\,.$$

Hence, by the maximum principle for the linearized operator (Proposition 2.1.4), we deduce that $u_{st} \geq 0$ in $\{s \geq t,\ \lambda \geq 0\}$, and the strong maximum principle yields $u_{st} > 0$ in $\{s > t > 0,\ \lambda \geq 0\}$. □

2.6 Stability of the saddle-shaped solution and the Simons cone in dimensions $2m \geq 14$

In this last section we prove our stability results. The first one is Theorem 2.1.6 and it establishes the stability of the saddle-shaped solution in dimensions $2m \geq 14$. The proof follows the strategy of its analogue in [34] and it is based on finding a positive supersolution to the linearized problem.

Proof of Theorem 2.1.6. Let us show that $\varphi = t^{-b}u_s - s^{-b}u_t$, with $b(b-m+2)+m-1 \leq 0$ and $b > 0$, is a positive supersolution of the linearized operator. That is, it satisfies

$$\varphi > 0 \quad \text{in } \overline{\mathbb{R}^{2m+1}_+} \setminus \{st = 0,\ \lambda > 0\}\,, \tag{2.6.1}$$

$$-\operatorname{div}(\lambda^a \nabla \varphi) \geq 0 \quad \text{in } \mathbb{R}^{2m+1}_+ \setminus \{st = 0,\ \lambda > 0\}\,, \tag{2.6.2}$$

and

$$\mathscr{L}_u \varphi \geq 0 \quad \text{on } \mathbb{R}^{2m} \setminus \{st = 0\}\,. \tag{2.6.3}$$

Indeed, note that $\varphi > 0$ in $\{s > t > 0,\ \lambda \geq 0\}$ by the monotonicity properties of u (Proposition 2.1.8). Since φ is even with respect to the Simons cone, i.e., $\varphi(t,s,\lambda) = \varphi(s,t,\lambda)$, it holds (2.6.1). Moreover, (2.6.3) follows readily, since φ satisfies

$$d_\gamma \frac{\partial \varphi}{\partial \nu^a} = f'(u)\,\varphi\,.$$

Let us now show (2.6.2). Since φ is even with respect to the Simons cone, it is enough to check that div $(\lambda^a \nabla\varphi) \leq 0$ in $\{s > t > 0,\ \lambda > 0\}$. By using that $\mathrm{div}(\lambda^a \nabla u) = 0$, we obtain by direct computation that

$$\begin{aligned}
\lambda^{-a}\,\mathrm{div}(\lambda^a \nabla\varphi) = {} & b(b-m+2)\left(t^{-b-2}\,u_s - s^{-b-2}\,u_t\right) \\
& + (m-1)\left(t^{-b}s^{-2}u_s - s^{-b}t^{-2}u_t\right) \\
& + 2b\left(t^{-b-1} - s^{-b-1}\right) u_{st}\,.
\end{aligned}$$

Now, by using that $u_{st} > 0$, $u_y > 0$ and $-u_t > 0$ in $\{s > t > 0,\ \lambda > 0\}$, and the fact that $b > 0$ satisfies $b(b-m+2) \leq -(m-1)$, we arrive at

$$\begin{aligned}
\lambda^{-a}\,\mathrm{div}(\lambda^a \nabla\varphi) &\leq t^{-b}(u_s+u_t)\left((m-1)s^{-2} + b(b-m+2)t^{-2}\right) \\
&\quad - t^{-b}u_t\left\{(m-1)s^{-2} + b(b-m+2)t^{-2}\right\} \\
&\quad - s^{-b}u_t\left\{(m-1)t^{-2} + b(b-m+2)s^{-2}\right\} \\
&= \sqrt{2}t^{-b}u_y\left((m-1)s^{-2} + b(b-m+2)t^{-2}\right) \\
&\quad + (-u_t)(m-1)\left(t^{-b}s^{-2} + s^{-b}t^{-2}\right) \\
&\quad + (-u_t)b(b-m+2)\left(t^{-2-b} + s^{-2-b}\right) \\
&= \sqrt{2}(m-1)t^{-b}u_y\left(s^{-2} - t^{-2}\right) \\
&\quad + (-u_t)(m-1)\left(t^{-b}s^{-2} + s^{-b}t^{-2} - t^{-2-b} - s^{-2-b}\right) \\
&\leq (-u_t)(m-1)(s^{-b} - t^{-b})(t^{-2} - s^{-2}) \\
&\leq 0\,.
\end{aligned}$$

Note that the existence of $b > 0$ such that $b(b-m+2) \leq -(m-1)$ is guaranteed by the assumption $2m \geq 14$.

Finally, let us show that since we have a positive supersolution to the linearized operator on $\mathbb{R}^{2m} \setminus \{st = 0\}$, the stability of u follows. We must check that (2.1.6) holds. To do it, let us first take nonnegative functions $\zeta \in C^1(\mathbb{R}^{2m+1}_+)$ with compact support in $\{st > 0,\ \lambda \geq 0\}$. Multiply (2.6.2) by ζ and integrate by parts. Using (2.6.3) we obtain

$$\int_{\{st>0\}} f'(u)\,\varphi\,\zeta\,\mathrm{d}x \leq d_\gamma \int_0^\infty \int_{\{st>0\}} \lambda^a \nabla\varphi\cdot\nabla\zeta\,\mathrm{d}x\,\mathrm{d}\lambda\,. \tag{2.6.4}$$

Now, let $\overline{\xi} \in C_c^\infty(\overline{\mathbb{R}^{2m+1}_+} \setminus \{st = 0,\ \lambda \geq 0\})$. Since $\varphi > 0$ in $\{st > 0, \lambda \geq 0\}$, taking $\zeta = \overline{\xi}^2/\varphi$ in (2.6.4) and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\int_{\{st>0\}} f'(u)\,\overline{\xi}^2\,\mathrm{d}x &= \int_{\{st>0\}} f'(u)\,\varphi\frac{\overline{\xi}^2}{\varphi}\,\mathrm{d}x \leq d_\gamma \int_0^\infty\int_{\{st>0\}} \lambda^a\nabla\varphi\cdot\nabla\left(\frac{\overline{\xi}^2}{\varphi}\right)\mathrm{d}x\,\mathrm{d}\lambda \\
&= d_\gamma \int_0^\infty\int_{\{st>0\}} \lambda^a\,\frac{2\overline{\xi}}{\varphi}\,\nabla\varphi\cdot\nabla\overline{\xi}\,\mathrm{d}x\,\mathrm{d}\lambda - d_\gamma\int_0^\infty\int_{\{st>0\}} \lambda^a\,\frac{\overline{\xi}^2}{\varphi^2}|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}\lambda \\
&\leq d_\gamma \int_0^\infty\int_{\{st>0\}} \lambda^a\,|\nabla\overline{\xi}|^2\,\mathrm{d}x\,\mathrm{d}\lambda\,.
\end{aligned}$$

To conclude the proof, let us show that the last inequality holds for every smooth function ξ with compact support in $\overline{\mathbb{R}^{2m+1}_+}$. This will yield the stability of u. Take $\eta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \leq \eta_\varepsilon \leq 1$ and

$$\eta_\varepsilon = \begin{cases} 1 & \text{in} \quad [\varepsilon, +\infty)\,, \\ 0 & \text{in} \quad [0, \varepsilon/2)\,. \end{cases}$$

Then, since $\xi\, \eta_\varepsilon(s)\, \eta_\varepsilon(t)$ has compact support in $\{st > 0,\ \lambda \geq 0\}$, we can replace $\overline{\xi}$ by $\xi\, \eta_\varepsilon(s)\, \eta_\varepsilon(t)$ in the previous inequality to get

$$\frac{1}{d_\gamma} \int_{\mathbb{R}^{2m}} f'(u)\, \xi^2\, \eta_\varepsilon^2(s)\, \eta_\varepsilon^2(t)\, \mathrm{d}x \leq \int_{\mathbb{R}^{2m+1}_+} \lambda^a\, |\nabla(\xi\, \eta_\varepsilon(s)\, \eta_\varepsilon(t))|^2\, \mathrm{d}x\, \mathrm{d}\lambda\,.$$

Now, we compute the terms in the right-hand side of this inequality. By using Cauchy-Schwarz, we see that to deduce the stability condition

$$\frac{1}{d_\gamma} \int_{\mathbb{R}^{2m}} f'(u)\, \xi^2\, \mathrm{d}x \leq \int_{\mathbb{R}^{2m+1}_+} \lambda^a |\nabla \xi|^2\, \mathrm{d}x\, \mathrm{d}\lambda$$

by letting $\varepsilon \to 0$, it is enough to show that

$$\int_{\mathbb{R}^{2m+1}_+} \lambda^a |\nabla \eta_\varepsilon(s)|^2\, \mathrm{d}x\, \mathrm{d}\lambda \to 0 \quad \text{as} \quad \varepsilon \to 0\,,$$

and the same with $\eta_\varepsilon(s)$ replaced by $\eta_\varepsilon(t)$. To see this, let $R > 0$ be such that $\operatorname{supp}(\xi) \subset \overline{B_R^+}$. Then, since $m \geq 3$,

$$\begin{aligned} \int_{\mathbb{R}^{2m+1}_+} \lambda^a |\nabla \eta_\varepsilon(s)|^2\, \mathrm{d}x\, \mathrm{d}\lambda &\leq \frac{C}{\varepsilon^2} \int_0^R \mathrm{d}\lambda\, \lambda^a \int_0^\varepsilon \mathrm{d}s\, s^{m-1} \int_0^R \mathrm{d}t\, t^{m-1} \\ &\leq C\, R^{m+a+1}\, \varepsilon^{m-2} \to 0 \quad \text{as} \quad \varepsilon \to 0\,, \end{aligned}$$

The computation is analogous for $\eta_\varepsilon(t)$. □

Finally, we present the proof of the stability of the Simons cone as a nonlocal (2γ)-minimal surface whenever $2m \geq 14$ and $\gamma \in (0, 1/2)$.

Proof of Corollary 2.1.7. Let u be the saddle-shaped solution of (2.1.1) in dimension $2m \geq 14$. Consider the blow-down sequence $u_k(x) = u(kx)$ with $k \in \mathbb{N}$. On the one hand, since u is stable in such dimensions and $\gamma \in (0, 1/2)$, by Theorem 2.6 in [39] there exists a subsequence k_j such that

$$u_{k_j} \to \chi_\Sigma - \chi_{\mathbb{R}^{2m} \setminus \Sigma} \quad \text{in } L^1(B_1) \quad \text{as } k_j \to +\infty\,,$$

for some cone Σ that is a stable set for the fractional perimeter.

On the other hand, by the asymptotic behavior of u (point (ii) in Theorem 2.1.2) it is clear that

$$u_k \to \chi_{\mathcal{O}} - \chi_{\mathbb{R}^{2m} \setminus \mathcal{O}} \quad \text{a.e. as } k \to +\infty\,.$$

Putting all together we conclude that $\mathcal{O}$ is a stable set for the fractional perimeter if $2m \geq 14$ and $\gamma \in (0, 1/2)$. This is the same as saying that the Simons cone is a stable nonlocal (2γ)-minimal surface in such dimensions. □

Chapter 3

Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone

This is the first of two papers concerning saddle-shaped solutions to the semilinear equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is a linear elliptic integro-differential operator and f is of Allen-Cahn type.

Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only on this set. By the odd symmetry, L_K coincides with a new operator $L_K^{\mathcal{O}}$ which acts on functions defined only on one side of the Simons cone, $\{|x'| > |x''|\}$, and that vanish on it. This operator $L_K^{\mathcal{O}}$, which corresponds to reflect a function oddly and then apply L_K, has a kernel on $\{|x'| > |x''|\}$ which is different from K.

In this first paper, we characterize the kernels K for which the new kernel is positive and therefore one can develop a theory on the saddle-shaped solution. The necessary and sufficient condition for this turns out to be that K is radially symmetric and $\tau \mapsto K(\sqrt{\tau})$ is a strictly convex function.

Assuming this, we prove an energy estimate for doubly radial odd minimizers and the existence of saddle-shaped solution. In a subsequent article, part II, further qualitative properties of saddle-shaped solutions will be established, such as their asymptotic behavior, a maximum principle for the linearized operator, and their uniqueness.

3.1 Introduction

In this paper we study solutions to the semilinear integro-differential equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m} \tag{3.1.1}$$

which are odd with respect to the Simons cone — defined in (3.1.5). The interest on these solutions, often called saddle-shaped solutions, is motivated by the nonlocal version of a conjecture by De Giorgi on the Allen-Cahn equation (see details below) with the aim of finding a counterexample in high dimensions. Moreover, this problem is related to the regularity theory of nonlocal minimal surfaces.

There are only three papers in the literature concerning saddle-shaped solutions to (3.1.1) with L_K being the fractional Laplacian: [60, 61] by Cinti and [88] by the authors.

In all of them the main tool is the extension problem. This paper, together with its second part [87], is the first one to study (3.1.1) without the extension. For this reason our arguments are purely nonlocal and hold for a more general class of kernels.

Equation (3.1.1) is driven by an integro-differential operator L_K of the form

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(y)\} K(x-y)\, dy, \tag{3.1.2}$$

where the kernel K satisfies

$$K \geq 0\,, \quad K(z) = K(-z) \quad \text{and} \quad \int_{\mathbb{R}^n} \min\left\{ |z|^2, 1 \right\} K(z)\, dz < +\infty\,. \tag{3.1.3}$$

The integral in (3.1.2) has to be understood in the principal value sense. The most canonical example of such operators is the fractional Laplacian, defined for $\gamma \in (0,1)$ as

$$(-\Delta)^\gamma u = c_{n,\gamma} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\gamma}}\, dy\,,$$

where $c_{n,\gamma}$ is a normalizing constant.

Recall that the fractional Laplacian has an associated extension problem (see [51]) that allows the use of local arguments to deal with equations such as (3.1.1). This is not the case for general operators L_K, and therefore some purely nonlocal techniques are developed along this work.

Throughout the paper, we assume that L_K is uniformly elliptic, that is,

$$\lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}} \leq K(z) \leq \Lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}}\,, \tag{3.1.4}$$

where λ and Λ are two positive constants. This condition is frequently adopted since it yields Hölder regularity of solutions (see [125, 139]). The family of linear operators satisfying conditions (3.1.3) and (3.1.4) is the so-called $\mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ ellipticity class. For short we will usually write $\mathcal{L}_0$ and we will make explicit the parameters only when needed.

Moreover, for many purposes we will need the operators to be invariant under rotations. This is equivalent to saying that the kernel is radially symmetric, $K(z) = K(|z|)$.

The Simons cone will be a central object along this paper. It is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} \ : \ |x'| = |x''| \right\}. \tag{3.1.5}$$

This cone is of special importance in the theory of local and nonlocal minimal surfaces, and its variational properties are related to the conjecture of De Giorgi (see the end of this introduction for more details). Through the whole article we will use $\mathcal{O}$ and $\mathcal{I}$ to denote each of the parts in which $\mathbb{R}^{2m}$ is divided by the cone $\mathscr{C}$:

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| > |x''| \right\} \text{ and } \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| < |x''| \right\}.$$

Both $\mathcal{O}$ and $\mathcal{I}$ belong to a family of sets in $\mathbb{R}^{2m}$ which are called of *double revolution*. These are sets that are invariant under orthogonal transformations in the first m variables, as well as under orthogonal transformations in the last m variables. That is, $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution if $R\Omega = \Omega$ for every given transformation $R \in O(m)^2 = O(m) \times O(m)$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$.

In this paper we deal with functions that are *doubly radial*. These are functions $w : \mathbb{R}^{2m} \to \mathbb{R}$ that only depend on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, $w(Rx) = w(x)$ for every $R \in O(m)^2$.

In order to define oddness and evenness of functions with respect to the Simons cone, we consider the following isometry, which will play a significant role in this article:

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \,. \end{array} \tag{3.1.6}$$

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathcal{I}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant —although not all points in $\mathscr{C}$ are fixed points of $(\cdot)^\star$. Taking into account this transformation, we say that a doubly radial function w is *odd with respect to the Simons cone* if $w(x) = -w(x^\star)$. Similarly, we say that a doubly radial function w is *even with respect to the Simons cone* if $w(x) = w(x^\star)$.

Regarding the doubly radial symmetry we define the following variables

$$s := |x'| \quad \text{and} \quad t := |x''| \,.$$

They are specially useful when dealing with the Laplacian in these coordinates, since

$$\Delta w = w_{ss} + w_{tt} + \frac{m-1}{s} w_s + \frac{m-1}{t} w_t \tag{3.1.7}$$

becomes an expression suitable to work with. A similar formula appears in the case of the fractional Laplacian thanks to the local extension problem. Having a PDE in the two variables $(s,t) \in \mathbb{R}^2$ is useful to perform certain computations (see [43, 44, 34, 45] for the local case and [60, 61, 88] for the fractional framework).

If we try to follow the same strategy by writing a rotation invariant operator L_K in (s,t) variables, the expression of the new operator is quite complex. Indeed, if $w : \mathbb{R}^{2m} \to \mathbb{R}$ is doubly radial and we define $\widetilde{w}(s,t) := w(s, 0, ..., 0, t, 0, ..., 0)$, it holds

$$L_K w(x) = \widetilde{L}_K \widetilde{w}(|x'|, |x''|)$$

with

$$\widetilde{L}_K \widetilde{w}(s,t) := \int_0^{+\infty} \int_0^{+\infty} \sigma^{m-1} \tau^{m-1} \big(\widetilde{w}(s,t) - \widetilde{w}(\sigma, \tau) \big) J(s,t,\sigma,\tau) \, d\sigma \, d\tau \tag{3.1.8}$$

and

$$J(s,t,\sigma,\tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1} \Big) \, d\omega \, d\tilde{\omega} \,.$$

Note that $\widetilde{L}_K$ is an integro-differential operator in $(0, +\infty) \times (0, +\infty)$, but the expression of its kernel is quite involved. Indeed, such an expression does not become simpler even when L_K is the fractional Laplacian. In this case, the kernel J involves hypergeometric functions of two variables, the so-called Appell functions (see Appendix 3.8 for more details on it), but this does not simplify computations.

Instead of working with the (s,t) variables, we follow another approach that we find more clear and concise. It consists on rewriting the operator L_K with a different kernel

$\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ that is doubly radial with respect to its both arguments, but in such a way that it still acts on functions defined in $\mathbb{R}^{2m}$ —and not in $(0, +\infty)^2$. As it is explained in detail in Section 3.2, if K is a radially symmetric kernel, then we can write L_K acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy \,, \tag{3.1.9}$$

where $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ is doubly radial in both arguments and is defined by

$$\overline{K}(x, y) := \fint_{O(m)^2} K(|Rx - y|) \, dR \,. \tag{3.1.10}$$

Here, dR denotes integration with respect to the Haar measure on $O(m)^2$ (see Section 3.2 for the details).

This new expression (3.1.9) has some advantages compared with (3.1.8). First, the computations in this new setting are shorter and more transparent than the analogous ones using (s, t) variables. This also makes the notation more concise. Furthermore we avoid some issues of the (s, t) variables such as the special treatment of the set $\{st = 0\}$. Although in this paper we do not work in (s, t) variables, we include an appendix at the end of the article with some computations using them (see Appendix 3.8). We think that this could be useful in future works.

Once we have rewritten L_K with a doubly radial kernel $\overline{K}$, as in (3.1.9), we shall find a suitable expression of the operator when acting on odd functions with respect to the Simons cone. Note that such functions are defined by their values in $\mathcal{O}$ and therefore we want to rewrite L_K taking this into account. To this purpose, we define the new operator

$$\begin{aligned} L_K^{\mathcal{O}} w(x) &:= \int_{\mathcal{O}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy + \int_{\mathcal{O}} \{w(x) + w(y)\} \overline{K}(x, y^\star) \, dy \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\} \{\overline{K}(x, y) - \overline{K}(x, y^\star)\} \, dy + 2w(x) \int_{\mathcal{O}} \overline{K}(x, y^\star) \, dy \,, \end{aligned} \tag{3.1.11}$$

where $(\cdot)^\star$ is defined in (3.1.6). As we show in Section 3.2, $L_K^{\mathcal{O}}$ acting on a doubly radial function $w : \mathcal{O} \to \mathbb{R}$ coincides with L_K acting on the odd extension of w with respect to the Simons cone.

Our first main result concerns necessary and sufficient conditions on the original kernel K for this operator to have a positive kernel. As we will stress through this paper, and also in the forthcoming work [87], the positivity of the kernel in (3.1.11) is crucial in order to develop a theory on the saddle-shaped solution. In particular, under this assumption a maximum principle for doubly radial odd functions will hold (see Proposition 3.1.2 below).

Theorem 3.1.1. *Let $K : (0, +\infty) \to (0, +\infty)$ and consider the radially symmetric kernel $K(|x - y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (3.1.10).*

If

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau \,, \tag{3.1.12}$$

then L_K has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x, y) > \overline{K}(x, y^\star) \quad \text{for every } x, y \in \mathcal{O} \,. \tag{3.1.13}$$

In addition, if $K \in C^2((0, +\infty))$, then (3.1.12) is not only a sufficient condition for (3.1.13) to hold, but also a necessary one.

This theorem is proved in Section 3.2 (see Propositions 3.2.4 and 3.2.5). Its proof is based on breaking the integral defining $\overline{K}$ in four clever regions —see (3.2.6)— that allow to compare the integrands for $y \in \mathcal{O}$ and for its reflected $y^* \in \mathcal{I}$. We will use a result on convex functions proved in Appendix 3.6 (Proposition 3.6.1). In the previous statement, by strict convexity in (3.1.12) we mean that

$$K(\sqrt{\tau_1}) + K(\sqrt{\tau_2}) > 2K(\sqrt{(\tau_1+\tau_2)/2})$$

for every $\tau_1, \tau_2 \in (0, +\infty)$.

In [105], Jarohs and Weth study solutions to general integro-differential equations which are odd with respect to a hyperplane. Here the natural sufficient condition on K to have a positive kernel when acting on odd functions is that K is decreasing in the orthogonal direction to the hyperplane. That this suffices is readily deduced after making a change of variables given by the symmetry with respect to such hyperplane. In our case, since we deal with a more complex symmetry, the kernel K is required to satisfy further assumptions than just monotonicity. Moreover, the proof of Theorem 3.1.1 is quite involved and requires a finer argument. Indeed, if we simply make the change $y \mapsto y^\star$ in (3.1.2), following [105], we should prove that $K(|x-y|) > K(|x-y^\star|)$ for every x and y in $\mathcal{O}$, but this is false even in the easiest case $L_K = (-\Delta)^\gamma$ and $2m = 2$. Instead, if we write L_K in the form (3.1.9) with the kernel $\overline{K}$, the analogous positivity condition (3.1.13) holds if we assume $K(\sqrt{\cdot})$ to be convex. Here the use of the (s,t) variables would not simplify the proof of Theorem 3.1.1. As mentioned in Appendix 3.8, an analogous result can be established for the kernel J in (3.1.8), but its proof presents exactly the same difficulties as the one for $\overline{K}$.

The first direct consequence of the positivity condition (3.1.13) is the following maximum principle.

Proposition 3.1.2 (Maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $\Omega \subset \mathcal{O}$ be an open set and let L_K be an integro-differential operator with a radially symmetric kernel K satisfying the positivity condition (3.1.13). Let $u \in C^\alpha(\Omega) \cap C^\gamma(\overline{\Omega}) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone.*

(i) (Weak maximum principle) Assume that

$$\begin{cases} L_K u + c(x)u & \geq & 0 & \text{in } \Omega\,, \\ u & \geq & 0 & \text{in } \mathcal{O}\setminus\Omega\,, \end{cases}$$

with $c \geq 0$, and that either

$$\Omega \text{ is bounded} \quad \text{or} \quad \liminf_{x\in\mathcal{O},\,|x|\to+\infty} u(x) \geq 0\,.$$

Then, $u \geq 0$ in Ω.

(ii) (Strong maximum principle) Assume that $L_K u + c(x)u \geq 0$ in Ω, with c any continuous function, and that $u \geq 0$ in $\mathcal{O}$. Then, either $u \equiv 0$ in $\mathcal{O}$ or $u > 0$ in Ω.

This statement differs from the usual maximum principle for L_K in the fact that we only assume that u is nonpositive in $\mathcal{O}\setminus\Omega$, instead of in $\mathbb{R}^{2m}\setminus\Omega$ (an assumption that makes no sense for odd functions). This form of maximum principle is analogous to the ones in [58, 105], where similar statements are considered for functions that are odd with respect to a hyperplane.

Since in this paper we will always consider doubly radial functions u which are odd with respect to the Simons cone, $L_K u = L_K^{\mathcal{O}} u$ in $\mathcal{O}$. Thus, to simplify the notation we will always write L_K for $L_K^{\mathcal{O}}$. To mean that Proposition 3.1.2 holds, we will say that L_K has a maximum principle in $\mathcal{O}$ when acting on doubly radial odd functions.

Let us now turn to the variational problem from which equation (3.1.1) arises. As it is well known, (3.1.1) is the Euler-Lagrange equation associated to the energy functional

$$\begin{aligned}\mathcal{E}(w,\Omega) := \frac{1}{4}\bigg\{ &\int_\Omega \int_\Omega |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2\int_\Omega \int_{\mathbb{R}^{2m}\setminus\Omega} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y \bigg\} + \int_\Omega G(w)\,\mathrm{d}x\,,\end{aligned} \tag{3.1.14}$$

where G a C^2 function satisfying $G' = -f$. In this paper, we assume the following conditions on G:

$$G \text{ is even and } G \geq G(\pm 1) = 0 \text{ in } \mathbb{R}\,. \tag{3.1.15}$$

Note that the previous conditions on G yield that f is a C^1 odd function with $f(0) = f(\pm 1) = 0$. In some cases, as in Theorem 3.1.4 below, we will further assume that $G(0) > 0$. In such situation, equation (3.1.1) can be seen as a model for phase transitions. The Allen-Cahn nonlinearity, $f(u) = u - u^3$, is the most typical example.

Using the same type of arguments as for the operator L_K, we can rewrite the energy of doubly radial odd functions with a suitable new expression that involves the kernel

$$\overline{K}(x,y) - \overline{K}(x,y^\star) > 0$$

and that only takes into account the values of the functions in $\mathcal{O}$. This will be extremely useful in many computations and estimates involving the nonlocal energy $\mathcal{E}$ (see Sections 3.3 and 3.4). To write this new expression, we introduce the following notation. For A, $B \subset \mathcal{O}$, two sets of double revolution, we define

$$\begin{aligned} I_w(A,B) := 2\int_A\int_B &|w(x)-w(y)|^2 \left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\ &+ 4\int_A\int_B \left\{w^2(x)+w^2(y)\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Then, as proved in Section 3.3 (see Lemma 3.3.2), we can rewrite the energy of a doubly radial odd function w as

$$\mathcal{E}(w,\Omega) = \frac{1}{4}\left\{I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\right\} + 2\int_{\Omega\cap\mathcal{O}} G(w)\,\mathrm{d}x\,. \tag{3.1.16}$$

Thanks to this new expression for the energy, we are able to establish the second main result of this paper. It is the following energy estimate for doubly radial odd minimizers of $\mathcal{E}$. To define such minimizers properly, we denote by $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ the space of doubly radial odd functions that vanish outside B_R and for which the energy $\mathcal{E}$ is well defined (see Section 3.3 for the precise definition). We say that $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ is a doubly radial odd minimizers of $\mathcal{E}$ in B_R if

$$\mathcal{E}(u,B_R) \leq \mathcal{E}(w,B_R)$$

for every $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$.

Theorem 3.1.3. *Let K be a radially symmetric kernel satisfying the positivity condition* (3.1.13) *and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Assume that G is a potential satisfying* (3.1.15). *Let $S \geq 2$ and let $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ be a doubly radial odd minimizer of $\mathcal{E}$ in B_R, with $R > S + 4$.*

Then

$$\mathcal{E}(u, B_S) \leq \begin{cases} C\, S^{2m-2\gamma} & \text{if } \gamma \in (0, 1/2), \\ C\, S^{2m-1} \log S & \text{if } \gamma = 1/2, \\ C\, S^{2m-1} & \text{if } \gamma \in (1/2, 1), \end{cases} \tag{3.1.17}$$

where C is a positive constant depending only on m, γ, λ, Λ, and $||G||_{C^2([-1,1])}$.

In the proof of this result, a first basic ingredient is that $-1 \leq u \leq 1$, as provided by Lemma 3.3.3. This information, $|u| \leq 1$, is also of importance for a solution of an Allen-Cahn equation, as in the existence Theorem 3.1.4 below. That $|u| \leq 1$ is proved with a variational cutting argument: cutting above 1 and below -1 reduces de energy. We believe that this property requires $\overline{K}(x, y) - \overline{K}(x, y^\star)$ to be nonnegative. In addition, the proof of Lemma 3.3.3 is a priori not simple since it involves a nonlocal energy of functions with symmetries. We succeeded to greatly simplify the computations by writing the energy as in (3.3.4), obtaining a short proof.

Note that Theorem 3.1.3 does not follow from the energy estimate for general minimizers stated in [137] by Savin and Valdinoci. The minimizers that they consider do not have any type of symmetry. In our case, the function u in the previous statement minimizes the energy in a smaller class of functions and the result in [137] cannot be applied. Nevertheless, we are able to adapt the arguments of Savin and Valdinoci to our setting. The strategy they follow is to compare the energy of u with the one of a suitable competitor which is constructed by taking the minimum between u and a radially symmetric auxiliary function —see (3.4.5) below. Such competitor is not permitted in our case, since it is not odd with respect to the Simons cone. Nevertheless, we show in Section 3.4 how to modify the auxiliary functions of [137] to carry out the same type of arguments. The assumption (3.1.13) will be crucial to guarantee that $0 \leq u \leq 1$ in $\mathcal{O}$.

The particular result of Theorem 3.1.3 for the fractional Laplacian has been proved by Cabré and Cinti [36] in the case of the half-Laplacian, and extended to all the powers $0 < \gamma < 1$ by Cinti [61] (see [37] for an extension to non-doubly radial minimizers). These papers use the local extension problem and therefore their proofs cannot be extended to general operators like L_K. Our proof, following [137], overcomes this issue.

As an application of the previous results, we prove, by using standard variational methods, the existence of saddle-shaped solution to (3.1.1) when f is of Allen-Cahn type. We say that a bounded solution u to (3.1.1) is a *saddle-shaped* solution if u is doubly radial, odd with respect to the Simons cone, and positive in $\mathcal{O}$.

Theorem 3.1.4 (Existence of saddle-shaped solution)**.** *Let G satisfy* (3.1.15), *$G(0) > 0$, and let $f = -G'$. Let K be a radially symmetric kernel satisfying the positivity condition* (3.1.13) *and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$.*

Then, for every even dimension $2m \geq 2$, there exists a saddle-shaped solution u to (3.1.1). *In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.*

We are interested in the study of this type of solutions since they are relevant in connection with a famous conjecture for the (classical) Allen-Cahn equation raised by De Giorgi, that reads as follows. Let u be a bounded monotone (in some direction) solution to $-\Delta u = u - u^3$ in $\mathbb{R}^n$, then, if $n \leq 8$, u depends only on one Euclidean variable, that is, all its level sets are hyperplanes. This conjecture is not completely closed (see [84] and references therein) but a counterexample in dimension $n = 9$ was build in [74] by using the so-called gluing method. Saddle-shaped solutions are natural objects to build a counterexample in a simpler way, as explained next. On the one hand, Jerison and Monneau [106] showed that a counterexample to the conjecture of De Giorgi in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate, but is not one-dimensional. On the other hand, by the Γ-converge results from Modica and Mortola (see [118, 119]) and the fact that the Simons cone is the simplest nonplanar minimizing minimal surface, saddle-shaped solutions are expected to be global minimizers of the Allen-Cahn equation in dimensions $2m \geq 8$ (this is still an open problem).

Similar facts happen in the nonlocal setting (see the introduction of [88] for further details). For this reason, saddle-shaped solutions are of interest in the study of the nonlocal version of the conjecture of De Giorgi for equation (3.1.1).

Saddle-shaped solutions to the local Allen-Cahn equation involving the Laplacian were studied in [69, 138, 43, 44, 34]. In these works, it is established the existence, uniqueness, and some qualitative properties of this type of solutions, such as their instability when $2m \leq 6$ and their stability if $2m \geq 14$. Stability in dimensions $8, 10$, and 12 is still an open problem, as well as minimality in dimensions $2m \geq 8$.

In the fractional framework, there are only three works concerning saddle-shaped solutions to the equation $(-\Delta)^\gamma u = f(u)$. In [60, 61], Cinti proved the existence of saddle-shaped solution as well as some qualitative properties such as their asymptotic behavior, some monotonicity properties, and their instability in low dimensions. In a previous paper by the authors [88], further properties of these solutions have been established, the main ones being uniqueness and, when $2m \geq 14$, stability. The present paper together with its second part [87] are the first ones studying saddle-shaped solutions for general integro-differential equations of the form (3.1.1). In the three previous papers [60, 61, 88], the main tool used is the extension problem for the fractional Laplacian (see [51]). As mentioned, this technique cannot be carried out for general integro-differential operators different from the fractional Laplacian. Therefore, some purely nonlocal techniques are developed through both papers.

In the forthcoming paper [87], we study saddle-shaped solutions to (3.1.1) in more detail taking advantage of the setting for odd functions built in the present article. We give an alternative proof for the existence of a saddle-shaped solution by using monotone iteration and maximum principle techniques. As in the proof of Theorem 3.1.4, the assumtion (3.1.13) is crucial. Furthermore, we prove the asymptotic behaviour of this type of solutions by using some symmetry and Liouville type results for general integro-differential operators that we establish in the same paper. Finally, we also show in [87] the uniqueness of the saddle-shaped solution through a maximum principle for the linearized operator, which we also prove in that article.

Let us make some final remarks on the minimality and stability properties of the Simons cone. Recall that, in the classical theory of minimal surfaces, it is well known that the Simons cone has zero mean curvature at every point $x \in \mathscr{C} \setminus \{0\}$, in all even dimensions, and it is a minimizer of the perimeter functional when $2m \geq 8$. Concerning the nonlocal setting, $\mathscr{C}$ has also zero nonlocal mean curvature in all even dimensions, although it is not known if it is a minimizer of the nonlocal perimeter in any dimension. If $2m = 2$ it cannot be a minimizer since in [136] it is proven that all minimizing nonlocal minimal cones in $\mathbb{R}^2$ are flat. In higher dimensions, the only available results appear in [70, 88] but concern stability, a weaker property than minimality. In [70], Dávila, del

Pino, and Wei characterize the stability of Lawson cones —a more general class of cones that includes $\mathscr{C}$— through an inequality involving only two hypergeometric constants which depend only on γ and the dimension n. This inequality is checked numerically in [70], finding that, in dimensions $n \leq 6$ and for γ close to zero, no Lawson cone with zero nonlocal mean curvature is stable. Numerics also shows that all Lawson cones in dimension 7 are stable if γ is close to zero. These results for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem. In [88], we proved, by using the saddle-shaped solution to the fractional Allen-Cahn equation and a Γ-convergence result of [39], that the Simons cone is a stable (2γ)-minimal cone in dimensions $2m \geq 14$. To the best of our knowledge, this is the first analytical proof of a stability result for the Simons cone in any dimension.

This paper is organized as follows. Section 3.2 is devoted to study the operator L_K acting on doubly radial odd functions. We deduce the expression of the kernel $\overline{K}$ and rewrite the operator acting on doubly radial odd functions, finding the expression (3.1.11). We also show Theorem 3.1.1 and Proposition 3.1.2. In Section 3.3 we study the energy functional associated to (3.1.1) and in Section 3.4 we establish the energy estimate stated in Theorem 3.1.3. Finally, in Section 3.5 we prove the existence of a saddle-shaped solution to the integro-differential Allen-Cahn equation. At the end of the paper there are three appendices. Appendix 3.6 is devoted to some results on convex functions, and Appendix 3.7 contains some auxiliary computations. Both are used in the proof of Theorem 3.1.1. In Appendix 3.8 we include some results and expressions in (s, t) variables for future reference.

3.2 Rotation invariant operators acting on doubly radial odd functions

This section is devoted to study rotation invariant operators of the class $\mathcal{L}_0$ when they act on doubly radial odd functions. First, we deduce an alternative expression for the operator in terms of a doubly radial kernel $\overline{K}$. Then, we present necessary and sufficient conditions on the kernel K in order to (3.1.13) hold (we establish Theorem 3.1.1). Finally, we show two maximum principles for doubly radial odd functions (Proposition 3.1.2).

3.2.1 Alternative expressions for the operator L_K

The main purpose of this subsection is to deduce an alternative expression for a rotation invariant operator $L_K \in \mathcal{L}_0$ acting on doubly radial functions. This expression is more suitable to work with and it will be used throughout the paper. Our first remark is that if w is invariant by $O(m)^2$, the same holds for $L_K w$. Indeed, for every $R \in O(m)^2$,

$$\begin{aligned} L_K w(Rx) &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(y)\} K(|Rx - y|)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(R\tilde{y})\} K(|Rx - R\tilde{y}|)\, \mathrm{d}\tilde{y} \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(\tilde{y})\} K(|x - \tilde{y}|)\, \mathrm{d}\tilde{y} \\ &= L_K w(x)\,. \end{aligned}$$

Here we have used the change $y = R\tilde{y}$ and the fact that $w(R\cdot) = w(\cdot)$ for every $R \in O(m)^2$.

Next, we present an alternative expression for the operator L_K acting on doubly radial functions. This expression involves the new kernel $\overline{K}$, which is also doubly radial.

Lemma 3.2.1. *Let $L_K \in \mathcal{L}_0(2m, \gamma)$ have a radially symmetric kernel K, and let w be a doubly radial function such that $L_K w$ is well-defined. Then, $L_K w$ can be expressed as*

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y)\, \mathrm{d}y$$

where $\overline{K}$ is symmetric, invariant by $O(m)^2$ in both arguments, and it is defined by

$$\overline{K}(x, y) := \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\,.$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$.

Recall (see for instance [120]) that the Haar measure on $O(m)^2$ exists and it is unique up to a multiplicative constant. Let us state next the properties of this measure that will be used in the rest of the paper. In the following, the Haar measure is denoted by μ. First, since $O(m)^2$ is a compact group, it is unimodular (see Chapter II, Proposition 13 of [120]). As a consequence, the measure μ is left and right invariant, that is, $\mu(R\Sigma) = \mu(\Sigma) = \mu(\Sigma R)$ for every subset $\Sigma \subset O(m)^2$ and every $R \in O(m)^2$. Moreover, it holds

$$\fint_{O(m)^2} g(R^{-1})\, \mathrm{d}R = \fint_{O(m)^2} g(R)\, \mathrm{d}R \tag{3.2.1}$$

for every $g \in L^1(O(m)^2)$ —see [120] for the details.

Proof of Lemma 3.2.1. Since $L_K w(x) = L_K w(Rx)$ for every $R \in O(m)^2$, by taking the mean over all the transformations in $O(m)^2$, we get

$$\begin{aligned} L_K w(x) = \fint_{O(m)^2} L_K w(Rx)\, \mathrm{d}R &= \fint_{O(m)^2} \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} K(|Rx - y|)\, \mathrm{d}y\, \mathrm{d}R \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\, \mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y)\, \mathrm{d}y\,. \end{aligned}$$

Now, we show that $\overline{K}$ is symmetric. Using property (3.2.1), we get

$$\begin{aligned}\overline{K}(y,x) &= \fint_{O(m)^2} K(|Ry-x|)\,\mathrm{d}R = \fint_{O(m)^2} K(|R^{-1}(Ry-x)|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|R^{-1}x-y)|)\,\mathrm{d}R = \overline{K}(x,y)\,.\end{aligned}$$

It remains to show that $\overline{K}$ is invariant by $O(m)^2$ in its two arguments. By the symmetry, it is enough to check it for the first one. Let $\tilde{R} \in O(m)^2$. Then,

$$\overline{K}(\tilde{R}x,y) = \fint_{O(m)^2} K(|R\tilde{R}x-y|)\,\mathrm{d}R = \fint_{O(m)^2} K(|Rx-y|)\,\mathrm{d}R = \overline{K}(x,y)\,,$$

where we have used the right invariance of the Haar measure. $\square$

In the following lemma we present some properties of the involution $(\cdot)^\star$ defined by (3.1.6) and its relation with the doubly radial kernel $\overline{K}$ and the transformations of $O(m)^2$. In particular, in the proof of Theorem 3.1.1 it will be useful to consider the following transformation. For every $R \in O(m)^2$, we define $R_\star \in O(m)^2$ by

$$R_\star := (R(\cdot)^\star)^\star\,. \tag{3.2.2}$$

Equivalently, if $R = (R_1, R_2)$ with $R_1, R_2 \in O(m)$, then $R_\star = (R_2, R_1)$.

Lemma 3.2.2. *Let $(\cdot)^\star : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be the involution defined by $x^\star = (x', x'')^\star = (x'', x')$ —see (3.1.6). Then,*

1. *The Haar integral on $O(m)^2$ has the following invariance:*

$$\int_{O(m)^2} g(R_\star)\,\mathrm{d}R = \int_{O(m)^2} g(R)\,\mathrm{d}R\,, \tag{3.2.3}$$

   for every $g \in L^1(O(m)^2)$.

2. *$\overline{K}(x^\star, y) = \overline{K}(x, y^\star)$. As a consequence, $\overline{K}(x^\star, y^\star) = \overline{K}(x,y)$.*

Proof. The first statement is easy to check by using Fubini:

$$\begin{aligned}\int_{O(m)^2} g(R_\star)\,\mathrm{d}R &= \int_{O(m)} \mathrm{d}R_1 \int_{O(m)} \mathrm{d}R_2\;\; g(R_2,R_1) = \int_{O(m)} \mathrm{d}R_2 \int_{O(m)} \mathrm{d}R_1\;\; g(R_2,R_1) \\ &= \int_{O(m)} \mathrm{d}R_1 \int_{O(m)} \mathrm{d}R_2\;\; g(R_1,R_2) = \int_{O(m)^2} g(R)\,\mathrm{d}R\,.\end{aligned}$$

To show the second statement, we use the definition of $R_\star$ and (3.2.3) to see that

$$\begin{aligned}\overline{K}(x^\star,y) &= \fint_{O(m)^2} K(|Rx^\star-y|)\,\mathrm{d}R = \fint_{O(m)^2} K(|(Rx^\star-y)^\star|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|(Rx^\star)^\star-y^\star|)\,\mathrm{d}R = \fint_{O(m)^2} K(|R_\star x-y^\star|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|Rx-y^\star|)\,\mathrm{d}R = \overline{K}(x,y^\star)\,.\end{aligned}$$

As a consequence, we have that $\overline{K}(x^\star,y^\star) = \overline{K}(x,(y^\star)^\star) = \overline{K}(x,y)$. $\square$

To conclude this subsection, we present two alternative expressions for the operator L_K when it acts on doubly radial odd functions. These expressions are suitable in the rest of the paper and also in the forthcoming one [87], since the integrals appearing in the expression are computed only in $\mathcal{O}$, and this is important to prove maximum principle and other properties.

Lemma 3.2.3. *Let w be a doubly radial function which is odd with respect to the Simons cone. Let $L_K \in \mathcal{L}_0(2m,\gamma,\lambda,\Lambda)$ be a rotation invariant operator and let $L_K^{\mathcal{O}}$ be defined by (3.1.11).*

Then,

$$\begin{aligned} L_K w(x) &= \int_{\mathcal{O}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{O}} \{w(x)+w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x)-w(y)\}\{\overline{K}(x,y)-\overline{K}(x,y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y\,. \end{aligned}$$

In particular, the second equality shows that $L_K w(x) = L_K^{\mathcal{O}} w(x)$. Moreover,

$$\frac{1}{C}\operatorname{dist}(x,\mathscr{C})^{-2\gamma} \le \int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y \le C\operatorname{dist}(x,\mathscr{C})^{-2\gamma}, \tag{3.2.4}$$

where $C>0$ is a constant depending only on m,γ,λ, and Λ.

Proof. The first statement is just a computation. Indeed, using the change of variables $\bar{y} = y^\star$ and the odd symmetry of w, we see that

$$\begin{aligned} \int_{\mathcal{I}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y &= \int_{\mathcal{O}} \{w(x)-w(y^\star)\}\overline{K}(x,y^\star)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x)+w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y\,. \end{aligned}$$

Hence,

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}^{2m}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{I}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x)-w(y)\}\overline{K}(x,y)\,\mathrm{d}y + \int_{\mathcal{O}} \{w(x)+w(y)\}\overline{K}(x,y^\star)\,\mathrm{d}y\,. \end{aligned}$$

By adding and subtracting $w(x)\overline{K}(x,y^\star)$ in the last integrand, we immediately deduce

$$L_K w(x) = \int_{\mathcal{O}} \{w(x)-w(y)\}\{\overline{K}(x,y)-\overline{K}(x,y^\star)\}\,\mathrm{d}y + 2w(x)\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y\,.$$

Note that we can add and subtract the term $w(x)\overline{K}(x,y^\star)$ since it is integrable with respect to y in $\mathcal{O}$. This is a consequence of (3.2.4).

Let us show now (3.2.4). In the following arguments we will use the letters C and c to denote positive constants, depending only on m,γ,λ, and Λ, that may change its value in each inequality.

On the one hand, for the upper bound in (3.2.4) we only need to use the ellipticity of the kernel and the inclusion $\mathcal{I} \subset \{y \in \mathbb{R}^{2m} : |x-y| \geq \operatorname{dist}(x,\mathscr{C})\}$ for every $x \in \mathcal{O}$. Indeed,

$$\begin{aligned}
\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y &= \int_{\mathcal{O}} K(|x-y^\star|)\,\mathrm{d}y = \int_{\mathcal{I}} K(|x-y|)\,\mathrm{d}y \leq \int_{|x-y|\geq \operatorname{dist}(x,\mathscr{C})} K(|x-y|)\,\mathrm{d}y \\
&\leq C \int_{|x-y|\geq \operatorname{dist}(x,\mathscr{C})} |x-y|^{-2m-2\gamma}\,\mathrm{d}y = C\int_{\operatorname{dist}(x,\mathscr{C})}^{\infty} \rho^{-1-2s}\,\mathrm{d}\rho \\
&= C\operatorname{dist}(x,\mathscr{C})^{-2s}.
\end{aligned}$$

On the other hand, for the lower bound in (3.2.4), let be $\overline{x} \in \mathscr{C}$ such that $|x-\overline{x}| = \operatorname{dist}(x,\mathscr{C})$. Then, given $y \in B_{\operatorname{dist}(x,\mathscr{C})}(\overline{x})$, it is clear that $|x-y| \leq |x-\overline{x}| + |\overline{x}-y| \leq 2\operatorname{dist}(x,\mathscr{C})$. Therefore, we have

$$\begin{aligned}
\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y &= \int_{\mathcal{I}} K(|x-y|)\,\mathrm{d}y \geq c\int_{\mathcal{I}} |x-y|^{-2m-2\gamma}\,\mathrm{d}y \\
&\geq c\int_{B_{\operatorname{dist}(x,\mathscr{C})}(\overline{x})\cap\mathcal{I}} |x-y|^{-2m-2\gamma}\,\mathrm{d}y \\
&\geq c\,(2\operatorname{dist}(x,\mathscr{C}))^{-2m-2\gamma}|B_{\operatorname{dist}(x,\mathscr{C})}(\overline{x})\cap\mathcal{I}| = c\,\operatorname{dist}(x,\mathscr{C})^{-2\gamma}.
\end{aligned}$$

Here we have used a property of the Simons cone: $|B_R(z)\cap\mathcal{I}| = 1/2|B_R|$ for every $z \in \mathscr{C}$ (see Lemma 2.5 in [88] for the proof). □

3.2.2 Necessary and sufficient conditions for ellipticity

In this subsection, we establish Theorem 3.1.1. As we have mentioned in the introduction, the kernel inequality (3.1.13) is crucial in the rest of the results of this paper, as well as in the ones in [87]. We will see in the next subsection that this inequality guarantees that the operator L_K has a maximum principle for odd functions (see Proposition 3.1.2).

First, we give a sufficient condition on a radially symmetric kernel K so that $\overline{K}$ satisfies (3.1.13). It is the following result.

Proposition 3.2.4. *Let $K : (0,+\infty) \to \mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m}\times\mathbb{R}^{2m} \to \mathbb{R}$ by (3.1.10). Assume that $K(\sqrt{\cdot})$ is strictly convex in $(0,+\infty)$. Then, the associated kernel $\overline{K}$ satisfies*

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \textit{for every } x,y \in \mathcal{O}\,. \tag{3.2.5}$$

Proof. Since $\overline{K}$ is invariant by $O(m)^2$, it is enough to choose a unitary vector $e \in \mathbb{S}^{m-1}$ and show (3.2.5) for points $x,y \in \mathcal{O}$ of the form $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$.

Now, define

$$\begin{aligned}
Q_1 &:= \left\{R = (R_1,R_2) \in O(m)^2 \ : \ e\cdot R_1 e > |e\cdot R_2 e|\right\}, \\
Q_2 &:= \left\{R = (R_1,R_2) \in O(m)^2 \ : \ e\cdot R_2 e > |e\cdot R_1 e|\right\} = (Q_1)_\star, \\
Q_3 &:= \left\{R = (R_1,R_2) \in O(m)^2 \ : \ e\cdot R_1 e < -|e\cdot R_2 e|\right\} = -Q_1, \\
Q_4 &:= \left\{R = (R_1,R_2) \in O(m)^2 \ : \ e\cdot R_2 e < -|e\cdot R_1 e|\right\} = -(Q_1)_\star.
\end{aligned} \tag{3.2.6}$$

Recall that given $R = (R_1, R_2) \in O(m)^2$, then $R_\star = (R_2, R_1) \in O(m)^2$ —see (3.2.2). Moreover, note that the sets Q_i are disjoint, have the same measure and cover all $O(m)^2$ up to a set of measure zero.

Therefore,

$$
\begin{aligned}
4\overline{K}(x,y) &= 4 \fint_{O(m)^2} K(|x - Ry|)\,\mathrm{d}R \\
&= \fint_{Q_1} K(|x - Ry|)\,\mathrm{d}R + \fint_{Q_2} K(|x - Ry|)\,\mathrm{d}R \\
&\quad + \fint_{Q_3} K(|x - Ry|)\,\mathrm{d}R + \fint_{Q_4} K(|x - Ry|)\,\mathrm{d}R \\
&= \fint_{Q_1} \{K(|x - Ry|) + K(|x + Ry|) \\
&\quad + K(|x - R_\star y|) + K(|x + R_\star y|)\}\,\mathrm{d}R
\end{aligned}
$$

and

$$
\begin{aligned}
4\overline{K}(x,y^\star) &= 4 \fint_{O(m)^2} K(|x - Ry^\star|)\,\mathrm{d}R \\
&= \fint_{Q_1} \{K(|x - Ry^\star|) + K(|x + Ry^\star|) \\
&\quad + K(|x - R_\star y^\star|) + K(|x + R_\star y^\star|)\}\,\mathrm{d}R.
\end{aligned}
$$

Thus, if we prove

$$
\begin{aligned}
&K(|x - Ry|) + K(|x + Ry|) + K(|x - R_\star y|) + K(|x + R_\star y|) \\
&\quad \geq K(|x - Ry^\star|) + K(|x + Ry^\star|) + K(|x - R_\star y^\star|) + K(|x + R_\star y^\star|)\,,
\end{aligned}
\tag{3.2.7}
$$

for every $R \in Q_1$, we immediately deduce (3.2.5) with a non strict inequality. To see that it is indeed a strict one, we will show that the inequality in (3.2.7) is strict for every $R \in Q_1$.

For a short notation, we call

$$
\alpha := e \cdot R_1 e \quad \text{and} \quad \beta := e \cdot R_2 e\,. \tag{3.2.8}
$$

Now, note that since $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, we have

$$
\begin{aligned}
|x \pm Ry|^2 &= |x' \pm R_1 y'|^2 + |x'' \pm R_2 y''|^2 \\
&= |x'|^2 + |y'|^2 \pm 2x' \cdot R_1 y' + |x''|^2 + |y''|^2 \pm 2x'' \cdot R_2 y'' \\
&= |x|^2 + |y|^2 \pm 2|x'||y'|\alpha \pm 2|x''||y''|\beta.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|x \pm R_\star y|^2 &= |x|^2 + |y|^2 \pm 2|x'||y'|\beta \pm 2|x''||y''|\alpha, \\
|x \pm Ry^\star|^2 &= |x|^2 + |y|^2 \pm 2|x'||y''|\alpha \pm 2|x''||y'|\beta,
\end{aligned}
$$

and

$$
|x \pm R_\star y^\star|^2 = |x|^2 + |y|^2 \pm 2|x'||y''|\beta \pm 2|x''||y'|\alpha.
$$

We define now

$$g(\tau) := K\left(\sqrt{|x|^2+|y|^2+2\tau}\right) + K\left(\sqrt{|x|^2+|y|^2-2\tau}\right).$$

Thus, proving (3.2.7) is equivalent to show that, for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$, it holds

$$\begin{aligned} g\Big(|x'||y'|\alpha + |x''||y''|\beta\Big) + g\Big(|x'||y'|\beta + |x''||y''|\alpha\Big) \\ \geq g\Big(|x'||y''|\alpha + |x''||y'|\beta\Big) + g\Big(|x'||y''|\beta + |x''||y'|\alpha\Big). \end{aligned} \tag{3.2.9}$$

Let

$$\begin{aligned} A_{\alpha,\beta} &:= |x'||y'|\alpha + |x''||y''|\beta, \quad B_{\alpha,\beta} := |x'||y''|\alpha + |x''||y'|\beta, \\ C_{\alpha,\beta} &:= |x''||y'|\alpha + |x'||y''|\beta, \quad D_{\alpha,\beta} := |x''||y''|\alpha + |x'||y'|\beta. \end{aligned}$$

With this notation and taking into account that g is even, (3.2.9) is equivalent to

$$g(|A_{\alpha,\beta}|) + g(|D_{\alpha,\beta}|) \geq g(|C_{\alpha,\beta}|) + g(|B_{\alpha,\beta}|), \tag{3.2.10}$$

for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$. Note that g is defined in the open interval $I = (-(|x|^2+|y|^2)/2,\ (|x|^2+|y|^2)/2)$ and that $A_{\alpha,\beta}$, $B_{\alpha,\beta}$, $C_{\alpha,\beta}$, $D_{\alpha,\beta} \in I$.

To show (3.2.10), we use Proposition 3.6.1 of the Appendix 3.6. There, it is stated that in order to establish (3.2.10) it is enough to check that

$$\begin{cases} |A_{\alpha,\beta}| \geq |B_{\alpha,\beta}|, \ \ |A_{\alpha,\beta}| \geq |C_{\alpha,\beta}|, \ \ |A_{\alpha,\beta}| \geq |D_{\alpha,\beta}|, \\ |A_{\alpha,\beta}| + |D_{\alpha,\beta}| \geq |B_{\alpha,\beta}| + |C_{\alpha,\beta}|. \end{cases}$$

The verification of these inequalities is a simple but tedious computation and it is presented in Appendix 3.7 —see point (1) of Lemma 3.7.1. Once this is proved, we deduce (3.2.10) by Proposition 3.6.1.

To finish, we see that the inequality in (3.2.10) is always strict for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$ (that corresponds to Q_1). By contradiction, assume that equality holds in (3.2.10). Thus, by Proposition 3.6.1, if follows that the sets $\{|A_{\alpha,\beta}|, |D_{\alpha,\beta}|\}$ and $\{|B_{\alpha,\beta}|, |C_{\alpha,\beta}|\}$ coincide. This fact and point (2) of Lemma 3.7.1 yield $\alpha = \beta = 0$, a contradiction. Thus, the inequality in (3.2.10) is strict, as well as the inequality in (3.2.7). This leads to (3.2.5). □

Now, we give a necessary condition on the kernel K so that inequality (3.1.13) holds.

Proposition 3.2.5. *Let $K : (0,+\infty) \to \mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (3.1.10).*

If

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \text{for almost every } x,y \in \mathcal{O}, \tag{3.2.11}$$

then $K(\sqrt{\cdot})$ cannot be concave in any open interval $I \subset [0,+\infty)$.

Proof. It suffices to show that if there exists an open interval where $K(\sqrt{\cdot})$ is concave, then we can find a nonempty open set in $\mathcal{O} \times \mathcal{O}$ where (3.2.11) is not satisfied.

Let $\ell_2 > \ell_1 > 0$ be such that $K(\sqrt{\cdot})$ is concave in (ℓ_1, ℓ_2) and define the set $\Omega_{\ell_1,\ell_2} \subset \mathbb{R}^{4m}$ as the points $(x,y) \in \mathcal{O} \times \mathcal{O}$ satisfying

$$\begin{cases} (|x'|-|y'|)^2 + (|x''|-|y''|)^2 > \ell_1, \\ (|x'|+|y'|)^2 + (|x''|+|y''|)^2 < \ell_2. \end{cases} \tag{3.2.12}$$

First, it is easy to see that Ω_{ℓ_1,ℓ_2} is a nonempty open set. In fact, points of the form $(x', 0, y', 0) \in (\mathbb{R}^m)^4$ such that $(|x'| - |y'|)^2 > \ell_1$ and $(|x'| + |y'|)^2 < \ell_2$ belong to Ω_{ℓ_1,ℓ_2}.

We need to prove that $\overline{K}(x,y) \le \overline{K}(x,y^\star)$ in Ω_{ℓ_1,ℓ_2} for any $(x,y) \in \mathcal{O} \times \mathcal{O}$ satisfying (3.2.12). For such points, we are going to show, as in the previous proof, that

$$\begin{aligned} &K(|x - Ry|) + K(|x + Ry|) + K(|x - R_\star y|) + K(|x + R_\star y|) \\ &\qquad \le K(|x - Ry^\star|) + K(|x + Ry^\star|) + K(|x - R_\star y^\star|) + K(|x + R_\star y^\star|)\,, \end{aligned} \tag{3.2.13}$$

for any $R \in Q_1$, where Q_1 is defined in (3.2.6) (see the proof of Proposition 3.2.4). As before, we can assume that x and y are of the form $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, with $e \in \mathbb{S}^{m-1}$ an arbitrary unitary vector. Then, by defining α and β as in (3.2.8), we see that proving (3.2.13) is equivalent to establish that

$$g(A_{\alpha,\beta}) + g(D_{\alpha,\beta}) \le g(B_{\alpha,\beta}) + g(C_{\alpha,\beta})\,, \tag{3.2.14}$$

for every $\alpha, \beta \in [-1,1]$ such that $\alpha > |\beta|$, where

$$\begin{aligned} A_{\alpha,\beta} &= |x'||y'|\alpha + |x''||y''|\beta\,, \quad & B_{\alpha,\beta} &= |x'||y''|\alpha + |x''||y'|\beta\,, \\ C_{\alpha,\beta} &= |x''||y'|\alpha + |x'||y''|\beta\,, \quad & D_{\alpha,\beta} &= |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

and

$$g(\tau) = K\left(\sqrt{|x|^2 + |y|^2 + 2\tau}\right) + K\left(\sqrt{|x|^2 + |y|^2 - 2\tau}\right).$$

Now, by (3.2.12), we have $\ell_1 < |x|^2 + |y|^2 < \ell_2$. As a consequence of this and the concavity of $K(\sqrt{\cdot})$ in (ℓ_1, ℓ_2), it is easy to see (by using Lemma 3.6.2 stated for $-h$, a concave function, instead of h) that g is concave in $\left(-\overline{\ell}, \overline{\ell}\right)$, and decreasing in $(0, \overline{\ell})$, where

$$\overline{\ell} := \min\left\{\frac{\ell_2 - |x|^2 - |y|^2}{2}, \frac{|x|^2 + |y|^2 - \ell_1}{2}\right\}.$$

Note that, since $\ell_1 < |x|^2 + |y|^2 < \ell_2$, we have $\overline{\ell} > 0$.

We claim that $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$, and $D_{\alpha,\beta}$ belong to $(-\overline{\ell}, \overline{\ell})$ for every $\alpha, \beta \in [-1,1]$ such that $\alpha > |\beta|$. Indeed, it is easy to check that for every $\alpha, \beta \in [-1,1]$ such that $\alpha > |\beta|$, the numbers $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$, and $D_{\alpha,\beta}$ belong to the open interval $(-|x'||y'| - |x''||y''|, |x'||y'| + |x''||y''|)$. Furthermore, since $x, y \in \Omega_{\ell_1,\ell_2}$, we obtain from (3.2.12) that

$$\begin{cases} |x'||y'| + |x''||y''| & < \dfrac{\ell_2 - |x|^2 - |y|^2}{2} \\ |x'||y'| + |x''||y''| & < \dfrac{|x|^2 + |y|^2 - \ell_1}{2} \end{cases}$$

and thus $|x'||y'| + |x''||y''| < \overline{\ell}$ and the claim is proved.

Finally, by applying Lemma 3.6.2 to the function $-g$ in $(0, \overline{\ell})$ (using again point (1) of Lemma 3.7.1), we obtain that inequality (3.2.14) is satisfied, which yields (3.2.13). Finally, by integrating (3.2.13) with respect to all the rotations $R \in Q_1$ we get

$$\overline{K}(x,y) \le \overline{K}(x,y^\star),$$

for every $(x,y) \in \Omega_{\ell_1,\ell_2}$, contradicting (3.2.11). □

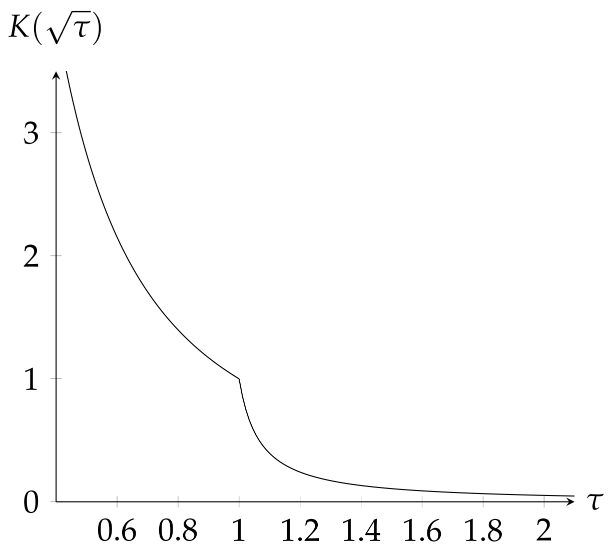

Figure 3.1: An example of kernel $K(\sqrt{\tau})$ ($m = 1$ and $\gamma = 1/2$) which is not strictly convex in $(0, +\infty)$ but does not have any interval of concavity.

From the two previous results, Theorem 3.1.1 follows immediately.

Proof of Theorem 3.1.1. The first statement is exactly the same as Proposition 3.2.4. Assume now that K is a C^2 function and that (3.1.13) holds. Then, by Proposition 3.2.5, $h(\cdot) := K(\sqrt{\cdot})$ is not concave in any interval of $[0, +\infty)$. Therefore, we cannot have $h'' < 0$ at any point. Thus, $h'' \geq 0$ in $[0, +\infty)$ or, in other words, h' is nondecreasing. Using again that h is not concave in any interval, we deduce that h' must be, in fact, increasing. It follows that $h(\cdot) = K(\sqrt{\cdot})$ is strictly convex as defined after the statement of Theorem 3.1.1. $\square$

Remark 3.2.6. Note that a priori we cannot relax the $K \in C^2$ assumption in the necessary condition of Theorem 3.1.1, since there are C^1 functions that are neither convex nor concave in any interval (they can be constructed as a primitive of a Weierstrass function, whose graph is a non rectifiable curve with fractal dimension). Besides these "exotic" examples, there are also simple radially symmetric kernels K that are not C^1 for which we do not know if the positivity condition (3.1.13) holds. For instance, given $0 < \gamma < 1$, if we consider the kernel

$$K(\tau) = \frac{1}{\tau^{2m+2\gamma}} \chi_{(0,1)}(\tau) + \frac{1}{10\tau^{2m+2\gamma} - 9} \chi_{[1,+\infty)}(\tau),$$

it is easy to check that K is continuous and decreasing but $K(\sqrt{\tau})$ is not convex in $(0, +\infty)$ even though it does not have any interval of concavity (see Figure 3.1).

3.2.3 Maximum principles for doubly radial odd functions

In this subsection we prove Proposition 3.1.2, a weak and a strong maximum principles for doubly radial functions that are odd with respect to the Simons cone. The formulation of these maximum principles is very suitable since all the hypotheses refer to the set $\mathcal{O}$ and not $\mathbb{R}^{2m}$. The key ingredient in the proofs is the kernel inequality (3.1.13).

Proof of Proposition 3.1.2. (i) By contradiction, suppose that u takes negative values in Ω. Under the hypotheses we are assuming, a negative minimum must be achieved. Thus, there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{\Omega} u =: m < 0\,.$$

Then, using the expression of L_K for odd functions (see Lemma 3.2.3), we have

$$L_K u(x_0) = \int_{\mathcal{O}} \{m - u(y)\}\{\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\}\, \mathrm{d}y + 2m \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\, \mathrm{d}y\,.$$

Now, since $m - u(y) \leq 0$ in $\mathcal{O}$, $m < 0$, $c \geq 0$, and $\overline{K}(x_0, y) \geq \overline{K}(x_0, y^\star) > 0$ —by (3.1.13)—, we get

$$0 \leq L_K u(x_0) + c(x_0) u(x_0) \leq m \left(2 \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\, \mathrm{d}y + c(x_0) \right) < 0\,,$$

a contradiction.

(ii) Assume that $u \not\equiv 0$ in $\mathcal{O}$. We shall prove that $u > 0$ in Ω. By contradiction, assume that there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$. Then, using the expression of L_K for odd functions given in Lemma 3.2.3, the kernel inequality (3.1.13), and the fact that $u \geq 0$ in $\mathcal{O}$, we obtain

$$0 \leq L_K u(x_0) + c(x_0) u(x_0) = - \int_{\mathcal{O}} u(y) \{\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\}\, \mathrm{d}y < 0\,,$$

a contradiction. $\square$

Remark 3.2.7. Note that since the operator L_K includes a term of order zero with positive coefficient in addition to the integro-differential part, the condition $c \geq 0$ in point (i) of the previous proposition can be slightly relaxed. Indeed, following the proof of the result, we see that

$$c(x) > -2 \int_{\mathcal{O}} \overline{K}(x, y^\star)\, \mathrm{d}y$$

suffices. This hypothesis seems hard to be checked for applications apart from the case $c \geq 0$. Nevertheless, recall that by Lemma 3.2.3 we have an explicit lower bound for the quantity $\int_{\mathcal{O}} \overline{K}(x, y^\star)\, \mathrm{d}y$ in terms of the function $\operatorname{dist}(x, \mathscr{C})$. This fact will be crucial for establishing a maximum principle in "narrow" sets close to the Simons cone —see Proposition 6.2 in part II [87].

3.3 The energy functional for doubly radial odd functions

This section is devoted to the energy functional associated to the semilinear equation (3.1.1). We first define appropriately the functional spaces where we are going to apply classic techniques of calculus of variations. Next we rewrite the energy in terms of the new kernel $\overline{K}$ and we give an alternative expression for the energy of doubly radial odd functions. Finally, we establish some results that are useful when using variational techniques, and that will be exploited in the next section.

Let us start by defining the functional spaces that we are going to consider in the rest of the paper. Given a set $\Omega \subset \mathbb{R}^n$ and a translation invariant and positive kernel K satisfying (3.1.3), we define the Hilbert space

$$\mathbb{H}^K(\Omega) := \left\{ w \in L^2(\Omega) \ : \ [w]^2_{\mathbb{H}^K(\Omega)} < +\infty \right\},$$

where

$$[w]^2_{\mathbb{H}^K(\Omega)} := \frac{1}{2} \int\int_{(\mathbb{R}^n)^2 \setminus (\mathbb{R}^n \setminus \Omega)^2} |w(x) - w(y)|^2 K(x-y)\, dx\, dy\,.$$

We also define

$$\begin{aligned} \mathbb{H}^K_0(\Omega) &:= \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\} \\ &= \left\{ w \in \mathbb{H}^K(\mathbb{R}^n) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \right\}. \end{aligned}$$

Assume that $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution. Then, we define

$$\widetilde{\mathbb{H}}^K(\Omega) := \left\{ w \in \mathbb{H}^K(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\}.$$

and

$$\widetilde{\mathbb{H}}^K_0(\Omega) := \left\{ w \in \mathbb{H}^K_0(\Omega) \ : \ w \text{ is doubly radial a.e.} \right\}.$$

We will add the subscript 'odd' and 'even' to these spaces to consider only functions that are odd (respectively even) with respect to the Simons cone.

Remark 3.3.1. If $\widetilde{\mathbb{H}}^K_0(\Omega)$ is equipped with the scalar product

$$\langle v, w \rangle_{\widetilde{\mathbb{H}}^K_0(\Omega)} := \frac{1}{2} \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \big(v(x) - v(y)\big)\big(w(x) - w(y)\big) K(x-y)\, dx\, dy\,,$$

then it is easy to check that $\widetilde{\mathbb{H}}^K_0(\Omega)$ can be decomposed as the orthogonal direct sum of $\widetilde{\mathbb{H}}^K_{0,\mathrm{even}}(\Omega)$ and $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$.

Note that when K satisfies (3.1.4), then $\mathbb{H}^K_0(\Omega) = \mathbb{H}^\gamma_0(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(z) = c_{n,\gamma}|z|^{-n-2\gamma}$. Furthermore, $\mathbb{H}^\gamma(\Omega) \subset H^\gamma(\Omega)$, where $\mathbb{H}^\gamma(\Omega)$ is the usual fractional Sobolev space where interactions of x and y are only computed in $\Omega \times \Omega$ (see [75]). For more comments on this, see [67], and the references therein.

Once presented the functional setting of our problem, we proceed with the study of the energy functional associated to equation (3.1.1).

Given a kernel K satisfying (3.1.3), a potential G, and a function $w \in \mathbb{H}^K(\Omega)$, with $\Omega \subset \mathbb{R}^n$, we write the energy defined in (3.1.14) as

$$\mathcal{E}(w, \Omega) = \mathcal{E}_{\mathrm{K}}(w, \Omega) + \mathcal{E}_{\mathrm{P}}(w, \Omega)\,,$$

where

$$\mathcal{E}_{\mathrm{K}}(w, \Omega) := \frac{1}{2}[w]^2_{\mathbb{H}^K(\Omega)} \quad \text{and} \quad \mathcal{E}_{\mathrm{P}}(w, \Omega) := \int_\Omega G(w)\, dx\,.$$

We will call $\mathcal{E}_{\mathrm{K}}$ and $\mathcal{E}_{\mathrm{P}}$ the *kinetic* and *potential* energies respectively. Recall that sometimes it is useful to rewrite the kinetic energy as

$$\begin{aligned}\mathcal{E}_{\mathrm{K}}(w,\Omega) = \frac{1}{4}\bigg\{ \int_{\Omega}\int_{\Omega} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ +2\int_{\Omega}\int_{\mathbb{R}^n\setminus\Omega} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\bigg\}.\end{aligned} \tag{3.3.1}$$

Roughly speaking, we have $\mathcal{E}_{\mathrm{K}}$ split into two parts: "interactions inside-inside" and "interactions inside-outside".

Note that, for functions $w \in \mathbb{H}_0^K(\Omega)$, it holds $\mathcal{E}_{\mathrm{K}}(w,\Omega) = \mathcal{E}_{\mathrm{K}}(w,\mathbb{R}^n)$. Moreover, if $G \geq 0$, the energy satisfies $\mathcal{E}(w,\Omega) \leq \mathcal{E}(w,\Omega')$ whenever $\Omega \subset \Omega'$.

Our goal is to rewrite the kinetic energy of doubly radial odd functions in terms of the kernel $\overline{K}$ and with integrals computed only in $\mathcal{O}$, in the spirit of the previous section with the operator L_K. In particular, we are interested in finding an expression similar to (3.3.1), where the positive kernel $\overline{K}(x,y) - \overline{K}(x,y^\star)$ appears. To do this, we introduce the following notation for the interaction. Given A, $B \subset \mathcal{O}$ sets of double revolution, we define

$$\begin{aligned}I_w(A,B) := 2\int_A\int_B |w(x)-w(y)|^2 \left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\ +4\int_A\int_B \Big\{w^2(x)+w^2(y)\Big\}\,\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned} \tag{3.3.2}$$

Thanks to this notation, we rewrite the kinetic energy as follows.

Lemma 3.3.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{2m}$ be two sets of double revolution that are symmetric with respect to the Simons cone, i.e., $\Omega_i^\star = \Omega_i$, and let $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\mathbb{R}^n)$. Let K be a radially symmetric kernel satisfying (3.1.3). Then,*

$$\int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 K(x-y) = I_w(\Omega_1\cap\mathcal{O}, \Omega_2\cap\mathcal{O}), \tag{3.3.3}$$

where $I_w(\cdot,\cdot)$ is the interaction defined in (3.3.2).

As a consequence, given a doubly radial set $\Omega \subset \mathbb{R}^{2m}$ with $\Omega^\star = \Omega$, and a function $v \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$, we can write the kinetic energy as

$$\mathcal{E}_{\mathrm{K}}(v,\Omega) = \frac{1}{4}\big\{I_v(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_v(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\big\}. \tag{3.3.4}$$

Proof. First, note that equality (3.3.4) for the kinetic energy follows directly combining expressions (3.3.1) and (3.3.3). Hence, we only need to prove (3.3.3).

Now, since w is doubly radial and Ω_1, Ω_2 are sets of double revolution, we obtain

$$\int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

once we consider the change $y = R\tilde{y}$ and take the average among all $R \in O(m)^2$ as in Lemma 3.2.1.

Finally, we split Ω_i into $\Omega_i \cap \mathcal{O}$ and $\Omega_i \setminus \mathcal{O} = (\Omega_i \cap \mathcal{O})^\star$. By using the change of variables given by $(\cdot)^\star$ and the symmetries of Ω_i and w, we get

$$
\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = 2\int_{\Omega_1\cap\mathcal{O}}\int_{\Omega_2\cap\mathcal{O}} |w(x)-w(y)|^2\overline{K}(x,y) + |w(x)+w(y)|^2\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = 2\int_{\Omega_1\cap\mathcal{O}}\int_{\Omega_2\cap\mathcal{O}} |w(x)-w(y)|^2\left\{\overline{K}(x,y)-\overline{K}(x,y^\star)\right\}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\quad +4\int_{\Omega_1\cap\mathcal{O}}\int_{\Omega_2\cap\mathcal{O}} \left\{w^2(x)+w^2(y)\right\}\overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = I_w(\Omega_1\cap\mathcal{O},\Omega_2\cap\mathcal{O}).
\end{aligned}
$$

Here we have used that $\overline{K}(x^\star,y^\star) = \overline{K}(x,y)$ —see Lemma 3.2.2. □

Using the previous expression for the energy, we can establish now the following lemma regarding the decrease of the energy under some operations. This result will be crucial in the next section, since it will allow us to assume that the minimizers of the energy are bounded by 1 by above (respectively by -1 by below) and that are nonnegative in $\mathcal{O}$.

Lemma 3.3.3. *Let $\Omega \subset \mathbb{R}^{2m}$ be a set of double revolution that is symmetric with respect to the Simons cone, and let K be a radially symmetric kernel satisfying the positivity condition* (3.1.13)*. Given $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$, we define*

$$
v(x) = \begin{cases} |u(x)| & \text{if } x\in\mathcal{O}, \\ -|u(x)| & \text{if } x\in\mathcal{I}, \end{cases} \quad \text{and} \quad w(x) = \begin{cases} \min\{1,u(x)\} & \text{if } x\in\mathcal{O}, \\ \max\{-1,u(x)\} & \text{if } x\in\mathcal{I}. \end{cases}
$$

If G satisfies (3.1.15)*, then*

$$
\mathcal{E}(v,\Omega) \le \mathcal{E}(u,\Omega) \quad \text{and} \quad \mathcal{E}(w,\Omega) \le \mathcal{E}(u,\Omega)\,.
$$

Proof. We first establish the result for v. Let us show that $\mathcal{E}_{\mathrm{K}}(v) \le \mathcal{E}_{\mathrm{K}}(u)$. Note that $v \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$. Thus, by using the expression of the kinetic energy given in (3.3.4) and the fact that $\overline{K}(x,y) > \overline{K}(x,y^\star) > 0$ if $x,y\in\mathcal{O}$ —see (3.1.13)—, we only need to check that $|v(x)-v(y)|^2 \le |u(x)-u(y)|^2$ and $v^2(x) \le u^2(x)$ whenever $x,y\in\mathcal{O}$. The first condition follows from the equivalence

$$
\big||u(x)|-|u(y)|\big|^2 \le |u(x)-u(y)|^2 \iff u(x)u(y) \le |u(x)u(y)|,
$$

while the second one is trivial and it is in fact an equality. Concerning the potential energy, since G is an even function we have that $\mathcal{E}_{\mathrm{P}}(v) = \mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result for v by adding the kinetic and potential energies.

We show now the result for w. Let us show that $\mathcal{E}_{\mathrm{K}}(w) \le \mathcal{E}_{\mathrm{K}}(u)$. As before, $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$ and thus, in view of (3.3.4) and the kernel inequality (3.1.13), we only need to check that $|w(x)-w(y)|^2 \le |u(x)-u(y)|^2$ and $w^2(x) \le u^2(x)$ whenever $x,y\in\mathcal{O}$. The first inequality is trivial whenever $u(x)\le 1$ and $u(y)\le 1$, or $u(x)\ge 1$ and $u(y)\ge 1$. If $u(x)\ge 1$ and $u(y)\le 1$, then $|u(x)-u(y)|^2 - |w(x)-w(y)|^2 = |u(x)-u(y)|^2 - |1-$

$u(y)|^2 = (u(x)-1))^2 + 2(u(x)-1)(1-u(y)) \geq 0$. The second inequality follows from the fact that $w^2(x) = u^2(x)$ when $u(x) \leq 1$, while $w^2(x) = 1 \leq u^2(x)$ if $u(x) \geq 1$. Concerning the potential energy, since G is such that $G(x) \geq G(1) = G(-1) = 0$ if $|x| \leq 1$, then clearly $\mathcal{E}_{\mathrm{P}}(w) \leq \mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result by adding the kinetic and potential energies. □

Next we present a result that will be used later, and concerns weak solutions to semilinear Dirichlet problems. Its main consequence is that a function $u \in \widetilde{\mathbb{H}}_0^K(\Omega)$ that minimizes the energy $\mathcal{E}$, but only among doubly radial functions, is actually a weak solution to a semilinear Dirichlet problem in Ω. We remark that to show the following result we do not need to use the kernel $\overline{K}$.

Proposition 3.3.4. *Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded set of double revolution and let $L_K \in \mathcal{L}_0$ with kernel K radially symmetric. Let $u \in \widetilde{\mathbb{H}}_0^K(\Omega)$ be such that*

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\xi(x)-\xi(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\xi(x)\,\mathrm{d}x$$

for every $\xi \in C_c^\infty(\Omega)$ that is doubly radial. Then, u is a weak solution to

$$\begin{cases} L_K u &= f(u) \quad \text{in } \Omega\,, \\ u &= 0 \qquad \text{in } \mathbb{R}^{2m}\setminus\Omega\,, \end{cases} \tag{3.3.5}$$

i.e.,

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\eta(x)\,\mathrm{d}x$$

for every $\eta \in C_c^\infty(\Omega)$ (not necessarily doubly radial).

As a consequence, if $u \in \widetilde{\mathbb{H}}_0^K(\Omega)$ is a doubly radial odd minimizer of the energy $\mathcal{E}(u,\Omega)$, then it is a weak solution to (3.3.5).

Proof. Let $\eta \in C_c^\infty(\Omega)$. We define an associated doubly radial function by

$$\overline{\eta}(x) := \fint_{O(m)^2} \eta(Rx)\,\mathrm{d}R\,.$$

Now, on the one hand, given $R \in O(m)^2$ and using the change $x = R\tilde{x}$, $y = R\tilde{y}$ and the fact that u is doubly radial, we get

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} &\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Taking the average in the previous equality among all $R \in O(m)^2$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} &\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \fint_{O(m)^2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}R \\ &= \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\,\{\overline{\eta}(x)-\overline{\eta}(y)\}\,K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

On the other hand, using also the change $x = R\tilde{x}$, we have

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x = \int_\Omega f(u(R^{-1}x))\eta(x)\,\mathrm{d}x = \int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,.$$

Similarly as before, taking the average among all $R \in O(m)^2$, we get

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x = \fint_{O(m)^2}\int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,\mathrm{d}R = \int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\,.$$

Hence, since $\overline{\eta} \in C_c^\infty(\Omega)$ is doubly radial, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f(u(x))\eta(x)\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\\
&\quad= 0\,,
\end{aligned}$$

and thus the first result is proved.

We next show that if u is a doubly radial odd minimizer, then it is a weak solution to (3.3.5). To see this, we consider perturbations $u + \varepsilon\xi$ with $\varepsilon \in \mathbb{R}$ and $\xi \in \widetilde{\mathbb{H}}_0^K(\Omega)$. By Remark 3.3.1, it suffices to consider only even and odd functions ξ. Let first $\xi \in \widetilde{\mathbb{H}}_{0,\mathrm{odd}}^K(\Omega)$. Then, since u is a minimizer among functions in $\widetilde{\mathbb{H}}_{0,\mathrm{odd}}^K(\Omega)$, we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(u+\varepsilon\xi,\Omega) = \langle u,\xi\rangle_{\widetilde{\mathbb{H}}_0^K(\Omega)} - \langle f(u),\xi\rangle_{L^2(\Omega)}\,.$$

Next, take $\xi \in \widetilde{\mathbb{H}}_{0,\mathrm{even}}^K(\Omega)$. Since u is odd with respect to the Simons cone, the same holds for $f(u)$ —recall that f is odd. Thus,

$$\langle u,\xi\rangle_{\widetilde{\mathbb{H}}_0^K(\Omega)} = 0 \quad \text{and} \quad \langle f(u),\xi\rangle_{L^2(\Omega)} = 0\,.$$

Therefore,

$$\langle u,\xi\rangle_{\widetilde{\mathbb{H}}_0^K(\Omega)} = \langle f(u_R),\xi\rangle_{L^2(\Omega)}$$

for every $\xi \in \widetilde{\mathbb{H}}_0^K(\Omega)$ with compact support in Ω. In particular,

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u_R(x)-u_R(y)\}\{\xi(x)-\xi(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u_R(x))\xi(x)\,\mathrm{d}x$$

for every $\xi \in C_c^\infty(\Omega)$ that is doubly radial. Finally, by the first statement of the proposition, that we just proved, we obtain that u is a weak solution to (3.3.5). □

Remark 3.3.5. This proposition combined with some regularity results for operators in the class $\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ yield that bounded minimizers among doubly radial functions of the energy $\mathcal{E}(\cdot,\Omega)$ are classical solutions to $L_K u = f(u)$ in Ω. Indeed, if $w \in L^\infty(\mathbb{R}^n)$ is a weak solution to $L_K w = h$ in $B_1 \subset \mathbb{R}^n$, then

$$||w||_{C^{2\gamma}(\overline{B_{1/2}})} \le C\left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)}\right)\,. \tag{3.3.6}$$

If, in addition, $w \in C^\alpha(\mathbb{R}^n)$ with $\alpha + 2\gamma$ not an integer, then

$$||w||_{C^{\alpha+2\gamma}(\overline{B_{1/2}})} \leq C \left(||h||_{C^\alpha(\overline{B_1})} + ||w||_{C^\alpha(\mathbb{R}^n)} \right), \tag{3.3.7}$$

where the previous two constants C depend only on n, γ, λ, and Λ (see [125, 139] and the references therein).

Note that in some situations these estimates are not suitable enough to be applied repeatedly, mainly due to the term $||w||_{C^\alpha(\mathbb{R}^n)}$ in (3.3.7) —recall that in the case of a Dirichlet problem with zero exterior data the optimal regularity in $\mathbb{R}^n$ is C^γ and not more. Using a cut-off argument as in Corollaries 2.4 and 2.5 of [126], we can modify (3.3.6) and (3.3.7) to obtain, respectively, the estimates

$$||w||_{C^{2\gamma}(\overline{B_{1/4}})} \leq C \left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(B_1)} + \left|\left| \frac{w(x)}{(1+|x|)^{n+2\gamma}} \right|\right|_{L^1(\mathbb{R}^n)} \right), \tag{3.3.8}$$

and

$$||w||_{C^{\alpha+2\gamma}(\overline{B_{1/4}})} \leq C \left(||h||_{C^\alpha(\overline{B_1})} + ||w||_{C^\alpha(\overline{B_1})} + \left|\left| \frac{w(x)}{(1+|x|)^{n+2\gamma}} \right|\right|_{L^1(\mathbb{R}^n)} \right). \tag{3.3.9}$$

Therefore, by applying these estimates (maybe a translated and rescaled version of them) to a weak solution $u \in L^\infty(\mathbb{R}^{2m})$ of $L_K u = f(u)$ in Ω, with f a C^1 nonlinearity, we easily conclude that u is a classical solution, that is, the equation makes sense pointwise.

3.4 An energy estimate for doubly radial odd minimizers

In this section we present an estimate for the energy in the ball B_S of minimizers in the space $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ with $R > S + 4$. That is, we prove Theorem 3.1.3. In order to establish this result, we follow the ideas of Savin and Valdinoci in [137], where they show the same estimate but for minimizers without any restriction on their symmetry.

First of all, let us comment briefly the strategy used in [137]. The argument is based on comparing the energy of the minimizer u in $B_R \subset \mathbb{R}^n$ with the energy of a suitable competitor v. This function v satisfies, in $B_{S+2} \subset B_R \subset \mathbb{R}^n$, the following assumptions:

(i) $-1 \leq v \leq 1$.

(ii) $v = u$ in ∂B_{S+2}.

(iii) The set $\{v \not\equiv -1\} \cap B_{S+2}$ has measure bounded by CS^{n-1} for some constant C.

(iv) $v \in \mathrm{Lip}(\overline{B_{S+2}})$ with a Lipschitz constant independent of R and S.

By the second property, v can be extended to coincide with u outside B_{S+2}, becoming an admissible competitor. Then, the desired estimate follows by finding precise bounds on the energy of v in B_{S+2}. The function v is constructed in B_{S+2} as $v = \min\{u, \phi_S\}$, where $\phi_S(x) = -1 + 2\min\{(|x| - S - 1)_+, 1\}$ —we will also use this function below, see (3.4.5).

In our case, the strategy will be the same but adapting some ingredients, namely, the competitor v. First, note that the previous construction for v cannot be used in our

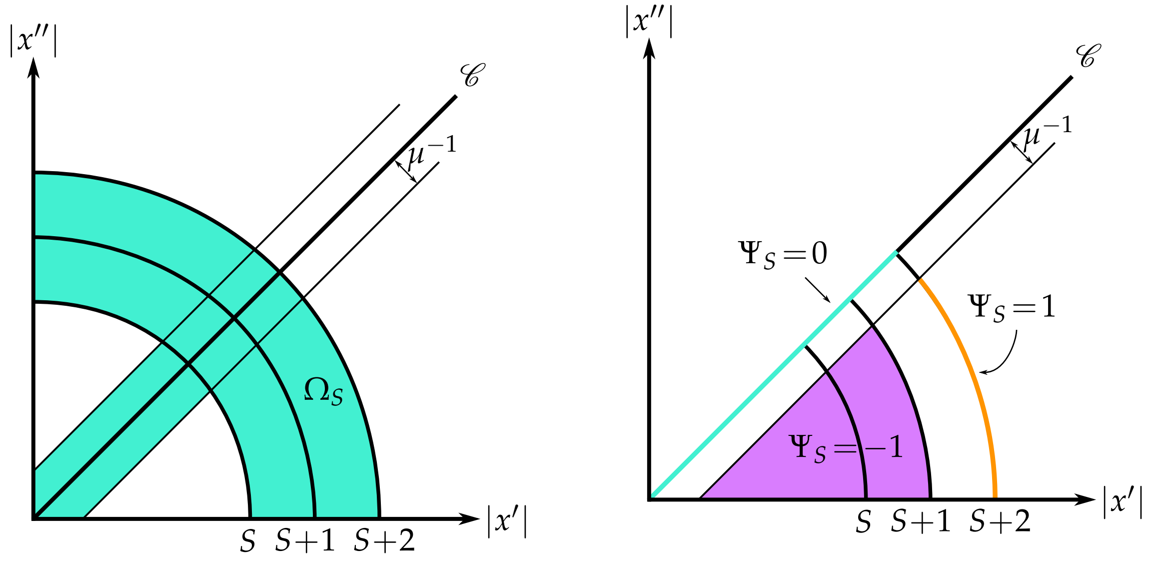

Figure 3.2: (a) The set Ω_S. (b) The 1 and -1 level sets of Ψ_S in $\overline{B_{S+2}} \cap \mathcal{O}$.

setting, since it would not produce a doubly radial odd function. To overcome this problem, we will construct a function w defined in $B_{S+2} \cap \mathcal{O}$ and satisfying the four previous assumptions on v. In addition, we will require w to be doubly radial and to vanish on the Simons cone (then we will consider its odd extension through $\mathscr{C}$).

To state the precise properties of w, we need to consider the Lipschitz constant of u in $\overline{B_{S+3}}$, namely

$$\mu := [u]_{\mathrm{Lip}(\overline{B_{S+3}})} . \tag{3.4.1}$$

By Proposition 3.3.4, we know that u solves $L_K u = f(u)$ in B_R with $R > S+4$. Moreover, by Lemma 3.3.3 we know that u is bounded. Therefore, by applying repeatedly the estimates (3.3.8) and (3.3.9) in balls centered at points in B_{S+3}, it is easy to see that $\mu \leq C$ with a positive constant C depending only on m, γ, λ, Λ, and $||f||_{C^1([-1,1])}$ (and thus, independent of R and S). Recall that $G' = -f$ and hence $||f||_{C^1([-1,1])} \leq ||G||_{C^2([-1,1])}$.

We can now define the set

$$\Omega_S := \left(\overline{B_{S+2}} \setminus B_S\right) \cup \left(\overline{B_{S+2}} \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) \leq 1\}\right), \tag{3.4.2}$$

—see Figure 3.2 (a). It is easy to see that

$$|\Omega_S| \leq C\, S^{2m-1}, \tag{3.4.3}$$

with a constant C depending only on m and μ. This can be checked following the computations in the proof of the energy estimate for the local equation in Theorem 1.3 of [43].

In the following lemma we state the precise properties for the competitor w that suffice to establish the energy estimate given by the right-hand side of (3.1.17) for $\mathcal{E}(w, B_{S+2})$.

Lemma 3.4.1. *Let $S \geq 2$ and $R > S+4$. Let $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ be a doubly radial odd minimizer of the energy (3.1.14) and let μ be defined by (3.4.1). Then, there exists a function $w : \overline{B_{S+2}} \cap \mathcal{O} \to \mathbb{R}$ satisfying the following:*

(H1) $-1 \leq w \leq 1$.

(H2) *w doubly radial and $w = 0$ in $\mathscr{C}$.*

(H3) *$w = u$ on $\partial B_{S+2} \cap \mathcal{O}$.*

(H4) *$w \equiv -1$ on $(B_{S+2} \cap \mathcal{O}) \setminus \Omega_S = B_S \cap \{\mu \operatorname{dist}(\cdot, \mathscr{C}) > 1\}$.*

(H5) *$w \in \operatorname{Lip}(\overline{B_{S+2}})$ with a Lipschitz constant independent of R and S. In addition,*

$$|w(x) - w(y)| \leq \frac{C}{\operatorname{dist}(x, \mathscr{C})} |x - y| \tag{3.4.4}$$

whenever $x, y \in B_{S+1} \cap \mathcal{O}$, $\mu \operatorname{dist}(x, \mathscr{C}) \geq 1$ and $\mu \operatorname{dist}(y, \mathscr{C}) \leq 1$, and with C a constant independent of R and S.

Proof. To construct the function w we first define

$$\phi_S(x) := \begin{cases} -1 & \text{if } \ |x| \leq S+1\,, \\ -1 + 2(|x| - S - 1) & \text{if } \ S+1 \leq |x| \leq S+2\,, \\ 1 & \text{if } \ S+2 \leq |x|\,, \end{cases} \tag{3.4.5}$$

which is the function used in [137]. Now, we modify it in order to vanish at $\mathscr{C}$. We define

$$\Psi_S(x) := \begin{cases} \phi_S(x) \mu \operatorname{dist}(x, \mathscr{C}) & \text{if } \ \mu \operatorname{dist}(x, \mathscr{C}) \leq 1\,, \\ \phi_S(x) & \text{if } \ \mu \operatorname{dist}(x, \mathscr{C}) \geq 1\,, \end{cases}$$

—see Figure 3.2 (b) for an schematic representation.

With this function on hand, we construct the competitor $w : \overline{B_{S+2} \cap \mathcal{O}} \to \mathbb{R}$ as

$$w := \min\{u, \Psi_S\}.$$

We check next that (H1)-(H5) hold.

First of all, recall that by Lemma 3.3.3, $0 \leq u \leq 1$ in $\mathcal{O}$. Since $-1 \leq \Psi_S \leq 1$ in $B_{S+2} \cap \mathcal{O}$, (H1) holds trivially. Moreover, since both functions are doubly radial and vanish on $\mathscr{C}$, (H2) follows —recall that the distance to the cone, in $\mathcal{O}$, is the doubly radial function given by $(|x'| - |x''|)/\sqrt{2}$. The verification of (H4) is easy, since $\Psi_S \equiv -1 \leq u$ in $(B_{S+2} \cap \mathcal{O}) \setminus \Omega_S$.

Now, we check that (H3) holds. On the one hand, if $x \in \partial B_{S+2} \cap \mathcal{O}$ and $\mu \operatorname{dist}(x, \mathscr{C}) \geq 1$, we have $\Psi_S(x) = \phi_S(x) = 1 \geq u(x)$, and therefore $w(x) = u(x)$. On the other hand, for $x \in \partial B_{S+2} \cap \mathcal{O}$ with $\mu \operatorname{dist}(x, \mathscr{C}) \leq 1$, we have $\Psi_S(x) = \mu \operatorname{dist}(x, \mathscr{C})$. By (3.4.1),

$$|u(y) - u(z)| \leq \mu |y - z| \quad \text{ for every } y,\ z \in \overline{B_{S+3}},$$

and thus, by taking $y = x$ and $z \in \mathscr{C}$ to be a point realizing $\operatorname{dist}(x, \mathscr{C})$, we obtain that

$$u(x) = |u(x)| \leq \mu |x - z| = \mu \operatorname{dist}(x, \mathscr{C}) = \Psi_S(x).$$

Thus, $w(x) = u(x)$ and (H3) holds.

Finally, we verify (H5). Obviously, w is Lipschitz in $\overline{B_{S+2}}$ since it is the minimum of two Lipschitz functions —with Lipschitz constants depending only on μ. From this it follows that (3.4.4) also holds, for a large constant C depending on μ, at points where $\operatorname{dist}(x, \mathscr{C}) \leq 2/\mu$. Finally, assume that $\operatorname{dist}(x, \mathscr{C}) \geq 2/\mu$. Then, by using the triangular inequality and the definition of distance to the Simons cone, we have

$$|x - y| \geq \operatorname{dist}(x, \mathscr{C}) - \operatorname{dist}(y, \mathscr{C}) \geq \frac{1}{2} \operatorname{dist}(x, \mathscr{C}).$$

From this and (H1), we readily deduce that (3.4.4) holds for a large constant C. □

To estimate the energy of w in B_{S+2}, it will be important to control the double integrals in the nonlocal energy first in the set where $|x-y| \geq d_S(x)$, and then in $\{|x-y| \leq d_S(x)\}$, where

$$d_S(x) := \min\{\operatorname{dist}(x, \partial B_{S+1}), \mu \operatorname{dist}(x, \mathscr{C})\} \quad \text{for } x \in B_S .$$

A similar technicality was used by Savin and Valdinoci in [137] but with the function $\operatorname{dist}(x, \partial B_{S+1})$, and it is the key point to get (3.1.17). We can now establish the energy estimate of Theorem 3.1.3.

Proof of Theorem 3.1.3. Take w constructed in Lemma 3.4.1 and extend it oddly through $\mathscr{C}$ and then to coincide with u outside B_{S+2}. Hence, since u is a doubly radial odd minimizer in B_R, and w an admissible competitor, $\mathcal{E}(u, B_R) \leq \mathcal{E}(w, B_R)$. Moreover, $u \equiv w$ in $\mathbb{R}^{2m} \setminus B_{S+2}$, and thus it follows that

$$\mathcal{E}(u, B_{S+2}) \leq \mathcal{E}(w, B_{S+2}).$$

By the monotonicity of the energy $\mathcal{E}$ by inclusions we get

$$\mathcal{E}(u, B_S) \leq \mathcal{E}(w, B_{S+2}).$$

Therefore, to obtain the desired result it remains to estimate $\mathcal{E}(w, B_{S+2})$.

In the following inequalities, the letter C will be a constant depending only on m, γ, λ, Λ, and $||G||_{C^2([-1,1])}$. Recall that μ defined in (3.4.1) depends only on these quantities.

First, note that using the upper bound for the kernel K —(3.1.4)— and the change of variables given by $(\cdot)^\star$, it follows that

$$\mathcal{E}(w, B_{S+2}) \leq C \int_{B_{S+2} \cap \mathcal{O}} dx \int_{\mathbb{R}^{2m}} dy \frac{|w(x) - w(y)|^2}{|x-y|^{2m+2\gamma}} + \int_{B_{S+2}} G(w)\, dx.$$

Now we estimate separately the potential and kinetic energies.

Estimate for the potential energy. Since, $w = \pm 1$ in $B_{S+2} \setminus \Omega_S$ by (H4), $-1 \leq w \leq 1$ by (H1), and $G(1) = G(-1) = 0$, it is clear that

$$\int_{B_{S+2}} G(w) = \int_{\Omega_S} G(w) \leq C|\Omega_S| \leq CS^{2m-1}.$$

Here we have used (3.4.3).

Estimate for the kinetic energy. We split the integral in three terms, as follows.

$$\begin{aligned}
\int_{B_{S+2} \cap \mathcal{O}} dx \int_{\mathbb{R}^{2m}} dy \frac{|w(x) - w(y)|^2}{|x-y|^{2m+2\gamma}} &= \int_{\Omega_S \cap \mathcal{O}} dx \int_{\mathbb{R}^{2m}} dy \, \frac{|w(x) - w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \int_{(B_S \setminus \Omega_S) \cap \mathcal{O}} dx \int_{B_{S+1} \cap \mathcal{O}} dy \, \frac{|w(x) - w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \int_{(B_S \setminus \Omega_S) \cap \mathcal{O}} dx \int_{(B_{S+1} \cap \mathcal{O})^c} dy \, \frac{|w(x) - w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Here $(\cdot)^c$ denotes the complementary set. Now we control each term separately.

We estimate the first integral:

$$
\begin{aligned}
I_1 &= \int_{\Omega_S\cap\mathcal{O}} dx \int_{\mathbb{R}^{2m}} dy \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&= \int_{\Omega_S\cap\mathcal{O}} dx \int_{|x-y|\le 1} dy \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} + \int_{\Omega_S\cap\mathcal{O}} dx \int_{|x-y|\ge 1} dy \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le C \int_{\Omega_S\cap\mathcal{O}} dx \int_0^1 dr \, r^{1-2\gamma} + C \int_{\Omega_S\cap\mathcal{O}} dx \int_1^\infty dr \, r^{-1-2\gamma} \\
&= C|\Omega_S| \le C\, S^{2m-1}.
\end{aligned}
$$

We have used that w is Lipschitz in $\overline{B_{S+3}}$ —see (H5) and (3.4.1)— to bound the first integral, while the second one is controlled using only that w is bounded, by (H1).

Next, we estimate I_2. To do it, we first claim that, if $|x-y| \le d_S(x)$, then

$$
|w(x)-w(y)| \le \frac{C}{d_S(x)}|x-y| \tag{3.4.6}
$$

for every $x \in (B_S \setminus \Omega_S)\cap\mathcal{O}$ and $y \in B_{S+1}\cap\mathcal{O}$. Recall that d_S is defined as $d_S(x) = \min\{\operatorname{dist}(x,\partial B_{S+1}), \mu \operatorname{dist}(x,\mathscr{C})\}$, and therefore it suffices to show that

$$
|w(x)-w(y)| \le \frac{C}{\operatorname{dist}(x,\mathscr{C})}|x-y|
$$

for $x \in B_S \cap \mathcal{O}$ with $\mu \operatorname{dist}(x,\mathscr{C}) \ge 1$ and $y \in B_{S+1}\cap\mathcal{O}$ (recall that C may depend on μ). Now, if we assume that $\mu \operatorname{dist}(y,\mathscr{C}) \ge 1$, it follows that $w(x) = w(y) = -1$ and (3.4.6) is trivially true. On the other hand, if we assume that $\mu \operatorname{dist}(y,\mathscr{C}) \le 1$, then (3.4.6) follows from (H5). Therefore, the claim is proved.

Using (3.4.6), we proceed as before splitting the integrals to obtain

$$
\begin{aligned}
I_2 &= \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} dx \int_{B_{S+1}\cap\mathcal{O}} dy \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} dx \int_{\{|x-y|\le d_S(x)\}\cap B_{S+1}\cap\mathcal{O}} dy \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} dx \int_{|x-y|\ge d_S(x)} dy \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le C \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} d_S(x)^{-2\gamma}\, dx.
\end{aligned}
$$

Here we have used (3.4.6) to estimate the first term, while for the second one we have only used that w is bounded, by (H1). The last integral for $d_S(x)^{-2\gamma}$ will be bounded later on.

Next, we estimate I_3. To do it, we first claim that if $x \in (B_S \setminus \Omega_S)\cap\mathcal{O}$ and $y \in (B_{S+1}\cap\mathcal{O})^c = \mathcal{I} \cup B_{S+1}^c$, then $|x-y| \ge c\,d_S(x)$ for some constant $c > 0$ depending only on μ. Indeed, on the one hand it is clear that, if $y \in B_{S+1}^c$, then $|x-y| \ge \operatorname{dist}(x,\partial B_{S+1}) \ge d_S(x)$. On the other hand, if $y \in \mathcal{I}$, then $|x-y| \ge \operatorname{dist}(x,\mathscr{C}) \ge d_S(x)/\mu$.

By the previous claim, since w is bounded, we obtain

$$\begin{aligned}
I_3 &= \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{(B_{S+1}\cap\mathcal{O})^c} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{|x-y|\ge C d_S(x)} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le C \int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x.
\end{aligned}$$

Now, we add up I_1, I_2, and I_3 to get

$$\mathcal{E}(w, B_{S+2}) \le C\left(\int_{(B_S\setminus\Omega_S)\cap\mathcal{O}} d_S(x)^{-2\gamma}\,\mathrm{d}x + S^{2m-1}\right). \tag{3.4.7}$$

We conclude the proof by estimating the integral of $d_S(x)^{-2\gamma}$, as follows.

$$\begin{aligned}
\int_{B_{S+2}\setminus\Omega_S} d_S(x)^{-2\gamma}\,\mathrm{d}x &= \int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} \max\{\mathrm{dist}(x,\partial B_{S+1})^{-2\gamma}, (\mu\,\mathrm{dist}(x,\mathscr{C}))^{-2\gamma}\}\,\mathrm{d}x \\
&\le \int_{B_S} (S+1-|x|)^{-2\gamma}\,\mathrm{d}x \\
&\qquad + C\int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} \mathrm{dist}(x,\mathscr{C})^{-2\gamma}\,\mathrm{d}x.
\end{aligned} \tag{3.4.8}$$

We next control these two integrals.

The first integral can be estimated by using spherical coordinates and the change $\tau = r/(S+1)$. Indeed,

$$\begin{aligned}
\int_{B_S} (S+1-|x|)^{-2\gamma}\,\mathrm{d}x &= C\int_0^S \frac{r^{2m-1}}{(S+1-r)^{2\gamma}}\,\mathrm{d}r \\
&\le C(S+1)^{2m-2\gamma}\int_0^{1-\frac{1}{S+1}} \frac{\tau^{2m-1}}{(1-\tau)^{2\gamma}}\,\mathrm{d}\tau \\
&\le C(S+1)^{2m-2\gamma}\int_0^{1-\frac{1}{S+1}} (1-\tau)^{-2\gamma}\,\mathrm{d}\tau \\
&\le \begin{cases} C\,S^{2m-2\gamma} & \text{if } \gamma\in(0,1/2), \\ C\,S^{2m-1}\log S & \text{if } \gamma = 1/2, \\ C\,S^{2m-1} & \text{if } \gamma\in(1/2,1). \end{cases}
\end{aligned}$$

To bound the second integral (note that it only appears in the proof when $1/\mu \le S$), we write it in the $(\bar{s},\bar{t})$ variables in $\mathbb{R}^2$, where

$$\bar{s} := \frac{|x'|+|x''|}{\sqrt{2}} \quad \text{and} \quad \bar{t} := \frac{|x'|-|x''|}{\sqrt{2}}.$$

Note that $\bar{t}$ is the signed distance to the cone (see Lemma 4.2 in [43]). Thus, still denoting by B_S the ball of radius S in $\mathbb{R}^2$,

$$\begin{aligned}\int_{B_S\cap\{\mu\,\mathrm{dist}(x,\mathscr{C})>1\}} \mathrm{dist}(x,\mathscr{C})^{-2\gamma}\,dx &\le C\int\int_{B_S\cap\{\bar s\ge|\bar t|>1/\mu\}} |\bar t|^{-2\gamma}\,(\bar s^2-\bar t^2)^{m-1}\,d\bar s\,d\bar t\\ &\le C\int\int_{B_S\cap\{\bar s\ge|\bar t|>1/\mu\}} |\bar t|^{-2\gamma}\,\bar s^{2m-2}\,d\bar s\,d\bar t\\ &\le C\int_{1/\mu}^{S} d\bar t\;\bar t^{-2\gamma}\int_0^S d\bar s\;\bar s^{2m-2}\\ &\le \begin{cases} C\,S^{2m-2\gamma} & \text{if}\quad \gamma\in(0,1/2),\\ C\,S^{2m-1}\log S & \text{if}\quad \gamma=1/2,\\ C\,S^{2m-1} & \text{if}\quad \gamma\in(1/2,1).\end{cases}\end{aligned}$$

Using these two estimates, combined with (3.4.7) and (3.4.8), the desired result follows by noticing that the term CS^{2m-1} in (3.4.7) is of lower order when $\gamma\le 1/2$. $\square$

3.5 Existence of saddle-shaped solution: variational method

In this section we establish the existence of saddle-shaped solutions to the integro-differential Allen-Cahn equation. The proof is based on the direct method of the calculus of variations, and it uses most of the results appearing in the previous sections.

Proof of Theorem 3.1.4. Since $\mathcal{E}(w,B_R)$ is bounded from below —by 0—, we can take a minimizing sequence in $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$, that we call u_R^j with $j\in\mathbb{Z}^+$. Note that, by Lemma 3.3.3 we can assume that $-1\le u_R^j\le 1$ and that $u_R^j\ge 0$ in $\mathcal{O}$ and $u_R^j\le 0$ in $\mathcal{I}$.

Now, using (3.1.4), $G\ge 0$, and the fact that u_R^j is a minimizing sequence, we deduce using (3.1.14) that

$$[u_R^j]^2_{H^\gamma(B_R)}\le \frac{c_{n,\gamma}}{\lambda}[u_R^j]^2_{\mathbb{H}^K(B_R)}\le\frac{2c_{n,\gamma}}{\lambda}\mathcal{E}(u_R^j,B_R)\le C$$

for a constant C that does not depend on j. Therefore, by combining this with the fractional Poincaré inequality (recall that $u_R^j\equiv 0$ in $\mathbb{R}^{2m}\setminus B_R$) we get that the sequence $\{u_R^j\}$ is bounded in $H^\gamma(B_R)$. Hence, by the compact embedding $H^\gamma(B_R)\subset\subset L^2(B_R)$ (see Theorem 7.1 of [75]), there exists a subsequence, still denoted by u_R^j, that converges to some doubly radial $u_R\in L^2(B_R)$, and thus, a.e. in B_R. By Fatou's lemma, we have

$$\mathcal{E}(u_R,B_R)\le\liminf_{j\to\infty}\mathcal{E}(u_R^j,B_R)=\inf\left\{\mathcal{E}(w,B_R)\;:\;w\in\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)\right\}.$$

Therefore, $u_R\in\widetilde{\mathbb{H}}^K(B_R)$. In addition, $u_R(x)=-u_R(x^\star)$ for every $x\in\mathbb{R}^{2m}$, and $u_R\equiv 0$ in $\mathbb{R}^{2m}\setminus B_R$. Thus, u_R is a minimizer of $\mathcal{E}(\cdot,B_R)$ in $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Moreover, it satisfies $-1\le u_R\le 1$ in B_R and $u_R\ge 0$ in $\mathcal{O}$. As a consequence, by Proposition 3.3.4 and the regularity for operators in L_K (see Remark 3.3.5), we have that u_R is a classical solution to

$$\begin{cases} L_K u_R = f(u_R) & \text{in } B_R,\\ u_R = 0 & \text{in } \mathbb{R}^{2m}\setminus B_R.\end{cases}$$

The next step is to pass to the limit in R to obtain a solution in $\mathbb{R}^{2m}$. This is done using a compactness argument. Let $S > 0$ and consider the family $\{u_R\}$, for $R > S+1$, of solutions to $L_K u_R = f(u_R)$ in B_S. Note first that, if w solves $L_K w = f(w)$ in B_ρ and $|w| \leq 1$ in $\mathbb{R}^{2m}$ with $f \in C^\alpha([-1,1])$ for some $\alpha > 0$, the combination of the estimates (3.3.6) and (3.3.9) yields

$$||w||_{C^{2\gamma+\varepsilon}(B_{\rho/8})} \leq C\left(n,\ \gamma,\ \lambda,\ \Lambda,\ ||f||_{C^\alpha([-1,1])}\right).$$

for some $\varepsilon > 0$. By applying this to u_R in balls of radius $\rho = 1$ and centered at points in $\overline{B_S}$, we obtain a uniform $C^{2\gamma+\varepsilon}(\overline{B_S})$ bound for u_R. By the Arzelà-Ascoli theorem, as $R \to +\infty$, a subsequence of $\{u_R\}$ converges in $C^{2\gamma+\varepsilon/2}(\overline{B_S})$ to a (pointwise) solution in B_S. Taking now $S = 1, 2, 3, \ldots$ and using a diagonal argument, we obtain a sequence u_{R_j} converging uniformly on compacts in the $C^{2\gamma+\varepsilon/2}$ norm to a solution $u \in C^{2\gamma+\varepsilon/2}(\mathbb{R}^{2m})$ of (3.1.1).

Therefore, we have obtained a solution u to $L_K u = f(u)$ in $\mathbb{R}^{2m}$ which is doubly radial. Furthermore, u is odd with respect to the Simons cone $\mathscr{C}$, i.e., $u(x) = -u(x^\star)$ for $x \in \mathbb{R}^{2m}$, and $0 \leq u \leq 1$ in $\mathcal{O}$.

Finally, we show that $0 < u < 1$ in $\mathcal{O}$. This will ensure that u is a saddle-shaped solution. First, note that $|u| < 1$ by the usual strong maximum principle (since u vanishes at $\mathscr{C}$ and is continuous, we have $u \not\equiv 1$ and $u \not\equiv -1$ in $\mathbb{R}^{2m}$). Let us show now that $u \not\equiv 0$. To do this, we use the energy estimate of Theorem 3.1.3. That is, if we consider u_R the minimizer of $\mathcal{E}(\cdot, B_R)$ with $R > 8$, we have

$$\mathcal{E}(u_R, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ S^{2m-2\gamma}\log S & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

for every $2 < S < R-5$ and with a constant C independent of R and S. By letting $R \to \infty$ we obtain the same estimate for u. By contradiction, assume $u \equiv 0$. Then, the previous estimate leads to

$$c_m G(0) S^{2m} = \mathcal{E}(0, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0,1/2), \\ C\ S^{2m-2\gamma}\log S & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2,1), \end{cases}$$

and, since $G(0) > 0$, this is a contradiction for S large enough. Therefore, $u \not\equiv 0$ and the strong maximum principle for odd functions (see Proposition 3.1.2) yields that $u > 0$ in $\mathcal{O}$. □

3.6 Appendix: Some auxiliary results on convex functions

In this appendix we present some auxiliary results concerning convex functions. The main result, used in the proof of Theorem 3.1.1, is the following.

Proposition 3.6.1. *Let $K : (0,+\infty) \to (0,+\infty)$ be a measurable function. Then, the following statements are equivalent:*

i) $K(\sqrt{\cdot})$ *is strictly convex in* $(0,+\infty)$.

ii) For every positive constants c_1 and c_2, the function $g : (0, 1/c_2) \to \mathbb{R}$ defined by

$$g(z) := K(c_1\sqrt{1 + c_2 z}) + K(c_1\sqrt{1 - c_2 z}) \tag{3.6.1}$$

satisfies

$$g(A) + g(D) \geq g(B) + g(C) \tag{3.6.2}$$

whenever A, B, C, and D belong to $(0, 1/c_2)$ and satisfy

$$A = \max\{A,\ B,\ C,\ D\} \quad \textit{and} \quad A + D \geq B + C.$$

In addition, still assuming $A = \max\{A,\ B,\ C,\ D\}$ and $A + D \geq B + C$, equality holds in (3.6.2) if and only if the sets $\{A, D\}$ and $\{B, C\}$ coincide.

To prove this proposition, we need a lemma on convex functions.

Lemma 3.6.2. *Let $0 < M \leq +\infty$ and let $h : (0, M) \to \mathbb{R}$ be a measurable nondecreasing function. Then, the following statements are equivalent.*

(a) h is convex in $(0, M)$.

(b) For every $0 \leq L \leq 2M$, the function $h_L(x) := h(x) + h(L - x)$ is convex in $(\max\{L - M, 0\}, \min\{L, M\})$.

(c) For every A, B, C, D in the interval $(0, M)$ such that

$$A = \max\{A,\ B,\ C,\ D\} \quad \textit{and} \quad A + D \geq B + C\,,$$

it holds

$$h(A) + h(D) \geq h(B) + h(C)\,. \tag{3.6.3}$$

Proof. $(a) \Rightarrow (c)$. Since B and C are interchangeable and h is nondecreasing, we may assume that $A \geq B \geq C \geq D$. Now, let M_C be the maximum slope of the supporting lines of h at C, and let m_B be the minimum slope of the supporting lines of h at B. By the convexity and monotonicity of h, it holds $m_B \geq M_C \geq 0$ and also

$$h(x) \geq h(B) + m_B(x - B) \quad \text{and} \quad h(x) \geq h(C) + M_C(x - C)$$

for every $x \in (0, M)$.

Hence, since $A - B \geq C - D \geq 0$, we have

$$h(A) - h(B) \geq m_B(A - B) \geq M_C(C - D) \geq h(C) - h(D)\,.$$

$(c) \Rightarrow (b)$. Let $x, y \in (\max\{L - M, 0\}, \min\{L, M\})$ and assume that $x > y$. By taking $A = x$, $B = C = (x + y)/2$, and $D = y$ in (3.6.3), we get

$$\frac{h(x) + h(y)}{2} \geq h\left(\frac{x + y}{2}\right).$$

Similarly, by taking $A = L - y$, $B = C = L - (x + y)/2$, and $D = L - x$ in (3.6.2), we get

$$\frac{h(L - x) + h(L - y)}{2} \geq h\left(L - \frac{x + y}{2}\right).$$

By adding up the previous two inequalities we obtain

$$\frac{h_L(x)+h_L(y)}{2} \geq h_L\left(\frac{x+y}{2}\right).$$

$(b) \Rightarrow (a)$. Let $x_0, y_0 \in (0, M)$ and choose $L = x_0 + y_0 \leq 2M$. By (b) we have

$$\frac{h(x)+h(x_0+y_0-x)+h(y)+h(x_0+y_0-y)}{2} \geq h\left(\frac{x+y}{2}\right) + h\left(x_0+y_0-\frac{x+y}{2}\right),$$

for every x and y in the interval $(\max\{L-M, 0\}, \min\{L, M\})$. By choosing $x = x_0$ and $y = y_0$ we obtain

$$h(x_0) + h(y_0) \geq 2h\left(\frac{x_0+y_0}{2}\right).$$

□

Remark 3.6.3. We can replace convexity by strict convexity in (a) and (b), and then the inequality in (3.6.3) is strict unless the sets $\{A, D\}$ and $\{B, C\}$ coincide.

Remark 3.6.4. Note that the function h_L is even with respect to $L/2$. Thus, if it is convex, it is nondecreasing in $(L/2, \min\{L, M\})$.

Remark 3.6.5. The assumption of h being nondecreasing is only used to deduce (c) from (a). It is not required to show the equivalence between (a) and (b), neither to deduce (a) from (c).

With this result available we can show now Proposition 3.6.1

Proof. $i) \Rightarrow ii)$ We take $M = +\infty$ and $h(\cdot) = K(\sqrt{\cdot})$ in Lemma 3.6.2. Since h is strictly convex, the function h_L is strictly convex in $(0, L)$ for every $L > 0$ (recall that we do not need to assume that h is monotone to deduce this, see Remark 3.6.5). Moreover, by Remark 3.6.4, h_L is nondecreasing in $(L/2, L)$. Thus, the function $\phi(\cdot) = h_L(\cdot + L/2)$ is strictly convex in $(-L/2, L/2)$ and nondecreasing in $(0, L/2)$. If we choose $L = 2c_1^2$, we have that $\phi((L/2)c_2\cdot) = g(\cdot)$, where g is defined by (3.6.1). Therefore, g is strictly convex in $(-1/c_2, 1/c_2)$ and nondecreasing in $(0, 1/c_2)$. Thus, the result follows by applying Lemma 3.6.2 to g in $(0, 1/c_2)$ (taking into account Remark 3.6.3).

$ii) \Rightarrow i)$ By Lemma 3.6.2 applied to g we deduce that g is strictly convex and nondecreasing in $(0, 1/c_2)$ —take $C = D$ to see that g is monotone. Thus, since g is even and nondecreasing, g is strictly convex in $(-1/c_2, 1/c_2)$ and $\varphi(\cdot) = g(\cdot/(c_1^2 c_2))$ is strictly convex in $(-c_1^2, c_1^2)$. Hence, if we call $h(\cdot) := K(\sqrt{\cdot})$ and $L := 2c_1^2$, we have that $\varphi(\cdot - c_1^2) = h(\cdot) + h(L - \cdot) =: h_L(\cdot)$, and thus h_L is strictly convex in $(0, L)$. Note that since $c_1 > 0$ is arbitrary, h_L is strictly convex in $(0, L)$ for all $L > 0$. Therefore, by Lemma 3.6.2, with $M = +\infty$, we conclude that $h(\cdot) = K(\sqrt{\cdot})$ is strictly convex in $(0, +\infty)$. □

3.7 Appendix: An auxiliary computation

In this appendix we present an auxiliary computation that is needed in Section 3.2 in order to complete the proof of Proposition 3.2.4.

Lemma 3.7.1. *Let α, β be two real numbers satisfying $\alpha \geq |\beta|$. Let $x = (x', x'')$, $y = (y', y'') \in \mathcal{O} \subset \mathbb{R}^{2m}$. Define*

$$\begin{aligned} A &= |x'||y'|\alpha + |x''||y''|\beta\,, \qquad & B &= |x'||y''|\alpha + |x''||y'|\beta\,, \\ C &= |x''||y'|\alpha + |x'||y''|\beta\,, \qquad & D &= |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

Then,

1. *It holds*
$$\begin{cases} |A| \geq |B|,\ |A| \geq |C|,\ |A| \geq |D|\,, \\ |A| + |D| \geq |B| + |C|\,. \end{cases}$$

2. *If the sets $\{|A|, |D|\}$ and $\{|B|, |C|\}$ coincide, then necessarily $\alpha = \beta = 0$.*

Proof. The proof is elementary but requires to check some cases. In all of them we will use the following inequalities. Since $\alpha \geq |\beta|$,

$$\alpha \geq 0 \quad \text{and} \quad -\alpha \leq \beta \leq \alpha\,.$$

Moreover, since $x, y \in \mathcal{O}$, it holds

$$|x'| > |x''| \quad \text{and} \quad |y'| > |y''|\,.$$

We start establishing the first statement. We show next that $A \geq 0$ and that

$$A \geq |B|,\ A \geq |C|,\ A \geq |D|\,.$$

- $A \geq 0$:
$$A = |x'||y'|\alpha + |x''||y''|\beta \geq (|x'||y'| - |x''||y''|)\alpha \geq 0\,.$$
- $A \geq |B|$:
$$A \pm B = (|x'|\alpha - |x''|\beta)(|y'| \pm |y''|) \geq 0\,.$$
- $A \geq |C|$:
$$A \pm C = (|y'|\alpha - |y''|\beta)(|x'| \pm |x''|) \geq 0\,.$$
- $A \geq |D|$:
$$A \pm D = (|x'||y'| \pm |x''||y''|)(\alpha \pm \beta) \geq 0\,.$$

It remains to show
$$A + |D| \geq |B| + |C|\,.$$

The proof of this fact is just a computation considering all the eight possible configurations of the signs of B, C, and D. Since the roles of B and C are completely interchangeable, we may assume that $B \geq C$ and we only need to check six cases. To do it, note first that

$$A + D - B - C = (|x'| - |x''|)(|y'| - |y''|)(\alpha + \beta) \geq 0\,, \tag{3.7.1}$$

$$A - D - B + C = (|x'| + |x''|)(|y'| - |y''|)(\alpha - \beta) \geq 0\,, \tag{3.7.2}$$

and

$$A + D + B + C = (|x'| + |x''|)(|y'| + |y''|)(\alpha + \beta) \geq 0\,, \tag{3.7.3}$$

With these three relations at hand we check the six cases.

- If $B \geq 0$, $C \geq 0$, and $D \geq 0$, then by (3.7.1) we have

$$A + |D| - |B| - |C| = A + D - B - C \geq 0\,.$$

- If $B \geq 0$, $C \geq 0$, and $D \leq 0$, we use the sign of D and (3.7.1) to see that

$$A + |D| - |B| - |C| = A - D - B - C = (A + D - B - C) + (-2D) \geq 0\,.$$

- If $B \geq 0$, $C \leq 0$, and $D \geq 0$, we use the sign of D and (3.7.2) to see that

$$A + |D| - |B| - |C| = A + D - B + C = (A - D - B + C) + 2D \geq 0\,.$$

- If $B \geq 0$, $C \leq 0$, and $D \leq 0$, then by (3.7.2) we have

$$A + |D| - |B| - |C| = A - D - B + C \geq 0\,.$$

- If $B \leq 0$, $C \leq 0$, and $D \geq 0$, then by (3.7.3) we have

$$A + |D| - |B| - |C| = A + D + B + C \geq 0\,.$$

- If $B \leq 0$, $C \leq 0$, and $D \leq 0$, we use the sign of D and (3.7.3) to see that

$$A + |D| - |B| - |C| = A - D + B + C = (A + D + B + C) + (-2D) \geq 0\,.$$

This concludes the proof of the first statement.

We prove now the second point of the lemma. Since the roles of B and C are completely interchangeable, we only need to show the result in the case $|A| = |B|$ and $|C| = |D|$.

Recall that $A \geq 0$. Hence, since $A = |B|$ and $|C| = |D|$, a simple computation shows that

$$\alpha = \operatorname{sign}(B)\frac{|x''|}{|x'|}\beta \quad \text{and} \quad \beta = \operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|}{|x'|}\alpha\,.$$

Hence, combining both equalities we obtain

$$\alpha = \operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|^2}{|x'|^2}\alpha.$$

Finally, if we assume $\alpha \neq 0$, then necessarily $\operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D) = 1$ and $|x'| = |x''|$, but this is a contradiction with $x \in \mathcal{O}$. Therefore, $\alpha = 0$ and thus $\beta = 0$. $\square$

3.8 Appendix: The integro-differential operator L_K in the (s,t) variables

The goal of this appendix is to take advantage of the doubly radial symmetry of the functions we are dealing with to write equation (3.1.1) in (s,t) variables, passing from an equation in $\mathbb{R}^{2m}$ to an equation in $(0,+\infty)\times(0,+\infty) \subset \mathbb{R}^2$. Although we do not use these computations in this paper, we include them here to show the usefulness of having introduced the $\overline{K}$ kernel obtained after integration with respect to the Haar measure on $O(m)^2$. Moreover, the following expressions could be useful for future reference. In the case of the fractional Laplacian, the kernel that we obtain involves essentially an hypergeometric function which is the so-called Appell function F_2 (see [6] for its definition).

Lemma 3.8.1. *Let $m \geq 1$, $\gamma \in (0,1)$, and let $w \in C^{\alpha}(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function, i.e., depending only on the variables s and t. Let L_K be a rotation invariant operator, that is, $K(z) = K(|z|)$, of the form (3.1.2). Then, if we define $\tilde{w} : (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ by $\tilde{w}(s,t) = w(s,0,...,0,t,0,...,0)$, it holds*

$$L_K w(x) = \tilde{L}_K \tilde{w}(|x'|, |x''|),$$

with

$$\widetilde{L}_K \tilde{w}(s,t) := \int_0^{+\infty} \int_0^{+\infty} \sigma^{m-1} \tau^{m-1} \big(\tilde{w}(s,t) - \tilde{w}(\sigma,\tau)\big) J(s,t,\sigma,\tau) \, \mathrm{d}\sigma \, \mathrm{d}\tau \,,$$

where:

1. *If $m = 1$,*

$$J(s,t,\sigma,\tau) := \sum_{i=0}^{1} \sum_{j=0}^{1} K\Big(\sqrt{s^2 + t^2 + \sigma^2 + \tau^2 - 2s\sigma(-1)^i - 2t\tau(-1)^j}\Big) \,. \tag{3.8.1}$$

2. *If $m \geq 2$,*

$$\begin{aligned} J(s,t,\sigma,\tau) := c_m^2 \int_{-1}^{1} \int_{-1}^{1} & (1-\theta^2)^{\frac{m-2}{2}} (1-\overline{\theta}^2)^{\frac{m-2}{2}} \\ & K\Big(\sqrt{s^2 + t^2 + \sigma^2 + \tau^2 - 2s\sigma\theta - 2t\tau\overline{\theta}}\Big) \, \mathrm{d}\theta \, \mathrm{d}\overline{\theta} \,, \end{aligned} \tag{3.8.2}$$

with

$$c_m = \frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}.$$

Proof. Let $x = (sx_s, tx_t)$ with $x_s, x_t \in \mathbb{S}^{m-1}$ and $y = (\sigma y_\sigma, \tau y_\tau)$ with $y_\sigma, y_\tau \in \mathbb{S}^{m-1}$. Then, decomposing $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ and using spherical coordinates in each $\mathbb{R}^m$ we obtain

$$\begin{aligned} L_K u(x) &= \int_{\mathbb{R}^{2m}} \big(u(x) - u(y)\big) K(|x-y|) \, \mathrm{d}y \\ &= \int_0^{+\infty} \int_0^{+\infty} \sigma^{m-1} \tau^{m-1} \big(u(s,t) - u(\sigma,\tau)\big) \\ &\qquad \left(\int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma y_\sigma|^2 + |tx_t - \tau y_\tau|^2}\Big) \, \mathrm{d}y_\sigma \, \mathrm{d}y_\tau \right) \mathrm{d}\sigma \, \mathrm{d}\tau. \end{aligned}$$

Now, we define the kernel

$$J(x_s, x_t, s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma y_\sigma|^2 + |tx_t - \tau y_\tau|^2}\Big) \, \mathrm{d}y_\sigma \, \mathrm{d}y_\tau \,. \tag{3.8.3}$$

First of all, it is easy to see that J does not depend on x_s nor x_t. Indeed, consider a different point $(z_s, z_t) \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1}$ and let M_s and M_t be two orthogonal transformations such that $M_s(x_s) = z_s$ and $M_t(x_t) = z_t$. Then, making the change of variables

$y_\sigma = M_s(\tilde{y}_\sigma)$ and $y_\tau = M_t(\tilde{y}_\tau)$, and using that $M_s(\mathbb{S}^{m-1}) = M_t(\mathbb{S}^{m-1}) = \mathbb{S}^{m-1}$, we find out that

$$\begin{aligned}
&J(z_s, z_t, s, t, \sigma, \tau) = \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sM_s(x_s) - \sigma y_\sigma|^2 + |tM_t(x_t) - \tau y_\tau|^2}\Big)\, \mathrm{d}y_\sigma\, \mathrm{d}y_\tau \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sM_s(x_s) - \sigma M_s(\tilde{y}_\sigma)|^2 + |tM_t(x_t) - \tau M_t(\tilde{y}_\tau)|^2}\Big)\, \mathrm{d}\tilde{y}_\sigma\, \mathrm{d}\tilde{y}_\tau \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|M_s(sx_s - \sigma \tilde{y}_\sigma)|^2 + |M_t(tx_t - \tau \tilde{y}_\tau)|^2}\Big)\, \mathrm{d}\tilde{y}_\sigma\, \mathrm{d}\tilde{y}_\tau \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|sx_s - \sigma \tilde{y}_\sigma|^2 + |tx_t - \tau \tilde{y}_\tau|^2}\Big)\, \mathrm{d}\tilde{y}_\sigma\, \mathrm{d}\tilde{y}_\tau \\
&\quad = J(x_s, x_t, s, t, \sigma, \tau)\,.
\end{aligned}$$

Therefore, we can replace x_s and x_t in (3.8.3) by $e = (1, 0, \ldots, 0) \in \mathbb{S}^{m-1}$. Thus, we have

$$J(s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2}\Big)\, \mathrm{d}y_\sigma\, \mathrm{d}y_\tau\,.$$

For an easier notation, we rename $\omega = y_\sigma$ and $\tilde{\omega} = y_\tau$, and thus we have

$$\begin{aligned}
|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2 &= |se - \sigma\omega|^2 + |te - \tau\tilde{\omega}|^2 \\
&= s^2 + \sigma^2 - 2s\sigma e\cdot\omega + t^2 + \tau^2 - 2t\tau e\cdot\tilde{\omega} \\
&= s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1\,.
\end{aligned}$$

Then, we can rewrite J as

$$J(s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big)\, \mathrm{d}\omega\, \mathrm{d}\tilde{\omega}\,.$$

At this point we have to distinguish the cases $m = 1$ and $m \geq 2$. For the fist one, since $\mathbb{S}^0 = \{-1, 1\}$ we directly obtain (3.8.1). For the second one, since the integrand only depends on ω_1 and $\tilde{\omega}_1$, defining $\rho(r) = \sqrt{1 - r^2}$ we proceed as follows

$$\begin{aligned}
J(s, t, \sigma, \tau) &= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big)\, \mathrm{d}\omega\, \mathrm{d}\tilde{\omega} \\
&= \int_{-1}^{1} \mathrm{d}\omega_1 \int_{\partial B_{\rho(\omega_1)}} \mathrm{d}\omega_2 \cdots \mathrm{d}\omega_m \int_{-1}^{1} \mathrm{d}\tilde{\omega}_1 \int_{\partial B_{\rho(\tilde{\omega}_1)}} \mathrm{d}\tilde{\omega}_2 \cdots \mathrm{d}\tilde{\omega}_m \\
&\qquad\qquad K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big) \\
&= \int_{-1}^{1} \int_{-1}^{1} |\partial B_{\rho(\omega_1)}| |\partial B_{\rho(\tilde{\omega}_1)}| \\
&\qquad\qquad K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\Big)\, \mathrm{d}\omega_1\, \mathrm{d}\tilde{\omega}_1.
\end{aligned}$$

Finally, we obtain (3.8.2) once we replace $|\partial B_r| = c_m\, r^{m-2}$, where c_m is the measure of the boundary of the ball of radius one in $\mathbb{R}^{m-1}$. $\square$

If the operator L_K is the fractional Laplacian, the previous expression of the kernel J can be rewritten in terms of a hypergeometric function of two variables, the so-called Appell function F_2 (see [6]). This expression does not simplify any of the arguments of this paper. Nevertheless, we think that it is worthy to point out the relation between J and F_2, since the known properties of the last one could provide some information about the kernel J.

Lemma 3.8.2. *Let F_2 be the Appell hypergeometric function defined in [6]. If $L_K = (-\Delta)^\gamma$ and $m \geq 2$, then*

$$J(s,t,\sigma,\tau) = \frac{c_{2m,\gamma}\pi^m \Gamma\left(\frac{m}{2}\right)^2}{\Gamma\left(\frac{m-1}{2}\right)^2 \Gamma\left(\frac{m+1}{2}\right)^2} \frac{F_2\left(m+\gamma; \frac{m}{2}, m; \frac{m}{2}, m; \frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}, \frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\right)}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}}. \tag{3.8.4}$$

Proof. If we take $K(z) = c_{2m,\gamma}|z|^{-2m-2\gamma}$ in (3.8.2) we get

$$J(s,t,\sigma,\tau) = c_{2m,\gamma}c_m^2 \int_{-1}^{1}\int_{-1}^{1} \frac{(1-\theta^2)^{\frac{m-2}{2}}(1-\overline{\theta}^2)^{\frac{m-2}{2}}}{(s^2+t^2+\sigma^2+\tau^2-2s\sigma\theta-2t\tau\overline{\theta})^{m+\gamma}}\, d\theta\, d\overline{\theta}\,.$$

Then, if we make the change of variables $\theta = 2\varpi_1 - 1$ and $\overline{\theta} = 2\varpi_2 - 1$ we arrive at

$$\begin{aligned}
J(s,t,\sigma,\tau) &= \frac{c_{2m,\gamma}2^{2m-4}c_m^2}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}}\cdot \\
&\qquad \int_0^1\int_0^1 \frac{\varpi_1^{\frac{m-2}{2}}(1-\varpi_1)^{\frac{m-2}{2}}\varpi_2^{\frac{m-2}{2}}(1-\varpi_2)^{\frac{m-2}{2}}}{\left(1-\frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}\varpi_1 - \frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\varpi_2\right)^{m+\gamma}}\, d\varpi_1\, d\varpi_2 \\
&= \frac{c_{2m,\gamma}2^{2m-4}c_m^2}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}} \frac{\Gamma\left(\frac{m}{2}\right)^4}{\Gamma(m)^2}\cdot \\
&\qquad F_2\left(m+\gamma; \frac{m}{2}, m; \frac{m}{2}, m; \frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}, \frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\right).
\end{aligned}$$

We finally obtain (3.8.4) by using the duplication formula for the Γ-function. □

To conclude the appendix, we rewrite the kernel inequality (3.1.13) in (s,t) variables and in terms of the kernel J. We do not present a proof of this result since it is identical to the one of Proposition 3.2.4 but changing the notation.

Lemma 3.8.3. *Let $m \geq 1$ and let J the kernel defined in (3.8.1)-(3.8.2) with $K(\sqrt{\cdot})$ strictly convex. Then, if $s > t$ and $\sigma > \tau$, we have*

$$J(s,t,\sigma,\tau) > J(s,t,\tau,\sigma)\,.$$

Chapter 4

Semilinear integro-differential equations, II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation

This paper, which is the follow-up to part I, concerns saddle-shaped solutions to the semilinear equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is a linear elliptic integro-differential operator with a radially symmetric kernel and f is of Allen-Cahn type. Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only in this set.

Following the setting established in part I for doubly radial odd functions, we show existence, asymptotic behavior, and uniqueness of the saddle-shaped solution. For this, we prove, among others, a Liouville type result, the one-dimensional symmetry of positive solutions to semilinear problems in a half-space, and maximum principles in "narrow" sets.

4.1 Introduction

In this paper, which is the second part of [86], we study saddle-shaped solutions to the semilinear equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{4.1.1}$$

where L_K is a linear integro-differential operator of the form (4.1.2) and f is of Allen-Cahn type. These solutions (see Definition 4.1.1 below) are particularly interesting in relation to the nonlocal version of a conjecture by De Giorgi, with the aim of finding a counterexample in high dimensions. Moreover, this problem is related to the regularity theory of nonlocal minimal surfaces. For more comments on this we refer to Subsection 4.1.3 and the references therein.

Previous to this article and its first part [86], there are only three works devoted to saddle-shaped solutions to the equation (4.1.1) with L_K being the fractional Laplacian. In [60, 61], Cinti proved the existence of a saddle-shaped solution as well as some qualitative properties such as asymptotic behavior, monotonicity properties, and instability whenever $2m \leq 6$. In a previous paper by the authors [88], further properties of these solutions were proved, the main ones being uniqueness and, when $2m \geq 14$, stability.

Concerning saddle-shaped solutions to the classical Allen-Cahn equation $-\Delta u = f(u)$, the same results were proved in [69, 138, 43, 44, 34]. The possible stability in dimensions 8, 10, and 12 is still an open problem (both in the local and fractional frameworks), as well as the possible minimality of this solution in dimensions $2m \geq 8$.

The present paper together with its first part [86] are the first ones studying saddle-shaped solutions for general integro-differential equations of the form (4.1.1). In the three previous papers [60, 61, 88] the main tool used was the extension problem for the fractional Laplacian (see [51]). Nevertheless, this technique has the limitation that it cannot be carried out for general integro-differential operators different from the fractional Laplacian. Therefore, some purely nonlocal techniques were developed in the previous paper [86] to study saddle-shaped solutions, and we exploit them in the present one.

In part I [86], we established an appropriate setting to study solutions to (4.1.1) that are doubly radial and odd with respect to the Simons cone, a property that is satisfied by saddle-shaped solutions (see Subsection 4.1.1). In that paper we deduced an alternative and very useful expression for the operator L_K when acting on doubly radial odd functions —see (4.1.6). This was used to deduce some maximum principles for odd functions under certain assumptions on the kernel K of the operator L_K. Moreover, we proved an energy estimate for doubly radial and odd minimizers of the energy associated to the equation, as well as the existence of saddle-shaped solutions to (4.1.1).

In the present paper, we further study saddle-shaped solutions to (4.1.1) by using the results obtained in part I [86]. First, we prove existence of this type of solutions by using the monotone iteration method (as an alternative to the proof in [86] where variational methods were used). After this, we establish the asymptotic behavior of saddle-shaped solutions, Theorem 4.1.4. To do it, we use two ingredients: a Liouville type theorem and a one-dimensional symmetry result, both for semilinear equations like (4.1.1) under some hypotheses on f. These are Theorems 4.1.6 and 4.1.7, proved in Section 4.4. In the study of the asymptotic behavior of saddle-shaped solutions we establish further properties of the so-called *layer solution* u_0 (see Section 4.5). Finally, we show the uniqueness of the saddle-shaped solution by using a maximum principle for the linearized operator $L_K - f'(u)$ (Proposition 4.1.5).

As in part I [86], equation (4.1.1) is driven by a linear integro-differential operator L_K of the form

$$L_K w(x) = \int_{\mathbb{R}^n} \{w(x) - w(y)\} K(x-y)\, dy. \tag{4.1.2}$$

The most canonical example of such operators is the fractional Laplacian, which corresponds to the kernel $K(z) = c_{n,\gamma}|z|^{-n-2\gamma}$, where $\gamma \in (0,1)$ and $c_{n,\gamma}$ is a normalizing positive constant —see (4.5.2).

Throughout the paper, we assume that K is symmetric, i.e.,

$$K(z) = K(-z), \tag{4.1.3}$$

and that L_K is uniformly elliptic, that is,

$$\lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}} \leq K(z) \leq \Lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}}, \tag{4.1.4}$$

where λ and Λ are two positive constants. Conditions (4.1.3) and (4.1.4) are frequently adopted since they yield Hölder regularity of solutions (see [125, 139]). The family of linear operators satisfying these two conditions is the so-called $\mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ ellipticity class. For short we will usually write $\mathcal{L}_0$ and we will make explicit the parameters only when needed.

Following the previous article [86], when dealing with doubly radial functions we will assume that the operator L_K is rotation invariant, that is, K is radially symmetric. This extra assumption allows us to rewrite the operator in a suitable form when acting on doubly radial odd functions, as explained below.

4.1.1 Integro-differential setting for odd functions with respect to the Simons cone

In this subsection we recall the basic definitions and results established in part I [86]. First, we present the Simons cone, which is a central object along this paper. It is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} \ : \ |x'| = |x''| \right\} .$$

This cone is of importance in the theory of (local and nonlocal) minimal surfaces (see Subsection 4.1.3). We will use the letters $\mathcal{O}$ and $\mathcal{I}$ to denote each of the parts in which $\mathbb{R}^{2m}$ is divided by the cone $\mathscr{C}$:

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| > |x''| \right\} \ \text{and} \ \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| < |x''| \right\} .$$

Both $\mathcal{O}$ and $\mathcal{I}$ belong to a family of sets in $\mathbb{R}^{2m}$ which are called of *double revolution*. These are sets that are invariant under orthogonal transformations in the first m variables, as well as under orthogonal transformations in the last m variables. That is, $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution if $R\Omega = \Omega$ for every given transformation $R \in O(m)^2 = O(m) \times O(m)$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$.

We say that a function $w : \mathbb{R}^{2m} \to \mathbb{R}$ is *doubly radial* if it depends only on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, $w(Rx) = w(x)$ for every $R \in O(m)^2$.

We recall now the definition of $(\cdot)^\star$, an isometry that played a significant role in part I [86]. It is defined by

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \, . \end{array}$$

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathcal{I}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant —although not all points in $\mathscr{C}$ are fixed points of $(\cdot)^\star$. Taking into account this transformation, we say that a doubly radial function w is *odd with respect to the Simons cone* if $w(x) = -w(x^\star)$. Similarly, we say that a doubly radial function w is *even with respect to the Simons cone* if $w(x) = w(x^\star)$.

With these definitions at hand we can precisely define saddle-shaped solutions.

Definition 4.1.1. We say that a bounded solution u to (4.1.1) is a *saddle-shaped solution* (or simply *saddle solution*) if

1. u is doubly radial.
2. u is odd with respect to the Simons cone.
3. $u > 0$ in $\mathcal{O} = \{|x'| > |x''|\}$.

Note that these solutions are even with respect to the coordinate axes and that their zero level set is the Simons cone $\mathscr{C} = \{|x'| = |x''|\}$.

Let us collect now the main results of the previous paper [86] that will be used in the present one. Recall that if K is a radially symmetric kernel we can rewrite the operator L_K acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x,y)\, \mathrm{d}y\,,$$

where $\overline{K}$ is doubly radial in both variables and is defined by

$$\overline{K}(x,y) := \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\,. \tag{4.1.5}$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$ (see Section 2 of [86] for the details). It is important to notice that, in contrast with $K = K(x-y)$, $\overline{K}$ is no longer translation invariant (i.e., it is a function of x and y but not of the difference $x - y$).

If we consider doubly radial functions that are, in addition, odd with respect to the Simons cone, we can use the involution $(\cdot)^\star$ to find that

$$L_K w(x) = \int_{\mathcal{O}} \{w(x) - w(y)\}\{\overline{K}(x,y) - \overline{K}(x,y^\star)\}\, \mathrm{d}y + 2w(x) \int_{\mathcal{O}} \overline{K}(x,y^\star)\, \mathrm{d}y\,. \tag{4.1.6}$$

Furthermore,

$$\frac{1}{C} \operatorname{dist}(x,\mathscr{C})^{-2\gamma} \le \int_{\mathcal{O}} \overline{K}(x,y^\star)\, \mathrm{d}y \le C \operatorname{dist}(x,\mathscr{C})^{-2\gamma}, \tag{4.1.7}$$

with $C > 0$ depending only on m, γ, λ, and Λ (see the details in part I [86]).

Note that the expression (4.1.6) has an integro-differential part plus a term of order zero with a positive coefficient. Thus, the most natural assumption to make in order to have an elliptic operator (when acting on doubly radial odd functions) is that the kernel of the integro-differential term is positive. That is, $\overline{K}(x,y) - \overline{K}(x,y^\star) > 0$. One of the main results in part I [86], stated next, established a necessary and sufficient condition on the original kernel K for L_K to have a positive kernel when acting on doubly radial odd functions.

Theorem 4.1.2 ([86]). *Let $K : (0,+\infty) \to (0,+\infty)$ and consider the radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (4.1.5).*

If

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau\,, \tag{4.1.8}$$

then L_K has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \text{for every } x,y \in \mathcal{O}\,. \tag{4.1.9}$$

In addition, if $K \in C^2((0,+\infty))$, then (4.1.8) is not only a sufficient condition for (4.1.9) to hold, but also a necessary one.

4.1.2 Main results

Through all the paper we will assume that f, the nonlinearity in (4.1.1), is a C^1 function satisfying

$$f \text{ is odd}, \quad f(\pm 1) = 0, \quad \text{and} \quad f \text{ is strictly concave in } (0,1). \tag{4.1.10}$$

It is easy to see that these properties yield $f > 0$ in $(0,1)$, $f'(0) > 0$ and $f'(\pm 1) < 0$.

The first main result of this paper concerns the existence and uniqueness of saddle-shaped solution.

Theorem 4.1.3 (Existence and uniqueness of the saddle-shaped solution)**.** *Let f satisfy* (4.1.10)*. Let K be a radially symmetric kernel satisfying the positivity condition* (4.1.9) *and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$.*

Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution u to (4.1.1)*. In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.*

The existence of saddle-shaped solutions was already proved in part I [86] using variational techniques. Here, we show that it can also be proved using, instead, the monotone iteration method. Let us remark that in both methods it is crucial to have the positivity condition (4.1.9). To establish the uniqueness of the saddle-shaped solution we will need two ingredients: the asymptotic behavior of saddle-shaped solutions and a maximum principle for the linearized operator in $\mathcal{O}$. Both results will be described later on.

The second main result of this paper is Theorem 4.1.4 below, on the asymptotic behavior of a saddle-shaped solution at infinity. To state it, let us introduce an important type of solutions in the study of the integro-differential Allen-Cahn equation: the layer solutions.

We say that a solution v to $L_K v = f(v)$ in $\mathbb{R}^n$ is a *layer solution* if v is increasing in one direction, say $e \in \mathbb{S}^{n-1}$ and $v(x) \to \pm 1$ as $x \cdot e \to \pm\infty$ (not necessarily uniform). When $n = 1$, a result of Cozzi and Passalacqua (Theorem 1 in [67]) establishes the existence and uniqueness (up to translations) of a layer solution to $L_{K_1} w = f(w)$ in $\mathbb{R}$. In addition, this solution is odd with respect to some point. They assume K_1 such that $L_{K_1} \in \mathcal{L}_0(1, \gamma, \lambda, \Lambda)$ and f satisfying (4.1.10). In the case of the fractional Laplacian this result was proved in [42, 41] by using the extension problem.

In $\mathbb{R}^n$, a special case of layer solutions are the one-dimensional ones. Actually, in relation with the available results concerning a conjecture by De Giorgi, in low dimensions all layer solutions are one-dimensional (see Subsection 4.1.3). One-dimensional layer solutions in $\mathbb{R}^n$ are in correspondence with the ones in $\mathbb{R}$ as follows —see also [67]. Let v be a layer solution to $L_K v = f(v)$ in $\mathbb{R}^n$ depending only on one direction, say $v(x) = w(x_n)$, and assume that $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$. Then w is a layer solution to $L_{K_1} w = f(w)$ in $\mathbb{R}$ with K_1 given by

$$K_1(t) := \int_{\mathbb{R}^{n-1}} K(\theta, t)\, d\theta = |t|^{n-1} \int_{\mathbb{R}^{n-1}} K(t\sigma, t)\, d\sigma.$$

Moreover $L_{K_1} \in \mathcal{L}_0(1, \gamma, \lambda, \Lambda)$. For more details see Proposition 4.5.1 in Section 4.5 and [67].

The layer solution in $\mathbb{R}$ that vanishes at the origin, denoted by u_0, therefore solves

$$\begin{cases} L_{K_1} u_0 &= f(u_0) & \text{in } \mathbb{R}, \\ \dot{u}_0 &> 0 & \text{in } \mathbb{R}, \\ u_0(x) &= -u_0(-x) & \text{in } \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} u_0(x) &= \pm 1, \end{cases} \tag{4.1.11}$$

and will play an important role in this paper. Note that, by the previous comments, $v(x) = u_0(x_n)$ is a one-dimensional layer solution to $L_K v = f(v)$ in $\mathbb{R}^n$. Moreover, the same holds for $u_0(x \cdot e)$ for every $e \in \mathbb{S}^{n-1}$ whenever the kernel K is radially symmetric.

The importance of the layer solution u_0 in relation with saddle-shaped solutions lies in that the associated function

$$U(x) := u_0 \left(\frac{|x'| - |x''|}{\sqrt{2}} \right) \tag{4.1.12}$$

will describe the asymptotic behavior of saddle solutions at infinity. Note that $(|x'| - |x''|)/\sqrt{2}$ is the signed distance to the Simons cone (see Lemma 4.2 in [44]). Therefore, the function U consists of "copies" of the layer solution u_0 centered at each point of the Simons cone and oriented in the normal direction to the cone.

The precise statement on the asymptotic behavior of saddle-shaped solutions at infinity is the following.

Theorem 4.1.4. *Let $f \in C^2(\mathbb{R})$ satisfy (4.1.10). Let K be a radially symmetric kernel satisfying the positivity condition (4.1.9) and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Let u be a saddle-shaped solution to (4.1.1) and let U be the function defined by (4.1.12).*

Then,

$$||u - U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + \left|\left| D^2 u - D^2 U \right|\right|_{L^\infty(\mathbb{R}^n \setminus B_R)} \to 0$$

as $R \to +\infty$.

Let us now describe some of the main ingredients that are used to prove Theorems 4.1.3 and 4.1.4. Concerning the uniqueness of the saddle-shaped solution, besides the asymptotic behavior described in Theorem 4.1.4 we also need to have on hand the following maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$.

Proposition 4.1.5. *Let $\Omega \subset \mathcal{O}$ be an open set (not necessarily bounded) and let K be a radially symmetric kernel satisfying the positivity condition (4.1.9) and such that $L_K \in \mathcal{L}_0$. Let u be a saddle-shaped solution to (4.1.1), and let $v \in C^\gamma(\overline{\Omega}) \cap C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, for some $\alpha > 2\gamma$, be a doubly radial function satisfying*

$$\begin{cases} L_K v - f'(u) v - c(x) v &\leq 0 & \text{in } \Omega, \\ v &\leq 0 & \text{in } \mathcal{O} \setminus \Omega, \\ -v(x^\star) &= v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x \in \Omega,\ |x| \to \infty} v(x) &\leq 0, \end{cases}$$

with $c \leq 0$ in Ω.

Then, $v \leq 0$ in Ω.

To establish it, the key tool is to use a maximum principle in "narrow" sets, also proved in Section 4.6. The proof of this result is much simpler than analogue maximum principles for the Laplacian. Indeed, this is an example of how the nonlocality of the operator makes the arguments easier and less technical (informally speaking, L_K "sees more", or 'further", than the Laplacian). Needless to mention, the proof of Proposition 4.1.5 is by far simpler than the one using the extension problem (Proposition 1.4 in [88]). In the proof, it is crucial again the positivity condition (4.1.9) together with the bounds (4.1.7).

Regarding the proof of Theorem 4.1.4, to establish the asymptotic behavior of saddle-shaped solutions we use a compactness argument as in [44, 60, 61], together with two important results established in Section 4.4. The first one, Theorem 4.1.6, is a Liouville type result for nonnegative solutions to a semilinear equation in the whole space. This result, in contrast with the previous ones, does not require the kernel to be radially symmetric, but only to satisfy (4.1.3) and (4.1.4).

Theorem 4.1.6. *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let v be a bounded solution to*

$$\begin{cases} L_K v = f(v) & \text{in } \mathbb{R}^n, \\ v \geq 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{4.1.13}$$

with a nonlinearity $f \in C^1$ satisfying

- $f(0) = f(1) = 0$,
- $f'(0) > 0$,
- $f > 0$ *in* $(0, 1)$, *and* $f < 0$ *in* $(1, +\infty)$.

Then, $v \equiv 0$ or $v \equiv 1$.

Similar classification results have been proved for the fractional Laplacian in [59, 112] (either using the extension problem or not) with the method of moving spheres, which uses crucially the scale invariance of the operator $(-\Delta)^\gamma$. To the best of our knowledge, there is no similar result available in the literature for general kernels in the ellipticity class $\mathcal{L}_0$ (which are not necessarily scale invariant). Thus, we present here a proof based on the techniques introduced by Berestycki, Hamel, and Nadirashvili [15] for the local equation with the classical Laplacian. It relies on the maximum principle, the translation invariance of the operator, a Harnack inequality, and a stability argument.

The second ingredient needed to prove the asymptotic behavior of saddle-shaped solutions is a symmetry result for equations in a half-space, stated next. Here and in the rest of the paper we use the notation $\mathbb{R}^n_+ = \{(x_H, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \, : \, x_n > 0\}$.

Theorem 4.1.7. *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let v be a bounded solution to one of the following two problems: either to*

$$\begin{cases} L_K v = f(v) & \text{in } \mathbb{R}^n_+, \\ v > 0 & \text{in } \mathbb{R}^n_+, \\ v(x_H, x_n) = -v(x_H, -x_n) & \text{in } \mathbb{R}^n, \end{cases} \tag{P1}$$

or to

$$\begin{cases} L_K v = f(v) & \text{in } \mathbb{R}^n_+, \\ v > 0 & \text{in } \mathbb{R}^n_+, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases} \tag{P2}$$

Assume that, in $\mathbb{R}^n_+$, the kernel K of the operator L_K is decreasing in the direction of x_n, i.e., it satisfies

$$K(x_H - y_H, x_n - y_n) \geq K(x_H - y_H, x_n + y_n) \quad \textit{for all} \quad x, y \in \mathbb{R}^n_+.$$

Suppose that $f \in C^1$ and

- $f(0) = f(1) = 0$,
- $f'(0) > 0$, *and* $f'(\tau) \leq 0$ *for all* $\tau \in [1-\delta, 1]$ *for some* $\delta > 0$,
- $f > 0$ *in* $(0,1)$, *and*
- f *is odd in the case of* (P1).

Then, v depends only on x_n and it is increasing in this direction.

The result for (P2) has been proved for the fractional Laplacian under some assumptions on f (weaker than the ones in Theorem 4.1.7) in [123, 7, 8, 83]. Instead, no result was available for general integro-differential operators. To the best of our knowledge, problem (P1) on odd solutions with respect to a hyperplane has not been treated even for the fractional Laplacian. In our case, the fact that f is of Allen-Cahn type allows us to use rather simple arguments that work for both problems (P1) and (P2) —moving planes and sliding methods. Moreover, the fact that the kernel of the operator is $|\cdot|^{-n-2\gamma}$ or a general K satisfying uniform ellipticity bounds does not affect significantly the proof. Although (P2) will not be used in this paper, since the proof for this problem is analogous to the one for (P1), we include it here for future reference.

4.1.3 Saddle-shaped solutions in the context of a conjecture by De Giorgi

To conclude this introduction, let us make some comments on the importance of problem (4.1.1) and its relation with the theory of (classical and nonlocal) minimal surfaces and a famous conjecture raised by De Giorgi.

A main open problem (even in the local case) is to determine whether the saddle-shaped solution is a minimizer of the energy functional associated to the equation, depending on the dimension $2m$. This question is deeply related to the regularity theory of local and nonlocal minimal surfaces, as explained next.

In the seventies, Modica and Mortola (see [118, 119]) proved that, considering an appropriately rescaled version of the (local) Allen-Cahn equation, the corresponding energy functionals Γ-converge to the perimeter functional. Thus, the blow-down sequence of minimizers of the Allen-Cahn energy converge to the characteristic function of a set of minimal perimeter. This same fact holds for the equation with the fractional Laplacian, though we have two different scenarios depending on the parameter $\gamma \in (0,1)$. If $\gamma \geq 1/2$, the rescaled energy functionals associated to the equation Γ-converge to the classical perimeter (see [**?**, 102]), while in the case $\gamma \in (0, 1/2)$, they Γ-converge to the fractional perimeter (see [135]).

In the recent years there has been an increasing interest in developing a regularity theory for nonlocal minimal surfaces, although very few results are known for the moment. It is beyond the scope of this article to describe all of them in detail, and we refer the interested reader to [66, 30] and the references therein. Let us just make some comments on the scarce available results concerning the possible minimality of the Simons cone as a nonlocal minimal surface, since this is connected to our work on saddle-shaped solutions. Note first that, by all its symmetries, it is easy to check that the Simons cone $\mathscr{C}$ is stationary for the fractional perimeter. If $2m = 2$, it cannot be a minimizer since in [136] Savin and Valdinoci proved that all minimizing nonlocal minimal cones in $\mathbb{R}^2$ are flat (indeed, dimension $n = 2$ is the only one where a complete classification of minimizing nonlocal minimal cones is available). In higher dimensions, the only available results regarding the possible minimality of $\mathscr{C}$ appear in [70] and in our paper [88], but they concern stability, a weaker property than minimality.

A very interesting characterization of the stability of Lawson cones —a more general class of cones that includes $\mathscr{C}$— has been found by Dávila, del Pino, and Wei [70]. It consists of an inequality involving two hypergeometric constants which depend only on γ and the dimension n. This inequality is checked numerically in [70], finding that, in dimensions $n \leq 6$ and for γ close to zero, no Lawson cone with zero nonlocal mean curvature is stable. Numerics also show that all Lawson cones in dimension 7 are stable if γ is close to zero. These two results for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem.

In contrast with the numeric computations in [70], our proof in [88] establishing the stability of $\mathscr{C}$ in dimensions $2m \geq 14$ is the first analytical proof of a stability result for the Simons cone in any dimension (in the nonlocal setting). This shows that the saddle-shaped solution does not only have its interest in the context of the Allen-Cahn equation, but it can also provide strategies to prove stability and minimality results in the theory of nonlocal minimal surfaces.

In addition to all this, saddle-shaped solutions are natural objects to build a counterexample to a famous conjecture raised by De Giorgi, as explained below. In 1978, De Giorgi [73] conjectured that bounded solutions to $-\Delta u = u - u^3$ in $\mathbb{R}^n$ which are monotone in one direction, say $\partial_{x_n} u > 0$, are one-dimensional if $n \leq 8$. This was proved to be true in dimensions $n = 2$ and $n = 3$ (see [96, 4]), and in dimensions $4 \leq n \leq 8$ with the extra assumption

$$\lim_{x_n \to \pm\infty} u(x_H, x_n) = \pm 1 \quad \text{for all } x_H \in \mathbb{R}^{n-1}\,, \tag{4.1.14}$$

(see [132]). A counterexample to the conjecture in dimensions $n \geq 9$ was given in [74] by using the gluing method.

An alternative approach to the one of [74] to construct a counterexample to the conjecture was given by Jerison and Monneau in [106]. They showed that a counterexample in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate but is not one-dimensional. The saddle-shaped solution is of special interest in search of this counterexample, since it is even with respect to all the coordinate axis and it is canonically associated to the Simons cone, which in turn is the simplest nonplanar minimizing minimal surface. Therefore, by proving that the saddle solution to the classical Allen-Cahn equation is a minimizer in some dimension $2m$, one would obtain automatically a counterexample to the conjecture in $\mathbb{R}^{2m+1}$.

The corresponding conjecture in the fractional setting, where one replaces the operator $-\Delta$ by $(-\Delta)^\gamma$, has been widely studied in the last years. In this framework, the conjecture has been proven to be true for all $\gamma \in (0,1)$ in dimensions $n = 2$ (see [42, 40, 143]) and $n = 3$ (see [36, 37, 76]). The conjecture is also true in dimension $n = 4$ in the case of $\gamma = 1/2$ (see [90]) and if $\gamma \in (0,1/2)$ is close to $1/2$ (see [39]). Assuming the additional hypothesis (4.1.14), the conjecture is true in dimensions $4 \leq n \leq 8$ for $1/2 \leq \gamma < 1$ (see [133, 134]), and also for $\gamma \in (0,1/2)$ if γ is close to $1/2$ (see [78]). A counterexample to the De Giorgi conjecture for the fractional Allen-Cahn equation in dimensions $n \geq 9$ for $\gamma \in (1/2,1)$ has been very recently announced in [57].

Concerning the conjecture with more general operators like L_K, fewer results are known. In dimension $n = 2$ the conjecture is proved in [103, 29, 85], under different assumptions on the kernel K and even for more general nonlinear operators. Note also that the results of [78] also hold for a particular class of kernels in $\mathcal{L}_0$.

A related issue to the conjecture by De Giorgi concerns the one-dimensional symmetry of minimizers to the Allen-Cahn equation. In the local case, a deep result of Savin [132] states that minimizers of the Allen-Cahn equation $-\Delta u = f(u)$ in $\mathbb{R}^n$ are one-dimensional if $n \leq 7$. On the other hand, Liu, Wang, and Wei [113] have constructed minimizers in dimensions $n \geq 8$ which are not one-dimensional. We should mention that the same question for stable solutions (instead of minimizers) is still largely open, only solved in dimension $n = 2$ (see [96, 11]).

Let us make a brief remark on the recent result of Liu, Wang, and Wei [113] concerning the existence of minimizers in $\mathbb{R}^8$ which are not one-dimensional. The authors proved that there exists an ordered family of solutions W_λ with their zero level set being asymptotic to the cone $\mathscr{C}$. From this ordering, they can establish that each solution W_λ is a minimizer of the Allen-Cahn equation. However, their construction only gives solutions W_λ for which $\{W_\lambda = 0\}$ is far from the origin of $\mathbb{R}^8$ (even if this set is asymptotic to the Simons cone at infinity). Therefore, this family does not include the saddle-shaped solution.

Concerning the same questions in the fractional setting, Savin [133, 134] extended his results for the Laplacian to the powers $\gamma \in [1/2,1)$, by proving that minimizers of the equation $(-\Delta)^\gamma u = f(u)$ in $\mathbb{R}^n$ are one-dimensional if $n \leq 7$. In the case $\gamma \in (0,1/2)$, Dipierro, Serra, and Valdinoci [78] proved that minimizers are one-dimensional provided that their level sets are asymptotically flat. Therefore, if one could prove a classification result for nonlocal minimal cones in some dimension n, this would entail the one-dimensional symmetry of minimizers to $(-\Delta)^\gamma u = f(u)$ in $\mathbb{R}^{n-1}$. As mentioned above, the classification of stable nonlocal minimal cones is still a fundamental open problem in dimensions $n \geq 3$. The one-dimensional symmetry of stable solutions is also largely open, only solved in dimension $n = 2$ (see [42, 41]).

4.1.4 Plan of the article

The paper is organized as follows. In Section 4.2 we present some preliminary results that will be used in the rest of the article. Section 4.3 contains the proof of the existence of a saddle-shaped solution via the monotone iteration method. In Section 4.4 we establish the Liouville type and symmetry results, Theorems 4.1.6 and 4.1.7. Section 4.5 is devoted to the layer solution u_0 of problem (4.1.1) and the proof of the asymptotic behavior of saddle-shaped solutions, Theorem 4.1.4. Finally, Section 4.6 concerns the proof of a maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$ (Proposition 4.1.5), as well as the proof of the uniqueness of the saddle-shaped solution.

4.2 Preliminaries

In this section we collect some preliminary results that will be used in the rest of this paper. First, we state the regularity results needed in the forthcoming sections. Then, we state a remark on stability that will be used later in this paper, and finally we recall the basic maximum principles for doubly radial odd functions proved in [86].

4.2.1 Regularity theory for nonlocal operators in the class $\mathcal{L}_0$

In this subsection we present the regularity results that will be used in the paper. For further details, see [125, 139] and the references therein.

First, note that for operators in the class $\mathcal{L}_0$, the minimal assumption on w so that $L_K w$ is well defined in an open set Ω is that $w \in C^\alpha(\Omega) \cap L^1_\gamma(\mathbb{R}^n)$ for some $\alpha > 2\gamma$, where $w \in L^1_\gamma(\mathbb{R}^n)$ means that

$$\int_{\mathbb{R}^n} \frac{|w(x)|}{1+|x|^{n+2\gamma}}\, dx < +\infty .$$

Now, we give a result on the interior regularity for linear equations.

Proposition 4.2.1 ([125, 139]). *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ and let $w \in L^\infty(\mathbb{R}^n)$ be a weak solution to $L_K w = h$ in B_1. Then,*

$$||w||_{C^{2\gamma}(B_{1/2})} \le C\left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)}\right). \tag{4.2.1}$$

Moreover, let $\alpha > 0$ and assume additionally that $w \in C^\alpha(\mathbb{R}^n)$. Then, if $\alpha + 2\gamma$ is not an integer,

$$||w||_{C^{\alpha+2\gamma}(B_{1/2})} \le C\left(||h||_{C^\alpha(B_1)} + ||w||_{C^\alpha(\mathbb{R}^n)}\right), \tag{4.2.2}$$

where C is a constant that depends only on n, γ, λ and Λ.

Throughout the paper we consider u to be a saddle solution to (4.1.1) that satisfies $|u| \le 1$ in $\mathbb{R}^n$. Hence, by applying (4.2.1) we find that for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} ||u||_{C^{2\gamma}(B_{1/2}(x_0))} &\le C\left(||f(u)||_{L^\infty(B_1(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)}\right) \\ &\le C\left(1 + ||f||_{L^\infty([-1,1])}\right). \end{aligned}$$

Note that the estimate is independent of the point x_0, and thus since the equation is satisfied in the whole $\mathbb{R}^n$,

$$||u||_{C^{2\gamma}(\mathbb{R}^n)} \le C\left(1 + ||f||_{L^\infty([-1,1])}\right).$$

Then, we use estimate (4.2.2) repeatedly and the same kind of arguments yield that, if $f \in C^k([-1,1])$, then $u \in C^\alpha(\mathbb{R}^n)$ for all $\alpha < k + 2\gamma$. Moreover, the following estimate holds:

$$||u||_{C^\alpha(\mathbb{R}^n)} \le C,$$

for some constant C depending only on n, γ, λ, Λ, k, and $||f||_{C^k([-1,1])}$.

4.2.2 A remark on stability

Recall that we say that a bounded solution w to $L_K w = f(w)$ in $\Omega \subset \mathbb{R}^n$ is *stable* in Ω if the second variation of the energy at w is nonnegative. That is, if

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f'(w)\xi^2\,\mathrm{d}x \geq 0$$

for every $\xi \in C_c^\infty(\Omega)$.

Here we prove that if $w \leq 1$ is a positive solution to $L_K w = f(w)$ in a set $\Omega \subset \mathbb{R}^n$, with f satisfying (4.1.10), then w is stable in Ω. We will use this in Sections 4.4 and 4.5. The proof of this fact is rather simple and we present it next. It is a consequence of the fact that, under these assumptions, w is a positive supersolution of the linearized operator $L_K - f'(w)$ (a more detailed discussion can be found in [103]).

On the one hand, since f is strictly concave in $(0,1)$ and $f(0) = 0$, then $f'(w)w < f(w)$ in Ω (recall that w is positive there). On the other hand, the following inequality holds for all functions φ and ξ, with $\varphi > 0$:

$$(\varphi(x)-\varphi(y))\left(\frac{\xi^2(x)}{\varphi(x)} - \frac{\xi^2(y)}{\varphi(y)}\right) \leq |\xi(x)-\xi(y)|^2\,. \tag{4.2.3}$$

Indeed, developing the square and the products, this last inequality is equivalent to $2\xi(x)\xi(y) \leq \xi^2(y)\varphi(x)/\varphi(y) + \xi^2(x)\varphi(y)/\varphi(x)$, which in turn is equivalent to

$$\left(\xi(x)\sqrt{\varphi(y)/\varphi(x)} - \xi(y)\sqrt{\varphi(x)/\varphi(y)}\right)^2 \geq 0\,.$$

Using these two facts and the symmetry of K, for every $\xi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}
\int_\Omega f'(w)\xi^2\,\mathrm{d}x &\leq \int_\Omega \frac{\xi^2}{w} f(w)\,\mathrm{d}x = \int_\Omega \frac{\xi^2}{w} L_K w\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} (w(x)-w(y))\left(\frac{\xi^2(x)}{w(x)} - \frac{\xi^2(y)}{w(y)}\right) K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\xi(x)-\xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Thus, w is stable in Ω.

4.2.3 Maximum principles for doubly radial odd functions

In this last subsection, we state the basic maximum principles for doubly radial odd functions. Note that in the following result we only need assumptions on the functions at one side of the Simons cone thanks to their symmetry. This was proved in part I [86] and follows readily from the expression (4.1.6) by using the key inequality (4.1.9) for the kernel $\overline{K}$.

Proposition 4.2.2 (Maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $\Omega \subset \mathcal{O}$ an open set and let L_K be an integro-differential operator with a radially symmetric kernel K satisfying the positivity condition (4.1.9). Let $w \in C^\alpha(\Omega) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone.*

(i) (Weak maximum principle) Assume that

$$\begin{cases} L_K w + c(x)w & \geq & 0 & \text{in } \Omega\,, \\ w & \geq & 0 & \text{in } \mathcal{O} \setminus \Omega\,, \end{cases}$$

with $c \geq 0$, and that either

$$\Omega \text{ is bounded} \quad \text{or} \quad \liminf_{x \in \mathcal{O},\, |x| \to +\infty} w(x) \geq 0\,.$$

Then, $w \geq 0$ in Ω.

(ii) (Strong maximum principle) Assume that $L_K w + c(x)w \geq 0$ in Ω, with $c(x)$ any function, and that $w \geq 0$ in $\mathcal{O}$. Then, either $w \equiv 0$ in $\mathcal{O}$ or $w > 0$ in Ω.

Remark 4.2.3. Following the proof of this result in [86] it is easy to see that the regularity assumptions on w in the previous results can be weakened. Indeed, we may allow $L_K w$ to take the value $+\infty$ at the points of Ω where w is not regular enough for $L_K w$ to be finite. This will be used in the proof of Theorem 4.1.3 in order to apply this maximum principle with a function that is no more regular than C^γ in the interior of Ω (see Remark 4.3.3)

4.3 Existence of saddle-shaped solution: monotone iteration method

In this section we give a proof of the existence result in Theorem 4.1.3 based on the maximum principle and the existence of a positive subsolution. To do this, we need a version of the monotone iteration procedure for doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. Along this section we will call odd sub/supersolutions to problem (4.3.2) the functions that are doubly radial, odd with respect to the Simons cone and satisfy the corresponding problem in (4.3.1). In view of Remark 4.2.3, we do not need the operator to be finite in the whole set when applied to a subsolution (respectively supersolution), it can take the value $-\infty$ (respectively $+\infty$) at some points.

Proposition 4.3.1. *Let $\gamma \in (0,1)$ and let K be a radially symmetric kernel satisfying the positivity condition (4.1.9) and such that $L_K \in \mathcal{L}_0$ and such that $L_K \in \mathcal{L}_0$. Assume that $\underline{v} \leq \overline{v}$ are two bounded functions which are doubly radial and odd with respect to the Simons cone. Furthermore, assume that $\underline{v} \in C^\gamma(\mathbb{R}^{2m})$ and that $\underline{v}$ and $\overline{v}$ satisfy respectively*

$$\begin{cases} L_K \underline{v} & \leq & f(\underline{v}) & \text{in } B_R \cap \mathcal{O}\,, \\ \underline{v} & \leq & \varphi & \text{in } \mathcal{O} \setminus B_R\,, \end{cases} \quad \text{and} \quad \begin{cases} L_K \overline{v} & \geq & f(\overline{v}) & \text{in } B_R \cap \mathcal{O}\,, \\ \overline{v} & \geq & \varphi & \text{in } \mathcal{O} \setminus B_R\,, \end{cases} \tag{4.3.1}$$

with f a C^1 odd function and φ a bounded doubly radial odd function.

Then, there exists a classical solution v to the problem

$$\begin{cases} L_K v & = & f(v) & \text{in } B_R\,, \\ v & = & \varphi & \text{in } \mathbb{R}^{2m} \setminus B_R\,, \end{cases} \tag{4.3.2}$$

such that $v \in C^{2\gamma+\varepsilon}(B_R)$ for some $\varepsilon > 0$, it is doubly radial, odd with respect to the Simons cone, and $\underline{v} \leq v \leq \overline{v}$ in $\mathcal{O}$.

Proof. The proof follows the classical monotone iteration method for elliptic equations (see for instance [82]). We just give here a sketch of the proof. First, let $M \geq 0$ be such that $-M \leq \underline{v} \leq \overline{v} \leq M$ and set

$$b := \max\left\{0, -\min_{[-M,M]} f'\right\} \geq 0\,.$$

Then one defines

$$\widetilde{L}_K w := L_K w + bw \quad \text{and} \quad g(\tau) := f(\tau) + b\tau\,.$$

Therefore, our problem is equivalent to find a solution to

$$\begin{cases} \widetilde{L}_K v &= g(v) \quad \text{in } B_R\,, \\ v &= \varphi \quad \text{in } \mathbb{R}^{2m} \setminus B_R\,, \end{cases}$$

such that v is doubly radial, odd with respect to the Simons cone and $\underline{v} \leq v \leq \overline{v}$ in $\mathcal{O}$. Here the main point is that g is also odd but satisfies $g'(\tau) \geq 0$ for $\tau \in [-M, M]$. Moreover, since $b \geq 0$, $\widetilde{L}_K$ satisfies the maximum principle for odd functions in $\mathcal{O}$ (as in Proposition 4.2.2).

We define $v_0 = \underline{v}$ and, for $k \geq 1$, let v_k be the solution to the linear problem

$$\begin{cases} \widetilde{L}_K v_k &= g(v_{k-1}) \quad \text{in } B_R\,, \\ v_k &= \varphi \quad \text{in } \mathbb{R}^{2m} \setminus B_R\,. \end{cases}$$

It is easy to see by induction and the regularity results from Proposition 4.2.1 that $v_k \in L^\infty(\mathbb{R}^n) \cap C^{2\gamma+2\varepsilon}(B_R)$ for some $\varepsilon > 0$. Moreover, given $\Omega \subset B_R$ a compact set, then $||v_k||_{C^{2\gamma+2\varepsilon}(\Omega)}$ is uniformly bounded in k.

Then, using the maximum principle it is not difficult to show by induction that

$$\underline{v} = v_0 \leq v_1 \leq \ldots \leq v_k \leq v_{k+1} \leq \ldots \overline{v} \quad \text{in } \mathcal{O}\,,$$

and that each function v_k is doubly radial and odd with respect to $\mathscr{C}$. Finally, by Arzelà-Ascoli theorem and the compact embedding of Hölder spaces we see that, up to a subsequence, v_k converges uniformly on compacts in $C^{2\gamma+\varepsilon}$ norm to the desired solution. □

In order to construct a positive subsolution, we also need a characterization and some properties of the first odd eigenfunction and eigenvalue for the operator L_K, which are presented next. This eigenfunction is obtained though a minimization of the Rayleigh quotient in the appropriate space, defined next.

Given a set $\Omega \subset \mathbb{R}^{2m}$ and a translation invariant and positive kernel K, we define the space

$$\mathbb{H}_0^K(\Omega) := \left\{w \in L^2(\Omega) \;:\; w = 0 \quad \text{a.e. in } \mathbb{R}^{2m} \setminus \Omega \quad \text{and } [w]^2_{\mathbb{H}^K(\mathbb{R}^{2m})} < +\infty\right\},$$

where

$$[w]^2_{\mathbb{H}^K(\mathbb{R}^{2m})} := \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x) - w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{4.3.3}$$

Recall also that when K satisfies the ellipticity assumption (4.1.4), then $\mathbb{H}_0^K(\Omega) = \mathbb{H}_0^\gamma(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(y) = c_{n,\gamma}|y|^{-n-2\gamma}$. We also define

$$\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega) := \left\{w \in \mathbb{H}_0^K(\Omega) \;:\; w \text{ is doubly radial a.e. and odd with respect to } \mathscr{C}\right\}.$$

Recall that when K is radially symmetric and w is doubly radial, we can replace the kernel $K(x-y)$ in the definition (4.3.3) by the kernel $\overline{K}(x,y)$. This is readily deduced after a change of variables and taking the mean among all $R \in O(m)^2$ (see the details in Secton 3 of [86]).

Lemma 4.3.2. *Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded set of double revolution and let K be a radially symmetric kernel satisfying the positivity condition (4.1.9) and such that $L_K \in \mathcal{L}_0(2m,\gamma,\lambda,\Lambda)$. Let us define*

$$\lambda_{1,\mathrm{odd}}(\Omega, L_K) := \inf_{w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_\Omega w(x)^2\,\mathrm{d}x}.$$

Then, such infimum is attained at a function $\phi_1 \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega) \cap L^\infty(\Omega)$ which solves

$$\begin{cases} L_K\phi_1 &= \lambda_{1,\mathrm{odd}}(\Omega, L_K)\phi_1 & \text{in } \Omega, \\ \phi_1 &= 0 & \text{in } \mathbb{R}^{2m}\setminus\Omega, \end{cases}$$

and satisfies that $\phi_1 > 0$ in $\Omega \cap \mathcal{O}$. We call this function ϕ_1 the first odd eigenfunction of L_K in Ω, *and $\lambda_{1,\mathrm{odd}}(\Omega, L_K)$, the* first odd eigenvalue.

Moreover, in the case $\Omega = B_R$, there exists a constant C depending only on n, γ, and Λ, such that

$$\lambda_{1,\mathrm{odd}}(B_R, L_K) \le CR^{-2\gamma}.$$

Proof. The first two statements are deduced exactly as in Proposition 9 of [140], using the same arguments as in Lemma 3.4 of [86] to guarantee that ϕ_1 is nonnegative in $\mathcal{O}$. The fact that $\phi_1 > 0$ in $\Omega \cap \mathcal{O}$ follows from the strong maximum principle (see Proposition 4.2.2).

We show the third statement. Let $\widetilde{w}(x) := w(Rx)$ for every $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Then,

$$\begin{aligned}
&\min_{w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2 \overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} w(x)^2\,\mathrm{d}x} \\
&\qquad \le \min_{\widetilde{w} \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x/R)-\widetilde{w}(y/R)|^2 |x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} \widetilde{w}(x/R)^2\,\mathrm{d}x} \\
&\qquad = R^{-2\gamma} \min_{\widetilde{w} \in \widetilde{\mathbb{H}}^s_{0,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x)-\widetilde{w}(y)|^2 |x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_1} \widetilde{w}(x)^2\,\mathrm{d}x} \\
&\qquad = \lambda_{1,\mathrm{odd}}(B_1, (-\Delta)^\gamma)\Lambda R^{-2\gamma}.
\end{aligned}$$

□

Remark 4.3.3. Note that, by standard regularity results for L_K, we have that $\phi_1 \in C^\gamma(\overline{\Omega}) \cap C^\infty(\Omega)$, and the regularity up to the boundary is optimal (by the Hopf lemma, see [125] and the references therein for the details). Due to this and the fact that $\phi_1 > 0$ in $\Omega \cap \mathcal{O}$ while $\phi_1 = 0$ in $\mathbb{R}^{2m} \setminus \Omega$, it is easy to check by using (4.1.6) that $-\infty < L_K\phi_1 < 0$ in $\mathcal{O} \setminus \overline{\Omega}$ and $L_K\phi_1 = -\infty$ in $\partial\Omega \cap \mathcal{O}$.

With these ingredients, we can proceed with the proof of the existence statement in Theorem 4.1.3.

Proof of Theorem 4.1.3: Existence. The strategy is to build a suitable solution u_R of

$$\begin{cases} L_K u_R = f(u_R) & \text{in } B_R\,, \\ u_R = 0 & \text{in } \mathbb{R}^{2m} \setminus B_R\,, \end{cases} \tag{4.3.4}$$

and then let $R \to +\infty$ to get a saddle-shaped solution.

Let $\phi_1^{R_0}$ be the first odd eigenfunction of L_K in $B_{R_0} \subset \mathbb{R}^{2m}$, given by Lemma 4.3.2, and let $\lambda_1^{R_0} := \lambda_{1,\mathrm{odd}}(B_{R_0}, L_K)$. Then, we claim that for R_0 big enough and $\varepsilon > 0$ small enough, $\underline{u}_R = \varepsilon\phi_1^{R_0}$ is an odd subsolution of (4.3.4) for every $R \geq R_0$. To see this, first note that, without loss of generality, we can assume that $\left|\left|\phi_1^{R_0}\right|\right|_{L^\infty(B_R)} = 1$. Now, since f is strictly concave in $(0,1)$ and $f(0) = 0$, then $f'(\tau)\tau < f(\tau)$ for all $\tau > 0$. Thus, using that $\varepsilon\phi_1^{R_0} > 0$ in $B_{R_0} \cap \mathcal{O}$, we see that for every $x \in B_{R_0} \cap \mathcal{O}$,

$$\frac{f(\varepsilon\phi_1^{R_0}(x))}{\varepsilon\phi_1^{R_0}(x)} > f'(\varepsilon\phi_1^{R_0}(x)) \geq f'(0)/2$$

if ε is small enough, independently of x (recall that we assumed $|\phi_1| \leq 1$). Therefore, since $f'(0) > 0$, taking R_0 big enough so that $\lambda_1^{R_0} < f'(0)/2$ (this can be achieved thanks to the last statement of Lemma 4.3.2), we have that for every $x \in B_{R_0} \cap \mathcal{O}$, $f(\varepsilon\phi_1^{R_0}(x)) > \lambda_1\varepsilon\phi_1^R(x)$. Thus,

$$L_K\underline{u}_R = \lambda_1^{R_0}\varepsilon\phi_1^{R_0} < f(\varepsilon\phi_1^{R_0}) = f(\underline{u}_R) \quad \text{in } B_{R_0} \cap \mathcal{O}\,.$$

In addition, if $x \in (B_R \setminus B_{R_0}) \cap \mathcal{O}$, by Remark 4.3.3 we have that

$$L_K\underline{u}_R < 0 = f(0) = f(\underline{u}_R) \quad \text{in } (B_R \setminus B_{R_0}) \cap \mathcal{O}\,.$$

Note that in ∂B_{R_0} we have $L_K\underline{u}_R = -\infty$. Hence, the claim is proved.

Now, if we define $\overline{u}_R := \chi_{\mathcal{O}\cap B_R} - \chi_{\mathcal{I}\cap B_R}$, a simple computation shows that it is an odd supersolution to (4.3.4). Therefore, using the monotone iteration procedure given in Proposition 4.3.1 (taking into account Remarks 4.2.3 and 4.3.3 when using the maximum principle), we obtain a solution u_R to (4.3.4) such that it is doubly radial, odd with respect to the Simons cone, and $\varepsilon\phi_1^{R_0} = \underline{u}_R \leq u_R \leq \overline{u}_R$ in $\mathcal{O}$. Note that, since $\underline{u}_R > 0$ in $\mathcal{O} \cap B_{R_0}$, the same holds for u_R.

Using a standard compactness argument as in [86], we let $R \to +\infty$ to obtain a sequence u_{R_j} converging on compacts in $C^{2\gamma+\eta}(\mathbb{R}^{2m})$ norm, for some $\eta > 0$, to a solution $u \in C^{2\gamma+\eta}(\mathbb{R}^{2m})$ of $L_K u = f(u)$ in $\mathbb{R}^{2m}$. Note that u is doubly radial, odd with respect to the Simons cone and $0 \leq u \leq 1$ in $\mathcal{O}$. Let us show that $0 < u < 1$ in $\mathcal{O}$ and hence u is a saddle-shaped solution. Indeed, the usual strong maximum principle yields $u < 1$ in $\mathcal{O}$. Moreover, since $u_R \geq \varepsilon\phi_1^{R_0} > 0$ in $\mathcal{O} \cap B_{R_0}$ for $R > R_0$, also the limit $u \geq \varepsilon\phi_1^{R_0} > 0$ in $\mathcal{O} \cap B_{R_0}$. Therefore, by applying the strong maximum principle for odd functions (see Proposition 4.2.2) we obtain that $0 < u < 1$ in $\mathcal{O}$. □

Remark 4.3.4. The fact of being u positive in $\mathcal{O}$ yields that u is stable in this set, as explained in Section 4.2.

4.4 Symmetry and Liouville type results

This section is devoted to prove the Liouville type result of 4.1.6 and the one-dimensional symmetry result of 4.1.7. Both of them will be needed in the following section to establish the asymptotic behavior of the saddle-shaped solution.

4.4.1 A Liouville type result for positive solutions in the whole space

In the proof of Theorem 4.1.6 we will need two main ingredients, that we present next. The first one is a Harnack inequality for solutions to the semilinear problem (4.1.13). This inequality follows readily from the results of Cozzi in [64], although the precise result that we need is not stated there. For the reader's convenience and for future reference, we present the result here and indicate how to deduce it from the results in [64].

Proposition 4.4.1. *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let w be a solution to (4.1.13) with f a Lipschitz nonlinearity such that $f(0) = 0$. Then, for every $x_0 \in \mathbb{R}^n$ and every $R > 0$, it holds*

$$\sup_{B_R(x_0)} w \leq C \inf_{B_R(x_0)} w,$$

with $C > 0$ depending only on $n, \gamma, \lambda, \Lambda$, and R.

Proof. Following the notation of [64], since f is Lipschitz and $f(0) = 0$, we have

$$|f(u)| \leq d_1 + d_2|u|^{q-1} \quad \text{in } \mathbb{R}^n,$$

with $d_1 = 0$, $d_2 = ||f||_{\mathrm{Lip}}$ and $q = 2$. With this choice of the parameters, we only need to repeat the proof of Proposition 8.5 from [64] (with $p = 2$ and $\Omega = \mathbb{R}^n$) in order to obtain that u belongs to the fractional De Giorgi class $\mathrm{DG}^{\gamma,2}(\mathbb{R}^n, 0, H, -\infty, 2\gamma/n, 2\gamma, +\infty)$ for some constant $H > 0$ (see [64] for the precise definition of these classes). Therefore, the Harnack inequality follows from Theorem 6.9 in [64]. □

The second ingredient that we need in the proof of Theorem 4.1.6 is the following parabolic maximum principle in the unbounded set $\mathbb{R}^n \times (0, +\infty)$.

Proposition 4.4.2. *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let v be a bounded function, C^α with $\alpha > 2\gamma$ in space and C^1 in time, such that*

$$\begin{cases} \partial_t v + L_K v + c(x)\, v & \leq & 0 & \text{in } \mathbb{R}^n \times (0, +\infty)\,, \\ v_0 := v(x, 0) & \leq & 0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

with $c(x)$ a continuous and bounded function. Then,

$$v(x, t) \leq 0 \quad \text{in} \quad \mathbb{R}^n \times [0, +\infty).$$

This result can be deduced from the usual parabolic maximum principle in a bounded (in space and time) set with a rather simple argument. Since we have not found a specific reference where such result is stated, let us present its proof with full detail for the sake of clarity. First of all, we present the usual parabolic maximum principle in a bounded set in $\mathbb{R}^n \times (0, +\infty)$. The proof for cylindrical sets $\Omega \times (0, T)$ can be found for instance in [9]. Although the argument for general bounded sets is essentially the same, we include here a short proof for the sake of completeness.

Lemma 4.4.3. *Let L_K be an integro-differential operator of the form (4.1.2) with kernel symmetric and satisfying (4.1.4), and let v be a bounded function, C^α with $\alpha > 2\gamma$ in space and C^1 in time, satisfying*

$$\begin{cases} \partial_t v + L_K v & \leq & 0 & \text{in } \Omega \subset B_R \times (0,T)\,, \\ v_0 := v(x,0) & \leq & 0 & \text{in } \overline{\Omega} \cap \{t=0\} \subset B_R\,, \\ v & \leq & 0 & \text{in } (\mathbb{R}^n \times (0,T)) \setminus \Omega\,. \end{cases}$$

Then, $v \leq 0$ in $\mathbb{R}^n \times [0,T]$.

Proof. By contradiction, for every $\varepsilon > 0$ assume that

$$M := \sup_{\mathbb{R}^n \times (0,T-\varepsilon)} v > 0$$

By the sign of the initial condition and since $v \leq 0$ in $(\mathbb{R}^n \times (0,T)) \setminus \Omega$, v attains this positive value M at a point $(x_0,t_0) \in \Omega$ with $t_0 \leq T-\varepsilon$. If $t_0 \in (0,T-\varepsilon)$, then (x_0,t_0) is an interior global maximum (in $\mathbb{R}^n \times (0,T-\varepsilon)$) and it must satisfy $v_t(x_0,t_0) = 0$ and $L_K v(x_0,t_0) > 0$, which contradicts the equation. If $t_0 = T-\varepsilon$, then $v_t(x_0,t_0) \geq 0$ and $L_K v(x_0,t_0) > 0$, which is also a contradiction with the equation. Thus, $v \leq 0$ in $\mathbb{R}^n \times [0,T-\varepsilon)$ and since this holds for $\varepsilon > 0$ arbitrarily small, we deduce $v \leq 0$ in $\mathbb{R}^n \times [0,T)$, and by continuity, in $\mathbb{R}^n \times [0,T]$. □

To establish Proposition 4.4.2 from Lemma 4.4.3, we need to introduce an auxiliary function enjoying certain properties (see Lemma 4.4.5 below). Before presenting it, we need the following result.

Lemma 4.4.4. *There is no bounded solution to $L_K v = 1$ in $\mathbb{R}^n$ for any $L_K \in \mathcal{L}_0$.*

Proof. Assume by contradiction that such solution exists. Then, by interior regularity (see Section 4.2) $v \in C^1(\mathbb{R}^n)$ and $|\nabla v| \leq C$ in $\mathbb{R}^n$. For every $i = 1,\ldots,n$, we differentiate the equation with respect to x_i to obtain

$$\begin{cases} L_K v_{x_i} & = & 0 & \text{in } \mathbb{R}^n\,, \\ |v_{x_i}| & \leq & C & \text{in } \mathbb{R}^n\,. \end{cases}$$

By the Liouville theorem for the operator L_K (it is proved exactly as in [129], see also [139]), v_{x_i} is constant. Hence, ∇v is constant, and thus v is affine. But since u is bounded, v must be constant, and we arrive at a contradiction with $L_K v = 1$. □

With this result we can introduce the auxiliary function that we will use to prove the parabolic maximum principle of Proposition 4.4.2.

Lemma 4.4.5. *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$. Then, for every $R > 0$ there exists a constant $M_R > 0$ and a continuous function $\psi_R \geq 0$ solution to*

$$\begin{cases} L_K \psi_R & = & -1/M_R & \text{in } B_R\,, \\ \psi_R & = & 1 & \text{in } \mathbb{R}^n \setminus B_R\,, \end{cases} \tag{4.4.1}$$

satisfying

$$\psi_R \to 0 \text{ uniformly} \quad \text{and} \quad M_R \to +\infty \text{ as } R \to +\infty\,.$$

Proof. First, consider ϕ_R the solution to

$$\begin{cases} L_K\phi_R &= 1 \quad \text{in } B_R\,, \\ \phi_R &= 0 \quad \text{in } \mathbb{R}^n \setminus B_R\,. \end{cases}$$

Note that the existence of a weak solution to the previous problem is given by the Riesz representation theorem. Moreover, by standard regularity results (see Section 4.2.1), ϕ_R is in fact a classical solution and by the maximum principle, $\phi_R > 0$ in B_R.

Define $M_R := \sup_{B_R} \phi_R$. Since M_R is increasing (to check this use the maximum principle to compare ϕ_R and $\phi_{R'}$ with $R > R'$), it must have a limit $M \in \mathbb{R} \cup \{+\infty\}$. Assume by contradiction that $M < +\infty$. To see this, consider the new function $\varphi_R := \phi_R / M_R$, which satisfies

$$\begin{cases} L_K\varphi_R &= 1/M_R \quad \text{in } B_R\,, \\ \varphi_R &= 0 \quad \text{in } \mathbb{R}^n \setminus B_R\,, \\ \varphi_R &\le 1\,. \end{cases} \tag{4.4.2}$$

By a standard compactness argument, we deduce that as $R \to +\infty$, φ_R converges (up to a subsequence) to a function φ that solves $L_K\varphi = 1/M$ in $\mathbb{R}^n$ and satisfies $|\varphi| \le 1$. This contradicts Lemma 4.4.4 and therefore, $M_R \to +\infty$ as $R \to +\infty$.

Define now $\psi_R := 1 - \phi_R/M_R = 1 - \varphi_R$, which solves trivially (4.4.1). Thus, it only remains to show that $\phi_R \to 0$ as $R \to +\infty$. We will see that $\varphi_R \to 1$ uniformly as $R \to +\infty$. Recall that φ_R solves problem (4.4.2), and by the previous arguments, by letting $R \to +\infty$ we have that a subsequence of φ_R converges uniformly in compact sets to a bounded function $\varphi \ge 0$ that solves $L_K\varphi = 0$ in $\mathbb{R}^n$. By the Liouville theorem, φ must be constant, and since its L^∞ norm is 1 and $\varphi \ge 0$, we conclude $\varphi \equiv 1$. $\square$

With these ingredients, we establish now the parabolic maximum principle in $\mathbb{R}^n \times (0, +\infty)$.

Proof of Proposition 4.4.2. First of all, note that with the change of function $\tilde{v}(x,t) = e^{-\alpha t} v(x,t)$ we can reduce the initial problem to

$$\begin{cases} \partial_t \tilde{v} + L_K \tilde{v} &\le 0 \quad \text{in } \Omega \subset \mathbb{R}^n \times (0,+\infty)\,, \\ \tilde{v} &\le 0 \quad \text{in } (\mathbb{R}^n \times (0,+\infty)) \setminus \Omega\,, \\ \tilde{v}_0 &\le 0 \quad \text{in } \mathbb{R}^n\,, \end{cases}$$

if we take $\alpha > ||c||_{L^\infty}$ and $\Omega := \{(x,t) \in \mathbb{R}^n \times (0,+\infty) \,:\, v(x,t) > 0\}$.

Now, consider the function

$$w_R(x,t) := ||\tilde{v}||_{L^\infty(\mathbb{R}^n \times (0,+\infty))} \left(\psi_R + \frac{t}{M_R} \right),$$

where ψ_R and M_R are defined in Lemma 4.4.5. Then, it is easy to check that w_R satisfies

$$\begin{cases} \partial_t w_R + L_K w_R &= 0 \quad \text{in } B_R \times (0,T)\,, \\ w_R(x,0) &\ge 0 \quad \text{in } B_R\,, \\ w_R(x,t) &\ge ||\tilde{v}||_{L^\infty(\mathbb{R}^n \times (0,+\infty))} \quad \text{in } (\mathbb{R}^n \setminus B_R) \times (0,T)\,, \end{cases}$$

for every $T > 0$ and $R > 0$. Since $w_R \ge 0 \ge \tilde{v}$ in $(\mathbb{R}^n \times (0,+\infty)) \setminus \Omega$, by the maximum principle in $(B_R \times (0,T)) \cap \Omega$ (see Lemma 4.4.3) we can easily deduce that $w_R \ge \tilde{v}$ in $B_R \times (0,T)$.

Finally, given an arbitrary point $(x_0, t_0) \in \Omega$, take $R_0 > 0$ and $T > 0$ such that $(x_0, t_0) \in B_{R_0} \times (0, T)$. Thus,

$$\tilde{v}(x_0, t_0) \leq w_R(x_0, t_0) = ||\tilde{v}||_{L^\infty(\mathbb{R}^n \times (0,+\infty))} \left(\psi_R(x_0) + \frac{t_0}{M_R} \right), \quad \text{for every} \quad R \geq R_0.$$

Letting $R \to +\infty$ and using that $\psi_R(x_0) \to 0$ and $M_R \to +\infty$ (see Lemma 4.4.5), we conclude $\tilde{v}(x_0, t_0) \leq 0$, and therefore $v(x_0, t_0) = \mathrm{e}^{\alpha t_0} \tilde{v}(x_0, t_0) \leq 0$. □

By using the Harnack inequality and the parabolic maximum principle we can now establish Theorem 4.1.6. The proof follows the ideas of Berestycki, Hamel, and Nadirashvili from Theorem 2.2 in [15] but adapted to the whole space and with an integro-differential operator.

Proof of Theorem 4.1.6. Assume $v \not\equiv 0$. Then, by the strong maximum principle $v > 0$. Our goal is to show that $v \equiv 1$, and this will be accomplished in two steps.

Step 1: We show that $m := \inf_{\mathbb{R}^n} v > 0$.

By contradiction, we assume $m = 0$. Then, there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that $v(x_k) \to 0$ as $k \to +\infty$.

On the one hand, by the Harnack inequality of Proposition 4.4.1, given any $R > 0$ we have

$$\sup_{B_R(x_k)} v \leq C_R \inf_{B_R(x_k)} v \leq C_R\, v(x_k) \to 0 \text{ as } k \to +\infty. \tag{4.4.3}$$

Moreover, since $f(0) = 0$ and $f'(0) > 0$, it is easy to show that $f(t) \geq f'(0)t/2$ if t is small enough. Therefore, from this and (4.4.3) we deduce that there exists $M(R) \in \mathbb{N}$ such that

$$L_K v - \frac{f'(0)}{2} v \geq 0 \ \text{ in } \ B_R(x_{M(R)}). \tag{4.4.4}$$

On the other hand, let us define

$$\lambda_R^{x_0} = \inf_{\substack{\varphi \in C_c^1(B_R(x_0)) \\ \varphi \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^2 K(x-y)\, \mathrm{d}x\, \mathrm{d}y}{\displaystyle\int_{\mathbb{R}^n} \varphi(x)^2\, \mathrm{d}x},$$

which decreases to zero uniformly in x_0 as $R \to +\infty$ from being $L_K \in \mathcal{L}_0$ (see the proof of Lemma 4.3.2 and also Proposition 9 of [140]). Therefore, there exists $R_0 > 0$ such that $\lambda_R^x < f'(0)/2$ for all $x \in \mathbb{R}^n$ and $R \geq R_0$. In particular, by choosing $x = x_{M(R_0)}$ there exists $w \in C_c^1(B_{R_0}(x_{M(R_0)}))$ such that $w \not\equiv 0$ and

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |w(x) - w(y)|^2 K(x-y)\, \mathrm{d}x\, \mathrm{d}y < \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x. \tag{4.4.5}$$

Finally, to get the contradiction, multiply (4.4.4) by $w^2/v \geq 0$ and integrate in $\mathbb{R}^n$. After symmetrizing the integral involving L_K we get

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^n} \frac{w^2}{v} L_K v\, \mathrm{d}x - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{v(x) - v(y)\} \left(\frac{w^2(x)}{v(x)} - \frac{w^2(y)}{v(y)} \right) K(x-y)\, \mathrm{d}x\, \mathrm{d}y - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |w(x) - w(y)|^2 K(x-y)\, \mathrm{d}x\, \mathrm{d}y - \frac{f'(0)}{2} \int_{\mathbb{R}^n} w^2\, \mathrm{d}x,
\end{aligned}$$

which contradicts (4.4.5). Here we have used that the kernel is positive and symmetric and the inequality (4.2.3). Therefore, $\inf_{\mathbb{R}^n} v > 0$.

Step 2: We show that $v \equiv 1$.

Choose $0 < \xi_0 < \min\{1, m\}$, which is well defined by Step 1, and let $\xi(t)$ be the solution of the ODE

$$\begin{cases} \dot{\xi}(t) &= f(\xi(t)) \quad \text{in } (0,+\infty)\,, \\ \xi(0) &= \xi_0\,. \end{cases}$$

Since $f > 0$ in $(0,1)$ and $f(1) = 0$ we have that $\dot{\xi}(t) > 0$ for all $t \geq 0$, and $\lim_{t\to+\infty} \xi(t) = 1$.

Now, note that both $v(x)$ and $\xi(t)$ solve the parabolic equation

$$\partial_t w + L_K w = f(w) \quad \text{in } \mathbb{R}^n \times (0,+\infty)\,,$$

and satisfy

$$v(x) \geq m \geq \xi_0 = \xi(0).$$

Thus, by the parabolic maximum principle (Proposition 4.4.2) applied to $v - \xi$, taking $c(x) = -\{f(v) - f(\xi)\}/(v - \xi)$, we deduce that $v(x) \geq \xi(t)$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. By letting $t \to +\infty$ we obtain

$$v(x) \geq 1 \quad \text{in } \mathbb{R}^n\,.$$

In a similar way, taking $\tilde{\xi}_0 > ||v||_{L^\infty} \geq 1$, using $f < 0$ in $(1,+\infty)$, $f(1) = 0$ and the parabolic maximum principle, we obtain the upper bound $v \leq 1$. $\square$

4.4.2 A one-dimensional symmetry result for positive solutions in a half-space

In this subsection we establish Theorem 4.1.7. To do it, we proceed in three steps. First, we show that the solution is monotone in the x_n direction by using a moving planes argument (see Proposition 4.4.6 below). Once this is shown, we can deduce that the solution v has uniform limits as $x_n \pm \to \infty$. Finally, by using the sliding method (see Proposition 4.4.12 below), we deduce the one-dimensional symmetry of the solution.

We proceed now with the details of the arguments. As we have said, the first step is to show that the solution is monotone. We establish the following result.

Proposition 4.4.6. *Let v be a bounded solution to one of the problems* (P1) *or* (P2)*, with $L_K \in \mathcal{L}_0$ such that the kernel K is decreasing in the direction of x_n in $\mathbb{R}^n_+$, that is,*

$$K(x_H - y_H, x_n - y_n) \geq K(x_H - y_H, x_n + y_n) \quad \textit{for all} \quad x, y \in \mathbb{R}^n_+.$$

Let f be a Lipschitz nonlinearity such that $f > 0$ in $(0, ||v||_{L^\infty(\mathbb{R}^n_+)})$.

Then,

$$\frac{\partial v}{\partial x_n} > 0 \quad \textit{in} \;\; \mathbb{R}^n_+.$$

To prove this monotonicity result, we use a moving planes argument, and for this reason we need a maximum principle in "narrow" sets for odd functions with respect to a hyperplane (see Proposition 4.4.10). Recall that for a set $\Omega \subset \mathbb{R}^n$, we define the quantity $R(\Omega)$ as the smallest positive R for which

$$\frac{|B_R(x)\setminus\Omega|}{|B_R(x)|}\geq\frac{1}{2}\quad\text{for every } x\in\Omega. \tag{4.4.6}$$

If no such radius exists, we define $R(\Omega)=+\infty$. We say that a set Ω is "narrow" if $R(\Omega)$ is small depending on certain quantities.

An important result needed to establish the maximum principle in "narrow" sets is the following ABP-type estimate. It is proved in [123] for the fractional Laplacian, following the arguments in [31] (see also [32]). The proof for a general operator L_K does not differ significantly from the one for the fractional Laplacian. Nevertheless, we include it here for the sake of completeness.

Theorem 4.4.7. *Let $\Omega\subset\mathbb{R}^n$ with $R(\Omega)<+\infty$. Let $L_K\in\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ and let $v\in L^1_\gamma(\mathbb{R}^n)\cap C^\alpha(\Omega)$, with $\alpha>2\gamma$, such that $\sup_\Omega v<+\infty$ and satisfying*

$$\begin{cases} L_Kv-c(x)v & \leq & h & \text{in }\Omega,\\ v & \leq & 0 & \text{in }\mathbb{R}^n\setminus\Omega,\end{cases}$$

with $c(x)\leq 0$ in Ω and $h\in L^\infty(\Omega)$.

Then,

$$\sup_\Omega v\leq CR(\Omega)^{2\gamma}\,||h||_{L^\infty(\Omega)}\,,$$

where C is a constant depending on n, γ, and Λ.

The only ingredient needed to show Theorem 4.4.7 is the following weak Harnack inequality proved in [65].

Proposition 4.4.8 (see Corollary 4.4 of [65])**.** *Let $\Omega\subset\mathbb{R}^n$ and $L_K\in(n,\gamma,\lambda,\Lambda)$. Let $w\in L^1_\gamma(\mathbb{R}^n)\cap C^\alpha(\Omega)$, with $\alpha>2\gamma$, such that $w\geq 0$ in $\mathbb{R}^n$. Assume that w satisfies weakly $L_Kw\geq h$ in Ω, for some $h\in L^\infty(\Omega)$. Then, there exists an exponent $\varepsilon>0$ and a constant $C>1$, both depending on n, γ and Λ, such that*

$$\left(\fint_{B_{R/2}(x_0)}w^\varepsilon\,dx\right)^{1/\varepsilon}\leq C\left(\inf_{B_R(x_0)}w+R^{2\gamma}\,||h||_{L^\infty(\Omega)}\right)$$

for every $x_0\in\Omega$ and $0<R<\operatorname{dist}(x_0,\partial\Omega)$.

With the previous weak Harnack inequality we can now establish the ABP estimate.

Proof of Theorem 4.4.7. First, note that it is enough to show it for $v>0$ in Ω satisfying

$$\begin{cases} L_Kv & \leq & h & \text{in }\Omega,\\ v & \leq & 0 & \text{in }\mathbb{R}^n\setminus\Omega.\end{cases}$$

Indeed, if we consider $\Omega_0=\{x\in\Omega\ :\ v>0\}$, then since $c\leq 0$ we have $L_Kv\leq L_Kv-c(x)v\leq h$ in Ω_0.

Define $M:=\sup_\Omega v$. Then, for every $\delta>0$ there exists a point $x_\delta\in\Omega$ such that $v(x_\delta)\geq M-\delta$. Consider now the function $w:=M-v^+$. Note that $0\leq w\leq M$, $w(x_\delta)\leq\delta$, and $w\equiv M$ in $\mathbb{R}^n\setminus\Omega$. If we extend h to be 0 outside Ω, we can easily verify that $L_Kw\geq -h$ in $B_R(x_\delta)$.

Now, by choosing $R = 2R(\Omega)$, and using the weak Harnack inequality of Proposition 4.4.8, we get

$$M\left(\frac{1}{2}\right)^{1/\varepsilon} \leq \left(M^\varepsilon \frac{|B_{R/2}(x_\delta) \setminus \Omega|}{|B_{R/2}(x_\delta)|}\right)^{1/\varepsilon} = \left(\frac{1}{|B_{R/2}(x_\delta)|} \int_{B_{R/2}(x_\delta)\setminus\Omega} w^\varepsilon \, \mathrm{d}x\right)^{1/\varepsilon}$$
$$\leq \left(\fint_{B_{R/2}(x_\delta)} w^\varepsilon \, \mathrm{d}x\right)^{1/\varepsilon} \leq C\left(\inf_{B_R(x_\delta)} w + R^{2\gamma} ||h||_{L^\infty(\Omega)}\right)$$
$$\leq C\left(\delta + R^{2\gamma} ||h||_{L^\infty(\Omega)}\right).$$

The conclusion follows from letting $\delta \to 0$. □

As a consequence of this result, one can deduce easily a general maximum principle in "narrow" sets.

Corollary 4.4.9. *Let $\Omega \subset \mathbb{R}^n$ with $R(\Omega) < +\infty$. Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $\sup_\Omega v < +\infty$ and satisfying*

$$\begin{cases} L_K v + c(x)v \leq 0 & \text{in } \Omega\,, \\ v \leq 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with $c(x)$ bounded below.

Then, there exists a number $\bar{R} > 0$ such that $v \leq 0$ in Ω whenever $R(\Omega) < \bar{R}$.

Proof. We write $c = c^+ - c^-$, and therefore $L_K v - (-c^+)v \leq c^- v^+$. By Theorem 4.4.7 we get

$$\sup_\Omega v \leq CR(\Omega)^{2\gamma} ||c^- v^+||_{L^\infty(\Omega)} \leq CR(\Omega)^{2\gamma} ||c^-||_{L^\infty(\Omega)} \sup_\Omega v\,.$$

Hence, if $CR(\Omega)^{2\gamma} ||c^-||_{L^\infty(\Omega)} < 1$, we deduce that $v \leq 0$ in Ω. □

The previous maximum principle in "narrow" sets is not suitable enough to apply the moving planes method. In the argument, we would want to use a maximum principle in a "narrow" band and applied to an odd function with respect to a hyperplane. However, odd functions cannot have a constant sign in the exterior of a band and in the hypotheses of Corollary 4.4.9 there is a prescribed constant sign of a function outside the set Ω. Thus, we need another version of a maximum principle in "narrow" sets that applies to odd functions and only requires a constant sign of the function at one side of a hyperplane (in the spirit of the maximum principles of Proposition 4.2.2). This is accomplished with the following result.

Proposition 4.4.10. *Let H be a half-space in $\mathbb{R}^n$, and denote by $x^\#$ the reflection of any point x with respect to the hyperplane ∂H. Let $L_K \in \mathcal{L}_0$ with a positive kernel K satisfying*

$$K(x-y) \geq K(x-y^\#), \quad \text{for all } \; x, y \in H. \tag{4.4.7}$$

Assume that $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\beta(\Omega)$, with $\beta > 2\gamma$, satisfies

$$\begin{cases} L_K v \geq c(x)\, v & \text{in } \Omega \subset H, \\ v \geq 0 & \text{in } H \setminus \Omega, \\ v(x) = -v(x^\#) & \text{in } \mathbb{R}^n, \end{cases}$$

with $c(x)$ bounded below.

Then, there exist a number $\overline{R}$ such that $v \geq 0$ in H whenever $R(\Omega) \leq \overline{R}$.

Proof. Let us begin by defining $\Omega_- = \{x \in \Omega : v < 0\}$. We shall prove that Ω_- is empty. Assume by contradiction that it is not empty. Then, we split $v = v_1 + v_2$, where

$$v_1(x) = \begin{cases} v(x) & \text{in } \Omega_-, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_-, \end{cases} \quad \text{and} \quad v_2(x) = \begin{cases} 0 & \text{in } \Omega_-, \\ v(x) & \text{in } \mathbb{R}^n \setminus \Omega_-. \end{cases}$$

We first show that $L_K v_2 \le 0$ in Ω_-. To see this, take $x \in \Omega_-$ and thus

$$L_K v_2(x) = \int_{\mathbb{R}^n \setminus \Omega_-} -v_2(y) K(x-y)\, dy = -\int_{\mathbb{R}^n \setminus \Omega_-} v(y) K(x-y)\, dy.$$

Now, we split $\mathbb{R}^n \setminus \Omega_-$ into

$$A_1 = \Omega_-^{\#}, \quad \text{and} \quad A_2 = (H \setminus \Omega_-) \cup (H \setminus \Omega_-)^{\#},$$

and we compute the previous integral in these two sets separately using that v is odd. On the one hand, $v \le 0$ in Ω_- and $K \ge 0$ in $\mathbb{R}^n$, we have

$$-\int_{A_1} v(y) K(x-y)\, dy = -\int_{\Omega_-} v(y^{\#}) K(x-y^{\#})\, dy = \int_{\Omega_-} v(y) K(x-y^{\#})\, dy \le 0.$$

On the other hand, by the kernel inequality (4.4.7)

$$\begin{aligned} -\int_{A_2} v(y) K(x-y)\, dy &= -\int_{H\setminus\Omega_-} v(y) K(x-y)\, dy - \int_{H\setminus\Omega_-} v(y^{\#}) K(x-y^{\#})\, dy \\ &= -\int_{H\setminus\Omega_-} v(y) \left\{ K(x-y) - K(x-y^{\#}) \right\} dy \le 0. \end{aligned}$$

Thus, we get $L_K v_2 \le 0$ in Ω_-.

Finally, since $L_K v_2 \le 0$ in Ω_-, it holds

$$L_K v_1 = L_K v - L_K v_2 \ge L_K v \ge c(x)\, v = c(x)\, v_1 \quad \text{in} \;\; \Omega_-.$$

Therefore v_1 solves

$$\begin{cases} L_K v_1 \ \ge \ c(x)\, v_1 & \text{in } \Omega_-, \\ \quad\; v_1 \ = \ 0 & \text{in } \mathbb{R}^n \setminus \Omega_-, \end{cases}$$

and we can apply the usual maximum principle for "narrow" sets (Corollary 4.4.9) to v_1 in Ω_-. We deduce that $v_1 \ge 0$ in all $\mathbb{R}^n$ whenever $R(\Omega) \le \overline{R}$. This contradicts the definition of v_1 since we assumed that Ω_- was not empty. Thus, $\Omega_- = \varnothing$ and this yields $v \ge 0$ in Ω. □

Remark 4.4.11. A maximum principle such as Proposition 4.4.10 was already proved for the fractional Laplacian in [58], but with the additional hypothesis that either Ω is bounded or $\liminf_{x\in\Omega,\, |x|\to\infty} v(x) \ge 0$. In the proof of Theorem 3.1 in [123], Quaas and Xia use a suitable argument (the truncation used in the previous proof, previously used by Felmer and Wang in [89]) to avoid the requirement of such additional hypotheses on Ω or v.

With the maximum principle in "narrow" sets for odd functions with respect to a hyperplane we can use the moving plane argument. Now we establish Proposition 4.4.6.

Proof of Proposition 4.4.6. The proof is based on the moving planes method, and is exactly the same as the analogue proof of Theorem 3.1 in [123], where Quaas and Xia establish an equivalent result for the fractional Laplacian. For this reason, we give here just a sketch. As usual, for $\lambda > 0$ we define $w_\lambda(x) = v(x_H, 2\lambda - x_n) - v(x_H, x_n)$ (recall that $x_H \in \mathbb{R}^{n-1}$) and since the nonlinearity is Lipschitz, w_λ solves, in both cases —(P1) or (P2)—, the following problem:

$$\begin{cases} L_K w_\lambda &= c_\lambda(x)\, w_\lambda & \text{in } \Sigma_\lambda \subset H_\lambda, \\ w_\lambda &\geq 0 & \text{in } H_\lambda \setminus \Sigma_\lambda, \\ w_\lambda(x_H, 2\lambda - x_n) &= -w_\lambda(x_H, x_n) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\Sigma_\lambda := \{x = (x_H, x_n) : 0 < x_n < \lambda\}$ and $H_\lambda := \{x = (x_H, x_n) : x_n < \lambda\}$ and c_λ is a bounded function. Note that w_λ is odd with respect to ∂H_λ. Then, using the maximum principle in "narrow" sets for odd functions (Proposition 4.4.10) we deduce that, if λ is small enough, $w_\lambda > 0$ in Σ_λ.

To conclude the proof, we define

$$\lambda^* := \sup\{\lambda \ : \ w_\eta > 0 \ \text{ in } \ \Sigma_\lambda \ \text{ for all } \ \eta < \lambda\}.$$

Note that λ^* is well defined (but may be infinite) by the previous argument. To conclude the proof, one has to show that $\lambda^* = \infty$. This can be done by proving that, if λ^* is finite, then there exists a small $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ we have

$$w_{\lambda^*+\delta}(x) > 0 \quad \text{in } \Sigma_{\lambda^*-\varepsilon} \setminus \Sigma_\varepsilon$$

for some small ε. This can be established using a compactness argument exactly as in Lemma 3.1 of [123] and thus we omit the details. In the argument a Harnack inequality is needed, one can use for instance Proposition 4.4.1. Finally, by the maximum principle in "narrow" sets we deduce that $w_{\lambda^*+\delta}(x) > 0$ in $\Sigma_{\lambda^*+\delta}$ if δ is small enough, contradicting the definition of λ^*. □

Now, we present the other important ingredient needed in the proof of Theorem 4.1.7. It is the following symmetry result.

Proposition 4.4.12. *Let $L_K \in \mathcal{L}_0$ and let v be a bounded solution to one of the following problems:*

$$\begin{cases} L_K v &= f(v) & \text{in } \mathbb{R}^n, \\ \lim\limits_{x_n \to \pm\infty} v(x_H, x_n) &= \pm 1 & \text{uniformly}. \end{cases} \tag{P3}$$

$$\begin{cases} L_K v &= f(v) & \text{in } \mathbb{R}^n_+ = \{x_n > 0\}, \\ v &= 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+ = \{x_n \leq 0\}, \\ \lim\limits_{x_n \to +\infty} v(x_H, x_n) &= 1 & \text{uniformly}. \end{cases} \tag{P4}$$

Assume that there exists a $\delta > 0$ such that

$$f' \leq 0 \quad \text{in} \quad [-1, -1+\delta] \cup [1-\delta, 1],$$

for problem (P3) and

$$f' \leq 0 \quad \text{in} \quad [1-\delta, 1]$$

for problem (P4).

Then, v depends only on x_n and is increasing in that direction.

Proof. It is based on the sliding method, exactly as in the proof of Theorem 1 in [14]. The idea is, as usual, to define $v^\tau(x) := v(x+\nu\tau)$ for every $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ and $\nu_n > 0$, and the aim is to show that $v^\tau(x) - v(x) \geq 0$ for all $\tau \geq 0$. Despite the fact that L_K is a nonlocal operator, the proof is exactly the same as the one in [14] —it only relies on the maximum principle, the translation invariance of the operator and the Lioville type result of Theorem 4.1.6. Therefore, we do not include here the details. □

Finally, we can proceed with the proof of Theorem 4.1.7.

Proof of Theorem 4.1.7. Note that by Proposition 4.4.12 we only need to prove that

$$\lim_{x_n \to +\infty} v(x_H, x_n) = 1$$

uniformly. Therefore we divide the proof in two steps: first, we prove that the limit exists and is 1, and then we prove that it is uniform.

Step 1: Given $x_H \in \mathbb{R}^{n-1}$, then $\lim_{x_n \to +\infty} v(x_H, x_n) = 1$.

By Proposition 4.4.6 we know that v is strictly increasing in the direction x_n. Since v is also bounded by hypothesis, we know that, given $x_H \in \mathbb{R}^{n-1}$, the one variable function $v(x_H, \cdot)$ has a limit as $x_n \to +\infty$, which we call $\overline{v}(x_H)$. Note that, since $v(x_H, 0) = 0$ and $v_{x_n} > 0$, it follows that $\overline{v}(x_H) > 0$.

Let x_n^k be any increasing sequence tending to infinity. Define $v_k(x_H, x_n) := v(x_H, x_n + x_n^k)$. By the regularity theory of the operator L_K (see Section 4.2) and a standard compactness argument, we see that, up to a subsequence, v_k converge uniformly on compact sets to a function v_∞ which is a classical solution to

$$\left\{ \begin{array}{rcll} L_K v_\infty & = & f(v_\infty) & \text{in } \mathbb{R}^n, \\ v_\infty & \geq & 0 & \text{in } \mathbb{R}^n. \end{array} \right. \tag{4.4.8}$$

By Theorem 4.1.6, either $v_\infty \equiv 0$ or $v_\infty \equiv 1$. But, by construction,

$$v_\infty(x_H, 0) = \lim_{k \to +\infty} v_k(x_H, 0) = \lim_{k \to +\infty} v(x_H, x_n^k) = \overline{v}(x_H) > 0,$$

and therefore the only possibility is

$$\lim_{x_n \to \infty} v(x_H, x_n) = 1 \quad \text{for all } \; x_H \in \mathbb{R}^{n-1}.$$

Step 2: The limit is uniform in x_H.

Let us proceed by contradiction. Suppose that the limit is not uniform. This means that given any $\varepsilon > 0$ small enough, there exists a sequence of points (x_H^k, x_n^k) with $x_n^k \to +\infty$ such that $v(x_H^k, x_n^k) = 1 - \varepsilon$. Similarly as before, the sequence of functions $\tilde{v}_k(x_H, x_n) = v(x_H + x_H^k, x_n + x_n^k)$ converge uniformly on compact sets to a function $\tilde{v}_\infty$ that also solves (4.4.8). By Theorem 4.1.6, either $\tilde{v}_\infty \equiv 0$ or $\tilde{v}_\infty \equiv 1$. But, by construction

$$\tilde{v}_\infty(0,0) = \lim_{k \to +\infty} \tilde{v}_k(0,0) = \lim_{k \to +\infty} v(x_H^k, x_n^k) = 1 - \varepsilon,$$

which is a contradiction for $\varepsilon > 0$ small enough. Thus, the limit is uniform.

Finally, by applying Proposition 4.4.12, we get that v depends only on x_n and is increasing in that direction. □

4.5 Asymptotic behavior of a saddle-shaped solution

In this section, we establish Theorem 4.1.4, concerning the asymptotic behavior of the saddle-shaped solution.

In order to study this behavior, it is important to relate the Allen-Cahn equation in $\mathbb{R}^{2m}$ with the same equation in $\mathbb{R}$. In the local case, this is very easy, since if v is a solution to $-\ddot{v} = f(v)$ in $\mathbb{R}$, then $w(x) = v(x \cdot e)$ solves $-\Delta w = f(w)$ in $\mathbb{R}^n$ for every unitary vector $e \in \mathbb{R}^n$. The same fact also happens for the fractional Laplacian, that is, if v is a solution to $(-\Delta)^\gamma v = f(v)$ in $\mathbb{R}$, then $w(x) = v(x \cdot e)$ solves the same equation in $\mathbb{R}^n$. We can easily see this relation via the local extension problem.

Nevertheless, for a general operator L_K this is not true anymore and we need a way to relate a solution to a one-dimensional problem with a one-dimensional solution to a n-dimensional problem. This is given in the next result. Some of its points appear in [67] with a different notation but we state and prove them here for completeness.

Proposition 4.5.1. *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ be a symmetric and translation invariant integro-differential operator of the form (4.1.2) with kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$. Define the one dimensional kernel $K_1 : \mathbb{R} \setminus \{0\} \to (0, +\infty)$ by*

$$K_1(\tau) := \int_{\mathbb{R}^{n-1}} K(\theta, \tau) \, d\theta = |\tau|^{n-1} \int_{\mathbb{R}^{n-1}} K(\tau\sigma, \tau) \, d\sigma. \tag{4.5.1}$$

(i) Let $v : \mathbb{R} \to \mathbb{R}$ and consider $w : \mathbb{R}^n \to \mathbb{R}$ defined by $w(x) = v(x_n)$. Then, $L_K w(x) = L_{K_1} v(x_n)$. If we assume moreover that K is radially symmetric, then the same happens with $w(x) = v(x \cdot e)$ for every unitary vector $e \in \mathbb{S}^{n-1}$. That is, $L_K w(x) = L_{K_1} v(x \cdot e)$.

(ii) If K is nonincreasing/decreasing in the x_n-direction in $\{x_n > 0\}$, then $K_1(\tau)$ is nonincreasing/decreasing in $(0, +\infty)$.

(iii) $L_{K_1} \in \mathcal{L}_0(1, \gamma, \lambda, \Lambda)$, and moreover, if L_K is the fractional Laplacian in dimension n, then L_{K_1} is the fractional Laplacian in dimension 1.

Proof. We start proving point (i). We write $y = (y_H, y_n)$, with $y_H \in \mathbb{R}^{n-1}$.

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}^n} \{w(x) - w(y)\} K(x-y) \, dy \\ &= \int_{\mathbb{R}^n} \{v(x_n) - v(y_n)\} K(x_H - y_H, x_n - y_n) \, dy_H \, dy_n. \end{aligned}$$

Now we make the change of variables $\theta = x_H - y_H$. That is,

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} \int_{\mathbb{R}^{n-1}} K(\theta, x_n - y_n) \, d\theta \, dy_n \\ &= \int_{\mathbb{R}} \{v(x_n) - v(y_n)\} K_1(x_n - y_n) \, dy_n = L_{K_1} v(x_n). \end{aligned}$$

This shows the first equality in (4.5.1). The alternative expression of the kernel K_1, that is useful in some cases, can be obtained from the change of variables $\theta = \tau\sigma$. Furthermore, in the case of K radially symmetric, the result is valid for $u(x) = v(x \cdot e)$ for every unitary vector $e \in \mathbb{S}^{n-1}$ after a change of variables in the previous computations.

The proof of point (ii) follows directly from the first expression of the unidimensional kernel K_1. That is,

$$K_1(\tau_2) - K_1(\tau_1) = \int_{\mathbb{R}^{n-1}} \{K(\theta,\tau_2) - K(\theta,\tau_1)\}\, d\theta \geq 0 \quad \text{for any} \quad \tau_2 > \tau_1 > 0.$$

We establish now point (iii). To do it, we bound the kernel K_1 using the ellipticity condition on K:

$$\begin{aligned}
K_1(\tau) &= |\tau|^{n-1} \int_{\mathbb{R}^{n-1}} K\left(\tau(\sigma,1)\right) d\sigma \geq |\tau|^{n-1} \int_{\mathbb{R}^{n}} c_{n,\gamma} \frac{\lambda}{|\tau|^{n+2\gamma}(|\sigma|^2+1)^{\frac{n+2s}{2}}}\, d\sigma \\
&= c_{n,\gamma} \frac{\lambda}{|\tau|^{1+2\gamma}} \int_{\mathbb{R}^{n-1}} \frac{d\sigma}{(|\sigma|^2+1)^{\frac{n+2\gamma}{2}}} = c_{n,\gamma} \frac{\lambda}{|\tau|^{1+2\gamma}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty \frac{r^{n-2}}{(r^2+1)^{\frac{n+2\gamma}{2}}}\, dr \\
&= c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{n}{2}+\gamma)} = c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{c_{1,\gamma}}{c_{n,\gamma}} = c_{1,\gamma} \frac{\lambda}{|t|^{1+2\gamma}},
\end{aligned}$$

where we have used the explicit value of the normalizing constant for the fractional Laplacian,

$$c_{n,\gamma} = \gamma \frac{2^{2\gamma}\Gamma(\frac{n}{2}+\gamma)}{\pi^{n/2}\Gamma(1-\gamma)}, \tag{4.5.2}$$

and the definition of the Beta and Gamma functions. The upper bound for K_1 is obtained in the same way. Note that the previous computation is an equality with $\lambda = 1$ in the case of the fractional Laplacian. □

In the proof of Theorem 4.1.4 we will use some properties of the layer solution, which are presented next. First, in [67] it is proved that there exists a constant C such that

$$|u_0(x) - \operatorname{sign}(x)| \leq C|x|^{-2\gamma} \quad \text{and} \quad |\dot{u}_0(x)| \leq C|x|^{-1-2\gamma} \quad \text{for large } |x|. \tag{4.5.3}$$

In our arguments we need also to show that the second derivative of the layer goes to zero at infinity. This is the first statement of the following lemma.

Lemma 4.5.2. *Let $K_1 : \mathbb{R}\setminus\{0\} \to (0,+\infty)$ be a symmetric kernel satisfying (4.1.4) and assume that it is decreasing in $(0,+\infty)$. Let u_0 be the layer solution associated to the kernel K_1, that is, u_0 solving (4.1.11). Then,*

(i) $\ddot{u}_0(x) \to 0$ *as* $x \to \pm\infty$.

(ii) $\ddot{u}_0(x) < 0$ *in* $(0,+\infty)$.

We prove here the first statement of this lemma, and we postpone the proof of the second one until the next section, since we need to use a maximum principle for the linearized operator $L_{K_1} - f'(u_0)$.

Proof of point (i) of Lemma 4.5.2. By contradiction, suppose that there exists an unbounded sequence $\{x_j\}$ satisfying $|\ddot{u}_0(x_j)| > \varepsilon$ for some $\varepsilon > 0$. Note that by the symmetry of u_0 we may assume that $x_j \to +\infty$. Now define $w_j(x) := \ddot{u}_0(x + x_j)$. By differentiating twice the equation of the layer solution, we see that $\ddot{u}_0$ solves

$$L_{K_1}\ddot{u}_0 = f''(u_0)\dot{u}_0^2 + f'(u_0)\ddot{u}_0 \quad \text{in } \mathbb{R}.$$

Hence, as $x_j \to +\infty$ a standard compactness argument combined with the asymptotic behavior given by (4.5.3) yields that w_j converges on compact sets to a function w that solves

$$L_{K_1} w = f'(1) w \quad \text{in } \mathbb{R}.$$

In addition, since $|\ddot{u}_0(x_j)| > \varepsilon$ we have $|w(0)| \geq \varepsilon$.

At this point we use Lemma 4.3 of [67] to deduce that, since $f'(1) < 1$, then $w \to 0$ as $|x| \to +\infty$. Therefore, if w is not identically zero, it has either a positive maximum or a negative minimum, but this contradicts the maximum principle (recall that $f'(1) < 1$). We conclude that $w \equiv 0$ in $\mathbb{R}$, but this is a contradiction with $|w(0)| \geq \varepsilon$. □

Now we have all the ingredients to establish the asymptotic behavior of the saddle-solution. The proof follows exactly the same compactness arguments used to prove the analogous result in the local case (see [44]) and for the fractional Laplacian using the extension problem (see [60, 61]). Thus we will omit some details. The main ingredients too establish this results are the translation invariance of the operator, the Liouville type and symmetry results of Theorems 4.1.6 and 4.1.7 and a stability argument (recall the comments in Section 4.2).

Proof of Theorem 4.1.4. By contradiction, assume that the result does not hold. Then, there exists an $\varepsilon > 0$ and an unbounded sequence $\{x_k\}$, such that

$$|u(x_k) - U(x_k)| + |\nabla u(x_k) - \nabla U(x_k)| + |D^2 u(x_k) - D^2 U(x_k)| > \varepsilon. \tag{4.5.4}$$

By the symmetry of u, we may assume without loss of generality that $x_k \in \overline{\mathcal{O}}$, and by continuity we can further assume $x_k \notin \mathscr{C}$.

Let $d_k := \operatorname{dist}(x_k, \mathscr{C})$. We distinguish two cases:

Case 1: $\{d_k\}$ is an unbounded sequence. In this situation, we may assume that $d_k \geq 2k$. Define

$$w_k(x) := u(x + x_k),$$

which satisfies $0 < w_k < 1$ in $\overline{B_k}$ and

$$L_K w_k = f(w_k) \ \text{ in } B_k.$$

By letting $k \to +\infty$, by the uniform estimates for the operators of the class $\mathcal{L}_0$ and the Arzelà-Ascoli theorem, we have that, up to a subsequence, w_k converges on compact sets to a function w which is a pointwise solution to

$$\begin{cases} L_K w = f(w) & \text{in } \mathbb{R}^n, \\ w \geq 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Then, by Theorem 4.1.6, either $w \equiv 0$ or $w \equiv 1$. First, note that w cannot be zero. Indeed, since w_k are stable with respect to perturbations supported in B_k (see the comments in Section 4.2), w is stable in $\mathbb{R}^n$, which means that the linearized operator $L_K - f'(w)$ is a positive operator. Nevertheless, if $w \equiv 0$, then the linearized operator $L_K - f'(w) = L_K - f'(0)$ is negative for sufficiently large balls, since $f'(0) > 0$ and the first eigenvalue of L_K is of order $R^{-2\gamma}$ in balls of radius R (as in Lemma 4.3.2, see Proposition 9 of [140]). Therefore $w \equiv 1$.

On the other hand, since $d_k \to +\infty$ and $U(x_k) = u_0(d_k)$, we get by the properties of the layer solution that $U(x_k) \to 1$, $\nabla U(x_k) \to 0$ and $D^2 U(x_k) \to 0$ —see (4.5.3) and Lemma 4.5.2. From this and condition (4.5.4) we get

$$|u(x_k) - 1| + |\nabla u(x_k)| + |D^2 u(x_k)| > \varepsilon/2,$$

for k big enough. This yields that

$$|w_k(0) - 1| + |\nabla w_k(0)| + |D^2 w_k(0)| > \varepsilon/2,$$

and this contradicts $w \equiv 1$.

Case 2: $\{d_k\}$ is a bounded sequence. In this situation, at least for a subsequence, we have that $d_k \to d$. Now, for each x_k we define x_k^0 as its projection on $\mathscr{C}$. Therefore, we have that $\nu_k^0 := (x_k - x_k^0)/d_k$ is the unit normal to $\mathscr{C}$. Through a subsequence, $\nu_k^0 \to \nu$ with $|\nu| = 1$.

We define

$$w_k(x) := u(x + x_k^0),$$

which solves

$$L_K w_k = f(w_k) \text{ in } \mathbb{R}^n.$$

Similarly as before, by letting $k \to +\infty$, up to a subsequence w_k converges on compact sets to a function w which is a pointwise solution to

$$\begin{cases} L_K w &= f(w) \quad \text{in } H := \{x \cdot \nu > 0\}\,, \\ \quad w &\geq 0 \qquad \text{in } H\,, \\ w \text{ is odd with respect to } H. \end{cases}$$

For the details about the fact that $\mathcal{O} + x_k^0 \to H$, see [43].

As in the previous case, by stability w cannot be zero, and thus $w > 0$ in H (by the strong maximum principle for odd functions with respect to a hyperplane, see [58]). Hence, by Theorem 4.1.7, w only depends on $x \cdot \nu$ and is increasing. Finally, by the uniqueness of the layer solution, $w(x) = u_0(x \cdot \nu)$ and

$$\begin{aligned} u(x_k) &= w_k(x_k - x_k^0) = w(x_k - x_k^0) + \mathrm{o}(1) \\ &= u_0((x_k - x_k^0) \cdot \nu) + \mathrm{o}(1) = u_0((x_k - x_k^0) \cdot \nu_k^0) + \mathrm{o}(1) \\ &= u_0(d_k |\nu_k^0|^2) + \mathrm{o}(1) = u_0(d_k) + \mathrm{o}(1) = U(x_k) + \mathrm{o}(1), \end{aligned}$$

contradicting (4.5.4). The same is done for ∇u and $D^2 u$. □

Remark 4.5.3. The previous result yields that, for $\varepsilon > 0$ the saddle-shaped solution satisfies $u \geq \delta$ in the set $\mathcal{O}_\varepsilon := \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| + \varepsilon < |x'|\}$, for some positive constant δ. That is, thanks to the asymptotic result, and since $U(x) \geq u_0(\varepsilon/\sqrt{2})$ for $x \in \mathcal{O}_\varepsilon$, there exists a radius $R > 0$ such that $u(x) \geq U(x)/2 \geq u_0(\varepsilon/\sqrt{2})/2$ if $x \in \mathcal{O}_\varepsilon \setminus B_R$. Moreover, since u is positive in the compact set $\overline{\mathcal{O}_\varepsilon} \cap \overline{B_R}$ it has a positive minimum in this set, say $m > 0$. Therefore, if we choose $\delta = \min\{m, u_0(\varepsilon/\sqrt{2})/2\}$ we obtain the desired result.

4.6 Maximum principles for the linearized operator and uniqueness of the saddle-shaped solution

In this section we show that the linearized operator $L_K - f'(u)$ satisfies the maximum principle in $\mathcal{O}$. This combined with the asymptotic result of Theorem 4.1.4 yields the uniqueness of the saddle-shaped solution.

In order to prove the maximum principle of Proposition 4.1.5, we need a maximum principle in "narrow" sets, stated next.

Proposition 4.6.1. *Let $\varepsilon > 0$ and let*

$$\mathcal{N}_\varepsilon \subset \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| < |x'| < |x''| + \varepsilon\} \subset \mathcal{O}$$

be an open set (not necessarily bounded). Let K be a radially symmetric kernel satisfying the positivity condition (4.1.9) and such that $L_K \in \mathcal{L}_0$. Let $v \in C^\gamma(\overline{\mathcal{N}_\varepsilon}) \cap C^\alpha(\mathcal{N}_\varepsilon) \cap L^1_\gamma(\mathbb{R}^{2m})$, for some $\alpha > 2\gamma$, be a doubly radial function satisfying

$$\begin{cases} L_K v + c(x) v & \leq & 0 & \text{in } \mathcal{N}_\varepsilon\,, \\ v & \leq & 0 & \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x \in \mathcal{N}_\varepsilon,\ |x| \to \infty} v(x) & \leq & 0\,, \end{cases} \tag{4.6.1}$$

with c a function bounded by below.

Under these assumptions there exists $\overline{\varepsilon} > 0$ depending only on λ, m, γ and $||c_-||_{L^\infty}$ such that, if $\varepsilon < \overline{\varepsilon}$, then $v \leq 0$ in $\mathcal{N}_\varepsilon$.

Proof. Assume, by contradiction, that

$$M := \sup_{\mathcal{N}_\varepsilon} v > 0\,.$$

Under the assumptions (4.6.1), M must be attained at an interior point $x_0 \in \mathcal{N}_\varepsilon$. Then,

$$0 \geq L_K v(x_0) + c(x_0) v(x_0) \geq L_K v(x_0) - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)}\, M\,. \tag{4.6.2}$$

Now, we compute $L_K v(x_0)$. Since v is doubly radial and odd with respect to the Simons cone, we can use the expression (4.1.6) to write

$$\begin{aligned} L_K v(x_0) &= \int_{\mathcal{O}} (M - v(y)) \left(\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\right) \mathrm{d}y + 2M \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\, \mathrm{d}y \\ &\geq 2M \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\, \mathrm{d}y, \end{aligned}$$

where the inequality follows from being M the supremum of v in $\mathcal{O}$ and the kernel inequality (4.1.9). Combining this last inequality with (4.6.2), we obtain

$$0 \geq L_K v(x_0) + c(x_0) v(x_0) \geq M \left\{ 2 \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\, \mathrm{d}y - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)} \right\}.$$

Finally, if we use the lower bound of the integral term from (4.1.7) and the fact that $\operatorname{dist}(x_0, \mathscr{C}) \leq \varepsilon/\sqrt{2}$, we get

$$\begin{aligned} 0 &\geq M \left\{ 2 \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\, \mathrm{d}y - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)} \right\} \geq M \left(\frac{1}{C} \operatorname{dist}(x_0, \mathscr{C})^{-2\gamma} - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)} \right) \\ &\geq M \left(\frac{1}{C} \varepsilon^{-2\gamma} - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)} \right). \end{aligned}$$

Therefore, for ε small enough, we arrive at a contradiction that follows from assuming that the supremum is positive. $\square$

Remark 4.6.2. Proposition 4.6.1 can be extended to general doubly radial "narrow" sets —in the sense of (4.4.6)— and without requiring any assumption at infinity, just repeating the exact same arguments as in the proof of Proposition 4.4.10. Indeed, we only need to replace symmetry with respect to a hyperplane by symmetry with respect to the Simons cone and use the kernel inequality (4.1.9). Nevertheless, we preferred to present the result for sets that are contained in an ε-neighborhood of the Simons cone, since we are only going to use the maximum principle in such sets. In addition, the crucial fact that the sets are contained in $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| < |x'| < |x''| + \varepsilon\}$ makes the argument rather simple.

Once this maximum principle in "narrow" sets is available, we can proceed with the proof of Proposition 4.1.5.

Proof of Proposition 4.1.5. For the sake of simplicity, we will denote

$$\mathscr{L} w := L_K w - f'(u) w - c w \,.$$

A crucial point in this proof is that u is a positive supersolution of the operator $\mathscr{L}$. Indeed, since f is strictly concave in $(0, 1)$ and $f(0) = 0$, then $f'(\tau)\tau < f(\tau)$ for all $\tau > 0$, and thus

$$\mathscr{L} u = L_K u - f'(u) u - cu \geq f(u) - f'(u) u > 0 \quad \text{in } \Omega \subset \mathcal{O} \,, \tag{4.6.3}$$

where in the first inequality we have used that $u > 0$ in $\mathcal{O}$ and that $c \leq 0$.

By contradiction, assume that there exists $x_0 \in \Omega$ such that $v(x_0) > 0$. We will show next that, if we assume this, we deduce $v \leq 0$ in Ω, arriving at a contradiction.

Let $\varepsilon > 0$ be such that the maximum principle of Proposition 4.6.1 is valid and define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{|x'| > |x''| + \varepsilon\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \Omega \cap \{|x''| < |x'| < |x''| + \varepsilon\} \,.$$

Define also, for $\tau \geq 0$,

$$w := v - \tau u.$$

First, we claim that $w \leq 0$ in Ω if τ is big enough. To see this, note first that by the asymptotic behavior of the saddle-shaped solution, we have

$$u \geq \delta > 0 \quad \text{in } \overline{\Omega_\varepsilon} \,, \tag{4.6.4}$$

for some $\delta > 0$ (see Remark 4.5.3). Therefore, $w < 0$ in $\overline{\Omega_\varepsilon}$ if τ is big enough. Moreover, since $v \leq 0$ in $\mathcal{O} \setminus \Omega$, we have

$$w \leq 0 \quad \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon \,.$$

Furthermore, it also holds

$$\limsup_{x \in \mathcal{N}_\varepsilon, \ |x| \to \infty} w(x) \leq 0$$

and, by (4.6.3),

$$\mathscr{L} w = \mathscr{L} v - \tau \mathscr{L} u \leq 0 \text{ in } \mathcal{N}_\varepsilon \,.$$

Thus, since w is odd with respect to $\mathscr{C}$, we can apply Proposition 4.6.1 in $\mathcal{N}_\varepsilon$ to deduce that

$$w \leq 0 \quad \text{in } \Omega \,,$$

if τ is big enough.

Now, define

$$\tau_0 := \inf\{\tau > 0 \ : \ v - \tau u \leq 0 \ \text{ in } \Omega\}\,.$$

By the previous claim, τ_0 is well defined. Moreover, it is easy to see that $\tau_0 > 0$. Indeed, it is obvious $v - \tau_0 u \leq 0$ in Ω and thus, since $v(x_0) > 0$, we have $-\tau_0 u(x_0) < v(x_0) - \tau_0 u(x_0) \leq 0$. Using that $u(x_0) > 0$, it follows that $\tau_0 > 0$.

We claim that $v - \tau_0 u \not\equiv 0$. Indeed, if $v - \tau_0 u \equiv 0$ then $v = \tau_0 u$ and thus, by using (4.6.3), the equation for v, and the fact that $\tau_0 > 0$, we get

$$0 \geq \mathscr{L} v(x_0) = \tau_0 \mathscr{L} u(x_0) > 0\,,$$

which is a contradiction.

Then, since $v - \tau_0 u \not\equiv 0$, the strong maximum principle for odd functions (see Proposition 4.2.2) yields

$$v - \tau_0 u < 0 \quad \text{ in } \Omega\,.$$

Therefore, by continuity, the assumption on v at infinity and (4.6.4), there exists $0 < \eta < \tau_0$ such that

$$\tilde{w} := v - (\tau_0 - \eta) u < 0 \quad \text{ in } \overline{\Omega}_\varepsilon\,.$$

Note that here we used crucially (4.6.4), and this is the reason for which we needed to introduce the sets Ω_ε and $\mathcal{N}_\varepsilon$. Using again the maximum principle in "narrow" sets with $\tilde{w}$ in $\mathcal{N}_\varepsilon$, we deduce that

$$v - (\tau_0 - \eta) u \leq 0 \quad \text{ in } \Omega\,,$$

and this contradicts the definition of τ_0. Hence, $v \leq 0$ in Ω and, as we said, this contradicts our initial assumption on the existence of a point x_0 where $v(x_0) > 0$. □

Note that if in the previous result we assume that $\partial\Omega \cap \mathscr{C}$ is empty, then Ω is at a positive distance to the cone and the lower bound on u in (4.6.4) holds in Ω. In this case no maximum principle in "narrow" sets is required in the previous argument. Instead, if we want to consider sets with $\partial\Omega \cap \mathscr{C} \neq \varnothing$, we need to introduce the set Ω_ε to have the uniform lower bound (4.6.4) and be able to carry out the proof.

The same argument used in the previous proof can be used to establish the remaining statement of Lemma 4.5.2.

Proof of point (ii) of Lemma 4.5.2. Let $v = \ddot{u}_0$. First we show that $v \leq 0$ in $(0, +\infty)$. To see this, note that since f is concave and by point (i) of Lemma 4.5.2, we have that

$$\begin{cases} L_{K_1} v - f'(u_0) v & \leq & 0 & \text{in } (0, +\infty)\,. \\ v(x) & = & -v(-x) & \text{for every } x \in \mathbb{R}\,, \\ \limsup\limits_{x \to +\infty} v(x) & = & 0\,. & \end{cases}$$

Now, we follow the proof of Proposition 4.1.5 but with the previous problem, replacing u by u_0 and using that

$$L_{K_1} u_0 - f'(u_0) u_0 > 0 \quad \text{ in } (0, +\infty)\,.$$

All the arguments are the same, using the maximum principle of Proposition 4.4.10 in the set $(0, \varepsilon)$, and yield that $v \leq 0$ in $(0, +\infty)$.

The fact that $\dot{u}_0 = v < 0$ in $(0, +\infty)$ can be readily deduced from the strong maximum principle for odd functions in $\mathbb{R}$, as follows. Suppose by contradiction that there exists a point $x_0 \in (0, +\infty)$ such that $v(x_0) = 0$. Then,

$$\begin{aligned} 0 \geq L_{K_1} v(x_0) &= -\int_{-\infty}^{+\infty} v(y) K_1(x_0 - y)\, \mathrm{d}y \\ &= -\int_{-\infty}^{+\infty} v(y) \{K_1(x_0 - y) - K_1(x_0 + y)\}\, \mathrm{d}y > 0\,, \end{aligned}$$

arriving at a contradiction. Here we have used that $v \not\equiv 0$ and the fact that K_1 is decreasing in $(0, +\infty)$, which yields $K_1(x - y) \geq K_1(x + y)$ for every $x > 0$ and $y > 0$. $\square$

With these ingredients available, we can finally establish the uniqueness of the saddle-shaped solution.

Proof of Theorem 4.1.3: Uniqueness. Let u_1 and u_2 be two saddle-shaped solutions. Define $v := u_1 - u_2$ which is a doubly radial function that is odd with respect to $\mathscr{C}$. Then,

$$L_K v = f(u_1) - f(u_2) \leq f'(u_2)(u_1 - u_2) = f'(u_2) v \quad \text{in } \mathcal{O}\,,$$

since f is concave in $(0, 1)$. Moreover, by the asymptotic result (see Theorem 4.1.4), we have

$$\limsup_{x \in \mathcal{O},\ |x| \to \infty} v(x) = 0\,.$$

Then, by the maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u_2)$ (see Proposition 4.1.5), it follows that $v \leq 0$ in $\mathcal{O}$, which means $u_1 \leq u_2$ in $\mathcal{O}$. Repeating the argument with $-v = u_2 - u_1$ we deduce $u_1 \geq u_2$ in $\mathcal{O}$. Therefore, $u_1 = u_2$ in $\mathbb{R}^{2m}$. $\square$

Bibliography

[1] G. Alberti, G. Bouchitté, and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal. **144** (1998), 1–46.

[2] S. M. Allen and J. W. Cahn, *A microscopic theory of domain wall motion and its experimental verification in feal alloy domain growth kinetics*, J. de Physique **38** (1977), C7–51.

[3] ———, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metallurgica **27** (1979), 1085 – 1095.

[4] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), 725–739.

[5] C. J. Amick and J. F. Toland, *Uniqueness and related analytic properties for the Benjamin-Ono equation - a nonlinear Neumann problem in the plane*, Acta Math. **167** (1991), 107–126.

[6] P. Appell, *Sur les fonctions hypergéométriques de plusieurs variables*, Mémoir. Sci. Math. **3** (1925), 1–75.

[7] B. Barrios, L. Del Pezzo, J. García-Melián, and A. Quaas, *Monotonicity of solutions for some nonlocal elliptic problems in half-spaces*, Calc. Var. Partial Differential Equations **56** (2017), Art. 39, 16.

[8] ———, *Symmetry results in the half space for a semi-linear fractional Laplace equation through a one-dimensional analysis*, preprint (2017).

[9] B. Barrios, I. Peral, F. Soria, and E. Valdinoci, *A Widder's type theorem for the heat equation with nonlocal diffusion*, Arch. Ration. Mech. Anal. **213** (2014), 629–650.

[10] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. **29** (1967), no. 3, 559–592.

[11] H. Berestycki, L. Caffarelli, and L. Nirenberg, *Further qualitative properties for elliptic equations inunbounded domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 69–94 (1998), Dedicated to Ennio De Giorgi. MR 1655510

[12] ———, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), 1089–1111.

[13] H. Berestycki, A. C. Coulon, J. M. Roquejoffre, and L. Rossi, *The effect of a line with nonlocal diffusion on Fisher-KPP propagation*, Math. Models Methods Appl. Sci. **25** (2015), no. 13, 2519–2562. MR 3397542

[14] H. Berestycki, F. Hamel, and R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. **103** (2000), 375–396.

[15] H. Berestycki, F. Hamel, and N. Nadirashvili, *The speed of propagation for KPP type problems II: General domains*, J. Amer. Math. Soc. **23** (2010), 1–34.

[16] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), 47–92.

[17] H. Berestycki, J. M. Roquejoffre, and L. Rossi, *The influence of a line with fast diffusion on Fisher-KPP propagation*, J. Math. Biol. **66** (2013), no. 4-5, 743–766. MR 3020920

[18] ______, *The shape of expansion induced by a line with fast diffusion in Fisher-KPP equations*, Comm. Math. Phys. **343** (2016), no. 1, 207–232. MR 3475665

[19] M. Birkner, J. A. López-Mimbela, and A. Wakolbinger, *Comparison results and steady states for the Fujita equation with fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 83–97.

[20] K. Bogdan, T. Grzywny, and M. Ryznar, *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Ann. Probab. **38** (2010), 1901–1923.

[21] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Inv. Math. **7** (1969), 243–268.

[22] J. L. Bona, D. Lannes, and J. C. Saut, *Asymptotic models for internal waves*, J. Math. Pures Appl. (9) **89** (2008), no. 6, 538–566. MR 2424620

[23] H. Brezis, *Is there failure of the inverse function theorem? Morse theory, minimax theory and their applications to nonlinear differential equations*, New Stud. Adv. Math. **1** (2003), 23–33.

[24] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Differential Equations **1** (1996), no. 1, 73–90. MR 1357955

[25] H. Brezis and J. L. Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), 443–469.

[26] A. Buades, B. Coll, and J. M. Morel, *A non-local algorithm for image denoising*, 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05), vol. 2, IEEE, 2005, pp. 60–65.

[27] ______, *A review of image denoising algorithms, with a new one*, Multiscale Modeling & Simulation **4** (2005), no. 2, 490–530.

[28] ______, *Image denoising methods. a new nonlocal principle*, SIAM Review **52** (2010), 113–147.

[29] C. Bucur, *A symmetry result in $\mathbb{R}^2$ for global minimizers of a general type of nonlocal energy*, preprint (2017).

[30] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana, Springer International Publishing, 2016.

[31] X. Cabré, *On the Alexandroff–Bakel'man–Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations.*, Comm. Pure Appl. Math. **48** (1995), 539–570.

[32] ———, *Topics in regularity and qualitative properties of solutions of nonlinear elliptic equations*, Discrete Contin. Dyn. Syst. **8** (2002), 331–359.

[33] ———, *Regularity of minimizers of semilinear elliptic problems up to dimension 4*, Comm. Pure Appl. Math. **63** (2010), 1362–1380.

[34] ———, *Uniqueness and stability of saddle-shaped solutions to the Allen–Cahn equation*, J. Math. Pures Appl. **98** (2012), 239–256.

[35] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. **238** (2006), 709–733.

[36] X. Cabré and E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. **28** (2010), 1179–1206.

[37] ———, *Sharp energy estimates for nonlinear fractional diffusion equations*, Calc. Var. Partial Differential Equations **49** (2014), 233–269.

[38] X. Cabré, E. Cinti, and J. Serra, *Stable s-minimal cones in $\mathbb{R}^3$ are flat for $s \sim 1$*, preprint (2017).

[39] ———, *Stable solutions to the fractional Allen-Cahn equation*, preprint (2018).

[40] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 23–53.

[41] ———, *Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. **367** (2015), 911–941.

[42] X. Cabré and J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. **58** (2005), 1678–1732.

[43] X. Cabré and J. Terra, *Saddle-shaped solutions of bistable diffusion equations in all of $\mathbb{R}^{2m}$*, J. Eur. Math. Soc. **11** (2009), 819–843.

[44] ———, *Qualitative properties of saddle-shaped solutions to bistable diffusion equations*, Comm. Partial Differential Equations **35** (2010), 1923–1957.

[45] X. Cabré and X. Ros-Oton, *Regularity of stable solutions up to dimension 7 in domains of double revolution*, Comm. Partial Differential Equations **38** (2013), 135–154.

[46] X. Cabré and J. Serra, *An extension problem for sums of fractional Laplacians and 1-D symmetry of phase transitions*, Nonlinear Anal. **137** (2016), 246–265. MR 3485125

[47] L. Caffarelli and A. Figalli, *Regularity of solutions to the parabolic fractional obstacle problem*, J. Reine Angew. Math. **680** (2013), 191–233. MR 3100955

[48] L. Caffarelli, J. M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), 1111–1144.

[49] L. Caffarelli, J.-M. Roquejoffre, and Y. Sire, *Variational problems for free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. (JEMS) **12** (2010), 1151–1179. MR 2677613

[50] L. Caffarelli, S. Salsa, and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), 425–461. MR 2367025

[51] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[52] L. Caffarelli and A. Vasseur, *The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics*, Discrete Contin. Dyn. Syst. Ser. S **3** (2010), no. 3, 409–427. MR 2660718

[53] ______, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. (2) **171** (2010), no. 3, 1903–1930. MR 2680400

[54] ______, *The De Giorgi method for nonlocal fluid dynamics*, Nonlinear partial differential equations, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer Basel AG, Basel, 2012, pp. 1–38. MR 3059761

[55] A. Capella, J. Dávila, L. Dupaigne, and Y. Sire, *Regularity of radial extremal solutions for some non-local semilinear equations*, Comm. Partial Differential Equations **36** (2011), 1353–1384.

[56] G. Carbou, *Unicité et minimalité des solutions d'une équation de Ginzburg-Landau*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), 305–318.

[57] H. Chan, Y. Liu, and J. Wei, *A gluing construction for fractional elliptic equations. Part I: a model problem on the catenoid*, preprint (2017).

[58] W. Chen, C. Li, and Y. Li, *A direct method of moving planes for the fractional Laplacian*, Adv. Math. **308** (2017), 404–437.

[59] W. Chen, Y. Li, and R. Zhang, *A direct method of moving spheres on fractional order equations*, J. Funct. Anal. **272** (2017), 4131–4157.

[60] E. Cinti, *Saddle-shaped solutions of bistable elliptic equations involving the half-Laplacian*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), 623–664.

[61] ______, *Saddle-shaped solutions for the fractional Allen-Cahn equation*, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), 441–463.

[62] P. Constantin, *Euler equations, Navier-Stokes equations and turbulence*, Mathematical foundation of turbulent viscous flows, Lecture Notes in Math., vol. 1871, Springer, Berlin, 2006, pp. 1–43. MR 2196360

[63] R. Cont and P. Tankov, *Financial modelling with jump processes*, Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[64] M. Cozzi, *Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: A unified approach via fractional De Giorgi classes*, J. Funct. Anal. (2017), no. 11, 4762–4837.

[65] ———, *Fractional De Giorgi classes and applications to nonlocal regularity theory*, preprint (2018).

[66] M. Cozzi and A. Figalli, *Regularity theory for local and nonlocal minimal surfaces: an overview*, Nonlocal and nonlinear diffusions and interactions: new methods and directions, Lecture Notes in Math., vol. 2186, Springer, Cham, 2017, pp. 117–158. MR 3588123

[67] M. Cozzi and T. Passalacqua, *One-dimensional solutions of non-local Allen-Cahn-type equations with rough kernels*, J. Differential Equations **260** (2016), 6638–6696.

[68] M. G. Crandall and P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. **58** (1975), 207–218.

[69] H. Dang, P. C. Fife, and L. A. Peletier, *Saddle solutions of the bistable diffusion equation*, Z. Angew. Math. Phys. **43** (1992), 984–998.

[70] J. Dávila, M. del Pino, and J. Wei, *Nonlocal s-minimal surfaces and Lawson cones*, J. Differential Geom. **109** (2018), 111–175.

[71] E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61, Editrice Tecnico Scientifica, Pisa, 1961. MR 0179651

[72] ———, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa (3) **19** (1965), 79–85. MR 0178385

[73] ———, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, pp. 131–188. MR 533166

[74] M. del Pino, M. Kowalczyk, and J. Wei, *On De Giorgi's conjecture in dimension $N \geq 9$*, Ann. of Math. **174** (2011), 1485–1569.

[75] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[76] S. Dipierro, A. Farina, and E. Valdinoci, *A three-dimensional symmetry result for a phase transition equation in the genuinely nonlocal regime*, Calc. Var. Partial Differential Equations **57** (2018).

[77] S. Dipierro, G. Palatucci, and E. Valdinoci, *Dislocation Dynamics in Crystals: A Macroscopic Theory in a Fractional Laplace Setting*, Comm. Math. Phys. **333** (2015), 1061–1105.

[78] S. Dipierro, J. Serra, and E. Valdinoci, *Improvement of flatness for nonlocal phase transitions*, preprint (2017).

[79] S. Dipierro and E. Valdinoci, *Nonlocal minimal surfaces: interior regularity, quantitative estimates and boundary stickiness*, Recent developments in nonlocal theory, De Gruyter, Berlin, 2018, pp. 165–209. MR 3824212

[80] M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in $\mathbb{R}^3$ are planes*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 6, 903–906. MR 546314

[81] L. Dupaigne, *Stable solutions of elliptic partial differential equations*, Chapman and Hall/CRC, 2011.

[82] L. C. Evans, *Partial Differential Equations: Second Edition*, 2nd ed., Graduate Studies in Mathematics, AMS, 2010.

[83] M. Fall and T. Weth, *Monotonicity and nonexistence results for some fractional elliptic problems in the half-space*, Commun. Contemp. Math. **18** (2016), 1550012, 25.

[84] A. Farina and E. Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems*, Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 74–96.

[85] M. Fazly and Y. Sire, *Symmetry properties for solutions of nonlocal equations involving nonlinear operators*, preprint (2018).

[86] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone*, preprint (2018).

[87] ______, *Semilinear integro-differential equations, II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation*, preprint (2018).

[88] ______, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, preprint (2018).

[89] P. Felmer and Y. Wang, *Radial symmetry of positive solutions to equations involving the fractional Laplacian*, Commun. Contemp. Math. **16** (2014), 1350023, 24.

[90] A. Figalli and J. Serra, *On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4+1*, preprint (2017).

[91] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211. MR 562550

[92] R. L. Frank and E. Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$*, Acta Math. **210** (2013), 261–318.

[93] R. L. Frank, E. Lenzmann, and L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. **69** (2016), no. 9, 1671–1726. MR 3530361

[94] R. L. Frank, E. H. Lieb, and R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. **21** (2008), no. 4, 925–950. MR 2425175

[95] I. M. Gel'fand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. (2) **29** (1963), 295–381. MR 0153960

[96] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (1998), 481–491.

[97] G. W. Gibbons and P. K. Townsend, *Bogomol'nyi equation for inter-secting domain walls*, Phys. Rev. Lett. **83** (1999), 1727–1730.

[98] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer Berlin, New York, 2001.

[99] G. Gilboa and S. Osher, *Nonlocal operators with applications to image processing*, Multiscale Model. Simul. **7** (2008), 1005–1028. MR 2480109

[100] V. L. Ginzburg and L. D. Landau, *Towards the Superconductivity Theory*, J. Exp. Theor. Phys. **20** (1950), 1064.

[101] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhauser Verlag, 1984.

[102] M.d.M González, *Gamma convergence of an energy functional related to the fractional Laplacian*, Calc. Var. Partial Differential Equations **36** (2009), 173–210.

[103] F. Hamel, X. Ros-Oton, Y. Sire, and E. Valdinoci, *A one-dimensional symmetry result for a class of nonlocal semilinear equations in the plane*, Ann. Institut H. Poincaré **34** (2017), 469–482.

[104] N. E. Humphries, N. Queiroz, J. R. Dyer, N. G. Pade, M. K. Musyl, K. M. Schaefer, D. W. Fuller, J. M. Brunnschweiler, T. K. Doyle, J. D. Houghton, G. C. Hays, C. S. Jones, L. R. Noble, V. J. Wearmouth, E. J. Southall, and D. W. Sims, *Environmental context explains Lévy and Brownian movement patterns of marine predators*, Nature **465** (2010), 1066–1069.

[105] S. Jarohs and T. Weth, *Symmetry via antisymmetric maximum principles in nonlocal problems of variable order*, Ann. Mat. Pura Appl. (4) **195** (2016), 273–291.

[106] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions*, Ann. Mat. Pura Appl. (4) **183** (2004), 439–467.

[107] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1973), 241–269.

[108] V. Katkovnik, A. Foi, K. Egiazarian, and J Astola, *From local kernel to nonlocal multiple-model image denoising*, Int. J. Comput. Vis. **86** (2010), no. 1, 1–32. MR 2683762

[109] O. A. Ladyẑenskaja, *New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems*, Trudy Mat. Inst. Steklov. **102** (1967), 85–104.

[110] G. Lapeyre, *Surface quasi-geostrophy*, Fluids **2** (2017), 7.

[111] S. Levendorski, *Pricing of the american put under lévy processes*, Int. J.Theor. Appl. Finance **7** (2004), 303–335.

[112] Y. Li and L. Zhang, *Liouville-type theorems and harnack-type inequalities for semilinear elliptic equations*, J. Anal. Math. **90** (2003), 27–87.

[113] Y. Liu, K. Wang, and J. Wei, *Global minimizers of the Allen-Cahn equation in dimension $n \geq 8$*, J. Math. Pures Appl. (9) **108** (2017), no. 6, 818–840. MR 3723158

[114] G. Lu, *The peierls—nabarro model of dislocations: A venerable theory and its current development*, pp. 793–811, Springer Netherlands, Dordrecht, 2005.

[115] A. Massaccesi and E. Valdinoci, *Is a nonlocal diffusion strategy convenient for biological populations in competition?*, J. Math. Biol. **74** (2017), no. 1-2, 113–147. MR 3590678

[116] R. Merton, *Option pricing when underlying stock returns are discontinuous*, J. Finan. Econ. **3** (1976), 125 – 144.

[117] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), 1 – 77.

[118] L. Modica, *Γ-convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u^3 - u)$*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, pp. 223–244.

[119] L. Modica and S. Mortola, *Un esempio di Γ^--convergenza*, Boll. Un. Mat. Ital. B (5) **14** (1977), 285–299.

[120] L. Nachbin, *The Haar integral*, D. Van Nostrand Co., Inc., 1965.

[121] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 997–1002.

[122] ______, *Extremal solutions of semilinear elliptic equations*, (2001), Unpublished preprint.

[123] A. Quaas and A. Xia, *Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in the half space*, Calc. Var. Partial Differential Equations **52** (2015), 641–659.

[124] X. Ros-Oton, *Regularity for the fractional Gelfand problem up to dimension 7*, J. Math. Anal. Appl. **419** (2014), 10–19.

[125] ______, *Nonlocal elliptic equations in bounded domains: a survey*, Publ. Mat. **60** (2016), 3–26.

[126] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), 275–302.

[127] ______, *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Differential Equations **50** (2014), 723–750.

[128] ______, *The Pohozaev identity for the fractional Laplacian*, Arch. Ration. Mech. Anal. **213** (2014), 587–628.

[129] ______, *Regularity theory for general stable operators*, J. Differential Equations **260** (2016), 8675–8715.

[130] M. Sanchón, *Boundedness of the extremal solution of some p-Laplacian problems*, Nonlinear Anal. **67** (2007), 281–294.

[131] T. Sanz-Perela, *Regularity of radial stable solutions to semilinear elliptic equations for the fractional Laplacian*, Commun. Pure Appl. Anal. **17** (2018), 2547–2575.

[132] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. **169** (2009), 41–78.

[133] ———, *Rigidity of minimizers in nonlocal phase transitions*, Anal. PDE **11** (2018), 1881–1900.

[134] ———, *Rigidity of minimizers in nonlocal phase transitions II*, preprint (2018).

[135] O. Savin and E. Valdinoci, *Γ-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 479–500.

[136] ———, *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations **48** (2013), 33–39.

[137] ———, *Density estimates for a variational model driven by the Gagliardo norm*, J. Math. Pures Appl. **101** (2014), 1–26.

[138] M. Schatzman, *On the stability of the saddle solution of Allen–Cahn's equation*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 1241–1275.

[139] J. Serra, *$C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels*, Calc. Var. Partial Differential Equations **54** (2015), 3571–3601.

[140] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), 2105–2137.

[141] ———, *On the spectrum of two different fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), 831–855.

[142] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. MR 0233295

[143] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), 1842–1864.

[144] P. R. Stinga and J. L. Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), 2092–2122.

[145] J. Tan, *Positive solutions for non local elliptic problems*, Discrete Contin. Dyn. Syst. **33** (2013), 837–859.

[146] J. Tan and J. Xiong, *A Harnack inequality for fractional Laplace equations with lower order terms*, Discrete Contin. Dyn. Syst. **31** (2011), 975–983.

[147] J. F. Toland, *The Peierls-Nabarro and Benjamin-Ono equations*, J. Funct. Anal. **145** (1997), no. 1, 136–150. MR 1442163

[148] V. Vasan and B. Deconinck, *The Bernoulli boundary condition for traveling water waves*, Appl. Math. Lett. **26** (2013), no. 4, 515–519. MR 3019985

[149] S. Villegas, *Boundedness of extremal solutions in dimension 4*, Adv. Math. **235** (2013), 126–133.

[150] G. M. Viswanathan, E. P. Raposo, and M. G. E. da Luz, *Lévy flights and superdiffusion in the context of biological encounters and random searches*, Physics of Life Reviews **5** (2008), 133–150.