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## INPUT CONSTRAINED ADAPTIVE TRACKING WITH APPLICATIONS TO EXOTHERMIC CHEMICAL REACTION MODELS\*

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**Abstract.** We consider input constrained adaptive output feedback control for a class of nonlinear systems which are prototype models for controlled exothermic chemical reactions. Our objective is set-point control of the output, i.e., the temperature of the reaction. In the context of chemical reactions, practical considerations lead us to work in the presence of input constraints. We adopt an approach based on modified  $\lambda$ -tracking controllers, whereby prespecified asymptotic tracking accuracy, quantified by  $\lambda > 0$  set by the designer, is ensured. The adaptive control strategy does not require any knowledge of the system's parameters and does not invoke an internal model. Only a feasibility assumption in terms of the reference temperature and the input constraints is assumed.

**Key words.** adaptive control, exothermic chemical reaction models, global stabilization, input saturation, tracking

**AMS subject classifications.** 93C20, 93D40

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**1. Introduction.** In this paper, we consider input constrained adaptive output feedback control for a class of nonlinear systems which arise as prototype models for controlled exothermic chemical reactions. The output of the system is the reaction temperature, and primarily we control the rate of change of reaction temperature. Secondary control is achieved via dilution, specifically by feedrate control of reactants. Our objective is set-point control of the output, i.e., the temperature of the reaction. In the context of chemical reactions, since the rate of conversion of product into reactant should be economically profitable, this set-point temperature is often close to a hyperbolic equilibrium of the open-loop system. Additional practical considerations lead us also to work in the presence of input constraints. We adopt an approach based on modified  $\lambda$ -tracking controllers [7]. We are motivated by results obtained by Viel, Jadot, and Bastin [12] for similar prototype chemical reaction models. Our aims are two-fold: to show that the  $\lambda$ -tracking approach can be developed for this relevant class of nonlinear systems, and moreover to show that input constraints are allowed. Of particular interest is the interplay between the input constraints, the specific nature of the nonlinearities in chemical reaction models, and the set-point to be tracked.

In chemical engineering, the analysis and control of exothermic continuous stirred tank reactors (ExCSTRs) originated in [2]. They have subsequently been used extensively as models in several industries including continuous polymerization reactors, distillation columns, biochemical fermentation, and biological processes. More recently, for the prototype class of chemical reaction models used in this paper, various

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nonadaptive control theory approaches have been developed for the set-point control of temperature. Specifically, in [12] a state feedback controller, with observer, was proposed for globally stabilizing the temperature of ExCSTRs; in [9] (adaptive) dynamic output PI type controllers were derived, and similar stabilization results were obtained in [1].

Whilst we are motivated by the issues raised and the results in [12, 9, 1], we adopt a different approach based on adaptive  $\lambda$ -tracking. This means that asymptotically a prespecified, arbitrarily small accuracy  $\lambda > 0$  of the tracking error is ensured; see [6, 7]. This  $\lambda$ -tracking technique is well suited to classes of systems with “strict relative-degree” one, which include models for temperature control in the prototype exothermic reactions. It is therefore reasonable to expect that  $\lambda$ -trackers would be well-suited in this context of exothermic reactions. However, their direct application is not so straightforward because of the input constraints and also the need to find alternatives to the “minimum phase assumptions” typical in  $\lambda$ -tracking. In fact, instead of “minimum phase assumptions” we need a certain feasibility assumption which essentially captures the interplay between the input constraints, the specific nonlinearity in the exothermic reaction model, and the set-point (temperature) to be tracked. In the case of global set-point control, we also need to accommodate more ad hoc, non-relative-degree one, control action via dilution rates. To some extent, in modifying the  $\lambda$ -tracking technique, we are guided by the developments in [12]. However, our results actually go further in that we tolerate disturbances to the temperature measurement and also parameters of the system model are not invoked in the controller. We also overcome some of the drawbacks in the previous approaches in [12], and also [9] and [1], which need state feedback, or have complicated controller structure, or else require the system to be minimum phase (i.e., have exponentially stable zero dynamics).

We consider the following class of nonlinear systems:

$$(1) \quad \begin{cases} \dot{x}(t) = Cr(x(t), T(t)) + d[x^{\text{in}} - x(t)], & x(0) = x^0 \in \mathbb{R}_{\geq 0}^n, \\ \dot{T}(t) = b^T r(x(t), T(t)) - qT(t) + u(t), & T(0) = T^0 \in \mathbb{R}_{> 0}. \end{cases}$$

In (1),  $n \in \mathbb{N}$  and the constants and variables represent the following for  $m \in \mathbb{N}$  with  $n > m$ :

$x(t) \in \mathbb{R}_{\geq 0}^n$	concentrations of $n$ chemical species,
$T(t) \in \mathbb{R}_{> 0}$	temperature of the reactor,
$u(t) \in \mathbb{R}_{\geq 0}$	control, a combination of the temperatures of reactant feed and coolant,
$x^{\text{in}} \in \mathbb{R}_{\geq 0}^n$	constant feed concentrations,
$C = [c_1, \dots, c_m] \in \mathbb{R}^{n \times m}$	stoichiometric matrix,
$b \in \mathbb{R}_{\geq 0}^m$	coefficients of the exothermicity,
$d > 0$	dilution rate,
$q > 0$	heat transfer rate between heat exchanger and reactor.

The function

$$(2) \quad r(\cdot, \cdot) : \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{> 0} \rightarrow \mathbb{R}_{\geq 0}^m$$

is locally Lipschitz with  $r(0, T) = 0$  for all  $T > 0$  and models the reaction kinetics.

In the context of chemical reactions, practical considerations lead us to assume that the control input  $u(\cdot)$  is constrained so that there exist  $\underline{u}$  and  $\bar{u}$  with  $0 < \underline{u} < \bar{u}$

so that

$$(3) \quad \underline{u} \leq u(t) \leq \bar{u} \quad \text{for all} \quad t \geq 0.$$

*Remark 1.*

- (i) Nonlinear systems of the form (1) have been used extensively in the last thirty years as simplified models for ExCSTR models, both mathematically and in industrial applications. Their relevance was established in [3].
- (ii) The values of  $\underline{u}$  and  $\bar{u}$  will depend on the specific application. In our work, and also in [12] and [9], they are fixed numbers which then feature strongly in the assumptions needed so as to prove convergence for the control schemes.

To make sense of (1) as a model for exothermic reactions, we make the following assumptions.

- (A1)  $\mathbb{R}_{\geq 0}^n \times \mathbb{R}_{> 0}$  is positively invariant under (1) for any bounded, nonnegative, locally integrable  $u(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ .
- (A2) There exists  $\gamma \in \mathbb{R}_{> 0}^n$  such that  $\gamma^T c_i \leq 0$  for all columns  $c_1, \dots, c_m$  of the stoichiometric matrix  $C$ .
- (A3) For  $T^* > 0$  there exist  $0 \leq \underline{u} < \bar{u}$  such that

$$\underline{u} < qT^* - b^T r(x, T^*) < \bar{u} \quad \text{for all} \quad x \in \Omega(\gamma, x^{\text{in}}) := \{x \in \mathbb{R}_{\geq 0}^n \mid \gamma^T x < \gamma^T x^{\text{in}}\}.$$

*Remark 2.*

- (i) The system (1) and assumptions (A1)–(A3) capture the essential features of ExCSTRs. They give rise to a class of nonlinear systems for which a  $\lambda$ -tracking approach seems plausible, whilst the interplay between the nonlinearity, input constraints, and the feasibility assumption provides novelty in controller design and convergence proofs.
- (ii) The assumption (A1) is natural for exothermic reactions. Indeed, concentrations and temperature should not become zero once they are positive. In fact, since  $r(\cdot, \cdot)$  is nonnegative, if  $u(\cdot)$  is nonnegative, then it is clear that  $T(t) > 0$  whenever  $T^0 > 0$ . It is easy to show that the remainder of (A1) holds automatically when  $n = 2$ , i.e., in the case of a single reaction. For multiple reactions, there are various conditions (see, e.g., [8, Proposition 6]) in terms of specific rates which imply that (A1) holds. (A1) has been formulated for the closed positive orthant  $\mathbb{R}_{\geq 0}^n$  of the concentrations and the open half line for the temperature. The latter is natural since the reactor should not operate with zero or negative temperature; the former could also be assumed for the open positive orthant  $\mathbb{R}_{> 0}^n$ ; the analysis goes through without any changes.
- (iii) (A2) holds if (1) satisfies the law of conservation of mass, which means that there exists  $\gamma \in \mathbb{R}_{> 0}^n$  with  $\gamma^T C = 0$ . This can be found implicitly in [4], and it is also assumed in [12]. If  $C$  does not represent exactly the stoichiometric relationships between all species, then conservation of mass need not be satisfied. Nevertheless, the reaction model might still be relevant provided that all essential reactions are obeyed. This approach was adopted in [3] and also in [8]. In [8] a concept of a noncyclic process was developed and shown to ensure dissipativity of mass and hence that (A2) is satisfied.
- (iv) (A3) is simply a feasibility assumption arising because of the saturation of the nonnegative input  $u(\cdot)$  at  $\underline{u}$  and  $\bar{u}$ . Assumption (A3) coincides with (H3) in [12].

Note that, by continuity of  $r(\cdot, \cdot)$ , assumption (A3) implies, for some  $\underline{T}$ ,  $\bar{T}$ , and small enough  $\rho > 0$ , the assumption

(A3') For  $T^* > 0$  there exist  $0 < \underline{T} < T^* < \bar{T}$ ,  $\rho > 0$ ,  $0 < \underline{u} < \bar{u}$ , such that

$$0 < \underline{u} + \rho < qT - b^T r(x, T) < \bar{u} - \rho \quad \text{for all } (x, T) \in \Omega(\gamma, x^{\text{in}}) \times [\underline{T}, \bar{T}].$$

We will work with (A3') rather than with the weaker (A3) for the following two reasons: The explicit introduction of  $\rho$  makes the exposition in the proofs clearer, and in some of the results we need to use explicit knowledge of  $[\underline{T}, \bar{T}]$  so that (A3') holds for a given  $\rho$ .

The control objective is to regulate the temperature  $T(t)$  toward a prespecified neighborhood of a given reference temperature  $T^*$ . In specific applications,  $T^*$  would correspond to a desirable, but possibly unstable, set-point temperature.

The actual error between  $T^*$  and  $T(t)$  is denoted by

$$\hat{e}(t) = T^* - T(t),$$

and, since the temperature measurement may be corrupted by disturbances, we denote by  $e(t)$  the measured error, i.e.,

$$e(t) = T^* - T(t) + \xi(t).$$

We assume that the disturbance signal  $\xi(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a continuous bounded function.

To achieve the control objective, we use a  $\lambda$ -tracking controller

$$(4) \quad \begin{cases} e(t) &= T^* - T(t) + \xi(t), \\ u(t) &= \text{sat}_{[\underline{u}, \bar{u}]}(\beta(t)e(t) + u^*), \\ \dot{\beta}(t) &= \kappa \begin{cases} (|e(t)| - \lambda)^l & \text{if } |e(t)| > \lambda, \\ 0 & \text{if } |e(t)| \leq \lambda, \end{cases} \quad \beta(0) = \beta^0, \end{cases}$$

and variations thereof. Here  $\lambda > 0$  specifies the tolerance of the tracking error;  $l \geq 1$ ,  $\kappa, \beta^0 > 0$  are design parameters, and  $u^* \in (\underline{u}, \bar{u})$  is a constant offset. Significantly, the controller involves a saturation function

$$\text{sat}_{[\underline{u}, \bar{u}]}(\eta) := \begin{cases} \underline{u} & \text{if } \eta < \underline{u}, \\ \eta & \text{if } \eta \in [\underline{u}, \bar{u}], \\ \bar{u} & \text{if } \eta > \bar{u}. \end{cases}$$

*Remark 3.* Note the simplicity of the adaptive  $\lambda$ -tracker. It consists of a proportional error feedback with saturation and a time-varying proportional gain  $\beta(\cdot)$  determined adaptively by the error measurement alone. However, the design parameters should be carefully chosen when the feedback controller is applied to a real process. The upper bound  $\bar{u}$  depends not only on the feasibility condition (A3') but also on the physical limitations of the actuator. When both conditions are compatible, i.e., the actuator limit is higher than the bound in (A3'), one should choose  $\bar{u}$  close to the actuator upper bound to avoid unnecessary cut off by the saturation bound.

To specify  $\lambda$  appropriately, one needs to know in advance an estimate of the upper bound for the magnitude of the measurement accuracy and disturbance signal. The power  $l$  in the gain adaptation influences the speed of the adaptation. If the difference  $(|e(t)| - \lambda)$  is smaller than 1, then a bigger  $l \geq 1$  gives a slower increase in  $\beta(t)$ ; if the difference is bigger than 1, then the bigger  $l$  is the faster  $\beta(t)$  increases. Similar effects can be achieved by varying  $\kappa$  or the initial gain  $\beta^0$ . The constant  $u^*$  is an input reference, an appropriate choice for which might be known from experiments with constant feedback. Note also that in applications any information specific to the chemical reaction of interest would be used to make additional modifications to the  $\lambda$ -tracking controller so as to fine-tune the performance.

Although our emphasis is on the adaptive controller (4), we also consider the nonadaptive version

$$(5) \quad \boxed{u(t) = \text{sat}_{[\underline{u}, \bar{u}]}(\beta(t)e(t) + u^*), \quad \beta(\cdot) : \mathbb{R}_{\geq 0} \rightarrow [\beta^*, \infty) \text{ continuous.}}$$

Although the gain  $\beta(t) \geq \beta^*$  in this nonadaptive controller might be conservatively too large, this nonadaptive controller is useful because it is even simpler than the already simple (4). We give explicit lower bounds for  $\beta^*$  in terms of weak conditions on the system data.

Throughout the paper we assume that the saturation bounds, the offset, the temperature set-point, and  $\lambda$  satisfy

$$(6) \quad 0 < \underline{u} < u^* < \bar{u}, \quad 0 < \lambda < \bar{T} - T^*, \quad 0 < \underline{T} < T^* < \bar{T}.$$

The paper is organized as follows. In section 2 we consider local (adaptive and nonadaptive)  $\lambda$ -set-point control in the sense that the initial temperature  $T^0$  belongs to  $(0, \bar{T})$ . We prove additional properties of the closed-loop system in the special case of a single reactant and a single product. In section 3 we consider the global tracking problem in the sense that we assume only  $T^0 > 0$ . This problem is solved by introducing a feedback control for the feedrate of reactants which has the effect of reducing the concentration of the reactants if the temperature of the reaction is too high. We make some conclusions in section 4. To help the presentation flow, we prove most of the results in the appendix.

**2.  $\lambda$ -set-point control for  $T^0 \in (0, \bar{T})$ .** In this section we consider local  $\lambda$ -set-point control in the sense that the initial temperature  $T^0$  is constrained in the interval  $(0, \bar{T})$ . We present two feedback strategies which force the temperature into a  $\lambda$ -neighborhood of the given setpoint. The first is nonadaptive, whilst the second is adaptive.

**PROPOSITION 4.** *Suppose (6), (A1), (A2), (A3') hold, and the continuous disturbance satisfies*

$$(7) \quad \sup_{t \geq 0} \{|\xi(t)|\} =: \|\xi\|_\infty < \bar{T} - T^*.$$

*If the initial data of (1) satisfy  $(x^0, T^0) \in \Omega(\gamma, x^{\text{in}}) \times (0, \bar{T})$ , then the feedback (5) with*

$$(8) \quad \beta^* \geq [u^* - \underline{u}] / [\bar{T} - T^* - \|\xi\|_\infty]$$

*applied to (1) yields a unique solution*

$$(9) \quad (x(\cdot), T(\cdot)) : \mathbb{R}_{\geq 0} \rightarrow \Omega(\gamma, x^{\text{in}}) \times (0, \bar{T}), \quad t \mapsto (x(t), T(t)).$$

If (8) is strengthened to

$$(10) \quad \beta^* \geq \max \left\{ \frac{u^* - \underline{u}}{\bar{T} - T^* - \|\xi\|_\infty}, \frac{\bar{u} - u^*}{\lambda}, \frac{u^* - \underline{u}}{\lambda} \right\},$$

then there exists  $t' \geq 0$  such that

$$(11) \quad T(t) \in [T^* - \lambda - \|\xi\|_\infty, T^* + \lambda + \|\xi\|_\infty] \quad \text{for all } t \geq t'.$$

Proposition 4 is proved in the appendix.

*Remark 5.* In Proposition 4, it is ensured that the set  $\Omega(\gamma, x^{\text{in}}) \times (0, \bar{T})$  (where  $\Omega(\gamma, x^{\text{in}})$  denotes the generalized triangle as defined in (A3)) remains positively invariant under the closed-loop system (1), (5); more importantly, after some finite time, the temperature  $T(t)$  is within the  $(\lambda + \|\xi\|_\infty)$ -neighborhood of the reference temperature. The width  $\lambda > 0$  of the strip around the reference temperature is prespecified, but the neighborhood is corrupted by  $\|\xi\|_\infty$ . The condition in (7) requires that the amplitude of the measurement disturbance must be sufficiently small when compared to  $\bar{T} - T^*$ . Note also that the feedback gain  $\beta(\cdot)$  must be large enough.

The following remark provides some intuition behind the dynamics of the closed-loop system (1), (5).

*Remark 6.* Consider the closed-loop system (1), (5). For any initial condition  $(x^0, T^0) \in \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{> 0}$ , there exists a unique continuously differentiable solution on a maximally extended interval  $[0, \omega)$ ,  $\omega \in (0, \infty]$ . This is a standard result of the theory of ordinary differential equations following from (2).

In the following we show that  $\Omega(\gamma, x^{\text{in}}) \times (0, \bar{T})$ , where  $\Omega(\gamma, x^{\text{in}})$  denotes the generalized triangle as defined in (A3), is invariant under (1), (5). Therefore, boundedness of  $(x(\cdot), T(\cdot))$  yields  $\omega = \infty$ ; i.e., finite escape time cannot occur.

- (i) Suppose (A1), (A2) hold. We show that for any initial data  $(x^0, T^0) \in \Omega(\gamma, x^{\text{in}}) \times (0, \bar{T})$ , the  $x(t)$  component of the solution (1), (5) remains in  $\Omega(\gamma, x^{\text{in}})$  for all  $t \in [0, \omega)$ . In particular,  $x(\cdot)$  is bounded on  $[0, \omega)$ .

To see this, we note from (A1) that we need only to show that  $\gamma^T x(t) < \gamma^T x^{\text{in}}$  for all  $t \in [0, \omega)$ . This follows from integration of

$$\frac{d}{dt} \gamma^T x(t) = \gamma^T C r(x(t), T(t)) + d \gamma^T [x^{\text{in}} - x(t)],$$

which yields, by invoking (A2) and  $\gamma^T x(0) < \gamma^T x^{\text{in}}$ , for all  $t \in [0, \omega)$ ,

$$\gamma^T x(t) \leq e^{-dt} \gamma^T x(0) + d \int_0^t e^{-d(t-\tau)} d\tau \gamma^T x^{\text{in}} = \gamma^T x^{\text{in}} - e^{-dt} [\gamma^T x^{\text{in}} - \gamma^T x(0)] \leq \gamma^T x^{\text{in}}.$$

- (ii) From Remark 2(i), if  $T^0 > 0$ , then  $T(t) > 0$  for all  $t \in [0, \omega)$ . Now suppose that (A1), (A2), (A3'), (7), and (8) hold. Now to see that  $T(t) < \bar{T}$  for all  $t \in [0, \omega)$ , first note that from (i) we have that  $x(t) \in \Omega(\gamma, x^{\text{in}})$  for all  $t \in [0, \omega)$ . Seeking a contradiction, suppose there exists  $t' \in [0, \omega)$  such that  $T(t') = \bar{T}$  and  $T(t) < \bar{T}$  for all  $t \in [0, t')$ .

Then by (8) we have that

$$\beta(t') e(t') \leq \beta(t') [T^* - \bar{T} + \|\xi\|_\infty] \leq \underline{u} - u^*,$$

and hence  $u(t') = \underline{u}$ . Using the feasibility condition (A3') yields

$$\dot{T}(t') = b^T r(x(t'), \bar{T}) - q\bar{T} + \underline{u} < -\rho,$$

and this contradicts the assumption. It follows that if  $T^0 \in (0, \bar{T})$ , then  $T(t) \in (0, \bar{T})$  for all  $t \in [0, \omega)$ .

In the following theorem, we show that it is possible to determine a sufficiently large  $\beta(\cdot)$  in (5) adaptively.

**THEOREM 7.** *Suppose (6), (A1), (A2), (A3') hold, and the continuous disturbance satisfies*

$$(12) \quad \sup_{t \geq 0} \{|\xi(t)|\} =: \|\xi\|_\infty < \lambda/2.$$

*Then an application of the  $\lambda$ -tracker (4) to any system (1) yields, for any initial data*

$$(13) \quad (x^0, T^0) \in \Omega(\gamma, x^{\text{in}}) \times (0, \bar{T}), \quad \beta^0 \geq [u^* - \underline{u}]/[\bar{T} - T^* - \|\xi\|_\infty],$$

*a closed-loop system with unique solution*

$$(14) \quad (x(\cdot), T(\cdot), \beta(\cdot)) : \mathbb{R}_{\geq 0} \longrightarrow \Omega(\gamma, x^{\text{in}}) \times (0, \bar{T}) \times \mathbb{R}_{> 0}$$

*defined on the whole time axis  $\mathbb{R}_{\geq 0}$  and, moreover,*

- (i)  $\lim_{t \rightarrow \infty} \beta(t) = \beta_\infty \in \mathbb{R}_{> 0}$ , *i.e., adaptation of the gain is convergent,*
- (ii)  $\lim_{t \rightarrow \infty} \text{dist}(|T^* - T(t)|, [0, \lambda + \|\xi\|_\infty]) = 0$ , *i.e., the temperature  $T(t)$  tends to the  $[\lambda + \|\xi\|_\infty]$ -strip  $[T^* - [\lambda + \|\xi\|_\infty], T^* + [\lambda + \|\xi\|_\infty]]$  as  $t \rightarrow \infty$ .*

Theorem 7 is proved in the appendix.

Note that the only information needed for the  $\lambda$ -tracker (4) to work is that the initial gain parameter  $\beta(0)$  is sufficiently large as determined from knowledge of the upper feasibility bound  $\bar{T}$  and  $\|\xi\|_\infty$ ; see (13). This has advantages when compared to the nonadaptive controller (5) in Proposition 4, which requires the stronger condition (10). The nonadaptive result in Proposition 4 guarantees that the temperature  $T(t)$  remains in the  $[\lambda + \|\xi\|_\infty]$ -strip after some finite time, but this time is unknown, whereas Theorem 7 ensures that  $T(t)$  approaches the  $[\lambda + \|\xi\|_\infty]$ -strip asymptotically.

To conclude this section, we consider the special case of (1) with only a single reaction. Specifically, we assume a model for a single reaction of the form

$$(15) \quad \begin{cases} \dot{x}_1(t) &= -k(T(t))x_1(t) + d[x_1^{\text{in}} - x_1(t)], \\ \dot{x}_2(t) &= k(T(t))x_1(t) - dx_2(t), \\ \dot{T}(t) &= bk(T(t))x_1(t) - qT(t) + u(t). \end{cases}$$

Here  $b > 0$  denotes the exothermicity of a reaction  $A \longrightarrow B$ ,  $x^{\text{in}} = (x_1^{\text{in}}, 0)^T$ , where  $x_1^{\text{in}}$  is the constant feed rate of reactant  $A$ , and the reaction kinetics are given by a locally Lipschitz function  $k(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  with  $k(0) = 0$ . A typical example of  $k(\cdot)$  is the Arrhenius law  $k(T) = k_0 e^{-\frac{E}{RT}}$  (extended to zero by continuity), where  $k_0$  is a constant,  $E$  is the activation energy, and  $R$  is the Joule constant. The function  $k(\cdot)$  and the positive constants  $d$ ,  $q$ , and  $b$  are typically unknown.

In this case  $\gamma = (1, 1)^T$  and the feasibility assumption (A3') becomes the following:

$$(A3'') \quad \text{There exist } \rho > 0 \text{ and } 0 < \underline{T} < T^* < \bar{T} \text{ such that} \\ 0 < \underline{u} + \rho < qT - bk(T)x_1 < \bar{u} - \rho \quad \text{for all } (x_1, T) \in [0, x_1^{\text{in}}] \times [\underline{T}, \bar{T}].$$

In [12] it is shown that the nonadaptive feedback law (5) with “sufficiently large” and constant  $\beta(\cdot) \equiv \beta^*$  ensures that  $(x_1(t), x_2(t))$  tends to an asymptotically stable equilibrium of the closed-loop reactor dynamics. The corresponding result for  $\lambda$ -tracking is stated as follows.



PROPOSITION 8. Suppose (6), (13),  $\xi(\cdot) \equiv 0$ , and (A3'') hold. Define

$$x_1^* := \frac{d x_1^{\text{in}}}{k(T^*) + d} \quad \text{and} \quad x_2^* := \frac{k(T^*) x_1^{\text{in}}}{k(T^*) + d}.$$

Then the solution of the closed-loop system (4), (15), parametrized by  $\lambda$ , satisfies

$$(16) \quad \lim_{\lambda \rightarrow 0} \limsup_{t \rightarrow \infty} (|x_1(t) - x_1^*|) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \limsup_{t \rightarrow \infty} (|x_2(t) - x_2^*|) = 0;$$

i.e., the narrower the  $\lambda$ -strip (i.e., smaller  $\lambda$ ) is, then the closer  $(x_1(t), x_2(t))$  is, eventually, to  $(x_1^*, x_2^*)$ .

Note that  $(x_1^*, x_2^*)$  is an equilibrium of (15) for  $T(\cdot) \equiv T^*$  and so  $(x_1^*, x_2^*) \in \partial\Omega((1, 1), (x_1^{\text{in}}, 0))$ .

The proof of Proposition 8 is given in the appendix.

In the remainder of this section, we illustrate previous results by some simulations. In the simulations we use a prototype model for a single exothermic chemical reaction as was also used in [12]. By using the same model, we can at least check that the performance of the  $\lambda$ -tracker is not out of line with a controller which actually relies on more system information. Specifically we consider (15) with reaction kinetics modelled by the Arrhenius law  $k(T) = k_0 e^{-k_1 T}$ . As in [12], we use the following system parameters:

$$(17) \quad k_0 = e^{25}, \quad d = 1.1, \quad q = 1.25 \text{ [min}^{-1}\text{]}, \quad k_1 = 8700 \text{ [K]}, \quad x_i^{\text{in}} = 1 \text{ [mol/l]}, \quad b = 209.2 \text{ [Kl/mol]}.$$

These parameter values are consistent with a laboratory-scale reaction vessel of approximately 100 liters [5].

The objective is to regulate the temperature to a neighborhood of  $T^* = 337.1 \text{ [K]}$ . Our constraints for the input  $u(\cdot)$  are similar to those in [12]. Specifically we suppose that

$$(18) \quad \underline{u} = 295, \quad \bar{u} = 505.$$

It is easy to see that the feasibility assumption (A3'') is satisfied in this case if

$$(19) \quad \underline{T} = 240, \quad \bar{T} = 339.65 \text{ [K]}, \quad \rho = 5.$$

We assume in this simulation that the error is disturbance free, i.e.,  $\xi \equiv 0$ , and aim for a tracking error of within 1%. This leads us to choose the following parameters in the  $\lambda$ -tracker (4):

$$(20) \quad \lambda = 2.85, \quad u^* = 330, \quad T^* = 337.1 \text{ [K]}, \quad l = 2.$$

In the simulations we choose  $\beta^0 = 12$ , which satisfies (13), and we consider three different initial conditions  $T^0 = 270$ ,  $T^0 = 320$ , and  $T^0 = 390$ . As in [12], we choose  $x_1(0) = 0.02$  and  $x_2(0) = 1.07$  for the initial conditions of the single reactor (15).

For the two initial conditions  $T^0 = 270$  and  $T^0 = 320$ , we see from Figure 1 that  $\lambda$ -tracking of  $T^*$  by  $T(t)$  is achieved in 1 minute. Note that in both cases the transient behavior of the input hits the saturation values only for a short period at the beginning of the simulation. Otherwise the input behaves smoothly. The simulation results are similar to those in [12].

The  $\lambda$ -tracker (4) does not work for  $T^0 = 390$ , which is outside the interval  $[0, \bar{T}]$ . As shown by the dotted line in Figure 1, a thermal runaway occurs and the

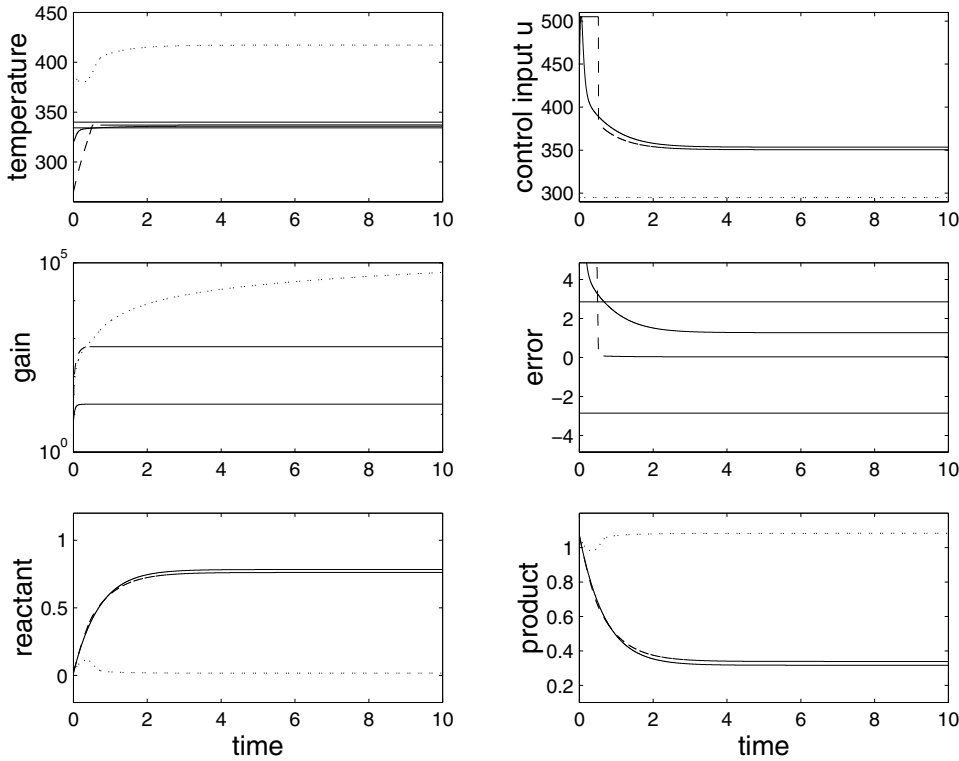


FIG. 1. Closed-loop behavior of the adaptive  $\lambda$ -tracker (4) for local set-point control without measurement disturbance with parameters (20) applied to the single reaction (15) with parameters (17), input constraints (18), feasibility bounds (19),  $T^0 = 270$  (dashed),  $T^0 = 320$  (solid),  $T^0 = 390$  (dotted). The latter exhibits a thermal runaway.

temperature is attracted to a stable but undesirably high temperature. As a result, the reaction becomes overheated, the reactant burns out, and there is a rapid growth of the product. Furthermore, the control input saturates at its lower limit throughout the simulation and the gain increases unboundedly.

**3. Global tracking.** The main result of the previous section, i.e., Theorem 7, has the shortcoming in that it is local in the sense that the initial temperature must lie inside  $(0, \bar{T})$ . This shortcoming can, under adverse temporary disturbances to the reaction, lead to a problem of thermal runaway in that the reaction dynamics are attracted to an undesirable equilibrium. See the simulations in Figure 1. Due to the given input saturations, it may even be impossible to reduce the temperature of the reaction from such equilibria by any type of control of the temperature alone. To overcome this problem, we borrow an idea from [12] and introduce an additional input action which has a cooling effect if the temperature is too large. To see the idea, consider the modification of the single reaction model (15) of the form

$$(21) \quad \begin{cases} \dot{x}_1(t) = -k(T(t))x_1(t) + d[v(t) - x_1(t)], \\ \dot{x}_2(t) = k(T(t))x_1(t) - dx_2(t), \\ \dot{T}(t) = bk(T(t))x_1(t) - qT(t) + u(t), \end{cases}$$

with constant feedrate  $x_1^{\text{in}}$  replaced by  $v(\cdot)$ , an additional open-loop control of the feedrate of reactant. In [12] a choice of  $v$  as feedback control is

$$(22) \quad v(T) = \begin{cases} x_1^{\text{in}} & \text{if } T \in (0, \bar{T}), \\ 0 & \text{if } T \in [\bar{T}, \infty). \end{cases}$$

The additional feedback (22) has the following beneficial effect: if  $T(t) \geq \bar{T}$ , then  $\dot{x}_1(t) \leq -d x_1(t)$  and hence  $x_1(\cdot)$  decreases; if  $T(t) \geq \bar{T}$  is maintained, then  $x_1(t)$  is eventually small enough to yield a decrease in temperature.

It is not clear to us whether the resulting discontinuous closed-loop system has a solution. It seems that the discontinuity should be harmless if the intervals in (22) are replaced by  $(0, \bar{T}]$  and  $(\bar{T}, \infty)$ . However, since we also assume that the temperature measurement is corrupted by measurement disturbance, this discontinuity will be difficult to handle rigorously. To circumvent this technical difficulty, we replace the discontinuity in (22) by a simple piecewise linear control for  $v(\cdot) : \mathbb{R} \rightarrow [0, x_1^{\text{in}}]$  given by

$$(23) \quad v(\beta e) = \begin{cases} 0 & \text{if } \beta e \in (-\infty, \underline{u} - u^*], \\ (\beta e + u^* - \underline{u}) x_1^{\text{in}} / \delta & \text{if } \beta e \in (\underline{u} - u^*, \underline{u} - u^* + \delta), \\ x_1^{\text{in}} & \text{if } \beta e \in [\underline{u} - u^* + \delta, \infty). \end{cases}$$

Here  $\delta > 0$  would be small.

The additional feedrate control action (23) can also be introduced for multiple reactions as follows. We divide the state  $x(t)$  into two substates  $x_1(t)$  and  $x_2(t)$  so that all reactants are collected in  $x_1$ . Applying a permutation of coordinates to (1) yields a system of the form

$$\begin{aligned} \dot{x}_1(t) &= C_1 r(x(t), T(t)) + d[x_1^{\text{in}} - x_1(t)], \\ \dot{x}_2(t) &= C_2 r(x(t), T(t)) + d[x_2^{\text{in}} - x_2(t)], \\ \dot{T}(t) &= b^T r(x(t), T(t)) - qT(t) + u(t), \end{aligned}$$

where  $C_1 \in \mathbb{R}^{(n-m) \times m}$ ,  $C_2 \in \mathbb{R}^{m \times m}$ ,  $x_1^{\text{in}} \in \mathbb{R}_{\geq 0}^{n-m}$ , and  $x_2^{\text{in}} \in \mathbb{R}_{\geq 0}^m$ . Since  $x_1$  represents the reactants of the chemical reactor, it follows that each entry of  $C_1$  is nonpositive, i.e.,  $C_1 \in \mathbb{R}_{\leq 0}^{(n-m) \times m}$ . In this multireaction global case, the assumption (2) on the reaction kinetics must be strengthened to

$$(A4) \quad \|r(x, T)\| \leq \hat{r}(x_1) T \quad \text{for all } (x, T) = (x_1^T, x_2^T)^T, T \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$$

for some locally Lipschitz function  $\hat{r} : \mathbb{R}_{\geq 0}^{n-m} \rightarrow \mathbb{R}_{\geq 0}$  with  $\lim_{x_1 \rightarrow 0} \hat{r}(x_1) = 0$ .

*Remark 9.* Note that (A4) encompasses the class of functions considered in [12], where  $b^T r(x, T) = \sum_{i=1}^m b_i k_i(T) \varphi_i(x)$ , each  $b_i > 0$ , each function  $T \mapsto k_i(T)$  is positive, bounded, and globally Lipschitz, and each function  $x \mapsto \varphi_i(x)$  is nonnegative and continuous and vanishes if any component of  $x$  is zero for  $i = 1, \dots, m$ , respectively.

The constant concentration of reactants in the feed flow  $x_1^{\text{in}}$  is replaced by an  $(n - m)$ -dimensional feedback term  $v(\beta(\cdot)e(\cdot))$  given by (23) and the overall model becomes (compare [12, eq. (20)])

$$(24) \quad \begin{cases} \dot{x}_1(t) = C_1 r(x(t), T(t)) + d[v(\beta(t)e(t)) - x_1(t)], \\ \dot{x}_2(t) = C_2 r(x(t), T(t)) + d[x_2^{\text{in}} - x_2(t)], \\ \dot{T}(t) = b^T r(x(t), T(t)) - qT(t) + u(t). \end{cases}$$

To proceed, we need to ensure that if the control  $u(\cdot)$  and concentration of reactant  $\nu(\cdot)$  in the feed of (24) are nonnegative, then the solution takes values in the positive orthant. To do this, we replace (A1) with the following:

$$(A1') \quad \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{> 0} \text{ is positively invariant under}$$

$$\begin{aligned} \dot{x}_1(t) &= C_1 r(x(t), T(t)) + d[\nu(t) - x_1(t)], \\ \dot{x}_2(t) &= C_2 r(x(t), T(t)) + d[x_2^{\text{in}} - x_2(t)], \\ \dot{T}(t) &= b^T r(x(t), T(t)) - qT(t) + u(t) \end{aligned}$$

for any bounded, nonnegative, locally integrable functions  $u(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and  $\nu(\cdot) : \mathbb{R} \rightarrow [0, x_1^{\text{in}}]$ .

As we pointed out in Remark 2(i),  $T(t) > 0$  for all  $t \geq 0$  is immediate and is only included in (A1') for a less technical presentation. For the same reason, we have stated (A1') for  $\nu(\cdot)$ , whereas it is only needed for  $t \mapsto v(\beta(t)e(t))$ .

Note that the comments we made for Assumption (A1) in Remark 2 apply here also. If we are in the situation described in Remark 9, then (A1') holds.

We are now in a position to state the main result of this paper.

**THEOREM 10** (adaptive tracking with measurement disturbance). *Suppose (6), (A1'), (A2), (A3'), (A4) hold and that the continuous disturbance satisfies*

$$(25) \quad \sup_{t \geq 0} \{|\xi(t)|\} =: \|\xi\|_\infty < \min \{T^* - \underline{T}, \lambda/2\}.$$

*Then an application of the  $\lambda$ -tracker (4) combined with (23) to any system (24) yields, for any initial data  $(x^0, T^0, \beta^0) \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{> 0}^2$ , a closed-loop system with unique solution  $(x(\cdot), T(\cdot), \beta(\cdot)) : \mathbb{R}_{\geq 0} \rightarrow \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{> 0}^2$  defined on the whole time axis  $\mathbb{R}_{\geq 0}$ . Moreover,*

- (i)  $\lim_{t \rightarrow \infty} \beta(t) = \beta_\infty \in \mathbb{R}_{> 0}$ , i.e., the gain adaptation is convergent,
- (ii)  $\lim_{t \rightarrow \infty} \text{dist}(|T^* - T(t)|, [0, \lambda + \|\xi\|_\infty]) = 0$ ; i.e., the temperature  $T(t)$  tends to the  $[\lambda + \|\xi\|_\infty]$ -strip  $[T^* - [\lambda + \|\xi\|_\infty], T^* + [\lambda + \|\xi\|_\infty]]$  as  $t \rightarrow \infty$ .

The proof of Theorem 10 relies on the following high-gain lemma. This lemma is of interest in its own right, as it also gives insight into essential structural properties of the system class (24). It also shows that for sufficiently large gain, after some finite time the error enters and remains in the  $\lambda$ -strip.

**LEMMA 11.** *Suppose (6), (A1'), (A2), (A3'), (A4), (25) hold. Then an application to any system (24) of the nonadaptive feedback*

$$(26) \quad u(t) = \text{sat}_{[\underline{u}, \bar{u}]}(\beta(t)e(t) + u^*), \quad e(t) = T^* - T(t) + \xi(t),$$

*combined with (23), yields, for any continuous  $\beta(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$  satisfying, for some  $t' \geq 0$ ,*

$$(27) \quad \beta(t) \geq \beta' := \max \left\{ \frac{\bar{u} - u^*}{\lambda - 2\|\xi\|_\infty}, \frac{u^* - \underline{u}}{\lambda - 2\|\xi\|_\infty} \right\} \quad \text{for all } t \geq t'$$

*and any initial data  $(x^0, T^0) \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{> 0}$ , a closed-loop system with unique solution*

$$(x(\cdot), T(\cdot)) : \mathbb{R}_{\geq 0} \rightarrow \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{> 0}$$

*on the whole time axis  $\mathbb{R}_{\geq 0}$ . Moreover, there exists a time  $t_1 \geq t'$  such that*

$$(28) \quad e(t) \in (-\lambda, \lambda) \quad \text{for all } t \geq t_1.$$

Theorem 10 and Lemma 11 are proved in the appendix.

A simple consequence of Lemma 11 is the following theorem, which shows that tracking can be achieved by the nonadaptive feedback (5) if the constant gain parameter  $\beta^*$  is sufficiently large (depending on the feasibility bounds). This feedback is simpler than (19) in [12], and we give an explicit lower bound for the gain in terms of weak conditions on the system.

**THEOREM 12** (nonadaptive tracking with measurement disturbance). *Suppose (6), (A1'), (A2), (A3'), (A4), (25) hold, and  $\beta^* \geq \beta'$  as defined in (27). Then an application of the nonadaptive output feedback*

$$(29) \quad u(t) = \text{sat}_{[\underline{u}, \bar{u}]}(\beta^* e(t) + u^*), \quad e(t) = T^* - T(t) + \xi(t),$$

combined with (23) to any system (24) yields, for any initial data  $(x^0, T^0) \in \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$ , a closed-loop system with a unique solution

$$(x(\cdot), T(\cdot)) : \mathbb{R}_{\geq 0} \longrightarrow \Omega(\gamma, x^{\text{in}}) \times \mathbb{R}_{>0}$$

on the whole time axis  $\mathbb{R}_{\geq 0}$ , and moreover, there exists a time  $t_1 \geq t'$  such that (28) is satisfied.

*Remark 13.* If  $\xi(\cdot) \equiv 0$ , then (25) holds trivially, and so the adaptive gain feedback controller (4) can be applied without restriction, whereas the constant gain feedback controller (29) needs  $\beta^* \geq \beta'$ . If  $\xi(\cdot) \not\equiv 0$ , then in applying either the nonadaptive or the adaptive controller, we need to check conditions involving  $\underline{T}$  and  $\bar{T}$ . Although this suggests that we might just as well use the simpler nonadaptive controller, in practice the adaptive gain is less conservative and the adaptive controller produces better results.

Figure 2 shows that the problem of thermal runaway above, exhibited for the local  $\lambda$ -set-point controller with  $T^0 = 390$ , is overcome by incorporating into the  $\lambda$ -tracker (4) the additional feedrate control via (23). Indeed, when  $T^0 = 390$  the input  $v$  is switched off, i.e.,  $v(0) = 0$ , and consumption of reactant is increased. This causes the temperature to drop, and  $\lambda$ -tracking is achieved. On the other hand, for  $T^0 = 270$  and  $T^0 = 320$ ,  $v(\cdot) \equiv x^{\text{in}}$  and the response curves are the same as in Figure 1.

To illustrate the effectiveness of the controller in the presence of temperature measurement disturbances, we consider a disturbance signal

$$(30) \quad \xi(t) = \frac{1}{12} q_1(t),$$

where  $q_1(\cdot)$  is the first component of the Lorenz equation

$$\begin{aligned} \dot{q}_1(t) &= 10[q_2(t) - q_1(t)], & q_1(0) &= 1, \\ \dot{q}_2(t) &= 28q_1(t) - q_2(t) - q_1(t)q_3(t), & q_2(0) &= 0, \\ \dot{q}_3(t) &= q_1(t)q_2(t) - \frac{8}{3}q_3(t), & q_3(0) &= 3. \end{aligned}$$

This Lorenz equation is known [11] to exhibit chaotic but bounded behavior. In this case  $|\xi(t)| \leq 1.42$  for all  $t \geq 0$ . Hence,  $\xi(\cdot)$  satisfies (25) for the data given in (18), (19), and (20).

Looking at Figures 3 and 4, we see that the error  $T^* - T(t)$  is forced into the  $[\lambda + \|\xi\|_\infty]$ -strip  $[-4.27, 4.27]$  despite the chaotic behavior of the disturbance signal (30). Since the constant gain in (5) is at all time equal to  $\beta' = 6619$ , unlike the adaptive gain in (4), which can be less than  $\beta' = 6619$ , the error in Figure 4 tends

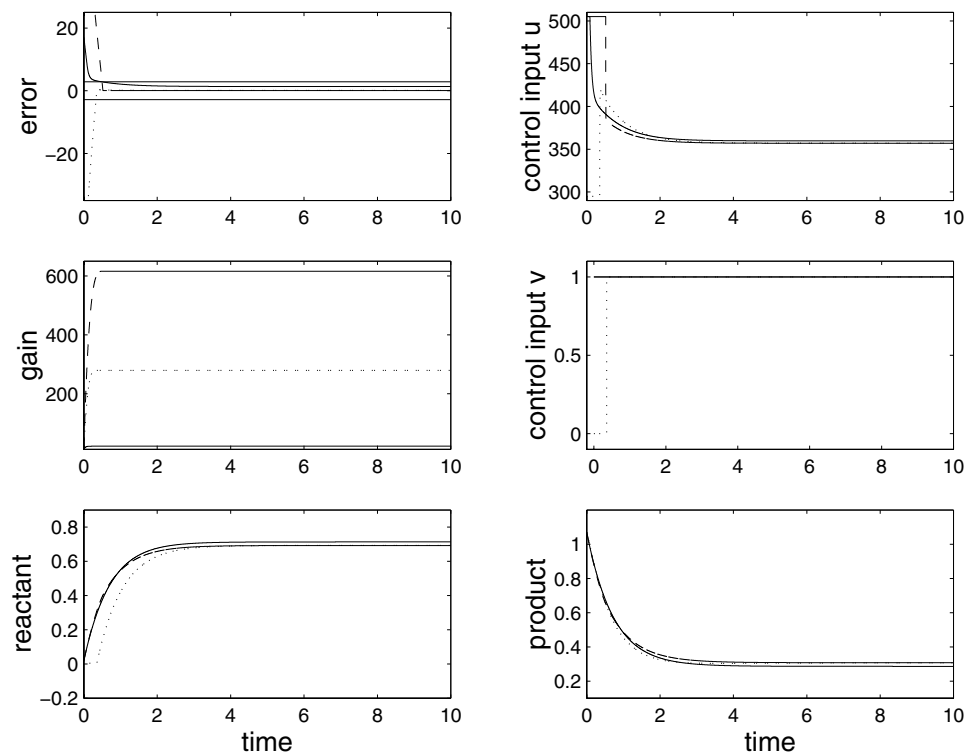


FIG. 2. Closed-loop behavior of the adaptive  $\lambda$ -tracker (4) combined with cooling action (23) for global setpoint control without disturbance with parameters (20) applied to the single reaction (21) with parameters (17), input constraints (18), feasibility bounds (19),  $T^0 = 270$  (dashed),  $T^0 = 320$  (solid),  $T^0 = 390$  (dotted). Here thermal runaway is overcome.

to a smaller strip than that of the error in Figure 4. Note that there is considerably more control action for the fixed gain controller than for the adaptive gain controller, even after the control objective has been met. This increased control action is bang-bang in nature, leads to a repeated switching on and off of the control action, and is therefore undesirable from a practical point of view. This observation provides some justification for the use of the adaptive gain controller in preference to the fixed gain controller. In this group of simulations, we have omitted the graphs corresponding to the initial temperature  $T^0 = 270$  since they are similar to those for  $T^0 = 320$ . Moreover, we have replaced the graph of the product in Figure 3 (which is close to that in Figure 4) by that of an error in a longer simulation time to show that  $\lambda$ -tracking is indeed achieved.

**4. Conclusion.** In the present paper we have developed a  $\lambda$ -tracking approach to the set-point control of the temperature for a class of nonlinear systems arising as models in chemical reactor control. The novelty in this development is the need to carefully consider the interplay between the reaction dynamics, input constraints, and feasibility. The application of  $\lambda$ -trackers requires only limited information concerning the system. In addition, the  $\lambda$ -trackers quite readily tolerate bounded temperature measurement disturbances. In many respects they generalize the controllers developed by [12]. It is worth noting that the minimum phase assumption usually needed for

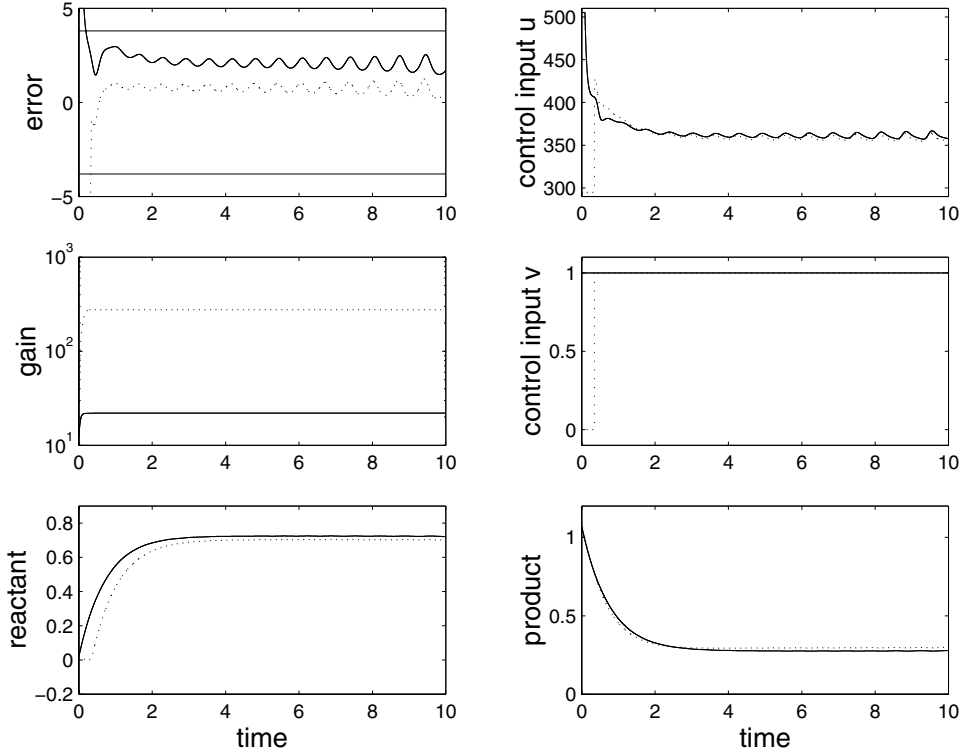


FIG. 3. Closed-loop behavior of the adaptive  $\lambda$ -tracker (4) combined with cooling action (23) for global setpoint control with measurement disturbance with parameters (20) and disturbance signal given by (30), applied to the single reaction (21) with parameters (17), input constraints (18), feasibility bounds (19),  $T^0 = 320$  (solid),  $T^0 = 390$  (dotted).

$\lambda$ -tracking is not needed here. Instead, we exploit the natural property of chemical reactions that the internal state, i.e., the concentrations, is bounded.

**Appendix. Proofs.** For the sake of presentation, we define, for arbitrary  $\Lambda > 0$ , the distance function

$$d_\Lambda(\eta) := \max\{|\eta| - \Lambda, 0\} \quad \text{for all } \eta \in \mathbb{R}.$$

Note that for every solution  $(x, T)$  of (1) or (21) on  $\mathbb{R}_{\geq 0}$  and  $\hat{e}(t) = T^* - T(t)$ , differentiation of

$$(31) \quad V_\Lambda(t) := d_\Lambda(\hat{e}(t))^2 \quad \text{for all } t \geq 0,$$

along (1) or (21) satisfies

$$(32) \quad \dot{V}_\Lambda(t) = \begin{cases} 2\sqrt{V_\Lambda(t)} [-b^T r(x(t), T(t)) + qT(t) - u(t)], & \hat{e}(t) > 0, \\ 0, & \hat{e}(t) = 0, \\ -2\sqrt{V_\Lambda(t)} [-b^T r(x(t), T(t)) + qT(t) - u(t)], & \hat{e}(t) < 0, \end{cases} \quad \text{for all } t \geq 0.$$

**Proof of Proposition 4.** Existence and uniqueness of the solution (9) follow from Remark 6.

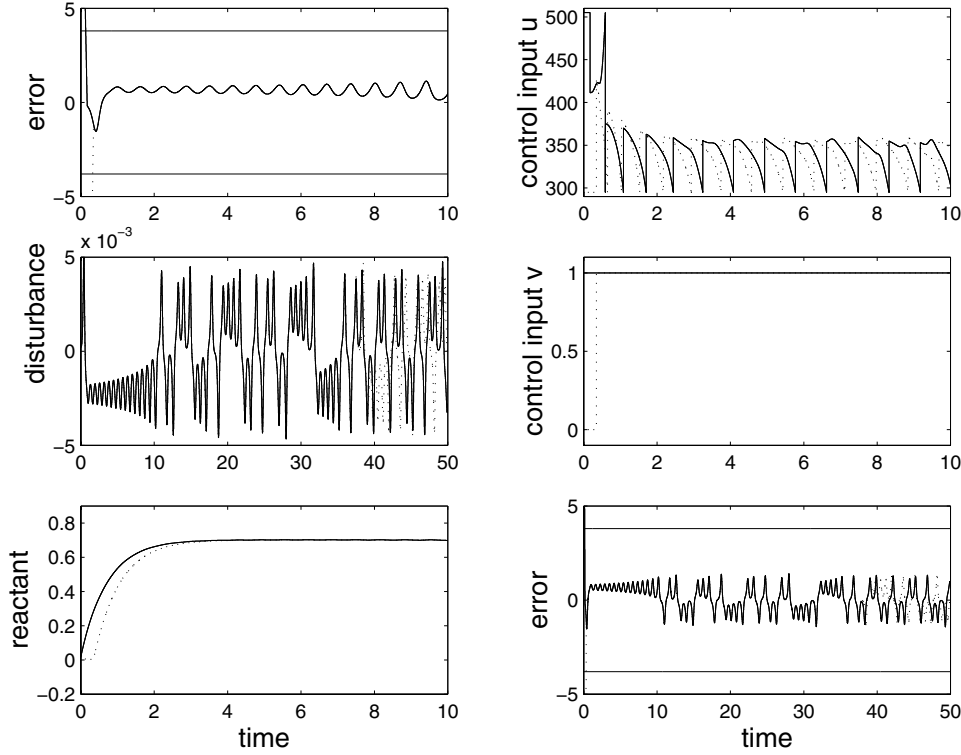


FIG. 4. Closed-loop behavior of the constant gain controller (5) combined with cooling action (23) for global set-point control with measurement disturbance with parameters (20) and disturbance signal given by (30), applied to the single reaction (21) with parameters (17), input constraints (18), feasibility bounds (19),  $T^0 = 320$  (solid),  $T^0 = 390$  (dotted).

Set  $\Lambda := \lambda + \|\xi\|_\infty$  and consider, for all  $t \geq 0$ , the evolution of the actual error  $\hat{e}(t) = T^* - T(t)$  with respect to  $V_\Lambda$  as in (31):

$$\begin{aligned}
 \hat{e}(t) \in [-\Lambda, \Lambda] &\implies V_\Lambda(t) = 0; \\
 \hat{e}(t) > \Lambda &\implies \beta(t)e(t) + u^* = \beta(t)[\hat{e}(t) + \xi(t)] + u^* \\
 &\stackrel{(5)}{>} \beta^*[\Lambda - \|\xi\|_\infty] + u^* = \beta^*\lambda + u^* \stackrel{(10)}{\geq} \bar{u} \stackrel{(5)}{=} u(t) \\
 &\stackrel{(A3') \& (32)}{\implies} \dot{V}_\Lambda(t) \leq -2\rho\sqrt{V_\Lambda(t)}; \\
 \hat{e}(t) < -\Lambda &\implies \beta(t)e(t) + u^* = \beta(t)[\hat{e}(t) + \xi(t)] + u^* \\
 &< \beta(t)[- \Lambda + \|\xi\|_\infty] + u^* \stackrel{(5)}{\leq} -\beta^*\lambda + u^* \stackrel{(10)}{\leq} \underline{u} \stackrel{(5)}{=} u(t) \\
 &\stackrel{(A3') \& (32)}{\implies} \dot{V}_\Lambda(t) \leq -2\rho\sqrt{V_\Lambda(t)}.
 \end{aligned}$$

Summarizing, we have, for all  $t \geq 0$ ,  $\dot{V}_\Lambda(t) \leq -2\rho\sqrt{V_\Lambda(t)}$ , and so there exists  $t' \geq 0$  such that  $\hat{e}(t) \in [-\Lambda, \Lambda]$  for all  $t \geq t'$ , whence (11).

**Proof of Theorem 7.** Existence and uniqueness of the initial value problem (1), (4), (13) on a maximally extended interval  $[0, \omega)$ ,  $\omega \in (0, \infty]$ , follows from the



theory of ordinary differential equations. Monotonicity of  $t \mapsto \beta(t)$  and (13) yield  $\beta(t) \geq \beta^*$  with  $\beta^*$  satisfying (8). Therefore Proposition 4 applies and  $\Omega(\gamma, x^{\text{in}}) \times (0, \bar{T})$  is positively invariant. Now the gain adaptation (4) yields that  $\beta(\cdot)$  cannot exhibit a finite escape time on  $[0, \omega)$  and hence  $\omega = \infty$ .

For the remainder of the proof, consider the unique solution  $(x(\cdot), T(\cdot), \beta(\cdot))$  of (14).

We show assertion (i). Seeking a contradiction, suppose that  $\beta$  is unbounded. Then

$$(33) \quad \text{there exists } \hat{t} \geq 0 : \text{ for all } t \geq \hat{t} \quad : \quad \beta(t) \geq \frac{\max\{u^* - \underline{u}, \bar{u} - u^*\}}{\lambda - 2\|\xi\|_\infty}.$$

Set  $\Lambda := \lambda - \|\xi\|_\infty$ . By (12)  $\Lambda > \|\xi\|_\infty$ . Now consider, for all  $t \geq \hat{t}$ , the evolution of the actual error  $\hat{e}(t) = T^* - T(t)$  with  $V_\Lambda$  as in (31):

$$\begin{aligned} \hat{e}(t) \in [-\Lambda, \Lambda] &\implies V_\Lambda(t) = 0; \\ \hat{e}(t) > \Lambda &\implies \begin{aligned} \beta(t)e(t) + u^* &= \beta(t)[\hat{e}(t) + \xi(t)] + u^* \\ &> \beta(t)[\Lambda - \|\xi\|_\infty] + u^* = \beta(t)[\lambda - 2\|\xi\|_\infty] + u^* \\ &\stackrel{(33)}{\geq} \beta^*[\lambda - 2\|\xi\|_\infty] + u^* \geq \bar{u} \stackrel{(5)}{=} u(t) \end{aligned} \\ &\stackrel{(A3') \& (32)}{\implies} \dot{V}_\Lambda(t) \leq -2\rho\sqrt{V_\Lambda(t)}; \\ \hat{e}(t) < -\Lambda &\implies \begin{aligned} \beta(t)e(t) + u^* &= \beta(t)[\hat{e}(t) + \xi(t)] + u^* \\ &< \beta(t)[- \Lambda + \|\xi\|_\infty] + u^* \\ &\stackrel{(33)}{\leq} -\beta^*[\lambda - 2\|\xi\|_\infty] + u^* \leq \underline{u} \stackrel{(5)}{=} u(t) \end{aligned} \\ &\stackrel{(A3') \& (32)}{\implies} \dot{V}_\Lambda(t) \leq -2\rho\sqrt{V_\Lambda(t)}. \end{aligned}$$

Summarizing, we have, for all  $t \geq \hat{t}$ ,  $\dot{V}_\Lambda(t) \leq -2\rho\sqrt{V_\Lambda(t)}$ , and so there exists  $t' \geq \hat{t}$  such that  $\hat{e}(t) \in [-\Lambda, \Lambda]$  for all  $t \geq t'$ , whence  $\beta(t) = 0$  for all  $t \geq t'$ , which contradicts the supposition of unboundedness of  $\beta$ . Therefore,  $\beta$  is bounded and assertion (i) follows by monotonicity of  $\beta$ .

Finally, we show assertion (ii). It is easy to see that

$$\kappa \int_0^t d_{\lambda + \|\xi\|_\infty}(\hat{e}(\tau))^l d\tau \leq \kappa \int_0^t d_\lambda(e(\tau))^l d\tau = \beta(t) - \beta^0 \quad \text{for all } t \geq 0,$$

and hence assertion (i) yields  $d_{\lambda + \|\xi\|_\infty}(\hat{e}(\cdot))^l \in \mathcal{L}^1([0, \infty); \mathbb{R})$ . Since continuity of  $\eta \mapsto d_\lambda(\eta)$ , together with boundedness and uniform continuity of  $t \mapsto \hat{e}(t)$ , yields uniform continuity of the composition  $t \mapsto d_\lambda(\hat{e}(t))$ , we may apply Barbălat's lemma (see, e.g., [10]) to conclude

$$\lim_{t \rightarrow \infty} \text{dist}\left(|\hat{e}(t)|, [0, \lambda + \|\xi\|_\infty]\right) = 0.$$

This proves assertion (ii) and completes the proof of the theorem.

**Proof of Proposition 8.** By Theorem 7, there exist  $t_0 \geq 0$  such that, for all  $t \geq t_0$ ,  $T(t) \in [T^* - 2\lambda, T^* + 2\lambda]$ , and so, for all  $t \geq t_0$ ,

$$(34) \quad \inf \left\{ k(T) \mid T \in [T^* - 2\lambda, T^* + 2\lambda] \right\} =: k_1(\lambda) \leq k(T(t)) \\ \leq k_2(\lambda) := \sup \left\{ k(T) \mid T \in [T^* - 2\lambda, T^* + 2\lambda] \right\},$$

whence, by the continuity of  $k(\cdot)$ ,

$$(35) \quad \lim_{\lambda \rightarrow 0} k_1(\lambda) = \lim_{\lambda \rightarrow 0} k_2(\lambda) = k(T^*).$$

Integrating the first equation in (15) yields

$$x_1(t) = e^{-\int_{t_0}^t (k(T(\tau)) + d) d\tau} x_1(t_0) + \int_{t_0}^t e^{-\int_s^t (k(T(\tau)) + d) d\tau} dx_1^{\text{in}} ds.$$

So, applying (34), we obtain

$$(36) \quad \frac{dx_1^{\text{in}}}{k_2(\lambda) + d} \leq \liminf_{t \rightarrow \infty} x_1(t) \leq \limsup_{t \rightarrow \infty} x_1(t) \leq \frac{dx_1^{\text{in}}}{k_1(\lambda) + d}.$$

Therefore the first equation in (16) follows from (36) and (35). Integrating the second equation in (15) yields

$$x_2(t) = e^{-d(t-t_0)} x_2(t_0) + \int_{t_0}^t e^{-d(t-s)} k(T(s)) x_1(s) ds.$$

Now applying (34) and (36), we obtain

$$(37) \quad \frac{k_1(\lambda) x_1^{\text{in}}}{k_2(\lambda) + d} \leq \liminf_{t \rightarrow \infty} x_2(t) \leq \limsup_{t \rightarrow \infty} x_2(t) \leq \frac{k_2(\lambda) x_1^{\text{in}}}{k_1(\lambda) + d},$$

and so the second equation in (16) follows from (37) and (35). This completes the proof.

**Proof of Lemma 11.** We proceed in several steps.

*Step 1.* The right-hand side of the closed-loop system is locally Lipschitz, and so the existence and uniqueness of the solution on a maximally extended interval  $[0, \omega)$ ,  $\omega \in (0, \infty]$ , follow from the theory of ordinary differential equations.

*Step 2.* We show positive invariance of  $\Omega(\gamma, x^{\text{in}}) \times (0, \infty)$ . Note that (A2) yields, for  $x^{\text{in}} = (x_1^{\text{in}T}, x_2^{\text{in}T})^T$ ,  $\frac{d}{dt} \gamma^T x(t) \leq -d\gamma^T x(t) + d\gamma^T x^{\text{in}}$ , and hence by integration

$$\gamma^T x(t) \leq e^{-dt} \gamma^T x(0) + \gamma^T x^{\text{in}} [1 - e^{-dt}] \quad \text{for all } t \in [0, \omega).$$

If  $x(0) \in \Omega(\gamma, x^{\text{in}})$ , then  $\gamma^T x(0) < \gamma^T x^{\text{in}}$ , and so this inequality together with assumption (A1') proves  $x(t) \in \Omega(\gamma, x^{\text{in}})$  for all  $t \in [0, \omega)$ . To see that  $T(t)$  is positive, note that if we had  $T(t) = 0$ , then, by (1) and (6),  $\dot{T}(t) \geq u(t) \geq \underline{u} > 0$ .

*Step 3.* We show  $\omega = \infty$ . Since  $x(\cdot)$  and  $u(\cdot)$  are bounded, (A4) ensures that the right-hand side of  $\dot{T}$  in (24) is affine linearly bounded in  $T$  and hence  $T(\cdot)$  cannot escape in finite time. Applying the boundedness of  $x(\cdot)$  and the maximality of  $\omega$  yields the claim.

*Step 4.* We show that there exists  $t_1 \geq t'$  such that  $T(t) \in (0, \bar{T})$  for all  $t \geq t_1$ . Recall that, by Step 2,  $T(t) > 0$  for all  $t \in [0, \omega)$ .

(4a) We claim that  $T(s) \leq \bar{T}$ , for some  $s \in [0, \omega)$ , implies  $T(t) \in (0, \bar{T})$  for all  $t \in (s, \omega)$ . This follows from (24) and (A3'), which give, in the case of  $T(t) = \bar{T}$ , that

$$\dot{T}(t) = b^T r(x(t), \bar{T}) - q\bar{T} + \underline{u} < -\rho.$$

(4b) It remains to be shown that if  $T(t') > \bar{T}$ , then  $T(t) = \bar{T}$  for some  $t > t'$ . Seeking a contradiction, suppose

$$(38) \quad T(t) > \bar{T} \quad \text{for all } t \geq t'.$$

Then (38) together with (27) gives

$$(39) \quad \beta(t)e(t) + u^* \leq \beta(t) [T^* + \|\xi\|_\infty - \bar{T}] + u^* \leq \underline{u} \quad \text{for all } t \geq t',$$

and hence, by (23),  $v(\beta(t)e(t)) = 0$  for all  $t \geq t'$ . Therefore, (24) and the fact that all entries of  $C_1$  are nonpositive yield

$$\frac{d}{dt} \|x_1(t)\|^2 = 2x_1(t)^T [C_1 r(x(t), T(t)) - dx_1(t)] \leq -2d \|x_1(t)\|^2,$$

and it follows that

$$(40) \quad \|x_1(t)\| \leq e^{-d(t-t')} \|x_1(t')\| \quad \text{for all } t \geq t'.$$

By (A3'), we may choose  $\varepsilon \in (0, q)$  sufficiently small so that

$$(41) \quad -[q - \varepsilon] \bar{T} + \underline{u} < -\rho/2.$$

By (A4) and (40), there exists  $t_1 \geq t'$  such that

$$(42) \quad \hat{r}(x_1(t)) \leq \varepsilon/\|b\| \quad \text{for all } t \geq t_1.$$

Finally, applying (39), (24), (A4), (42), (38), and (41) yields

$$\begin{aligned} \dot{T}(t) &\leq br(x(t), T(t)) - qT(t) + \underline{u} \leq \|b\| \hat{r}(x_1(t)) T(t) - qT(t) + \underline{u} \\ &\leq -[q - \varepsilon] T(t) + \underline{u} \leq -[q - \varepsilon] \bar{T} + \underline{u} < -\rho/2 \quad \text{for all } t \geq t_1. \end{aligned}$$

It then follows that there exists  $t_2 \geq t_1$  such that  $T(t_2) = \bar{T}$ , which contradicts (38). This completes the proof of Step 4.

*Step 5.* Finally, we prove the existence of some  $t_1 \geq t'$  such that (28) holds. Note that it suffices to show that there exists  $t_2 \geq t_1$ ,  $t_1$  as in Step 4, such that the actual error satisfies

$$(43) \quad \hat{e}(t) \in (-\lambda + \|\xi\|_\infty, \lambda - \|\xi\|_\infty) \quad \text{for all } t \geq t_2,$$

since then (25) yields (28).

Set  $\Lambda := \lambda - \|\xi\|_\infty$  and consider, for all  $t \geq t_1$ , the evolution of the actual error  $\hat{e}(t) = T^* - T(t)$  with respect to  $V_\Lambda$  as in (31). Then, for all  $t \geq t_1$ ,

$$\begin{aligned} \hat{e}(t) \in [-\Lambda, \Lambda] &\implies V_\Lambda(t) = 0; \\ \hat{e}(t) > \Lambda &\implies \begin{aligned} \beta(t)e(t) + u^* &= \beta(t) [\hat{e}(t) + \xi(t)] + u^* \\ &> \beta(t) [\Lambda - \|\xi\|_\infty] + u^* = \beta(t) [\lambda - 2\|\xi\|_\infty] + u^* \\ &\stackrel{(25,27)}{\geq} \beta' [\lambda - 2\|\xi\|_\infty] + u^* \stackrel{(5,27)}{\geq} \bar{u} = u(t) \end{aligned} \\ &\stackrel{(A3') \& (32)}{\implies} \dot{V}_\Lambda(t) \leq -2\rho \sqrt{V_\Lambda(t)}; \\ \hat{e}(t) < -\Lambda &\implies \begin{aligned} \beta(t)e(t) + u^* &= \beta(t) [\hat{e}(t) + \xi(t)] + u^* \\ &< \beta(t) [-\Lambda + \|\xi\|_\infty] + u^* = \beta(t) [\lambda - 2\|\xi\|_\infty] + u^* \\ &\stackrel{(25,27)}{\leq} -\beta' [\lambda - 2\|\xi\|_\infty] + u^* \stackrel{(5,27)}{\leq} \underline{u} = u(t) \end{aligned} \\ &\stackrel{(A3') \& (32)}{\implies} \dot{V}_\Lambda(t) \leq -2\rho \sqrt{V_\Lambda(t)}. \end{aligned}$$

Summarizing, we have, for all  $t \geq t_1$ ,  $\dot{V}_\Lambda(t) \leq -2\rho \sqrt{V_\Lambda(t)}$ , and so there exists  $t_2 \geq \hat{t}$  such that  $\hat{e}(t) \in [-\Lambda, \Lambda]$  for all  $t \geq t_2$ , whence (43). This completes the proof of the lemma.

**Proof of Theorem 10.**

*Steps 1–3.* These steps are the same as in the proof of Lemma 11. The only addition to the proofs is that  $\beta(\cdot)$  does not have a finite escape time if  $T(\cdot)$  does not have a finite escape time.

*Step 4.* We prove (i). Note that  $t \mapsto \beta(t)$  is monotonically nondecreasing. Then either (27) is not satisfied, in which case (i) is immediate, or (27) is satisfied. However, the latter yields by Lemma 11 that (28) holds, and thus the “dead zone” in the adaptation law (4) guarantees boundedness of  $\beta(\cdot)$ .

*Step 5.* We prove boundedness of  $T(\cdot)$ . Seeking a contradiction, suppose that  $T(\cdot)$  is unbounded. Then there exists a sequence of disjoint intervals  $I_m = (a_m, b_m)$ ,  $m \in \mathbb{N}$ , with

$$T(a_m) = T^0 + T^* + \|\xi\|_\infty + \lambda + m < T(t) < T(b_m) = T^0 + T^* + \|\xi\|_\infty + \lambda + m + 1$$

for all  $t \in I_m$ . It follows that

$$|e(t)| = |T(t) - T^* - \xi(t)| \geq T(t) - T^* - \|\xi\|_\infty > \lambda + 1,$$

and hence  $d_\lambda(e(t)) = |e(t)| - \lambda > 1$  for all  $t \in \bigcup_{m \in \mathbb{N}} I_m$ . Furthermore, we have, with  $d := c[T^0 + T^* + \|\xi\|_\infty + \lambda + 1] + \bar{u}$  and for all  $m \in \mathbb{N}$ ,

$$1 = T(b_m) - T(a_m) = \int_{a_m}^{b_m} \dot{T}(t) dt < \int_{a_m}^{b_m} [cT(b_m) + \bar{u}] dt = [cm + d] [b_m - a_m].$$

This leads to the contradiction

$$\infty = \sum_{m \in \mathbb{N}} \frac{1}{cm + d} < \sum_{m \in \mathbb{N}} |b_m - a_m| < \sum_{m \in \mathbb{N}} \int_{a_m}^{b_m} d_\lambda(|e(t)|)^l dt \leq \frac{\beta_\infty}{\kappa} < \infty.$$

*Step 6.* Since all variables of the closed-loop system are bounded, the proof of (ii) is identical to Step 6 in the proof of Theorem 7. This completes the proof of the theorem.

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