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Pricing Variance Swaps in a Hybrid Model of Stochastic Volatility and Interest Rate with Regime-Switching

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Abstract In this paper, we consider the problem of pricing discretely-sampled variance swaps based on a hybrid model of stochastic volatility and stochastic interest rate with regime-switching. Our modeling framework extends the Heston stochastic volatility model by including the Cox-Ingersoll-Ross (CIR) stochastic interest rate model. In addition, certain model parameters in our model switch according to a continuous-time observable Markov chain process. This enables our model to capture several macroeconomic issues such as alternating business cycles. A semi-closed form pricing formula for variance swaps is derived. The pricing formula is assessed through numerical implementation, where we validate our pricing formula against the Monte Carlo simulation. The impact of incorporating regime-switching for pricing variance swaps is also discussed, where variance swaps prices with and without regime-switching effects are examined in our model. We also explore the economic consequence for the prices of variance swaps by allowing the Heston-CIR model to switch across three different regimes.

Keywords Heston-CIR hybrid model · Regime-switching · Realized variance · Stochastic interest rate · Stochastic volatility · Variance swap

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1 Introduction

A *variance swap* is a forward contract on the future realized variance of returns of a specified asset. At maturity time $T > 0$, the variance swap rate $V(T)$ can be evaluated as

$$V(T) = (RV - K) \times L, \tag{1}$$

where K is the annualized delivery or strike price for the swap, RV is the realized variance of the swap and L is the notional amount of the swap in dollars. In the literature, typical formulae for measuring RV are

$$RV_{d1} = \frac{AF}{N} \sum_{j=1}^{N} \left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})} \right)^2 \times 100^2 \tag{2}$$

and

$$RV_{d2} = \frac{AF}{N} \sum_{j=1}^{N} \ln^2 \left(\frac{S(t_j)}{S(t_{j-1})} \right) \times 100^2, \tag{3}$$

where $S(t_j)$ is the closing price of the underlying asset at the j-th observation time t_j and N is the number of observations. The annualized factor AF follows the sampling frequency to convert the above evaluation to annualized variance points. Assuming there are 252 business days in a year, then AF is equal to 252 for daily sampling frequency. However, if the sampling frequency is monthly or weekly, then AF will be 12 or 52, respectively. The measure of realized variance requires monitoring the underlying price path discretely, usually at the end of a business day. For this purpose, we assume equally discrete observations to be compatible with the real market, which reduces to $AF = \frac{1}{\Delta t} = \frac{N}{T}$. The long position of variance swaps pays a fixed delivery price K at the expiration and receives the floating amounts of annualized realized variance, whereas the short position is the opposite. In the sequel, for convenience, we shall call RV_{d1} the *simple-return realized variance* and RV_{d2} the *log-return realized variance*.

Since variance swaps were first launched in 1998, the problem of how to price them has been an active research topic in mathematical and quantitative finance. Carr and Madan (1998) combined static replication using options with dynamic trading in the futures to price and hedge variance swaps without specifying the volatility process. Demeterfi et al. (1999) worked in the same direction by proving that a variance swap could be reproduced via a portfolio of standard options. A finite-difference method via dimension-reduction approach was explored in Little and Pant (2001) to obtain high efficiency and accuracy for pricing discretely-sampled variance swaps. In Zhu and Lian (2011, 2012), Zhu and Lian extended the work in Little and Pant (2001) by incorporating Heston two-factor stochastic volatility for pricing discretely-sampled variance swaps. Later, Rujivan and Zhu (2012) applied the Schwarts solution procedure to derive a simpler analytic approach for pricing discretely-sampled variance swaps with stochastic volatility. In Elliott et al. (2007), Elliott et al. proposed a continuous-time Markovian-regulated version of the Heston stochastic volatility model to distinguish different states of a business cycle. An analytical formula for pricing volatility swaps was obtained using the regime-switching Esscher transform and comparisons were made between models with and without switching regimes. The essence of incorporating regime-switching for pricing variance swaps under the Heston stochastic volatility model was illustrated in Elliott and Lian (2013) and Elliott et al. (2007), where a common assumption is “continuous sampling time”. It was shown that incorporating regime-switching into the Heston model had a significant impact on the price of volatility swaps. Note that in practice, a variance swap is written in the realized variance of a specified