Limiting embeddings, entropy numbers and envelopes in function spaces

Habilitationsschrift

vorgelegt am 9. Januar 2002

der Fakultät für Mathematik und Informatik

der Friedrich-Schiller-Universität Jena

von

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Öffentliche Verteidigung (wissenschaftliches Colloquium) am **26. April 2002**

Öffentlicher Vortrag (Lehrprobe) am **7. Mai 2002**

Erteilung der Lehrbefähigung am **15. Mai 2002**

To my parents

Acknowledgement.

First of all I want to thank Prof. Dr. Dr. Hans Triebel for all the time and energy he spent while discussing interesting problems and assertions with me, giving further valuable hints, but also – more than once – providing strong encouragement if necessary. I am convinced that everyone can deem oneself lucky who has had such an academic teacher like me. I am especially indebted to him with respect to the concept of envelopes, presented in Part II of this report; while he was preparing his most recent book [Tri01] (in which Chapter 2 is devoted to related questions) we could discuss the material at various occasions.

It is a real pleasure for me to give my special thanks to Prof. David E. Edmunds; he formulated the idea which led to the fruitful collaboration resulting in our joint papers [EH99] and [EH00]. I learned a lot on mathematics and also linguistic subtleties, though I am afraid that especially the latter are in danger to fall into oblivion gradually. In any case, I enjoyed my frequent stays in Sussex very much; it gave me the pleasant opportunity to devote my time almost exclusively to research.

Similarly, to a smaller extent, I am grateful to my colleagues in Cardiff, Madrid, Aveiro and Coimbra for their kind hospitality. They offered me their time for listening, discussing, and also introduced me to some interesting new concepts and ideas. In addition, I take this opportunity to gratefully acknowledge some helpful comments on the subject of envelopes studied in Part II, which I was given by Prof. V.I. Kolyada and Prof. M.L. Gol'dman.

I am especially indebted to my colleagues in the research group 'Function Spaces' in Jena for their kind assistance in many different respects over the last years; this facilitated my task essentially.

Last, but not least, my deep gratitude is owed to my family and friends; without their moral support and attentive care in difficult phases of my life, their patience, their understanding and endless encouragement this work had not yet been completed.

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Introduction

The principal object of this report is the study of limiting embeddings in function spaces. The history of such questions starts in the thirties of the last century with SOBOLEV'S famous embedding theorem [Sob38]

$$W_p^k(\Omega) \hookrightarrow L_r(\Omega) \ , \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, L_r, $1 < r \leq \infty$, stands for the usual Lebesgue space, and W_p^k, $k \in \mathbb{N}$, $1 < p < \infty$, are the classical Sobolev spaces. The latter have been widely accepted as one of the crucial instruments in functional analysis – in particular, in connection with PDE's – and have played a significant role in numerous parts of mathematics for many years. SOBOLEV's famous result (0.1) holds for $k \in \mathbb{N}$ with $k < \frac{n}{p}$, and r such that $\frac{k}{n} - \frac{1}{p} \geq -\frac{1}{r}$ (strictly speaking, [Sob38] covers the case $\frac{k}{n} - \frac{1}{p} > -\frac{1}{r}$, whereas the extension to $\frac{k}{n} - \frac{1}{p} = -\frac{1}{r}$ was achieved later). In the limiting case, when $k = \frac{n}{p} \in \mathbb{N}$, this inclusion (0.1) does not hold for $r = \infty$, whereas for all $r < \infty$

$$W_p^{n/p}(\Omega) \hookrightarrow L_r(\Omega) \ . \tag{0.2}$$

The theory of Sobolev type embeddings originates in classical inequalities from which integrability properties of a real function can be deduced from those of its derivatives. In that sense (0.2) can be understood simply as the impossibility to specify integrability conditions of a function $f \in W_p^{n/p}(\Omega)$ by means of L_r conditions merely. In order to obtain further refinements of the limiting case of (0.1) it becomes necessary to deal with a wider class of function spaces. Lorentz-Zygmund spaces $L_r(\log L)_a(\Omega)$, $1 < r \leq \infty$, $a \in \mathbb{R}$, being the set of all those functions f such that

$$\int_\Omega |f(x)|^r \ \log^{ar}(2 + |f(x)|) \ \mathrm{d}x \ < \ \infty \tag{0.3}$$

(with the usual modification if $r = \infty$) constitute a natural class to consider. In the late sixties of the last century PEETRE [Pee66], TRUDINGER [Tru67], and POHOŽAEV [Poh65] independently found refinements of (0.1) expressed in terms of Orlicz spaces of exponential type, see also [Str72] by STRICHARTZ; this was followed by a lot of contributions investigating problems related to (0.1) in detail in the last decades. In 1979 HANSSON [Han79] and BRÉZIS, WAINGER [BW80] showed independently that

$$W_p^{n/p}(\Omega) \hookrightarrow L_{\infty,p}\left(\log L\right)_{-1}(\Omega) \ , \tag{0.4}$$

where $1 < p < \infty$, and the spaces $L_{r,u}\left(\log L\right)_a(\Omega)$ appearing in (0.4) are derived from $L_r(\log L)_a(\Omega)$ given by (0.3) providing an even finer tuning. Recently we noticed a revival of interest in limiting embeddings of Sobolev spaces indicated by a considerable number of publications devoted to this subject; let us only mention a series of papers by EDMUNDS with different co-workers ([EGO96], [EGO97], [EGO00], [EK95], [EKP00]), by CWIKEL, PUSTYLNIK [CP98], and – also from the standpoint of applications to spectral theory – the publications [ET95], [ET99], [Tri93], [Tri99] by EDMUNDS and TRIEBEL. This list is by no means complete, but reflects the increased interest in related questions in the last years. There are a lot of different approaches how to modify (0.1) appropriately in order to get – in the adapted framework – optimal assertions. We return to this discussion after a short digression to entropy numbers.

The idea of the entropy of a set has attracted a great deal of attention over the years, connected with the concept of entropy numbers e_k, $k \in \mathbb{N}$, of embeddings between function spaces. The paper [KT59] by KOLMOGOROV and TIKHOMIROV is certainly one of the earliest significant contributions to this subject, stating that the $k-$th entropy number of the embedding $id_m : C^m\left([0,1]^n\right) \longrightarrow C\left([0,1]^n\right)$ asymptotically behaves like $k^{-m/n}$, written as

$$e_k\left(id_m : C^m\left([0,1]^n\right) \longrightarrow C\left([0,1]^n\right)\right) \ \sim \ k^{-\frac{m}{n}} \ , \quad k \in \mathbb{N}, \tag{0.5}$$

where the involved spaces consist of the (m-times differentiable) bounded uniformly continuous functions on the cube $[0,1]^n$ in $\mathbb{R}^n$. The next milestone in that development is unquestionable the paper [BS67] by Birman and Solomyak; in this pioneering work they introduced the method of piecewise polynomial approximation and established sharp estimates for the entropy numbers of the embedding (0.1),

$$e_\ell \left(id_S : W_p^k(\Omega) \longrightarrow L_r(\Omega) \right) \ \sim \ \ell^{-\frac{k}{n}} , \quad \ell \in \mathbb{N}, \tag{0.6}$$

where $1<p,r<\infty$, and $k>n\ \max(\frac{1}{p}-\frac{1}{r},0)$. It is essentially remarkable in this asymptotic characterisation that – apart from the restriction $k>n\ \max(\frac{1}{p}-\frac{1}{r},0)$ – the numbers p and r do not appear on the right-hand side of (0.6). Here as in the sequel we shall assume that Ω stands for the unit ball $U=\{x\in\mathbb{R}^n : \ |x|<1\}$ in $\mathbb{R}^n$, but this is for convenience and simplicity rather than necessity. It is furthermore hidden in our above description so far that entropy numbers are used to characterise (the quality of) *compact* embeddings only; however, for $k=n(\frac{1}{p}-\frac{1}{r})$ the embedding (0.1) is merely continuous, but not compact. It is natural to enquire into its nature by approaching this non-compact limiting situation by related (possibly) non-limiting compact ones. This was carried out in detail in [ET95] for the situation when (0.1) is replaced by

$$id_a : W_p^k(\Omega) \longrightarrow L_r\left(\log L\right)_a(\Omega) \ , \tag{0.7}$$

as id_a is compact for $a<0$, $k=n(\frac{1}{p}-\frac{1}{r})$. Though the target space in (0.7) is then slightly larger than $L_r(\Omega)$ originally, the modification is so gentle that we continue referring to (0.7) as a limiting embedding.

We consider generalisations of (0.1) in two directions : at first, we investigate the counterpart of (0.1) with $W_p^{n/p}$ replaced by the more general fractional Sobolev spaces $H_p^{n/p}$, or even by spaces of Besov or Triebel-Lizorkin type $B^s_{p,q}$ and $F^s_{p,q}$, respectively; secondly, we additionally study spaces defined on $\mathbb{R}^n$ with some weight function of type $w(x)=(1+|x|)^\alpha \ \log^\mu(2+|x|)$, $\alpha,\ \mu\in\mathbb{R}$. This leads to limiting assertions for spaces on Ω or on $\mathbb{R}^n$ with a weight $w(x)$, respectively, which have the form

$$F^s_{p,q} \ \hookrightarrow \ L_r \ , \quad s-\frac{n}{p}=-\frac{n}{r}, \quad 1<r<\infty, \quad 0<q\leq\infty, \quad s>0, \tag{0.8}$$

and

$$B^s_{p,q} \ \hookrightarrow \ L_r \ , \quad s-\frac{n}{p}=-\frac{n}{r}, \quad 1<r<\infty, \quad 0<q\leq r, \quad s>0, \tag{0.9}$$

complemented by their counterparts for $r=\infty$,

$$F^{n/p}_{p,q} \ \hookrightarrow \ L_\infty \qquad \text{if, and only if,} \qquad 0<p\leq 1 \quad \text{and} \quad 0<q\leq\infty, \tag{0.10}$$

and

$$B^{n/p}_{p,q} \ \hookrightarrow \ L_\infty \qquad \text{if, and only if,} \qquad 0<p\leq \infty \quad \text{and} \quad 0<q\leq 1 \ , \tag{0.11}$$

cf. [ET96, 2.3.3 (iii), p. 45]. Again we face the problem that, say, (0.8) is continuous, but not compact for $s-\frac{n}{p}=-\frac{n}{r}$ assuming that Ω or the weight is suitably chosen. Though adapting the weight function can shrink or extend the corresponding space, this is not sufficient to gain compactness of the underlying embedding. So, roughly speaking, we transfer the idea behind (0.7) and look for modifications of (0.8) – concerning the type of spaces, too – such that the embedding in the adapted setting becomes compact. This is presented in two versions : once for the counterpart of (0.7) on weighted spaces on $\mathbb{R}^n$, otherwise for embeddings of spaces on bounded domains similar to the situation (0.11). In either case we estimate the corresponding entropy numbers subsequently. These two examples together make up Part I of the report. A good deal of this work was motivated by the need for suitable embedding theorems more delicate than the classical ones and new as far as we are aware.
In Part II our goal is different : in contrast to recent approaches studying optimal source or target spaces of limiting embeddings within a certain context (of rearrangement-invariant spaces, for instance) we look for an original characterisation of the involved spaces (as appearing in, say, (0.3) or (0.4)). More precisely, in view of (0.10), (0.11) the question which suggests itself is in what sense the unboundedness of functions belonging to $F^s_{p,q}$ with $1<p<\infty$, and $B^s_{p,q}$ with $1<q\leq\infty$, respectively, can be qualified. Concentrating on this particular feature only we introduce the concept of growth envelope functions $\mathcal{E}^X_{\mathsf{G}}$ 'measuring' the

unboundedness of such functions belonging to some function space $X \subset L_1^{\mathrm{loc}}$, $f \in X$, by means of their non-increasing rearrangement $f^*(t)$,

$$\mathcal{E}_{\mathsf{G}}^{X}(t) \;=\; \sup_{\|f|X\|\leq 1} \; f^*(t) \;, \quad t > 0. \tag{0.12}$$

Surprisingly enough one finds rather simple and final answers characterising apparently complicated spaces like $B^s_{p,q}$ and $F^s_{p,q}$; in fact, the results contain an even finer description of this feature than measured by $\mathcal{E}_{\mathsf{G}}^{X}$ merely. Likewise we investigate parallel limiting situations when questions of (un)boundedness of functions are replaced by inquiries about (almost) Lipschitz continuity, for instance. This refers to limiting embeddings based on (0.10), (0.11), but lifted by smoothness 1,

$$F^{1+n/p}_{p,q} \;\hookrightarrow\; \mathrm{Lip}^1 \qquad \text{if, and only if,} \qquad 0 < p \leq 1 \quad \text{and} \quad 0 < q \leq \infty, \tag{0.13}$$

and

$$B^{1+n/p}_{p,q} \;\hookrightarrow\; \mathrm{Lip}^1 \qquad \text{if, and only if,} \qquad 0 < p \leq \infty \quad \text{and} \quad 0 < q \leq 1 \;, \tag{0.14}$$

see [ET96, (2.3.3/9,10), p. 45]. Dealing with spaces $B^{1+n/p}_{p,q}$ for $1 < q \leq \infty$ one finds that they contain 'almost' Lipschitz continuous functions in the sense that the Lipschitz continuity is spoilt by a logarithmic term of order $\frac{1}{q'} = 1 - \frac{1}{q}$. The associated concept of continuity envelope functions $\mathcal{E}_{\mathsf{C}}^{X}$ replaces (0.12) by

$$\mathcal{E}_{\mathsf{C}}^{X}(t) \;=\; \sup_{\|f|X\|\leq 1} \; \frac{\omega(f,t)}{t} \;, \quad t > 0, \tag{0.15}$$

stressing the same arguments as above afterwards. In (0.15) the function $\omega(f,t)$ stands for the well-known modulus of continuity of a function $f \in X \hookrightarrow C$.

This outlines some historic background as well as the main goals of our report. Further historic references are given at the corresponding places.

The report consists of an introductory Section 1 followed by two parts (as briefly mentioned above), Part I composed of Sections 2 and 3, and Part II containing the remaining four sections. We discuss the mathematical programme and structure of this report at the end of Section 1, that is, in Section 1.4 in greater detail. We preferred this probably unusual procedure because of the big advantage that we can explain the concept and formal structure subordinate to it more precisely then (compared with the rather vague terms as above).

We collect in this report a selection of results of our papers [Har98], [Har00a], [EH99], [EH00], [Har00b] and from the recent preprint [Har01]. Though the outcomes are thus not new essentially, the report tries a completely new way of linking model cases on the one hand, and more abstract approaches on the other hand, and focuses on their interdependence as well as striking differences. Only the totality of all these pieces together form the idea we want to present. In that sense this report intends to be not only the sum of its components (papers); it pursues the idea of passing the existing results in review from another viewpoint, as sometimes the welter of details makes it harder to see the connection (or distinction, respectively).

The motivation and guiding principles under which we selected and rearranged the material are explained in Section 1.4.

1 General concept, basic definitions

In this section we collect the necessary definitions and basic facts on function spaces, embeddings and entropy numbers. We shall rely on the notation introduced here throughout the whole report.

Afterwards, at the end of this section – and having thus all the necessary definitions and facts introduced –, we can precisely describe the structure of this report. This is done firstly from the mathematical point of view and subsequently from a more formal one, as the reasons for our selection – *why we have chosen to present just this material* – can hardly be understood without the preliminaries.

1.1 Function spaces

Let $\mathbb{R}^n$ be Euclidean n-space and

$$\langle x\rangle = (2+|x|^2)^{1/2}\ , \quad x\in\mathbb{R}^n\ . \tag{1.1.1}$$

In a slight abuse of notation we also use $\langle k\rangle$ to stand for $(2+k^2)^{1/2}$ when $k\in\mathbb{N}$. Given two (quasi-) Banach spaces X and Y, we write $X\hookrightarrow Y$ if $X\subset Y$ and the natural embedding of X in Y is continuous. For non-negative functions $f,g:\mathbb{N}\longrightarrow\mathbb{R}$, the symbol $f(k)\sim g(k)$ will mean that there are positive numbers c_1,c_2 such that for all $k\in\mathbb{N}$,

$$c_1\ f(k)\ \le\ g(k)\ \le\ c_2\ f(k).$$

All unimportant positive constants will be denoted by c, occasionally with subscripts. For any $a\in\mathbb{R}$ let

$$a_+ = \max(a,0) \quad\text{and}\quad [a]=\max\{k\in\mathbb{Z}: k\le\ a\}\,. \tag{1.1.2}$$

Moreover, for $0<r\le\infty$ the number r' is given by

$$\frac{1}{r'} := \left(1-\frac{1}{r}\right)_+\ , \quad 0<r\le\infty\ . \tag{1.1.3}$$

For convenience, let both $\mathrm{d}x$ and $|\cdot|$ also stand for the (n-dimensional) Lebesgue measure ℓ_n in the sequel.

1.1.1 Classical spaces

We briefly recall the definitions and properties of some well-known spaces which will be used below.

The Lebesgue space L_p and some relatives

Let $L_p(\Omega)$, $0<p\le\infty$, be the (quasi-) Banach space with respect to Lebesgue measure, normed by

$$\|f|L_p(\Omega)\| = \Big(\int_\Omega |f(x)|^p\ \mathrm{d}x\Big)^{1/p}\ , \tag{1.1.4}$$

(with the usual modification for $p=\infty$), where Ω in (1.1.4) may stand both for a bounded domain in $\mathbb{R}^n$, or $\mathbb{R}^n$ itself. A natural refinement of this scale of Lebesgue spaces are the spaces $L_p(\log L)_a(\Omega)$ being the set of all measurable functions $f:\Omega\longrightarrow\mathbb{C}$ such that

$$\int_\Omega |f(x)|^p\ \log^{ap}\big(2+|f(x)|\big)\,\mathrm{d}x<\infty\ . \tag{1.1.5}$$

This definition (1.1.5) for spaces $L_p(\log L)_a(\Omega)$ may be found in the book of Bennett and Sharpley in [BS88, Ch. 4, Def. 6.11, p. 252] where $1<p<\infty$, $a\in\mathbb{R}$, and $\Omega\subset\mathbb{R}^n$ with $|\Omega|<\infty$. They are called *Zygmund spaces* there. We give an alternative definition (admitting also parameters $0 < p \leq 1$ and $p = \infty$), in Definition 1.1.1 below.

In [BS88, Ch. 4, Lemma 6.12, p. 252] it is shown that $f \in L_p(\log L)_a(\Omega)$, $1 < p < \infty$, $a \in \mathbb{R}$, if, and only if,

$$\left(\int_0^{|\Omega|} \Big[(1 + |\log t|)^a f^*(t) \Big]^p \, \mathrm{d}t \right)^{1/p} < \infty \,, \tag{1.1.6}$$

where f^* denotes the non-increasing rearrangement of f, as usual,

$$f^*(t) = \inf \big\{ s > 0 : |\{ x \in \Omega : |f(x)| > s \}| \leq t \big\} \quad , \quad t > 0 \tag{1.1.7}$$

(with the agreement $\inf \emptyset = \infty$). Note that f^* is non-negative, decreasing and right-continuous on $[0, \infty)$. Moreover,

$$f^*(t) = \sup \{ s \geq 0 : |\{ x \in \Omega : |f(x)| > s \}| > t \} \quad , \quad t \geq 0 \,, \tag{1.1.8}$$

$(af)^* = |a| f^*$, $a \in \mathbb{R}$, $(|f|^p)^* = (f^*)^p$, $0 < p < \infty$, and $|g| \leq |f|$ a.e. implies $g^* \leq f^*$. One knows that f and f^* are equi-measurable, i.e.

$$\mu_f(s) := |\{ x \in \Omega : |f(x)| > s \}| = |\{ t \geq 0 : f^*(t) > s \}| = \nu_{f^*}(s) \,, \quad s \geq 0, \tag{1.1.9}$$

where $\nu(\cdot) = |\cdot|$ stands for the usual Lebesgue measure on $\mathbb{R}_+$. Furthermore, $f^*(0) = \|f|L_\infty(\Omega)\|$, and $f^*(t) = 0$ for $t > |\Omega|$. Note that f^* satisfies the weak form of sub-additivity only, that is,

$$(f+g)^*(t_1 + t_2) \;\leq\; f^*(t_1) + g^*(t_2) \,, \qquad t_1, t_2 \geq 0.$$

There is a plenty of literature on this topic; we refer to [BS88, Ch. 2, Prop. 1.7, p. 41] and [DL93, Ch. 2, §2], for instance. In view of (1.1.6) we come to an alternative definition of $L_p(\log L)_a(\Omega)$, which simultaneously extends it to parameters $0 < p \leq \infty$.

Definition 1.1.1 *Let $\Omega \subset \mathbb{R}^n$, and $0 < p, q \leq \infty$.*

(i) *The* Lorentz space $L_{p,q}(\Omega)$ *consists of all measurable functions $f : \Omega \longrightarrow \mathbb{C}$ for which the quantity*

$$\|f|L_{p,q}(\Omega)\| = \begin{cases} \left(\displaystyle\int_0^{|\Omega|} \Big[t^{\frac{1}{p}} f^*(t) \Big]^q \frac{\mathrm{d}t}{t} \right)^{1/q} & , \quad 0 < q < \infty, \\ \displaystyle\sup_{0<t<|\Omega|} t^{\frac{1}{p}} f^*(t) & , \quad q = \infty \end{cases} \tag{1.1.10}$$

is finite.

(ii) *Let $a \in \mathbb{R}$. The* Lorentz-Zygmund space $L_{p,q}(\log L)_a = L_{p,q}(\log L)_a(\Omega)$ *consists of all measurable functions $f : \Omega \longrightarrow \mathbb{C}$ for which*

$$\|f|L_{p,q}(\log L)_a(\Omega)\| = \begin{cases} \left(\displaystyle\int_0^{|\Omega|} \Big[t^{\frac{1}{p}} \left(1 + |\log t|\right)^a f^*(t) \Big]^q \frac{\mathrm{d}t}{t} \right)^{1/q} & , \quad 0 < q < \infty, \\ \displaystyle\sup_{0<t<|\Omega|} t^{\frac{1}{p}} \left(1 + |\log t|\right)^a f^*(t) & , \quad q = \infty \end{cases} \tag{1.1.11}$$

is finite.

The above definition given by Bennett and Sharpley may be found in [BS88, Ch. 4, Def. 6.13] and in [BR80, (1.4), (1.14)]. Note that $L_{p,p} = L_p$ are the usual Lebesgue spaces, $0 < p \leq \infty$, and $L_{p,q}\left(\log L\right)_0 =$

$L_{p,q}$. The spaces $L_{p,q}(\log L)_a$ are monotonically ordered in q (for fixed p and a) as well as in a (when p, q are fixed). In particular, for $a_1, a_2 \in \mathbb{R}$, $a_2 < a_1$,

$$L_p(\log L)_{a_1}(\Omega) \hookrightarrow L_p(\log L)_{a_2}(\Omega) \ . \tag{1.1.12}$$

Moreover, when $|\Omega| < \infty$, then there is also some monotonicity in p, i.e. we obtain for any $0 < \varepsilon < p$ and all $a > 0$,

$$L_{p+\varepsilon}(\Omega) \hookrightarrow L_p(\log L)_a(\Omega) \hookrightarrow L_p(\Omega) \hookrightarrow L_p(\log L)_{-a}(\Omega) \hookrightarrow L_{p-\varepsilon}(\Omega) \ , \tag{1.1.13}$$

see [ET95, Rem. 2.1/2] and [ET96, Prop. 2.6.1/1 (i)]. Otherwise, when $|\Omega| = \infty$, there is no monotonicity in p. Note that $L_{\infty,q}$, $0 < q < \infty$, is trivial; i.e. it contains the zero function only. The same happens for spaces of type $L_{p,q}(\log L)_a$ when $p = \infty$, $0 < q < \infty$, and $a + 1/q \geq 0$, or $p = q = \infty$, but $a > 0$. Thus when $p = \infty$ we only study spaces $L_{p,q}(\log L)_a$ in the sequel, where $a + 1/q < 0$ for $0 < q < \infty$, or $a \leq 0$ for $q = \infty$, respectively.
Moreover, when $|\Omega| < \infty$, say, $|\Omega| = 1$, and $p = q = \infty$, $a \geq 0$, one has $L_{\infty,\infty}(\log L)_{-a}(\Omega) = L_{\exp,a}(\Omega)$, where the latter are the *Zygmund spaces* consisting of all measurable functions f on Ω for which there is a constant $\lambda = \lambda(f) > 0$ such that

$$\int_\Omega \exp\left(\lambda |f(x)|\right)^{1/a} \,\mathrm{d}x \ < \ \infty, \tag{1.1.14}$$

(if $a = 0$, this is interpreted as f is bounded, i.e. $L_{\exp,0} = L_\infty$); see [BS88, Ch. 4, Def. 6.11, Lemma 6.12, p. 252].

Remark 1.1.2 Note that (1.1.10) and (1.1.11) do not give a norm in any case, not even for $p, q \geq 1$. However, replacing the non-increasing rearrangement f^* in (1.1.10) and (1.1.11) by its maximal function f^{**}, given by

$$\left(\mathcal{M}f^*\right)(t) \ = \ f^{**}(t) \ = \ \frac{1}{t}\int_0^t f^*(s) \,\mathrm{d}s \ , \quad t > 0, \tag{1.1.15}$$

one obtains for $1 < p < \infty$, $1 \leq q \leq \infty$, or $p = q = \infty$, a norm in that way, see [BS88, Ch. 4, Thm. 4.6, p. 219]. An essential advantage of the maximal function f^{**} – compared with f^* – is that it possesses a certain sub-additivity property,

$$(f+g)^{**}(t) \ \leq \ f^{**}(t) + g^{**}(t) \ , \quad t > 0, \tag{1.1.16}$$

cf. [BS88, Ch. 2, (3.10), p. 54]. Moreover, for $1 < p \leq \infty$ and $1 \leq q \leq \infty$, the corresponding expressions (1.1.10) with f^* and f^{**}, respectively, are equivalent; cf. [BS88, Ch. 4, Lemma 4.5, p. 219].

Banach function spaces

The spaces L_p, $1 \leq p \leq \infty$, belong to the category of *Banach function spaces* (or *lattices*); we briefly recall this notion and follow [BS88, Ch. 1, Sect. 1] closely. We assume the underlying measure space to be (a subset of) $\mathbb{R}^n$ equipped with the Lebesgue measure ℓ_n. Then these are Banach spaces X of locally integrable functions for which the norm $\|\cdot|X\|$ is related to the order by the property that $|f(x)| \leq |g(x)|$ a.e. for $g \in X$ implies $f \in X$ and $\|f|X\| \leq \|g|X\|$. One also assumes that X contains the characteristic functions χ_A of all subsets of $\mathbb{R}^n$ with finite measure $\ell_n(A) < \infty$. Finally one requires that X satisfies the *Fatou property* : if $f_n \geq 0$ is an increasing sequence in X, $0 \leq f_n \uparrow f$ a.e., then $\|f|X\| = \lim\limits_{n\to\infty} \|f_n|X\|$. Obviously one can extend this definition to quasi-Banach function spaces, if X is equipped with a quasi-norm only. Note that for Banach function spaces X and Y (over the same measure space $[\mathcal{R}, \mu]$) the condition $X \subset Y$ already implies $X \hookrightarrow Y$; cf. [BS88, Ch. 1, Thm. 1.8, p. 7].

Spaces of continuous functions

Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the $\sup$-norm as usual. If $m \in \mathbb{N}$, we define

$$C^m(\mathbb{R}^n) = \{f : D^\alpha f \in C(\mathbb{R}^n) \quad \text{for all} \quad |\alpha| \leq m\}.$$

Here D^α are classical derivatives and $C^m(\mathbb{R}^n)$ is endowed with the norm

$$\|f|C^m(\mathbb{R}^n)\| = \sum_{|\alpha|\leq m} \|D^\alpha f|L_\infty(\mathbb{R}^n)\|.$$

Recall the concept of the difference operator Δ_h^m, $m \in \mathbb{N}$, $h \in \mathbb{R}^n$. Let $f(x)$ be an arbitrary function on $\mathbb{R}^n$, then

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \ , \qquad (\Delta_h^{m+1} f)(x) = \Delta_h^1(\Delta_h^m f)(x), \tag{1.1.17}$$

where $x, h \in \mathbb{R}^n$. For convenience we may write Δ_h instead of Δ_h^1. Let $r \in \mathbb{N}$, the *r-th modulus of smoothness* (or *r-th order modulus of continuity*) of a function $f \in L_p(\mathbb{R}^n)$, $0 < p \leq \infty$, is defined by

$$\omega_r(f,t)_p = \sup_{|h|\leq t} \|\Delta_h^r f|L_p(\mathbb{R}^n)\| \ , \quad t > 0, \tag{1.1.18}$$

see [BS88, Ch. 5, Def. 4.2, p. 332] or [DL93, Ch. 2, §7, pp. 44-46]. Note that each modulus $\omega_r(f,t)_p$, $1 \leq p \leq \infty$, $r \in \mathbb{N}$, is a nonnegative, continuous, increasing function of $t > 0$. Moreover, $\omega_r(f,t)_p \searrow \omega_r(f,0)_p = 0$ for $t \downarrow 0$. We also have for $1 \leq p \leq \infty$,

$$\omega_r(f,t)_p \ \leq \ 2^r \ \|f|L_p\| \qquad \text{and} \qquad \omega_r(f,2t)_p \ \leq \ 2^r \ \omega_r(f,t)_p \ , \quad t>0, \quad f \in L_p;$$

there is some triangle inequality,

$$\omega_r(f+g,t)_p \ \leq \ \omega_r(f,t)_p + \omega_r(g,t)_p \ , \quad t > 0 \ , \quad f,g \in L_p \ .$$

We shall write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$ and omit the index $p = \infty$ if there is no danger of confusion, that is, $\omega(g,t)$ instead of $\omega(g,t)_\infty$. We refer to the literature mentioned above for further details.

Marchaud's inequality states the following : let $f \in L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $t > 0$, and $k \in \mathbb{N}$; then

$$\omega_k(f,t)_p \leq \frac{k}{\log 2} \, t^k \int_t^\infty \frac{\omega_{k+1}(f,u)_p}{u^k} \, \frac{\mathrm{d}u}{u}, \tag{1.1.19}$$

see [BS88, Ch. 5, (4.11), p. 334] or [DL93, Ch. 2, Thm. 8.1, p. 47] (for the one-dimensional case).

Definition 1.1.3 *Let* $0 < a \leq 1$. *The* Lipschitz space $\mathrm{Lip}^a(\mathbb{R}^n)$ *is defined as the set of all* $f \in C(\mathbb{R}^n)$ *such that*

$$\|f|\mathrm{Lip}^a(\mathbb{R}^n)\| := \|f|C(\mathbb{R}^n)\ \| + \sup_{0<t<1} \frac{\omega(f,t)}{t^a} \tag{1.1.20}$$

is finite.

Remark 1.1.4 Note that the restriction $0 < a \leq 1$ is quite natural, as otherwise the spaces contain only constants; when $a = 1$ one recovers the classical Lipschitz space $\mathrm{Lip}^1(\mathbb{R}^n)$,

$$\left\| f|\mathrm{Lip}^1(\mathbb{R}^n) \right\| = \|f|C(\mathbb{R}^n)\| + \sup_{0<t<1} \frac{\omega(f,t)}{t} \ . \tag{1.1.21}$$

1.1.2 Sequence spaces

Our estimation of the entropy numbers of embedding maps involves a reduction of the problem to the study of maps between finite-dimensional sequence spaces; this method has been efficiently used before in [ET96] and [Tri97]. Accordingly we introduce the sequence spaces ℓ_p^M, $M \in \mathbb{N}$, $0 < p \leq \infty$ and follow [ET96, 3.2.1, p. 97]. By ℓ_p^M we shall mean the linear space of all complex M-tuples $y = (y_j)$, endowed with the quasi-norm

$$\|y|\ell_p^M\| = \left(\sum_{j=1}^{M} |y_j|^p\right)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Moreover, we also need weighted ℓ_p-spaces in the following sense : Let $(M_j)_{j\in\mathbb{N}_0}$ be a sequence of natural numbers with $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. Let $0 < p \le \infty$ and $0 < q \le \infty$. Let $(w_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers (weights), mainly of the type

$$w_j = 2^{j\gamma} \quad \text{or} \quad w_j = \langle j\rangle^{\varkappa}, \qquad j \in \mathbb{N}_0, \quad \gamma > 0, \quad \varkappa \in \mathbb{R}.$$

We extend the definition of TRIEBEL given in [Tri97, 8.1, p. 38]. Then $\ell_q(w_j\,\ell_p^{M_j})$ stands for the linear space of all complex sequences $x = (x_{j,l} : j \in \mathbb{N}_0; l = 1, \dots, M_j)$ endowed with the quasi-norm

$$\|x|\ell_q\left(w_j\,\ell_p^{M_j}\right)\| = \left(\sum_{j=0}^{\infty} w_j^q \left(\sum_{l=1}^{M_j} |x_{j,l}|^p\right)^{q/p}\right)^{1/q} \tag{1.1.22}$$

(with the obvious modifications if $p = \infty$ or $q = \infty$). In case of $w_j \equiv 1$ we write $\ell_q(\ell_p^{M_j})$. The above notation was introduced in [EH99, (3.1)] and coincides with [Tri97, (8.2), p. 38] when $w_j = 2^{j\gamma}$, $\gamma > 0$.

In addition to the above notation of the spaces $\ell_q(w_j\,\ell_p^{M_j})$ endowed with the quasi-norm (1.1.22) we have to introduce spaces $\ell_u\left[2^{\mu m}\,\ell_q(w_j\,\ell_p^{M_j})\right]$, $0 < u \le \infty$, $\mu > 0$, as the linear space of all $\ell_q(w_j\,\ell_p^{M_j})$-valued sequences $x = (x^m)_{m\in\mathbb{N}_0}$ such that the quasi-norm

$$\|x|\ell_u\left[2^{\mu m}\,\ell_q(w_j\,\ell_p^{M_j})\right]\| = \left(\sum_{m=0}^{\infty} 2^{\mu m u}\,\|x^m|\ell_q(w_j\,\ell_p^{M_j})\|^u\right)^{1/u} \tag{1.1.23}$$

(with the obvious modification if $u = \infty$) is finite. In case of $w_j \equiv 1$ and $\mu = 0$ we write $\ell_u\left[\ell_q(\ell_p^{M_j})\right]$. The above notation coincides with [Tri97, (9.1)] when $w_j = 2^{j\gamma}$, $\gamma > 0$.

Let $Q_{\nu m}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. Furthermore, $\chi_{\nu m}^{(p)}$ is the p-normalised characteristic function of the cube $Q_{\nu m}$, that is

$$\chi_{\nu m}^{(p)}(x) = 2^{\frac{\nu n}{p}} \quad \text{if} \quad x \in Q_{\nu m} \quad \text{and} \quad \chi_{\nu m}^{(p)}(x) = 0 \quad \text{if } x \notin Q_{\nu m},$$

where $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and $0 < p \le \infty$. Plainly, $\|\chi_{\nu m}^{(p)} \,|\, L_p(\mathbb{R}^n)\| = 1$.

Definition 1.1.5 *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and* $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. *Then*

$$b_{pq} = \left\{\lambda : \|\lambda \,|\, b_{pq}\| = \left(\sum_{\nu=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\}$$

and

$$f_{pq} = \left\{\lambda : \|\lambda \,|\, f_{pq}\| = \left\|\,\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}\,\chi_{\nu m}^{(p)}(\cdot)|^q\Big)^{1/q} \,|\, L_p(\mathbb{R}^n)\right\| < \infty\right\}$$

(*with the usual modification if* $p = \infty$ *and/or* $q = \infty$).

This definition is a modification of the related one in [FJ90] and coincides with [Tri97, Def. 13.5, p. 74].

1.1.3 Spaces of type $B^s_{p,q}$, $F^s_{p,q}$

Function spaces of Besov or Triebel-Lizorkin type, $B^s_{p,q}$ and $F^s_{p,q}$, respectively, will serve both as essential motivation and as outstanding examples in the sequel. We recall briefly the basic ingredients needed to their introduction.

Fourier-analytical approach

The Schwartz space $S(\mathbb{R}^n)$ and its dual $S'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. We first need the notion of a *smooth dyadic resolution of unity*. Let

$$A_\ell = \left\{x \in \mathbb{R}^n : 2^{\ell-1} < |x| < 2^{\ell+1}\right\}, \quad \ell \in \mathbb{N}, \tag{1.1.24}$$

complemented by

$$A_0 = \{x \in \mathbb{R}^n : |x| < 2\}, \tag{1.1.25}$$

the usual dyadic annuli in $\mathbb{R}^n$. Let $\{\varphi_j\}_{j=0}^\infty$ be a sequence of C_0^∞ functions satisfying the following conditions :

(i) $\operatorname{supp}\varphi_j \subset \overline{A_j}, \quad j \in \mathbb{N}_0$,

(ii) for any multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$ there exists a positive constant c_γ such that

$$2^{j|\gamma|}\,|D^\gamma \varphi_j(x)| \le c_\gamma \quad \text{for all} \quad x \in \mathbb{R}^n, \quad |\gamma| = \gamma_1 + \ldots + \gamma_n,$$

(iii) $\displaystyle\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n$.

Then $\{\varphi_j\}_{j=0}^\infty$ is said to be a *smooth dyadic resolution of unity*. Such a smooth dyadic resolution of unity can be constructed, say, based on some $\varphi \in S(\mathbb{R}^n)$ with

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1\,. \tag{1.1.26}$$

Put $\varphi_0 = \varphi$ and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a smooth dyadic resolution of unity. Given any $f \in S'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively.

Definition 1.1.6 *Let $s \in \mathbb{R}$, $0 < q \le \infty$, and let $\{\varphi_j\}$ be a smooth dyadic resolution of unity.*
(i) *Let $0 < p \le \infty$. The space $B^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that*

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^\infty 2^{jsq}\,\big\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f|L_p(\mathbb{R}^n)\big\|^q\Big)^{1/q} \tag{1.1.27}$$

(with the usual modification if $q = \infty$) is finite.

(ii) *Let $0 < p < \infty$. The space $F^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that*

$$\big\|f|F^s_{p,q}(\mathbb{R}^n)\big\| = \Big\|\Big(\sum_{j=0}^\infty 2^{jsq}|\mathcal{F}^{-1}\varphi_j\mathcal{F}f(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| \tag{1.1.28}$$

(with the usual modification if $q = \infty$) is finite.

For later use we introduce numbers

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ \tag{1.1.29}$$

where $0 < p \le \infty$ and $0 < q \le \infty$, recall notation (1.1.2).

Remark 1.1.7 The theory of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ as given above has been developed in detail in [Tri83] and [Tri92] but has a longer history already including many contributors; we do not want to discuss this here. Let us mention instead that these two scales $B^s_{p,q}$ and $F^s_{p,q}$ cover (fractional) Sobolev spaces, Hölder-Zygmund spaces, local Hardy spaces, and classical Besov spaces – characterised via derivatives and differences : Let $0<p\leq\infty$, $s>\sigma_p$, $0<q\leq\infty$, and $r\in\mathbb{N}$ with $r>s$. Then with $\omega_r(f,t)_p$ given by (1.1.18),

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| \quad\sim\quad \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^{\frac{1}{2}} \left[t^{-s}\omega_r(f,t)_p\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \tag{1.1.30}$$

(with the usual modification if $q=\infty$), see [BS88, Ch. 5, Def. 4.3, p. 332], [DL93, Ch. 2, §10, pp. 54-56] (where the Besov spaces are defined in that way) for the Banach case, and [Tri83, Thm. 2.5.12, p. 110] for what concerns the equivalence of Definition 1.1.6 (i) and characterisation (1.1.30). In particular, with $p=q=\infty$, one recovers Hölder-Zygmund spaces $\mathcal{C}^s$. Let, say, $0<s<1$, then $B^s_{\infty,\infty}=\mathcal{C}^s$ (in the sense of equivalent norms),

$$\left\|f|B^s_{\infty,\infty}(\mathbb{R}^n)\right\| \sim \|f|C(\mathbb{R}^n)\| + \sup_{0<t<1} \frac{\omega(f,t)}{t^s}\ , \tag{1.1.31}$$

cf. [Tri83, Thm. 2.5.12, p. 110]. Concerning F–spaces, one has, for instance, $F^s_{p,2}(\mathbb{R}^n)=H^s_p(\mathbb{R}^n)$, $s\in\mathbb{R}$, $1<p<\infty$, the latter being the well-known (fractional) Sobolev spaces of all measurable functions $f:\mathbb{R}^n\longrightarrow\mathbb{C}$, normed by

$$\|f|H^s_p(\mathbb{R}^n)\| = \|I_s f|L_p(\mathbb{R}^n)\|\ , \tag{1.1.32}$$

where

$$I_\sigma f = \mathcal{F}^{-1}\langle\xi\rangle^\sigma\mathcal{F}f, \quad f\in S'(\mathbb{R}^n), \quad \sigma\in\mathbb{R}, \tag{1.1.33}$$

is the lift operator and $\langle\cdot\rangle$ is given by (1.1.1); in particular, in case of classical Sobolev spaces W^k_p it holds

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \quad k\in\mathbb{N}_0\,, \quad 1<p<\infty, \qquad \text{i.e.} \qquad F^0_{p,2}(\mathbb{R}^n)=L_p(\mathbb{R}^n)\ . \tag{1.1.34}$$

For later use we also recall the definition of the local (non-homogeneous) Hardy spaces h_p, $0<p<\infty$. Let $\varphi(x)$ be a test function on $\mathbb{R}^n$, $\varphi\in C_0^\infty(\mathbb{R}^n)$, with $\varphi(0)=1$. Put $\varphi_t(x)=\varphi(tx)$ for $x\in\mathbb{R}^n$ and $t>0$. Then

$$h_p(\mathbb{R}^n) = \left\{ f\in S'(\mathbb{R}^n) : \|f|h_p(\mathbb{R}^n)\| = \left\| \sup_{0<t<1} \left|\mathcal{F}^{-1}\varphi_t\mathcal{F}f\right| \,\Big|L_p(\mathbb{R}^n)\right\| < \infty \right\}. \tag{1.1.35}$$

This definition is due to Goldberg [Gol79b, Gol79a], see also [Tri83, Sect. 2.2.2, p. 37]. According to [Tri83, Thm. 2.5.8/1, p. 92] it holds

$$h_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)\ , \quad 0<p<\infty\ . \tag{1.1.36}$$

The local (non-homogeneous) space of functions of bounded mean oscillation, bmo, consists of all locally-integrable functions $f\in L_1^{\mathrm{loc}}$ satisfying the following condition,

$$\mathrm{bmo}\,(\mathbb{R}^n) = \Bigg\{ f\in L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad : \\ \|f|\mathrm{bmo}\,(\mathbb{R}^n)\| \;=\; \sup_{|Q|\leq 1}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x + \sup_{|Q|>1}\frac{1}{|Q|}\int_Q |f(x)|\,\mathrm{d}x < \infty \Bigg\}, \tag{1.1.37}$$

where Q are cubes in $\mathbb{R}^n$, and f_Q is the mean value of f with respect to Q, $f_Q=\frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$. This definition coincides with [Tri83, 2.2.2 (viii), p. 37]; see also [BS88, Ch. 5, Def. 7.6, (7.15), p. 380].

Atomic decompositions

It turns out that the following characterisation of function spaces of type $B^s_{p,q}$ or $F^s_{p,q}$ is sometimes preferred (compared with the above Fourier-analytical approach), in particular when arguments for entropy numbers of embeddings between such function spaces can thus be transferred to related questions of embeddings in (well-adapted) sequence spaces (as introduced in Section 1.1.2) which are sometimes easier to handle.
Concerning atomic decompositions of spaces $B^s_{p,q}$ and $F^s_{p,q}$, we closely follow the presentation in [Tri97, Sect. 13]. Recall our notation $Q_{\nu m}$, $\chi^{(p)}_{\nu m}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, given at the end of Section 1.1.2. For a cube Q in $\mathbb{R}^n$ and $r > 0$ we shall mean by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 1.1.8

(i) *Let $K \in \mathbb{N}_0$ and $d > 1$. A K times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K = 0$) is called a 1_K-atom if*

$$\operatorname{supp} a \subset dQ_{0m} \quad \text{for some} \quad m \in \mathbb{Z}^n \tag{1.1.38}$$

and

$$|D^\alpha a(x)| \le 1 \quad \text{for} \quad |\alpha| \le K. \tag{1.1.39}$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L + 1 \in \mathbb{N}_0$, and $d > 1$. A K times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K = 0$) is called an $(s,p)_{K,L}$- atom if for some $\nu \in \mathbb{N}_0$*

$$\operatorname{supp} a \subset d\,Q_{\nu m} \quad \text{for some} \quad m \in \mathbb{Z}^n, \tag{1.1.40}$$

$$|D^\alpha a(x)| \le 2^{-\nu(s-n/p)+|\alpha|\nu} \quad \text{for} \quad |\alpha| \le K \tag{1.1.41}$$

and

$$\int_{\mathbb{R}^n} x^\beta\, a(x)\, \mathrm{d}x = 0 \quad \text{if} \quad |\beta| \le L. \tag{1.1.42}$$

This definition coincides with [Tri97, Def. 13.3, p. 73]. The number d in (1.1.38) and (1.1.40) is unimportant in so far as it simply makes clear that at the level ν some controlled overlapping of the supports of $a_{\nu m}$ must be allowed. Assumption (1.1.42) is called a *moment condition*, where $L = -1$ means that there are no moment conditions. It is convenient to write $a_{\nu m}(x)$ instead of $a(x)$ if this atom is located at $Q_{\nu m}$ according to (1.1.38) and (1.1.40).
We come to the main theorem now, the atomic characterisation of function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, respectively, as obtained by TRIEBEL in [Tri97].

Theorem 1.1.9 [Tri97, Thm. 13.8, p. 75]

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K \ge (1 + [s])_+ \quad \text{and} \quad L \ge \max(-1, [\sigma_p - s]) \tag{1.1.43}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}(x), \quad \text{convergence being in } S'(\mathbb{R}^n), \tag{1.1.44}$$

where the $a_{\nu m}$ are 1_K-atoms ($\nu = 0$) or $(s,p)_{K,L}$-atoms ($\nu \in \mathbb{N}$) according to Definition 1.1.8, with

$$\operatorname{supp} a_{\nu m} \subset d\,Q_{\nu m}\,, \quad \nu \in \mathbb{N}_0\,.\, m \in \mathbb{Z}^n,\ d > 1, \tag{1.1.45}$$

and $\lambda \in b_{pq}$. Furthermore

$$\inf \|\lambda \,|\, b_{pq}\|\,, \tag{1.1.46}$$

where the infimum is taken over all admissible representations (1.1.44)*, is an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$.*

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$, *and* $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$ *and* $L + 1 \in \mathbb{N}_0$ *with*

$$K \geq (1 + [s])_+ \quad \textit{and} \quad L \geq \max(-1, [\sigma_{pq} - s]) \tag{1.1.47}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (1.1.44), *where the atoms* $a_{\nu m}$ *have the same meaning as in part* (i) (*now perhaps with a different value of* L) *and* $\lambda \in f_{pq}$. *Furthermore*

$$\inf \|\lambda \mid f_{pq}\| , \tag{1.1.48}$$

where the infimum is taken over all admissible representations (1.1.44), *is an equivalent quasi-norm in* $F^s_{p,q}(\mathbb{R}^n)$.

For the proof as well as further remarks and consequences we refer to [Tri97, Thm. 13.8, p. 75].

Convention. Note that we shall write $A^s_{p,q}$ occasionally, when both scales of spaces – either $A^s_{p,q} = B^s_{p,q}$ or $A^s_{p,q} = F^s_{p,q}$ – are concerned simultaneously and the particular choice does not matter.

Weighted function spaces

We recall the concept of 'admissible' weight functions and some basics about weighted functions spaces, see [HT94a], for instance.

Definition 1.1.10 *The class of* admissible weight functions *is the collection of all positive* C^∞ *functions* $w(x)$ *on* $\mathbb{R}^n$ *with the following properties:*

(i) *For any multi-index* γ *there exists a positive constant* c_γ *with*

$$|D^\gamma w(x)| \leq c_\gamma \, w(x) \quad \textit{for all} \quad x \in \mathbb{R}^n , \tag{1.1.49}$$

(ii) *there exist two constants* $c > 0$ *and* $\alpha \geq 0$ *such that*

$$0 < w(x) \leq c \, w(y) \langle x - y \rangle^\alpha \quad \textit{for all} \quad x \in \mathbb{R}^n \quad \textit{and all} \quad y \in \mathbb{R}^n . \tag{1.1.50}$$

In this paper we merely deal with special weight functions of type

$$w(x) = \langle x \rangle^\alpha \, \log^\mu \langle x \rangle , \quad \alpha \in \mathbb{R}, \; \mu \in \mathbb{R} . \tag{1.1.51}$$

Therefore we do not discuss the more general concept of weight functions, but details may be found in [HT94a, 2.1], for instance. Nevertheless we shall formulate the next results in the framework of *admissible weight functions* in the sense of Definition 1.1.10, but the special weights (1.1.51) may serve as typical examples.

We use the notation $L_p(w(\cdot), \Omega)$ for the weighted L_p spaces where $w(x)$ is some admissible weight function in the sense of Definition 1.1.10 and $\Omega \subseteq \mathbb{R}^n$, normed via

$$\|f | L_p(w(\cdot), \Omega)\| = \|wf | L_p(\Omega)\| . \tag{1.1.52}$$

The weighted Sobolev spaces $H^s_p(w(\cdot), \mathbb{R}^n)$ are defined in the following way : one has to replace the unweighted basic space $L_p(\mathbb{R}^n)$ in (1.1.32) by its weighted counterpart, i.e.

$$\|f | H^s_p(w(\cdot), \mathbb{R}^n)\| = \|I_s f | L_p(w(\cdot), \mathbb{R}^n)\| , \tag{1.1.53}$$

where $w(x)$ is an admissible weight function in the sense of Definition 1.1.10. In [HT94a, Thm. 2.2] we have shown that this definition (1.1.53) is completely consistent with that approach,

$$f \in H^s_p(w(\cdot), \mathbb{R}^n) \quad \Longleftrightarrow \quad wf \in H^s_p(\mathbb{R}^n) . \tag{1.1.54}$$

More precisely, we have proved there that the operator $f \mapsto wf$ is an isomorphic mapping from $H^s_p(w(\cdot), \mathbb{R}^n)$ onto $H^s_p(\mathbb{R}^n)$ and that $\|wf | H^s_p(\mathbb{R}^n)\|$ is an equivalent norm in $H^s_p(w(\cdot), \mathbb{R}^n)$.

Remark 1.1.11 Our paper [HT94a] is written in the framework of more general Besov and Triebel-Lizorkin spaces, $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively, where $s \in \mathbb{R}$, $0 < p \le \infty$ $(p < \infty$ in case of the F-spaces) and $0 < q \le \infty$. Assertion (1.1.54) is valid for more general spaces than Sobolev or Lebesgue spaces, but there is no need to pursue this point here.

Spaces on domains

We give the definition for the spaces $A^s_{p,q}(\Omega)$.

Definition 1.1.12 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $s \in \mathbb{R}$, $0 < p \le \infty$ $(p < \infty$ in case of $A^s_{p,q} = F^s_{p,q})$ and $0 < q \le \infty$. Then $A^s_{p,q}(\Omega)$ is the restriction of $A^s_{p,q}(\mathbb{R}^n)$ to Ω, i.e.*

$$A^s_{p,q}(\Omega) = \{f \in D'(\Omega) : \exists\ g \in A^s_{p,q}(\mathbb{R}^n), g_{|\Omega} = f\}\ . \tag{1.1.55}$$

Furthermore,

$$\|f|A^s_{p,q}(\Omega)\| = \inf \|g|A^s_{p,q}(\mathbb{R}^n)\|$$

where the infimum is taken over all $g \in A^s_{p,q}(\mathbb{R}^n)$, $g_{|\Omega} = f$.

1.2 Embeddings

The intention of this section is a short summary of results concerning embeddings of weighted function spaces on $\mathbb{R}^n$ or on bounded domains $\Omega \subset \mathbb{R}^n$; we begin with the so-called *non-limiting* case. Though this situation is well-known and not the main topic of our investigations, we think it at least convenient and helpful for a better understanding where the differences and analogues are comparing the *non-limiting case* and the *limiting* situation we deal with.

1.2.1 Non-limiting embeddings

As already mentioned in the introduction, SOBOLEV'S famous embedding theorem (0.1) led to a large number of further embedding results in more general function spaces, say, of type $A^s_{p,q}$. We briefly collect some of these well-known facts for further reference mainly. These results are originally $\mathbb{R}^n-$ results, but can be transferred to spaces on domains by the restriction procedure described in (1.1.55). Therefore we shall omit Ω or $\mathbb{R}^n$ in the formulation below. Let $A^s_{p,q}$ stand for $B^s_{p,q}$ or $F^s_{p,q}$, respectively, where we assume $s \in \mathbb{R}$, $0 < p \le \infty$ (with $p < \infty$ for $F-$spaces), and $0 < q \le \infty$. Then

$$A^s_{p,q} \hookrightarrow A^s_{p,r} \qquad \text{for} \quad q \le r \le \infty, \tag{1.2.1}$$

$$A^{s+\varepsilon}_{p,q} \hookrightarrow A^s_{p,r} \qquad \text{for all} \quad 0 < r \le \infty, \quad \varepsilon > 0, \tag{1.2.2}$$

and, for $0 < p < \infty$,

$$B^s_{p,\min(p,q)} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max(p,q)}\ , \tag{1.2.3}$$

see [Tri83, Prop. 2.3.2/2, p. 47]. Moreover, dealing with classical spaces such as L_p and C, one can complement (1.1.34) by

$$B^m_{p,1} \hookrightarrow W^m_p \hookrightarrow B^m_{p,\infty} \qquad \text{when} \quad 1 \le p < \infty, \quad m \in \mathbb{N}_0\,, \tag{1.2.4}$$

and

$$B^m_{\infty,1} \hookrightarrow C^m \hookrightarrow B^m_{\infty,\infty} \qquad \text{for} \quad m \in \mathbb{N}_0\,, \tag{1.2.5}$$

see [Tri83, Prop. 2.5.7, (2.5.7/10,11), pp. 89-90]. On the other hand, regarding spaces with different metrics, then not only the so-called 'differential dimension' $s - \frac{n}{p}$ of the involved spaces is important, but – in case of B- spaces – also the q- indices gain influence. Let $0 < p_1 < p_2 < \infty$, $0 < q_1,\ q_2,\ q \le \infty$ and $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, then

$$B^{s_1}_{p_1,q} \hookrightarrow B^{s_2}_{p_2,q} \qquad \text{and} \qquad F^{s_1}_{p_1,q_1} \hookrightarrow F^{s_2}_{p_2,q_2}\ , \tag{1.2.6}$$

cf. [Tri83, Thm. 2.7.1, p. 129]. Let us introduce the notation

$$\delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right) \ . \tag{1.2.7}$$

Together with (1.2.2) it then follows immediately that $A^{s_1}_{p_1,q_1} \hookrightarrow A^{s_2}_{p_2,q_2}$ for all admitted parameters $0 < q_1,\ q_2 \leq \infty$, assuming that $s_1 > s_2,\ 0 < p_1 \leq p_2 \leq \infty$ (with $p_2 < \infty$ in the $F-$ case), and $\delta > 0$, whereas this is not true for $\delta = 0$ and all $q-$ parameters in the $B-$ case, see (1.2.6). This is the first reason why $\delta = 0$ can be regarded as some *limiting case*. We give further arguments below. There is no compact embedding in case of (unweighted) spaces on $\mathbb{R}^n$.

Embeddings between weighted spaces

In Section 2 we consider a special limiting case where both source and target space are chosen as Sobolev spaces. For this reason we give the corresponding non-limiting result of weighted embeddings in this adapted special setting only though it is valid for much more general situations.

Theorem 1.2.1 [HT94a, Thm. 2.3] *Let* $-\infty < s_2 < s_1 < \infty,\ 1 < p_1 \leq p_2 < \infty$ *and* w_1, w_2 *be admissible weight functions according to Definition* 1.1.10.

(i) $H^{s_1}_{p_1}(w_1(\cdot), \mathbb{R}^n)$ *is continuously embedded in* $H^{s_2}_{p_2}(w_2(\cdot), \mathbb{R}^n)$,

$$H^{s_1}_{p_1}(w_1(\cdot), \mathbb{R}^n) \hookrightarrow H^{s_2}_{p_2}(w_2(\cdot), \mathbb{R}^n), \tag{1.2.8}$$

if, and only if,

$$\delta \ \geq \ 0 \quad \textit{and} \quad \frac{w_2(x)}{w_1(x)} \leq c < \infty \tag{1.2.9}$$

for some $c > 0$ *and all* $x \in \mathbb{R}^n$.

(ii) *The embedding* (1.2.8) *is compact, if, and only if,*

$$\delta \ > \ 0 \quad \textit{and} \quad \frac{w_2(x)}{w_1(x)} \longrightarrow 0 \quad \textit{if} \quad |x| \longrightarrow \infty. \tag{1.2.10}$$

Remark 1.2.2 Recall that when $w_1(x) = w_2(x) = 1$ one obtains the unweighted case and (1.2.9) is simply the well-known embedding theorem in $\mathbb{R}^n$. Furthermore one obviously has no compact embedding in the unweighted case, in view of (1.2.10). Let us mention that Theorem 1.2.1 has also been proved in the wider context of B- and F-spaces in [HT94a, Thm. 2.3] where more details can be found, too.

Note that by (1.1.54) and conditions (1.2.9), (1.2.10) it is completely sufficient to consider situations where only the source space is weighted, the target one unweighted. For later use we specify two embedding maps $id^{\alpha,\mu}$ and id^{β} as follows. In view of Theorem 1.2.1 it is obvious that both embedding operators

$$id^{\alpha,\mu} : H^{s_1}_{p_1}\left(\langle x\rangle^{\alpha}\ \log^{\mu}\langle x\rangle\right) \longrightarrow H^{s_2}_{p_2}\ , \quad \alpha > 0,\ \mu \in \mathbb{R}, \tag{1.2.11}$$

and

$$id^{\beta} : H^{s_1}_{p_1}\left(\log^{\beta}\langle x\rangle\right) \longrightarrow H^{s_2}_{p_2}\ , \quad \beta > 0, \tag{1.2.12}$$

are compact if $\delta > 0$, where we assume $s_2 < s_1$ and $1 < p_1 \leq p_2 < \infty$. (Note that there are extension to values $p_2 < p_1$ when $\alpha > 0$, but this is of no further interest in our context of limiting situations.)

Embeddings between spaces on domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain, assume $-\infty < s_2 \leq s_1 < \infty,\ 0 < p_1, p_2 \leq \infty$ ($p_1, p_2 < \infty$ in the F-case), $0 < q_1, q_2 \leq \infty$, and denote by id_Ω the natural embedding operator

$$id_\Omega = id_\Omega : A^{s_1}_{p_1,q_1}(\Omega) \longrightarrow A^{s_2}_{p_2,q_2}(\Omega)\ , \tag{1.2.13}$$

where the spaces $A^s_{p,q}(\Omega)$ are given by Definition 1.1.12. Then id_Ω is continuous when

$$\delta_+ := s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \geq 0 \tag{1.2.14}$$

and $q_1 \leq q_2$ if $\delta_+ = 0$ in the B- case. Furthermore, id_Ω becomes compact when $\delta_+ > 0$. The extension to values $p_2 < p_1$ – compared with the $\mathbb{R}^n$- setting – is due to Hölder's inequality and the finite measure $|\Omega| < \infty$.

1.2.2 Limiting embeddings

It is known that spaces of type $A^s_{p,q}$ can – roughly speaking – be embedded along lines of constant differential dimension $s - \frac{n}{p} \equiv const$; see (1.2.6). Moreover, by (1.2.10) and the situation described for id_Ω it is obvious, that the case $\delta = 0$ is not only more difficult to handle, but also refers to a different quality of embeddings – one has compactness of the corresponding embeddings only for $\delta > 0$. This led us to a separate study of continuous embeddings $A^{s_1}_{p_1,q_1} \hookrightarrow A^{s_2}_{p_2,q_2}$ (on $\mathbb{R}^n$ with weights, or on bounded $\Omega \subset \mathbb{R}^n$) in the so-called

$$\textbf{limiting case}, \quad \text{i.e.} \quad \delta = 0\ . \tag{1.2.15}$$

We shall retain this meaning of a *'limiting embedding'* throughout this report.

In the usual $(\frac{1}{p}, s)$-diagram, where any space of the above type is characterised by its parameters s and p (neglecting q for the moment), that is $A^s_{p,q} \leftrightarrow (\frac{1}{p}, s)$, these embeddings correspond to embeddings along lines with slope n, i.e. $s - \frac{n}{p} \equiv const$. In view of the historical background (0.1), that is, the question whether a space contains essentially unbounded functions, it is reasonable to call embeddings (or simply spaces) of type $A^s_{p,q}$ with $s - \frac{n}{p} = 0$ '*critical*', whereas situations with $s - \frac{n}{p} > 0$ and $s - \frac{n}{p} < 0$ are regarded as '*super-critical*' or '*sub-critical*', respectively. Moreover, as indicated in the diagram aside, we shall merely study spaces where $\sigma_p \leq s \leq \frac{n}{p} + 1$. The idea to focus on that set of parameters has essentially two reasons. It turns out that – in general – the concepts we study make sense only for spaces $A^s_{p,q} \subset L_1^{\mathrm{loc}}$, i.e. when we deal with locally integrable functions.

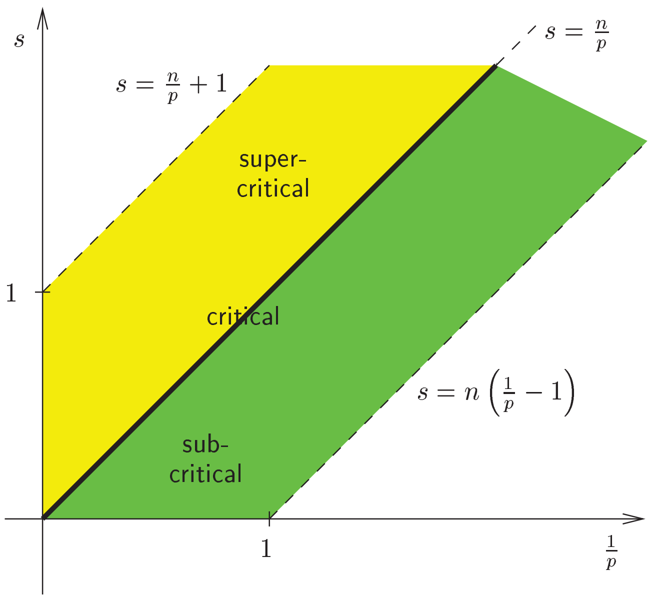


Figure 1

This implies that we have to assume $s \geq \sigma_p$; for a complete characterisation of $A^s_{p,q} \subset L_1^{\mathrm{loc}}$ see [ST95, Thm. 3.3.2] by Sickel and Triebel. We return to this point later. On the other hand, spaces with $s > \frac{n}{p} + 1$ are not very interesting in our context, we refer to our introductory remarks in Section 6.2 below. Thus we shall rely on the notation as indicated in Figure 1, where both, the *super-critical* and the *sub-critical* case are represented by the corresponding strips in the diagram.

For later use it is reasonable to complement (1.2.6) by its counterpart concerning the case when both, B- as well as F-spaces are involved (as source or target spaces, respectively). Having different smoothness parameters s_i in the spaces under consideration, then the situation (1.2.3) is improved as follows; we gain from a result of Sickel and Triebel in [ST95, Thm. 3.2.1]. Let $0 < p_0 < p < p_1 \leq \infty$, $s \in \mathbb{R}$, $s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}$, and $0 < q \leq \infty$, $0 < u \leq \infty$, $0 < v \leq \infty$, then

$$B^{s_0}_{p_0,u} \hookrightarrow F^s_{p,q} \hookrightarrow B^{s_1}_{p_1,v} \quad \text{if, and only if,} \quad 0 < u \leq p \leq v \leq \infty\ . \tag{1.2.16}$$

The 'if'-part of the right-hand embedding is due to JAWERTH [Jaw77], whereas the 'if'-part of the left-hand embedding was proved by FRANKE [Fra86]. The sharp assertion (1.2.16) is proved in [ST95, Sect. 5.2]. In particular, (1.2.16) yields

$$B^{s_0}_{p_0,p} \hookrightarrow F^s_{p,q} \hookrightarrow B^{s_1}_{p_1,p} \tag{1.2.17}$$

for $0 < p_0 < p < p_1 \le \infty$, $s \in \mathbb{R}$, $s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}$, and $0 < q \le \infty$. Further conclusions from (1.2.1), (1.2.5) and (1.2.17) playing a crucial role in the sequel are

$$F^{n/p}_{p,q} \hookrightarrow C \quad \text{if, and only if,} \quad 0 < p \le 1 \quad \text{and} \quad 0 < q \le \infty, \tag{1.2.18}$$

and

$$B^{n/p}_{p,q} \hookrightarrow C \quad \text{if, and only if,} \quad 0 < p \le \infty \quad \text{and} \quad 0 < q \le 1, \tag{1.2.19}$$

where C in (5.3.1) and (5.3.2) can be replaced by L_∞; see [ET96, 2.3.3 (iii), p. 45]. Its lifted counterpart reads as

$$F^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^1 \quad \text{if, and only if,} \quad 0 < p \le 1 \quad \text{and} \quad 0 < q \le \infty, \tag{1.2.20}$$

and

$$B^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^1 \quad \text{if, and only if,} \quad 0 < p \le \infty \quad \text{and} \quad 0 < q \le 1; \tag{1.2.21}$$

see [ET96, (2.3.3/9,10), p. 45].

1.3 Entropy numbers

1.3.1 Definition, elementary properties

Let us briefly recall the definition of entropy numbers. Let A_1 and A_2 be two complex (quasi-) Banach spaces and let T be a linear and continuous operator from A_1 into A_2. If T is compact then for any given $\varepsilon > 0$ there are finitely many balls in A_2 of radius ε which cover the image $T\,U_1$ of the unit ball $U_1 = \{a \in A_1 \,:\, \|a|A_1\| \le 1\}$.

Definition 1.3.1 *Let $k \in \mathbb{N}$ and let $T : A_1 \to A_2$ be the above continuous operator. The* kth entropy number e_k *of T is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in A_2 of radius ε which cover $T\,U_1$.*

For details and properties of entropy numbers we refer to [CS90], [EE87], [Kön86] and [Pie87] (always restricted to the case of Banach spaces). The extension of these properties to quasi-Banach spaces causes no problems. Among other features we only want to mention the multiplicativity of entropy numbers : let A_1, A_2 and A_3 be complex (quasi-) Banach spaces and $T_1 : A_1 \longrightarrow A_2$, $T_2 : A_2 \longrightarrow A_3$ two operators in the sense of Definition 1.3.1. Then

$$e_{k_1+k_2-1}(T_2 \circ T_1) \le e_{k_1}(T_1)\, e_{k_2}(T_2), \quad k_1, k_2 \in \mathbb{N}. \tag{1.3.1}$$

Note that one has in general that

$$\lim_{k\to\infty} e_k(T) = 0 \quad \Longleftrightarrow \quad T \quad \text{compact} . \tag{1.3.2}$$

The last equivalence justifies the saying that entropy numbers measure 'how compact' an operator acts. This is one reason to study the asymptotic behaviour of entropy numbers (that is, their decay) for compact operators in detail.

1.3.2 Related results in the non-limiting situation

We restrict ourselves to give the main results related to compact embeddings of function spaces on domains and in weighted spaces – always bound to the non-limiting setting. The famous forerunner of all these considerations is certainly the result of EDMUNDS and TRIEBEL [ET89, ET92] (see also [ET96, Thm. 3.3.3/2, p. 118]) : Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain and id_Ω the compact embedding operator given by (1.2.13), $id_\Omega : A^{s_1}_{p_1,q_1}(\Omega) \longrightarrow A^{s_2}_{p_2,q_2}(\Omega)$. Then

$$e_k(id_\Omega) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}, \tag{1.3.3}$$

where $s_1 \ge s_2$, $0 < p_1, p_2 \le \infty$ ($p_1, p_2 < \infty$ in the F-case), $0 < q_1, q_2 \le \infty$, and $\delta_+ > 0$. We come to the situation of weighted spaces now, where the weights are of type (1.1.51).

Proposition 1.3.2 [Har97a, Prop. 4.1] *Let* $s_1 > s_2$, $1 < p_1, p_2 < \infty$, *with* $\frac{1}{p_1} \le \frac{1}{p_2} \le \frac{1}{p_1} + \frac{\alpha}{n}$, *and* $\delta > 0$. *Assume* $\alpha > 0$, *and* $\mu \in \mathbb{R}$. *Then* $id^{\alpha,\mu}$ *from* (1.2.11) *is compact, and we have the following estimates for its entropy numbers.*

(i) *Let* $0 < \delta < \alpha$, $\mu \in \mathbb{R}$. *Then*

$$e_k(id^{\alpha,\mu}) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}. \tag{1.3.4}$$

(ii) *Let* $\delta > \alpha > 0$, $\mu \in \mathbb{R}$, *and* $\frac{1}{p_1} \le \frac{1}{p_2} \le \frac{1}{p_1} + \frac{\alpha}{n}$. *Then*

$$e_k(id^{\alpha,\mu}) \sim k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} (\log\langle k\rangle)^{-\mu}, \quad k \in \mathbb{N}. \tag{1.3.5}$$

(iii) *Let* $\delta > \alpha > 0$, $\mu \in \mathbb{R}$, *and* $p_1 < p_2$. *Then there exist a constant* $c > 0$ *and for any* $\varepsilon > 0$ *a constant* $c_\varepsilon > 0$ *such that for all* $k \in \mathbb{N}$

$$c\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} (\log\langle k\rangle)^{-\mu} \le e_k(id^{\alpha,\mu}) \le c_\varepsilon\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} (\log\langle k\rangle)^{-\mu+\varepsilon+\frac{1}{p_1}+1-\frac{1}{p_2}}. \tag{1.3.6}$$

(iv) *Let* $\delta = \alpha > 0$, *and* $\mu > \frac{s_1-s_2}{n} + 1$. *Then*

$$e_k(id^{\alpha,\mu}) \sim k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}. \tag{1.3.7}$$

Remark 1.3.3 We restricted ourselves in part (iv) of Proposition 1.3.2 to that situation concerning $\mu \in \mathbb{R}$ where a satisfying answer could be achieved. There are counterparts of (1.3.7) in case of $\mu \le \frac{s_1-s_2}{n} + 1$, but at the expense of a gap between upper and lower bound for the respective entropy numbers $e_k(id^{\alpha,\mu})$; the case $\mu = 0$ is covered by our more general result [HT94a, Thm. 4.2], complemented and partly improved in [Har97a].

Dealing with limiting situations in the sequel, we are mainly interested in situations related to (i), (iii) and (iv) where $p_1 < p_2$. Finally, when

$$w(x) = \log^\beta\langle x\rangle, \quad \beta > 0,$$

(the special weight we mainly want to use in the following) our estimate reads as follows.

Proposition 1.3.4 [Har97a, Prop. 4.4] *Let* $\beta > 0$, $1 < p_1 \le p_2 < \infty$, $s_2 < s_1$ *and* $\delta > 0$. *Denote by* $e_k(id^\beta)$, $k \in \mathbb{N}$, *the respective entropy numbers of the compact embedding operator*

$$id^\beta : H^{s_1}_{p_1}\left(\log^\beta\langle x\rangle\right) \longrightarrow H^{s_2}_{p_2}.$$

Then there are two constants $c_1 > 0$ *and* $c_2 > 0$ *such that for all* $k \in \mathbb{N}$

$$c_1\, k^{-\frac{1}{p_1}+\frac{1}{p_2}} (\log\langle k\rangle)^{-\beta} \le e_k(id^\beta) \le c_2\, (\log\langle k\rangle)^{-\beta}.$$

1.3.3 Connection with applications

The study of entropy numbers of embeddings between function spaces is closely related to the distribution of eigenvalues of (degenerate) elliptic operators, as the books [ET96] and [Tri97] show.

Carl's inequality

The motivation comes from CARL'S inequality giving an excellent link to possible applications, in particular, between entropy numbers and eigenvalues of some compact operator. The setting is the following. Let A be a complex (quasi-) Banach space and $T \in \mathcal{L}(A)$ compact. Then the spectrum of T (apart form the point 0) consists only of eigenvalues of finite algebraic multiplicity. Let $\{\mu_k(T)\}_{k\in\mathbb{N}}$ be the sequence of all non-zero eigenvalues of T, repeated according to algebraic multiplicity and ordered such that

$$|\mu_1(T)| \geq |\mu_2(T)| \geq \ldots \geq 0.$$

Then CARL's inequality states that

$$\left(\prod_{m=1}^{k} |\mu_m(T)|\right)^{1/k} \leq \inf_{n\in\mathbb{N}} 2^{\frac{n}{2k}}\, e_n(T)\,, \qquad k\in\mathbb{N}.$$

In particular, we have

$$|\mu_k(T)| \leq \sqrt{2}\, e_k(T). \tag{1.3.8}$$

This result was originally proved by CARL in [Car81] and CARL and TRIEBEL in [CT80] when A is a Banach space. An extension to quasi-Banach spaces is proved in [ET96, Thm. 1.3.4].

Eigenvalue distribution

We consider the operator

$$B = b_2 \circ b(\cdot, D) \circ b_1 \tag{1.3.9}$$

acting in some L_p space where $b(x,D)$ is in some Hörmander class $\Psi^{-\varkappa}_{1,\gamma}$, $\varkappa > 0$, $0 \leq \gamma \leq 1$, and the functions $b_i(x)$, $i=1,2$, belong to certain function spaces. Let $\{\mu_k\}$ be the sequence of the eigenvalues of B, counted according to their algebraic multiplicity and ordered by decreasing modulus as described above. In view of CARL's inequality (1.3.8) one arrives at $|\mu_k| \leq \sqrt{2}\, e_k(B)$; this problem can often be reduced further to the study of entropy numbers of suitable embeddings assuming that one has corresponding Hölder inequalities for b_1, b_2 available.

Negative spectrum

Another possible application is connected with the Birman-Schwinger principle as described in [Sch86, Ch. 8, Sect. 5, p. 193]. Let A be a self-adjoint operator acting in a Hilbert space $\mathcal{H}$ and let A be positive. Let V be a closable operator acting in $\mathcal{H}$ and suppose that $K:\mathcal{H}\to\mathcal{H}$ is a compact linear operator such that

$$Ku \;=\; VA^{-1}V^*u \qquad \text{for all} \quad u \in \mathrm{dom}(VA^{-1}V^*)$$

where V^* is the adjoint of V. Assume that $\mathrm{dom}(A) \cap \mathrm{dom}(V^*V)$ is dense in $\mathcal{H}$. Then the above-mentioned result provides : $A - V^*V$ has a self-adjoint extension H with pure point spectrum in $(-\infty, 0]$ such that

$$\#\,\{\sigma(H)\cap(-\infty,0]\,\} \;\leq\; \#\,\{k\in\mathbb{N} : |\lambda_k| \geq 1\}$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of K, counted according to their multiplicity and ordered by decreasing modulus. The number of elements of a finite set M is denoted by $\#\,M$, as usual. In particular, we consider the behaviour of the 'negative spectrum' $\sigma(H_\nu)\cap(-\infty,0]$ of the self-adjoint unbounded operator

$$H_\nu = a(x,D) - \nu b^2(x) \qquad \text{as} \quad \nu\to\infty \tag{1.3.10}$$

where

$$a(x,D) \in \Psi^{\varkappa}_{1,\gamma}\,, \quad \varkappa > 0\,, \quad 0 \leq \gamma < 1, \tag{1.3.11}$$

is assumed to be a positive-definite and self-adjoint operator in L_2 and $b(x)$ is a real-valued function. We know from former considerations, cf. [HT94b, 2.4, 5.2], that

$$\#\{\sigma(H_\nu)\cap(-\infty,0]\} \leq \#\left\{k\in\mathbb{N} : \sqrt{2}\; e_k \geq \nu^{-1}\right\} \tag{1.3.12}$$

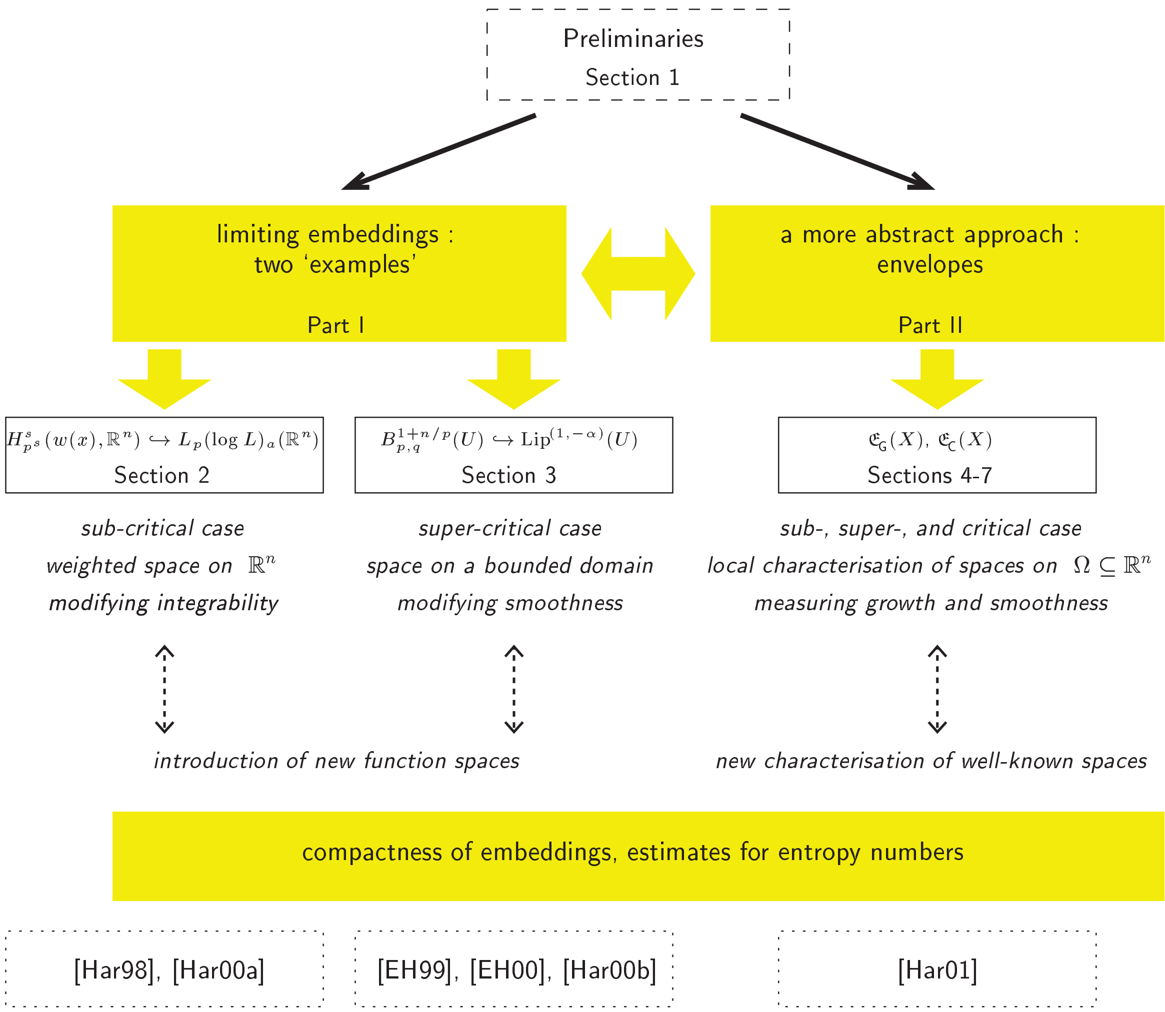
with $e_k = e_k\big(b(x)\, b(x,D)\, b(x)\big)$ and $b(x,D) = a^{-1}(x,D) \in \Psi^{-\varkappa}_{1,\gamma}$.

These are essentially the applications we have in mind for using our results on entropy numbers of compact embeddings. This programme was carried out in [HT94b], [ET96], first, and [Tri97], [Har98], [Har00a], [EH00] in different settings afterwards; we refer to these papers and books for details.

1.4 Concept and structure of this report

The idea of the present report is to collect material on limiting embeddings, entropy numbers and envelopes, mainly published already, and to arrange it in a more coherent form than the separate parts (papers) provide. We would like to qualify this immediately by confessing that we do not aim at a survey of these topics in the sense that the state of the art as well as all the historic background is reflected completely. Based on our own results we shall give all the related references we know of at the moment, but the selection of the presented material is guided by our own goals only.

We explain the structure of the report, see also the diagram below.



The report is divided in two parts which reflect different approaches to the topic of limiting embeddings. According to our philosophy explained in Section 1.3.3 we shall be concerned with embeddings of type

$$id : A^{s_1}_{p_1,q_1} \longrightarrow A^{s_2}_{p_2,q_2} \tag{1.4.1}$$

mainly, where $s_1 \geq s_2$, $0 < p_1, p_2 \leq \infty$ ($p_1, p_2 < \infty$ in the F-case), $0 < q_1, q_2 \leq \infty$, and

$$\delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right) = 0, \tag{1.4.2}$$

see (1.2.15). Furthermore, as the determination of entropy numbers is another objective of this report, we are especially interested in compact limiting embeddings; a comparison with the non-limiting situation described in Section 1.3.1 suggests that the setting should be adapted to either the study of weighted spaces on $\mathbb{R}^n$,

$$id_w : A^{s_1}_{p_1,q_1}(w(\cdot),\mathbb{R}^n) \longrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)\,, \tag{1.4.3}$$

or to spaces on bounded domains $\Omega \subset \mathbb{R}^n$,

$$id_\Omega : A^{s_1}_{p_1,q_1}(\Omega) \longrightarrow A^{s_2}_{p_2,q_2}(\Omega)\ . \tag{1.4.4}$$

This is exactly the programme followed in Part I, where Section 2 concerns a model case for id_w, and Section 3 is devoted to some question derived from (1.4.4). Moreover, both sections in Part I differ inasmuch as the problem posed in Section 2 leads to modification in the integrability of the regarded functions, whereas this is replaced in Section 3 by refined smoothness assertions. Linking it with the general setting described in Fig. 1, Section 2 refers to the sub-critical case and Section 3 to the super-critical one. Both model cases investigated in Sections 2 and 3 share, however, one essential feature : the 'repair' of the (original) loss of compactness in limiting embeddings $id_w,\ id_\Omega$, is achieved in either case by the introduction of new function spaces, especially adapted to the problem under consideration. Thus the disadvantage is obvious – the solution appears *hand-made* and can hardly be transferred to other problems. However, as the introduction of the new spaces relies in both cases on well-known concepts (such as the Lorentz-Zygmund spaces in Section 2 and the famous Brézis-Wainger inequality (3.1.3) in Section 3), the construction seems quite natural - *at least we would like to convince the reader of this claim* ! Moreover, the restriction to very special settings as in Part I permits subsequently a variety of results and applications. This briefly outlines the pros and cons of our approach in Part I; more details can also be found in the introductory Sections 2.1 and 3.1.
The method performed in Part II is now easy to explain : In contrast to Part I we concentrate on a more general, abstract approach, tackling all sub-, super- and critical cases given by Fig. 1. We do not seek for *new spaces*, but *new descriptions for well-known spaces*. The concept of envelopes is separated from (special) limiting embeddings, dealing instead with the (involved) spaces. Of course, the idea to introduce envelopes arose from the well-tilled field of limiting embeddings, too, and has thus inherited intrinsic features of this background; but this should rather be regarded as some *motivation* for studying envelopes, the corresponding definitions can be understood independently of it. We explain the idea first in simple, well-known terms and with very classical examples before climbing up to the peaks – the corresponding results in terms of spaces of type $A^s_{p,q}$; this is indeed technically more complicated, but their simple elegance undoubtedly compensates for the preceding efforts. This phenomenon can be experienced twice : what is first carried out in view of measuring local growth (unboundedness) of functions is afterwards presented in a parallel approach to characterise smoothness of functions. Roughly speaking, the most interesting spaces we deal with are such 'nearby' L_∞ (or other Lebesgue spaces L_u) and, secondly, those containing functions which are 'almost' Lipschitz continuous.
We emphasised the independence of the concept of envelopes from limiting embeddings, but already confessed that there are close (historic) links, too. In that sense our sharp assertions on envelopes imply a lot of interesting (new) inequalities; in our opinion, however, the essential advantage of this new approach rather results from its simplicity when establishing (so far final) answers to relatively difficult questions. There is only one exception of this statement explicitly to be mentioned : It concerns the very last part of this report when we study the interplay of envelops, lifts and compact embeddings, including first entropy numbers estimates obtained as *applications* of envelope results. This seems to be a promising new subject and worth to be investigated further; it is, however, left for future work.

We are thus immediately led to further confessions *what will not be contained in this report* :

- no approximation number results : Another tool to characterise compact embeddings more precisely is the concept of approximation numbers which can be used effectively for applications, too. We dealt with corresponding estimates in [EH00], that is in the model case described in Section 3, and in [Har01] briefly. But as we lack results for the first model case in Section 2 and have to restrict the length of the report anyway, we decided to skip this topic completely.
- no more general settings (measure spaces, homogeneous spaces) : Likewise we dealt in our papers [Har98] and [Har01] with slightly more general settings than presented here; the first model case given in Section 2

is considered in the framework of homogeneous type spaces in [Har98, Sect. 5] whereas the approach in [Har01] relies partially on more general measure spaces than $\mathbb{R}^n$ equipped with the Lebesgue measure only. For reasons of consistency (and length) we also omitted these extensions.

- no applications : Finally, we do not give any applications of our entropy number results in the sense indicated in Section 1.3.3. Although we pursued this line in both model cases, see [Har98, Sect. 4], [Har00a, Sect. 4], [EH00, Sect. 4], and consider it in fact for one of the strongest reasons to study entropy numbers in detail, we have to leave it out by means of restriction simply. Nevertheless we decided to outline the link between entropy numbers and possible applications in Section 1.3.3 briefly, as the motivation to study questions of compactness in limiting cases appears essentially weaker otherwise.

Formally the report is built upon our papers [Har98], [Har00a], [EH99], [EH00], [Har00b] and the recent preprint [Har01]. More precisely, in Section 2 we use results from [Har98] and [Har00a], whereas Section 3 relies on [EH99], [EH00] and [Har00b]; Part II consists of [Har01] mainly. All the material is selected and re-arranged under the above-described programme and restrictions. Moreover, we do not give any proofs of our results here (apart from very few original assertions); they can be found in the original papers according to the given references. We insert some sketches of proofs only when we think it indisputable for the comprehension of the background, for realising technical difficulties, or, conversely, the interaction of apparently separated components and methods. Certainly this reduces the comprehensibility of a mathematical report necessarily; but we found no reasonable alternative in view of its length. On the other hand we tried to lay more emphasis on the account *why* this and that solution or definition was chosen – correspondingly the presentation *how* it worked technically came second to it. This also explains that we conceded motivating arguments, examples and comparisons (with well-known facts) relatively large scope. We hope that this selection of the material and concentration on more descriptive and explanatory elements does not prevent but – quite the reverse – encourages the honourable reader !

Part I
Limiting embeddings, entropy numbers

2 Modified integrability

2.1 Introduction

We start with a model case for id_w from (1.4.3). It is known by Theorem 1.2.1 that the embedding operator

$$id_H : H^{s_1}_{p_1}(w(\cdot), \mathbb{R}^n) \quad \longrightarrow \quad H^{s_2}_{p_2}(\mathbb{R}^n) \tag{2.1.1}$$

is continuous if, and only if, the weight function $w(x)$ is bounded from below,

$$w(x) \geq c > 0, \quad x \in \mathbb{R}^n, \qquad \text{and} \qquad \delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right) \geq 0, \tag{2.1.2}$$

where $-\infty < s_2 < s_1 < \infty$, $1 < p_1 \leq p_2 < \infty$, and $w(x)$ is of type

$$w(x) = \langle x \rangle^{\alpha} \ \log^{\mu}\langle x \rangle \ , \quad \alpha \in \mathbb{R}, \ \mu \in \mathbb{R}.$$

Moreover, id_H from (2.1.1) is compact if, and only if, $w(x) \longrightarrow \infty$ as $|x| \to \infty$ and δ from (2.1.2) is positive, $\delta > 0$. We are thus led to the problem of characterising this compactness of id_H further in terms of entropy (or approximation) numbers. We studied this question in [HT94a], [Har97a] and obtained estimates for the respective entropy numbers $e_k(id_H)$ of the form $e_k(id_H) \sim k^{-\varkappa} \log^{\beta}\langle k \rangle$, $k \in \mathbb{N}$, where the numbers $\varkappa$, β depend upon the given parameters s_i, p_i, $i = 1, 2$, and the weight function, see also our survey [Har97b].

There are various possibilities to come to 'limiting embeddings' based on (2.1.1). According to the philosophy of this report (1.2.15) we stick at $\delta = 0$ now. Obviously compactness of id_H from (2.1.1) is then lost independently of the weight chosen. We handle a model case first and simplify the setting as much as possible from the very beginning. We assume for the target space $s_2 = 0$, i.e. a Lebesgue space $L_{p_2}(\mathbb{R}^n)$, and fix the weight by $w(x) = \log^{\beta}\langle x \rangle$, $\beta > 0$. Now the idea is clear : the source space $H^{s_1}_{p_1}\big(\log^{\beta}\langle x \rangle, \mathbb{R}^n\big)$ becomes smaller depending upon $\beta > 0$. Although this is not sufficient to gain compactness of

$$id^{\beta} : H^{s_1}_{p_1}\left(\log^{\beta}\langle x \rangle, \mathbb{R}^n\right) \longrightarrow L_{p_2}(\mathbb{R}^n)\, , \tag{2.1.3}$$

where $\beta > 0$, $s_1 > 0$, $1 < p_1 < p_2 < \infty$, and $s - \frac{n}{p_1} = -\frac{n}{p_2}$, one tries to enlarge the target space $L_{p_2}(\mathbb{R}^n)$ simultaneously to achieve compactness, but also keeping the integrability index p_2 fixed (that is, preserving $\delta = 0$). One needs reasonable extensions of $L_{p_2}(\mathbb{R}^n)$ as described above. Here we benefitted essentially from parallel work done for function spaces on bounded domains. Let $\Omega \subset \mathbb{R}^n$ be a bounded C^{∞} domain and denote by

$$id_{\Omega} : H^{s_1}_{p_1}(\Omega) \longrightarrow L_{p_2}(\Omega), \tag{2.1.4}$$

where the parameters are as above. Embeddings of that type (in particular, what concerns questions of compactness and corresponding entropy numbers) have been studied by Edmunds and Triebel in [ET89], [ET92] for the non-limiting case ($\delta > 0$) and in [ET95], [ET96] for the limiting one ($\delta = 0$), respectively; in the 'limiting situation' they led to the replacement of the target space $L_{p_2}(\Omega)$ by the logarithmic Lebesgue space $L_{p_2}(\log L)_a(\Omega)$, $a < 0$. Dealing with weighted spaces on $\mathbb{R}^n$ – being in some sense the 'natural counterpart' of spaces on bounded domains – we follow this idea, but immediately face the problem of a suitable definition for $L_{p_2}(\log L)_a(\mathbb{R}^n)$. The first main question to answer is to develop a reasonable definition of those spaces. We present some motivation and our approach in Section 2.2. In Section 2.3 we give some more features of the 'new' spaces, serving as some justification for their definition, too. Finally, we end this section with our results on the compactness of embeddings of type (2.1.4) (where $L_{p_2}(\mathbb{R}^n)$ is replaced by $L_{p_2}(\log L)_a(\mathbb{R}^n)$, $a < 0$), and on corresponding entropy numbers; this is always compared with the associated non-limiting outcome.

The material we present in this section is essentially based on our papers [Har98] and [Har00a]; we summarise in this report, however, only selected results – according to our general strategy to describe model cases only focused now under a certain point of view : '*how to handle limiting situations when non-limiting counterparts are well-understood*'. Likewise all related proofs and further minor results (which do not contribute to our above question directly) are to be found in these papers according to the references given below. Moreover, for the same reason we completely skip a discussion of possible applications as well as further extensions to homogeneous type spaces in this context; details can be found in [Har98, Sect. 4,5] and [Har00a, Sect. 4].

2.2 Spaces of type $L_p(\log L)_a$ and $H^s_p(\log H)_a$ on $\mathbb{R}^n$; basic properties

We introduce logarithmic spaces of type $L_p(\log L)_a$ and $H^s_p(\log H)_a$ on $\mathbb{R}^n$. One should always keep in mind that we study the embedding (2.1.3) with $\beta > 0$, $s_1 > 0$, $1 < p_1 < p_2 < \infty$, and $s - \frac{n}{p_1} = -\frac{n}{p_2}$. For that reason and a parallel study related to (unweighted) spaces on bounded domains Ω, where

$$id_{\Omega,a} : H^{s_1}_{p_1}(\Omega) \quad \longrightarrow \quad L_{p_2}(\log L)_{-a}(\Omega)$$

is compact for any $a > 0$, we try to enlarge $L_{p_2}(\mathbb{R}^n)$ slightly to some space $L_{p_2}(\log L)_{-a}(\mathbb{R}^n)$. The problem thus consists in finding a suitable counterpart on $\mathbb{R}^n$ of $L_p(\log L)_a(\Omega)$ given by Definition 1.1.1 (ii) with $p = q$, as usual. There are, however, different ways of extension depending upon the preceding decision which features should be kept in any case – and which might go lost. (At first glance one could hope, of course, to find the one extension which carries over all nice properties of $L_p(\log L)_a(\Omega)$ to $L_p(\log L)_a(\mathbb{R}^n)$; but – whether it appears disappointing or rather normal in life – this desideratum cannot exist.) One has to balance advantages and disadvantages of this or that approach – according to the purpose one has in mind. Let us only mention two different approaches of extending $L_p(\log L)_a(\Omega)$ to $\mathbb{R}^n$: firstly, a very natural way was to replace $|\Omega|$ by ∞ in (1.1.11), i.e. to require

$$\left(\int_0^\infty (1 + |\log t|)^{ap} \, f^*(t)^p \, \mathrm{d}t \right)^{1/p} < \infty$$

and to construct spaces on that basis. For later reason we shall call these spaces $L_p(\log L)^*_a(\mathbb{R}^n)$. Another – and from our point of view preferred – extension relies on a characterisation of spaces $L_p(\log L)_a$ by means of extrapolating L_p spaces (corresponding to non-limiting situations). The gain following that method was obvious – we could benefit from our exact knowledge on compact embeddings in non-limiting situation (as briefly mentioned in Subsection 1.2.1) when tackling the limiting one. The price to pay for this better adapted setting we choose is, for instance, that the spaces $L_p(\log L)_a(\mathbb{R}^n)$ and $L_p(\log L)^*_a(\mathbb{R}^n)$ differ – unlike in case of a bounded underlying domain Ω. We return to this point in Subsection 2.2.4 below.

2.2.1 Motivation – spaces on Ω revisited

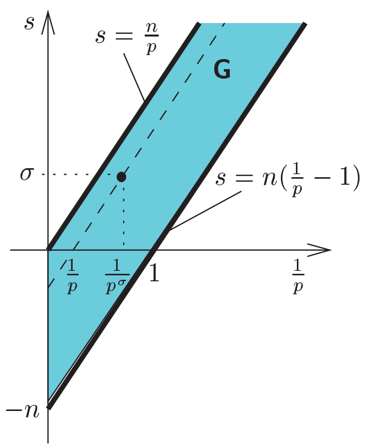


Figure 2

Recall the definition of spaces $L_p(\log L)_a(\Omega)$ by Definition 1.1.1 (ii) with $p = q$. As already announced we are more interested in characterisations of these spaces by extrapolation techniques as obtained by Edmunds and Triebel in [ET96, Thm. 2.6.2, p. 69]. We start with some notation. Introduce the strip

$$\mathbf{G} = \left\{ \left(\frac{1}{p}, s\right) : 0 < p < \infty, n\left(\frac{1}{p} - 1\right) < s < \frac{n}{p} \right\}$$

in the usual $(\frac{1}{p}, s)$ diagram, see Figure 1, where $H^s_p \leftrightarrow (\frac{1}{p}, s)$, $0 < p < \infty$, $s \in \mathbb{R}$. Any line of slope n is characterised by its 'foot point' where it meets the axis $s = 0$. For convenience we adopt the notation

$$\frac{1}{p^\sigma} = \frac{1}{p} + \frac{\sigma}{n}, \tag{2.2.1}$$

where $1 < p < \infty$, $\sigma \in \mathbb{R}$ and $1 < p^\sigma < \infty$.

Theorem 2.2.1 [ET96, Thm. 2.6.2, p. 69]

(i) *Let* $1 < p < \infty$ *and* $a > 0$. *Then* $L_p(\log L)_{-a}(\Omega)$ *is the set of all measurable functions* $f : \Omega \longrightarrow \mathbb{C}$ *such that*

$$\left(\int_0^{\varepsilon} \Big[\sigma^a \|f|L_{p^\sigma}(\Omega)\| \Big]^p \frac{\mathrm{d}\sigma}{\sigma} \right)^{1/p} < \infty \tag{2.2.2}$$

for small $\varepsilon > 0$, *and* (2.2.2) *defines an equivalent norm on* $L_p(\log L)_{-a}(\Omega)$. *Furthermore,* (2.2.2) *can be replaced by the equivalent norm*

$$\left(\sum_{j=J}^{\infty} 2^{-jap} \|f|L_{p^{\sigma(j)}}(\Omega)\|^p \right)^{1/p} < \infty \tag{2.2.3}$$

for large $J \in \mathbb{N}$ *and* $\sigma(j) = 2^{-j}$.

(ii) *Let* $1 < p < \infty$ *and* $a > 0$. *Then* $L_p(\log L)_a(\Omega)$ *is the set of all measurable functions* $g : \Omega \longrightarrow \mathbb{C}$ *which can be represented as*

$$g = \sum_{j=J}^{\infty} g_j \,, \quad g_j \in L_{p^{-\sigma(j)}}(\Omega) \,, \tag{2.2.4}$$

for large $J \in \mathbb{N}$, *with*

$$\left(\sum_{j=J}^{\infty} 2^{jap} \|g_j|L_{p^{-\sigma(j)}}(\Omega)\|^p \right)^{1/p} < \infty \,. \tag{2.2.5}$$

The infimum of the expression (2.2.5) *taken over all admissible representations* (2.2.4) *is an equivalent norm on* $L_p(\log L)_a(\Omega)$.

There is also an extension of this theorem to spaces $L_{p,q}(\log L)_a(\Omega)$ in [Har98, Prop. 2.5]. Note that assertion (ii) looks technically more complicated because we have (in the above notation)

$$L_{p^{-\sigma(j)}}(\Omega) \hookrightarrow L_p(\log L)_a(\Omega) \hookrightarrow L_p(\Omega) \hookrightarrow L_p(\log L)_{-a}(\Omega) \hookrightarrow L_{p^\sigma}(\Omega) \,, \quad |\Omega| < \infty, \tag{2.2.6}$$

where $1 < p < \infty$, $a > 0$, such that $f \in L_p(\log L)_{-a}(\Omega)$ belongs to all spaces $L_{p^\sigma}(\Omega)$, $\sigma > 0$, in (i), whereas this is not the case in situation (ii); see also (1.1.13). When $a \geq 0$, there is a similar result in [Sob98] by Sobukawa.

Remark 2.2.2 We want to discuss the use of the above theorem for our purposes a bit further. The idea of this characterisation is to 'approximate' spaces $L_p(\log L)_a(\Omega)$ by usual Lebesgue spaces in a precise way (rather than by (2.2.6) simply). The main reason for this in [ET96] was to make these spaces $L_p(\log L)_a(\Omega)$ (appearing in limiting embeddings) more handy, especially from the standpoint of entropy numbers. Denoting (non-limiting) embeddings $H^s_{p^s}(\Omega) \hookrightarrow L_{p^\sigma}(\Omega)$ by id_σ, that is,

$$id_\sigma : H^s_{p^s}(\Omega) \longrightarrow L_{p^\sigma}(\Omega), \tag{2.2.7}$$

where $s > 0$, $\sigma > 0$, $1 < p < \infty$, it is well-known that id_σ is compact, see (1.2.13) with $\delta_+ = \sigma > 0$. The asymptotic behaviour of its entropy numbers is determined by $e_k(id_\sigma) \sim k^{-\frac{s}{n}}$ for all $\sigma > 0$, see (1.3.3). So if one succeeds to control the dependence of the equivalence constants upon the number $\sigma > 0$, one can hope to benefit from the non-limiting case when treating the limiting one. We return to this point later in Section 2.4 when we study the entropy numbers of limiting embeddings in detail. In the course of this programme, Edmunds and Triebel needed the above characterisation of spaces $L_p(\log L)_a(\Omega)$ in terms of 'nearby' Lebesgue spaces $L_{p^\sigma}(\Omega)$ or $L_{p^{-\sigma(j)}}(\Omega)$, respectively.

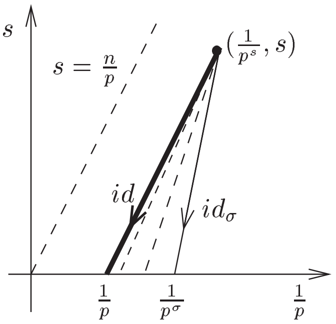


Figure 3

In Figure 3 we additionally illustrated this idea in the $(\frac{1}{p}, s)$-diagram, recall Figure 2. One is finally interested in the limiting embedding $id : H^s_{p^s}(\Omega) \longrightarrow L_p(\Omega)$, where $1 < p < \infty$, $s > 0$. This embedding is continuous, but not compact. However, replacing the target spaces $L_p(\Omega)$ by slightly larger spaces $L_p(\log L)_{-a}(\Omega)$, $a > 0$, one regains compactness and can further ask about the (asymptotic) behaviour of the corresponding entropy numbers. The essential trick of Edmunds and Triebel was now to study the same question, but taking into consideration that one has information about $e_k(id_\sigma)$ for all $\sigma > 0$.

2.2.2 Definition and elementary properties

We look for spaces larger than $L_p(\mathbb{R}^n)$ which additionally should be extensions of $L_p(\log L)_a(\Omega)$ in case of bounded $\Omega \subset \mathbb{R}^n$. In order to emphasise whether we are dealing with extensions (or restrictions) of the usual L_p space, we prefer the notation $L_p(\log L)_{-a}(\Omega)$ or $L_p(\log L)_a(\Omega)$, respectively, now always assuming $a > 0$. We retain this notation in this section.
In view of the norm expression (2.2.2) one immediately realises that in case of bounded domains Ω (or, at least, with finite measure $|\Omega| < \infty$) those spaces $L_{p^\sigma}(\Omega)$ are monotonically embedded,

$$L_p(\Omega) \hookrightarrow L_{p^\varrho}(\Omega) \hookrightarrow L_{p^\sigma}(\Omega)\ , \quad 0 \le \varrho \le \sigma, \tag{2.2.8}$$

which becomes false if Ω is replaced by $\mathbb{R}^n$. One has to find a reasonable substitution of that fact in the $\mathbb{R}^n$ situation. In a first step we slightly modify (2.2.2) in case of annuli $\Omega = A_\ell$, $\ell \in \mathbb{N}_0$, see (1.1.24), (1.1.25), by

$$\left(\int\limits_0^\varepsilon \sigma^{ap} \| f | L_{p^\sigma}(\langle x \rangle^{-\sigma}, A_\ell) \|^p \frac{\mathrm{d}\sigma}{\sigma} \right)^{1/p} , \tag{2.2.9}$$

where $\langle x \rangle$ is given by (1.1.1) and $\| \cdot | L_q(w(.), \Omega) \|$ is the weighted L_q norm, see (1.1.52). In view of (2.2.2) one recognises that (2.2.9) is an equivalent norm on $L_p(\log L)_{-a}(A_\ell)$ for any fixed $\ell \in \mathbb{N}_0$, because

$$\| f | L_{p^\sigma}(\langle x \rangle^{-\sigma}, A_\ell) \| \sim 2^{-\ell\sigma}\ \| f | L_{p^\sigma}(A_\ell) \|\ .$$

Furthermore, Hölder's inequality provides

$$L_p(A_\ell) \hookrightarrow L_{p^\varrho}(\langle x \rangle^{-\varrho}, A_\ell) \hookrightarrow L_{p^\sigma}(\langle x \rangle^{-\sigma}, A_\ell)\ , \quad 0 \le \varrho \le \sigma\ . \tag{2.2.10}$$

A simple replacement of $\| \cdot | L_{p^\sigma}(\langle x \rangle^{-\sigma}, A_\ell) \|$ by its $\mathbb{R}^n-$ counterpart still fails, but monotonicity as in (2.2.10) was important for the construction in (2.2.2). We may cope with these problems using interpolation arguments. In particular, one can prove that for $1 < p < \infty$ a Hölder inequality of type

$$L_{p^\varrho,p}(\mathbb{R}^n) \cdot L_{n/(\sigma-\varrho)\,,\infty}(\mathbb{R}^n) \hookrightarrow L_{p^\sigma,p}(\mathbb{R}^n)\ , \quad 0 \le \varrho < \sigma\ ,$$

holds, meaning that whenever $f \in L_{p^\varrho,p}(\mathbb{R}^n)$ and $g \in L_{n/(\sigma-\varrho)\,,\infty}(\mathbb{R}^n)$, then fg belongs to $L_{p^\sigma,p}(\mathbb{R}^n)$, i.e

$$\| fg | L_{p^\sigma,p}(\mathbb{R}^n) \| \ \le\ c\, \| f | L_{p^\varrho,p}(\mathbb{R}^n) \|\, \| g | L_{n/(\sigma-\varrho)\,,\infty}(\mathbb{R}^n) \|\ ;$$

cf. [Har98, Lemma 2.12]. Choosing $g(x) = \langle x \rangle^{-\sigma} \in L_{n/\sigma\,,\infty}(\mathbb{R}^n)$ we thus obtain

$$L_p(\mathbb{R}^n)\ \hookrightarrow\ L_{p^\varrho,p}(\langle x \rangle^{-\varrho}, \mathbb{R}^n)\ \hookrightarrow\ L_{p^\sigma,p}(\langle x \rangle^{-\sigma}, \mathbb{R}^n)\ , \quad 0 \le \varrho \le \sigma, \tag{2.2.11}$$

that is, the desired substitute of (2.2.8). So replacing $L_{p^\sigma}(\Omega)$ in (2.2.2) by $L_{p^\sigma,p}(\langle x \rangle^{-\sigma}, \mathbb{R}^n)$ as basic spaces, $\sigma > 0$, we arrive at the definition for spaces $L_p(\log L)_{-a}(\mathbb{R}^n)$; in view of Theorem 2.2.1 (ii) it is complemented with the definition of $L_p(\log L)_a(\mathbb{R}^n)$.

Definition 2.2.3 [Har98, Defs. 2.13, 2.20] *Let* $1 < p < \infty$ *and* $a > 0$.

(i) *The space* $L_p(\log L)_{-a}(\mathbb{R}^n)$ *is the set of all measurable functions* $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ *such that* $\langle x\rangle^{-\sigma} f \in L_{p^\sigma,p}(\mathbb{R}^n)$ *for small* $\sigma > 0$, *and*

$$\|f|L_p(\log L)_{-a}(\mathbb{R}^n)\| := \left(\int_0^\varepsilon \sigma^{ap} \,\|\langle x\rangle^{-\sigma} f|L_{p^\sigma,p}(\mathbb{R}^n)\|^p \,\frac{\mathrm{d}\sigma}{\sigma}\right)^{1/p} < \infty \tag{2.2.12}$$

for small $\varepsilon > 0$.

(ii) *The space* $L_p(\log L)_a(\mathbb{R}^n)$ *is the set of all measurable functions* $g : \mathbb{R}^n \longrightarrow \mathbb{C}$ *which can be represented as*

$$g = \sum_{j=J}^{\infty} g_j, \quad \langle x\rangle^{\sigma(j)}\, g_j \in L_{p^{-\sigma(j)},p}(\mathbb{R}^n) \tag{2.2.13}$$

for large $J \in \mathbb{N}$, $\sigma(j) = 2^{-j}$, *and*

$$\left(\sum_{j=J}^{\infty} 2^{jap} \left\|\langle x\rangle^{\sigma(j)}\, g_j|L_{p^{-\sigma(j)},p}(\mathbb{R}^n)\right\|^p\right)^{1/p} < \infty. \tag{2.2.14}$$

The infimum of expression (2.2.14), *taken over all admissible representations* (2.2.13) *is defined as* $\|g|L_p(\log L)_a(\mathbb{R}^n)\|$.

In the above definition we have introduced spaces $L_p(\log L)_a(\mathbb{R}^n)$, $1 < p < \infty$, $a \in \mathbb{R}$, $a \neq 0$. For convenience we adopt the following notation,

$$L_p(\log L)_0(\mathbb{R}^n) := L_p(\mathbb{R}^n)\,, \quad 1 < p < \infty. \tag{2.2.15}$$

Clearly, Definition 2.2.3 gives the desired $\mathbb{R}^n-$ counterpart of Theorem 2.2.1 characterising spaces $L_p(\log L)_a$, $a \in \mathbb{R}$, $1 < p < \infty$, by extrapolation techniques based on (weighted) Lorentz or Lebesgue spaces, respectively.

Remark 2.2.4 The above definition can also be extended to the cases $0 < p \leq 1$ or $p = \infty$, resp., but we omit these generalisations here. Moreover, let us additionally assume that $\varepsilon > 0$ and $J \in \mathbb{N}$ are chosen such that all involved spaces $L_{p^\sigma,p}$ and $L_{p^{-\sigma(j)},p}$ are Banach spaces. In view of Theorem 2.2.1 (i) expression (2.2.12) can be complemented by its discrete counterpart :

$$\|f|L_p(\log L)_{-a}(\mathbb{R}^n)\| \sim \left(\sum_{j=J}^{\infty} 2^{-jap} \left\|\langle x\rangle^{-\sigma(j)} f|L_{p^{\sigma(j)},p}(\mathbb{R}^n)\right\|^p\right)^{1/p} \tag{2.2.16}$$

where $\sigma(j) = 2^{-j}$ and $J \in \mathbb{N}$ is large.

One can introduce spaces $L_{p,q}(\log L)_a(\mathbb{R}^n)$, $1 < p < \infty$, $1 \leq q \leq \infty$, and $a \in \mathbb{R}$, completely analogous, cf. [Har98, Defs. 2.15, 2.21]. We come to the definition of spaces $H^s_p(\log H)_a(\mathbb{R}^n)$ now.

Let $\sigma \in \mathbb{R}$, recall that I_σ is the usual lift operator, mapping $H^s_p(\mathbb{R}^n)$ isomorphically onto $H^{s-\sigma}_p(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$. In particular, $I_{-s}L_p(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$, see (1.1.32) and (1.1.53), respectively.

Definition 2.2.5 [Har00a, Def. 2.1] *Let* $s \in \mathbb{R}$, $1 < p < \infty$ *and* $a \in \mathbb{R}$. *Then* $H^s_p(\log H)_a(\mathbb{R}^n)$ *is the set of all* $f \in S'(\mathbb{R}^n)$ *such that* $I_s f \in L_p(\log L)_a(\mathbb{R}^n)$, *with*

$$\|f|H^s_p(\log H)_a(\mathbb{R}^n)\| := \|I_s f|L_p(\log L)_a(\mathbb{R}^n)\|. \tag{2.2.17}$$

Note that by (1.1.32) we extend our convention (2.2.15) by

$$H^s_p(\log H)_0(\mathbb{R}^n) = H^s_p(\mathbb{R}^n) \ ; \qquad H^0_p(\log H)_a(\mathbb{R}^n) = L_p(\log L)_a(\mathbb{R}^n) \ , \tag{2.2.18}$$

where $1 < p < \infty$, $s \in \mathbb{R}$, $a \in \mathbb{R}$. Moreover, using Definition 2.2.3 (i) we obtain that for $1 < p < \infty$, $a > 0$,

$$\|f|H^s_p(\log H)_{-a}(\mathbb{R}^n)\| = \left(\int_0^\varepsilon \sigma^{ap} \, \|\langle x\rangle^{-\sigma} I_s f|L_{p^\sigma,p}(\mathbb{R}^n)\|^p \ \frac{\mathrm{d}\sigma}{\sigma} \right)^{1/p} . \tag{2.2.19}$$

Likewise the counterpart for $H^s_p(\log H)_a(\mathbb{R}^n)$ can be given by Definition 2.2.3 (ii) : Some $g \in S'(\mathbb{R}^n)$ belongs to $H^s_p(\log H)_a(\mathbb{R}^n)$, $1 < p < \infty$, $a > 0$, if, and only if, it can be represented as

$$g = \sum_{j=J}^\infty g_j \ , \quad \langle x\rangle^{\sigma(j)}(I_s g_j) \in L_{p^{-\sigma(j)},p}(\mathbb{R}^n), \tag{2.2.20}$$

for large $J \in \mathbb{N}$, $\sigma(j) = 2^{-j}$, (convergence in S') and

$$\left(\sum_{j=J}^\infty 2^{jap} \left\| I_s g_j | L_{p^{-\sigma(j)},p}(\langle x\rangle^{\sigma(j)}, \mathbb{R}^n) \right\|^p \right)^{1/p} < \infty. \tag{2.2.21}$$

Now $\|g|H^s_p(\log H)_a(\mathbb{R}^n)\|$ is the infimum of expression (2.2.21) taken over all admissible representations (2.2.20).

We end this subsection with some elementary properties of the above-defined spaces with the conventions (2.2.15), (2.2.18).

Proposition 2.2.6 [Har98, Props. 2.16, 2.22], [Har00a, Prop. 2.2] *Let* $s \in \mathbb{R}$, $1 < p < \infty$.

(i) *Let* $a \in \mathbb{R}$. *Then* $H^s_p(\log H)_a(\mathbb{R}^n)$ *is a Banach space (using equivalent quasi-norms).*

(ii) *Let* $a > 0$. *Then* $H^s_p(\log H)_a(\mathbb{R}^n) \hookrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow H^s_p(\log H)_{-a}(\mathbb{R}^n)$.

(iii) *Let* $a_1 > a_2$, *then* $H^s_p(\log H)_{a_1}(\mathbb{R}^n) \hookrightarrow H^s_p(\log H)_{a_2}(\mathbb{R}^n)$.

(iv) *Let* $a \in \mathbb{R}$, *then* $I_{-s}L_p(\log L)_a(\mathbb{R}^n) = H^s_p(\log H)_a(\mathbb{R}^n)$.

Taking our convention (2.2.15) into consideration we thus arrive at some analogue of (1.1.12), (1.1.13), now in case of $\mathbb{R}^n$:

$$L_p(\log L)_a(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \hookrightarrow L_p(\log L)_{-a}(\mathbb{R}^n) \tag{2.2.22}$$

for $a > 0$, $1 < p < \infty$.

2.2.3 Examples

We look for some 'typical' function belonging to $L_p(\log L)_a(\mathbb{R}^n)$, $1 < p < \infty$, $a \in \mathbb{R}$; recall our convention (2.2.15). All spaces in this subsection are defined on $\mathbb{R}^n$ unless otherwise stated. We briefly recall the situation of bounded domains first. Let

$$\Omega = \left\{ y \in \mathbb{R}^n : |y| < \frac{1}{2} \right\} \tag{2.2.23}$$

and $1 < p < \infty$, $\lambda \in \mathbb{R}$. Then

$$b(x) = |x|^{-\frac{n}{p}} \big| \log|x| \big|^{-\lambda} \ \in L_p(\log L)_a(\Omega) \quad \text{if, and only if,} \quad \lambda > \frac{1}{p} + a, \tag{2.2.24}$$

see [ET96, Ex. 5.3.3, p. 215]. The idea is to 'extend' these functions to $\mathbb{R}^n$, i.e. to investigate functions

$$g(x) = |x|^{-\frac{n}{p}} \big| \log |x| \big|^{-\lambda} , \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}^n . \tag{2.2.25}$$

But this direct counterpart to (2.2.24) does not fit our needs as one easily verifies that g from (2.2.25) belongs to L_p if, and only if, $\lambda > \frac{1}{p}$. There is no way to find some 'better' bound for λ (relying on the parameter a additionally), such that, say, $g \in L_p(\log L)_{-a}$ for some $\lambda \le \frac{1}{p}$. This necessarily fails as the restriction to λ is due to the global behaviour of g belonging to some space 'nearby' L_p, whereas the local (logarithmic) structure is neglected. But this does not correspond to the structure of the spaces $L_p(\log L)_a$ as introduced in Subsection 2.2.2. One has rather to concentrate on local-global characterisations, i.e. searching functions $f(x)$ which locally behave like $b(x)$ from (2.2.24) but additionally satisfy convergence conditions (in the sense of L_p). Let $\psi \in C_0^\infty$, supp $\psi \subset \overline{\Omega}$, where Ω is given by (2.2.23), $0 \le \psi \le 1$, and $\psi(x) = 1$ if $|x| \le \frac{1}{4}$. Put

$$\begin{aligned} f(x) &= \langle x\rangle^{-\frac{n}{p}} \left(\log\langle x\rangle\right)^{-(\gamma+\frac{1}{p})} \sum_{k\in\mathbb{Z}^n} |x-k|^{-\frac{n}{p}} \big|\log|x-k|\big|^{-\lambda}\ \psi(x-k) \\ &\sim \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{-\frac{n}{p}} \left(\log\langle k\rangle\right)^{-(\gamma+\frac{1}{p})} |x-k|^{-\frac{n}{p}} \big|\log|x-k|\big|^{-\lambda}\ \psi(x-k) \end{aligned} \tag{2.2.26}$$

where $x \in \mathbb{R}^n$ and $\gamma > 0$. Obviously the first multiplicative term on the right-hand side of (2.2.26), i.e. $\langle x\rangle^{-\frac{n}{p}} \left(\log\langle x\rangle\right)^{-(\gamma+\frac{1}{p})}$, belongs to L_p itself whereas the sum refers to the local structure, see (2.2.24).
We have shown in [Har98, 2.5] that f from (2.2.26) belongs to $L_p(\log L)_{-a}$, $a > 0$, if, and only if, $\lambda > \frac{1}{p} - a$, and $f \in L_p$ if, and only if, $\lambda > \frac{1}{p}$. In other words, $f \in L_p(\log L)_{-a} \setminus L_p$ if, and only if, $\frac{1}{p} - a < \lambda \le \frac{1}{p}$, meaning that the spaces $L_p(\log L)_{-a}$, $a > 0$, are in fact extensions of L_p. Furthermore, application of Hölder inequalities (as presented in Section 2.3.3 below) yields that f from (2.2.26) does not belong to $L_p(\log L)_a$ in case of $\lambda \le \frac{1}{p} + a$, $a > 0$; hence $f \in L_p \setminus L_p(\log L)_a$ when $\frac{1}{p} < \lambda \le \frac{1}{p} + a$. Consequently th spaces $L_p(\log L)_a$ are properly contained in L_p for $a > 0$.

2.2.4 An alternative approach

Obviously the spaces $L_p(\log L)_{-a}(\mathbb{R}^n)$ introduced above by an extrapolation approach possess those (basic) properties we had in mind; that is, they extend the already known $L_p(\log L)_{-a}$ spaces on domains in a reasonable sense and as much monotonicity is preserved as could be expected in case of $\mathbb{R}^n$, see (1.1.12), (1.1.13) and parts (ii) and (iii) of Proposition 2.2.6, resp. We shall derive further useful features in Section 2.3 below, but briefly present another approach first.

In view of (1.1.11) the following extension of $L_p(\log L)_a(\Omega)$ appears natural. Let $1 < p < \infty$, and $a \in \mathbb{R}$. Denote by $L_p(\log L)^*_a(\mathbb{R}^n)$ the set of all measurable functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

$$\|f|L_p(\log L)^*_a(\mathbb{R}^n)\| = \left(\int_0^\infty (1 + |\log t|)^{ap} f^*(t)^p \,\mathrm{d}t \right)^{1/p} \tag{2.2.27}$$

is finite, where f^* is given by (1.1.7). These spaces have been introduced as *Lorentz-Zygmund spaces* by Bennett and Rudnick in [BR80, (1.4)]. Obviously

$$L_p(\log L)^*_a(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \hookrightarrow L_p(\log L)^*_{-a}(\mathbb{R}^n) , \quad a > 0.$$

In contrast to the situation on bounded domains, see Section 2.2.1, those spaces do not coincide with the spaces given by Definition 2.2.3. Moreover, these spaces are different in the sense that there is no general inclusion relation between, say, $L_p(\log L)^*_{-a}(\mathbb{R}^n)$ and $L_p(\log L)_{-b}(\mathbb{R}^n)$ for $a, b > 0$ – though they have a non-empty intersection containing $L_p(\mathbb{R}^n)$: We return to example $g(x)$ given by (2.2.25). One verifies $g \in L_p(\log L)^*_{-a}(\mathbb{R}^n) \setminus L_p(\log L)_{-b}(\mathbb{R}^n)$ for $\frac{1}{p} - a < \lambda \le \frac{1}{p}$ and all $b > 0$, see [Har98, Sect. 2.5]. Conversely, in case of $b > a > \frac{1}{p}$ take $f(x)$ given by (2.2.26). Then f belongs to $L_p(\log L)_{-b}(\mathbb{R}^n)$ if

$\lambda > \frac{1}{p} - b$, but $f \notin L_p(\log L)^*_{-a}(\mathbb{R}^n)$ for $\lambda \leq \frac{1}{p} - a < 0$, see [Har98, 2.2]. Now for $b > a > \frac{1}{p}$ one can always choose λ with $\frac{1}{p} - b < \lambda \leq \frac{1}{p} - a < 0$ such that f belongs to $L_p(\log L)_{-b}(\mathbb{R}^n) \setminus L_p(\log L)^*_{-a}(\mathbb{R}^n)$. Similar arguments can be stressed concerning $L_p(\log L)_a(\mathbb{R}^n)$, $L_p(\log L)^*_a(\mathbb{R}^n)$, $a > 0$.

In other words, extending $L_p(\log L)_a(\Omega)$ to $\mathbb{R}^n$ by Definition 2.2.3 or (2.2.27), respectively, leads to different concepts of spaces. The spaces $L_p(\log L)^*_a(\mathbb{R}^n)$ have been thoroughly investigated in a series of papers by Edmunds, Gurka and Opic [EGO95a], [EGO95b], [EGO96], [EGO97], [EGO98],[EGO00], [GO98], and by Evans, Opic and Pick in [EOP96], [EO00], and [OP99]. However, in our opinion the spaces $L_p(\log L)_a(\mathbb{R}^n)$ seem to represent the needed extensions to $\mathbb{R}^n$ in the context of entropy numbers we aim at.

Turning to logarithmic Sobolev spaces $H^s_p(\log H)_a$ on $\mathbb{R}^n$, there is also a parallel approach, based on $L_p(\log L)^*_a(\mathbb{R}^n)$ instead of $L_p(\log L)_a(\mathbb{R}^n)$. Denoting these spaces by $H^s_p(\log H)^*_a(\mathbb{R}^n)$ accordingly, $a \in \mathbb{R}$, $1 < p < \infty$, $s > 0$, one can define them in a parallel way to Definition 2.2.5, i.e.

$$f \in H^s_p(\log H)^*_a(\mathbb{R}^n) \quad \Longleftrightarrow \quad I_s f \in L_p(\log L)^*_a(\mathbb{R}^n).$$

This has been done, for instance, in [EGO97, (2.8)]. It follows by our above remarks about $L_p(\log L)^*_a(\mathbb{R}^n)$ and $L_p(\log L)_a(\mathbb{R}^n)$ that also $H^s_p(\log H)^*_a(\mathbb{R}^n)$ and $H^s_p(\log H)_a(\mathbb{R}^n)$ cannot coincide. Opic and Trebels followed a similar line when introducing their spaces $H^{\frac{n}{p},\alpha}(L_p)$ in [OT00] : the basic space L_p is lifted by a logarithmically adapted version of (1.1.33).

2.3 Further properties

We briefly discuss some more features of $L_p(\log L)_a(\mathbb{R}^n)$ and $H^s_p(\log H)_a(\mathbb{R}^n)$ as introduced in Definitions 2.2.3 and 2.2.5. The intention is twofold : a better illustration of the new members in the already well-equipped world of function spaces on the one hand, and, secondly – and more important – to expound our grounds for introducing *new* spaces rather than studying existing concepts (like $L_p(\log L)^*_a(\mathbb{R}^n)$) further.

2.3.1 Local versions

An important tool when studying entropy numbers on $\mathbb{R}^n$ is to reduce this problem essentially to the related question of embeddings of function spaces on (particular) bounded domains, say, annuli $\{A_\ell\}_{\ell \in \mathbb{N}_0}$, granted that the dependence of appearing constants upon that special domain can be controlled. Here the annuli are given by (1.1.24) and (1.1.25). We introduce subspaces $L_p(\widetilde{\log L})_a(A_\ell)$ of $L_p(\log L)_a(\mathbb{R}^n)$ by

$$L_p(\widetilde{\log L})_a(A_\ell) = \{f \in L_p(\log L)_a(\mathbb{R}^n) : \mathrm{supp}\ f \subset \overline{A_\ell}\}\ , \tag{2.3.1}$$

where $1 < p < \infty$, $a \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$.

Proposition 2.3.1 [Har98, Lemmata 2.17, 2.23] *Let* $1 < p < \infty$, $a \in \mathbb{R}$; *let* A_ℓ, $\ell \in \mathbb{N}_0$, *be the above annuli. Then*

$$\| f \,| L_p(\widetilde{\log L})_a(A_\ell)\| \;\sim\; 2^{\ell\frac{n}{p}}\ \| f(2^\ell\cdot) \,|\, L_p(\widetilde{\log L})_a(A_0)\| \tag{2.3.2}$$

for all $f \in L_p(\log L)_a(\mathbb{R}^n)$ *with* $\mathrm{supp}\ f \subset \overline{A_\ell}$.

Remark 2.3.2 In case of A_0 (or any fixed bounded $\Omega \subset \mathbb{R}^n$) we have $L_p(\log L)_a(A_0) = L_p(\widetilde{\log L})_a(A_0)$ (appropriately interpreted), where the spaces $L_p(\log L)_a(A_0)$ are given by Definition 1.1.1 (i) and (1.1.6), see [Har98, (2.52), (2.70)]. Furthermore, having the lift operator I_s available in spaces of type $H^s_p(\log H)_a$, see (1.1.33) and Proposition 2.2.6 (iv), we do not need a counterpart of (2.3.2) when dealing with entropy numbers and spaces of type $H^s_p(\log H)_a(\mathbb{R}^n)$.

In addition to the extrapolation matter already explained this local behaviour is the second main reason for us to extend $L_p(\log L)_a(\Omega)$ as given in Subsection 2.2.2 (unlike $L_p(\log L)^*_a(\mathbb{R}^n)$) : an easy calculation shows that using the (quasi-) norm (2.2.27) there is no counterpart of (2.3.2). But this property is essentially needed in our argument when dealing with entropy numbers below.

2.3.2 Duality

When defining spaces $L_p(\log L)_a$ on $\mathbb{R}^n$ one naturally wants to keep duality assertions – known from the case of bounded domains. Moreover, duality can also be used to extend results on entropy numbers, relying on an important paper by Bourgain, Pajor, Szarek and Tomczak-Jaegermann [BPSTJ89].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $L_p(\log L)_a(\Omega)$ as given in Definition 1.1.1 (i) with $p = q$. Then one has

$$\big[L_p(\log L)_{-a}(\Omega)\big]' = L_{p'}(\log L)_a(\Omega)\ , \quad 1 < p < \infty, \quad a \in \mathbb{R}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.3.3}$$

where the dash $'$ denotes the dual space; see [BR80, Thm. 8.4, p. 30], [ET96, Prop. 2.6.1/2 (i), p. 68]. The counterpart on $\mathbb{R}^n$ reads as follows.

Proposition 2.3.3 [Har98, Prop. 2.26] *Let $1 < p < \infty$, and $a \in \mathbb{R}$. Then*

$$[L_p(\log L)_a(\mathbb{R}^n)]' = L_{p'}(\log L)_{-a}(\mathbb{R}^n)\ . \tag{2.3.4}$$

We come to spaces $H^s_p(\log H)_a(\mathbb{R}^n)$, $1 < p < \infty$, $a \in \mathbb{R}$, $s \in \mathbb{R}$. Recall that $\big[H^s_p(\mathbb{R}^n)\big]' = H^{-s}_{p'}(\mathbb{R}^n)$, where $s \in \mathbb{R}$, and $1 < p < \infty$; cf. [Tri78a, Thm. 2.6.1(a), p. 198]. Here the duality is understood in the sense of the $[S, S']$ pairing, as usual.

Proposition 2.3.4 [Har00a, Prop. 2.4] *Let $s \in \mathbb{R}$, $1 < p < \infty$, and $a \in \mathbb{R}$. Then*

$$\big[H^s_p(\log H)_a(\mathbb{R}^n)\big]' = H^{-s}_{p'}(\log H)_{-a}(\mathbb{R}^n)\ . \tag{2.3.5}$$

Edmunds and Triebel proved in [ET96, Thm. 2.6.3 (iii), p. 79] a counterpart for spaces $H^s_p(\log H)_a(\Omega)$ defined on a bounded C^∞ domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, $a \in \mathbb{R}$ and $s \geq 0$.

2.3.3 Hölder inequalities

It is often very useful to have special Hölder inequalities available when using results on entropy numbers in order to estimate eigenvalues of compact operators acting in, say, (weighted) L_p- spaces. This has been carried out in detail in [Har98, Sect. 4] and [Har00a, Sect. 4]. Though applications (of that type) are out of the scope of the present report we want to mention some results on Hölder inequalities in spaces of type $L_p(\log L)_a(\mathbb{R}^n)$ and $H^s_p(\log H)_a(\mathbb{R}^n)$. Besides Hölder inequalities serve for the extension of our example in Section 2.2.3, too. Note that all spaces are defined on $\mathbb{R}^n$ unless otherwise stated.

Proposition 2.3.5 [Har00a, Prop. 2.6] *Let $1 < p, q < \infty$ with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$. Let $a, b \in \mathbb{R}$ and $c < a + b$. Then*

$$L_p(\log L)_a \ \cdot\ L_q(\log L)_b \ \hookrightarrow\ L_r(\log L)_c\ . \tag{2.3.6}$$

Note that (2.3.6) has to be understood in the sense that whenever $f \in L_p(\log L)_a$ and $g \in L_q(\log L)_b$, then fg belongs to $L_r(\log L)_c$, i.e

$$\|fg|L_r(\log L)_c\| \ \leq\ c\, \|f|L_p(\log L)_a\|\, \|g|L_q(\log L)_b\|.$$

The result in the form given above coincides with [Har00a, Prop. 2.6] and extends in that way [Har98, Cor. 2.29]. There is, however, an even sharper version in [Har98, Prop. 2.27] where we obtained that

$$\|fg|L_r\| \ \leq\ c\, \|f|L_p(\log L)_{-a}\|\, \|g|L_q(\log L)_a\|\ ,$$

and $1 < p, q < \infty$ with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$, $a > 0$. But this improvement is achieved at the expense of some additional assumptions imposed on $g \in L_q(\log L)_a$; we refer to [Har98, Prop. 2.27] for details.

Remark 2.3.6 It is obvious that the outcome (2.3.6) is sub-optimal in the sense that one would like to have $c = a + b$ in view of the classical Hölder inequality (when $a = b = c = 0$ in our notation). On the other hand it turned out, that in applications this version (with $c < a + b$ instead of $c \le a + b$) is completely sufficient.
Let us mention the parallel result when dealing with spaces $L_p(\log L)^*_a$ instead of $L_p(\log L)_a$; see Subsection 2.2.4. Recall that $L_p(\log L)^*_a$ are Orlicz spaces when $a \ge \frac{1}{p}$, see [BS88, Ch. 4, Sect. 8, pp. 265-280] for the notion of an Orlicz space and [BS88, Ch. 4, Ex. 8.3(e), p. 266] for this fact. Using YOUNG's inequality, cf. [BS88, Ch. 4, Thm. 8.12, Lemma 8.16, pp. 271-276] – one can conclude that

$$L_p(\log L)^*_a \cdot L_q(\log L)^*_b \hookrightarrow L_r(\log L)^*_c \tag{2.3.7}$$

holds with $c = a + b$, where $1 < p, q < \infty$, $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$ and $a \ge \frac{1}{p}$, $b \ge \frac{1}{q}$. I thank this hint my colleagues L. PICK and A. CIANCHI.

We seek for some counterpart of Proposition 2.3.5 in case of $H^s_p(\log H)_a-$ spaces, $a \in \mathbb{R}$, $1 < p < \infty$, $s > 0$. We briefly mention what is known when $a = 0$; recall notation (2.2.1). In [ST95] SICKEL and TRIEBEL studied Hölder inequalities in the wider framework of Besov- and Triebel-Lizorkin spaces. In our case of (fractional) Sobolev spaces their result reads as

$$H^s_{p^s} \cdot H^s_{q^s} \quad \hookrightarrow \quad H^s_{r^s}, \tag{2.3.8}$$

where $s > 0$, $1 < p, q < \infty$ with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$, see [ST95, Thm. 4.2.1]. Note that (2.3.8) is the classical Hölder inequality when $s = 0$. Moreover, there is an extension of (2.3.8) to some negative $s \in \mathbb{R}$ by EDMUNDS and TRIEBEL in [ET96, Thm. 2.4.5, p. 56] :

$$H^s_{p^s} \cdot H^{|s|}_{q^{|s|}} \quad \hookrightarrow \quad H^s_{r^s},$$

where $s \in \mathbb{R}$, $1 < p, q < \infty$ with $\frac{1}{p^s} = \frac{1}{p} + \frac{s}{n} > 0$ and $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$. In case of logarithmic Sobolev spaces on bounded domains, $H^s_p(\log H)_a(\Omega)$, EDMUNDS and TRIEBEL obtained in [ET96, (5.3.3/30), p. 219]

$$H^s_{p^s}(\log H)_a(\Omega) \cdot H^s_{q^s}(\log H)_b(\Omega) \quad \hookrightarrow \quad H^s_{r^s}(\Omega), \tag{2.3.9}$$

where $s > 0$, $1 < p, q < \infty$ with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$, and $b > -a > 0$. Here Ω is a bounded C^∞ domain in $\mathbb{R}^n$. In view of (2.3.6), (2.3.8) as well as (2.3.9) the desired result in our case was

$$H^s_{p^s}(\log H)_a \cdot H^s_{q^s}(\log H)_b \quad \hookrightarrow \quad H^s_{r^s}(\log H)_c \tag{2.3.10}$$

with $s > 0$, $1 < p, q < \infty$, $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$ and $c < a + b$. But we are not yet able to prove (or disprove) an assertion of that type. However, in some special case we may verify (2.3.10) and give the counterpart of [Har98, Prop. 2.27].

Proposition 2.3.7 [Har00a, Prop. 2.10] *Let $s > 0$, $a > 0$, $1 < q < \infty$ with $q^s > 1$. Let $1 < p < \infty$ be such that $p^s > 1$ and $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} < 1$. Let $g \in H^s_{p^s}(\log H)_a$ and assume that $\{g_j = \varphi_j g\}_{j=J}^\infty$ is an admissible representation of g according to* (2.2.20), (2.2.21), *i.e.*

$$\left(\sum_{j=J}^{\infty} 2^{jap^s} \| I_s(\varphi_j g) | L_{p^s - \sigma(j), p^s}(\langle x \rangle^{\sigma(j)}) \|^{p^s} \right)^{1/p^s} < \infty, \tag{2.3.11}$$

where $J \in \mathbb{N}$ is large, $\sigma(j) = 2^{-j}$, and $\{\varphi_j\}_{j=J}^\infty$ is a smooth dyadic resolution of unity. Then $fg \in H^s_{r^s}$ for any $f \in H^s_{q^s}(\log H)_{-a}$,

$$\| fg | H^s_{r^s} \| \le c \, \left\| f | H^s_{q^s}(\log H)_{-a} \right\| \, \left\| g | H^s_{p^s}(\log H)_a \right\| .$$

In contrast to the situation $s = 0$ we cannot yet replace the probably rather technical assumption (2.3.11) by the more convenient one $g \in H^s_{p^s}(\log H)_b$ for $b > a$.

2.3.4 Equivalent norms

Let $s \in \mathbb{N}$, $1 < p < \infty$. It is well-known, that $H_p^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n)$, i.e.

$$\|f|H_p^s(\mathbb{R}^n)\| \quad \sim \quad \sum_{|\alpha|\leq s} \|D^\alpha f|L_p(\mathbb{R}^n)\| \,, \tag{2.3.12}$$

where $\alpha \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $f \in S'(\mathbb{R}^n)$, see [Tri78a, Thm. 2.3.3, p. 177], for instance. In case of logarithmic Sobolev spaces on bounded C^∞ domains one has a parallel result for $s \in \mathbb{N}_0$, $1 < p < \infty$, and $a \in \mathbb{R}$:

$$\|f|H_p^s(\log H)_a(\Omega)\| \quad \sim \quad \sum_{|\alpha|\leq s} \|D^\alpha f|L_p(\log L)_a(\Omega)\| \,,$$

see [ET96, Thm. 2.6.3, p. 79]. Thus it is reasonable to ask whether a similar assertion is true in case of logarithmic Sobolev spaces on $\mathbb{R}^n$. We obtain the following.

Proposition 2.3.8 [Har00a, Prop. 2.5] *Let $m \in \mathbb{N}_0$, $1 < p < \infty$ and $a \in \mathbb{R}$. Then $f \in H_p^m(\log H)_a(\mathbb{R}^n)$ if, and only if, $D^\alpha f \in L_p(\log L)_a(\mathbb{R}^n)$, $|\alpha| \leq m$, and*

$$\|f|H_p^m(\log H)_a(\mathbb{R}^n)\| \sim \sum_{|\alpha|\leq m} \|D^\alpha f|L_p(\log L)_a(\mathbb{R}^n)\| \,. \tag{2.3.13}$$

Remark 2.3.9 A parallel result for spaces $L_p(\log L)_a^*(\mathbb{R}^n)$, $H_p^s(\log H)_a^*(\mathbb{R}^n)$ was obtained by Edmunds, Gurka and Opic in [EGO97, Thm. 4.2].

2.4 Compact embeddings, and entropy numbers

We return to our initial problem (2.1.1) and study the following embedding map in the sequel,

$$id_a^\beta : H_{p_1}^{s_1}(\log H)_{a_1}(\log^\beta\langle x\rangle, \mathbb{R}^n) \quad \longrightarrow \quad H_{p_2}^{s_2}(\log H)_{a_2}(\mathbb{R}^n), \tag{2.4.1}$$

where $-\infty < s_2 \leq s_1 < \infty$, $1 < p_1 \leq p_2 < \infty$, with $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, and $a_1, a_2 \in \mathbb{R}$, $\beta \in \mathbb{R}$. All spaces are defined on $\mathbb{R}^n$ in the sequel unless otherwise stated.

2.4.1 Embeddings

We first investigate when the above embedding (2.4.1) is continuous or even compact. Thus we always assume in the sequel that s_i, p_i, $i = 1, 2$, are given as above and fixed now. We concentrate on the remaining parameters $a_1, a_2, \beta \in \mathbb{R}$ and their influence upon continuity or compactness of id_a^β. So we also use (a_1, a_2)-diagrams now, sometimes additionally depending upon $\beta \in \mathbb{R}$.

Proposition 2.4.1 [Har00a, Prop. 3.1]

Let $s_1 \geq s_2$, $1 < p_1 \leq p_2 < \infty$ with $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ and id_a^β be given by (2.4.1).

- (i) *id_a^β is continuous if $a_1 \geq a_2$, $\beta \geq 0$.*
- (ii) *id_a^β is compact if $s_1 > s_2$, $\beta > 0$ and $a_1 > a_2$, $a_1 \geq 0$, $a_2 \leq 0$.*

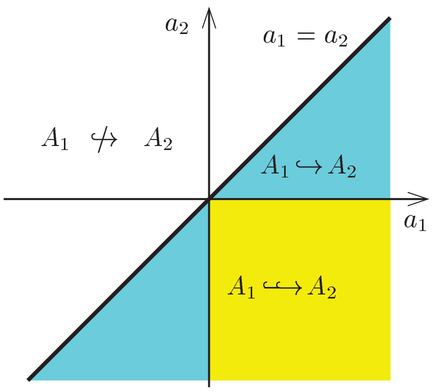


Figure 4

We illustrated Proposition 2.4.1 in Figure 4, where A_1, A_2 temporarily denote the spaces involved in (2.4.1), and $A_1 \hookrightarrow\hookrightarrow A_2$ stands for the compact embedding id_a^β. Note that the result (i) is known when $a_1 = a_2 = 0$; it follows from our more general result Theorem 1.2.1. Furthermore, by Theorem 1.2.1 (ii) the assumption

$\beta > 0$ for the weight function in (ii) appears reasonable, though we cannot have a compact embedding in the situation covered by Theorem 1.2.1, i.e. $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ and $a_1 = a_2 = 0$. Furthermore, there is no continuous embedding for $a_1 \leq 0$, $a_2 \geq 0$ and $a_1 < a_2$, $\beta \geq 0$.This can be disproved easily by (ii) combined with Theorem 1.2.1 (ii); see also the argument in [Har00a, Cor. 3.2]. In the remaining cases with $a_1 < a_2$ the assumption is that there is no continuous embedding, too, but the proof in [Har00a, Cor. 3.2] covers the case $s_1 = s_2$ only.

Remark 2.4.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, then one can similarly ask for which parameters

$$id_{\Omega,a} : H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \quad \longrightarrow \quad H^{s_2}_{p_2}(\log H)_{a_2}(\Omega) \tag{2.4.2}$$

is continuous or even compact. In that case Edmunds and Triebel [ET96] as well as Edmunds and Netrusov [EN98] have proved that $id_{\Omega,a}$ is compact when $s_1 > s_2$, $1 < p_1 < p_2 < \infty$, with $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, and $a_1 > a_2$. From that point of view an extension of Proposition 2.4.1 (ii) – concerning the parameters a_1, a_2 – might be true, but is not yet clear.

In the framework of different spaces $L_p(\log L)^*_a(\mathbb{R}^n)$ and $H^s_p(\log H)^*_a(\mathbb{R}^n)$ Edmunds, Gurka and Opic obtained in [EGO97], [EGO00] parallel results on continuous or compact embeddings of type (2.4.1).

2.4.2 Entropy numbers

We investigate compact embeddings as given by Proposition 2.4.1 (ii); in particular, we study the asymptotic behaviour of the corresponding entropy numbers. Clearly, by (1.3.2), this can be reformulated as to characterise their rate of decay more precisely.

We postpone a discussion of related known results – in particular those for embeddings of spaces on domains – to Section 2.4.3 below and come immediately to our results for limiting embeddings on $\mathbb{R}^n$. Let

$$\begin{aligned} -\infty < s_2 < s_1 < \infty, \quad 1 < p_1 < p_2 < \infty, \qquad &\text{with} \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \\ a_1 \geq 0, \quad a_2 \leq 0 \quad \text{with} \quad a_1 > a_2, \qquad &\text{and} \quad \beta > 0. \end{aligned} \tag{2.4.3}$$

For later reason we also introduce the number

$$\mu_* := \min\left(a_1 - a_2, \frac{s_1 - s_2}{n}\right) > 0. \tag{2.4.4}$$

Recall our notation for id^β_a given by (2.4.1).

Theorem 2.4.3 [Har00a, Thm. 3.7] *Let assumptions* (2.4.3) *be satisfied, then* id^β_a *is compact. Assume that* $a_1 - a_2 \neq \frac{s_1 - s_2}{n}$. *Then there are constants* $c_1, c_2 > 0$ *such that for all* $k \in \mathbb{N}$

$$c_1\, k^{-\mu_*} \;\leq\; e_k(id^\beta_a) \;\leq\; c_2 \begin{cases} k^{-\mu_*} & , \quad \beta > \mu_* + 1 \\ k^{-\mu_*} \log^\beta\langle k\rangle & , \quad \beta = \mu_* + 1 \\ k^{-\mu_* \frac{\beta}{1+\mu_*}} & , \quad \beta < \mu_* + 1\,. \end{cases} \tag{2.4.5}$$

Remark 2.4.4 We may complement Theorem 2.4.3 by the estimates related to the case $a_1 - a_2 = \frac{s_1 - s_2}{n}$. The counterpart of (2.4.5) reads then as

$$c\, k^{-\mu_*} \;\leq\; e_k(id^\beta_a) \;\leq\; c_\varepsilon \begin{cases} k^{-\mu_* + \varepsilon} & , \quad \beta \geq \mu_* + 1 \\ k^{-\mu_* \frac{\beta}{1+\mu_*} + \varepsilon} & , \quad \beta < \mu_* + 1\,. \end{cases} \tag{2.4.6}$$

There are forerunners of the above theorem given in [Har98]; in particular, the case $s_2 = 0$, $a_1 = 0$, $a_2 < -2\frac{s_1}{n}$ refers to [Har98, Thm. 3.5], whereas the setting $s_1 = 0$, $a_2 = 0$, $a_1 > -2\frac{s_2}{n}$ is related to [Har98, Cor. 3.7].

We briefly illustrate the meaning of the restrictions in (2.4.5) concerning the interplay of a_1, a_2 and β. All other parameters are assumed to be fixed for the moment; thus (2.4.5) and (2.4.6) provide upper and lower estimates for the corresponding entropy numbers of the form

$$c_1\, k^{-\nu_1} \;\leq\; e_k \;\leq\; c_2\, k^{-\nu_2}$$

(neglecting ε-perturbations for the moment). Similarly to Figure 4 we have indicated in the (a_1, a_2)-diagram below these (usually different) exponents according to the areas given by (2.4.5) and (2.4.6). It is obvious that for 'strong' weights $w(x) = \log^\beta\langle x\rangle$, that is where β is large enough ($\beta > \mu_* + 1$), the asymptotic behaviour of the entropy numbers is determined (up to constants) by $e_k\left(id_a^\beta\right) \sim k^{-\mu_*}$, whereas in the more interesting case of small $\beta > 0$ we have no general result. But one may observe that in any case the different behaviour is determined both by the a-parameters (contributing to μ_*) as well as the weight β (taking the other parameters $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ as fixed for the moment).

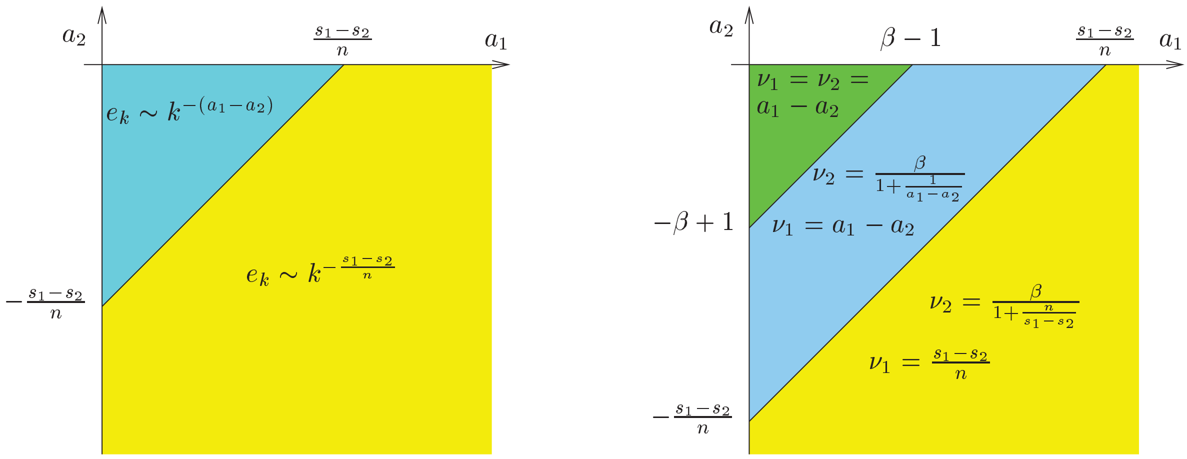


Figure 5 : *The case* $\beta > \dfrac{s_1 - s_2}{n} + 1.$ *The case* $0 < \beta \leq \dfrac{s_1 - s_2}{n} + 1.$

Using interpolation arguments for entropy numbers as presented in [HT94a, Thm. 3.2 (i)] together with [Har00a, Cor. 2.9], the upper estimate in (2.4.5) for $\beta < \mu_* + 1$ can be improved slightly.

Proposition 2.4.5 [Har00a, Prop. 3.11] *Let assumptions* (2.4.3) *be satisfied with* $0 < \beta < \frac{s_1-s_2}{n} + 1$ *and*

$$a_1 - \frac{s_1 - s_2}{n} < a_2 \leq \min(a_1 - \beta + 1, 0). \tag{2.4.7}$$

Then for any $\varepsilon > 0$ *there is some* $c_\varepsilon > 0$ *such that for all* $k \in \mathbb{N}$,

$$e_k\left(id_a^\beta\right) \;\leq\; c_\varepsilon\, k^{-\nu+\varepsilon} \qquad \textit{with} \quad \nu = \min\left(a_1 - a_2\,,\, \frac{\beta}{1 + \frac{n}{s_1 - s_2}}\right). \tag{2.4.8}$$

Obviously (2.4.5) as well as (2.4.6) give (2.4.8) when $a_2 \leq a_1 - \frac{s_1-s_2}{n}$, i.e. $\beta < \frac{s_1-s_2}{n} + 1 \leq a_1 - a_2 + 1$. Returning to our above diagrams in Figure 5, in particular, the right-hand side, Proposition 2.4.5 concerns the upper exponent ν_2 in the intermediate strip. So, roughly speaking, the achievement of (2.4.8) consists in the removal of this strip (indicated by the two broken lines in Figure 6) and its replacement by the line L; more exactly, we could extend both areas (where either $\nu_2 = a_1 - a_2$ or $\nu_2 = \beta/(1 + \frac{n}{s_1-s_2})$ is a correct upper exponent) from the corresponding broken lines to the line L – neglecting ε-terms for the moment.

In the diagram aside we sketched those areas in the (a_1, a_2)-diagram where the corresponding 'upper' exponents ν_2 are of the same type. The lower exponents ν_1 are only given for completeness, where $\nu_1 = a_1 - a_2$ is 'responsible' for the area $a_1 - \frac{s_1-s_2}{n} \le a_2 \le 0$, whereas $\nu_1 = \frac{s_1-s_2}{n}$ concerns the remaining part $a_2 \le a_1 - \frac{s_1-s_2}{n}$. There is no improvement in view of Theorem 2.4.3. However, concerning the 'upper' exponent ν_2, we could remove the strip

$$a_1 - \frac{s_1 - s_2}{n} \le a_2 \le \min(a_1 - \beta + 1, 0)$$

(indicated by the two broken lines); compare the right-hand side of Figure 5 and Figure 6.

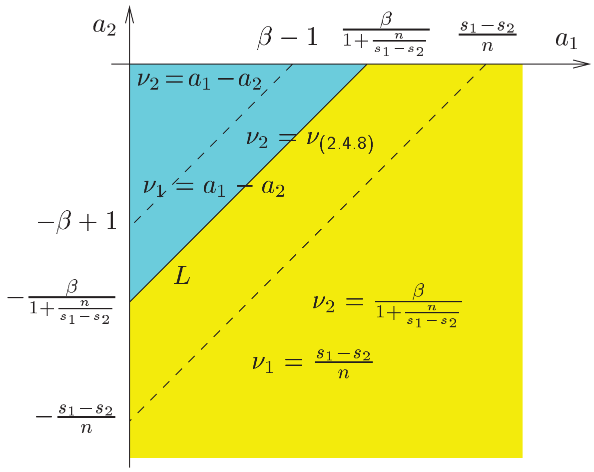


Figure 6

We may summarise Theorem 2.4.3 and Proposition 2.4.5 in the following sense. Note that $0 < \beta < \frac{s_1-s_2}{n} + 1$ and (2.4.7) imply $\nu < \frac{s_1-s_2}{n}$ in (2.4.8). Recall our notation μ_* given by (2.4.4). We complement it by

$$\mu^* := \min\left(\mu_*, \frac{\beta}{\frac{n}{s_1-s_2}+1}\right) = \min\left(a_1 - a_2, \frac{s_1 - s_2}{n}, \frac{\beta}{\frac{n}{s_1-s_2}+1}\right) \le \mu_*. \tag{2.4.9}$$

Corollary 2.4.6 [Har00a, Cor. 3.13] *Let assumptions* (2.4.3) *be satisfied; we make use of the above notation. Then there is some* $c > 0$ *and for any* $\varepsilon > 0$ *some* $c_\varepsilon > 0$ *such that for all* $k \in \mathbb{N}$,

$$c\, k^{-\mu_*} \le e_k\left(id_a^\beta\right) \le c_\varepsilon\, k^{-\mu^*+\varepsilon} \tag{2.4.10}$$

with $\varepsilon = 0$ *if* $a_1 - a_2 \neq \frac{s_1-s_2}{n}$ *and* $\beta > \mu_* + 1$ *or* $0 < \beta < \frac{s_1-s_2}{n} + 1 < a_1 - a_2 + 1$.

At the moment, we have no better ('sharper') result to characterise the asymptotic behaviour of the entropy numbers of embedding id_a^β, given by (2.4.1) and (2.4.3). We do not even claim that the upper bound in (2.4.10) (apart from ε-terms) is the correct one. However, in some formal sense the number μ^* given by (2.4.9) looks very reasonable in so far as the interplay between the 'non-limiting' exponent $\frac{s_1-s_2}{n}$ (see (a), (c) in Section 2.4.3 below, respectively), and the auxiliary parameters a_1, a_2 and β in that limiting situation is concerned. In other words, if we can manage to shrink the original space and/or to extend the target space sufficiently well (by means of a_1, a_2 and β) then we regain the 'non-limiting' behaviour of the corresponding entropy numbers, that is, when $\beta > 0$ and/or $a_1 - a_2$ are sufficiently large. Certainly these quantities should have some influence on the 'quality' of the compactness (measured in terms of entropy numbers), see also (b) in Section 2.4.3 below, for instance.

Remark 2.4.7 By the same technique as presented above one can prove similar estimates for the entropy numbers when (2.4.1) is replaced by

$$id^\alpha : H^{s_1}_{p_1}(\log H)_{a_1}(\langle x\rangle^\alpha, \mathbb{R}^n) \longrightarrow H^{s_2}_{p_2}(\log H)_{a_2}(\mathbb{R}^n),$$

where $-\infty < s_2 < s_1 < \infty$, $1 < p_1 < p_2 < \infty$, with $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, and $\alpha > 0$. Let $a_1 \ge 0$, $a_2 \le 0$ with $a_1 > a_2$, and assume $a_1 - a_2 \neq \frac{s_1-s_2}{n}$. Recall (2.4.4) and (2.4.9). Now $\mu^* = \mu_*$ and hence

$$e_k(id^\alpha) \sim k^{-\mu_*}. \tag{2.4.11}$$

2.4.3 Comparison of limiting and non-limiting results

We described in Remark 2.2.2 the idea of approximating the limiting embedding $id_{\Omega,a} : H^s_{p^s}(\Omega) \longrightarrow L_p(\log L)_{-a}(\Omega)$ by means of non-limiting embeddings $id_\sigma : H^s_{p^s}(\Omega) \longrightarrow L_{p^\sigma}(\Omega)$, $\sigma \downarrow 0$, which were thoroughly investigated in the past; see also Figure 3. We return to this point and – after a short review of related results for spaces on bounded domains – focus especially on the behaviour of the corresponding entropy numbers under this approximation procedure. Let us always assume now

$$s_1 > s_2, \quad 1 < p_1 < p_2 < \infty \ , \quad \text{with} \quad \delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \geq 0, \tag{2.4.12}$$

for simplicity.

Embeddings of spaces on a bounded domain Ω

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain.

(a) *Non-limiting case* : Let (2.4.12) be satisfied with $\delta > 0$ and $id_\Omega : H^{s_1}_{p_1}(\Omega) \longrightarrow H^{s_2}_{p_2}(\Omega)$. This situation is covered by the general result of Edmunds and Triebel in [ET89, ET92] :

$$e_k(id_\Omega) \quad \sim \quad k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}, \tag{2.4.13}$$

see also (1.3.3).

(b) *Limiting case* : Let (2.4.12) be satisfied with $\delta = 0$ and let $id_{\Omega,a}$ be the natural embedding given by (2.4.2), where we additionally assume $a_1 > a_2$ and $a_1 - a_2 \neq \frac{s_1-s_2}{n}$. In this situation studied by Edmunds and Triebel in [ET96, 3.4, p. 128-151], Edmunds and Netrusov in [EN98], and further extended by Caetano in [Cae00], one obtains for the corresponding entropy numbers

$$e_k(id_{\Omega,a}) \quad \sim \quad k^{-\mu_*} \ , \quad k \in \mathbb{N}, \tag{2.4.14}$$

where μ_* is given by (2.4.4). In particular, when $s = s_1 > 0$, $s_2 = 0$, $p = p_2$, $p_1 = p^s$, $a_1 = 0$, $a := -a_2 > 0$, and $a \neq \frac{s}{n}$, then (2.4.14) implies,

$$e_k\left(id : H^s_{p^s}(\Omega) \to L_p(\log L)_{-a}(\Omega)\right) \quad \sim \quad k^{-\min\left(a,\frac{s}{n}\right)}. \tag{2.4.15}$$

We want to link this with our Remark 2.2.2 briefly. Recall that by Theorem 2.2.1 (i) one can characterise the target space $L_p(\log L)_{-a}(\Omega)$ by extrapolating spaces $L_{p^\sigma}(\Omega)$, $\sigma > 0$. On the other hand, (1.3.3) yields for id_σ, given by (2.2.7),

$$e_k\left(id_\sigma : H^s_{p^s}(\Omega) \to L_{p^\sigma}(\Omega)\right) \quad \sim \quad k^{-\frac{s}{n}} \qquad \text{for all} \quad \sigma > 0, \tag{2.4.16}$$

cf. [ET96, Thm. 3.3.3/2, p. 118]. Comparing the limiting result (2.4.15) with the non-limiting one (2.4.16), one observes that the non-limiting exponent '$-\frac{s}{n}$' survives when the additional parameter $a > 0$ is large enough, $a > \frac{s}{n}$; otherwise it determines the behaviour of the entropy numbers.

One can obviously derive further estimates of entropy numbers in the case of non-limiting compact embeddings when either source or target space is of logarithmic type; this can be obtained by decomposition techniques, for instance, but is omitted here.

Embeddings of weighted spaces on $\mathbb{R}^n$

Clearly there are no compact embeddings in unweighted spaces on $\mathbb{R}^n$, thus we return to the setting described in Propositions 1.3.2, 1.3.4. We link the situation studied now with our more general results in Sections 1.1.3, 1.3.2, where the weights are of type (1.1.51), $w(x) = \langle x\rangle^\alpha \ \log^\mu\langle x\rangle$, $\alpha, \mu \in \mathbb{R}$.

(c) *Non-limiting case, weighted spaces on* $\mathbb{R}^n$: Let (2.4.12) be satisfied with $\delta > 0$ and assume first $\alpha > 0$, $\mu \in \mathbb{R}$ for the corresponding weight function. Then we have by Proposition 1.3.2 for $id^{\alpha,\mu}$ given by (1.2.11),

$$e_k(id^{\alpha,\mu}) \quad \sim \quad k^{-\frac{s_1-s_2}{n}} \qquad \text{if} \quad \begin{cases} \alpha > \delta \quad , \quad \mu \in \mathbb{R} \\ \alpha = \delta \quad , \quad \mu > \frac{s_1-s_2}{n} + 1 \end{cases} \tag{2.4.17}$$

and for $0<\alpha<\delta$ and any number $\varepsilon>0$,

$$c\,k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}\left(\log\langle k\rangle\right)^{-\mu} \le e_k(id^{\alpha,\mu}) \le c_\varepsilon\,k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}\left(\log\langle k\rangle\right)^{-\mu+\varepsilon+\frac{1}{p_1}+1-\frac{1}{p_2}} . \tag{2.4.18}$$

When $\alpha=0$, $\mu=\beta>0$, then Proposition 1.3.4 gives

$$c_1\,k^{-\frac{1}{p_1}+\frac{1}{p_2}}\left(\log\langle k\rangle\right)^{-\beta} \le e_k(id^{\beta}) \le c_2\,\left(\log\langle k\rangle\right)^{-\beta} . \tag{2.4.19}$$

Note that (2.4.11) and (2.4.17) (with $\mu=0$) coincide – also with the non-limiting situation on Ω, see (2.4.13) – assuming that *in the limiting case* a_1-a_2 is sufficiently large, i.e. $a_1-a_2>\frac{s_1-s_2}{n}$, and in its *non-limiting counterpart* the weight is strong enough, $\alpha>\delta$ (which is always the case in (2.4.11) as $\delta=0$). Though otherwise, when the weight is of purely logarithmic type, we have no sharp results in (2.4.10) and (2.4.19), there are grounds for the supposition that the decay in (2.4.19) should be of power type, too – in contrast to the upper bound in (2.4.19) so far.

Let us finally give the counterpart of (2.4.16), but related to the situation of weighted embeddings on $\mathbb{R}^n$. For simplicity we assume $s=s_1>0$, $s_2=0$, $p=p_2$, $p_1=p^s$, $a_1=0$ and $a=-a_2>0$. We first compare (2.4.10) with some non-limiting counterpart, i.e. we deal with the weight function $w(x)=\log^\beta\langle x\rangle$, $\beta>0$. We shall consider only the case $\beta>1+\frac{s}{n}$ and $a\neq\frac{s}{n}$; then Theorem 2.4.3, in particular (2.4.5) provides

$$e_k\left(id_a^\beta : H^s_{p^s}(\log^\beta\langle x\rangle,\mathbb{R}^n)\to L_p(\log L)_{-a}(\mathbb{R}^n)\right) \quad\sim\quad k^{-\min\left(a,\frac{s}{n}\right)} .$$

In view of (2.2.11) and Definition 2.2.3 (i) the counterpart of (2.2.7) is given by

$$id_{\sigma,\mathbb{R}^n} : H^s_{p^s}(\log^\beta\langle x\rangle,\mathbb{R}^n) \longrightarrow L_{p^\sigma,p}(\langle x\rangle^{-\sigma},\mathbb{R}^n)\,,\quad \sigma>0, \tag{2.4.20}$$

and we have by (2.4.17) with $\alpha=\sigma=\delta>0$, $\mu=\beta>1+\frac{s}{n}=1+\frac{s_1-s_2}{n}$ (and interpolation arguments concerning the target space) that

$$e_k\left(id_{\sigma,\mathbb{R}^n} : H^s_{p^s}(\log^\beta\langle x\rangle,\mathbb{R}^n)\to L_{p^\sigma,p}(\langle x\rangle^{-\sigma},\mathbb{R}^n)\right) \quad\sim\quad k^{-\frac{s}{n}}\,,\quad \sigma>0.$$

Consequently the non-limiting exponent '$-\frac{s}{n}$' survives in that situation, too (like when studying limiting embeddings of function spaces on bounded domains), supposed that $a>0$ and $\beta>0$ are large enough. The situation is even nicer when dealing with the weight $w(x)=\langle x\rangle^\alpha$, $\alpha>0$: (2.4.11) implies

$$e_k\left(id^\alpha : H^s_{p^s}(\langle x\rangle^\alpha,\mathbb{R}^n)\to L_p(\log L)_{-a}(\mathbb{R}^n)\right) \quad\sim\quad k^{-\min\left(a,\frac{s}{n}\right)}$$

if $a\neq\frac{s}{n}$, whereas for the counterpart of (2.4.20)

$$id^\alpha_{\sigma,\mathbb{R}^n} : H^s_{p^s}(\langle x\rangle^\alpha,\mathbb{R}^n)\longrightarrow L_{p^\sigma,p}(\langle x\rangle^{-\sigma},\mathbb{R}^n)\,,\quad\sigma>0,$$

it follows by (2.4.17) with $\alpha'=\alpha+\sigma>\delta=\sigma>0$, $\mu=0$, $s=s_1-s_2$ (and interpolation arguments)

$$e_k\left(id^\alpha_{\sigma,\mathbb{R}^n} : H^s_{p^s}(\langle x\rangle^\alpha,\mathbb{R}^n)\to L_{p^\sigma,p}(\langle x\rangle^{-\sigma},\mathbb{R}^n)\right) \quad\sim\quad k^{-\frac{s}{n}} .$$

The conclusion is the same again : turning from the non-limiting situation – with $L_{p^\sigma,p}(\langle x\rangle^{-\sigma},\mathbb{R}^n)$, $\sigma>0$, as target spaces – to the limiting one – now embedding into $L_p(\log L)_{-a}(\mathbb{R}^n)$ – the asymptotic behaviour of the corresponding entropy numbers changes from $k^{-\frac{s}{n}}$ to $k^{-\min\left(a,\frac{s}{n}\right)}$, assuming that $a\neq\frac{s}{n}$ and the weight is strong enough (either $w(x)=\log^\beta\langle x\rangle$ with $\beta>\frac{s}{n}+1$ or $w(x)=\langle x\rangle^\alpha$, $\alpha>0$). Otherwise the weight gains additional influence, as expected.

Conclusion. We briefly summarise this short discussion. It is obvious that – even when dealing with limiting situations – there are settings such that the non-limiting behaviour of the corresponding entropy numbers is preserved. The prize to pay for this achievement is some compensation measured in additional fine indices $a_1,a_2\in\mathbb{R}$. Moreover, following that process, new 'limiting situations' naturally arise, e.g. $a_1-a_2=\frac{s_1-s_2}{n}$;

we leave this refinement process here. More important from our point of view was to close the gaps in (2.4.10); but this is left for future work and – possibly – some even stronger motivation (than the aim for completeness merely).

We have presented an obviously reasonable opportunity how to cope with limiting embeddings of weighted spaces on $\mathbb{R}^n$ of the type studied above. The introduction of spaces $L_p(\log L)_a(\mathbb{R}^n)$ according to Definition 2.2.3 led to a number of features which appear desirable in view of further investigations. The most essential disadvantage is in our opinion the resulting diversity of spaces $L_p(\log L)_a(\mathbb{R}^n)$ and $L_p(\log L)_a^*(\mathbb{R}^n)$, being in sharp contrast to their counterparts on bounded domains $\Omega \subset \mathbb{R}^n$. The reward for our deviation from the 'standard' approach $L_p(\log L)_a^*(\mathbb{R}^n)$ lies in the outcome finally, permitting not only entropy number estimates for related limiting embeddings but also a comparison with closely linked non-limiting assertions. We do not know of parallel results when $L_p(\log L)_a(\mathbb{R}^n)$ is replaced by $L_p(\log L)_a^*(\mathbb{R}^n)$ as target space.

As applications are out of the scope of the present report we end our discussion of this first example here.

3 Modified smoothness

3.1 Introduction

We present a model case for id_Ω from (1.4.4) and study the embedding

$$id : B^{s_1}_{p_1,q_1}(\Omega) \longrightarrow B^{s_2}_{p_2,q_2}(\Omega)\ , \tag{3.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain, $0 < p_1,\ p_2,\ q_1,\ q_2 \le \infty$, $s_1,\ s_2 \in \mathbb{R}$. The embedding (3.1.1) is compact if

$$0 < p_1, p_2 \le \infty, \quad s_1 - s_2 > n \max\left(\frac{1}{p_1} - \frac{1}{p_2}, 0\right), \quad 0 < q_1, q_2 \le \infty, \tag{3.1.2}$$

see Section 1.2.1. Posing the question what happens when (3.1.2) is replaced by $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2 \le \infty$, one firstly observes that the embedding (3.1.1) is no longer compact. However, modifying the setting in this so-called *limiting case* by enlarging the target space sufficiently carefully (where the initial space is assumed to be fixed now), this leads to compact *limiting* embeddings.
In contrast to Section 2 we shall recover compactness of (3.1.1) with $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ now by decreasing the smoothness of the target space in such a way, that the *smoothness* s_2 is preserved and the embedding becomes compact. In that way one quite naturally arrives at the introduction of new spaces with additional 'logarithmic smoothness'. As an example one may consider the case $s_2 = 1$ and $p_2 = \infty$. It turns out that in case of the B-spaces there is an interplay between the (usually neglected) q-parameters and the additional logarithmic smoothness. This result is somewhat surprising in our opinion, though similar results were obtained before; cf. [EOP96].
The second reason to deal with spaces of 'logarithmic smoothness' in more detail, is the well-known and celebrated result of Brézis and Wainger [BW80] in which it was shown that every function u in $H_p^{1+n/p}(\mathbb{R}^n)$ is 'almost' Lipschitz-continuous, in the sense that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < \frac{1}{2}$,

$$|u(x) - u(y)| \ \le\ c\,|x-y| \,\Big|\log|x-y|\Big|^{1/p'} \ \|u|H_p^{1+n/p}(\mathbb{R}^n)\|\,. \tag{3.1.3}$$

Here c is a constant independent of x, y and u, and $\frac{1}{p'} + \frac{1}{p} = 1$. Our aim in [EH99] was to investigate how 'sharp' this result is (concerning the exponent of the log-term), as well as to look for possible extensions to the wider scale of F-spaces and parallel results for B-spaces. We found that the exponent $\frac{1}{p'}$ is sharp in the F-setting, whereas in case of B-spaces the sharp exponent turned out to be $\frac{1}{q'}$. As already mentioned above, this important role played by the q-parameter is rather unusual. In that way (3.1.3) suggests some definition of 'logarithmic' Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \ge 0$, as the collection of all $f \in C(\mathbb{R}^n)$ such that

$$\left\|f|\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\right\| := \|f|L_\infty(\mathbb{R}^n)\| + \sup_{0<|h|<1/2}\ \sup_{x\in\mathbb{R}^n} \frac{|(\Delta_h f)(x)|}{|h|\,|\log|h||^\alpha} \tag{3.1.4}$$

is finite. Then the BRÉZIS-WAINGER result (3.1.3) can be extended to $H_p^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ if, and only if, $\alpha \geq \frac{1}{p'}$ where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, generalising the spaces $\mathrm{Lip}^{(1,-\alpha)}$ given by (3.1.4) further, one arrives at spaces $\mathrm{Lip}_{p,q}^{(1,-\alpha)}$, $1 \leq p \leq \infty$, $0 < q \leq \infty$, $\alpha > \frac{1}{q}$. Likewise one asks which embedding results can be derived for such spaces when $p < \infty$, $q < \infty$, and compares the outcome with the case already studied, i.e. for $p = q = \infty$.
Secondly, we discuss the compactness of embeddings into spaces of Lipschitz type and analyse these embeddings from the standpoint of entropy numbers : we consider the embedding

$$id : B_{p,q}^{1+n/p}(U) \quad \longrightarrow \quad \mathrm{Lip}^{(1,-\alpha)}(U) \ ,$$

where $0 < p, q \leq \infty$, $\alpha > \frac{1}{q'}$, and U being the unit ball in $\mathbb{R}^n$, and determine the asymptotic behaviour of its entropy numbers $e_k(id)$ for $k \in \mathbb{N}$ large.

Finally, let us briefly mention that these logarithmic Lipschitz spaces also appear in other connections, e.g. when studying (generalised) moduli of smoothness and related inequalities, see [BS88], [DL93]. Furthermore, these spaces are involved when characterising the regularity of solutions in stationary problems (see [Lio98]) and when investigating hydrodynamics in Besov spaces (cf. [Vis98]). Thus it is not only of inner-mathematical interest to study such spaces in greater detail, but also in view of applications. They are, however, out of the scope of the present report.

3.2 Spaces of additional logarithmic smoothness

Spaces of generalised smoothness have been studied from different points of view, coming from the interpolation side (with a function parameter) we refer to MERUCCI [Mer84] and COBOS, FERNANDEZ [CF88], whereas the rather abstract approach (approximation by series of entire analytic functions and coverings) was independently developed by GOL'DMAN and KALYABIN, see [Gol81], [Gol83], [Gol87a], and [Kal77], [Kal83]. Furthermore, the survey by KALYABIN and LIZORKIN [KL87] and the appendix [Liz86] cover the extensive (Russian) literature at that time. More recently, we mention the contributions of GOL'DMAN [Gol87b], [Gol94] and NETRUSOV [Net88], [Net92] and of BURENKOV [Bur99]. We give further references below in connection with special topics. One of the latest works is certainly that one of FARKAS and LEOPOLD [FL01] linking function spaces of generalised smoothness with negative definite functions – and thus opening another scene : the application to pseudo-differential operators (as generators of sub-Markovian semi-groups). Plainly all this is out of the scope of the present report; it may, however, serve as some explanation that function spaces of generalised smoothness have long been of interested already, but are far from being 'old-fashioned'.

3.2.1 Motivation

We were led to this subject quite naturally when dealing with (particular) limiting situations : It is well-known that functions in the (fractional) Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, when $1 < p < \infty$, are Hölder-continuous with exponent α for any $\alpha \in (0,1)$ but need not be Lipschitz-continuous. This limiting situation was clarified in an important paper by BRÉZIS and WAINGER [BW80] in which it was shown that every function u in $H_p^{1+n/p}(\mathbb{R}^n)$ is 'almost' Lipschitz-continuous, in the sense that for all $x, y \in \mathbb{R}^n$, $x \neq y$, $|x-y| < 1/2$,

$$|u(x) - u(y)| \ \leq \ c\,|x-y| \,\Big|\log|x-y|\Big|^{1/p'} \ \|u|H_p^{1+n/p}(\mathbb{R}^n)\| \,. \tag{3.2.1}$$

Here c is a constant independent of x, y and u, and $\frac{1}{p'} + \frac{1}{p} = 1$. Reformulating this fact in terms of (limiting) embeddings (3.2.1) immediately suggests the definition of 'logarithmically' spoilt spaces of Lipschitz type $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \geq 0$, as the space of all functions $f \in C(\mathbb{R}^n)$ such that

$$\|f|\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)\| := \|f|L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,y\,\in\,\mathbb{R}^n \\ 0<|x-y|<1/2}} \frac{|f(x)-f(y)|}{|x-y|\,|\log|x-y||^{\alpha}} \tag{3.2.2}$$

is finite. Parallel studies (of limiting situations) led LEOPOLD in [Leo98], [Leo00b] to the introduction of spaces $B^{(s,b)}_{p,q}$ of type $B^s_{p,q}$, but with additional logarithmic smoothness. We give the related definitions here and derive some basic properties of the spaces.

3.2.2 Definition

Recall our notation for the difference operator Δ^m_h in (1.1.17) and for $\omega_r(f,t)_p$ in (1.1.18).

<u>*Spaces of Lipschitz type*</u>

Definition 3.2.1 [Har00b, Def. 1] *Let* $1 \le p \le \infty$, $0 < q \le \infty$, $\alpha > \frac{1}{q}$ (*with* $\alpha \ge 0$ *if* $q = \infty$). *Then* $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}(\mathbb{R}^n)$ *is defined as the set of all* $f \in L_p(\mathbb{R}^n)$ *such that*

$$\left\| f|\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}(\mathbb{R}^n)\right\| := \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)_p}{t\,|\log t|^\alpha}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \tag{3.2.3}$$

(*with the usual modification if* $q = \infty$) *is finite.*

Note that Definition 3.2.1 coincides with [EH00, Def. 4.1] when $q = \infty$, and in case of $p = q = \infty$, $\alpha \ge 0$, we recover the logarithmic Lipschitz spaces, $\mathrm{Lip}^{(1,-\alpha)} = \mathrm{Lip}^{(1,-\alpha)}_{\infty,\,\infty}$ introduced by (3.1.4) in [EH99, Def. 1.1]. For $\alpha = 0$ they collapse to the classical Lipschitz spaces $\mathrm{Lip}^1(\mathbb{R}^n)$; as long as there is no danger of confusion we shall write $\mathrm{Lip}^{(1,-\alpha)}$ instead of $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\,\infty}$. The restriction $\alpha > \frac{1}{q}$ is quite natural as otherwise we have $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q} = \{0\}$ only, see [Har00b, Rem. 18]. However, when $q = \infty$ we may also admit $\alpha = 0$, whereas $\mathrm{Lip}^{(1,-\alpha)}$ would consist only of constants were α allowed to be negative. The somehow unusual notation using $-\alpha$ (instead of α) is simply due to the fact that we want to emphasise that the additional smoothness parameter α acts in such a way that the usual spaces $\mathrm{Lip}^1(\mathbb{R}^n)$ are extended : $\mathrm{Lip}^1(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ for all $\alpha \ge 0$, i.e. the spaces become larger when less smoothness is assumed – as it should be in some reasonable notation. Definition (3.1.4) was suggested first by TRIEBEL in some unpublished notes.

Remark 3.2.2 The spaces $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\,\infty}(\mathbb{R}^n)$, $\alpha \ge 0$, can also be obtained as a special case of the more general spaces $C^{0,\,\sigma(t)}(\overline{\Omega})$, $\Omega \subseteq \mathbb{R}^n$, which were introduced by KUFNER, JOHN and FUČÍK; see [KJF77, Def. 7.2.12, p. 361]. Moreover, spaces of type $\mathrm{Lip}^{(1,-\alpha)}_{p,\,\infty}$, $\alpha = 0$, are given as $\mathrm{Lip}(1, L_p)$ by DEVORE and LORENTZ in [DL93, Ch. 2, §9, p. 51], where $\mathbb{R}^n$ is being replaced by some interval $[a,b] \subset \mathbb{R}$ and $0 < p \le \infty$. Similarly, spaces $\mathrm{Lip}(\alpha, p)$ were studied by KOLYADA in [Kol89]; see also the end of Section 3.3.3 for further references.

We introduce the *Zygmund spaces* $\mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)$, $\alpha \ge 0$, as some counterparts of the spaces $\mathrm{Lip}^{(1,-\alpha)}$; this definition also relies on some unpublished notes by TRIEBEL.

Definition 3.2.3 [EH99, Def. 4.1] *Let* $\alpha \ge 0$. *Then the space* $\mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)$ *is defined as the set of all* $f \in C(\mathbb{R}^n)$ *such that*

$$\|f|\mathcal{C}^{(1,-\alpha)}(\mathbb{R}^n)\| = \|f|L_\infty(\mathbb{R}^n)\| + \sup_{\substack{x,\,h\,\in\,\mathbb{R}^n\\ 0<|h|<1/2}} \frac{|(\Delta^2_h f)(x)|}{|h|\,|\log|h||^\alpha} < \infty. \tag{3.2.4}$$

Though it might not be obvious at first glance there is an essential difference between spaces of type, say, $\mathrm{Lip}^{(1,-\alpha)}$ and $\mathcal{C}^{(1,-\alpha)}$, $\alpha \ge 0$ – concerning their compatibility with spaces of type $B^s_{p,q}$ introduced by LEOPOLD.

Spaces of type $B^s_{p,q}$

As already mentioned, spaces of generalised smoothness have been intensively studied for long; in our context we concentrate on the following generalisations of spaces $B^s_{p,q}$ merely, where some additional (logarithmic) smoothness is incorporated. Recently, an important contribution to this subject was achieved by MOURA in [Mou01].

Definition 3.2.4 [Leo00b, Def. 1] *Let* $s \in \mathbb{R}$, $b \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, *and let* $\{\varphi_j\}$ *be a smooth dyadic partition of unity. The space* $B^{(s,b)}_{p,q}(\mathbb{R}^n)$ *is the collection of all* $f \in S'(\mathbb{R}^n)$ *such that*

$$\|f|B^{(s,b)}_{p,q}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}(1+j)^{bq}\left\|\mathcal{F}^{-1}\varphi_j\mathcal{F}f|L_p(\mathbb{R}^n)\right\|^q\Big)^{1/q} \tag{3.2.5}$$

(*with the usual modification if* $q=\infty$) *is finite.*

When $b=0$ this definition coincides with the usual one, see Definition 1.1.6 (i) or [Tri83, Def. 2.3.1/2, p. 45]. On the other hand, spaces of type $B^{(s,b)}_{p,q}$ are special cases of $B^{(s,\Psi)}_{p,q}$, $F^{(s,\Psi)}_{p,q}$, introduced by MOURA in [Mou01, Def. 1.5], where Ψ is an 'admissible' function (including $\Psi(x) = (1+|\log x|)^b$, $b \in \mathbb{R}$); for details we refer to [Mou01].

Spaces on domains

Let Ω be a bounded domain in $\mathbb{R}^n$; for simplicity we shall mainly assume

$$\Omega \;=\; U \;=\; \{x \in \mathbb{R}^n : |x| < 1\} \tag{3.2.6}$$

throughout this paper, i.e. that Ω is the unit ball in $\mathbb{R}^n$. One can easily check that our results remain true when U is replaced by some arbitrary bounded C^∞ domain $\Omega \subset \mathbb{R}^n$ (meant in the sense of [EE87, Def. V.4.1, p. 244], say), but at the expense of some constants (depending on Ω).

Definition 3.2.5 *Let* $\alpha \geq 0$. *The space* $\mathrm{Lip}^{(1,-\alpha)}(U)$ *is defined as the set of all* $f \in C(\overline{U})$ *such that*

$$\left\|f|\mathrm{Lip}^{(1,-\alpha)}(U)\right\| = \|f|L_\infty(U)\| + \sup_{\substack{x,\,x+h \in U\\ 0<|h|<1/2}} \frac{|(\Delta_h f)(x)|}{|h|\,|\log|h||^\alpha} \tag{3.2.7}$$

is finite.

Standard procedures (see, for example, [EE87, pp. 250-251]) show that there is a bounded extension map from $\mathrm{Lip}^{(1,-\alpha)}(U)$ to $\mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$. Spaces of type $B^{(s,b)}_{p,q}(\Omega)$ are defined by restriction completely parallel to Definition 1.1.12. This approach coincides with the one of LEOPOLD in [Leo00b, Sect. 3].

In spite of the different approach to spaces on Ω (intrinsic characterisation in case of $\mathrm{Lip}^{(1,-\alpha)}$ and by restriction for $B^{(s,b)}_{p,q}(\Omega)$) one can cope with that technicality by extension procedures. Clearly, one could avoid it from the very beginning by introducing both spaces on Ω in the same way (either by restriction or by intrinsic characterisation) but the respective definitions given above are the more natural ones in our opinion.

3.2.3 Properties, equivalent norms

All spaces are defined on $\mathbb{R}^n$ unless otherwise stated. In view of applications suitably adapted Hölder inequalities are often needed; we give an example for spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\infty}$.

Proposition 3.2.6 [EH00, Prop. 4.3, Rem. 4.4] *Let* $1 \leq p,q \leq \infty$ *such that* $0 \leq \frac{1}{r} = \frac{1}{p}+\frac{1}{q} \leq 1$. *Let* $\alpha,\beta \geq 0$. *Then*

$$\mathrm{Lip}^{(1,-\alpha)}_{p,\infty} \cdot \mathrm{Lip}^{(1,-\beta)}_{q,\infty} \hookrightarrow \mathrm{Lip}^{(1,-\max(\alpha,\beta))}_{r,\infty} \hookrightarrow \mathrm{Lip}^{(1,-(\alpha+\beta))}_{r,\infty}. \tag{3.2.8}$$

We consider spaces of type $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ and $B^{(s,b)}_{p,q}$, see Definitions 3.2.1 and 3.2.4, and give some equivalent characterisations. Recall that we have in B-spaces the equivalent norm (1.1.30). The following extrapolation type result for spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ is known; for details about extrapolation techniques we refer to [Mil94].

Proposition 3.2.7 [EH00, Prop. 4.2 (i)], [Har00b, Prop. 7] *Let* $1 \leq p \leq \infty$.

(i) *Let* $q = \infty$, $\alpha > 0$. *Then* $f \in \mathrm{Lip}^{(1,-\alpha)}_{p,\,\infty}$ *if, and only if,* f *belongs to* L_p *and there is some* $c > 0$ *such that for all* λ, $0 < \lambda < 1$,

$$\sup_{0<t<1/2} \frac{\omega(f,t)_p}{t^{1-\lambda}} \leq c\,\lambda^{-\alpha}.$$

Moreover, we obtain as an equivalent norm in $\mathrm{Lip}^{(1,-\alpha)}_{p,\,\infty}$,

$$\left\| f|\mathrm{Lip}^{(1,-\alpha)}_{p,\,\infty} \right\| \sim \|f|L_p\| + \sup_{0<\lambda<1} \lambda^{\alpha} \sup_{0<t<1/2} \frac{\omega(f,t)_p}{t^{1-\lambda}}. \tag{3.2.9}$$

(ii) *Let* $0 < q < \infty$, $\alpha > \frac{1}{q}$. *Then* $f \in \mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ *if, and only if,* f *belongs to* L_p *and there is some* $c > 0$ *such that*

$$\int_0^1 \lambda^{\alpha q} \int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)_p}{t^{1-\lambda}}\right]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}\lambda}{\lambda} \leq c\,.$$

Moreover,

$$\left\| f|\mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \right\| \sim \|f|L_p\| + \left(\int_0^1 \lambda^{\alpha q} \int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)_p}{t^{1-\lambda}}\right]^q \frac{\mathrm{d}t}{t} \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}. \tag{3.2.10}$$

Remark 3.2.8 When $p = \infty$ Proposition 3.2.7 (i) coincides with the result of Krbec and Schmeisser in [KS01a, Prop. 2.5] which was also our motivation for the above extension; part (i) was already presented in [EH00, Prop. 4.2 (i)].

We want to mention some apparently elegant, but dangerous notation replacing (3.2.9). In view of (1.1.30) with $r = 1$ and $s = 1 - \lambda$, $q = \infty$, i.e

$$\|f|B^{1-\lambda}_{p,\infty}\| \quad \sim \quad \|f|L_p\| + \sup_{0<t<1/2} \frac{\omega(f,t)_p}{t^{1-\lambda}}, \tag{3.2.11}$$

one might be tempted to shorten (3.2.9) by

$$\left\| f|\mathrm{Lip}^{(1,-\alpha)}_{p,\,\infty} \right\| \sim \sup_{0<\lambda<1} \lambda^{\alpha} \ \|f|B^{1-\lambda}_{p,\infty}\|, \tag{3.2.12}$$

or – likewise – to replace (3.2.10) by

$$\left\| f|\mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \right\| \sim \left(\int_0^1 \lambda^{\alpha q} \ \|f|B^{1-\lambda}_{p,q}\|^q \ \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}. \tag{3.2.13}$$

However, the (hidden) equivalence constants in (3.2.11) depend upon λ, especially for $\lambda \downarrow 0$, thus one either has to calculate this dependence explicitly, or has to note that the B-spaces in (3.2.12), (3.2.13) are defined via first differences only (in contrast to the usual Fourier-analytical approach). Hence we prefer the slightly more complicated but correct formulation as in Proposition 3.2.7.

Note that the idea of the characterisations (3.2.12), (3.2.13) resembles in some sense the argument given in Theorem 2.2.1 (i) concerning spaces $L_p(\log L)_{-a}$, $1 < p < \infty$, $a > 0$.

We come to some counterpart of (1.1.30) when dealing with spaces of type $B^{(s,b)}_{p,q}$, $b \in \mathbb{R}$.

Proposition 3.2.9 [Har00b, Prop. 7] *Let* $1 \leq p \leq \infty$, $0 < q \leq \infty$, $b \geq 0$. *Then*

$$\left\| f|B^{(1,-b)}_{p,q} \right\| \;\sim\; \|f|L_p\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega_2(f,t)_p}{t\,|\log t|^b} \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{3.2.14}$$

(*with the usual modification for* $q = \infty$).

Remark 3.2.10 In view of [Tri83, Thm. 2.5.12 (i)] one can extend (3.2.14) to spaces $B^{(s,b)}_{p,q}$ with $0 < p \leq \infty$, $s > \sigma_p$, $b \in \mathbb{R}$, $0 < q \leq \infty$, where $\omega_2(f,t)_p$ has to be replaced by $\omega_r(f,t)_p$ with $r > s$, $r \in \mathbb{N}$,

$$\left\| f|B^{(s,-b)}_{p,q} \right\| \sim \|f|L_p\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega_r(f,t)_p}{t^s\,|\log t|^b} \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \,, \tag{3.2.15}$$

see also [Nev01a, Thm. 4.2] by Neves. In particular, for $p = q = \infty$ we arrive at spaces of Zygmund type, $\mathcal{C}^{(s,-\alpha)} = B^{(s,-\alpha)}_{\infty,\infty}$, $s > 0$, $\alpha \in \mathbb{R}$,

$$\|f|\mathcal{C}^{(s,-\alpha)}\| = \|f|L_\infty\| + \sup_{0<t<1/2} \frac{\omega_r(f,t)}{t^s\,|\log t|^\alpha} \,, \tag{3.2.16}$$

where $r \in \mathbb{N}$, $r > s$.

3.3 Sharp embeddings

We have already reserved the expression 'limiting' (in connection with embeddings) for situations described by (1.2.15). Now we shall adopt the saying '*sharp embedding*' when – at least for one parameter – there cannot be chosen any 'better' (smaller or larger, respectively) value such that the embedding still holds. For instance, returning to the famous result of Brézis and Wainger [BW80], see (3.2.1) and rewritten now as

$$H^{1+n/p}_p(\mathbb{R}^n) \;\hookrightarrow\; \mathrm{Lip}^{(1,-1/p')}(\mathbb{R}^n) \,, \tag{3.3.1}$$

one asks whether the embedding (3.3.1) is *sharp* in the sense that

$$H^{1+n/p}_p(\mathbb{R}^n) \quad \not\hookrightarrow \quad \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$$

if $\alpha < \frac{1}{p'}$ (by the monotonicity of spaces $\mathrm{Lip}^{(1,-\alpha)}$ in α one clearly looks for the smallest value of α). All spaces in this section are defined on $\mathbb{R}^n$ unless otherwise stated.

3.3.1 Sharp embeddings into spaces of Lipschitz type

We care for the question posed above, i.e. the *sharpness* of $\alpha = \frac{1}{p'}$ in (3.3.1), and extend it simultaneously : $H^{1+n/p}_p$ will be replaced by $A^{1+n/p}_{p,q}$. Moreover, turning to spaces defined on bounded domains, it then becomes reasonable to ask for which parameters embeddings of the above type (3.3.1) (suitably adapted to function spaces on domains) become compact, but this is postponed to Section 3.4. Our result is the following.

Theorem 3.3.1 [EH99, Thm. 2.1] *Let* $0 < p \leq \infty$ ($p < \infty$ *in* F*-case*), $0 < q \leq \infty$ *and* $\alpha \geq 0$. *Then*

$$B^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)} \qquad \textit{if, and only if,} \qquad \alpha \geq \frac{1}{q'}, \tag{3.3.2}$$

and

$$F^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)} \qquad \textit{if, and only if,} \qquad \alpha \geq \frac{1}{p'}. \tag{3.3.3}$$

Note that Theorem 3.3.1 was already known for $0 < q \leq 1$ in $B-$ case and $0 < p \leq 1$ in $F-$ case, see (1.2.20) and (1.2.21).

Remark 3.3.2 We proved our result [EH99, Thm. 2.1] using (sub-)atomic decompositions of function spaces, interpolation arguments and extremal functions. We are indebted to H. Triebel in what concerns this result. He stated it together with a sketch of its proof in some unpublished notes and encouraged us to publish it in [EH99].
Another way to prove (3.3.2) when $p = \infty$ and $1 \le q \le \infty$ (apart from the sharpness assertion) is given by Marchaud's inequality : One uses equivalent characterisations of $\mathrm{Lip}^{(1,-\alpha)}$, $B^s_{p,q}$, via the modulus of continuity; recall (1.1.30) with $p = \infty$, i.e.

$$\|f|B^1_{\infty,q}\| \quad \sim \quad \|f|L_\infty\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega_2(f,t)_\infty}{t}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}. \tag{3.3.4}$$

On the other hand, (3.1.4) implies

$$\|f|\mathrm{Lip}^{(1,-\alpha)}\| \quad \sim \quad \|f|L_\infty\| + \sup_{0<t<1/2} \frac{\omega_1(f,t)_\infty}{t|\log t|^\alpha}. \tag{3.3.5}$$

An application of Marchaud's inequality (1.1.19) with $k = 1$ and $p = \infty$,

$$\omega_1(f,t)_\infty \le c\, t \int_t^\infty \frac{\omega_2(f,u)_\infty}{u} \, \frac{\mathrm{d}u}{u}, \tag{3.3.6}$$

for some $c > 0$ and all $f \in L_\infty$ and $t > 0$ results in

$$\|f|\mathrm{Lip}^{(1,-\alpha)}\| \le C\, \|f|B^1_{\infty,q}\| \quad \text{if} \quad \alpha \ge \frac{1}{q'} \,,$$

which yields (3.3.2) for $p = \infty$ and $1 \le q \le \infty$. The extension to $0 < p < \infty$ then comes from the elementary embedding

$$B^{1+n/p}_{p,q} \hookrightarrow B^1_{\infty,q} \,. \tag{3.3.7}$$

We thank this hint our colleague V. Rychkov and refer to [EH99, Rem. 2.4] for further details.

Remark 3.3.3 In view of our introductory remarks, in particular (3.2.1), the theorem implies that for $1 < p < \infty$ and $0 < q \le \infty$ there is some $c > 0$ such that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < \frac{1}{2}$, and all $f \in F^{1+n/p}_{p,q}$,

$$|f(x) - f(y)| \le c\, |x-y| \, \Big|\log|x-y|\Big|^{1/p'} \, \Big\|f|F^{1+n/p}_{p,q}\Big\| \,, \tag{3.3.8}$$

where the exponent $\frac{1}{p'}$ is sharp. Similarly, for $0 < p \le \infty$ and $1 < q \le \infty$ there is some $c > 0$ such that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < \frac{1}{2}$, and all $f \in B^{1+n/p}_{p,q}$,

$$|f(x) - f(y)| \le c\, |x-y| \, \Big|\log|x-y|\Big|^{1/q'} \, \Big\|f|B^{1+n/p}_{p,q}\Big\| \,, \tag{3.3.9}$$

where the exponent $\frac{1}{q'}$ is sharp. Recall $F^s_{p,2} = H^s_p$, $s \in \mathbb{R}$, $1 < p < \infty$. Thus we regain by (3.3.8) the original Brézis-Wainger result (3.3.1); for other works on sharpness of related embeddings see [EGO97], [EGO00] and [EK95]. On the other hand, (3.3.9) gives for $p = q = \infty$ that there is some $c > 0$ such that for all f belonging to the Hölder-Zygmund space $\mathcal{C}^1 = B^1_{\infty,\infty}$, cf. [Tri83, Thm. 2.5.7, p. 90],

$$|f(x) - f(0)| \;\le\; c\, |x| \, \Big|\log|x|\Big| \, \|f|\mathcal{C}^1\|, \tag{3.3.10}$$

for all x, $0 < |x| < \frac{1}{2}$. The exponent 1 of $|\log|x||$ in (3.3.10) is sharp. Further consequences of Theorem 3.3.1 (in terms of sharp inequalities) are discussed in [EH99, Rem. 2.5].

The sharpness assertion essentially relies on results on extremal functions as presented below.

Proposition 3.3.4 [EH99, Prop. 2.2] *Let* $1 < p < \infty$ *and* $\sigma > \frac{1}{p}$. *There is a function* $g_{p\sigma}$ *with*

$$g_{p\sigma} \in B^{1+n/p}_{p,p} \ , \quad g_{p\sigma}(0) = 0,$$

$$|g_{p\sigma}(x)| \ \geq \ c\,|x| \left|\log|x|\right|^{1/p'} \left(\log\left|\log\varepsilon|x|\right|\right)^{-\sigma}$$

for some $c > 0$, *small* $\varepsilon > 0$ *and* $x = (x_1, 0, \ldots, 0)$, $0 < x_1 < \delta$, $\delta > 0$ *small.*

This is essentially the 'lifted' version of an example given by Triebel in [Tri93, Thms. 3.1.2, 4.2.2]; see also [ET96, Thm. 2.7.1, p. 82].

We give the counterpart of Theorem 3.3.1 where the target spaces $\mathrm{Lip}^{(1,-\alpha)}$ in (3.3.2), (3.3.3) are replaced by $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\,v}$, $0 < v \leq \infty$.

Proposition 3.3.5 *Let* $0 < q, v \leq \infty$, $\alpha > \frac{1}{v}$ (*with* $\alpha \geq 0$ *if* $v = \infty$).

(i) *Let* $0 < p \leq \infty$. *Then*

$$B^{1+n/p}_{p,q} \ \hookrightarrow \ \mathrm{Lip}^{(1,-\alpha)}_{\infty,\,v} \qquad \text{if, and only if,} \qquad \alpha \geq \frac{1}{v} + \frac{1}{q'} \ . \tag{3.3.11}$$

In particular, for $v = q$,

$$B^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{\infty,\,q} \qquad \text{if, and only if,} \qquad \alpha \geq 1 \ . \tag{3.3.12}$$

(ii) *Let* $0 < p < \infty$. *Then*

$$F^{1+n/p}_{p,q} \ \hookrightarrow \ \mathrm{Lip}^{(1,-\alpha)}_{\infty,\,v} \qquad \text{if, and only if,} \qquad \alpha \geq \frac{1}{v} + \frac{1}{p'} \ . \tag{3.3.13}$$

In particular, for $v = p$,

$$F^{1+n/p}_{p,q} \ \hookrightarrow \ \mathrm{Lip}^{(1,-\alpha)}_{\infty,\,p} \qquad \text{if, and only if,} \qquad \alpha \geq 1 \ . \tag{3.3.14}$$

Proof : As this result is new in this formulation we insert a short proof. Note that (3.3.11) as well as (3.3.13) with $v = \infty$ are already covered by Theorem 3.3.1. Our results [Har00b, Prop. 11, Cor. 13, Cor. 20] provided, however, weaker assertions only than above (when $v < \infty$) whereas the sharpness of $\alpha = \frac{1}{q'} + \frac{1}{v}$ in (3.3.11) is already covered by [Har00b, Cor. 20]. The essential contribution now comes from our recent studies on envelopes in function spaces which are the main subject of Part II of this report. We do not go into further detail, but refer to our results in [Har01] (described in detail in Sections 4, 5 and 6). There we obtain by Theorem 6.2.5 that there is some $c > 0$ such that for all $f \in A^{1+n/p}_{p,q}$,

$$\left(\int_0^{\varepsilon} \left[\frac{\omega(f,t)}{t\ |\log t|}\right]^u \ \frac{\mathrm{d}t}{t}\right)^{1/u} \ \leq \ c \left\| f | A^{1+n/p}_{p,q} \right\| \qquad \text{if, and only if,} \qquad \begin{cases} u \geq p \ , & A^{1+n/p}_{p,q} = F^{1+n/p}_{p,q}, \\ u \geq q \ , & A^{1+n/p}_{p,q} = B^{1+n/p}_{p,q}; \end{cases} \tag{3.3.15}$$

see in particular [Har01, (5.46), (5.47), (5.56)]. To verify the sharpness in (3.3.12), (3.3.14), as well as to show (3.3.11) and (3.3.13) is then a consequence of the above mentioned result (3.3.15) and Proposition 3.3.6 below. □

3.3.2 Sharp embeddings between spaces with additional logarithmic smoothness

We first deal with (sharp) embeddings between logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}$ and Zygmund spaces $\mathcal{C}^{(1,-\alpha)}$, both of which are defined by differences. Our first result is of 'purely Lipschitzian' type.

Proposition 3.3.6 [Har00b, Prop. 16] *Let* $1 \le p \le \infty$, $0 < q, v \le \infty$, $\alpha > \frac{1}{q}$, $\beta > \frac{1}{v}$. *Then*

$$\mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \hookrightarrow \mathrm{Lip}^{(1,-\beta)}_{p,\,v} \qquad \textit{if, and only if,} \quad \begin{cases} \beta - \frac{1}{v} \ge \alpha - \frac{1}{q}\,, & v \ge q, \\ \beta - \frac{1}{v} > \alpha - \frac{1}{q}\,, & v < q. \end{cases} \tag{3.3.16}$$

Remark 3.3.7 One recognises that our result (3.3.16) resembles the outcome of Bennett and Rudnick concerning spaces $L_{\infty,q}(\log L)_a$:

$$L_{\infty,q}(\log L)_a \hookrightarrow L_{\infty,v}(\log L)_b \qquad \text{if} \qquad \begin{cases} a + \frac{1}{q} \ \ge \ b + \frac{1}{v} & , \quad v \ge q \\ a + \frac{1}{q} \ > \ b + \frac{1}{v} & , \quad v < q \end{cases}\,, \tag{3.3.17}$$

see [BR80, Thms. 9.3, 9.5]. Let us especially point out the somehow astonishing result that concerning the embedding $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ into $\mathrm{Lip}^{(1,-\beta)}_{p,\,v}$ one can 'compensate' some gain of logarithmic smoothness $-\beta > -\alpha$ by 'paying' with the additional index q, that is, as long as $(-\beta) - (-\alpha) \le \frac{1}{q} - \frac{1}{v}$, $v \ge q$.

This situation is essentially different from the related one when dealing with spaces $B^{(s,b)}_{p,q}$ exclusively, see Proposition 3.3.10 below.

We investigate the situation when Zygmund spaces $\mathcal{C}^{(1,-\alpha)}$ are involved additionally.

Proposition 3.3.8 [EH00, Prop. 2.7] *Let* α, β, γ *be non-negative real numbers. Then*

$$\mathrm{Lip}^{(1,-\alpha)} \hookrightarrow \mathcal{C}^{(1,-\beta)} \hookrightarrow \mathrm{Lip}^{(1,-\gamma)} \tag{3.3.18}$$

if, and only if,

$$\beta \ge \alpha, \quad \textit{and} \quad \gamma \ge \beta + 1.$$

We give the counterpart of Proposition 3.3.5 for $p < \infty$.

Proposition 3.3.9 [Har00b, Prop. 11] *Let* $1 \le p < \infty$, $0 < q, v \le \infty$, $\alpha > \frac{1}{v}$ (*with* $\alpha \ge 0$ *if* $v = \infty$). *Then*

$$B^1_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,\,v} \qquad \textit{if} \qquad \begin{cases} \alpha \ge \frac{1}{q'} & , \quad v = \infty, \\ \alpha > \frac{1}{v} + \frac{1}{q'} & , \quad v < \infty\,. \end{cases} \tag{3.3.19}$$

Note that we proved [Har00b, Prop. 11] for all p, $1 \le p \le \infty$, but the case $p = \infty$ is now replaced by the better result Proposition 3.3.5. Furthermore, for $p < \infty$ and $v = \infty$ (3.3.19) is covered by [EH00, Prop. 4.2 (ii)] already. Comparing (3.3.19) and (3.3.11) the question naturally arises whether $B^1_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,\,v}$ remains true for $\alpha = \frac{1}{v} + \frac{1}{q'}$ and $v < \infty$, $p < \infty$. This is not so clear at the moment, at least not covered by our recent studies on envelopes. However, when $p = \infty$ [Har00b, Cor. 20] implies that there cannot be an embedding like (3.3.19) for $\alpha < \frac{1}{v} + \frac{1}{q'}$. Otherwise, for $1 \le p < \infty$, there is an improved version of (3.3.19) by Neves in [Nev01a, Prop. 5.2] based upon Timan's inequality [DL93, Ch. 2, Thm. 8.4, p. 49] instead of Marchaud's (1.1.19).

We showed in [EH99, Prop. 4.2] that $\mathcal{C}^{(1,-\alpha)} = B^{(1,-\alpha)}_{\infty,\infty}$, $\alpha \ge 0$. In that sense Proposition 3.3.8 also leads to the question what else can be said about the relation between spaces of type $B^{(s,b)}_{p,q}$ (defined in the Fourier-analytical way, see Definition 3.2.4) and spaces defined by differences, in particular, $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$. We try to clarify this interplay by some more results and a subsequent discussion in Section 3.3.3. We begin with a result of Leopold obtained in [Leo98, Thm. 1] which is closely linked to Theorem 3.3.1 as well as to Proposition 3.3.6.

Proposition 3.3.10 [Leo98, Thm. 1] *Let* $s_1 \geq s_2$, $b_1, b_2 \in \mathbb{R}$, $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$, *and assume* $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$. *Then*

$$B^{(s_1,b_1)}_{p_1,q_1} \hookrightarrow B^{(s_2,b_2)}_{p_2,q_2} \quad \text{if, and only if,} \quad \begin{cases} b_1 - b_2 \geq 0 & , \quad q_1 \leq q_2 \\ b_1 - b_2 > \dfrac{1}{q_2} - \dfrac{1}{q_1} & , \quad q_1 > q_2 \, . \end{cases} \tag{3.3.20}$$

The above assertion can also be found (as some special case) in [Mou01].

As already mentioned, we are interested in the interplay between both scales of spaces especially. Recall that for $\alpha = 0$ it is known that $\mathcal{C}^s = B^s_{\infty,\infty}$, $s > 0$, see [Tri83, Thm. 2.5.7 (ii), p. 90], and $B^1_{\infty,1} \hookrightarrow \mathrm{Lip}^1 \hookrightarrow B^1_{\infty,\infty}$, see [Tri83, (2.5.7/2), (2.5.7/11), p. 89/90]. In [EH99] we proved that there are extensions to $\alpha \geq 0$.

Proposition 3.3.11 [EH99, Props. 4.2, 4.4] *Let* $\alpha \geq 0$. *Then*

$$B^{(1,-\alpha)}_{\infty,1} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)} \hookrightarrow \mathcal{C}^{(1,-\alpha)} = B^{(1,-\alpha)}_{\infty,\infty} \, . \tag{3.3.21}$$

Moreover,

$$B^{(1,-\alpha)}_{\infty,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)} \quad \text{if, and only if,} \quad 0 < q \leq 1.$$

Note that also the latter assertion is well-known for $\alpha = 0$, see [ET96, (2.3.3/9,10), p. 45].

Before we come to compare spaces of type $B^{(s,b)}_{p,q}$ and $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ in Section 3.3.3, we derive a few more, rather elementary embeddings between both scales of spaces. In view of characterisation (3.2.14) and MARCHAUD's inequality we may extend Proposition 3.3.11 to spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$.

Corollary 3.3.12 [Har00b, Prop. 23, Cor. 26] *Let* $1 \leq p \leq \infty$, $0 < q \leq \infty$, $\alpha > \frac{1}{q}$.

(i) *Then*

$$B^{(1,-\beta)}_{p,1} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \quad \text{if} \quad \begin{cases} \beta < \alpha - \frac{1}{q} & , \quad 0 < q < \infty \, , \\ \beta \leq \alpha & , \quad q = \infty \, . \end{cases} \tag{3.3.22}$$

Moreover,

$$B^{(1,-(\alpha-\frac{1}{q}))}_{p,\min(q,1)} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,\,q}. \tag{3.3.23}$$

(ii) *Let* $1 \leq q \leq \infty$, $\alpha > 1$. *Then*

$$B^{(1,-(\alpha-1))}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \, . \tag{3.3.24}$$

Recall the notation for spaces $\mathcal{C}^{(1,-\alpha)}$, $\alpha \geq 0$, see (3.2.16) with $s = 1$, $r = 2$. Then by Proposition 3.3.8 assertion (3.3.24) coincides with (the right-hand embedding in) (3.3.18) when $p = q = \infty$.

Corollary 3.3.13 [Har00b, Cor. 25] *Let* $1 \leq p \leq \infty$, $0 < q, v \leq \infty$, $\alpha > \frac{1}{q}$, $\beta > \frac{1}{v}$. *Then*

$$\mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \hookrightarrow B^{(1,-\beta)}_{p,v} \quad \text{if} \quad \begin{cases} \beta - \frac{1}{v} \geq \alpha - \frac{1}{q} & , \quad v \geq q \, , \\ \beta - \frac{1}{v} > \alpha - \frac{1}{q} & , \quad v < q \, . \end{cases} \tag{3.3.25}$$

Remark 3.3.14 For $p = q = \infty$ assertions (3.3.22), (3.3.23) and (3.3.25) coincide with Proposition 3.3.11.

3.3.3 Some discussion

We examine the relation between 'logarithmically smooth' Besov spaces $B^{(s,b)}_{p,q}$, introduced by LEOPOLD in [Leo98], and 'logarithmic' Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$. From the point of dealing with these spaces in view of atomic decompositions etc. it is essential that the logarithmic B-spaces, that is $B^{(s,b)}_{p,q}$, arise by a Fourier-analytical approach (like the usual spaces $B^{s}_{p,q}$), see (3.2.5), whereas the logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$, defined via first differences, see (3.2.3), remain as 'Fourier-unfriendly' as were their classical forerunners (with $p=q=\infty$, $\alpha=0$). In fact, the almost inconspicuous modification in (3.2.3) compared with (3.2.14), namely the substitution of $\omega_2(f,t)_p$ by $\omega_1(f,t)_p$, causes a striking difference in the features of the corresponding spaces (as it does for $\alpha=b=0$).

We return to Proposition 3.3.10 obtained by LEOPOLD in [Leo98, Thm. 1]. Plainly, it implies

$$B^{(1,-\alpha)}_{p,q} \hookrightarrow B^{(1,-\beta)}_{p,v} \qquad \text{if, and only if,} \qquad \begin{cases} \beta-\alpha\geq 0 & ,\ q\leq v \\ \beta-\frac{1}{v}>\alpha-\frac{1}{q} & ,\ q>v. \end{cases} \tag{3.3.26}$$

It is obvious, that – though (3.3.16) and (3.3.26) appear related somehow – the role played by the parameter q in either case is different. The 'diagonal argument' (essentially used in Step 3 of the proof of Proposition 3.3.6 and borrowed from BENNETT and RUDNICK) does not apply in (3.3.26). In other words, the parallel notation (taking the same parameter q) in both cases $B^{(s,b)}_{p,q}$ and $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$, respectively, is a dangerous one (though suggestive in either case), possibly pretending at first glance that the construction with respect to q might be the same; however, it is not. On the other hand, it is nevertheless surprising that the 'fine index' q in these limiting cases becomes so important.

We study the question now '*where*' the Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ can be found within the scale of Besov spaces $B^{(s,b)}_{p,q}$. Let $1\leq p\leq\infty$ and $0<q\leq\infty$. Concerning the scale of logarithmic Besov spaces $B^{(1,b)}_{p,q}$ for fixed p and q, but arbitrary $b\in\mathbb{R}$, we may locate the Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ as follows. Denote by $q^*:=\min(q,1)$ and assume $\alpha>\frac{1}{q^*}$. Then

$$B^{(1,-(\alpha-\frac{1}{q^*}))}_{p,q} \quad\hookrightarrow\quad \mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \quad\hookrightarrow\quad B^{(1,-\alpha)}_{p,q}\ , \tag{3.3.27}$$

see (3.3.23), (3.3.25) and (3.3.24). Insisting, however, on the same (logarithmic) smoothness in both nestling spaces of type $B^{(1,b)}_{p,q}$, that is, for fixed p and b, but varying q, we found

$$B^{(1,-(\alpha-\frac{1}{q}))}_{p,q^*} \quad\hookrightarrow\quad \mathrm{Lip}^{(1,-\alpha)}_{p,\,q} \quad\hookrightarrow\quad B^{(1,-(\alpha-\frac{1}{q}))}_{p,\infty}\ , \tag{3.3.28}$$

recall (3.3.23) and (3.3.25). One verifies that for $1<q<\infty$ the respective initial spaces and endpoint spaces in (3.3.27) and (3.3.28) are incomparable in the sense that neither of them is contained in the corresponding other one; this refers to $B^{(1,-(\alpha-\frac{1}{q^*}))}_{p,q}$ and $B^{(1,-(\alpha-\frac{1}{q}))}_{p,q^*}$ as well as to $B^{(1,-\alpha)}_{p,q}$ and $B^{(1,-(\alpha-\frac{1}{q}))}_{p,\infty}$, respectively. Obviously they coincide, respectively, when $0<q\leq 1$ (in case of the initial spaces) and when $q=\infty$ (concerning the endpoint spaces). Thus we have the general situation that

$$\underbrace{\begin{array}{ccc} & \nearrow & B^{(1,-(\alpha-\frac{1}{q^*}))}_{p,q} \\ B^{(1,-(\alpha-\frac{1}{q^*}))}_{p,q^*} & & \\ & \searrow & B^{(1,-(\alpha-\frac{1}{q}))}_{p,q^*} \end{array}}_{\equiv\ B^{(1,-(\alpha-\frac{1}{q}))}_{p,q},\ \ 0<q\leq 1} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} \boxed{\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}} \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \underbrace{\begin{array}{ccc} B^{(1,-\alpha)}_{p,q} & & \\ & \searrow & \\ & & B^{(1,-\alpha)}_{p,\infty} \\ & \nearrow & \\ B^{(1,-(\alpha-\frac{1}{q}))}_{p,\infty} & & \end{array}}_{\equiv\ B^{(1,-\alpha)}_{p,\infty},\ \ q=\infty}\ . \tag{3.3.29}$$

Moreover, we have the same diagram with $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ replaced by $B^{(1,-(\alpha-\frac{1}{q}))}_{p,q}$. These spaces, however, are not comparable (in the above sense) when $1<q<\infty$, whereas $\mathrm{Lip}^{(1,-\alpha)}_{p,\,q}\hookrightarrow B^{(1,-(\alpha-\frac{1}{q}))}_{p,q}$ when $q=\infty$, and $B^{(1,-(\alpha-\frac{1}{q}))}_{p,q}\hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{p,\,q}$ for $0<q\leq 1$, see (3.3.23); we also refer to [Har00b, Sect. 4].

There are a lot of further related approaches to spaces of Lipschitz type, recall Remark 3.2.2, see also [Her68] by HERZ, the books of STEIN [Ste70] and PEETRE [Pee76], and the papers [Tai64], [Tai65], [Tai66] by TAIBLESON and [Tri73] by TRIEBEL. Let us finally mention only a few, more recent papers : AKSOY and MALIGRANDA, see [AM96], studied descriptions of spaces of Lipschitz-Orlicz type $\mathrm{Lip}(\alpha, L_M)$ and $\mathrm{Zyg}(\alpha, L_M)$ in terms of Poisson integrals; BRANDOLINI, [Bra98], introduced *generalised Lipschitz spaces*, i.e. spaces of type $\Lambda^\alpha_X(\mathbb{R}^n)$, $\alpha > 0$ and X being either $L_{p,\infty}(\mathbb{R}^n)$ or $L_p(\mathbb{R}^n)$; in particular, for $X = L_p(\mathbb{R}^n)$ and $\alpha = 1$ these are the above spaces $\mathrm{Lip}^{(1,0)}_{p,\infty}$. The closest approach we found in the literature so far – really dealing with *logarithmic* or similar modifications of the usual Lipschitz spaces – is given in the paper [BS94] by BLOOM and DE SOUZA. They concentrated on weighted Lipschitz spaces of type $\mathrm{Lip}\,\varrho$, where $\varrho : [0, 2\pi] \to [0, \infty)$ is a nondecreasing weight function with $\varrho(0) = 0$. With a slight modification we may regard $\varrho_\alpha(t) \sim t|\log t|^\alpha$, $t > 0$ small, as such a weight, and – in their notation – we obtain that $\mathrm{Lip}\,\varrho_\alpha = \mathrm{Lip}^{(1,-\alpha)}_{\infty,\infty}$ and for the Zygmund spaces $\Lambda(\varrho_\alpha) = \mathcal{C}^{(1,-\alpha)}$.
In a wider context – dealing with spaces of generalised smoothness – there is a variety of literature, recall our introductory remarks at the beginning of Section 3.2.

3.4 Compact embeddings, and entropy numbers

3.4.1 Entropy numbers in sequence spaces

As in [ET96] and [Tri97], our estimation of the entropy numbers of embedding maps involves a reduction of the problem to the study of maps between finite-dimensional sequence spaces. Accordingly we study the situation in sequence spaces – as defined in Section 1.1.2 – first. Concerning entropy numbers of the embedding map $id : \ell^M_{p_1} \longrightarrow \ell^M_{p_2}$, $0 < p_1 \le p_2 \le \infty$, we make use of the results [ET96, Prop. 3.2.2, p. 98] as well as [Tri97, Prop. 7.2, p. 36]. Note that in the Banach space setting estimates for the entropy numbers in finite-dimensional sequence spaces have been studied in great detail for a long time. We refer to [Sch84] as well as [Kön86, Sect. 3.c.8] for further details and references.

We consider the embedding

$$id_{p_1,p_2} : \ell_q\left(\ell^{M_j}_{p_1}\right) \to \ell_q\left(\langle j\rangle^{-\varkappa}\ell^{M_j}_{p_2}\right), \tag{3.4.1}$$

where $0 < p_1 \le p_2 \le \infty$, $0 < q \le \infty$, $\varkappa > 0$, and $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. We have shown in [EH99, Prop. 3.1] that id_{p_1,p_2} is compact for $\varkappa > 0$ and $p_1 \le p_2$ (see also Proposition 3.4.1 below which implies the compactness, too). We study (the asymptotic behaviour of) the corresponding entropy numbers $e_k(id_{p_1,p_2})$ in the sequel. Note that in case of entropy numbers parallel results – i.e. when dealing with dyadic weights of type $w_j = 2^{j\delta}$, $\delta > 0$, – were obtained by KÜHN in [Küh84] and TRIEBEL in [Tri97, Sect. 8].

It turns out that for later application we need only deal with the cases when $p = p_1 = p_2$ and $p = p_1$, $p_2 = \infty$, respectively. We begin with the setting when $0 < p = p_1 = p_2 \le \infty$ and adopt the notation

$$id_{p,p} : \ell_q\left(\ell^{M_j}_{p}\right) \to \ell_q\left(\langle j\rangle^{-\varkappa}\ell^{M_j}_{p}\right), \tag{3.4.2}$$

where $0 < p \le \infty$, $0 < q \le \infty$, $\varkappa > 0$, and $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. As a first result we obtained in [EH99] the following.

Proposition 3.4.1 [EH99, Prop. 3.1] *Let* $\varkappa > 0$, $0 < p \le \infty$, $0 < q \le \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$. *Then*

$$e_k\left(id_{p,p}\right) \quad \sim \quad \left(\log\langle k\rangle\right)^{-\varkappa}, \quad k \in \mathbb{N}. \tag{3.4.3}$$

Remark 3.4.2 When $w_j = 2^{j\delta}$, $\delta > 0$, our notation (1.1.22) coincides with [Tri97, (8.2)]. The result parallel to Proposition 3.4.1, assuming $0 < p \le \infty$, $0 < q \le \infty$, $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$, is then a special case of [Tri97, Thm. 8.2, p. 39] and reads as

$$e_k\left(id : \ell_q\left(2^{j\delta}\ell^{M_j}_p\right) \to \ell_q\left(\ell^{M_j}_p\right)\right) \quad \sim \quad k^{-\frac{\delta}{n}}, \quad k \in \mathbb{N}.$$

Furthermore, as will be clarified later, we do need some generalisation of Proposition 3.4.1 in the context of spaces $\ell_u\left[2^{\mu m}\ell_q\left(\ell_p^{M_j}\right)\right]$, $0<u\le\infty$; see (1.1.23) for the definition. This is covered by [EH99, Cor. 3.3]; it yields, in particular, for $\varkappa>0$, $0<p,q\le\infty$, $M_j\sim 2^{jn}$, $j\in\mathbb{N}_0$,

$$e_k\left(id:\ell_\infty\left[2^{\varrho_1 m}\ell_q\left(\ell_p^{M_j}\right)\right] \quad\longrightarrow\quad \ell_\infty\left[2^{\varrho_2 m}\ell_q\left(\langle j\rangle^{-\varkappa}\ell_p^{M_j}\right)\right]\right) \sim (\log\langle k\rangle)^{-\varkappa} \tag{3.4.4}$$

for all $k\in\mathbb{N}$, where $\varrho_1>\varrho_2$. The parallel result to (3.4.4) with $w_j=2^{j\delta}$, $\delta>0$, is given in [Tri97, Thm. 9.2, p. 47].

We study the embedding

$$id_{p,\infty}:\ell_q\left(\ell_p^{M_j}\right)\to\ell_q\left(\langle j\rangle^{-\varkappa}\ell_\infty^{M_j}\right) \tag{3.4.5}$$

now, where $0<p<\infty$, $0<q\le\infty$ and $\varkappa>0$. Note that the compactness of $id_{p,\infty}$ is a consequence of the compactness of $id_{p,p}$. We estimate the corresponding entropy numbers.

Proposition 3.4.3 [EH00, Props. 3.4, 3.5] *Let* $\varkappa>0$, $0<p<\infty$, $0<q\le\infty$, $M_j\sim 2^{jn}$, $j\in\mathbb{N}_0$.

(i) *There is some* $c>0$ *such that for all* $k\in\mathbb{N}$,

$$e_k\left(id_{p,\infty}\right) \quad\ge\quad c\begin{cases} k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\varkappa} &, \quad \varkappa>\frac{1}{p} \\ k^{-\varkappa} &, \quad \varkappa\le\frac{1}{p}\end{cases}. \tag{3.4.6}$$

Moreover, if we additionally have $1\le p<\infty$, *then* (3.4.6) *can be replaced by*

$$e_k\left(id_{p,\infty}\right) \quad\ge\quad c\begin{cases} k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\varkappa+\frac{1}{p}} &, \quad \varkappa>\frac{1}{p} \\ k^{-\varkappa} &, \quad \varkappa\le\frac{1}{p}\end{cases}. \tag{3.4.7}$$

(ii) *Let* $\varrho:=\min(q,1)$. *There is some* $c>0$ *such that for all* $k\in\mathbb{N}$,

$$e_k\left(id_{p,\infty}\right) \quad\le\quad c\begin{cases} k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\varkappa+\frac{1}{\varrho}+\frac{2}{p}} &, \quad \varkappa>\frac{1}{\varrho}+\frac{2}{p} \\ k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{\frac{1}{\varrho}+\frac{1}{p}} &, \quad \varkappa=\frac{1}{\varrho}+\frac{2}{p} \\ k^{-\frac{1}{p}\frac{\varkappa}{\frac{1}{\varrho}+\frac{2}{p}}} &, \quad \varkappa<\frac{1}{\varrho}+\frac{2}{p}\end{cases}. \tag{3.4.8}$$

A major improvement of Proposition 3.4.3 was obtained in a recent paper by Cobos and Kühn [CK01] : They showed that (3.4.7) and (3.4.8) can be improved using tricky combinatorial arguments, complex interpolation and an extended knowledge on the ℓ-norm and related results for Kolmogorov- and entropy numbers; we refer to the book of Pisier [Pis89, Ch. 5] for an excellent presentation of all the necessary background material as well as details, and to the papers of Gluskin [Glu83], Sudakov [Sud71], and Pajor and Tomczak-Jaegermann [PTJ86], [PTJ89]. We already discussed this possibility briefly in [EH00, Rem. 3.6]. The result of Cobos and Kühn is the following.

Proposition 3.4.4 [CK01, Thms. 1, 2] *Let* $\varkappa>0$, $1\le p<\infty$, $1\le q\le\infty$, $M_j\sim 2^{jn}$, $j\in\mathbb{N}_0$.

(i) *There is some* $c>0$ *such that for all* $k\in\mathbb{N}$,

$$e_k\left(id_{p,\infty}\right) \quad\ge\quad c\begin{cases} k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\varkappa+\frac{1}{p}} &, \quad \varkappa>\frac{2}{p} \\ k^{-\frac{\varkappa}{2}} &, \quad \varkappa\le\frac{2}{p}\end{cases}. \tag{3.4.9}$$

(ii) *There is some* $c>0$ *such that for all* $k\in\mathbb{N}$,

$$e_k\left(id_{p,\infty}\right) \quad\le\quad c\begin{cases} k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\varkappa+\frac{2}{p}} &, \quad \varkappa>\frac{2}{p} \\ k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{\frac{1}{p}} &, \quad \varkappa=\frac{2}{p} \\ k^{-\frac{\varkappa}{2}} &, \quad \varkappa<\frac{2}{p}\end{cases}. \tag{3.4.10}$$

One has a sharp result now for small $\varkappa < \frac{2}{p}$, i.e.

$$e_k\left(id_{p,\infty}\right) \sim k^{-\frac{\varkappa}{2}}, \quad k \in \mathbb{N},$$

and the gap between lower and upper estimate in the remaining cases became much smaller compared with Proposition 3.4.3, at least in the Banach case situation. Moreover, Cobos and Kühn conjecture in their paper that the upper bound is sharp for $\varkappa \geq \frac{2}{p}$, too. This is based on two reasons : firstly, when $q = \infty$, then by [CK01, Prop. 1]

$$e_k\left(id_{p,\infty}\right) \sim k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\varkappa+\frac{2}{p}}$$

for all $1 \leq p < \infty$ and $\varkappa > \frac{2}{p}$. Secondly, they briefly mention a brand-new result by Belinsky [Bel01] verifying the upper bound as sharp even in the quasi-Banach setting.

Remark 3.4.5 Note that Leopold obtained in [Leo00c, Thm. 3] similar results when dealing with the more general setting

$$id_p^q : \ell_{q_1}\left(\ell_{p_1}^{M_j}\right) \to \ell_{q_2}\left(\langle j\rangle^{-\varkappa}\ell_{p_2}^{M_j}\right),$$

where $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$, $\varkappa > (\frac{1}{q_2} - \frac{1}{q_1})_+$, and $M_j \sim 2^{jn}$, $j \in \mathbb{N}_0$; see also [Leo00a]. These results were sharpened in a recent paper by Kühn and Schonbek [KS01b].

3.4.2 Compact embeddings and entropy numbers

We are prepared now to tackle the problem of estimating the entropy numbers of our limiting embedding.

Compact embeddings

Clearly it makes no sense to study compactness of natural embeddings like

$$id : B_{p,q}^{1+n/p}(\mathbb{R}^n) \to \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$$

in (unweighted) $\mathbb{R}^n$−setting : we have for any $\alpha \geq \frac{1}{q'}$ and any $\varepsilon > 0$, $0 < u \leq \infty$, the embeddings

$$B_{p,q}^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{(1,-\alpha)}(\mathbb{R}^n) \hookrightarrow B_{\infty,u}^{1-\varepsilon}(\mathbb{R}^n),$$

which are all continuous by our results in Section 3.3 and in view of Definitions 1.1.6 (i) and 3.2.4 (referring to the last embedding). Assuming $B_{p,q}^{1+n/p}(\mathbb{R}^n) \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}(\mathbb{R}^n)$ was compact for some $\alpha > \frac{1}{q'}$, then this implied compactness of $B_{p,q}^{1+n/p}(\mathbb{R}^n) \hookrightarrow B_{\infty,u}^{1-\varepsilon}(\mathbb{R}^n)$ immediately, but this is not true; cf. Theorem 1.2.1 (ii) and its more general version [HT94a, Thm. 2.3].

A gentle modification of our setting surmounting the above-described difficulty consists in the introduction of additional weight functions (as presented in the first example in Section 2) or, alternatively, to reduce the problem to spaces on domains. We follow the latter concept here. By our remarks in Section 3.2.2 concerning spaces on domains it is clear that our embedding results in Section 3.3 remain valid. Let $U = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$.

Proposition 3.4.6 [EH00, Prop. 2.5, Cor. 2.8] *Let* $0 < q \leq \infty$.

(i) *Assume* $0 < p \leq \infty$, $\alpha > \frac{1}{q'}$. *Then* $\quad id^B : B_{p,q}^{1+n/p}(U) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(U) \quad$ *is compact.*

(ii) *Assume* $0 < p < \infty$, $\alpha > \frac{1}{p'}$. *Then* $\quad id^F : F_{p,q}^{1+n/p}(U) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(U) \quad$ *is compact.*

In view of our embedding results in Section 3.3 and Proposition 3.4.7 below this result is obvious. We collect two further results dealing with either Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}$ or spaces of type $B_{p,q}^{(s,b)}$ exclusively.

Proposition 3.4.7 [EH00, Prop. 2.6] *Let* $\beta > \alpha > 0$. *Then* $\quad id_{\alpha\beta} : \mathrm{Lip}^{(1,-\alpha)}(U) \longrightarrow \mathrm{Lip}^{(1,-\beta)}(U)$ *is compact.*

Remark 3.4.8 In Remark 3.2.2 we identified $\mathrm{Lip}^{(1,-\alpha)}(\Omega)$ as a special case of the more general $C^{0,\sigma(t)}(\overline{\Omega})$ spaces introduced in [KJF77, Def. 7.2.12, p. 361]. The above proposition can also be found as a special case of a related result for $C^{0,\sigma(t)}(\overline{\Omega})$ spaces, that is [KJF77, Lemma 7.4.3, p. 368].

Leopold obtained in [Leo98] a similar result. We present it in a simplified version (adapted to our setting) only. Recall notation (1.1.2).

Proposition 3.4.9 [Leo98, Thm. 2] *Let* $s \in \mathbb{R}$, $0 < p,\ q_1,\ q_2 \leq \infty$, *and* $b > \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+$. *Then*

$$id : B^{(s,b)}_{p,q_1}(U) \longrightarrow B^s_{p,q_2}(U)$$

is compact.

This result can also be identified as a special case of [Mou01, Thm. 3.13, p. 78].

Our intention was to deal with some model cases only; however, in view of (1.3.2) more compactness results can be easily obtained from our results below when we deal with estimates for entropy numbers.

<u>Entropy numbers</u>

Recall our notation id^B for the embedding

$$id^B : B^{1+n/p}_{p,q}(U) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(U)\,, \tag{3.4.11}$$

where $0 < p \leq \infty$, $0 < q \leq \infty$, $\alpha > \frac{1}{q'}$. According to Proposition 3.4.6 (i) id^B is compact and it makes sense to study its entropy numbers.

Theorem 3.4.10 [EH99, Thm. 4.10] *Let* $0 < q \leq \infty$ *and* $\alpha > \frac{1}{q'}$. *Then there are positive numbers* c_1 *and* c_2 *such that for all* $k \in \mathbb{N}$,

$$c_1 \left(\log\langle k\rangle\right)^{-\alpha} \leq\ e_k\left(id : B^1_{\infty,q}(U) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(U)\right) \leq\ c_2 \left(\log\langle k\rangle\right)^{-\alpha+\frac{1}{q'}}. \tag{3.4.12}$$

In particular, when $0 < q \leq 1$ *and thus* $\alpha > 0$, *we obtain*

$$e_k\Big(id : B^1_{\infty,q}(U) \longrightarrow \mathrm{Lip}^{(1,-\alpha)}(U)\Big) \quad \sim \quad \left(\log\langle k\rangle\right)^{-\alpha}. \tag{3.4.13}$$

Due to the embedding $B^{1+n/p}_{p,q}(U) \hookrightarrow B^1_{\infty,q}(U)$ and the multiplicativity of entropy numbers the upper estimate is true for all id^B, $0 < p \leq \infty$, whereas we already showed in [EH99, Thm. 4.10] that the lower bound (3.4.12) has to be replaced by $c_1\, k^{-\frac{1}{p}} \left(\log\langle k\rangle\right)^{-\alpha}$ when $p < \infty$. Our result for id^B and $0 < p < \infty$ is the following.

Theorem 3.4.11 [EH00, Thm. 3.11] *Let* $0 < p < \infty$, $0 < q \leq \infty$, $\alpha > \frac{1}{q'}$. *Let* $\varrho = \min(q,1)$. *There are positive numbers* c_1 *and* c_2 *such that for all* $k \in \mathbb{N}$,

$$c_1\, k^{-\frac{1}{p}} \left(\log\langle k\rangle\right)^{-\alpha} \leq e_k\left(id^B\right) \leq c_2 \begin{cases} k^{-\frac{1}{p}} \left(\log\langle k\rangle\right)^{-\alpha+\frac{1}{q'}+\frac{1}{\varrho}+\frac{2}{p}} & , \quad \alpha > \frac{1}{q'}+\frac{1}{\varrho}+\frac{2}{p} \\ k^{-\frac{1}{p}} \left(\log\langle k\rangle\right)^{\frac{1}{\varrho}+\frac{2}{p}} & , \quad \alpha = \frac{1}{q'}+\frac{1}{\varrho}+\frac{2}{p} \\ k^{-\frac{\alpha-1/q'}{2+p/\varrho}} & , \quad \alpha < \frac{1}{q'}+\frac{1}{\varrho}+\frac{2}{p}\,. \end{cases} \tag{3.4.14}$$

We briefly sketch the main ideas of our proof in [EH00]. It indicates the way in which our preceding results are used for that purpose. We start with the estimate from below, essentially using the characterisation (3.2.9), (3.2.12) for $p = \infty$, and our complete knowledge about the non-limiting case, see (1.3.3)

$$e_k\left(id : B^{s_1}_{p_1,q_1}(U) \longrightarrow B^{s_2}_{p_2,q_2}(U)\right) \quad \sim \quad k^{-\frac{s_1-s_2}{n}}, \quad k \in \mathbb{N}, \tag{3.4.15}$$

where $s_1 > s_2$, $0 < p_1, p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$, and $s_1 - s_2 > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+$. We apply (3.4.15) with $s_1 = 1 + \frac{n}{p}$, $s_2 = 1 - \lambda$, $\lambda > 0$, $p_1 = p$, $q_1 = q$, $p_2 = q_2 = \infty$. A straightforward (and nevertheless careful) calculation of the extremal problem in $\lambda > 0$ completes the argument for the lower estimate.
We care about the upper estimate in (3.4.14). Here we benefit from our results on entropy numbers in sequence spaces in Section 3.3.1. We outline the main points, for details we refer to [EH99], [EH00]. The crucial trick is to find a (non-)linear bounded operator S and a linear operator T such that we obtain the following commutative diagram,

$$\begin{array}{ccc} B^{1+n/p}_{p,q}(U) & \xrightarrow{\quad S \quad} & \ell_q\big(\ell_p^{M_k}\big) \\ id^B \Big\downarrow & & \Big\downarrow id_{p,\infty} \\ \mathrm{Lip}^{(1,-\alpha)}(U) & \xleftarrow{\quad T \quad} & \ell_q\big(\langle k\rangle^{-(\alpha-1/q')}\,\ell_\infty^{M_k}\big) \end{array} \tag{3.4.16}$$

This is done via atomic (or, strictly speaking, even quarkonial) decompositions of function spaces, but we do not propose to go into further details here; we remind the reader of Section 1.1.3, in particular, Theorem 1.1.9 and [Tri97, Sect. 13]. In [Tri97, Th. 13.8, p. 75] there is a mechanism established by which distributions $f \in B^s_{p,q}(\mathbb{R}^n)$ can be transformed into a sequence of complex numbers belonging to some space $\ell_q(\ell_p^{M_k})$, simultaneously controlling the corresponding norms. This provides the boundedness of the operator S. Concerning the independence of the 'inverse' operator T from the used atomic decomposition, one has to involve even 'smaller' building blocks than atoms, i.e. 'quarks'; cf. [Tri97, Sect. 14] for all necessary details. Moreover, one also needs some 'quarkonial version' of Propositions 3.4.1 and 3.4.3 (ii) then, but this can be obtained without difficulties; cf. [Tri97, Sect. 9], [EH99, Cor. 3.3] and Remark 3.4.2. One verifies that $T : \ell_q\big(\langle k\rangle^{-(\alpha-\frac{1}{q'})}\,\ell_\infty^{M_k}\big) \to \mathrm{Lip}^{(1,-\alpha)}(U)$ is bounded. Thus by the multiplicativity of entropy numbers and $id^B = T \circ id_{p,\infty} \circ S$, Proposition 3.4.3 (ii) concludes the proof.

Remark 3.4.12 Due to their improved estimates for $e_k(id_{p,\infty})$, see Proposition 3.4.4, Cobos and Kühn achieved in [CK01, Thm. 3] a replacement for the upper estimates in (3.4.14) as follows :

$$e_k\left(id^B\right) \leq c_2 \begin{cases} k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{-\alpha+\frac{1}{q'}+\frac{2}{p}} & , \quad \alpha > \frac{1}{q'} + \frac{2}{p} \\ k^{-\frac{1}{p}}\left(\log\langle k\rangle\right)^{\frac{1}{p}} & , \quad \alpha = \frac{1}{q'} + \frac{2}{p} \\ k^{-\frac{\alpha-1/q'}{2}} & , \quad \alpha < \frac{1}{q'} + \frac{2}{p}, \end{cases}$$

where $1 \leq p < \infty$, $1 \leq q \leq \infty$, $\alpha > \frac{1}{q'}$.

We return to the situation of Proposition 3.4.7 and give our result on the (asymptotic behaviour of the) entropy numbers of the compact embedding $id_{\alpha\beta}$, $\beta > \alpha > 0$.

Theorem 3.4.13 [EH00, Thm. 3.17] *Let $\beta > \alpha > 0$. Then*

$$e_k\left(id_{\alpha\beta} : \mathrm{Lip}^{(1,-\alpha)}(U) \longrightarrow \mathrm{Lip}^{(1,-\beta)}(U)\right) \sim \left(\log\langle k\rangle\right)^{-(\beta-\alpha)}, \quad k \in \mathbb{N}. \tag{3.4.17}$$

Combining Propositions 3.3.8 and 3.4.7 the compactness of $id : \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathcal{C}^{(1,-\beta)}(U)$ for $\beta > \alpha > 0$ is obvious. We proceed with the corresponding result on entropy numbers.

Corollary 3.4.14 [EH00, Cor. 3.18]

(i) *Let $\beta > \alpha > 0$. Then*

$$e_k\left(id : \mathrm{Lip}^{(1,-\alpha)}(U) \to \mathcal{C}^{(1,-\beta)}(U)\right) \sim \left(\log\langle k\rangle\right)^{-(\beta-\alpha)}, \quad k \in \mathbb{N}. \tag{3.4.18}$$

(ii) *Assume* $\gamma - 1 > \beta > 0$. *Then there are positive numbers* c_1, c_2 *such that for all* $k \in \mathbb{N}$,

$$c_1 \left(\log\langle k\rangle\right)^{-(\gamma-\beta)} \leq e_k\left(id : \mathcal{C}^{(1,-\beta)}(U) \to \mathrm{Lip}^{(1,-\gamma)}(U)\right) \leq c_2 \left(\log\langle k\rangle\right)^{-(\gamma-\beta)+1}.$$

We finally give LEOPOLD's related result [Leo98, Thm. 2] for the case mentioned in Proposition 3.4.9. We could slightly improve it in [EH99].

Proposition 3.4.15 [Leo00b, Thm. 1], [EH99, Prop. 4.7, Cor. 4.9]

Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $1 \leq q_1, q_2 \leq \infty$, *and* $b_1, b_2 \in \mathbb{R}$ *with* $b_1 - b_2 > \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+$. *There are numbers* $c_1, c_2 > 0$ *such that for all* $k \in \mathbb{N}$,

$$c_1 \left(\log\langle k\rangle\right)^{-(b_1-b_2)} \leq e_k(B^{(s,b_1)}_{p,q_1}(U) \hookrightarrow B^{(s,b_2)}_{p,q_2}(U)) \leq c_2 \left(\log\langle k\rangle\right)^{-(b_1-b_2)+(\frac{1}{q_2}-\frac{1}{q_1})_+} .$$

In particular, if $q_1 \leq q_2$, *then*

$$e_k(B^{(s,b_1)}_{p,q_1}(U) \hookrightarrow B^{(s,b_2)}_{p,q_2}(U)) \quad \sim \quad \left(\log\langle k\rangle\right)^{-(b_1-b_2)} . \tag{3.4.19}$$

Another related result concerning entropy numbers of $id : B^{(s_1,b_1)}_{p_1,q_1}(U) \to B^{(s_2,b_2)}_{p_2,q_2}(U)$, $s_1 > s_2$, can be found in [Leo00b, Thm. 2]; see also [Mou01, Thm. 3.13, p. 78].

Remark 3.4.16 We want to mention some (in our opinion) peculiar and very interesting consequences which might shed some light on the place of Lipschitz spaces in between the Fourier-analytically based B-spaces, see our discussion in Section 3.3.3. We contribute to these considerations with the following observation : let $0 < q \leq 1$, $\alpha > 0$, then by (3.4.13) and (3.4.19),

$$\left.\begin{array}{ccc} & & B^{(1,-\alpha)}_{\infty,1}(U) \\ & \nearrow & \curvearrowright \\ id \;:\; B^1_{\infty,q}(U) & \longrightarrow & \mathrm{Lip}^{(1,-\alpha)}(U) \\ & \searrow & \curvearrowright \\ & & B^{(1,-\alpha)}_{\infty,\infty}(U) \end{array}\right\} \quad e_k \;\sim\; \left(\log\langle k\rangle\right)^{-\alpha}$$

So at least in that particular situation it turns out that the entropy numbers for the corresponding embeddings behave 'equally well' (meaning that the compactness of the underlying embedding is seen by the entropy numbers as of the same quality) independent of whether the respective target spaces are rather 'Fourier-unfriendly' (as it is with $\mathrm{Lip}^{(1,-\alpha)}$) or not.

3.4.3 Comparison with the non-limiting setting

We briefly want to compare our *limiting* results, i.e. Theorems 3.4.10 and 3.4.11, with their non-limiting counterparts; we refer to Remark 2.2.2 and Figure 3 for a parallel discussion referring to our first described example (in Section 2).

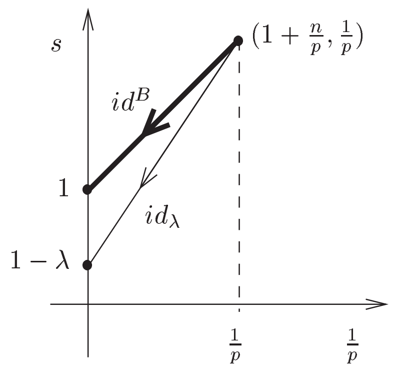


Figure 7

One possibility to 'approximate' our limiting embedding id^B by non-limiting embeddings of a similar type is shown in the $(\frac{1}{p}, s)$-diagram aside. Any space $A^s_{p,q}$ is characterised there by its pair of parameters $(\frac{1}{p}, s)$ (independent of q, $0 < q \le \infty$), as usual. In that (rough) sense our target space $\mathrm{Lip}^{(1,-\alpha)}(U)$ can be found at the point $(0,1)$, too (neglecting the additional smoothness provided by the $\log$-exponent $\alpha \ge 0$). In our situation described above we stick at the parameter $p_2 = \infty$ for the target space, but have less smoothness, say, $s_2 = 1-\lambda < 1$, $\lambda > 0$. Thus we are interested in assertions about $e_k(id_\lambda)$ when $\lambda \downarrow 0$ and id_λ is given by

$$id_\lambda : B^{1+n/p}_{p,q}(U) \longrightarrow B^{1-\lambda}_{\infty,\infty}(U) , \tag{3.4.20}$$

where $0 < p \le \infty$, $0 < q \le \infty$, and $\lambda > 0$.

Note that one has for any $k \in \mathbb{N}$ and $\lambda > 0$,

$$e_k(id_\lambda) \ \sim \ k^{-\frac{1}{p}-\frac{\lambda}{n}} , \tag{3.4.21}$$

cf. [ET96, Thm. 2, p. 118] and (1.3.3). In view of (3.4.21) (for $\lambda \downarrow 0$) it is thus rather natural that the extra term $k^{-\frac{1}{p}}$ survives the limiting procedure, see Theorem 3.4.11, whereas the loss of $k^{-\frac{\lambda}{n}}$ has to be compensated by some additional ($\log -$) term (depending on the particular kind of extension of the target space in (3.4.20) when $\lambda = 0$), as clearly id_λ is no longer compact for $\lambda = 0$.

We stick at the non-limiting situation, i.e. $s_1 - s_2 > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, and give some related results when the 'new' spaces appear as source or target spaces, respectively. This is of great help when having applications in mind (excluded in this report from the very beginning), but it also illustrates the influence of the parameter α in $\mathrm{Lip}^{(1,-\alpha)}$ a bit further.

Corollary 3.4.17 [EH00, Cor. 3.19]

(i) *Let* $\alpha \ge 0$, $s > 0$, $0 < q \le \infty$. *Then for all* $k \in \mathbb{N}$,

$$e_k\left(id : \mathrm{Lip}^{(1,-\alpha)}(U) \to B^{1-s}_{\infty,q}(U)\right) \sim \ k^{-\frac{s}{n}} \left(\log\langle k\rangle\right)^{\alpha} . \tag{3.4.22}$$

(ii) *Let* $\alpha \ge 0$, $0 < p,q \le \infty$, *and* $s > 1 + \frac{n}{p}$. *Then for all* $k \in \mathbb{N}$,

$$e_k\left(id : B^s_{p,q}(U) \to \mathrm{Lip}^{(1,-\alpha)}(U)\right) \sim \ k^{-\frac{s-1}{n}} \left(\log\langle k\rangle\right)^{-\alpha} , \tag{3.4.23}$$

where $B^s_{p,q}$ *in* (3.4.23) *may be replaced by* $F^s_{p,q}$ (*when* $p < \infty$).

Remark 3.4.18 Leopold obtained in [Leo00b, Thm. 3] estimates for the entropy numbers in the non-limiting situation $id : B^{(s_1,b_1)}_{p_1,q_1}(U) \to B^{s_2}_{p_2,q_2}(U)$, where $s_1 - s_2 > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, $0 < p_1 \le p_2 \le \infty$, $0 < q_1$, $q_2 \le \infty$, $b_1 \in \mathbb{R}$.

Part II
Envelopes

4 Envelope functions $\mathcal{E}_{\mathsf{G}}^X$ and $\mathcal{E}_{\mathsf{C}}^X$: definition, and basic properties

4.1 Introduction

We present our recently developed concept of envelopes in function spaces – a relatively simple tool for the study of rather complicated spaces, say, of Besov or Triebel-Lizorkin type, $B^s_{p,q}$ or $F^s_{p,q}$, respectively, in so-called '*limiting*' situations. It is well-known, for instance, that $B^{n/p}_{p,q} \hookrightarrow L_\infty$ if, and only if, $0 < p < \infty$, $0 < q \leq 1$ – but what can be said about the growth of functions $f \in B^{n/p}_{p,q}$ otherwise, i.e. when $B^{n/p}_{p,q}$ contains essentially unbounded functions ? Edmunds and Triebel proved that one can characterise such spaces by sharp inequalities involving the non-increasing rearrangement f^* of a function f : Let $\varkappa$ be a bounded, continuous, decreasing function on $(0,1]$ and $1 < p < \infty$. Then there is a constant $c > 0$ such that

$$\left(\int_0^1 \left(\frac{f^*(t)\varkappa(t)}{1+|\log t|} \right)^p \frac{\mathrm{d}t}{t} \right)^{1/p} \leq c \left\| f|H^{n/p}_p \right\| \tag{4.1.1}$$

for all $f \in H^{n/p}_p$ if, and only if, $\varkappa$ is bounded, cf. [ET99, Thm. 2.5]. Parallel studies in the '*sub-critical*' case, i.e. for embeddings $B^s_{p,q} \hookrightarrow L_r$, $s > 0$, $1 < r < \infty$, $0 < q \leq \infty$, and $0 < p < \infty$ such that $\frac{n}{p} = s + \frac{n}{r}$, led Triebel in [Tri99] to similar results.

As already explained in Section 1.4 we are looking now for some feature only 'belonging' to the spaces under consideration, but not bound to a certain context of embedding (with original or target spaces within a prescribed scale), defined as elementary as possible (using classical approaches) – and gaining from the many forerunners essentially. In view of the above-mentioned papers and our results in Section 3 the choice of f^* (the non-increasing rearrangement) and $\omega(f,t)$ (the classical modulus of continuity) was apparently suggested as basic concept our new tool should be built on. This led us to the introduction of the *growth envelope function* of a function space $X \subset L_1^{\mathrm{loc}}$,

$$\mathcal{E}_{\mathsf{G}}^X(t) := \sup_{\|f|X\| \leq 1} f^*(t) , \qquad 0 < t < 1 .$$

It turns out that in rearrangement-invariant spaces there is a connection between $\mathcal{E}_{\mathsf{G}}^X$ and the fundamental function φ_X; we derive further properties and give some examples. The pair $\mathfrak{E}_{\mathsf{G}}(X) = (\mathcal{E}_{\mathsf{G}}^X(t), u_X)$ is called *growth envelope* of X, where u_X, $0 < u_X \leq \infty$, is the infimum of all numbers v satisfying

$$\left(\int_0^\varepsilon \left[\frac{f^*(t)}{\mathcal{E}_{\mathsf{G}}^X(t)} \right]^v \mu_{\mathsf{G}}(\mathrm{d}t) \right)^{1/v} \leq c \, \|f|X\|$$

for some $c > 0$ and all $f \in X$, and μ_{G} is the Borel measure associated with $-\log \mathcal{E}_{\mathsf{G}}^X$. One verifies for the Lorentz spaces $\mathfrak{E}_{\mathsf{G}}(L_{pq}) = (t^{-1/p}, q)$, but we also obtain characterisations for spaces of type $A^s_{p,q}$, where $\sigma_p \leq s \leq \frac{n}{p}$; this is contained in Section 5. Instead of investigating the growth of functions one can also focus on their smoothness, i.e. when $X \hookrightarrow C$ it makes sense to replace $f^*(t)$ by $\frac{\omega(f,t)}{t}$, where $\omega(f,t)$ is the modulus of continuity. Now the *continuity envelope function* $\mathcal{E}_{\mathsf{C}}^X$ and the continuity envelope $\mathfrak{E}_{\mathsf{C}}$ are introduced completely parallel to $\mathcal{E}_{\mathsf{G}}^X$ and $\mathfrak{E}_{\mathsf{G}}$, respectively, and similar questions are studied in Section 6.We finally present in Section 7 some more, rather astonishing consequences of our recent studies on envelopes in view of lifting arguments and compactness.

4.2 The growth envelope function $\mathcal{E}_{\mathsf{G}}^X$, and the index u_{G}^X

We already mentioned that characterisations like (4.1.1) gave reason to study the behaviour of the non-increasing rearrangement f^* of a function $f \in A^s_{p,q}$, in particular, when these spaces contain essentially unbounded functions. Regarding this problem in a more general context this leads to the introduction of growth envelopes, and in particular, to growth envelope functions. Our results for spaces of type $A^s_{p,q}$ are postponed to Sections 5.2, 5.3; we start with some simple features to give a better feeling what is really 'measured' by growth envelopes. For that reason we test our new envelope tool on rather classical spaces like Lorentz(-Zygmund) spaces; these examples are to be found in Section 5.1. Of course, there was no big gain to develop a theory for, say, $L_{p,q}$ spaces only – had we not achieved more surprising results in Sections 5.2, 5.3. Finally, there is also some astonishing outcome in Section 4.2.2 : the recognition of growth envelope functions in terms of fundamental functions in rearrangement-invariant spaces.

We shall only consider (quasi-) normed function spaces $X \subset L_1^{\mathrm{loc}}$ in the sequel.

4.2.1 Definition and basic properties

Definition 4.2.1 [Har01, Def. 2.2] *Let X be some (quasi-) normed function space on $\mathbb{R}^n$. The* growth envelope function $\mathcal{E}_{\mathsf{G}}^X : (0,\infty) \to [0,\infty]$ *is defined by*

$$\mathcal{E}_{\mathsf{G}}^X(t) := \sup_{\|f|X\|\leq 1} f^*(t) , \quad t > 0. \tag{4.2.1}$$

We shall adopt the usual convention to put $\mathcal{E}_{\mathsf{G}}^X(\tau) := \infty$ when $\{f^*(\tau) \ : \ \|f|X\| \leq 1\}$ is not bounded from above for some $\tau > 0$.

Remark 4.2.2 Note that (4.2.1) immediately causes some problem when taking into account that we shall always deal with equivalent (quasi-) norms in the underlying function space (rather than a fixed one) : Assume we have two different, but equivalent (quasi-) norms $\|\cdot|X\|_1$ and $\|\cdot|X\|_2$ in X. Then every function $f \in X$ with $\|f|X\|_1 \leq 1$, $f \not\equiv 0$, is connected with some $g_f := cf$, where $c = \|f|X\|_1/\|f|X\|_2$, $\|g_f|X\|_2 \leq 1$, and $g_f^* = cf^*$, leading to a different, but equivalent expression for $\mathcal{E}_{\mathsf{G}}^X$. So, strictly speaking, we are concerned with equivalence classes of growth envelope functions, where we choose one representative

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup_{\|f|X\|\leq 1} f^*(t) , \quad t > 0.$$

However, we shall not make this difference between equivalence class and representative in the sequel – but return to this point in Subsection 4.2.3 below.

Furthermore, by (4.2.1) the growth envelope $\mathcal{E}_{\mathsf{G}}^X(t)$ is defined for all values $t > 0$, but at the moment we are only interested in local characterisations (singularities) of the spaces referring to small values of $t > 0$, say, $0 < t < 1$. Nevertheless questions of global behaviour ($t \to \infty$) as well as the comparison with their local counterparts are certainly of interest and will be tackled in the future. This preference of local studies also implies that we can transfer a lot of our results from spaces on $\mathbb{R}^n$ to their counterparts on bounded domains formally. The necessary modifications in case of our examples in Section 5.1 below are obvious; concerning spaces of type $A^s_{p,q}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain, they are defined by restriction from their $\mathbb{R}^n$-counterparts, so that the local behaviour of functions is not 'spoilt'. Conversely we may conclude that in most cases (apart from a few explicitly mentioned) the study of spaces on domains does not contribute very much to our results. This justifies that we shall mainly deal with function spaces on $\mathbb{R}^n$ in the sequel.

We briefly discuss the obvious question whether the growth envelope function $\mathcal{E}_{\mathsf{G}}^X$ is always finite for $t > 0$ or what necessary / sufficient conditions on X (or the underlying measure space) imply this; recall notation (1.1.9).

Lemma 4.2.3 [Har01, Lemmata 2.8, 2.9]

(i) *There are function spaces X on $\mathbb{R}^n$ which do not have a growth envelope function in the sense that $\mathcal{E}_{\mathsf{G}}^X(t)$ is not finite for $t > 0$.*

(ii) *Let X be some (quasi-) normed function space on $\mathbb{R}^n$. Then $\mathcal{E}_{\mathsf{G}}^{X}(t)$ is finite for any $t > 0$ if, and only if,*

$$\sup_{\|f|X\| \le 1} \mu_f(\lambda) \longrightarrow 0 \qquad \textit{for} \quad \lambda \to \infty \,. \tag{4.2.2}$$

Hence the definition of $\mathcal{E}_{\mathsf{G}}^{X}$ is non-trivial and reasonable. We now collect a few elementary properties of it. Simplifying technical matters in the sequel we introduce the number τ_0 by

$$\tau_0 = \tau_0^{\mathsf{G}}(X) := \sup \left\{ t > 0 \ : \ \mathcal{E}_{\mathsf{G}}^{X}(t) > 0 \right\}. \tag{4.2.3}$$

Note that $\mathcal{E}_{\mathsf{G}}^{X}(t) = 0$ for some $t > 0$ implies $f^*(t) = 0$ for all $f \in X$, $\|f|X\| \le 1$; thus – by some scaling argument – $g^*(t) = 0$ for all $g \in X$. But then (1.1.8) yields that X contains only functions having a support with finite measure, i.e. $|\{x \in \mathbb{R}^n : |g(x)| > 0\}| \le t$ for all $g \in X$. This is in particular true, when X is defined on $\Omega \subset \mathbb{R}^n$ with $|\Omega| \le t$. On the other hand, as already mentioned above, we are only interested in the local behaviour of functions $g \in X$, so we shall not focus on larger values of $t > 0$, that is, say, when $t > \tau_0$.

Proposition 4.2.4 [Har01, Prop. 2.4] *Let X be a (quasi-) normed function space on $\Omega \subset \mathbb{R}^n$.*

(i) *$\mathcal{E}_{\mathsf{G}}^{X}$ is monotonically decreasing and right-continuous. We have $\mathcal{E}_{\mathsf{G}}^{X} = \left(\mathcal{E}_{\mathsf{G}}^{X}\right)^*$.*

(ii) *If $|\Omega| < \infty$, then $\mathcal{E}_{\mathsf{G}}^{X}(t) = 0$ for $t > |\Omega|$ and any function space X on Ω.*

(iii) *We have $X \hookrightarrow L_\infty$ if, and only if, $\mathcal{E}_{\mathsf{G}}^{X}(\cdot)$ is bounded, i.e. $\sup_{t>0} \mathcal{E}_{\mathsf{G}}^{X}(t) = \lim_{t \downarrow 0} \mathcal{E}_{\mathsf{G}}^{X}(t)$ is finite. In that case it holds*

$$\mathcal{E}_{\mathsf{G}}^{X}(0) := \lim_{t \downarrow 0} \mathcal{E}_{\mathsf{G}}^{X}(t) = \|id : X \to L_\infty\| \,.$$

(iv) *Let X_1, X_2 be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t > 0$,*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \ \le \ c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t).$$

One may choose $c = \|id : X_1 \to X_2\|$ in that case.

(v) *Let $\varkappa : (0,\infty) \to [0,\infty)$ be some non-negative function, assume that* (4.2.2) *is satisfied. Then $\varkappa(\cdot)$ is bounded on $(0, \tau_0)$ if, and only if, there is some $c > 0$ such that for all $f \in X$, $\|f|X\| \le 1$,*

$$\sup_{0<t<\tau_0} \frac{\varkappa(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)}\, f^*(t) \ \le \ c\,. \tag{4.2.4}$$

(vi) *Assume that X additionally satisfies*

$$\left\| f\left(2^{-\frac{1}{n}}\cdot\right) |X \right\| \ \le \ c \ \|f|X\| \tag{4.2.5}$$

for some $c > 0$ and all $f \in X$. Then

$$\mathcal{E}_{\mathsf{G}}^{X}\left(2^{-j}\right) \ \sim \ \mathcal{E}_{\mathsf{G}}^{X}\left(2^{-j+1}\right) \tag{4.2.6}$$

for some $j_0 \in \mathbb{N}$ and all $j \ge j_0$.

Parts (i)-(v) are covered by [Har01, Prop. 2.4] whereas (vi) is a generalisation of [Tri01, (12.38), p. 190]; the monotonicity (i) of $\mathcal{E}_{\mathsf{G}}^{X}$ immediately yields '$\ge$' in (4.2.6), whereas the converse inequality uses functions $f_n(x) := f(2^{-\frac{1}{n}}\, x)$ built upon $f \in X$, say, with $\|f|X\| \le 1$. Plainly $f_n^*(2t) = f^*(t)$, the rest is covered by (4.2.5). Note that all spaces of type $A^s_{p,q}$, $L_{p,q}(\log L)_a$ studied below satisfy (4.2.5).

Remark 4.2.5 We have shown in [Har01, Rem. 2.14] that some counterpart of (iv) in the sense of (iii), i.e. that some relation of the envelope functions implied some (continuous) embedding for the corresponding spaces, cannot hold in general; see also Section 5.1. Concerning (v), we proved in [Har01, Cor. 2.6] even more, namely that in some sense $\mathcal{E}_{\mathsf{G}}^{X}$ is the only such function with the property described above.

In contrast to [Har01] we postpone examples to Section 5.1.

4.2.2 Connection with the fundamental function

In rearrangement-invariant function spaces X one has the concept of the '*fundamental function*' φ_X; we investigate its connection with the growth envelope function $\mathcal{E}_{\mathsf{G}}^{X}$. All function spaces are considered on $\mathbb{R}^n$ (equipped with the Lebesgue measure ℓ_n). We closely follow the presentation in [BS88, Ch. 2, §5].

Recall the notion of a (quasi-) Banach function space as presented in Section 1.1.1. A function (quasi-) norm $\|\cdot|X\|$ over $\mathbb{R}^n$ is said to be *rearrangement-invariant*, if $\|f|X\| = \|g|X\|$ for every pair of equimeasurable functions f and g , i.e. if for all non-negative measurable functions f, g, finite a.e., with $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$ this implies $\|f|X\| = \|g|X\|$. A (quasi-) Banach function space X generated by a rearrangement-invariant (quasi-) norm is called *rearrangement-invariant* (*quasi*–) *Banach function space* or simply rearrangement-invariant space. Recall that we have for such spaces always $\chi_{_A} \in X$ when $A \subset \mathbb{R}^n$, $\ell_n(A) < \infty$.

Definition 4.2.6 *Let X be a rearrangement-invariant Banach function space over $\mathbb{R}^n$. For each $t > 0$, let $A_t \subset \mathbb{R}^n$ be such that $\ell_n(A_t) = t$, and let*

$$\varphi_X(t) = \left\| \chi_{_{A_t}} |X \right\| . \tag{4.2.7}$$

The function φ_X so defined is called fundamental function *of X.*

Note that the particular choice of the set A_t with $\ell_n(A_t) = t$ is immaterial since if B_t is another subset $B_t \subset \mathbb{R}^n$ with $\ell_n(B_t) = t$, then $\chi_{_{A_t}}$ and $\chi_{_{B_t}}$ are equi-measurable, and so $\|\chi_{_{A_t}}|X\| = \|\chi_{_{B_t}}|X\|$ because of the rearrangement-invariance of X. Hence φ_X is well-defined. We give some well-known examples.

Let $1 \leq p \leq \infty$, and $L_p = L_p(\mathbb{R}^n)$; then for $t \geq 0$,

$$\varphi_{L_p}(t) = t^{\frac{1}{p}} , \quad 1 \leq p < \infty, \qquad \text{and} \qquad \varphi_{L_\infty}(t) = \begin{cases} 0 & , \quad t = 0 \\ 1 & , \quad t > 0 \end{cases} \tag{4.2.8}$$

cf. [BS88, p. 65]. Moreover, when $1 \leq q \leq p < \infty$ or $p = q = \infty$, then $L_{p,q}$ is rearrangement-invariant and

$$\varphi_{L_{p,q}}(t) = t^{\frac{1}{p}} , \tag{4.2.9}$$

see [BS88, Ch. 4, Thm. 4.3, p. 218]. (In view of Remark 1.1.2 one can further prove that $L_{p,q}$ is a rearrangement-invariant Banach space for $1 < p < \infty$, $1 \leq q \leq \infty$, or $p = q = \infty$, when f^* in (1.1.10) is replaced by f^{**}; cf. [BS88, Ch. 4, Thm. 4.6, p. 219].) Likewise, let $\Omega \subset \mathbb{R}^n$ have finite measure, say, $\ell_n(\Omega) = 1$. Then it is known that $L_1 (\log L)_1 (\Omega)$ and $L_{\exp,1}(\Omega)$ are rearrangement-invariant with fundamental functions

$$\varphi_{L_1(\log L)_1}(t) = t\,(1 + |\log t|) , \qquad \text{and} \qquad \varphi_{L_{\exp,1}}(t) = (1 + |\log t|)^{-1} , \tag{4.2.10}$$

for $0 < t < 1$, see [BS88, Ch. 4, Thm. 6.4, p. 246]. So in view of our examples in Section 5.1, i.e. Propositions 5.1.2, 5.1.4, where we calculated $\mathcal{E}_{\mathsf{G}}^{X}$ for the same spaces as involved in (4.2.8)-(4.2.10), the following assertion is naturally suggested.

Proposition 4.2.7 [Har01, Prop. 2.22] *Let X be a rearrangement-invariant Banach function space over $\mathbb{R}^n$, and φ_X the corresponding fundamental function. Then*

$$\mathcal{E}_{\mathsf{G}}^{X}(t) = \frac{1}{\varphi_X(t)} , \quad t > 0. \tag{4.2.11}$$

Remark 4.2.8 One can prove a counterpart of Proposition 4.2.7 when the underlying measure space $[\mathcal{R}, \mu] = [\mathbb{R}^n, \ell_n]$ is replaced by some non-atomic finite measure space $[\mathcal{R}, \mu]$.

After completing [Har01] we found that Carro, Pick, Soria and Stepanov studied related questions in [CPSS01]; in particular, [CPSS01, Rem. 2.5 (ii)] essentially coincides with (4.2.11), where the function $\varrho_X(t)$ used there corresponds to $\mathcal{E}_{\mathsf{G}}^X(t)$. Moreover, when X is a rearrangement-invariant Banach function space then by [CPSS01, Thm. 2.8 (iii)] there is a counterpart of Proposition 4.2.4 (iii) as follows :

$$X \hookrightarrow L_{q,\infty} \iff \sup_{t>0} t^{\frac{1}{q}} \, \mathcal{E}_{\mathsf{G}}^X(t) < \infty , \quad 0<q<\infty .$$

4.2.3 The index u_{G}^X

We shall need a finer characterisation than provided by the growth envelope functions solely. By Proposition 5.1.2 below it is obvious, for instance, that $\mathcal{E}_{\mathsf{G}}^X$ cannot distinguish between different spaces like $L_{p,q_1}(\log L)_a$ and $L_{p,q_2}(\log L)_a$, $q_1 \neq q_2$. So it appears desirable to complement $\mathcal{E}_{\mathsf{G}}^X$ by some expression, naturally belonging to $\mathcal{E}_{\mathsf{G}}^X$, but yielding – as a test – the number q (or a related quantity) in case of $L_{p,q}(\log L)_a$ spaces. Again a more substantial justification for complementing $\mathcal{E}_{\mathsf{G}}^X$ by this additional expression results from more complicated spaces (like $A^s_{p,q}$) than $L_{p,q}(\log L)_a$; but in these classical cases the outcome can be checked immediately.
The missing link is obtained by the introduction of some 'characteristic' index u_{G}^X, which gives a finer measure of the (local) integrability of functions belonging to X. Moreover, the definition below is also motivated by (sharp) inequalities of type (4.1.1) with $\varkappa \equiv 1$.

We start with some preliminaries. Let ψ be a real continuous monotonically increasing function on the interval $[0,\varepsilon]$ for some small $\varepsilon>0$. Assume $\psi(0)=0$ and $\psi(t)>0$ if $0<t\leq\varepsilon$. Let $\mu_{\log\psi}$ be the associated Borel measure with respect to the distribution function $\log\psi$; if, in addition, ψ is differentiable in $(0,\varepsilon)$ then

$$\mu_{\log\psi}(\,\mathrm{d}t) = \frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t \tag{4.2.12}$$

in $(0,\varepsilon)$; cf. [Lan93, p. 285] or [Hal74, §15(9), p. 67]. The following result of Triebel is essential for our argument below.

Proposition 4.2.9 [Tri01, Prop. 12.2, p. 183]

(i) *Let ψ and $\mu_{\log\psi}$ be as above, and $0<r_0\leq r_1<\infty$. Then there are numbers $c_2>c_1>0$ such that*

$$\sup_{0<t<\varepsilon} \psi(t)g(t) \leq c_1 \left(\int_0^\varepsilon [\psi(t)g(t)]^{r_1}\,\mu_{\log\psi}(\,\mathrm{d}t)\right)^{1/r_1} \leq c_2 \left(\int_0^\varepsilon [\psi(t)g(t)]^{r_0}\,\mu_{\log\psi}(\,\mathrm{d}t)\right)^{1/r_0} \tag{4.2.13}$$

for all functions $g(t)\geq 0$, which are monotonically decreasing.

(ii) *Let ψ_1, ψ_2 be two equivalent functions as above and $\mu_{\log\psi_1}$, $\mu_{\log\psi_2}$ the corresponding measures. Assume $0<r\leq\infty$. Then*

$$\left(\int_0^\varepsilon [\psi_1(t)\,g(t)]^r\,\mu_{\log\psi_1}(\,\mathrm{d}t)\right)^{1/r} \sim \left(\int_0^\varepsilon [\psi_2(t)\,g(t)]^r\,\mu_{\log\psi_2}(\,\mathrm{d}t)\right)^{1/r} \tag{4.2.14}$$

(usual modification if $r=\infty$) for all functions $g(t)\geq 0$, which are monotonically decreasing.

In a slight abuse of notation we shall mean by μ_{G} the Borel measure associated with a function ψ (as described above and) equivalent to $1/\mathcal{E}_{\mathsf{G}}^X$, where X is some function space satisfying (4.2.5) and $X \not\hookrightarrow L_\infty$; that is, $\psi(t)\sim 1/\mathcal{E}_{\mathsf{G}}^X(t)$, $0<t<\varepsilon$. Note that all growth envelope functions $\mathcal{E}_{\mathsf{G}}^X$ of a space X with (4.2.5) belong to the same equivalence class which contains moreover a continuous representative. If $\mathcal{E}_{\mathsf{G}}^X$ is differentiable, then

$$\mu_{\mathsf{G}}(\,\mathrm{d}t) \sim -\frac{\left(\mathcal{E}_{\mathsf{G}}^X\right)'(t)}{\mathcal{E}_{\mathsf{G}}^X(t)}\ \mathrm{d}t \tag{4.2.15}$$

for small $t > 0$. This approach coincides with the one presented by TRIEBEL in [Tri01, Sect. 12.1, pp. 181-183] and [Tri01, Sect. 12.8, p. 192]. Recall our notation τ_0 in (4.2.3).

Definition 4.2.10 [Har01, Def. 3.1] *Let $X \not\hookrightarrow L_\infty$ be some (quasi-) normed function space on $\mathbb{R}^n$ with* (4.2.5) *and growth envelope function $\mathcal{E}_{\mathsf{G}}^X$. Assume $0 < \varepsilon < \tau_0$. The index u_{G}^X, $0 < u_{\mathsf{G}}^X \le \infty$, is defined as the infimum of all numbers v, $0 < v \le \infty$, such that*

$$\left(\int\limits_0^\varepsilon \left[\frac{f^*(t)}{\mathcal{E}_{\mathsf{G}}^X(t)}\right]^v \mu_{\mathsf{G}}(\,\mathrm{d}t)\right)^{1/v} \le c\,\|f|X\| \tag{4.2.16}$$

(*with the usual modification if* $v = \infty$) *holds for some* $c > 0$ *and all* $f \in X$.

Remark 4.2.11 It is clear by Proposition 4.2.4 (v) (with $\varkappa \equiv 1$) that (4.2.16) holds with $v = \infty$ in any case. Thus the question arises whether (depending upon the underlying function space X) there is some smaller v such that (4.2.16) is still satisfied. Moreover, it is reasonable to ask for the *smallest* parameter v satisfying (4.2.16) as the corresponding expressions on the left-hand side are monotonically ordered in v by Proposition 4.2.9 (i) with $g = f^*$ and $\psi \sim 1/\mathcal{E}_{\mathsf{G}}^X$.

The number u_{G}^X in Definition 4.2.10 is defined as the *infimum* of all numbers v satisfying (4.2.16); however, it is not clear at the moment, whether this infimum (4.2.16) is in fact always a minimum. More precisely, one can study the question what assumptions (on the function space X and the underlying measure space) imply that u_{G}^X satisfies (4.2.16), too. So far we only know that all cases we studied (as presented below) are examples for the latter case (when u_{G}^X happens to be a minimum), but lack a general answer.

Remark 4.2.12 We explicitly excluded the case $X \hookrightarrow L_\infty$ (in particular, $X = L_\infty$) in Definition 4.2.10 above. One may, however, adopt the (reasonable) opinion that – in case of bounded growth functions $\mathcal{E}_{\mathsf{G}}^X$ (that is, according to Proposition 4.2.4 (iii), when $X \hookrightarrow L_\infty$) – (4.2.16) is replaced by

$$\sup_{0<t<\varepsilon} f^*(t) \le c\,\|f|X\|\,,$$

for some $c > 0$ and all $f \in X$; thus $u_{\mathsf{G}}^X := \infty$.

The following assertion is not very complicated to prove – relying on Proposition 4.2.9 essentially – but quite effective in application later on.

Proposition 4.2.13 [Har01, Prop. 3.5] *Let X_1, X_2 be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their growth envelope functions*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \sim \mathcal{E}_{\mathsf{G}}^{X_2}(t), \quad 0 < t < \varepsilon\,. \tag{4.2.17}$$

Then we obtain for the corresponding indices

$$u_{\mathsf{G}}^{X_1} \le u_{\mathsf{G}}^{X_2}\,. \tag{4.2.18}$$

Remark 4.2.14 We give another interpretation of the meaning of (4.2.16) in terms of sharp embeddings. Assume that $\mathcal{E}_{\mathsf{G}}^X(t) \sim t^{-\alpha}\,|\log t|^{\mu}$ for small $t > 0$ with $\alpha > 0$, $\mu \in \mathbb{R}$, or $\alpha = 0$, $\mu > 0$ (recall the monotonicity of $\mathcal{E}_{\mathsf{G}}^X$ near 0). Then

$$\mu_{\mathsf{G}}(\,\mathrm{d}t) \sim \frac{\mathrm{d}t}{t} \quad \text{if} \quad \alpha > 0\ , \qquad \text{and} \qquad \mu_{\mathsf{G}}(\,\mathrm{d}t) \sim \frac{\mathrm{d}t}{t\ |\log t|} \quad \text{if} \quad \alpha = 0\ ,$$

and (4.2.16) can be reformulated as follows : What is the smallest space of type

$$L_{\frac{1}{\alpha},v}\,(\log L)_{-\mu} \quad \text{if} \quad \alpha > 0\ , \qquad \text{or} \qquad L_{\infty,v}\,(\log L)_{-\left(\mu+\frac{1}{v}\right)} \quad \text{if} \quad \alpha = 0\ ,$$

respectively, such that X can be embedded into it continuously ? Having this idea in mind the results in Section 5.1 are not very astonishing. However, this is only some *interpretation* of (4.2.16); the definition itself is independent of any scale of Lorentz spaces as target spaces.

4.3 The continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$, and the index u_{C}^X

We introduce the continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$ and derive some elementary properties. The method is parallel to that in the preceding section.

4.3.1 Definition and basic properties

Recall that $C(\mathbb{R}^n)$ is the space of all complex-valued bounded uniformly continuous functions equipped with the $\sup$-norm as usual.

Definition 4.3.1 *Let* $X \hookrightarrow C$ *be some function space on* $\mathbb{R}^n$. *The* continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$: $(0,\infty) \to [0,\infty)$ *is defined by*

$$\mathcal{E}_{\mathsf{C}}^X(t) := \sup_{\|f|X\|\leq 1} \frac{\omega(f,t)}{t}\ , \quad t > 0. \tag{4.3.1}$$

Remark 4.3.2 An adapted version of Remark 4.2.2 holds here, too, concerning the equivalence classes of continuity envelope functions as well as the question of local (instead of global) behaviour of functions, implying our restriction on function spaces on $\mathbb{R}^n$ rather than function spaces on domains. We do not want to repeat the arguments in detail.

In view of Section 4.2.1, in particular Lemma 4.2.3 (i), one may ask whether any space X of the above type possesses a continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$, that is, whether in any admissible situation $\mathcal{E}_{\mathsf{C}}^X(t)$ is finite for any $t > 0$. In contrast to $\mathcal{E}_{\mathsf{G}}^X$, see Lemma 4.2.3 (i), our assumption $X \hookrightarrow C$ already implies

$$\mathcal{E}_{\mathsf{C}}^X(t) \;=\; \sup_{\|f|X\|\leq 1} \frac{\omega(f,t)}{t} \;\leq\; \sup_{\|f|X\|\leq 1} \frac{2\,\|f|C\|}{t} \;\leq\; 2\ \|id : X \to C\|\ \frac{1}{t}\ , \quad t > 0, \tag{4.3.2}$$

i.e. there is some $c > 0$ such that for all $t > 0$, $\mathcal{E}_{\mathsf{C}}^X(t) \;\leq\; \frac{c}{t}$. In that sense any space $X \hookrightarrow C$ has a continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$.

We collect a few elementary properties of $\mathcal{E}_{\mathsf{C}}^X(t)$. Note that $\mathcal{E}_{\mathsf{C}}^X(t)$ cannot be too small for $t \downarrow 0$, for $\mathcal{E}_{\mathsf{C}}^X(t) \searrow 0$ as $t \downarrow 0$ implies that X contains constants only. Furthermore, one introduces a number τ_0^{C} – parallel to (4.2.3) – by

$$\tau_0^{\mathsf{C}} = \tau_0^{\mathsf{C}}(X) := \sup\Big\{t > 0\ :\ \mathcal{E}_{\mathsf{C}}^X(t) > 0\Big\}\,. \tag{4.3.3}$$

However, as $\mathcal{E}_{\mathsf{C}}^X(t) = 0$ for some $t > 0$ means $\omega(f,t) = 0$ for all $f \in X$ (i.e. X consists of constants merely) we are mainly interested in spaces X with $\tau_0^{\mathsf{C}}(X) = \infty$; investigating the local behaviour (small $t > 0$) at the moment, it was even sufficient to assume, say, $\sup\{0 < t < 1\ :\ \mathcal{E}_{\mathsf{C}}^X(t) > 0\} \;=\; 1.$

Proposition 4.3.3 [Har01, Prop. 4.3] *Let $X \hookrightarrow C$ be some function space on $\mathbb{R}^n$.*

(i) *$\mathcal{E}_{\mathsf{C}}^{X}$ is continuous and 'essentially monotonically decreasing', that is, $\mathcal{E}_{\mathsf{C}}^{X}$ is equivalent to some monotonically decreasing function.*

(ii) *We have $X \hookrightarrow \mathrm{Lip}^1$ if, and only if, $\mathcal{E}_{\mathsf{C}}^{X}(\cdot)$ is bounded, i.e. $\sup\limits_{t>0} \mathcal{E}_{\mathsf{C}}^{X}(t) = \limsup\limits_{t\downarrow 0} \mathcal{E}_{\mathsf{C}}^{X}(t)$ is finite. In that case it holds*
$$\mathcal{E}_{\mathsf{C}}^{X}(0) := \limsup_{t\downarrow 0} \mathcal{E}_{\mathsf{C}}^{X}(t) = \left\| id : X \to \mathrm{Lip}^1 \right\|.$$

(iii) *Let $X_i \hookrightarrow C$, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t > 0$,*
$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \;\le\; c\, \mathcal{E}_{\mathsf{C}}^{X_2}(t).$$
One may choose $c = \|id : X_1 \to X_2\|$ in that case.

(iv) *Let $X \hookrightarrow C$ be non-trivial, i.e. $\tau_0^{\mathsf{C}}(X) = \infty$. Let $\varkappa : (0,\infty) \to [0,\infty)$ be some non-negative function. Then $\varkappa(\cdot)$ is bounded if, and only if, there is some $c > 0$ such that for all $f \in X$, $\|f|X\| \le 1$,*
$$\sup_{t>0} \frac{\varkappa(t)}{\mathcal{E}_{\mathsf{C}}^{X}(t)} \frac{\omega(f,t)}{t} \;\le\; c\,. \tag{4.3.4}$$

(v) *Assume that X additionally satisfies*
$$\left\| f\left(2^{-1}\cdot\right) |X \right\| \;\le\; c\; \|f|X\| \tag{4.3.5}$$
for some $c > 0$ and all $f \in X$. Then
$$\mathcal{E}_{\mathsf{C}}^{X}\left(2^{-j}\right) \;\sim\; \mathcal{E}_{\mathsf{C}}^{X}\left(2^{-j+1}\right) \tag{4.3.6}$$
for some $j_0 \in \mathbb{N}$ and all $j \ge j_0$.

Parts (i)-(iv) are covered by [Har01, Prop. 4.3] whereas (v) generalises [Tri01, (12.78), p. 197]; see the similar argument following Proposition 4.2.4. The somehow clumsy formulation in (i) results from the fact that $\omega(f,t)$ is not necessarily concave itself, but equivalent to its least concave majorant $\overline{\omega}(f,t)$,
$$\frac{1}{2}\,\overline{\omega}(f,t) \;\le\; \omega(f,t) \;\le\; \overline{\omega}(f,t)\,, \quad t > 0, \tag{4.3.7}$$
for any $f \in C$; cf. [DL93, Ch. 2, Lemma 6.1, p. 43].

Remark 4.3.4 In analogy to Remark 4.2.5 we mention that we proved in [Har01, Cor. 4.4] more than (iv), namely that in some sense $\mathcal{E}_{\mathsf{C}}^{X}$ is the only such function with the property described above.

4.3.2 The index u_{C}^{X}

Recall our introductory remarks at the beginning of Section 4.2.3. Analogously to the situation described there we shall introduce the Borel measure μ_{C} associated with the function ψ as described in Section 4.2.3, and equivalent to $1/\mathcal{E}_{\mathsf{C}}^{X}$ for some function space X with (4.3.5) and $X \not\hookrightarrow \mathrm{Lip}^1$, $\psi(t) \sim 1/\mathcal{E}_{\mathsf{C}}^{X}(t)$, $0 < t < \varepsilon$. Then (granted that $\mathcal{E}_{\mathsf{C}}^{X}$ was differentiable) we obtain
$$\mu_{\mathsf{C}}(\,\mathrm{d}t) \sim \; - \frac{\left(\mathcal{E}_{\mathsf{C}}^{X}\right)'(t)}{\mathcal{E}_{\mathsf{C}}^{X}(t)}\;\mathrm{d}t \tag{4.3.8}$$
for small $t > 0$.

Definition 4.3.5 [Har01, Def. 5.1] *Let $X \hookrightarrow C$ be some function space on $\mathbb{R}^n$ with* (4.3.5)*, $X \not\hookrightarrow \mathrm{Lip}^1$ and continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$. Assume $\varepsilon > 0$. The index u_{C}^X, $0 < u_{\mathsf{C}}^X \leq \infty$, is defined as the infimum of all numbers v, $0 < v \leq \infty$, such that*

$$\left(\int_0^{\varepsilon} \left[\frac{\omega(f,t)}{t\, \mathcal{E}_{\mathsf{C}}^X(t)} \right]^v \mu_{\mathsf{C}}(\mathrm{d}t) \right)^{1/v} \leq c\, \|f|X\| \tag{4.3.9}$$

(*with the usual modification if $v = \infty$*) *holds for some $c > 0$ and all $f \in X$.*

Remark 4.3.6 Proposition 4.3.3 (iv) (with $\varkappa \equiv 1$) implies that (4.3.9) holds with $v = \infty$ in any case; but – depending upon the underlying function space X – there might be some smaller v such that (4.3.9) is still satisfied. As Proposition 4.2.9 (i) can be applied to the above case, that is, $\psi \sim 1/\mathcal{E}_{\mathsf{C}}$ and $g(t) \sim \frac{\omega(f,t)}{t}$, without any difficulties, we have the monotonicity of (4.3.9) in v.

The question posed in Section 4.2.3, that is, under which assumptions

$$u_{\mathsf{C}}^X = \inf \{ v : 0 < v \leq \infty,\ v \quad \text{satisfies} \quad (4.3.9) \} \tag{4.3.10}$$

is in fact a *minimum*, makes sense in that context, too, but is likewise open in general. Again, all the examples studied below are such (possibly special) cases where u_{C}^X satisfies (4.3.9).

Remark 4.3.7 In analogy to Remark 4.2.12 we handle the case when $X \hookrightarrow \mathrm{Lip}^1$ separately. Parallel to Remark 4.2.12 we can include this situation by putting $u_{\mathsf{C}}^X := \infty$ as for bounded $\mathcal{E}_{\mathsf{C}}^X$, that is, by Proposition 4.3.3 (ii), when $X \hookrightarrow \mathrm{Lip}^1$, (4.3.9) can be replaced by

$$\sup_{0<t<\varepsilon} \frac{\omega(f,t)}{t} \leq c\, \|f|X\|,$$

for some $c > 0$ and all $f \in X$. We give the counterpart of Proposition 4.2.13 in terms of continuity envelopes.

Proposition 4.3.8 [Har01, Prop. 5.4] *Let $X_i \hookrightarrow C$, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their continuity envelope functions*

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \sim \mathcal{E}_{\mathsf{C}}^{X_2}(t), \quad 0 < t < \varepsilon\ . \tag{4.3.11}$$

Then we get for the corresponding indices

$$u_{\mathsf{C}}^{X_1} \leq u_{\mathsf{C}}^{X_2}\ . \tag{4.3.12}$$

5 Growth envelopes $\mathfrak{E}_{\mathsf{G}}$

We introduce the concept of growth envelopes, followed by our corresponding results; first we shall deal with classical spaces such as Lebesgue and Lorentz spaces whereas afterwards the (sub-) critical case for spaces $A^s_{p,q}$ is considered. All spaces are defined on $\mathbb{R}^n$ unless otherwise stated.

5.1 Definition and first examples

Let X be some (quasi-) normed function space, recall the definitions for $\mathcal{E}_{\mathsf{G}}^X$ and u_{G}^X as given in Definitions 4.2.1 and 4.2.10, respectively.

Definition 5.1.1 [Har01, Def. 3.1] *Let* $X \not\hookrightarrow L_\infty$ *be some function space on* $\mathbb{R}^n$ *with* (4.2.5) *and growth envelope function* $\mathcal{E}_{\mathsf{G}}^X(t)$, $0 < t < \varepsilon$, *and index* u_{G}^X. *Then*

$$\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^X(\cdot),\ u_{\mathsf{G}}^X\right) \tag{5.1.1}$$

is called growth envelope *for the function space* X.

We claim that the growth envelope $\mathfrak{E}_{\mathsf{G}}(X)$ of some function space X gives some characteristic feature of X in the sense that it indicates the 'quality' of the unboundedness of functions contained in X. We start with some easy examples to illustrate the concept of the *growth envelope* introduced above, though the more surprising results are obtained when dealing with spaces of type $A^s_{p,q}$; this is postponed to Sections 5.2, 5.3. Recall the definition for Lorentz (-Zygmund) spaces $L_{p,q}$, $L_{p,q}(\log L)_a$ in Definition 1.1.1.

Proposition 5.1.2 [Har01, Props. 2.13, 2.16, 2.18, 3.7, 3.8, 3.9]

(i) *Let* $0 < p, q \le \infty$ (*with* $q = \infty$ *when* $p = \infty$). *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(L_{p,q}\right) = \left(t^{-\frac{1}{p}},\ q\right). \tag{5.1.2}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, *and* $a \in \mathbb{R}$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(L_{p,q}(\log L)_a\right) = \left(t^{-\frac{1}{p}}\,|\log t|^{-a},\ q\right). \tag{5.1.3}$$

(iii) *Let* $0 < q < \infty$, $a \in \mathbb{R}$, *with* $a + \frac{1}{q} < 0$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(L_{\infty,q}(\log L)_a\right) = \left(|\log t|^{-(a+\frac{1}{q})},\ q\right). \tag{5.1.4}$$

Plainly, we obtain in, say, (i) that $\mathcal{E}_{\mathsf{G}}^{L_{p,q}}(t) \sim t^{-\frac{1}{p}}$ for all admitted q, $0 < q \le \infty$. Hence there cannot exist a direct counterpart of Proposition 4.2.4 (iv), because otherwise all $L_{p,q}-$ spaces were contained in each other. Moreover, it becomes clear that only the index u_{G}^X can distinguish between L_{p,q_1} and L_{p,q_2}, whereas, of course, u_{G}^X solely carries not enough information on the spaces as well; but the pair $\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^X,\ u_{\mathsf{G}}^X\right)$ does. This justifies the introduction of the growth envelope again.

Remark 5.1.3 As already announced in Remark 4.2.14, the above results were to expect in view of the reformulation of (4.2.16). The value of Proposition 5.1.2 rather lies in the verification of our method to recover the fine index q in case of Lorentz (-Zygmund) spaces $L_{p,q}(\log L)_a$; this was our aim as announced before.

Looking back on Section 4.2.2 the question arises naturally whether u_{G}^X can also be identified as some quantity, known for a long time (and in possibly another context) in Banach space theory. By Proposition 5.1.2 we have to look for expressions only which take the value q when, say, $X = L_{p,q}(\log L)_a$; we were not yet successful in this task.

Let $|\Omega| < \infty$, say, $|\Omega| = 1$; recall that $L_{\infty,\infty}(\log L)_{-a}(\Omega) = L_{\exp,a}(\Omega)$ for $a \ge 0$ and $L_{\exp,a}$ being the Zygmund spaces given by (1.1.14).

Proposition 5.1.4 [Har01, Props. 2.19, 3.11] *Let* $\Omega \subset \mathbb{R}^n$ *with* $|\Omega| = 1$, *and* $a \ge 0$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(L_{\exp,a}(\Omega)\right) = \left(|\log t|^a,\ \infty\right). \tag{5.1.5}$$

Note that we determined the growth envelope function $\mathcal{E}_{\mathsf{G}}^X(t)$ in [Har01, Props. 2.13, 2.16, 2.18, 2.19] directly, not relying on results about the fundamental function φ_X and Proposition 4.2.7. In fact, it happened just the other way round in [Har01] : we took our results [Har01, Props. 2.13, 2.16, 2.18, 2.19] together with (4.2.8)-(4.2.10) as motivation for Proposition 4.2.7. The result remains true when $[\mathbb{R}^n, \ell_n]$ is replaced by some $\sigma-$finite measure space $[\mathcal{R}, \mu]$ satisfying that for every number $s \in [0, \mu(\mathcal{R})]$ there is some $A_s \subset \mathcal{R}$ in the $\sigma-$algebra of $\mathcal{R}$ with $\mu(A_s) = s$; likewise one can assume $[\mathcal{R}, \mu]$ to be a finite non-atomic measure space.

5.2 Growth envelopes in the sub-critical case

In this section we deal with spaces of type $A^s_{p,q}$, as introduced in Definition 1.1.6. Let $s \geq 0$, $0 < p < \infty$, and $0 < q \leq \infty$. Then according to our notation in Figure 1 (and the explanations given there) we call spaces *sub-critical* when $-n \leq s - \frac{n}{p} < 0$. As usual, the borderline case $s = \sigma_p$, that is, $s = 0$ when $1 \leq p < \infty$, and $s = n(\frac{1}{p} - 1)$ for $0 < p < 1$, needs some additional care concerning the corresponding spaces. This refers to the thick lines in Figure 8. We shall deal with that situation separately, but postpone it to the end of this subsection.

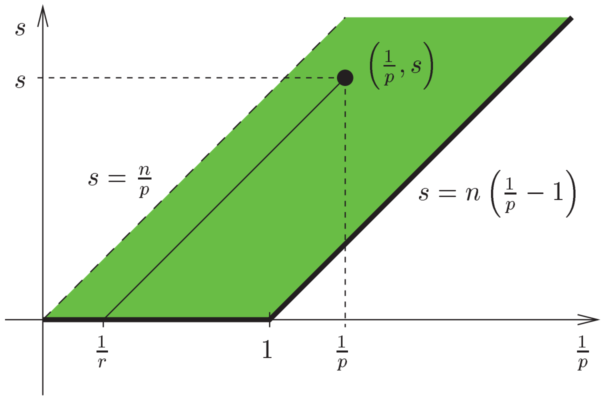


Figure 8

First we consider the '*sub-critical strip*' where $\frac{n}{p} > s > \sigma_p$, $0 < p < \infty$ and $0 < q \leq \infty$. Let $1 < r < \infty$, then all spaces on the line with slope n and 'foot-point' $\frac{1}{r}$ (see Figure 8) belong to this *sub-critical* area. Moreover, as all spaces of type $A^s_{p,q}$ (with such parameters) can be embedded in, say, suitable Lebesgue spaces L_u, it makes sense to study their growth envelopes, see the previous section.

Theorem 5.2.1 [Har01, Thm. 3.12], [Tri01, Thm. 15.2, p. 230] *Let* $0 < q \leq \infty$, $s > 0$, $1 < r < \infty$ *and* p *with* $0 < p < \infty$ *be such that* $s - \frac{n}{p} = -\frac{n}{r}$. *Then*

$$\mathfrak{E}_\mathsf{G}\left(F^s_{p,q}\right) = \left(t^{-\frac{1}{r}},\ p\right) \tag{5.2.1}$$

and

$$\mathfrak{E}_\mathsf{G}\left(B^s_{p,q}\right) = \left(t^{-\frac{1}{r}},\ q\right)\ . \tag{5.2.2}$$

We briefly explain the main ideas of the proof, starting with the determination of the growth envelope functions. By (1.1.34) and (1.2.6) we have

$$F^s_{p,q} \hookrightarrow F^0_{r,2} \ = \ L_r\,, \quad 1 < r < \infty\ . \tag{5.2.3}$$

Now Propositions 4.2.4 (iv) and 5.1.2 immediately imply $\mathcal{E}_\mathsf{G}^{F^s_{p,q}}(t) \ \leq \ c\, t^{-\frac{1}{r}}$. Stressing real interpolation arguments we obtain not only the corresponding estimate for B-spaces, but also a sharper result in the F-case,

$$F^s_{p,q} \hookrightarrow L_{r,p} \tag{5.2.4}$$

and

$$B^s_{p,q} \hookrightarrow L_{r,q}\ ; \tag{5.2.5}$$

we refer to [BL76, Thm. 5.3.1, p. 113], [Tri78a, Thm. 1.18.6/2, p. 134; (2.4.2/6), p. 185], [FJ90, Cor. 6.7 and §12] and [Tri83, Thm. 2.4.2, p. 64] for details on the interpolation results. Application of Propositions 4.2.4 (iv) and 5.1.2 leads to $\mathcal{E}_\mathsf{G}^{B^s_{p,q}}(t) \ \leq \ c\, t^{-\frac{1}{r}}$. Conversely, we use an example given in [Tri99, 3.2]. Let $\psi(x)$ be some compactly supported C^∞-function in $\mathbb{R}^n$ given by

$$\psi(x) \ = \begin{cases} e^{-\frac{1}{1-|x|^2}} & , \quad |x| < 1\ , \\ 0 & , \quad |x| \geq 1\ . \end{cases} \tag{5.2.6}$$

Let $j \in \mathbb{N}$, then the functions

$$f_j(x) \ := \ 2^{j\frac{n}{r}}\ \psi\left(2^j x\right)\ , \quad x \in \mathbb{R}^n, \tag{5.2.7}$$

are atoms in $B^s_{p,q}$ in the sub-critical case; we refer to Section 1.1.3, in particular Theorem 1.1.9 (i). Besides, these atoms satisfy

$$f_j^*\left(2^{-jn}\right) \sim 2^{j\frac{n}{r}}, \quad j\in\mathbb{N},$$

implying

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}\left(2^{-jn}\right) \geq f_j^*\left(2^{-jn}\right) \sim 2^{j\frac{n}{r}}, \quad j\in\mathbb{N}.$$

This yields not only the desired B-result, $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t) \sim c\, t^{-\frac{1}{r}}$, $0<t<1$, but also its counterpart for $F-$spaces, due to the embedding $B^\sigma_{v,p} \hookrightarrow F^s_{p,q}$ for $\sigma>s$ and $1<v<p$ such that $\sigma-\frac{n}{v}=s-\frac{n}{p}=-\frac{n}{r}$; see (1.2.17). Proposition 4.2.4 (iv) completes the proof in as far as envelope functions are concerned. Turning to the indices $u_{\mathsf{G}}^{A^s_{p,q}}$ we benefit from Propositions 4.2.13 and 5.1.2 together with (5.2.5) and (5.2.4) providing $u_{\mathsf{G}}^{F^s_{p,q}} \leq p$ and $u_{\mathsf{G}}^{B^s_{p,q}} \leq q$, respectively. The sharpness is a consequence of [Tri99, Cor. 2.5].

Remark 5.2.2 Note that (5.2.1) together with Proposition 5.1.2 implies

$$\mathfrak{E}_{\mathsf{G}}\Big(L_{r,p}\Big) = \Big(t^{-\frac{1}{r}},\, p\Big) = \mathfrak{E}_{\mathsf{G}}\Big(F^s_{p,q}\Big), \tag{5.2.8}$$

where $0<q\leq\infty$, $s>0$, $1<r<\infty$ and $0<p<\infty$ with $s-\frac{n}{p}=-\frac{n}{r}$; that is, we have by (5.2.4) the embedding $F^s_{p,q}\hookrightarrow L_{r,p}$ only, whereas the corresponding envelopes even coincide. This can be interpreted as $L_{r,p}$ being indeed the best possible space within the Lorentz scale in which $F^s_{p,q}$ can be embedded continuously. On the other hand this is to be understood in the sense that $L_{r,p}$ is '*as good as*' $F^s_{p,q}$ – as far as only the growth of the unbounded functions belonging to the spaces under consideration is concerned; (additional) smoothness features (making a big difference between the spaces $L_{r,p}$ and $F^s_{p,q}$, for instance) are obviously '*ignored*' by the growth envelope. This is not really astonishing in view of its construction, but worth to be noticed. The parallel assertion for the B-case, i.e. (5.2.2) together with Proposition 5.1.2 provide

$$\mathfrak{E}_{\mathsf{G}}\Big(L_{r,q}\Big) = \Big(t^{-\frac{1}{r}},\, q\Big) = \mathfrak{E}_{\mathsf{G}}\Big(B^s_{p,q}\Big), \tag{5.2.9}$$

the parameters being as above. Again we note by (5.2.5) that $B^s_{p,q}$ can be embedded in $L_{r,q}$, whereas their envelopes even coincide. We return to this phenomenon in Section 7.2.

The embedding result (5.2.5) can (in the Banach space situation) also be found in [Gol87a] and [Kol98]. Moreover, GOL'DMAN's result [Gol87b, Thm. 2.1, Cor. 5.1] can be disclosed as the fact that $L_{r,q}$ is the best possible space within the Lorentz scale in which $B^s_{p,q}$ can be embedded continuously – coinciding with our above interpretation of (5.2.8).

Remark 5.2.3 Forerunners of this result – formulated in a different context – are presented in [Tri99]. This is extended and generalised in [Tri01, Sect. 15]. There one also finds a lot of remarks and references on the long history of related studies; thus we shall only mention some of the most important names and papers briefly : essential contributions were achieved by PEETRE [Pee66], STRICHARTZ [Str67], HERZ [Her68], as well as in the Russian school by BRUDNYI [Bru72], GOL'DMAN [Gol87c, Gol87b], LIZORKIN [Liz86], KALYABIN, LIZORKIN [KL87], NETRUSOV [Net87, Net89], see also the book by ZIEMER [Zie89]. More recent treatments are, for instance, [CP98] by CWIKEL, PUSTYLNIK, [EKP00] by EDMUNDS, KERMAN, PICK and the surveys [Kol98] by KOLYADA, [Tar98] by TARTAR. There are far more investigations connected in some sense with limiting embeddings, we refer to the survey papers for detailed information.

Remark 5.2.4 Recently CAETANO and MOURA obtained parallel results in the sub-critical case when studying spaces of generalised smoothness of type $B^{(s,\Psi)}_{p,q}$, $F^{(s,\Psi)}_{p,q}$, introduced by MOURA in [Mou01], see our remark after Definition 3.2.4. In particular, $\Psi(x)=(1+|\log x|)^b$, $b\in\mathbb{R}$, is admitted in this context. The result of CAETANO and MOURA in [CM01, Thm. 4.4] completely characterises the influence of the additional smoothness function Ψ by

$$\mathfrak{E}_{\mathsf{G}}\left(B^{(s,\Psi)}_{p,q}\right) = \left(t^{-\frac{1}{r}}\Psi(t)^{-1},\, q\right), \qquad \mathfrak{E}_{\mathsf{G}}\left(F^{(s,\Psi)}_{p,q}\right) = \left(t^{-\frac{1}{r}}\Psi(t)^{-1},\, p\right),$$

where Ψ is an admissible function, $0<p<\infty$, $0<q\leq\infty$, $s\in\mathbb{R}$ with $\sigma_p<s<\frac{n}{p}$, and $1<r<\infty$ such that $s-\frac{n}{p}=-\frac{n}{r}$. Thus with $\Psi_a(x)=(1+|\log x|)^a$, $a\in\mathbb{R}$, one concludes by Proposition 5.1.2 (ii)

$$\mathfrak{E}_{\mathsf{G}}\left(B^{(s,\Psi_a)}_{p,q}\right)=\left(t^{-\frac{1}{r}}\left|\log t\right|^{-a},\ q\right)=\ \mathfrak{E}_{\mathsf{G}}\left(L_{r,q}\left(\log L\right)_a\right)$$

and

$$\mathfrak{E}_{\mathsf{G}}\left(F^{(s,\Psi_a)}_{p,q}\right)=\left(t^{-\frac{1}{r}}\left|\log t\right|^{-a},\ p\right)=\ \mathfrak{E}_{\mathsf{G}}\left(L_{r,p}\left(\log L\right)_a\right)\ ,$$

where the parameters are as above. This seems in some sense the counterpart of (5.2.8) and (5.2.9), whereas some more qualified discussion is still missing; this refers in particular to assertions like (5.2.4) and (5.2.5) adapted to this more general setting.

Borderline cases

We study the situation $s=\sigma_p=n(\frac{1}{p}-1)_+$ now; recall that this refers to the thick lines in Figure 8. However, in this situation additional care is needed, because not all spaces in question are contained in L_1^{loc} . A complete treatment of this problem $A^s_{p,q}\subset L_1^{\mathrm{loc}}$ can be found in [ST95], where SICKEL and TRIEBEL obtained in [ST95, Thm. 3.3.2] the following result – related to the case $s=\sigma_p$ we are interested now :

$$F^{\sigma_p}_{p,q}\ \subset\ L_1^{\mathrm{loc}}\qquad\text{if, and only if,}\qquad\begin{cases}\text{either}\quad 0<p<1\ ,\quad 0<q\leq\infty\,,\\ \text{or}\quad 1\leq p<\infty\ ,\quad 0<q\leq 2\,.\end{cases}\tag{5.2.10}$$

The parallel assertion for B-spaces reads as

$$B^{\sigma_p}_{p,q}\ \subset\ L_1^{\mathrm{loc}}\qquad\text{if, and only if,}\qquad\begin{cases}\text{either}\quad 0<p\leq 1\ ,\quad 0<q\leq 1\,,\\ \text{or}\quad 1<p\leq\infty\ ,\quad 0<q\leq\min(p,2)\ .\end{cases}\tag{5.2.11}$$

We first consider the '*bottom line*' of the sub-critical strip in Figure 8; that is, where $1<p<\infty$, and $s=0$.

Proposition 5.2.5 [Har01, Prop. 3.15] *Let* $1<p<\infty$.

(i) *Assume* $0<q\leq 2$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(F^0_{p,q}\right)=\left(t^{-\frac{1}{p}},\ p\right).\tag{5.2.12}$$

(ii) *Assume* $0<q\leq\min(p,2)$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(B^0_{p,q}\right)=\left(t^{-\frac{1}{p}},\ u_{\mathsf{G}}^{B^0_{p,q}}\right)\qquad\textit{with}\qquad q\ \leq\ u_{\mathsf{G}}^{B^0_{p,q}}\ \leq\ p\ .\tag{5.2.13}$$

In particular,

$$\mathfrak{E}_{\mathsf{G}}\left(B^0_{p,p}\right)=\left(t^{-\frac{1}{p}},\ p\right)\ ,\quad 1<p\leq 2.$$

The assertion for the envelope functions and the upper bounds for $u_{\mathsf{G}}^{A^s_{p,q}}$ are proved via embeddings $A^s_{u,q}\hookrightarrow A^0_{p,q}\hookrightarrow L_p$, where the parameters are as above, see (1.2.6), (1.1.34), and (1.2.3), and application of Theorem 5.2.1 together with Propositions 4.2.4 (iv), 4.2.13, 5.1.2. The lower bounds for $u_{\mathsf{G}}^{A^s_{p,q}}$ rely on modified extremal functions

$$f(x)=\sum_{j=1}^{\infty}\ b_j\ 2^{j\frac{n}{p}}\ \left(\psi\left(2^jx\right)-\psi\left(2^jx-x^0\right)\right)\ ,\quad x\in\mathbb{R}^n\ ,\tag{5.2.14}$$

which is an adapted version of [Tri99, Sect. 3.2]. Assume $x^0\in\mathbb{R}^n$ with $\left|x^0\right|>4$ (one needs first moment conditions now). Choosing the sequence $b=\{b_j\}_{j\in\mathbb{N}}$ in a clever way one verifies (5.2.12) and (5.2.13), respectively. Obviously, $\mathfrak{E}_{\mathsf{G}}(F^0_{p,q})=\mathfrak{E}_{\mathsf{G}}(L_p)=(t^{-\frac{1}{p}},\ p)$, $1<p<\infty$, $0<q\leq 2$, and $\mathfrak{E}_{\mathsf{G}}(B^0_{p,p})=\mathfrak{E}_{\mathsf{G}}(L_p)=(t^{-\frac{1}{p}},\ p)$, $1<p\leq 2$. We add a remark on the gap in (5.2.13) at the end of this section.

We study the line $s=\sigma_p=n(\frac{1}{p}-1)$, where $0<p\leq 1$.

Proposition 5.2.6 [Har01, Prop. 3.17] *Let* $0 < p \le 1$ *and* $s = n\left(\frac{1}{p} - 1\right)$.

(i) *Assume* $0 < q \le \infty$, *and* $0 < q \le 2$ *if* $p = 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(F^s_{p,q}\right) = \left(t^{-1},\ u_{\mathsf{G}}^{F^s_{p,q}}\right) \qquad \textit{with} \qquad p \ \le\ u_{\mathsf{G}}^{F^s_{p,q}} \ \le\ 1\ . \tag{5.2.15}$$

In particular,

$$\mathfrak{E}_{\mathsf{G}}\left(F^0_{1,q}\right) = \left(t^{-1},\ 1\right)\ , \quad 0 < q \le 2\ .$$

(ii) *Assume* $0 < q \le 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\left(B^s_{p,q}\right) = \left(t^{-1},\ u_{\mathsf{G}}^{B^s_{p,q}}\right) \qquad \textit{with} \qquad q \ \le\ u_{\mathsf{G}}^{B^s_{p,q}} \ \le\ 1\ . \tag{5.2.16}$$

In particular,

$$\mathfrak{E}_{\mathsf{G}}\left(B^s_{p,1}\right) = \left(t^{-1},\ 1\right)\ , \quad 0 < p \le 1,\ s = n\left(\frac{1}{p} - 1\right)\ .$$

The ideas of the corresponding proof in [Har01] resemble those discussed above briefly, i.e. embeddings as well as extremal functions.

Clearly Propositions 5.2.5 and 5.2.6 show that the borderline situation, in particular, the determination of the corresponding indices u_{G}^X, is rather complicated to handle and not yet solved completely (apart from some special cases). Even worse, a reasonable guess what the correct outcome could be, is also missing. Concerning the '*bottom line*' – referring to Proposition 5.2.5 – one asks whether $B-$spaces with $s = 0$ show their 'usual' behaviour, i.e. $u_{\mathsf{G}}^{B^0_{p,q}} = q$, independently of the delicate limiting situation, or if they 'suffer' from this setting and tend to behave like the $F-$spaces, that is $u_{\mathsf{G}}^{B^0_{p,q}} = p$ – or something in between. The situation is even more obscure on the line $s = n(\frac{1}{p} - 1)$, $0 < p \le 1$: here also the $F-$spaces keep silence about their indices (so far). There was a good assumption that $u_{\mathsf{G}}^{F^s_{p,q}} = p$ – simply as this happens in all other cases we studied; on the other hand, also $u_{\mathsf{G}}^{F^s_{p,q}} = 1$ was some good choice in view of the borderline situation ($F^s_{p,q} \subset L_1^{\mathrm{loc}}$), not to speak of the $B-$setting.

5.3 Growth envelopes in the critical case

We deal with spaces $A^s_{p,q}$, where $s = \frac{n}{p}$, see Figure 1. We recall the limiting embeddings (1.2.18) and (1.2.19) : Let $0 < p \le \infty$ (with $p < \infty$ for F spaces), and $0 < q \le \infty$. Then

$$F^{n/p}_{p,q} \hookrightarrow L_\infty \qquad \text{if, and only if,} \qquad 0 < p \le 1 \quad \text{and} \quad 0 < q \le \infty, \tag{5.3.1}$$

and

$$B^{n/p}_{p,q} \hookrightarrow L_\infty \qquad \text{if, and only if,} \qquad 0 < p \le \infty \quad \text{and} \quad 0 < q \le 1, \tag{5.3.2}$$

see [ET96, 2.3.3 (iii), p. 45]. In view of Proposition 4.2.4 (iii) it is clear that spaces given by (5.3.1) and (5.3.2), respectively, are of no further interest in our context, because the corresponding (growth) envelope functions are bounded. We shall study the remaining cases now.

Theorem 5.3.1 [Har01, Thm. 3.19], [Tri01, Thm. 13.2, p. 203] *Let* $0 < p < \infty$ *and* $0 < q \le \infty$.

(i) *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *as usual. Then*

$$\mathfrak{E}_{\mathsf{G}}\left(F^{n/p}_{p,q}\right) = \left(|\log t|^{\frac{1}{p'}}\ ,\ p\right)\ . \tag{5.3.3}$$

(ii) *Let* $1 < q \le \infty$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$, *as usual. Then*

$$\mathfrak{E}_{\mathsf{G}}\left(B^{n/p}_{p,q}\right) = \left(|\log t|^{\frac{1}{q'}}\ ,\ q\right)\ . \tag{5.3.4}$$

The proof is essentially based on ideas of H. Triebel and also relies on [ET99]; we give its main ideas. One starts with $\mathcal{E}_{\mathsf{G}}^{F^{n/p}_{p,q}}$ and applies [ET99, Thm. 2.5] with $\varkappa \equiv 1$, that is,

$$\left(\int_0^{\frac12} \left[\frac{g^*(t)}{|\log t|} \right]^p \frac{\mathrm{d}t}{t} \right)^{1/p} \leq c \, \left\| g | H^{n/p}_p \right\| \tag{5.3.5}$$

for any $g \in H^{n/p}_p$. Let $f \in F^{n/p}_{p,\infty}$, then by a result of NETRUSOV [Net89, Thm. 1.1] there is some $g \in H^{n/p}_p$, $|f(x)| \leq g(x)$ a.e. in $\mathbb{R}^n$ (implying $f^* \leq g^*$), with $\left\| g | H^{n/p}_p \right\| \leq c \left\| f | F^{n/p}_{p,\infty} \right\|$, and hence (5.3.5) leads to

$$\left(\int_0^{\frac12} \left[\frac{f^*(t)}{|\log t|} \right]^p \frac{\mathrm{d}t}{t} \right)^{1/p} \leq \left(\int_0^{\frac12} \left[\frac{g^*(t)}{|\log t|} \right]^p \frac{\mathrm{d}t}{t} \right)^{1/p} \leq c \, \left\| g | H^{n/p}_p \right\| \leq c' \, \left\| f | F^{n/p}_{p,\infty} \right\| \tag{5.3.6}$$

for all $f \in F^{n/p}_{p,\infty}$. By monotonicity this follows for $F^{n/p}_{p,q}$, $0 < q \leq \infty$, and application of (3.3.17) results in

$$\sup_{0<t<\frac12} \frac{f^*(t)}{|\log t|^{1/p'}} \leq c \left(\int_0^{\frac12} \left[\frac{f^*(t)}{|\log t|} \right]^p \frac{\mathrm{d}t}{t} \right)^{1/p} \leq c' \left\| f | F^{n/p}_{p,q} \right\| \tag{5.3.7}$$

for all $f \in F^{n/p}_{p,q}$. This gives $\mathcal{E}_{\mathsf{G}}^{F^{n/p}_{p,q}}(t) \leq c \, |\log t|^{\frac{1}{p'}}$, $0 < t < \frac12$. Concerning the B-counterpart of (5.3.7), that is

$$\sup_{0<t<\frac12} \frac{f^*(t)}{|\log t|^{1/q'}} \leq c \, \left\| f | B^{n/p}_{p,q} \right\| \tag{5.3.8}$$

for all $f \in B^{n/p}_{p,q}$, $0 < p < \infty$, $1 < q \leq \infty$, we exploit the following embeddings

$$B^{n/p}_{p,\infty} \hookrightarrow \mathrm{bmo} \hookrightarrow L_{\exp,1} , \quad 0 < p < \infty ; \tag{5.3.9}$$

see (1.1.37) for the definition of bmo. The latter embedding is covered by [BS88, (7.22), p. 383] (locally, but this is sufficient for our purpose), whereas the first one is verified by means of embeddings (1.2.17) and duality results,

$$B^{n/p}_{p,\infty} = \left(B^{-n\left(1-\frac{1}{p'}\right)}_{p',1} \right)' \hookrightarrow \left(F^0_{1,2} \right)' = (h_1)' = \mathrm{bmo} , \quad 1 < p < \infty, \tag{5.3.10}$$

where h_p are the local Hardy spaces, see (1.1.35) and (1.1.36). Here we use the duality result $\mathrm{bmo} = (h_1)'$, see [Gol79b], and [Tri83, Thm. 2.11.2, p. 178] for the duality of $B-$spaces. This gives

$$\sup_{0<t<\frac12} \frac{f^*(t)}{|\log t|} \leq c \, \left\| f | B^{n/p}_{p,\infty} \right\| \tag{5.3.11}$$

for all $f \in B^{n/p}_{p,\infty}$, because of (5.3.9) and Propositions 5.1.4, 4.2.4 (iv), i.e. the desired upper estimate for $\mathcal{E}_{\mathsf{G}}^{B^{n/p}_{p,\infty}}(t)$. The extension to $1 < q < \infty$ is achieved by some (non-linear) real interpolation argument for $T : f \longmapsto f^{**}$ mapping from suitably chosen $B-$ spaces into weighted L_∞ spaces; note that the sub-additivity of f^{**} (1.1.16) immediately gives the Lipschitz-continuity of T which allows us to apply TARTAR's result [Tar72, Thm. 4, p. 476]. We end up with (5.3.8) when $1 < p < \infty$, the remaining case $0 < p \leq 1$ follows by the monotonicity of B-spaces simply. The converse estimates for $\mathcal{E}_{\mathsf{G}}^{A^{n/p}_{p,q}}(t)$ are proved with extremal functions similar to (5.2.14), i.e.

$$f_b(x) = \sum_{j=1}^{\infty} b_j \, \psi\left(2^{j-1}x\right) , \quad x \in \mathbb{R}^n ; \tag{5.3.12}$$

this construction of extremal functions goes back to [ET99] by Edmunds and Triebel. Choosing the sequence $b = \{b_j\}_{j=1}^{\infty}$ properly, the $B-$result is completed, whereas the $F-$counterpart directly results from embedding (1.2.17) via

$$B^{n/r}_{r,p} \hookrightarrow F^{n/p}_{p,q} \tag{5.3.13}$$

for $r < p$, and Proposition 4.2.4 (iv).

Determining the correct indices $u_{\mathsf{G}}^{A^{n/p}_{p,q}}$ needs much more effort, at least when upper bounds are concerned. Clearly, (5.3.7) gives $u_{\mathsf{G}}^{F^{n/p}_{p,q}} \leq p$ already, leading via (5.3.13) to $u_{\mathsf{G}}^{B^{n/p}_{p,q}} \leq q$, but only for $p \leq q$. In general one has to cope with the atomic decomposition of $f \in B^{n/p}_{p,q}$, we refer to Section 1.1.3, in particular, Theorem 1.1.9 for details. One finally arrives at

$$\left(\int_0^{\varepsilon} \left[\frac{f^*(t)}{|\log t|} \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \leq c \left\| f | B^{n/p}_{p,q} \right\|, \qquad 1 < q < \infty , \tag{5.3.14}$$

such that $u_{\mathsf{G}}^{B^{n/p}_{p,q}} \leq q$, but now for all admitted q. It remains to verify the converse inequalities for $u_{\mathsf{G}}^{A^{n/p}_{p,q}}$. In $B-$case this is a matter of extremal functions (5.3.12) where the sequence $b = \{b_j\}_{j=1}^{\infty}$ has to be chosen suitably; the $F-$case follows by (5.3.13) and Proposition 4.2.13.

Remark 5.3.2 In analogy to (5.2.8) and (5.2.9) in Remark 5.2.2 we see that

$$\mathfrak{E}_{\mathsf{G}}\Big(L_{\infty,p}(\log L)_{-1}\Big) = \Big(|\log t|^{\frac{1}{p'}}\ ,\ p\Big) = \mathfrak{E}_{\mathsf{G}}\left(F^{n/p}_{p,q}\right), \tag{5.3.15}$$

where $0 < q \leq \infty$ and $1 < p < \infty$; cf. Proposition 5.1.2 (iii) and (5.3.3). This also refers to [BW80] in case of Sobolev spaces. Correspondingly the situation in B-case reads as

$$\mathfrak{E}_{\mathsf{G}}\Big(L_{\infty,q}(\log L)_{-1}\Big) = \Big(|\log t|^{\frac{1}{q'}}\ ,\ q\Big) = \mathfrak{E}_{\mathsf{G}}\left(B^{n/p}_{p,q}\right), \tag{5.3.16}$$

where $0 < p < \infty$ and $1 < q \leq \infty$. This follows by Proposition 5.1.2 (iii) and (5.3.4).

Remark 5.3.3 Studying spaces on a bounded domain $\Omega \subset \mathbb{R}^n$, say with $|\Omega| < 1$, (5.3.7) and (5.3.8) can be rewritten as $F^{n/p}_{p,q}(\Omega) \hookrightarrow L_{\exp,1/p'}(\Omega)$, $1 < p < \infty$, $0 < q \leq \infty$, and $B^{n/p}_{p,q}(\Omega) \hookrightarrow L_{\exp,1/q'}(\Omega)$, $0 < p < \infty$, $1 < q \leq \infty$, see Definition 1.1.1 (ii) with $L_{\infty,\infty}(\log L)_{-a} = L_{\exp,a}$, $a \geq 0$. In view of (5.3.1), (5.3.2) and our notation (1.1.3) this can be summarised as follows (see [Har01, Cor. 3.23]); recall $L_{\exp,0} = L_{\infty}$. Then $F^{n/p}_{p,q}(\Omega) \hookrightarrow L_{\exp,a}(\Omega)$ if, and only if, $a \geq \frac{1}{p'}$, and $B^{n/p}_{p,q}(\Omega) \hookrightarrow L_{\exp,a}(\Omega)$ if, and only if, $a \geq \frac{1}{q'}$, where $0 < p \leq \infty$ ($p < \infty$ for $F-$spaces), $0 < q \leq \infty$, and $\Omega \subset \mathbb{R}^n$ with $|\Omega| < 1$. Note that this is the classical result by Pohožaev, Peetre, Trudinger, Strichartz extended to all reasonable cases in the context of B- or F-spaces. Moreover, the history of papers devoted to critical embeddings in the above sense is very long already; we mentioned in Remark 5.2.3 some of the relevant papers. Additionally we shall refer to Strichartz [Str72], Trudinger [Tru67], Yudovich [Yud61], Pohožaev [Poh65], Hansson [Han79], Brézis, Wainger [BW80], Bennett, Sharpley [BS88, Ch. 4] and Triebel in [Tri93]. We refer to [ET99, Rem. 2.6] for an extensive discussion of the history of embeddings of that 'critical' type.

Remark 5.3.4 Obviously assertions (5.3.1) and (5.3.2), together with elementary embedding properties of spaces $A^s_{p,q}$ given in Section 1.2.1 imply that $A^s_{p,q} \hookrightarrow L_{\infty}$ in the super-critical case, see Figure 1. Thus we know that $\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t)$ is bounded in the super-critical case and so, by our convention, $\mathfrak{E}_{\mathsf{G}}(A^s_{p,q}) = (1,\ \infty)$ where $0 < p \leq \infty$ ($p < \infty$ in the F-case), $s > \frac{n}{p}$, and $0 < q \leq \infty$.

We excluded in the above theorem the study of $B^0_{\infty,q}$, $0 < q \leq \infty$, whereas we clearly have by (5.3.2) that $B^0_{\infty,q} \hookrightarrow L_{\infty}$ when $0 < q \leq 1$. On the other hand, (5.2.11) implies that $B^0_{\infty,q} \subset L_1^{\mathrm{loc}}$ for $0 < q \leq 2$. So it remains to consider the case $1 < q \leq 2$.

Proposition 5.3.5 [Har01, Prop. 3.25] *Assume* $1 < q \leq 2$. *Then there are positive constants* c_1, c_2 *such that for all small* $t > 0$

$$c_1 \ |\log t|^{\frac{1}{q'}} \ \leq \ \mathcal{E}_{\mathsf{G}}^{B^0_{\infty,q}}(t) \ \leq \ c_2 \ |\log t| \ . \tag{5.3.17}$$

Clearly the result for $B^0_{\infty,q}$, $1 < q \leq 2$, is not yet satisfactory and needs further effort; in compensation for this we end this section with some complete result which is in some sense also surprising. We promised in Remark 4.2.2 that growth envelopes for spaces on bounded domains and for the corresponding spaces on $\mathbb{R}^n$ are essentially the same – *apart from a few explicitly mentioned cases*. Clearly, Proposition 5.1.4 already deals with such an exception as $L_{\exp,a}$ does not make sense otherwise. Even more peculiar is the following situation when dealing with bmo; for a definition we refer to (1.1.37). Starting with the situation on $\mathbb{R}^n$ it can be easily checked that functions like

$$\sum_{m \in \mathbb{Z}^n} \psi(x-m) \ |\log|x-m|| \ ,$$

where $\psi(x)$ is given by (5.2.6), belong to $\mathrm{bmo}(\mathbb{R}^n)$; see [BS88, Ch. 5, Sect. 7, p. 376] for the local matter, and [Tri01, Sect. 13.7]. On the other hand, these members of $\mathrm{bmo}(\mathbb{R}^n)$ immediately lead to $\mathcal{E}_{\mathsf{G}}^{\mathrm{bmo}}(t) = \infty$ for all $t > 0$ (representing another example for Lemma 4.2.3 (i)). Restricting, however, the space bmo to a bounded domain $\Omega \subset \mathbb{R}^n$, say, with $|\Omega| < 1$, then spaces $\mathrm{bmo}(\Omega)$, defined by restriction from $\mathrm{bmo}(\mathbb{R}^n)$, possess a much more interesting growth envelope function.

Proposition 5.3.6 [Har01, Prop. 3.26], [Tri01, Sect. 13.7] *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, say, with* $|\Omega| < 1$. *Then*

$$\mathfrak{E}_{\mathsf{G}}\Big(\mathrm{bmo}\ (\Omega)\Big) \ = \ (|\log t|\,, \ \infty)\,. \tag{5.3.18}$$

The proof easily follows from our (local) assertion (5.3.9) together with Propositions 4.2.4 (iv), 4.2.13, 5.1.4 and Theorem 5.3.1 (ii).

Let us finally mention that there is a connection between spaces of type $F^0_{\infty,q}$ and bmo, appearing (though secretly hidden) in (5.3.10) already. Spaces $F^s_{p,q}$ with $p = \infty$ are excluded in our considerations usually; however they were introduced already in [Tri78b, 2.5.1, p. 118] for $1 < q < \infty$, see also [Tri83, Sect. 2.3.4, p. 50]. This definition was modified and extended to $0 < q < \infty$ in [FJ90, Sect. 5]. In the critical case $s = 0$, $p = \infty$, one has for $0 < q \leq 2$,

$$F^0_{\infty,q} \ \hookrightarrow \ F^0_{\infty,2} \ = \ \mathrm{bmo} \ \hookrightarrow \ L_1^{\mathrm{loc}} \ . \tag{5.3.19}$$

Conversely, $\textsc{Marschall}$ proved in [Mar95, Lemma 16], that $B^{s+n/p}_{p,\infty} \ \hookrightarrow \ F^s_{\infty,q}$ for all $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$; in particular,

$$B^{n/p}_{p,\infty} \ \hookrightarrow \ F^0_{\infty,q} \ , \quad 0 < p < \infty, \ 0 < q \leq \infty \ ; \tag{5.3.20}$$

the case $q \geq 1$ is already covered by [Mar87, Cor. 4]. Combining Proposition 5.3.6, Theorem 5.3.1 (ii), and (5.3.19), (5.3.20) we arrive at

$$\mathfrak{E}_{\mathsf{G}}\left(F^0_{\infty,q}\right) \ = \ (|\log t|\,, \ \infty) \ , \quad 0 < q \leq 2 \ . \tag{5.3.21}$$

6 Continuity envelopes $\mathfrak{E}_{\mathsf{C}}$

The programme for this section is similar to the previous one, where now questions of growth of functions are replaced by smoothness assertions. All spaces are defined on $\mathbb{R}^n$ unless otherwise stated.

6.1 Definition and first examples

Recall that C stands for the space of all bounded and uniformly continuous functions on $\mathbb{R}^n$, as usual.

Definition 6.1.1 *Let $X \hookrightarrow C$ be some function space on $\mathbb{R}^n$ with* (4.3.5), $X \not\hookrightarrow \mathrm{Lip}^1$ *and continuity envelope function $\mathcal{E}_{\mathsf{C}}^X$, $0 < t < \varepsilon$, and index u_{C}^X. Then*

$$\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^X(\cdot),\ u_{\mathsf{C}}^X\right) \tag{6.1.1}$$

is called continuity envelope *for the function space X.*

We begin with Lipschitz spaces Lip^a, $0 < a \le 1$, and $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\, q}$, $0 < q \le \infty$, $\alpha > \frac{1}{q}$ (with $\alpha \ge 0$ if $q = \infty$), see Definitions 1.1.3 and 3.2.1, respectively, as examples. Recall that $f \in C$ belongs to $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\, q}$ if

$$\left\| f|\mathrm{Lip}^{(1,-\alpha)}_{\infty,\, q}\right\| := \|f|C\,\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)}{t\,|\log t|^\alpha}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \tag{6.1.2}$$

(with the usual modification if $q = \infty$) is finite; see (3.2.3). Combining Definitions 1.1.3 for Lip^a and (6.1.2) for $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\, q}$ one can introduce spaces $\mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}$, $0 < a < 1$, $\alpha \in \mathbb{R}$, $0 < q \le \infty$. We add this consideration by matter of completeness.

Definition 6.1.2 *Let $0 < a < 1$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$. The space $\mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}$ is defined as the set of all $f \in C$ such that*

$$\left\| f|\mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}\right\| := \|f|C\,\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)}{t^a\,|\log t|^\alpha}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \tag{6.1.3}$$

(*with the usual modification if $q = \infty$*) *is finite.*

Remark 6.1.3 One can easily verify that there is a counterpart of Proposition 3.2.7 for spaces $\mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}$, $0 < a < 1$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$. In particular, $f \in \mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}$ if, and only if, f belongs to C (or L_∞) and there is some $c > 0$ such that

$$\int_0^a \lambda^{\alpha q} \int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)}{t^{a-\lambda}}\right]^q \frac{\mathrm{d}t}{t}\,\frac{\mathrm{d}\lambda}{\lambda} \le c\,.$$

Moreover,

$$\left\| f|\mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}\right\| \sim \|f|C\| + \left(\int_0^a \lambda^{\alpha q} \int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)}{t^{a-\lambda}}\right]^q \frac{\mathrm{d}t}{t}\,\frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}. \tag{6.1.4}$$

In contrast to (3.2.12), (3.2.13) there is no problem now writing this as

$$\left\| f|\mathrm{Lip}^{(a,-\alpha)}_{\infty,\, q}\right\| \sim \left(\int_0^a \lambda^{\alpha q}\ \|f|B^{a-\lambda}_{\infty,q}\|^q\ \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}. \tag{6.1.5}$$

as long as $0 < a < 1$.

Westerhoff dealt in [Wes01] with spaces $C^{(a,u)}$ coinciding with $\mathrm{Lip}^{(a,-u)}_{\infty,\,\infty}$, estimating also entropy numbers for related embeddings.

Proposition 6.1.4 [Har01, Props. 4.9, 4.12, 4.14, 4.15, 5.5, 5.6, 5.7, 5.8]

(i) *Let $0 < a \le 1$. Then*

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^a\right) = \left(t^{-(1-a)},\ \infty\right). \tag{6.1.6}$$

(ii) *Let* $0 < q \leq \infty$, $\alpha > \frac{1}{q}$ (*with* $\alpha \geq 0$ *if* $q = \infty$). *Then*

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(1,-\alpha)}_{\infty,\,q}\right) = \left(|\log t|^{\alpha-\frac{1}{q}}\,,\ q\right). \tag{6.1.7}$$

(iii) *Let* $0 < a < 1$, $0 < q \leq \infty$, *and* $\alpha \in \mathbb{R}$. *Then*

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(a,-\alpha)}_{\infty,\,q}\right) = \left(t^{-(1-a)}\,|\log t|^{\alpha}\,,\ q\right). \tag{6.1.8}$$

(iv) *We have*

$$\mathfrak{E}_{\mathsf{C}}(\ C\ ) = \left(t^{-1},\ \infty\right). \tag{6.1.9}$$

6.2 Continuity envelopes in the super-critical case

We finally deal with the *super-critical* case of spaces of type $A^s_{p,q}$ as introduced in Figure 1, i.e. let $0 < p \leq \infty$ (with $p < \infty$ in the F-case), $0 < q \leq \infty$, and $\frac{n}{p} < s \leq \frac{n}{p} + 1$. Obviously (5.3.1), (5.3.2) and some elementary embedding argument for B- and F-spaces imply that such spaces can be embedded into C. Hence it is reasonable to study their continuity envelope function. On the other hand, when $s > \frac{n}{p}+1$, we may conclude that $A^s_{p,q}$ is continuously embedded in Lip^1, so that by Proposition 4.3.3 (ii) the corresponding continuity envelope functions are bounded and thus of no further interest.

First we study spaces $A^s_{p,q}$ belonging to the '*super-critical strip*' (without the border-lines so far), that is, $0 < s - \frac{n}{p} < 1$, $0 < p \leq \infty$, see Figure 9 aside.

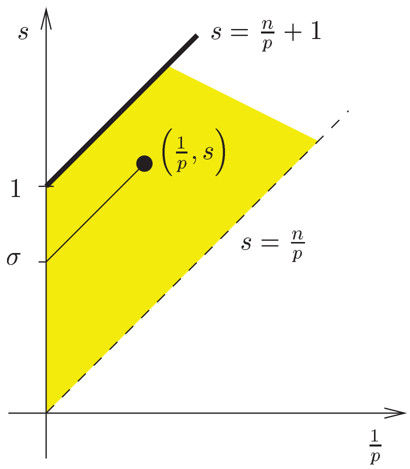


Figure 9

Theorem 6.2.1 [Har01, Prop. 5.9, Thm. 5.10] *Let* $0 < p \leq \infty$ (*with* $p < \infty$ *in the* F*-case*), $0 < q \leq \infty$, $0 < \sigma < 1$ *and* $s = \frac{n}{p} + \sigma$. *Then*

$$\mathfrak{E}_{\mathsf{C}}\left(B^s_{p,q}\right) = \left(t^{-(1-\sigma)},\ q\right) \tag{6.2.1}$$

and

$$\mathfrak{E}_{\mathsf{C}}\left(F^s_{p,q}\right) = \left(t^{-(1-\sigma)},\ p\right). \tag{6.2.2}$$

We outline the main ideas of the proof. First one deals with the case $p = \infty$ in (6.2.1); here one gains from characterisation (1.1.30) with $0 < s = \sigma < 1$, $r = 1$ and $p = \infty$. This yields the upper estimates for $\mathcal{E}_{\mathsf{C}}^{B^s_{\infty,q}}(t)$ and $u_{\mathsf{C}}^{B^s_{\infty,q}}$ almost immediately; the converse is done by extremal functions. Furthermore, the elementary embedding $B^s_{p,q} \hookrightarrow B^\sigma_{\infty,q}$ implies then $\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}(t) \leq c\, t^{-(1-\sigma)}$ for small $t > 0$. Conversely, let

$$f_j(x) = 2^{-j\sigma}\ \varphi\left(2^j x\right)\ , \quad j \in \mathbb{N}, \tag{6.2.3}$$

where φ is a smooth function like

$$\varphi(x) \ \sim\ \begin{cases} 0 & , \quad |x| \geq 1, \\ 1 - |x| & , \quad |x| \leq 1, \end{cases} \qquad x \in \mathbb{R}^n .$$

Then f_j given by (6.2.3) is a $B^s_{p,q}-$ atom (as we do not need moment conditions), $\|f_j|B^s_{p,q}\| \sim 1$, and

$$\frac{\omega(f_j,t)}{t} \sim 2^{j(1-\sigma)}, \qquad t \sim 2^{-j}, \quad j \in \mathbb{N}.$$

Note that f_j given by (6.2.3) is in some sense the substitute of construction (5.2.7) for the subcritical case (and growth envelopes). This leads to the desired $B-$ result, $\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}(t) \geq c\, t^{-(1-\sigma)}$, $0 < t < 1$, and by the elementary embedding $B^s_{p,\min(p,q)} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max(p,q)}$ also to the assertion for the F-case. Concerning the indices $u_{\mathsf{C}}^{A^s_{p,q}}$ we benefit from the preceding argument for $B^s_{\infty,q}$: Let $s_0 > s > \sigma$, $s_0 - \frac{n}{p_0} = s - \frac{n}{p} = \sigma$, then (1.2.17) implies $B^{s_0}_{p_0,p} \hookrightarrow F^s_{p,q} \hookrightarrow B^\sigma_{\infty,p}$, and Proposition 4.3.8 leads to the correct upper estimates for $u_{\mathsf{C}}^{A^s_{p,q}}$. In view of elementary embeddings as above it remains to verify $u_{\mathsf{C}}^{B^s_{p,q}} \geq q$. This is achieved with extremal functions based on a combination of the functions f_j given by (6.2.3). Put

$$f(x) := \sum_{j=1}^{\infty} b_j\, 2^{-j\sigma}\, \varphi\left(2^j x - y^j\right), \qquad x \in \mathbb{R}^n, \tag{6.2.4}$$

where $b_j > 0$, $j \in \mathbb{N}$. A clever choice of $y^j \in \mathbb{R}^n$, $j \in \mathbb{N}$, (such that the supports of $\varphi\left(2^j \cdot - y^j\right)$ and $\varphi\left(2^k \cdot - y^k\right)$ are disjoint for $k \neq j$) and the sequence $b = \{b_j\}_{j\in\mathbb{N}} \in \ell_q$ results in $\|b|\ell_v\| \leq c\ \|b|\ell_q\|$ for any number v satisfying (4.3.9), which reads in our setting now as

$$\left(\int_0^\varepsilon \left[\frac{\omega(f,t)}{t^\sigma}\right]^v \frac{\mathrm{d}t}{t}\right)^{1/v} \leq c\ \left\|f|B^s_{p,q}\right\|. \tag{6.2.5}$$

Hence $v \geq q$ is obvious and the proof is finished. Note that this argument resembles the construction for the sub-critical case given in [Tri99, Sect. 3.2] and its adapted version presented in (5.2.14).

Remark 6.2.2 Parallel to Remarks 5.2.2 and 5.3.2 we mention that Proposition 6.1.4 (iii) and Theorem 6.2.1 lead to

$$\mathfrak{E}_{\mathsf{C}}\left(B^{\sigma+n/p}_{p,q}\right) = \left(t^{-(1-\sigma)},\, q\right) = \mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(\sigma,0)}_{\infty,q}\right), \quad 0<p\leq\infty,\ 0<q\leq\infty,\ 0<\sigma<1,$$

and

$$\mathfrak{E}_{\mathsf{C}}\left(F^{\sigma+n/p}_{p,q}\right) = \left(t^{-(1-\sigma)},\, p\right) = \mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(\sigma,0)}_{\infty,p}\right), \quad 0<p<\infty,\ 0<q\leq\infty,\ 0<\sigma<1.$$

It remains to study the borderline case $s = \frac{n}{p} + 1$, referring to the thick line in Figure 9. First observe that for $0 < p \leq \infty$ (with $p < \infty$ for F-spaces), $0 < q \leq \infty$ and $\alpha \geq 0$

$$F^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{\infty,\infty} \quad \text{if, and only if,} \quad \alpha \geq \frac{1}{p'}, \tag{6.2.6}$$

and

$$B^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{\infty,\infty} \quad \text{if, and only if,} \quad \alpha \geq \frac{1}{q'}, \tag{6.2.7}$$

see [ET96, (2.3.3/9,10), p. 45], [EH99, Thm. 2.1]; in particular, with $\alpha = 0$ we regain (1.2.20) and (1.2.21),

$$F^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^1 \quad \text{if, and only if,} \quad 0<p\leq 1 \quad \text{and} \quad 0<q\leq\infty, \tag{6.2.8}$$

and

$$B^{1+n/p}_{p,q} \hookrightarrow \mathrm{Lip}^1 \quad \text{if, and only if,} \quad 0<p\leq\infty \quad \text{and} \quad 0<q\leq 1\,; \tag{6.2.9}$$

these are the '*super-critical*' counterparts of (5.3.1) and (5.3.2). Hence, in view of Proposition 4.3.3 (ii) it is clear that spaces given by (6.2.8) and (6.2.9), respectively, are of no further interest in our context, because the corresponding envelope functions are bounded. We are concerned with the remaining cases now.

We start with some 'lifting' assertion which turns out to be an essential key in the later argument. It provides some relation between the modulus of continuity of some (sufficiently smooth) function and the non-increasing rearrangement of its gradient. The idea is to gain from results obtained in spaces of (sub-)critical type (and hence in terms of growth envelopes) when dealing with (super-)critical spaces (and continuity envelopes). Roughly speaking, we want to 'lift' our (sub-)critical results by smoothness 1 to (super-)critical ones. This is at least partly possible. We return to this point later in Section 7.2.1 and discuss it in more detail. Recall $(\nabla f)(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$, $x \in \mathbb{R}^n$, with

$$|\nabla f(x)| = \left(\sum_{l=1}^{n} \left|\frac{\partial f}{\partial x_l}(x)\right|^2\right)^{\frac{1}{2}} \sim \sum_{l=1}^{n} \left|\frac{\partial f}{\partial x_l}(x)\right|. \tag{6.2.10}$$

Proposition 6.2.3 [Har01, Prop. 5.12]

(i) *There is some* $c>0$ *such that for all* $t>0$ *and all* $f \in C^1(\mathbb{R}^n)$

$$\omega(f,t) \le c \int_0^{t^n} s^{\frac{1}{n}-1} \, |\nabla f|^*(s)\,\mathrm{d}s \sim \int_0^t |\nabla f|^*(\sigma^n)\,\mathrm{d}\sigma\ . \tag{6.2.11}$$

(ii) *Let* $0<r\le\infty$, $u>\frac{1}{r}$, *and* $0<\varepsilon<1$. *Then there is some number* $c>0$ *such that*

$$\int_0^\varepsilon \left[\frac{\omega(f,t)}{t\,|\log t|^u}\right]^r \frac{\mathrm{d}t}{t} \le c \int_0^\varepsilon \left[\frac{|\nabla f|^*(t)}{|\log t|^u}\right]^r \frac{\mathrm{d}t}{t} \tag{6.2.12}$$

(*with the obvious modification when* $r=\infty$) *for all* $f \in C^1(\mathbb{R}^n)$.

(iii) *Let* $0<r\le\infty$, $0<\varkappa<1$, *and* $0<\varepsilon<1$. *Then there is some number* $c>0$ *such that*

$$\int_0^\varepsilon \left[\frac{\omega(f,t)}{t^\varkappa}\right]^r \frac{\mathrm{d}t}{t} \le c \int_0^\varepsilon \left[t^{\frac{1}{n}(1-\varkappa)} \, |\nabla f|^*(t)\right]^r \frac{\mathrm{d}t}{t} \tag{6.2.13}$$

(*with the obvious modification when* $r=\infty$) *for all* $f \in C^1(\mathbb{R}^n)$.

We thank the idea to estimate (6.2.11) Prof. V. Kolyada; assertions (6.2.12) and (6.2.13) can then be derived from (6.2.11) using an extended version of HARDY's inequality obtained by BENNETT and RUDNICK in [BR80, Thm. 6.4].

Remark 6.2.4 Note that TRIEBEL obtained in [Tri01, Prop. 12.16, p. 199] assertion (6.2.12), too, but based on a different estimate replacing (6.2.11) by

$$\frac{\omega(f,t)}{t} \le c\, |\nabla f|^{**}\left(t^{2n-1}\right) + 3 \sup_{0<\tau\le t^2} \tau^{-\frac{1}{2}}\, \omega(f,\tau) \tag{6.2.14}$$

for some small $\varepsilon>0$ and all $0<t<\varepsilon$ and all $f\in C^1(\mathbb{R}^n)$, cf. [Tri01, Prop. 12.16, p. 199]. We discuss these results in Section 7.2.1 below. The exponent $2n-1$ (instead of n) in the first term on the right-hand side of (6.2.14) prevented a result like (6.2.13) in that case, in contrast to (6.2.12) where the $\log$-term takes no notice of exponents. Besides, with the help of (6.2.13) and Theorem 5.2.1 one easily derives $u_\mathsf{C}^{B^s_{p,q}} \le q$ and $u_\mathsf{C}^{F^s_{p,q}} \le p$, respectively, in Theorem 6.2.1 : simply put $\varkappa=\sigma$, $s=\sigma+\frac{n}{p}$, $\frac{1}{r}=\frac{1-\sigma}{n}$, and make use of the lifting property for $A^s_{p,q}$- spaces, cf. [Tri83, Thm. 2.3.8, p. 58]. NEVES derived some counterpart to (6.2.13) from (6.2.14), see [Nev01b, Prop. 4.2.28].

We are prepared now to give our result in the 'borderline' super-critical case when $s = \frac{n}{p} + 1$. Recall that we are only interested in the cases not covered by (6.2.8) and (6.2.9), respectively.

Theorem 6.2.5 [Har01, Thm. 5.14, Prop. 5.16], [Tri01, Thm. 14.2, p. 218/219; (14.44), p. 226]

(i) *Let* $1 < p < \infty$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$, *and* $0 < q \leq \infty$. *Then*

$$\mathfrak{E}_{\mathsf{C}}\left(F^{1+n/p}_{p,q}\right) = \left(|\log t|^{\frac{1}{p'}}\ ,\ p\right)\ . \tag{6.2.15}$$

(ii) *Let* $0 < p \leq \infty$, *and* $1 < q \leq \infty$ *with* $\frac{1}{q} + \frac{1}{q'} = 1$. *Then*

$$\mathfrak{E}_{\mathsf{C}}\left(B^{1+n/p}_{p,q}\right) = \left(|\log t|^{\frac{1}{q'}}\ ,\ q\right). \tag{6.2.16}$$

We briefly sketch the proof. Clearly, Propositions 4.3.3 (iii) and 6.1.4 (ii), together with (6.2.6), (6.2.7), give the upper estimates $\mathcal{E}_{\mathsf{C}}^{F^{1+n/p}_{p,q}}(t) \leq c\,|\log t|^{\frac{1}{p'}}$, $\mathcal{E}_{\mathsf{C}}^{B^{1+n/p}_{p,q}}(t) \leq c'\,|\log t|^{\frac{1}{q'}}$. For the converse inequalities we use extremal functions f_b as constructed by TRIEBEL in [Tri01, (14.15)-(14.19), pp. 220/221]; these are in some sense 'lifted' counterparts of (5.3.12), satisfying (in the $B-$ case)

$$\left(\sum_{j=1}^{\infty} b_j^q\right)^{1/q} \sim \left(\int_0^{\varepsilon}\left[\frac{\omega(f_b,t)}{t\,|\log t|}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \sim \left\|f_b|B^{1+n/p}_{p,q}(\mathbb{R}^n)\right\|, \tag{6.2.17}$$

where $b = \{b_j\}_{j\in\mathbb{N}}$ is a sequence of non-negative numbers. Moreover, those functions f_b are atoms in $B^{1+n/p}_{p,q}$ or $F^{1+n/p}_{p,q}$, respectively, see [Tri01, Cor. 13.4, p. 213]. For a clever choice of the sequence $b = \{b_j\}_{j\in\mathbb{N}}$ one obtains in that way functions f_J, $J \in \mathbb{N}$, with $\left\|f_J|B^{1+n/p}_{p,q}\right\| \sim J^{1/q}$, and

$$\frac{\omega\left(f_J, 2^{-J}\right)}{2^{-J}} \sim J\ . \tag{6.2.18}$$

This implies for any $J \in \mathbb{N}$, $\mathcal{E}_{\mathsf{C}}^{B^{1+n/p}_{p,q}}\left(2^{-J}\right) \geq J^{-\frac{1}{q}}\ \frac{\omega\left(f_J,2^{-J}\right)}{2^{-J}} \sim J^{\frac{1}{q'}}$, completing the argument in the B-case. The F-case can be handled in analogy to (5.3.13); in particular, (1.2.17) implies $B^{1+n/r}_{r,p} \hookrightarrow F^{1+n/p}_{p,q}$ for $0 < r < p$ leading to $\mathcal{E}_{\mathsf{C}}^{F^{1+n/p}_{p,q}} \sim |\log t|^{\frac{1}{p'}}$ finally.

Concerning the indices $u_{\mathsf{C}}^{A^{1+n/p}_{p,q}}$, in particular their upper bounds, we essentially gain from Proposition 6.2.3 and our preceding results in Section 5.3 now. Assume $p < \infty$ first; recall

$$\left\|f|B^{1+n/p}_{p,q}(\mathbb{R}^n)\right\| \sim \left\|f|B^{n/p}_{p,q}(\mathbb{R}^n)\right\| + \sum_{k=1}^{n}\left\|\frac{\partial f}{\partial x_k}\ \Big|\ B^{n/p}_{p,q}(\mathbb{R}^n)\right\|, \tag{6.2.19}$$

and similarly for $F^{1+n/p}_{p,q}$, see [Tri83, Thm. 2.3.8, p. 58]. We start with the $B-$ case, that is, $0 < p < \infty$, $1 < q \leq \infty$. Let first $q < \infty$. Apply (6.2.12) with $r = q$, $u = 1$ (recall our assumption $q > 1$, that is $u = 1 > \frac{1}{q} = \frac{1}{r}$), then (5.3.14) and (6.2.19) yield

$$\left(\int_0^{\varepsilon}\left[\frac{\omega(f,t)}{t\,|\log t|}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \leq c\left(\int_0^{\varepsilon}\left[\frac{|\nabla f|^*(t)}{|\log t|}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q} \leq c'\ \left\|f|B^{1+n/p}_{p,q}(\mathbb{R}^n)\right\| \tag{6.2.20}$$

for all $f \in C^1 \cap B^{1+n/p}_{p,q}$; the rest is done by completion. The same method applies in the F-case when $q < \infty$, now using (6.2.12) and (5.3.7). When $q = \infty$, one has to modify the above argument slightly and work with a sequence of functions which converge pointwise to f and satisfy the corresponding estimates uniformly. We deal with the case $u_{\mathsf{C}}^{B^1_{\infty,q}}$ separately, i.e. we have to show

$$\left(\int_0^{\varepsilon} \left[\frac{\omega(f,t)}{t\,|\log t|} \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \leq c \, \left\| f | B^1_{\infty,q} \right\| , \tag{6.2.21}$$

or, in other words, $B^1_{\infty,q} \hookrightarrow \mathrm{Lip}^{(1,-1)}_{\infty,\,q}$ (locally). When $q = \infty$, then [Har00b, Cor. 13, Rem. 19] covers this case. Assume $1 < q < \infty$ now; here we obtained in [Har00b, Prop. 11] only $B^1_{\infty,q} \hookrightarrow \mathrm{Lip}^{(1,-\alpha)}_{\infty,\,q}$ for $\alpha > 1$. However, a very simple and elegant proof of (6.2.21), for $1 < q < \infty$ was obtained by BOURDAUD and LANZA in [BL00, Prop. 1], combining MARCHAUD's and HARDY's inequality, see (1.1.19) and [BS88, Ch. 3, Lemma 3.9, p. 124] for the latter one. We thank this hint our colleague W. Sickel.

It remains to show the sharpness of $u_{\mathsf{C}}^{B^{1+n/p}_{p,q}} \geq q$ and $u_{\mathsf{C}}^{F^{1+n/p}_{p,q}} \geq p$, whereas it is again sufficient to deal with the $B-$ case only by elementary embeddings. This works exactly as in the proof of Theorem 5.3.1, now with the extremal functions given by (6.2.17).

Remark 6.2.6 Combining Proposition 6.1.4 (ii) and (6.2.15), (6.2.16), we arrive at

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(1,-1)}_{\infty,\,p}\right) = \left(|\log t|^{\frac{1}{p'}} ,\ p\right) = \mathfrak{E}_{\mathsf{C}}\left(F^{1+n/p}_{p,q}\right), \tag{6.2.22}$$

with $1 < p < \infty$, $0 < q \leq \infty$, and

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(1,-1)}_{\infty,\,q}\right) = \left(|\log t|^{\frac{1}{q'}} ,\ q\right) = \mathfrak{E}_{\mathsf{C}}\left(B^{1+n/p}_{p,q}\right), \tag{6.2.23}$$

with $0 < p \leq \infty$, $1 < q \leq \infty$, respectively. This situation is similar to Remarks 5.2.2 and 5.3.2 when dealing with growth envelopes; the corresponding envelopes coincide whereas the underlying spaces do not; cf. [Har00b, Cor. 13, 20] and its extension by NEVES [Nev01a]. In addition to the more or less historic references we gave in Remarks 5.2.3 and 5.3.3 already, which are partly connected with the super-critical case, too, we shall mention the results by BRÉZIS, WAINGER [BW80], the above-mentioned by BOURDAUD and LANZA [BL00], approaches based on extrapolation by EDMUNDS, KRBEC [EK95], KRBEC, SCHMEISSER [KS01a], and recently by NEVES [Nev01a]. The borderline case was already studied by ZYGMUND [Zyg45, Zyg77].

Remark 6.2.7 Note that LEOPOLD introduced in [Leo98] spaces of type $B^{(s,b)}_{p,q}$, $b \in \mathbb{R}$, which generalise spaces of type $B^s_{p,q}$, see Definition 1.1.6 (i), in terms of some additional logarithmic smoothness, we refer to Definition 3.2.4. For our purposes the characterisation (3.2.15) is sufficient, see Remark 3.2.10; then

$$\left\| f | B^{(s,-b)}_{\infty,q} \right\| \sim \|f|C\| + \left(\int_0^{\frac{1}{2}} \left[\frac{\omega_r(f,t)}{t^s |\log t|^b} \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{6.2.24}$$

(usual modification if $q = \infty$), where $s > 0$, $b \in \mathbb{R}$, $0 < q \leq \infty$, and $r \in \mathbb{N}$ such that $r > s$. Plainly, by the definition of $\mathcal{E}^X_{\mathsf{C}}$ and Proposition 4.3.3 only spaces $B^{(s,-b)}_{\infty,q}$ with $0 < s < 1$ (and arbitrary $b \in \mathbb{R}$), or $s = 1$, $b \geq 0$ are of interest in this context. When $0 < s < 1$, $B^{(s,-b)}_{\infty,q}$ coincides with $\mathrm{Lip}^{(s,-b)}_{\infty,q}$, see Definition 6.1.2; thus Proposition 6.1.4 (iii) covers this case. Let $s = 1$, $b \geq 0$. In view of the close relation between spaces $B^{(1,-b)}_{\infty,q}$ and $\mathrm{Lip}^{(1,-\alpha)}_{\infty,\,q}$, $0 < q \leq \infty$, $\alpha > \frac{1}{q}$, – studied in Section 3.3.2 in some detail – one is naturally led to the study of $\mathfrak{E}_{\mathsf{C}}\left(B^{(1,-b)}_{\infty,q}\right)$. So by Corollaries 3.3.12, 3.3.13 and Proposition 6.1.4 (ii), as well as Proposition 3.3.10 and Theorem 6.2.5 (ii) (with $p = \infty$) we immediately derive the following bounds for $\mathcal{E}_{\mathsf{C}}^{B^{(1,-b)}_{\infty,q}}(t)$: there is some $c > 0$ and for any $\varepsilon > 0$ some $c_\varepsilon > 0$ such that for (small) $t > 0$,

$$c_\varepsilon \ |\log t|^{b + \frac{1}{q'} - \varepsilon} \ \leq \ \mathcal{E}_{\mathsf{C}}^{B^{(1,-b)}_{\infty,q}}(t) \ \leq \ c\,|\log t|^{b+\frac{1}{q'}} \ , \tag{6.2.25}$$

where $1 \leq q < \infty$, $b > 0$. The exact asymptotic behaviour of $\mathcal{E}_{\mathsf{C}}^{B^{(1,-b)}_{\infty,q}}(t)$ in all cases $0 < q \leq \infty$, $b > 0$, could not be obtained yet; we refer to some forthcoming research of our colleagues A. Caetano and S.D. Moura dealing with situations described above, but in a more general setting.

6.3 Continuity envelopes in the critical case

We return to the *critical* case, already studied in Section 5.3; that is, we consider spaces $A^{n/p}_{p,q}$, see Figure 1. In view of (5.3.1) and (5.3.2) (where L_∞ can be replaced by C) we deal with the remaining cases now, not covered by Theorem 5.3.1 (in terms of growth envelopes $\mathfrak{E}_{\mathsf{G}}$).

Theorem 6.3.1 [Har01, Thm. 5.18] *Let* $0<p\leq\infty$ *and* $0<q\leq\infty$.

(i) *Assume* $0<p\leq 1$. *Then*

$$\mathcal{E}_{\mathsf{C}}^{F^{n/p}_{p,q}} \sim t^{-1} \quad , \qquad 0<t<1 \, , \tag{6.3.1}$$

and

$$p \leq u_{\mathsf{C}}^{F^{n/p}_{p,q}} \leq \infty \, . \tag{6.3.2}$$

(ii) *Assume* $0<q\leq 1$. *Then*

$$\mathcal{E}_{\mathsf{C}}^{B^{n/p}_{p,q}} \sim t^{-1} \quad , \qquad 0<t<1 \, , \tag{6.3.3}$$

and

$$q \leq u_{\mathsf{C}}^{B^{n/p}_{p,q}} \leq \infty \, . \tag{6.3.4}$$

We outline the main ideas of the proof. Firstly, (5.3.1) and (5.3.2) give $A^{n/p}_{p,q} \hookrightarrow C$ for the admitted parameters, thus Proposition 6.1.4 (iv) immediately provides $\mathcal{E}_{\mathsf{C}}^{A^{n/p}_{p,q}}(t) \leq c\, t^{-1}$, $0<t<1$. Conversely, note that our construction of the functions f_j in the proof of Theorem 6.2.1, that is, in (6.2.3), works for $\sigma=0$, too. This yields the lower estimate in the B- case (no moment conditions), and – by (5.3.13) for $0<r<p\leq 1$, $0<q\leq\infty$ – also in the F- case. It remains to verify $u_{\mathsf{C}}^{B^{n/p}_{p,q}} \geq q$, whereas $u_{\mathsf{C}}^{F^{n/p}_{p,q}} \geq p$ follows then by (5.3.13) again. Note that the extremal functions (6.2.4) work also for $\sigma=0$, leading to the desired $B-$ result.

Remark 6.3.2 We briefly discuss the obvious gaps in (6.3.2) and (6.3.4). At first glance one is certainly tempted to assume that $u_{\mathsf{C}}^{B^{n/p}_{p,q}}=q$, $u_{\mathsf{C}}^{F^{n/p}_{p,q}}=p$ was a good choice in that situation, too – simply 'as it always happens'. However, our methods presented so far fail necessarily in this *limiting case* : assume we would like to prove that

$$\left(\int_0^\varepsilon [\omega(f,t)]^q \ \frac{\mathrm{d}t}{t}\right)^{1/q} \leq c \left\|f|B^{n/p}_{p,q}\right\| \tag{6.3.5}$$

holds for all $f\in B^{n/p}_{p,q}$, $0<p\leq\infty$, $0<q\leq 1$. The 'lifting argument', however, as used in Step 2 of the proof of Theorem 6.2.5 quite effectively, cannot be used as our setting now refers to Proposition 6.2.3 (iii), but with $\varkappa=0$. This is probably not true in general, but at least not covered by Proposition 6.2.3. Still tackling (6.3.5) one could also try to verify

$$\left(\int_0^\varepsilon [\omega(f,t)]^q \ \frac{\mathrm{d}t}{t}\right)^{1/q} \leq c\ \|f|L_p\| + c \left(\int_0^1 \left[\frac{\omega(f,t)_p}{t^{n/p}}\right]^q \ \frac{\mathrm{d}t}{t}\right)^{1/q} , \tag{6.3.6}$$

which at least for large values of p, $p>n$, is an equivalent reformulation of (6.3.5); cf. [Tri83, Thm. 2.5.12, p. 110]. But the estimate

$$\omega(f,t) \leq c \int_0^t \frac{\omega(f,s)_p}{s^{n/p}} \ \frac{\mathrm{d}s}{s} \ , \quad 0<t<1,$$

(which can be shown similarly to [BS88, Ch. 5, Cor. 4.21, p. 346]) does not imply (6.3.6).

Quite the reverse we rather question now the suggestion $u_{\mathsf{C}}^{B^{n/p}_{p,q}}=q$; these doubts do not rely on the present situation (lacking of a proof, dead-ends as described above), but on a more general point of view. Note that, on the one hand, we have $B^{n/p}_{p,q} \hookrightarrow C$ if, and only if, $0 < q \le 1$, where $0 < p < \infty$, see (5.3.2). Keeping, on the other hand, Proposition 6.1.4 (iv) in mind, it was indeed rather astonishing (though, of course, not impossible) that the apparently 'small' change from $B^{n/p}_{p,1}$ to C causes a rather 'huge' jump from $u_{\mathsf{C}}^{B^{n/p}_{p,1}} = 1$ to $u_{\mathsf{C}}^{C} = \infty$, whereas both spaces share the same continuity envelope function

$$\mathcal{E}_{\mathsf{C}}^{B^{n/p}_{p,1}}(t) \sim \mathcal{E}_{\mathsf{C}}^{C}(t) \sim t^{-1} \quad , \quad 0 < t < 1 \; .$$

So from that point of view an expression for $u_{\mathsf{C}}^{B^{n/p}_{p,q}}$ which tends to ∞ when $q \uparrow 1$ was very much reasonable, too. Following that line further one needs of course 'better' extremal functions than involved in Step 2 of the proof of Theorem 6.3.1. One discovers, for instance, the extremal functions

$$f_{p\sigma}(x) = |\log|x||^{1-\frac{1}{p}} \left(\log\left(1 - \log|x|\right)\right)^{-\sigma} \psi(x) \;\in\; B^{n/p}_{p,p} \cap F^{n/p}_{p,2} \; ,$$

constructed in [ET96, Thm. 2.7.1, p. 82] by Edmunds and Triebel for $1 < p < \infty$ and $\sigma > \frac{1}{p}$, where $\psi(x)$ is a cut-off function supported near the origin. Plainly the functions do not serve in the above-given form as extremal functions in our situation ($B^{n/p}_{p,p} \not\hookrightarrow C$ for $p > 1$); but there might be a clever modification adapted for our purpose. At the moment we have to content ourselves with the 'less exciting' state of the art, i.e. estimates (6.3.2) and (6.3.4) in Theorem 6.3.1.

7 An outlook : envelopes and related questions

7.1 The envelope functions $\mathcal{E}_{\mathsf{G}}$ and $\mathcal{E}_{\mathsf{C}}$ revisited

In this concluding part we return to some more general features of envelopes and additionally collect some open problems, phenomena, desiderata. We do not aim at completeness of the posed questions (concerning possible extensions of known facts, say), and rather intend to give an outlook on future work. We study the interplay between envelopes and lifting properties as well as envelopes and related questions of compactness. The idea is twofold : firstly, of course, to find out what potential this new tool will show in the near future when tackling already 'familiar' or even new problems; secondly, we try to find as many 'interfaces' to well-established theory as possible. The latter means, for instance, that connections with related results for entropy and approximation numbers are very much welcome, because one of the starting points for introducing the concept of envelopes was the study of limiting embeddings, for instance. These problems are often connected with questions of continuity or compactness of embeddings, implying subsequent investigations of entropy numbers as performed in Sections 2, 3. Another very desirable link would be the one to more abstract Banach space theory, say. As we already explained in case of the fundamental function together with growth envelopes and questions concerning the (geometric) meaning of u_{G}, u_{C}, we are interested in further 'identifications' in that sense, at least in special cases.

7.1.1 Further properties

We summarise some features which naturally appeared as consequences of earlier observations, but were not needed before. All spaces are defined on $\mathbb{R}^n$ unless otherwise stated.

In Subsection 4.2.2 we recalled the notion of a fundamental function φ_X of a rearrangement-invariant Banach function space X . Some further property (in addition to the already mentioned in Subsection 4.2.2) is its *quasi-concavity* by which the following is meant : A non-negative function φ defined on $\mathbb{R}_+$ is called *quasi-concave*, if $\varphi(t)$ is increasing on $(0,\infty)$, $\varphi(t) = 0$ if, and only if, $t = 0$, and $\frac{\varphi(t)}{t}$ is decreasing on $(0,\infty)$; see [BS88, Ch. 2, Def. 5.6, p. 69]. Observe that every nonnegative concave function on $\mathbb{R}_+$, that vanishes only at the origin, is quasi-concave; the converse, however, is not true. Any quasi-concave function φ is equivalent to its least concave majorant $\widetilde{\varphi}$, cf. [BS88, Ch. 2, Prop. 5.10, p. 71]. Thus Proposition 4.2.7 implies the following result.

Corollary 7.1.1 [Har01, Cor. 6.1] *Let X be a rearrangement-invariant Banach function space, put*

$$\psi_{\mathsf{G}}(t) \;=\; t\,\mathcal{E}_{\mathsf{G}}^{X}(t)\,, \quad t>0\,. \tag{7.1.1}$$

(i) *The function $\psi_{\mathsf{G}}(t)$ is monotonically increasing in $t>0$.*

(ii) *Assume that $\lim\limits_{t\downarrow 0}\psi_{\mathsf{G}}(t)=0$, then $\psi_{\mathsf{G}}(t)$ is equivalent to some concave function for $t>0$.*

(iii) *The growth envelope function $\mathcal{E}_{\mathsf{G}}^{X}(t)$ is equivalent to some convex function for $t>0$.*

The question whether the rearrangement-invariance of X is really necessary or to what extent this assumption can be weakened suggests itself. Obviously in all cases we studied in the previous sections, i.e. spaces of type $L_{p,q}(\log L)_a$ and $A^s_{p,q}$, respectively, we obtained the above-described behaviour of $\mathcal{E}_{\mathsf{G}}^{X}$ and ψ_{G} whenever $X\subset L_1^{\mathrm{loc}}$ was satisfied (incorporating in a slight abuse of notation the case of constant functions ψ_{G} in (i), too; then also $X=L_1$ with $\mathcal{E}_{\mathsf{G}}^{X}(t)\sim t^{-1}$ and thus $\psi_{\mathsf{G}}(t)\sim 1$ is covered) : functions of type

$$\mathcal{E}_{\mathsf{G}}^{X}(t)\;\sim t^{-\varkappa}\;|\log t|^{\mu}\;,\quad t>0\quad \text{small},$$

with $0<\varkappa<1$, $\mu\in\mathbb{R}$, or $\varkappa=0$, $\mu>0$, lead to functions $\psi_{\mathsf{G}}(t)$ clearly satisfying Corollary 7.1.1 (with the above-mentioned extension to $\varkappa=1$, $\mu\le 0$). On the other hand, as we did not observe a direct application of (an extended version of) Corollary 7.1.1 so far we studied this question of a more general setting than X being rearrangement-invariant not yet further.

Corollary 7.1.2 [Har01, Cor. 6.3] *Let $X\hookrightarrow C$ be a function space, put*

$$\psi_{\mathsf{C}}(t) \;=\; t\,\mathcal{E}_{\mathsf{C}}^{X}(t)\,, \quad t>0\,. \tag{7.1.2}$$

(i) *The function $\psi_{\mathsf{C}}(t)$ is monotonically increasing in $t>0$ with $\lim\limits_{t\downarrow 0}\psi_{\mathsf{C}}(t)=0$.*

(ii) *The function $\psi_{\mathsf{C}}(t)$ is equivalent to some concave function for $t>0$.*

The coincidences as well as differences between Corollaries 7.1.1 and 7.1.2 are obvious. Note that in all cases we studied we have the counterpart of Corollary 7.1.1 (iii), too, i.e. $\mathcal{E}_{\mathsf{C}}^{X}$ is (equivalent to) some convex function.

More important from our point of view, however, is the observation that obviously the (different) envelope functions $\mathcal{E}_{\mathsf{G}}^{X}$ and $\mathcal{E}_{\mathsf{C}}^{X}$ show similar behaviour; we merely take it as some kind of (delayed) justification that the definition of the two envelope functions – arising in completely different problems when measuring smoothness or unboundedness, respectively, – led to parallel concepts, though each one of them separately was motivated by suitable classical settings initially. In Subsection 7.2.1 we return to this point in the sense, that there are in fact deeper connections between both envelope functions than those already discussed.

7.1.2 Spaces on $\mathbb{R}_+$

In this subsection we insert a short digression to (envelopes of) spaces on $\mathbb{R}_+=[0,\infty)$. We pose the question whether, say,

$$\mathcal{E}_{\mathsf{G}}^{X}\in X\;,$$

and this makes sense only in such spaces. We simplify the setting further and regard only spaces X on $\Omega=\left[0,\frac{1}{2}\right]$ in the sequel. First we collect some immediate consequences of our results in Section 5.1. Recall the definition for Lorentz and Zygmund spaces $L_{p,q}(\log L)_a$, $L_{\exp,a}$, in Definition 1.1.1 and (1.1.14).

Corollary 7.1.3 [Har01, Cors. 2.15, 2.17, 2.20] *Let all spaces be defined on $\Omega=\left[0,\frac{1}{2}\right]$.*
(i) *Let $0<p,q\le\infty$ (with $q=\infty$ when $p=\infty$). Then*

$$\mathcal{E}_{\mathsf{G}}^{L_{p,q}}\in L_{p,q}\qquad \text{if, and only if,}\qquad q=\infty. \tag{7.1.3}$$

(ii) *Let* $0<p<\infty$, $0<q\le\infty$, *and* $a\in\mathbb{R}$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{L_{p,q}(\log L)_a} \in L_{p,q}(\log L)_a \qquad \text{if, and only if,} \qquad q=\infty. \tag{7.1.4}$$

(iii) *Let* $a\ge 0$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{L_{\exp,a}} \in L_{\exp,a} . \tag{7.1.5}$$

(iv) *We have*

$$\mathcal{E}_{\mathsf{G}}^{\mathrm{bmo}} \in \mathrm{bmo} . \tag{7.1.6}$$

Parts (i)-(iii) are covered by [Har01, Cors. 2.15, 2.17, 2.20], whereas (iv) follows from Proposition 5.3.6, i.e. $\mathcal{E}_{\mathsf{G}}^{\mathrm{bmo}}(t) \sim |\log t|$, $t>0$ small, and [BS88, Ch. 5, Sect. 7, p. 376]. We obtain as a direct consequence of Corollary 7.1.3 that there are examples of spaces X with $\mathcal{E}_{\mathsf{G}}^{X}\in X$ as well as such where this is not the case. Moreover, taking also the index u_{G}^{X}, see Definition 4.2.10, into account, one observes the following peculiarity : whenever

$$X = \left\{ \begin{array}{lll} L_{p,\infty} & , & 0<p\le\infty \\ L_{p,\infty}(\log L)_a & , & 0<p<\infty, \quad a\in\mathbb{R} \\ L_{\exp,a} & , & a\ge 0 \\ \mathrm{bmo} & & \end{array} \right\} \implies \mathcal{E}_{\mathsf{G}}^{X}\in X, \quad u_{\mathsf{G}}^{X}=\infty \; ; \tag{7.1.7}$$

we refer to Corollary 7.1.3 and Propositions 5.1.2, 5.1.4, 5.3.6. Thus the following assertion seems natural.

Proposition 7.1.4 [Har01, Props. 6.5, 6.6] *Let* $X\hookrightarrow L_1^{\mathrm{loc}}$ *be some function space on* $\Omega=\left[0,\frac12\right]$ *with*

$$\mathcal{E}_{\mathsf{G}}^{X}\in X$$

and $\mathcal{E}_{\mathsf{G}}^{X}\not\equiv 0$. *Then this implies* $u_{\mathsf{G}}^{X}=\infty$, *i.e.* $\mathfrak{E}_{\mathsf{G}}(X)=\left(\mathcal{E}_{\mathsf{G}}^{X},\ \infty\right)$, *and* $\left\|\mathcal{E}_{\mathsf{G}}^{X}\,|\,X\right\|\ge 1.$

The second assertion, $\|\mathcal{E}_{\mathsf{G}}^{X}\,|\,X\|\ge 1$, is obviously a direct consequence of $\mathcal{E}_{\mathsf{G}}^{X}\in X$ and the definition and basic properties of $\mathcal{E}_{\mathsf{G}}^{X}$, we refer to Section 4.2.1. Besides, we have in all examples given in (7.1.7) even equivalence, that is

$$\left\|\mathcal{E}_{\mathsf{G}}^{X}\,|\,X\right\| \sim 1 . \tag{7.1.8}$$

This is due to the fact that all these spaces are rearrangement-invariant spaces which can be equivalently renormed to rearrangement-invariant spaces of type $M(X)$, $\|f|M(X)\| = \sup_{t>0} f^{**}(t)\varphi_X(t)$, for the definition of the maximal function $f^{**}(t)$ and the fundamental function $\varphi_X(t)$ we refer to (1.1.15) and (4.2.7), respectively; for Lorentz spaces of type $M(X)$ see [BS88, Ch. 2, Sect. 5, pp. 69-72]. In view of Propositions 4.2.4 (i), 4.2.7 and the fact that $\left(\mathcal{E}_{\mathsf{G}}^{X}\right)^{**}(t) \sim (\mathcal{M}\mathcal{E}_{\mathsf{G}}^{X})(t) \sim \mathcal{E}_{\mathsf{G}}^{X}(t)$ in all above-mentioned examples, we immediately obtain (7.1.8).

We return to the situations studied in Sections 5.2, 5.3 in detail.

Corollary 7.1.5 *Let all spaces be defined on* $\Omega=\left[0,\frac12\right]$.
(i) *Let* $0<q\le\infty$, $s>0$, $1<r<\infty$ *and* $0<p<\infty$ *be such that* $s-\frac1p=-\frac1r$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}} \in B^s_{p,q} \qquad \text{if, and only if,} \qquad q=\infty. \tag{7.1.9}$$

(ii) *Let* $1<q\le\infty$, *and* $0<p<\infty$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{B^{1/p}_{p,q}} \in B^{1/p}_{p,q} \qquad \text{if, and only if,} \qquad q=\infty. \tag{7.1.10}$$

By Theorems 5.2.1 and 5.3.1 together with Proposition 7.1.4 it is immediately clear that only B–spaces with $q=\infty$ can satisfy $\mathcal{E}_{\mathsf{G}}^X \in X$ as otherwise $u_{\mathsf{G}}^{A^s_{p,q}} < \infty$ which contradicts $\mathcal{E}_{\mathsf{G}}^X \in X$. So it remains to verify that in the sub-critical case $t^{-1/r} \in B^s_{p,\infty}$, $s-\frac{1}{p} = -\frac{1}{r}$ (locally), and $|\log t| \in B^{1/p}_{p,\infty}$, $0<p<\infty$, referring to the critical case. For $p \geq 1$ a straightforward calculation based on (1.1.30) was sufficient, but otherwise the atomic characterisation seems to be better adapted : we start with the sub-critical case, i.e. $s-\frac{1}{p} = -\frac{1}{r}$. Let φ be some smooth cut-off function supported near $t=0$, take, for instance, the standard one from (1.1.26). Let $\psi_j(t) = \varphi(2^j t) - \varphi(2^{j+1}t)$, $j\in\mathbb{N}_0$, $0<t<1$, build a partition of unity, then

$$t^{-\frac{1}{r}} \;=\; \varphi(t)\, t^{-\frac{1}{r}} \;\sim\; \sum_{j=0}^{\infty} 2^{-j\left(s-\frac{1}{p}\right)} \underbrace{\psi_j(t)\,\varphi(t)\,t^{-\frac{1}{r}}\,2^{j\left(s-\frac{1}{p}\right)}}_{:=\,a_j(t)}, \quad 0<t<1, \tag{7.1.11}$$

where the $a_j(t)$, $j\in\mathbb{N}_0$, are supported near $\left\{s\in[0,1] : s\sim 2^{-j}\right\}$, such that $t^{-\frac{1}{r}} \sim 2^{\frac{j}{r}} \sim 2^{-j\left(s-\frac{1}{p}\right)}$, $t\in\operatorname{supp} a_j$. Hence (7.1.11) can be understood as an atomic decomposition of $t^{-\frac{1}{r}}$ (near 0, no moment conditions) with coefficients $\lambda_j \equiv 1$, i.e. $\|\lambda|\ell_\infty\| = 1$. Theorem 1.1.9 (i) then implies $t^{-\frac{1}{r}} \in B^s_{p,\infty}$. Concerning the critical case we return to our construction (5.3.12); in particular, with $\varphi(t)$ as above, and $\psi(t)$ the (one-dimensional version of the) function given by (5.2.6), we consider

$$\sum_{j=1}^{\infty} \psi\left(2^{j-1}t\right)\varphi(2t)\ , \tag{7.1.12}$$

supported near $t=0$. Then for small $t>0$,

$$\sum_{j=1}^{\infty} \psi\left(2^{j-1}t\right)\varphi(2t) \;\sim\; \sum_{j=1}^{[|\log t|]} 1 \;\sim\; |\log t|\ ,$$

i.e. (7.1.12) can be interpreted as an atomic decomposition for $|\log t|$ near 0 (no moment conditions) with $\lambda_j \sim 1$ and thus $\|\lambda|\ell_\infty\| \sim 1$. Consequently $|\log t| \in B^{1/p}_{p,\infty}$, $0<p<\infty$ (locally).

Remark 7.1.6 Note that a different, but related question is that one asking for the boundedness of the operator $* : u \mapsto u^*$, $u\in X(0,1)$. Cianchi proved in [Cia01] that this $*-$ operator is bounded in $B^s_{p,q}(0,1)$, where $1\leq p<\infty$, $1\leq q\leq\infty$, $0<s<1+\frac{1}{p}$. In another context this means an extension of the Pólya - Szegö principle known for W^1_p, L_p already. Clearly the additional supremum in the definition of $\mathcal{E}_{\mathsf{G}}^X$ causes an essential distinction between the corresponding assertion for any u, say, with $\|u|X\|\leq 1$, and $\mathcal{E}_{\mathsf{G}}^X$.

Concerning $\mathcal{E}_{\mathsf{C}}^X$ it obviously makes no sense to ask whether $\mathcal{E}_{\mathsf{C}}^X \in X$ with X being a function space on $\Omega = \left[0,\frac{1}{2}\right]$, for – apart from the not very interesting case when $\mathcal{E}_{\mathsf{C}}^X$ is bounded, i.e. $X \hookrightarrow \mathrm{Lip}^1$ – we know that $\mathcal{E}_{\mathsf{C}}^X(t) \nearrow \infty$ when $t\downarrow 0$, such that $\mathcal{E}_{\mathsf{C}}^X \notin X$ for all $X\hookrightarrow C$. However, one may replace this question by

$$\mathfrak{e}^X(t) := t\,\mathcal{E}_{\mathsf{C}}^X(t) \;\in\; X \quad ? \tag{7.1.13}$$

It is clear by Corollary 7.1.2 (i) that $\mathfrak{e}^X(t)$ is uniformly bounded, recall (4.3.2) and $X\hookrightarrow C$. Looking for a counterpart of (7.1.7) we first collect some examples. In a slight abuse of notation we put $\mathrm{Lip}^0 = C$.

Corollary 7.1.7 [Har01, Lemmata 6.7, 6.8, 6.9, 6.10] *Let all spaces be defined on* $\Omega = \left[0,\frac{1}{2}\right]$.

(i) *Let* $0\leq a\leq 1$. *Then*

$$\mathfrak{e}^{\mathrm{Lip}^a} \in \mathrm{Lip}^a\ . \tag{7.1.14}$$

(ii) *Let* $0 < q \le \infty$, $\alpha > \frac{1}{q}$ (*with* $\alpha \ge 0$ *when* $q = \infty$). *Then*

$$\mathfrak{e}^{\mathrm{Lip}^{(1,-\alpha)}_{q,\infty}} \in \mathrm{Lip}^{(1,-\alpha)}_{q,\infty} \qquad \textit{if, and only if,} \qquad q = \infty. \tag{7.1.15}$$

(iii) *Let* $0 < a < 1$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$. *Then*

$$\mathfrak{e}^{\mathrm{Lip}^{(a,-\alpha)}_{\infty,q}} \in \mathrm{Lip}^{(a,-\alpha)}_{\infty,q} \qquad \textit{if, and only if,} \qquad q = \infty. \tag{7.1.16}$$

So we can summarise Proposition 6.1.4 and Corollary 7.1.7 as follows,

$$X = \left\{ \begin{array}{lll} \mathrm{Lip}^{a} & , & 0 \le a \le 1 \\ \mathrm{Lip}^{(a,-\alpha)}_{\infty,\infty} & , & 0 < a < 1, \quad \alpha \in \mathbb{R} \\ & \text{or} & a = 1, \quad \alpha \ge 0 \end{array} \right\} \implies \mathfrak{e}^X \in X\ , \quad u^X_{\mathsf{C}} = \infty\ . \tag{7.1.17}$$

This suggests the counterpart of Proposition 7.1.4.

Proposition 7.1.8 [Har01, Props. 6.11, 6.12] *Let* $X \hookrightarrow C$ *be some non-trivial function space on* $\Omega = \left[0, \frac{1}{2}\right]$ *with*

$$\mathfrak{e}^X \in X\ .$$

Then (*unless* $\mathfrak{e}^X$ *is a constant*) *this implies* $u^X_{\mathsf{C}} = \infty$, *i.e.* $\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}^X_{\mathsf{C}},\ \infty\right)$, *and* $\left\|\mathfrak{e}^X \mid X\right\| \ge 1$.

One observes that for our examples (7.1.17) it always holds $\left\|\mathfrak{e}^X \mid X\right\| \sim 1$. We review our results in Section 6.2.

Corollary 7.1.9 *Let all spaces be defined on* $\Omega = \left[0, \frac{1}{2}\right]$.

(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $0 < \sigma < 1$, *and* $s = \sigma + \frac{1}{p}$. *Then*

$$\mathfrak{e}^{B^s_{p,q}} \in B^s_{p,q} \qquad \textit{if, and only if,} \qquad q = \infty. \tag{7.1.18}$$

(ii) *Let* $0 < p \le \infty$, *and* $1 < q \le \infty$. *Then*

$$\mathfrak{e}^{B^{1+1/p}_{p,q}} \in B^{1+1/p}_{p,q} \qquad \textit{if, and only if,} \qquad q = \infty. \tag{7.1.19}$$

Theorems 6.2.1, 6.2.5 imply that only $B-$ spaces with $q = \infty$ can satisfy $\mathfrak{e}^X \in X$, see Proposition 7.1.8. So we have to show that $t^\sigma \in B^{\sigma+1/p}_{p,\infty}$ for $0 < \sigma < 1$, $0 < p \le \infty$ (at least locally), and $t|\log t| \in B^{1+1/p}_{p,\infty}$, $0 < p \le \infty$. For the super-critical case we proceed parallel to the sub-critical one in Corollary 7.1.5 (i), where (7.1.11) is now being replaced by

$$t^\sigma = \varphi(t)\ t^\sigma \sim \sum_{j=0}^{\infty} 2^{-j\left(\sigma+\frac{1}{p}-\frac{1}{p}\right)}\ \psi_j(t)\ \varphi(t)\ t^\sigma\ 2^{j\sigma}\ , \quad 0 < t < 1, \tag{7.1.20}$$

the rest is similar. Concerning (ii) we return to the extremal functions f_b as constructed by Triebel in [Tri01, (14.15)-(14.19), pp. 220/221]; see also (6.2.17). Put $b_j \equiv 1$, then this is essentially the integrated version of (7.1.12),

$$\sum_{j=1}^{\infty} 2^{-j+1}\Psi\left(2^{j-1}t\right)\varphi(2t)\ , \qquad \Psi(z) = \int_{-\infty}^{z} \psi(u)\,\mathrm{d}u\ , \tag{7.1.21}$$

where $\psi(t)$, $\varphi(t)$ are as above; note that we need no moment conditions. One checks that

$$\sum_{j=1}^{\infty} 2^{-j+1}\Psi\left(2^{j-1}t\right)\varphi(2t) \sim t\,|\log t|\ , \quad 0 < t < \frac{1}{2},$$

and (7.1.21) can be understood as the atomic decomposition of $t\,|\log t|$ (near 0). Now (6.2.17) and the particular choice of the sequence $b \in \ell_\infty$ imply $t\,|\log t| \in B^{1+1/p}_{p,\infty}$, $0 < p \le \infty$.

Remark 7.1.10 TRIEBEL studied a related question in [Tri01, Sect. 17.1, pp. 243-246], asking under what conditions there are functions $f \in A^s_{p,q}$ such that $f^*(t)$ or $\frac{\omega(f,t)}{t}$ are equivalent to the corresponding growth or continuity envelope functions. By the same arguments as above only $B-$ spaces with $q = \infty$ are left to consider; TRIEBEL applies these outcomes showing that certain Green's functions (of $(id - \Delta)^{-\frac{n}{2}}$ for the critical case, for instance) materialise the corresponding envelope functions.

7.2 Envelopes, lifts, and compact embeddings

We discover some links and consequences of the above topics which seem both surprising and promising. In future, there is certainly more fruit to be reaped of our previous studies.

7.2.1 Envelopes and lifts

Recall that $\mathcal{E}^X_{\mathsf{G}}(t)$ is bounded when $X \hookrightarrow L_\infty$, see Proposition 4.2.4 (iii), whereas $\mathcal{E}^X_{\mathsf{C}}(t)$ is only defined for $X \hookrightarrow C$. Thus it might not appear very interesting at first glance to study the interplay of $\mathcal{E}^{X_1}_{\mathsf{G}}$ and $\mathcal{E}^{X_2}_{\mathsf{C}}$ in general – at least not when the spaces X_1 and X_2 coincide, $X_1 = X_2$. We may, however, observe some phenomena granted that X_1 and X_2 are connected in a suitable way; we shall try to interprete and generalise this afterwards.

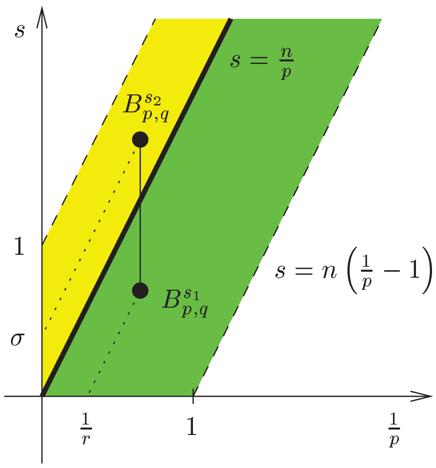


Figure 10

We consider the following situation. Let $0 < p < \infty$ and $0 < q \le \infty$. Assume (as indicated in Figure 10) that $s_1 = \frac{n}{p} - \frac{n}{r}$ for some r, $1 < r < \infty$, and $s_2 = \sigma + \frac{n}{p}$ for some σ with $0 < \sigma < 1$. We consider the case that $s_2 = s_1 + 1$; that is, where $\sigma = 1 - \frac{n}{r}$. (Note that the assumptions on σ thus imply $r > n$.) Furthermore, by Theorem 5.2.1 we know $\mathcal{E}^{A^s_{p,q}}_{\mathsf{G}}(t) \sim t^{-\frac{1}{r}}$, whereas Theorem 6.2.1 yields $\mathcal{E}^{A^{s+1}_{p,q}}_{\mathsf{C}}(t) \sim t^{-(1-\sigma)}$. Consequently we obtain in that case

$$\mathcal{E}^{A^{s+1}_{p,q}}_{\mathsf{C}}(t) \sim t^{-(1-\sigma)} = (t^n)^{-\frac{1}{r}} \sim \mathcal{E}^{A^s_{p,q}}_{\mathsf{G}}(t^n).$$

Likewise, for $0 < p < n$ and $0 < q \le 1$ Theorems 6.3.1 and 5.2.1 (with $r = n$) lead to

$$\mathcal{E}^{A^{n/p}_{p,q}}_{\mathsf{C}}(t) \sim t^{-1} = (t^n)^{-\frac{1}{r}} \sim \mathcal{E}^{A^{n/p-1}_{p,q}}_{\mathsf{G}}(t^n).$$

A similar behaviour can be observed when dealing with the borderline cases, $B^{n/p}_{p,q}$ and $B^{1+n/p}_{p,q}$, respectively,

$$\mathcal{E}^{B^{1+n/p}_{p,q}}_{\mathsf{C}}(t) \sim |\log t|^{\frac{1}{q'}} \sim \mathcal{E}^{B^{n/p}_{p,q}}_{\mathsf{G}}(t^n),$$

and a parallel result for the F-case. However, the $\log$-function spoils the interplay of t and t^n in that case. Turning to the envelopes $\mathfrak{E}_{\mathsf{G}}$ or $\mathfrak{E}_{\mathsf{C}}$, it thus appears reasonable to define

$$\mathfrak{E}^n_{\mathsf{G}}(X) := \left(\mathcal{E}^X_{\mathsf{G}}(t^n), u^X_{\mathsf{G}}\right),$$

where u^X_{G} is given as in Definition 4.2.10. Then Theorems 5.2.1 and 6.2.1, as well as Theorems 5.3.1 and 6.2.5 lead to

$$\mathfrak{E}^n_{\mathsf{G}}(A^s_{p,q}) = \mathfrak{E}_{\mathsf{C}}\left(A^{s+1}_{p,q}\right) \quad \text{if} \quad \begin{cases} 0 < p < \infty, & 0 < q \le \infty, & s = \frac{n}{p} - \frac{n}{r}, & \text{and} & n < r < \infty \\ 1 < p < \infty, & 0 < q \le \infty, & s = \frac{n}{p}, & \text{and} & A^s_{p,q} = F^s_{p,q} \\ 0 < p \le \infty, & 1 < q \le \infty, & s = \frac{n}{p}, & \text{and} & A^s_{p,q} = B^s_{p,q} \end{cases} \tag{7.2.1}$$

When $r = n$, i.e. $s = \frac{n}{p} - 1$, we have at least the corresponding result for the envelope functions,

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t^n) \sim \mathcal{E}_{\mathsf{C}}^{A^{s+1}_{p,q}}(t) , \tag{7.2.2}$$

see Theorems 5.2.1 (with $r = n$) and 6.3.1.

Does this reflect a more general behaviour, that is, in what sense can this particular result be extended ?

So far we only collected results 'associated' in the above sense, but achieved (almost) independently of each other. The more desirable was a direct link between $\frac{\omega(f,t)}{t}$ and $|\nabla f|^*(t^n)$ or $|\nabla f|^{**}(t^n)$) for, say, $f \in X \hookrightarrow C^1$. We return to Proposition 6.2.3, in particular to estimate (6.2.11),

$$\omega(f,t) \leq c \int_0^{t^n} s^{\frac{1}{n}-1} |\nabla f|^*(s) \, \mathrm{d}s \tag{7.2.3}$$

for $t > 0$ and all $f \in C^1(\mathbb{R}^n)$. Plainly, this estimate plays an essential role in our subsequent study of $\mathcal{E}_{\mathsf{C}}^{X_1}$ and $\mathcal{E}_{\mathsf{G}}^{X_2}$, where $X_1 \hookrightarrow C$ and $X_2 \subset L_1^{\mathrm{loc}}$ are such that $|\nabla f| \in X_2$ for $f \in X_1$ (this setting is motivated by our above observations). We first discuss the 'optimality' of (7.2.3). Recall that we have by (7.2.3) for $n = 1$,

$$\frac{\omega(f,t)}{t} \leq c \, |f'|^{**}(t) , \quad 0 < t < \varepsilon , \quad f \in C^1(\mathbb{R}) . \tag{7.2.4}$$

So one can ask whether a replacement of (6.2.11) in the sense of (7.2.4), i.e.

$$\frac{\omega(f,t)}{t} \leq c \, |\nabla f|^{**}(t^n) , \quad 0 < t < \varepsilon , \tag{7.2.5}$$

was true for all $f \in C^1(\mathbb{R}^n)$ and dimension $n > 1$. Obviously, (7.2.5) was sharper than (7.2.3), and also implied TRIEBEL'S result [Tri01, Prop. 12.16, p. 199] mentioned in Remark 6.2.4,

$$\frac{\omega(f,t)}{t} \leq c \, |\nabla f|^{**}\left(t^{2n-1}\right) + 3 \sup_{0<\tau\leq t^2} \tau^{-\frac{1}{2}} \, \omega(f,\tau) \tag{7.2.6}$$

for some small $\varepsilon > 0$ and all $0 < t < \varepsilon$ and all $f \in C^1(\mathbb{R}^n)$; we refer to [Har01, Sect. 6.3]. However, (7.2.5) cannot hold in general when $n > 1$; we give some argument disproving (7.2.5).

Assume (7.2.5) was true for $n > 1$. Let $f \in W_n^1(\mathbb{R}^n) = F_{n,2}^1(\mathbb{R}^n)$; by density arguments we may furthermore suppose that $f \in F_{n,2}^1(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$. Then by [Tri83, Thm. 2.3.8, p. 58] $|\nabla f| \in F_{n,2}^0 = L_n$, leading to $|\nabla f|^{**}(\tau) \leq C_n \, \tau^{-\frac{1}{n}}$, $\tau > 0$, and (7.2.5) then implies

$$\omega(f,t) \leq c \, t \, |\nabla f|^{**}(t^n) \leq c' t \, (t^n)^{-\frac{1}{n}} = c'$$

for small $t > 0$. In other words, all $f \in F_{n,2}^1(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$ (and by the usual density arguments then all $f \in F_{n,2}^1(\mathbb{R}^n)$, too) are (locally) bounded. This, however, is wrong : recall (5.3.1) with $p = n > 1$; cf. [ET96, 2.3.3 (iii), p. 45]. On the other hand, one can also rely on a result of STEIN in [Ste81] stating that if a function f on $\mathbb{R}^n$ satisfies $\nabla f \in L_{n,1}$ locally, then f is equi-measurable with a continuous function. Moreover, there is a remark that the result is sharp in the following sense : taking $g \notin L_{n,1}$ with $f = |x|^{-(n-1)} * g$, then there is a positive $\widetilde{g}$, equi-measurable with $|g|$, such that the resulting f is unbounded near every point; see also [Ste70, Ch. 8] and [Kol89, §5] for further details. So (7.2.3) – stating exactly that $|\nabla f|$ belongs to $L_{n,1}$ locally – is the best possible result (in that sense) and (7.2.5) – referring to $|\nabla f| \in L_n$ – cannot hold. The essential difference to the one-dimensional case is obvious in this setting as $L_{1,1} = L_1$, but $L_{p,1}(\mathbb{R}^n)$ is properly contained in $L_p(\mathbb{R}^n)$ for any $p > 1$.

Hence for $n > 1$ we are left with the two estimates (7.2.3) and (7.2.6) (instead of (7.2.5)) and try to compare them. At first glance it seems that our estimate (7.2.3) might be slightly sharper : though both estimates in question gave raise to the estimate (6.2.12), only (7.2.3) implies (6.2.13). The case $n = 1$ is clear : the second term in (7.2.6) disappears and we have (7.2.4) again.

Lemma 7.2.1 [Har01, Lemma 6.13] *Let $n > 1$. There is some $c > 0$ such that for all $0 < t < 1$ and all $f \in C^1(\mathbb{R}^n)$,*

$$\int_0^{t^n} s^{\frac{1}{n}-1} \; |\nabla f|^* (s) \, \mathrm{d}s \le \; c_1 \; t \; |\nabla f|^{**} \left(t^{2n-1}\right) + c_2 \; t^{2-\frac{1}{n}} \; \left\| f | C^1 \right\| \; . \tag{7.2.7}$$

Obviously the estimate for the second term on the right-hand side in (7.2.7) is very rough and can probably be improved. Following the proof in [Har01] one easily realizes, for instance, that the first term on the right-hand side of (7.2.7) can be reduced at the expense of the latter one,

$$\int_0^{t^n} s^{\frac{1}{n}-1} \; |\nabla f|^* (s) \, \mathrm{d}s \le \; c_\varepsilon \left[\, t \; |\nabla f|^{**} \left(t^{n+\varepsilon}\right) + \; t^{1+\varepsilon} \; \left\| f | C^1 \right\| \right] \; ;$$

this argument resembles [Tri01, Rem. 12.17, p. 202]. On the other hand, one verifies that a second term for 'compensation' is necessary in general; see [Har01, Sect. 6.3].

Comparing (7.2.3) and (7.2.6) we conclude that a combination of (7.2.3) and (7.2.7) results in an estimate less sharp than (7.2.6). On the other hand, due to the partly rather rough estimates in the proof of [Har01, Lemma 6.13] it is not yet clear, whether (7.2.6) or (7.2.3) are better in general. Nevertheless, for our purpose estimate (7.2.3) was completely sufficient; recall Proposition 6.2.3.

We come back to our 'lifting' problem for the envelopes. Let $X \subset L_1^{\mathrm{loc}}$ be some function space on $\mathbb{R}^n$ of regular distributions with, say, $X \not\hookrightarrow L_\infty$. Denote by $X^\nabla \subset X$ the following subspace

$$X^\nabla = \left\{ g \in L_1^{\mathrm{loc}} \; : \; g, \; |\nabla g| \in X \right\} \tag{7.2.8}$$

with

$$\left\| g | X^\nabla \right\| \; \sim \; \| g | X \| + \left\| \; |\nabla g| \; | X \right\| \; .$$

We assume that $X^\nabla \; \hookrightarrow \; C$; this setting is obviously motivated by $X = A^s_{p,q}$, see (6.2.19). In view of (7.2.1) and (7.2.2) we study the problem under which assumptions one has

$$\mathfrak{E}^n_{\mathsf{G}}(X) \; = \; \mathfrak{E}_{\mathsf{C}}\left(X^\nabla\right) \tag{7.2.9}$$

or, at least,

$$\mathcal{E}^X_{\mathsf{G}} \left(t^n\right) \; \sim \; \mathcal{E}^{X^\nabla}_{\mathsf{C}} (t) \; , \quad 0 < t < \varepsilon \; . \tag{7.2.10}$$

We have no complete answer, but a partial one.

Corollary 7.2.2 [Har01, Cor. 6.14] *Let the spaces X, X^∇ be given as above.*

(i) *There is some $c > 0$ such that*

$$\mathcal{E}^{X^\nabla}_{\mathsf{C}} (t) \; \le \; c \, \frac{1}{t} \int_0^{t^n} s^{\frac{1}{n}-1} \; \mathcal{E}^X_{\mathsf{G}}(s) \, \mathrm{d}s \; \sim \; \frac{1}{t} \int_0^t \mathcal{E}^X_{\mathsf{G}} \left(\sigma^n\right) \; \mathrm{d}\sigma \; = \; \left[\mathcal{M} \; \mathcal{E}^X_{\mathsf{G}} \left(\tau^n\right) \right] (t) \tag{7.2.11}$$

for all small t, $0 < t < \varepsilon$. Moreover, if there is some number $C > 0$ such that for all large $J \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \; 2^{-k} \; \frac{\mathcal{E}^X_{\mathsf{G}} \left(2^{-(k+J)n}\right)}{\mathcal{E}^X_{\mathsf{G}} \left(2^{-Jn}\right)} \; \le \; C \; , \tag{7.2.12}$$

then (7.2.11) *can be replaced by*

$$\mathcal{E}^{X^\nabla}_{\mathsf{C}} (t) \; \le \; c \, \mathcal{E}^X_{\mathsf{G}} \left(t^n\right) \; . \tag{7.2.13}$$

(ii) *Assume there is some number* $c > 0$ *such that for all* $k \in \mathbb{N}$,

$$\sum_{\nu=0}^{k} 2^{-\nu\varrho} \left[\frac{\mathcal{E}_{\mathsf{G}}^{X}\left(2^{-kn}\right)}{\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}\left(2^{-(k-\nu)}\right)} \right]^{r} \leq c \, , \tag{7.2.14}$$

where $\varrho < r = u_{\mathsf{G}}^{X}$ *(in case of* $r = u_{\mathsf{G}}^{X} \leq 1$ *we may admit* $\varrho = r$*). Then*

$$u_{\mathsf{C}}^{X^{\nabla}} \leq u_{\mathsf{G}}^{X} \, . \tag{7.2.15}$$

In particular, when $\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}(t) \sim \mathcal{E}_{\mathsf{G}}^{X}(t^n)$, (7.2.14) *can be replaced by*

$$\sum_{\nu=0}^{k} 2^{-\nu\varrho} \left[\frac{\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}\left(2^{-k}\right)}{\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}\left(2^{-(k-\nu)}\right)} \right]^{r} \leq c \, . \tag{7.2.16}$$

Clearly (7.2.12) is satisfied for

$$\mathcal{E}_{\mathsf{G}}^{X}(\tau) \sim \tau^{-\mu} \, |\log \tau|^{\varkappa} \qquad \text{with} \qquad \begin{cases} 0 < \mu < \frac{1}{n} & , \quad \varkappa \in \mathbb{R} \, , \\ \mu = 0 & , \quad \varkappa > 0 \, , \\ \mu = \frac{1}{n} & , \quad \varkappa < -1 \, ; \end{cases} \tag{7.2.17}$$

this covers all cases in (7.2.1) apart from the limiting case when $X = B^{n/p-1}_{p,q}$, $X^{\nabla} = B^{n/p}_{p,q}$, $0 < p < n$, $0 < q \leq 1$, – reflecting that (7.2.12) is only sufficient for (7.2.13). Concerning (ii), one observes that (6.2.12) and (6.2.13) are certain examples for (7.2.14) : the first one with $\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}(t) \sim \mathcal{E}_{\mathsf{G}}^{X}(t) \sim |\log t|^{u}$, $u > \frac{1}{r}$, whereas (6.2.13) is related to the setting $\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}(t) \sim t^{-(1-\varkappa)}$, $\mathcal{E}_{\mathsf{G}}^{X}(t) \sim t^{-\frac{1}{n}(1-\varkappa)}$, $0 < \varkappa < 1$; see [Har01, Sect. 6.3] for details. In view of Theorems 5.2.1 (with $r = n$) and 6.3.1 we have to check (7.2.16), reading now as the question whether

$$\sum_{\nu=0}^{k} 2^{-\nu\varrho} \left[\frac{\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}\left(2^{-k}\right)}{\mathcal{E}_{\mathsf{C}}^{X^{\nabla}}\left(2^{-(k-\nu)}\right)} \right]^{r} = \sum_{\nu=0}^{k} 2^{-\nu\varrho} \left[\frac{2^{k}}{2^{(k-\nu)}} \right]^{r} = \sum_{\nu=0}^{k} 2^{-\nu(\varrho-r)}$$

converges independently of $k \in \mathbb{N}$. This, however, fails because of $\varrho \leq r$. So condition (7.2.16) reflects the additional problems appearing on the critical line exactly.

Inequalities converse to (7.2.13) and (7.2.15) are missing so far; further studies in the sense of [JMP91] are necessary, and – in view of our results (7.2.1), (7.2.2) – also promising.

7.2.2 Envelopes and compactness

Finally we briefly discuss questions related to compactness (of certain embeddings). We already mentioned that – turning to spaces on bounded domains defined by restriction – most of our results for (growth or continuity) envelopes can be transferred immediately. Taking this for granted at the moment, it makes sense to study the following problem : Consider an embedding between two function spaces defined on a bounded domain, and ask whether there are consequences concerning its compactness (note that continuity is assumed) by means of their envelopes.
Let $X_i \subset L_1^{\mathrm{loc}}$ or $X_i \hookrightarrow C$, $i = 1, 2$, respectively, and denote by

$$\mathfrak{q}_{\mathsf{G}}^{(X_1,X_2)}(t) = \mathfrak{q}_{\mathsf{G}}(t) := \frac{\mathcal{E}_{\mathsf{G}}^{X_1}(t)}{\mathcal{E}_{\mathsf{G}}^{X_2}(t)} \quad , \qquad \mathfrak{q}_{\mathsf{C}}^{(X_1,X_2)}(t) = \mathfrak{q}_{\mathsf{C}}(t) := \frac{\mathcal{E}_{\mathsf{G}}^{X_1}(t)}{\mathcal{E}_{\mathsf{G}}^{X_2}(t)}, \qquad 0 < t < \varepsilon \, . \tag{7.2.18}$$

We may assume that $\varepsilon > 0$ is chosen sufficiently small, say, $\varepsilon < \tau_0^{\mathsf{G}}(X_2)$, given by (4.2.3), and $\varepsilon < \tau_0^{\mathsf{C}}(X_2)$, according to (4.3.3). Now Propositions 4.2.4 (iv) and 4.3.3 (iii) imply that there cannot be a continuous embedding $X_1 \hookrightarrow X_2$ at all whenever

$$\sup_{0<t<\varepsilon} \mathfrak{q}_{\mathsf{G}}(t) \;=\; \infty \;, \quad \text{or} \quad \sup_{0<t<\varepsilon} \mathfrak{q}_{\mathsf{C}}(t) \;=\; \infty \;. \tag{7.2.19}$$

So for a continuous embedding (not to speak of compactness so far) it is at least necessary that $\mathfrak{q}_{\mathsf{G}}(t)$ or $\mathfrak{q}_{\mathsf{C}}(t)$ are bounded. Moreover, granted the embedding $X_1 \hookrightarrow X_2$ was continuous, the boundedness of $\mathfrak{q}_{\mathsf{G}}(t)$, $\mathfrak{q}_{\mathsf{C}}(t)$ is not sufficient for its compactness : Triebel proved in [Tri01, 14.6, pp. 227/228] that, roughly speaking, some embedding cannot be compact when the envelopes of source and target spaces coincide, i.e. $\mathfrak{q}_{\mathsf{G}}(t) \;\sim\; 1$ or $\mathfrak{q}_{\mathsf{C}}(t) \;\sim\; 1$. Consequently the corresponding embedding

$$id : X_1(U) \;\longrightarrow\; X_2(U)$$

can only be compact when

$$\lim_{t\downarrow 0} \mathfrak{q}_{\mathsf{G}}(t) \;=\; 0 \;, \quad \text{or} \quad \lim_{t\downarrow 0} \mathfrak{q}_{\mathsf{C}}(t) \;=\; 0 \tag{7.2.20}$$

(if the corresponding limits exist). We return to this point after some digression linking entropy (and approximation) numbers and (continuity) envelopes more directly. This approach relies on a result of Carl and Stephani [CS90, Thm. 5.6.1, p. 178] estimating approximation numbers in terms of moduli of continuity. As we restricted ourselves in this report to the study of entropy numbers, we formulate the result below in this adapted setting. Moreover, we consider a simple example only and compare the outcome with already known results on entropy numbers.

We consider the following situation. Let U be the unit ball in $\mathbb{R}^n$, denote by id_X^1, id_X^2 the natural embedding operators

$$id_X^1 : X(U) \;\longrightarrow\; C(U) \;, \qquad id_X^2 : X(U) \;\longrightarrow\; B^{-1}_{\infty,\infty}(U)$$

where the spaces $X(U)$ are defined by restriction from their $\mathbb{R}^n-$ counterparts. We assume that the embeddings exist; in particular, we are mainly interested in the cases

$$X(U) \;=\; \begin{cases} A^s_{p,q}(U) & , \quad \frac{n}{p}+1 > s > \frac{n}{p} \;, \quad 0<p\le\infty, \quad 0<q\le\infty \\ & \text{or} \quad s = \frac{n}{p}+1, \quad 0<p<\infty, \; 1<q\le\infty \\ \mathrm{Lip}^{(1,-\alpha)}(U) & , \quad \alpha > 0 \end{cases} \tag{7.2.21}$$

concerning id_X^1, and

$$X(U) \;=\; \begin{cases} A^s_{p,q}(U) & , \quad \frac{n}{p} > s > \frac{n}{p}-1 \;, \quad s\ge 0, \quad 0<p\le\infty, \quad 0<q\le\infty \\ & \text{or} \quad s = \frac{n}{p} \;, \quad 0<p<\infty, \quad 1<q\le\infty \\ L_p(\log L)_a(U) & , \quad n<p<\infty, \; a\in\mathbb{R} \\ & \text{or} \quad p=n, \; a>0 \quad \text{or} \quad p=\infty, \; a\le 0 \end{cases} \tag{7.2.22}$$

in connection with id_X^2.

Then compactness of id_X^1 is guaranteed for spaces of type (7.2.21) : cf. [ET96, (2.5.1/10), p. 60], or (1.3.3) for the first assertion, and Corollary 3.4.17 (i) in connection with (1.3.2) for the second one. Likewise (1.3.3) and (1.3.2) cover the compactness of id_X^2 in the first line of (7.2.22), whereas it follows for the second one from (2.4.2) and another application of (1.3.3).

Corollary 7.2.3 *Let $X \subset L_1^{\mathrm{loc}}$ be some Banach-space defined on the unit ball U in $\mathbb{R}^n$. Let $f : \mathbb{N} \to \mathbb{R}$ be a positive and increasing function satisfying*

$$f\left(2^k\right) \;\le\; c\, f\left(2^{k-1}\right) \tag{7.2.23}$$

for some $c > 0$ and all $k \in \mathbb{N}$.

(i) *Assume* $X \hookrightarrow C$. *Then there is some* $C > 0$ *such that for all* $m \in \mathbb{N}$,

$$\sup_{1\le k\le m} f(k)\; e_k\left(id_X^1 : X(U) \longrightarrow C(U)\right) \;\le\; C \sup_{1\le k\le m} f(k)\; k^{-\frac{1}{n}}\; \mathcal{E}_{\mathsf{C}}^X\left(k^{-\frac{1}{n}}\right) . \tag{7.2.24}$$

(ii) *Assume* $X \not\hookrightarrow L_\infty$, *but with* $X \hookrightarrow B^{-1}_{\infty,\infty}$ *and* $X^\nabla \hookrightarrow C$. *Let* $\mathcal{E}_{\mathsf{G}}^X$ *satisfy* (7.2.12) *and assume that there is a bounded (linear) lift operator* L *mapping* $X(U)$ *into* $X^\nabla(U)$ *such that* L^{-1} *maps* $C(U)$ *into* $B^{-1}_{\infty,\infty}(U)$. *Then*

$$\sup_{1\le k\le m} f(k)\; e_k\left(id_X^2 : X(U) \longrightarrow B^{-1}_{\infty,\infty}(U)\right) \;\le\; C \sup_{1\le k\le m} f(k)\; k^{-\frac{1}{n}}\; \mathcal{E}_{\mathsf{G}}^X\left(k^{-1}\right) . \tag{7.2.25}$$

Part (i) is an immediate consequence of [CS90, Thm. 5.6.1, p. 178] and the definition of $\mathcal{E}_{\mathsf{C}}^X$ (leading directly to [Har01, Cor. 6.15] formulated for approximation numbers) and a general relation between entropy and approximation numbers; cf. [Car81, p. 294] and [CS90, p. 96] for the Banach case, and [ET96, Thm. 1.3.3, Rem. 2, pp. 15-17] for its extension as given above. The technicality dealing with $\overline{U}$ (in the original formulation in [CS90, Thm. 5.6.1, p. 178]) or U as above can be surmounted by extension procedures and further natural embeddings. Similarly one could also use [CS90, (5.7.1), Thm. 5.7.1, p. 185] leading to the same results in our cases. Concerning (ii) we stress lifting arguments, Corollary 7.2.2 (i) and (7.2.24). Having a bounded lift $L : X(U) \longrightarrow X^\nabla(U)$ with $L^{-1} : B^0_{\infty,\infty}(U) \longrightarrow B^{-1}_{\infty,\infty}(U)$, then the decomposition

$$id_X^2 \;=\; L^{-1} \;\circ\; \left(C(U) \hookrightarrow B^0_{\infty,\infty}(U)\right) \;\circ\; id^1_{X^\nabla} \;\circ\; L$$

together with the multiplicativity of entropy numbers, (1.2.5) and (7.2.24) yield

$$\sup_{1\le k\le m} f(k)\; e_k\left(id_X^2\right) \;\le\; C \sup_{1\le k\le m} f(k)\; k^{-\frac{1}{n}}\; \mathcal{E}_{\mathsf{C}}^{X^\nabla}\left(k^{-\frac{1}{n}}\right) ,$$

whereas the last step to (7.2.25) results from Corollary 7.2.2 (i).

Remark 7.2.4 In fact, Corollary 7.2.3 is rather an approximation number result (in its original intention), the transfer to entropy numbers causes the somewhat clumsy formulation; the spoilt elegance is due to our restriction on entropy numbers (instead of approximation numbers) from the very beginning of this report. The advantage of this procedure, however, lies in the possible comparisons of our results presented in both parts of this report – at least as far as entropy numbers are concerned. This was not possible to the same extent when dealing with approximation numbers exclusively.

Assume now that $\mathcal{E}_{\mathsf{C}}^X(t) \;\sim\; t^{-\mu}|\log t|^{\varkappa}$ with $\mu > 0$, $\varkappa \in \mathbb{R}$, or $\mu = 0$, $\varkappa > 0$; recall $\mathcal{E}_{\mathsf{C}}^C(t) \;\sim\; t^{-1}$ by (6.1.9). Thus by (7.2.20) we have to consider the cases $\mu < 1$, $\varkappa \in \mathbb{R}$, or $\mu = 1$, $\varkappa < 0$ only such that (7.2.24) eventually leads to

$$e_k(id_X^1) \;\le\; c \begin{cases} k^{-\frac{1-\mu}{n}}\left(\log\langle k\rangle\right)^{\varkappa} & , \quad 0 < \mu < 1 \quad , \quad \varkappa \in \mathbb{R} \\ k^{-\frac{1}{n}}\; \left(\log\langle k\rangle\right)^{\varkappa} & , \quad \mu = 0 \quad , \quad \varkappa > 0 \\ \left(\log\langle k\rangle\right)^{\varkappa} & , \quad \mu = 1 \quad , \quad \varkappa < 0 \end{cases} \tag{7.2.26}$$

Dealing with (ii) one firstly observes that $L_n(U) \hookrightarrow B^{-1}_{\infty,\infty}(U)$ continuously, and $\mathcal{E}_{\mathsf{G}}^{L_n}(t) \;\sim\; t^{-\frac{1}{n}}$; see (5.1.2). Moreover, all spaces $X(U)$ compactly embedded into $L_n(U)$ are then compactly embedded in $B^{-1}_{\infty,\infty}(U)$, too. Thus in view of (7.2.20) and (7.2.17) it makes at least sense to regard the following consequences of (7.2.25) when $\mathcal{E}_{\mathsf{G}}^X(t) \;\sim\; t^{-\mu}|\log t|^{\varkappa}$:

$$e_k(id_X^2) \;\le\; c \begin{cases} k^{-\frac{1}{n}+\mu}\left(\log\langle k\rangle\right)^{\varkappa} & , \quad 0 < \mu < \frac{1}{n} \quad , \quad \varkappa \in \mathbb{R} \\ k^{-\frac{1}{n}}\; \left(\log\langle k\rangle\right)^{\varkappa} & , \quad \mu = 0 \quad , \quad \varkappa > 0 \\ \left(\log\langle k\rangle\right)^{\varkappa} & , \quad \mu = \frac{1}{n} \quad , \quad \varkappa < -1 \end{cases} \tag{7.2.27}$$

Similarly one could argue that the right-hand side of (7.2.25) with $f \equiv 1$ cannot be finite otherwise.

We return to our examples (7.2.21), (7.2.22) and compare it with known results. We start with $X(U) = B^s_{p,q}(U)$, where $s > \frac{n}{p}$, $0 < p \leq \infty$, $0 < q \leq \infty$. (Strictly speaking, we had to restrict ourselves to $p, q \geq 1$ to meet exactly the Banach space assumption in the above corollary; however, as only estimates from above are concerned and the corresponding spaces with $0 < p, q < 1$ can be embedded in suitable Banach spaces, the multiplicativity of entropy numbers (1.3.1) covers all above cases.) In view of $B^0_{\infty,1} \hookrightarrow C \hookrightarrow B^0_{\infty,\infty}$, see (1.2.5), and (1.3.3) we have

$$e_k(id^1_B) \sim k^{-\frac{s}{n}} \tag{7.2.28}$$

for all $0 < p \leq \infty$, $s > \frac{n}{p}$, $0 < q \leq \infty$. On the other hand, (7.2.26) with $\mu = 1 - (s - \frac{n}{p})$ and Theorems 6.2.1, 6.2.5 lead to

$$e_k\left(id^1_B\right) \leq c \begin{cases} k^{-\frac{s}{n}+\frac{1}{p}} & , \quad 0 < s - \frac{n}{p} < 1, \quad 0 < q \leq \infty \\ k^{-\frac{s}{n}+\frac{1}{p}} \left(\log\langle k\rangle\right)^{\frac{1}{q'}} , & s = \frac{n}{p} + 1, \quad 1 < q \leq \infty \end{cases} \tag{7.2.29}$$

We briefly compare (7.2.28) and (7.2.29). One realizes that for $0 < s - \frac{n}{p} < 1$ (i.e. in the '*super-critical strip*') we are led to the correct upper estimates for $e_k\left(id^1_B\right)$ when $p = \infty$, whereas otherwise – on the '*super-critical* (*border–*)*line*' $s = \frac{n}{p} + 1$ – our method provides a less sharp upper bound only. The reasons, however, are obvious : firstly, our result Corollary 7.2.3 is originally a result for approximation numbers [Har01, Cor. 6.15], the transfer above gives usually satisfactory results in special cases only. In particular, in the super- or subcritical strips, respectively, we have the same envelopes for spaces with the same differential dimension δ; this corresponds exactly to the asymptotic behaviour of approximation numbers (unlike entropy numbers). On the other hand, as long as we are not in limiting situations (as it is the case with id^1_X and X given by (7.2.21), i.e. when $\delta > 0$), then the $q-$ index plays no role for the entropy numbers of the corresponding embeddings; however the continuity envelopes reflect this tricky 'almost' Lipschitzian continuity of functions $f \in B^s_{p,q}$ with $s = \frac{n}{p} + 1$, $1 < q \leq \infty$. Moreover, one could obviously complement (7.2.29) by $e_k\left(id^1_X\right) \leq c\, k^{-\frac{1}{n}}$ whenever $s = \frac{n}{p} + 1$ and $0 < q \leq 1$, or $s > \frac{n}{p} + 1$, $0 < q \leq \infty$. Clearly this is worse than (7.2.28) as our continuity envelope functions are 'made' for $0 \leq s - \frac{n}{p} \leq 1$ only; it is not at all surprising that we lose interesting information otherwise.
We study the second case in (7.2.21). Let $X(U) = \mathrm{Lip}^{(1,-\alpha)}(U)$, $\alpha > 0$. Then Proposition 6.1.4 (ii) and (7.2.26) yield

$$e_k\left(id^1_{\mathrm{Lip}}\right) \leq c\, k^{-\frac{1}{n}} \left(\log\langle k\rangle\right)^{\alpha} ;$$

this coincides with Corollary 3.4.17 (i) for that case, i.e. (3.4.22) with $s = 1$ (recall $B^0_{\infty,1} \hookrightarrow C \hookrightarrow B^0_{\infty,\infty}$).

Summarising these two examples, the rather astonishing observation from our point of view is the sharpness of the results in embedding situations 'well-adapted' to the context we studied with our envelopes : note that we combined a very general result of Carl and Stephani [CS90, Thm. 5.6.1, p. 178] with our envelope results, which grew up in absence of any compactness criteria. But at least in the above-described setting they meet exactly as they should !

We come to (ii) and our settings for X described in (7.2.22). When $X = B^s_{p,q}$, $\frac{n}{p} - 1 < s < \frac{n}{p}$, $s \geq 0$, $0 < p, q \leq \infty$, or $s = \frac{n}{p}$, $0 < p < \infty$, $1 < q \leq \infty$, then again by (1.3.3)

$$e_k(id^2_B) \sim k^{-\frac{s+1}{n}} \tag{7.2.30}$$

in all admitted cases. The counterpart of (7.2.29) is given by (7.2.27) with $\mu = \frac{1}{r} = \frac{1}{p} - \frac{s}{n}$ and Theorems 5.2.1, 5.3.1 such that (7.2.25) implies

$$e_k\left(id^2_B\right) \leq c \begin{cases} k^{-\frac{s+1}{n}+\frac{1}{p}} & , \quad \frac{n}{p} - 1 < s < \frac{n}{p}, \quad s \geq 0,\ 0 < p \leq \infty,\ 0 < q \leq \infty \\ k^{-\frac{s+1}{n}+\frac{1}{p}} \left(\log\langle k\rangle\right)^{\frac{1}{q'}} , & s = \frac{n}{p}, \quad 0 < p < \infty,\ 1 < q \leq \infty \end{cases} \tag{7.2.31}$$

Note that the existence of the lift operator L can be seen by applying usual restriction-extension procedures and the lift operator I_σ in $\mathbb{R}^n$ given by (1.1.33), which maps $B^s_{p,q}$ isomorphically onto $B^{s-\sigma}_{p,q}$ for all admitted parameters. Alternatively one can also use regular elliptic differential operators adapted to U; see [Tri78a, Thm. 4.9.2, p. 335] for the case $1<p<\infty$, $1\le q\le\infty$, and [Tri83, Thm. 4.3.4, p. 235] for the extensions to $0<p,q\le\infty$, which are based on more recent techniques of Fourier multipliers. The discussion of (7.2.30) and (7.2.31) copies the one related to (7.2.28) and (7.2.29); it is thus omitted. Finally, we come to $X=L_p(\log L)_a$ as given in (7.2.22). The existence of a bounded linear lift is covered by [ET96, Thm. 2.6.3, p. 79], at least for $n\le p<\infty$. Propositions 5.1.2 (ii) and 5.1.4 combined with (7.2.27) for $\mu=\frac{1}{p}$, $\varkappa=-a$, and (7.2.25) provide

$$e_k(id^2_{p;a}) \;\le\; c\begin{cases} k^{-\frac{1}{n}+\frac{1}{p}}\,(\log\langle k\rangle)^{-a} & , \quad n<p<\infty \quad , \quad a\in\mathbb{R} \\ k^{-\frac{1}{n}}\;(\log\langle k\rangle)^{-a} & , \quad p=\infty \quad , \quad a<0 \\ (\log\langle k\rangle)^{-a} & , \quad p=n \quad , \quad a>1 \end{cases} \tag{7.2.32}$$

We briefly compare it with known results. Clearly for $n<p<\infty$ and well-known embeddings like (2.2.6), i.e. $L_{p+\varepsilon}(U)\hookrightarrow L_p(\log L)_a(U)\hookrightarrow L_{p-\varepsilon}(U)$, we conclude in this non-limiting situation from (1.3.3) for all $a\in\mathbb{R}$ that

$$e_k(id^2_{p;a}) \;\sim\; k^{-\frac{1}{n}}\,,$$

which is obviously better than (7.2.32). Let $p=n$, then by (2.4.14)

$$e_k\left(id: L_n\left(\log L\right)_a(U)\to H^{-s}_r(U)\right) \;\sim\; k^{-\min(a,\frac{s}{n})}\,,\quad k\in\mathbb{N}, \tag{7.2.33}$$

assuming that $n=p<r<\infty$, $s>0$ with $\frac{1}{r}=\frac{1-s}{n}$, and $a\ne\frac{s}{n}$, $a>0$. Using the multiplicativity of entropy numbers as well as the embeddings $H^{-s}_r=F^{-s}_{r,2}\hookrightarrow B^{-s}_{r,\infty}\hookrightarrow B^{-1}_{\infty,\infty}$, see (1.1.34), (1.2.6), (1.2.3), this leads to

$$e_k(id^2_{n;a}) \;\le\; c_r\; k^{-\min(a,\frac{1}{n}-\frac{1}{r})}$$

for any number r, $n<r<\infty$, and $a\ne\frac{1}{n}-\frac{1}{r}$. Choosing r suitably, this can be reformulated into

$$e_k(id^2_{n;a}) \;\le\; \begin{cases} c_\varepsilon\; k^{-\frac{1}{n}+\varepsilon} & , \quad a\ge\frac{1}{n} \\ c\;k^{-a} & , \quad 0<a<\frac{1}{n} \end{cases}$$

for any small $\varepsilon>0$. Though no final result is achieved so far it is rather unlikely that the last line of (7.2.32) gives the correct upper bound, as (for sufficiently large a, say, $a>\frac{1}{n}$) one would rather guess a behaviour like $e_k(id^2_{n;a})\;\le\;c\,k^{-\frac{1}{n}}$ (with some additional term depending on a possibly) in view of (7.2.33). A similar argument held for the case $p=\infty$ where one has to care for the required linear lift (to apply Corollary 7.2.3 (ii)) additionally. We do not pursue this point further at the moment.

By the arguments stressed above it appears that the problem to determine $e_k(id^X_2)$ in Corollary 7.2.3 (ii) might not be well-adapted to our knowledge on growth envelopes which we want to apply. Though the target space $B^{-1}_{\infty,\infty}(U)$ cannot be avoided by our lifting procedure and the intended application of (i) of Corollary 7.2.3 we do not benefit enough from the continuous embedding $L_n(U)\hookrightarrow B^{-1}_{\infty,\infty}(U)$ – from the point of entropy numbers. In other words, the target space $B^{-1}_{\infty,\infty}(U)$ might be 'too far away' from the (sub-) critical strip where the (spaces having) growth envelopes live. This does not affect the approximation numbers very much as they show the same asymptotic behaviour along (compact embeddings of) spaces having the same differential dimension δ; we already mentioned this fact above. So for approximation numbers it does not matter whether the target space is $L_n(U)$, $B^{-1}_{\infty,\infty}(U)$, or something in between, i.e. $A^s_{p,q}$ with $\delta=s-\frac{1}{p}=-\frac{1}{n}$ (as long as one sticks with p at the same side of 2 compared with the source space, but for $n\ge 2$ and $A^s_{p,q}$ between $L_n(U)$ and $B^{-1}_{\infty,\infty}(U)$ this is satisfied); in contrast to that, entropy numbers distinguish between $L_n(U)$, $B^{-1}_{\infty,\infty}(U)$, and some intermediate $A^s_{p,q}$ with $s-\frac{1}{p}=-\frac{1}{n}$ as target spaces essentially, as they go with the difference in smoothness between source and target space asymptotically.

Remark 7.2.5 We already mentioned that the natural approach to (7.2.26) relies on approximation numbers instead of entropy numbers; here the results are even more convincing. Moreover, our results on envelopes can thus be applied to obtain related (upper) estimates for approximation numbers of compact embeddings in a rather elegant way. This works also in cases not studied separately before, say,

$$a_k \left(id : L_p(\log L)_a(U) \to B^{-1}_{\infty,\infty}(U) \right) ,$$

with $n < p < \infty$, $a \in \mathbb{R}$. A further study of related questions will be carried out in the near future. In that sense the difficulty mentioned in Remark 7.2.4 (that we do not have approximation number results in all cases we would like to compare) can immediately be turned into its contrary : it offers some interesting cases to apply our envelope results very effectively.

Finally we return to Corollary 7.2.3 from a more abstract point of view; i.e. we have a closer look on the structure of the right-hand sides of (7.2.24) and (7.2.25). Note that Proposition 6.1.4 (iv) together with the definition of $\mathfrak{q}_{\mathsf{C}}^{(X_1,X_2)}$ reveals that the entropy numbers of $id^1_X : X(U) \longrightarrow C(U)$ are estimated at the expense of $\mathfrak{q}_{\mathsf{C}}^{(X,C)}(t)$, i.e.

$$\sup_{1\leq k\leq m} f(k)\; e_k \left(id^1_X : X(U) \longrightarrow C(U) \right) \;\leq\; C \sup_{1\leq k\leq m} f(k)\; \mathfrak{q}_{\mathsf{C}}^{(X,C)} \left(k^{-\frac{1}{n}} \right) .$$

The counterpart for Corollary 7.2.3 (ii) is given by

$$\sup_{1\leq k\leq m} f(k)\; e_k \left(id^2_X : X(U) \longrightarrow B^{-1}_{\infty,\infty}(U) \right) \;\leq\; C \sup_{1\leq k\leq m} f(k)\; \mathfrak{q}_{\mathsf{G}}^{(X,L_n)} \left(k^{-1} \right) ,$$

where L_n may be replaced by any space $A^0_{n,q}$, $0 < q \leq \infty$, as long as $A^0_{n,q} \subset L_1^{\mathrm{loc}}$. So it appears reasonable to ask in what sense this can be generalised for embeddings $id : X_1(U) \longrightarrow X_2(U)$. This study promises to be interesting in future.

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Symbols

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Ehrenwörtliche Erklärung

Ich erkläre hiermit, daß mir die Habilitationsordnung der Friedrich-Schiller-Universität Jena vom 30. 4. 1997 bekannt ist.

Ferner erkläre ich, daß ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

Weitere Personen waren an der inhaltlich-materiellen Erstellung der Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Arbeit stehen.

Die Arbeit wurde bisher weder im In- noch Ausland in gleicher oder ähnlicher Form einer anderen Prüfbehörde vorgelegt.

Ich versichere, daß ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, den 7. Januar 2002

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