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Georgieva, Petia ; Ilchmann, Achim:

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Adaptive λ-tracking control of activated sludge processes

PETIA GEORGIEVA† and ACHIM ILCHMANN‡*

An adaptive controller for activated sludge processes is introduced. The control objective is to keep, in the presence of input constraints, the concentration of the biomass proportional to the influent flow rate, where a prespecified small tracking error of size λ is tolerated. This is achieved by the so called λ-tracker which is simple in its design, relies only on structural properties of the process and weak feasibility properties, and does not invoke any estimation or identification mechanism or probing signals. λ-Tracking is proved for a model of an activated sludge process with unknown reaction kinetics and including unknown time-varying process parameters. It is illustrated by simulations that the λ-tracker works successfully, and even under practical circumstances which go beyond what we can prove mathematically, it can cope with 'white noise' corrupting the measurement and periodically acting disturbances.

Nomenclature

V	bioreactor volume (l)
$F_{\text{in}}(t)$	influent flow rate (l/h)
$F_{\text{R}}(t)$	recycle flow rate (l/h)
S_{in}	influent substrate concentration (mgCOD/l)
$S(t)$	substrate concentration in the reactor (mgCOD/l)
$X(t)$	biomass concentration in the reactor (mg/l)
$X_{\text{R}}(t)$	concentration of the biomass in the recycle stream (mg/l)
$X_{\text{ref}}(t)$	reference signal for $X_{\text{R}}(t)$ (mg/l)
$X_{\text{m}}(t)$	measured biomass concentration (mg/l)
$\mu(S)$	specific growth rate (l/h)
$Y(t)$	yield coefficient
c_{ref}	reference coefficient
$r(t)$	output-input ratio of the settler
T_{m}	sensor time constant (h)
$n(t)$	measurement noise
$c_{\text{d}}(t)$	decay rate parameter of biomass concentration

1. Introduction

The purpose of the paper is twofold. On the one hand, we introduce an adaptive controller to regulate an activated sludge process. As opposed to other (adaptive) control strategies suggested in the literature, we do not linearize the system to design a controller, but take the non-linearities into account for controlling the process. The aim is to keep the concentration of biomass proportional to the influent flow rate, and this in the presence of measurement noise, model uncertainties, actuator constraints and disturbances.

On the other hand, the present activated sludge process serves as an example to show the practical relevance of the so called λ-tracker, which has been studied theoretically over the last five years but needs to be modified when it comes to practical constraints. We introduce and study modifications of the λ-tracker which preserve its simplicity and universiality, i.e. it functions for numerous processes as long as they satisfy certain structural properties and meet some feasibility conditions. In particular, the effect of certain design parameters of the controller to the dynamics of the closed-loop system are studied with respect to practical relevance.

The treatment of industrial and urban waste water becomes increasingly relevant. The process of water purifying goes through the two basic stages of liquid and solid line treatments. Liquid phase treatment includes preliminary treatment for removal of coarse solid material, primary treatment for separation of decantable material and biological treatment (activated sludge process and recirculation). The treatment of solid material includes anaerobic digestion and sludge mechanical dehydration. In this paper we design a control structure for the biological treatment stage.

Waste wate treatment is performed in an aeration tank, in which the contaminated water is mixed with biomass in suspension (activated sludge), and the biodegradation process is then triggered in the presence of oxygen. The tank is equipped with a surface aeration turbine which supplies oxygen to the biomass and, additionally, changes its suspension into a homogeneous mass.

After some period, the biomass mixture and the remaining substrate go to a separating chamber where the biologic flocks (biologic sludge) are separated from the treated effluent. The treated effluent is then led to a host environment. The maintenance of adequate concentration of active biomass in the aeration tank, which allows the aerobic degradation of the in-coming

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* Author for correspondence. e-mail: ilchmann@mathematik.tu-ilmenau.de

† Bulgarian Academy of Sciences, Institute of Control and System Research, P.O. Box 79, 1113 Sofia, Bulgaria.

‡ Ilmenau Technical University, Institute of Mathematics, PF 10 05 65, D-98684 Ilmenau, Germany.

waste water, is achieved by the recirculation of the sludge accumulated in the decanter.

The aim is good settling of the biomass in the settler and high conversion of the entering organic material in the bioreactor. The concentration of the biomass in the recycle stream serves as an indicator of both the sludge activity and the sludge settling characteristics, and is therefore considered as the controlled variable.

The main objective of the control system is to keep the recycle biomass concentration close to the reference signal, and this should be achieved in the presence of disturbances and measurement noise acting on the recycle flow rate. The control task is hampered by the strong non-linearity of the process dynamics, the variations in the reaction kinetics and by unknown and possibly time-varying process parameters. These considerations are well known and valid for all biochemical processes, but a typical peculiarity of the waste water treatment system is the (proportional) dependence of the recycle biomass concentration on the influent flow rate, acting as measured disturbance. Since the influent flow rate has generally periodic behaviour, the goal is not to keep the recycle biomass concentration constant, but to follow a desired time trajectory, a proportion of the influence flow rate.

The adaptive control strategy presented in this paper is in the spirit of the so called λ-tracker introduced by Ilchmann and Ryan (1994). λ-tracking was successfully applied to the pH-regulation of a Biogas Tower Reactor at a yeast production company (see Ilchmann and Pahl 1998), and has theoretically been studied mainly for non-linear systems with exponentially stable zero dynamics (or systems with input-to-state-stable subsystems) and strict relative degree one, see e.g. Ryan (1998) or Ilchmann (1998). The results of the aforementioned references are not applicable to models of activated sludge processes and have not been investigated in case of input constraints. In the present paper we will show that λ-tracking, even in the presence of input constraints, is possible for these processes. The controller is much simpler in its design than conventional adaptive controllers based on identification mechanism or robust controllers based on linearization and much more knowledge of the process.

The paper is organized as follows. In § 2 we introduce a mathematical model for the activated sludge process and prove that it captures properties which are expected from the real process. In § 3 we introduce the control objective and the feedback law and gain adaptation by which the control objectives are achieved. Section 4 contains two theorems which prove that λ-tracking is successful for the model if certain weak feasibility assumptions are satisfied. Finally, in § 5 the theoretical results are illustrated by simulations, the intuition is discussed in length, and in addition we compare how the λ-tracker works in practical situations which go beyond the theoretical results achieved so far.

2. The activated sludge process and its model

The exact model of the activated sludge process consists of several reactions, many variables (different biomass and substrate concentrations) and coefficients. Detailed description of bioreactor and settler dynamics involves several non-linear differential equations. Instead of using such a rigorous model, we adopt a modelling strategy based on a reasonable simplification of the description which preserves the essential structural properties of the process. Singular perturbation technique is applied for neglecting substrates and products with low solubility in the liquid phase and reduction of reactions, which are characterized by fast dynamics. Moreover, taking into account the particular aim of the control design, we are not interested in considering separately all different biomass and substrate concentrations in the bioreactor. It is assumed that the process is a single-substrate, single-biomass reaction and is modelled by the system of differential equations

$$\left.\begin{aligned} \dot{X}(t) &= \left[\mu(S(t)) - \frac{F_{\mathrm{in}}(t)+F_{\mathrm{R}}(t)}{V} - c_{\mathrm{d}}\right] X(t) \\ &\quad + \frac{F_{\mathrm{R}}(t)}{V} X_{\mathrm{R}}(t) \\ \dot{S}(t) &= -\frac{1}{Y}\mu(S(t))X(t) + \frac{F_{\mathrm{in}}(t)}{V} S_{\mathrm{in}} \\ &\quad - \frac{F_{\mathrm{in}}(t)+F_{\mathrm{R}}(t)}{V} S(t) \end{aligned}\right\} \tag{1}$$

where the state variables are: $X(t)$, biomass concentration in the reactor at time t; $S(t)$, substrate concentration in the reactor at time t; $X_{\mathrm{R}}(t)$, biomass concentration in the recycle stream at time t (the output); $F_{\mathrm{R}}(\cdot)$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is the recycle flow rate as the input, a piecewise continuous function; $F_{\mathrm{in}}(\cdot)$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is the influent flow rate, a piecewise continuous function which is bounded away from zero†; and $\mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$ denote the set of non-negative, positive real numbers, respectively. $\mu(\cdot)$ denotes the specific growth rate, which is the key parameter for description of biomass growth and substrate consumption of the reaction; a typical example may be Monod's growth rate $\mu(S) = \mu_{\mathrm{m}} S/(K_m + S)$, where μ_{m} is the maximum growth rate and K_m the Michaelis–Menten constant. $S \mapsto \mu(S)$ is continuous, bounded and $\mu(0) = 0$. $c_{\mathrm{d}} X$ denotes the decay rate of the biomass concentration (which is added in the model to simulate biomass

† A function $\varphi(\cdot)$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is called *bounded away from zero* if, and only if, there exists some $\hat{\varphi} > 0$ such that $\varphi(t) > \hat{\varphi}$ for all $t \geq 0$.

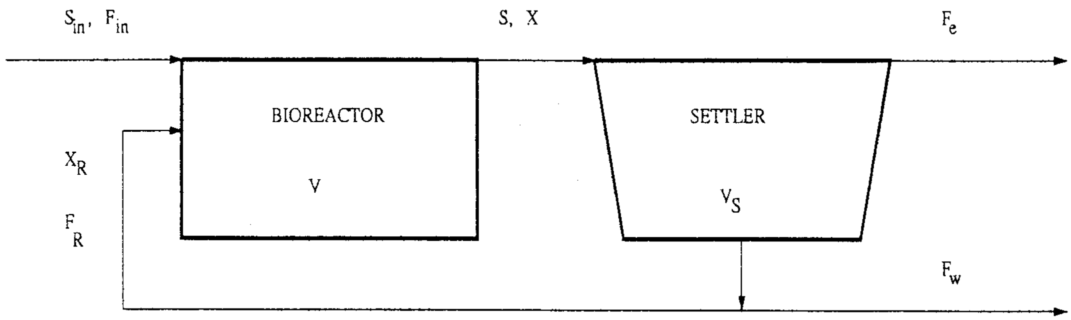

Figure 1. Activated sludge process with settler.

mortality) with $c_d > 0$ as the decay rate parameter, $Y > 0$ is the yield coefficient, V is the reactor volume, and S_{in} the influent substrate concentration.

It is supposed that none of the biomass is left in the effluent F_e of the settler (see figure 1), so that the whole biomass in the clarifier is settled. The concentration of the biomass in the recycle stream depends on the settler used. The dynamics of the concentration of the biomass in the settler, $X_R(t)$, can be described by the mass balance equation

$$\dot{X}_R(t) = \frac{F_{in}(t) + F_R(t)}{V_s} X(t) - \frac{F_w(t) + F_R(t)}{V_s} X_R(t) \quad (2)$$

where $F_w(\cdot)$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denotes the waste flow rate, a piecewise continuous function bounded away from zero, and V_s the volume of the settler.

Proposition 1 shows that the model (1) and (2) captures the properties which are expected.

Proposition 1: *For every set of positive initial conditions* $(X(0), S(0), X_R(0)) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ *there exists a unique solution* $(X(\cdot), S(\cdot), X_R(\cdot))$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ *of* (1), (2), *and this solution is bounded.*

Proof: Existence and uniqueness of a solution of (1) and (2) on a maximal interval of existence $[0, \omega)$, for some $\omega \in (0, \infty]$, follows from the theory of ordinary differential equations.

To prove positive invariance of the positive orthant suppose, on the contrary, that there exists some $t' \in (0, \omega)$ such that, for all $t \in (0, t')$

$$X(t')S(t')X_R(t') = 0 \quad \text{and} \quad X(t)S(t)X_R(t) > 0$$

If $S(t') = 0$, then $F_{in}(t') > 0$ yields $\dot{S}(t') > 0$ and by continuity this contradicts $S(t') = 0$. If $X(t') + X_R(t') \geq 0$, then integrating each of (1) and (2) yields that $X(t') > 0$ and $X_R(t') > 0$. Therefore, we have proved that all three components of the trajectory stay positive.

It remains to prove boundedness, which immediately gives $\omega = \infty$. Differentiation of

$$P(t) := X(t) + YS(t) + \frac{V_S}{V} X_R(t)$$

along (1) and (2) and setting

$$\alpha := \inf_{t \geq 0} \left\{ c_d, \frac{F_{in}(t) + F_R(t)}{V}, \frac{F_w(t)}{V_S} \right\}$$

yields

$$\begin{aligned} \frac{d}{dt} P(t) &= -\left[c_d X(t) + \frac{F_{in}(t) + F_R(t)}{V} YS(t) \right. \\ &\quad \left. + \frac{F_w(t)}{V_S} \frac{V_S}{V} X_R(t) \right] + \frac{Y}{V} F_{in}(t) S_{in} \\ &\leq -\alpha P(t) + \frac{Y}{V} F_{in}(t) S_{in} \end{aligned}$$

Since $F_{in}(\cdot)$ and $F_w(\cdot)$ are supposed to be bounded away from zero, it follows that $\alpha > 0$, and hence

$$\begin{aligned} P(t) &\leq e^{-\alpha t} P(0) + \int_0^t e^{-\alpha(t-\tau)} \frac{Y}{V} F_{in}(\tau) S_{in} \, d\tau \\ &\leq e^{-\alpha t} P(0) + \frac{Y}{\alpha V} S_{in} \sup_{t \geq 0} F_{in}(t) \end{aligned}$$

This completes the proof. □

Since the settler has first order dynamics which are much faster than the bioreactor dynamics, and since we assume that a constant ratio of output to input solids concentration is maintained, we may approximate the settler behaviour by

$$X_R(t) = r(t) X(t) \quad (3)$$

where $r(\cdot)$ is a continuously differentiable and bounded function with bounded inverse, bounded derivative and $r(t) > 1$ for all $t \geq 0$. The biomass concentration in the settler is higher than the biomass concentration in the reactor because it accumulates at the bottom of the vessel and good settling is only possible if the settler is designed such that $X_R(t) > X(t)$.

Moreover, in order to capture model uncertainties and disturbances we assume that the coefficients in (1) are time-varying, and instead of (1) and (2) we will consider (1) with time-varying coefficients and (3).

Therefore, the model considered in the present paper has the form

$$\left.\begin{aligned} \dot{X}_R(t) &= \left[\frac{\dot{r}(t)}{r(t)} + \mu(t, S(t)) - \frac{F_{in}(t)}{V} - c_d(t)\right. \\ &\quad \left. + \frac{r(t)-1}{V} F_R(t)\right] X_R(t) \\ \dot{S}(t) &= -\frac{1}{Y(t)} \mu(t, S(t)) \frac{1}{r(t)} X_R(t) \\ &\quad - \frac{F_{in}(t) F_R(t)}{V} S(t) + \frac{F_{in}(t)}{V} S_{in} \end{aligned}\right\} \tag{4}$$

where the functions $F_{in}(\cdot)$, $1/[F_{in}(\cdot)]$, $c_d(\cdot)$, $r(\cdot)$, $\dot{r}(\cdot)$, $1/[r(\cdot)]$, $Y(\cdot)$, $1/[Y(\cdot)]$: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are continuous and bounded. $\mu(\cdot,\cdot)\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and bounded and $\mu(t, 0) = 0$ for all $t \geq 0$. $r(\cdot)$ is continuously differentiable with bounded derivative and $r(t) > 1$ for all $t \geq 0$.

Note that we cannot apply to (4) the theory of λ-tracking for non-linear systems with exponentially stable zero dynamics and strict relative degree one as developed in Ilchmann (1998). One of the crucial assumptions, i.e. the high frequency gain must be bounded away from zero, is not satisfied in (4) for $[(r(t) - 1)/V] X_R(t) F_R(t)$.

3. Control objective and adaptive feedback strategy

The reference signal to be tracked is a proportion of the bounded influence flow rate

$$X_{ref}(t) = c_{ref} F_{in}(t), \qquad c_{ref} > 0$$

and this reference signal is assumed to be measurable. The output $X_R(t)$ is supposed to track asymptotically $X_{ref}(t)$, where a prespecified tracking error of size $\lambda > 0$ is tolerated.

Note that some $\lambda > 0$ has to be chosen since a steady state error is expected if there is no integration term in the control law. From a practical point of view it is justified to choose λ since the measurement cannot be that accurate anyway and 2–4% deviation of the controlled variable around the reference signal is most likely to be reached. Moreover, it is a reasonable compromise between control efforts and realistically possible process performance.

3.1. *Feedback control law*

The control objective will be achieved by a simple proportional output feedback of the error

$$e(t) = X_{ref}(t) - X_R(t) \tag{5}$$

taking into account input saturations

$$F_R(t) = \underset{[0, \hat{F}_R]}{\mathrm{sat}} (k(t) e(t)) \tag{6}$$

where, for $a < b$, we use the notation

$$\underset{[a,b]}{\mathrm{sat}}(\eta) := \begin{cases} a, & \eta < a \\ \eta, & \eta \in [a, b] \\ b, & \eta > b \end{cases}$$

and $\hat{F}_R > 0$ denotes the upper bound of the input constraint.

We stress the simplicity of the feedback law (6), in particular when compared to other controllers aiming to control the same process (see e.g. Schaper *et al.* 1990). In our approach we make use of the non-linear structure of the process rather then ignoring it by linearization. Equation (6) is a proportional output feedback with time-varying gain. The adaptation of the gain is crucial and will be introduced below.

3.2. *Feasibility assumptions*

Due to the practical restrictions of the process input saturations are invoked. These bounds certainly depend on the systems' parameters and the following feasibility assumptions are therefore necessary.

To simplify the notation, define for all $t \geq 0$ and all $S > 0$

$$\alpha(t, S) := -\frac{\dot{r}(t)}{r(t)} - \mu(t, S) + \frac{F_{in}(t)}{V} + c_d(t) \tag{7}$$

Assumption 1: *Suppose that there exist some constants* $\alpha_1, \alpha_2 > 0$ *such that*

$$\alpha_1 \geq \alpha(t, S) \geq \alpha_2 \qquad \text{for all } t \geq 0 \text{ and all } S > 0 \tag{8}$$

That $\alpha(\cdot,\cdot)$ is uniformly bounded away from zero ensures that $X_R(t)$ in (4) decreases as long as $F_R(t) = 0$. Note that the uniform bound α_1 follows already from the restrictions on the functions in (4), these assumptions correspond to the physical meaning of the functions involved.

Assumption 2: *The saturation bound* $\hat{F}_R > 0$ *in* (6) *must be sufficiently large so that there exists some* $\varepsilon > 0$ *such that*

$$\frac{r(t)-1}{V} \hat{F}_R \geq \alpha_1 + \varepsilon \qquad \text{for all } t \geq 0 \tag{9}$$

We may rewrite the first equation in (4) as

$$\dot{X}_R(t) = \left[-\alpha(t, S(t)) + \frac{r(t)-1}{V} F_R(t)\right] X_R(t)$$

Now it is easy to see that the input $F_R(t)$ has roughly to be chosen as follows. If $X_R(t)$ needs to decrease in order to follow the reference signal, then put $F_R(\cdot) \equiv 0$ so that the system will force $X_R(t)$ to decrease by 'itself' since $\alpha(t, S(t))$ is positive. The physical interpretation of this phenomenon is that if the biomass concentration in the reactor exceeds some reference level, i.e. $X_R(t)$ large, then this will cause undernourishment of the micro-organism community (because of not sufficient feeding substrate) and therefore increase the biomass mortality. The way to prevent this is to stop the sludge recirculation, which leads to less biomass concentration in the reactor and consequently to a decrease of $X_R(t)$.

If $X_R(t)$ needs to increase to follow the reference signal, then $(r(t)-1)F_R(t)/V$ needs to dominate $\alpha(t, S(t))$, and this is ensured by (9).

3.3. *Gain adaptation*

Now we are in a position to introduce the gain adaptation of the feedback law. We will use two phases. In the first phase we set

$$\dot{k}(t) := \gamma \begin{cases} (|e(t)| - \lambda)^{\beta}, & |e(t)| \geq \lambda \\ 0, & |e(t)| < \lambda \end{cases} \tag{10}$$

where $\gamma > 0$ and $\beta \geq 1$, $k(0) > 0$ are design parameters. The design parameters are crucial to the dynamics of the closed-loop system as will be illustrated in the simulations in § 5. $\lambda > 0$ is the prespecified tolerated tracking error. $k(t)$, which is governed by (10), increases as long as the error is outside the λ-strip. Once the gain is sufficiently large, (6) will achieve tracking, so that finally the error is kept inside the λ-strip and thus $\dot{k}(t) = 0$, or in other words, $k(t)$ keeps constant.

However, the gain $k(t)$ defined by (10) turns out to be unbounded in many practical situations which is due to noise corrupting the measurements. For this reason we choose the modified gain adaptation

$$\dot{k}(t) = -\sigma[k(t) - k^*] + \gamma \begin{cases} (|e(t)| - \lambda)^{\beta}, & |e(t)| \geq \lambda \\ 0, & |e(t)| < \lambda \end{cases} \tag{11}$$

with design parameters $\sigma, \gamma > 0$ and $\beta \geq 1, k(0), k^* > 0$.

To give some intuition behind this gain adaptation first note the following obvious properties.

Remark 1: Let $e(\cdot): \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be arbitrary but fixed. Then $k(\cdot)$ in (11) satisfies:

(i) If $k(0) > 0$, then $k(t) > 0$ for all $t \geq 0$.
(ii) If $k(t) < k^*$, then $\dot{k}(t) > 0$.
(iii) For every $\varepsilon > 0$ there exists a $t' > 0$ such that $k(t) \geq k^* - \varepsilon$ for all $t \geq t'$.
(iv) If $e(\cdot)$ is bounded, then $k(\cdot)$ is bounded.

From a mathematical point of view, the main difference between the two gain adaptations is that $k(\cdot)$ in (10) is monotonically non-decreasing whilst $k(\cdot)$ in (11) might oscillate. But the motivation for introducing (11) is as follows: If (10), combined with the feedback law (6), is applied to the process (4) and some feasibility assumptions are satisfied, then according to Theorem 1 λ-tracking is ensured. It can be shown that λ-tracking also works in the presence of noise corrupting the output, provided the noise is bounded and its derivative is bounded. This has been done in different ways (see e.g. Ilchmann (1998)). We do not consider this in the present paper for two reasons. One is to avoid technicalities, and the other, more important reason, is that we like to introduce the gain adaptation (11) which copes with disturbances in a different way. Suppose the process is run in two phases. In Phase I, which should contain 'little' noise, the gain adaptation (10) is applied. Eventually the error is within the λ-strip and the increase of the gain $k(t)$ is negligible, say at finite time T. One might have a good guess for k^* being slightly larger than $k(T)$. Then one switches to Phase II where possibly disturbances are corrupting the output which drive the error outside the λ-strip, the gain $k(t)$ in (11) will increase, until finally the error is forced back into the λ-strip and, in the absence of further noise, $k(t)$ tends back to k^*.

Hence (6) and (10) is only used in Phase I, where the disturbances are low. k^* should be sufficiently large so that the non-adaptive feedback

$$F_R(t) = \underset{[0,\hat{F}_R]}{\text{sat}} (k^* e(t))$$

achieves λ-tracking. If $k(t)$ tends, over a time interval of 'considerable' length, to a constant value k^*, then switch to Phase II and apply (6), (11) and k^* as found in Phase I. See the illustrations in § 5.

It might be worth mentioning that (11) has to be treated carefully. If k^* is too small to ensure tracking, then the closed-loop system might exhibit oscillatory and even chaotic behaviour. This has already been observed for two-dimensional linear minimum phase systems of the form

$$\dot{y}(t) = -dy(t) + u(t)$$
$$\dot{k}(t) = -\sigma k(t) + y(t)^2$$
$$u(t) = -k(t)y(t) + h$$

In Mareels *et al.* (1999) it is proved that the dynamical behaviour depends crucially on the parameters $\sigma > 0$, $d, h \in \mathbb{R}$, and might exhibit oscillatory behaviour and limit cycles. However, when k^* in (11) is chosen sufficiently large, then oscillatory or chaotic behaviour is not possible.

4. Adaptive λ-tracking

In this section we will present our main results, showing that (6), (10) or (6), (11) achieve λ-tracking of a proportion of the influent flow rate if some weak feasibility assumptions are satisfied.

Theorem 1: *Consider the model* (4) *of an activated sludge process and suppose* (8) *and* (9) *are satisfied. Suppose furthermore that there exists some* $\delta \in (0, \lambda)$ *such that*

$$\alpha_1 < \frac{r(t)-1}{V} k(0)[\lambda - \delta] \qquad \text{for all } t \geq 0 \tag{12}$$

If a reference signal $X_{\text{ref}}(\cdot)$ *is proportional to the influence flow rate, i.e.* $X_{\text{ref}}(\cdot) \equiv c_{\text{ref}} F_{\text{in}}(\cdot)$, *for some* $c_{\text{ref}} > 0$, *and if* $\lambda \in (0, X_{\text{ref}}(t))$ *for all* $t \geq 0$, *and*

$$-\alpha_2[\lambda + X_{\text{ref}}(t)] < \dot{X}_{\text{ref}}(t) < \left[-\alpha_1 + \frac{r(t)-1}{V}\hat{F}_{\text{R}}\right] \times [X_{\text{ref}}(t) - \lambda] \qquad \text{for all } t \geq 0 \tag{13}$$

then the proportional error feedback (6) *with gain adaptation* (10) *applied to the waste water process* (4), *for arbitrary initial condition* $(X_{\text{R}}(0), S(0), k(0)) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, *yields a closed-loop system which has a unique solution*

$$(X_{\text{R}}(\cdot), S(\cdot), k(\cdot)) : \mathbb{R}_{\geq 0} \quad \to \quad \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$$

on the whole of $\mathbb{R}_{\geq 0}$ *and, moreover:*

(i) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}_{\geq 0}$ *exists;*
(ii) $X_{\text{R}}(\cdot)$ *and* $S(\cdot)$ *are bounded;*
(iii) $\lim_{t\to\infty} \operatorname{dist}(|X_{\text{ref}}(t) - X_{\text{R}}(t)|, [0, \lambda]) = 0.$

Proof: Existence and uniqueness of the solution follows from the theory of ordinary differential equations, and it is easy to see that finite escape time cannot occur.

We proceed in several steps.

Step 1. We prove positivity of $X_{\text{R}}(\cdot)$ and $S(\cdot)$.

To see that $S(\cdot)$ stays positive, suppose $S(t) = 0$. Then (4) yields

$$\dot{S}(t) = \frac{F_{\text{in}}(t)}{V} S_{\text{in}} > 0$$

and by continuity it follows that $S(\cdot) \equiv 0$ is repelling from below, i.e. $\dot{S}(t) > 0$ for 'small' $S(t)$. To establish positivity of $X_{\text{R}}(\cdot)$ note that by (8)

$$\dot{X}_{\text{R}}(t) \geq \left[-\alpha_1 + \frac{r(t)-1}{V} F_{\text{R}}(t)\right] X_{\text{R}}(t)$$

If $F_{\text{R}}(t) = \hat{F}_{\text{R}}$, then (9) yields $\dot{X}_{\text{R}}(t) \geq \varepsilon X_{\text{R}}(t)$. If $F_{\text{R}}(t) < \hat{F}_{\text{R}}$ and $X_{\text{R}}(t)$ is small, say $X_{\text{R}}(t) \in (0, \delta]$, then $e(t) \geq X_{\text{ref}}(t) - \delta > \lambda - \delta > 0$ and $k(t)e(t) > k(0)[\lambda - \delta]$. Hence in view of (12),

$$-\alpha_1 + \frac{r(t)-1}{V} F_{\text{R}}(t) > -\alpha_1 + \frac{r(t)-1}{V} k(0)[\lambda - \delta] > 0$$

whence $\dot{X}_{\text{R}}(t) > 0$. Therefore $X_{\text{R}}(t)$ stays positive for all $t \geq 0$.

Step 2. We prove that if there exists some $t_0 \geq 0$ such that

$$\lambda k(t) \geq \hat{F}_{\text{R}} \qquad \text{for all } t \geq t_0 \tag{14}$$

then there exists some $t' \geq t_0$ such that $|e(t)| < \lambda$ for all $t \geq t'$.

Suppose (14) holds. Then the following three claims are true:

Claim 1: There exists $t_1 \geq t_0$ such that $e(t_1) < \lambda$.

Seeking a contradiction, suppose $e(t) \geq \lambda$ for all $t \geq t_0$. Then by (14) $F_{\text{R}}(t) = \hat{F}_{\text{R}}$ for all $t \geq t_0$ and, in view of (4), (8) and (9) we have $\lim_{t\to\infty} X_{\text{R}}(t) = \infty$, hence $\lim_{t\to\infty} e(t) = -\infty$. This is a contradiction to $e(t) \geq \lambda$ for all $t \geq t_0$, and therefore Claim 1 has been proved.

Claim 2: There exists $t_2 \geq t_0$ such that $e(t_2) > -\lambda$.

Suppose the contrary. Then $F_{\text{R}}(t) = 0$ for all $t \geq t_0$ and, in view of (4) and (8) we have $\lim_{t\to\infty} X_{\text{R}}(t) = 0$, whence $\lim_{t\to\infty} e(t) = c_{\text{ref}} F_{\text{in}}(t)$. This is a contradiction to the assumption and so Claim 2 is proved.

Claim 3: There exists $t_3 \geq t_0$ such that $e(t) \in (-\lambda, \lambda)$ for all $t \geq t_3$.

To prove this we have to study the dynamics of the error. To this end set

$$\psi(t) := -\left[-\alpha(t, S(t)) + \frac{r(t)-1}{V} F_{\text{R}}(t)\right] X_{\text{ref}}(t) + \dot{X}_{\text{ref}}(t)$$

and thus

$$\dot{e}(t) = \left[-\alpha(t, S(t)) + \frac{r(t)-1}{V} F_{\text{R}}(t)\right] e(t) + \psi(t) \tag{15}$$

It follows from Claims 1 and 2 that $e(t') \in (-\lambda, \lambda)$ for some $t' \geq t_0$. We consider the two cases that $e(t)$ hits the boundaries.

If $e(t) = \lambda$ for some $t > t'$, then in view of (14) and (6) $F_{\text{R}}(t) = \hat{F}_{\text{R}}$ and hence, in view of (4), (8) and (13)

$$\dot{e}(t) \leq \left[-\alpha_1 + \frac{r(t)-1}{V}\hat{F}_{\text{R}}\right][\lambda - X_{\text{ref}}(t)] + \dot{X}_{\text{ref}}(t) < 0$$

By continuity of $e(\cdot)$ we conclude that $e(\cdot) \equiv \lambda$ is repelling from above.

If $e(t) = -\lambda$ for some $t > t'$, then $F_{\text{R}}(t) = 0$ and hence, in view of (4), (8) and (13)

$$\dot{e}(t) \geq \alpha_2[\lambda + X_{\text{ref}}(t)] + \dot{X}_{\text{ref}}(t) > 0$$

By continuity of $e(\cdot)$ we conclude that $-\lambda$ is repelling. This completes the proof of Step 2.

Step 3. $k(\cdot)$ is bounded. If it was unbounded, then (14) would hold for some $t' \geq t_0$ and hence $\dot{k}(t) = 0$ for all $t \geq t'$, contradicting unboundedness of $k(\cdot)$.

Since $t \mapsto k(t)$ is monotonically non-decreasing, (i) follows.

Step 4. We prove boundedness of $X_R(\cdot)$ and $S(\cdot)$.

Let $\hat{\psi} := \sup_{t\geq 0} \|\psi(t)\|$. If $e(t) < -(\hat{\psi}+1)/\ \alpha_2$, then $F_R(t) = 0$ and by (4) and (8) we have

$$\dot{e}(t) = -\alpha(t, S(t))e(t) + \psi(t) > \alpha_2(\hat{\psi}+1)/\alpha_2 + \psi(t) > 1$$

and hence $e(\cdot)$ is bounded from below.

Now boundedness of $F_{\mathrm{in}}(\cdot)$ respectively $X_{\mathrm{ref}}(\cdot)$ yields boundedness of $X_R(\cdot)$ from above, and boundedness from below follows from positivity.

Integrating the second equation in (4) yields

$$S(t) \leq \exp\left(-\int_0^t \frac{F_{\mathrm{in}}(\tau) + F_R(\tau)}{V}\,\mathrm{d}\tau\right) S(0) + \int_0^t \exp\left(-\int_s^t \frac{F_{\mathrm{in}}(\tau) + F_R(\tau)}{V}\,\mathrm{d}\tau\right) \frac{F_{\mathrm{in}}(s)}{V} S_{\mathrm{in}}\,\mathrm{d}s$$

and hence, boundedness of $F_{\mathrm{in}}(\cdot)$ from above and away from zero shows boundedness of $S(\cdot)$ from above. Since $S(\cdot)$ is positive, (ii) is proved.

Step 5. We prove (iii).

Define

$$d_\lambda(e) = \max\{e - \lambda, 0\}$$

Since $k(\cdot)$ and $e(\cdot)$ are bounded we conclude from (4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r} d_\lambda(e(t))^\beta\right) = d_\lambda(e(t))^{\beta-1} \frac{e(t)}{|e(t)|} \dot{e}(t)$$

is bounded as well. Boundedness of $k(\cdot)$ is equivalent to $d_\lambda(e(\cdot))^\beta$ is integrable over $[0, \infty)$, and so it follows from Barbălat's Lemma (see, e.g. Khalil (1996)) that $\lim_{t\to\infty} d_\lambda(e(t))^\beta = 0$. This proves (iii). □

The following theorem is similar to Theorem 1 but now the modified gain adaptation (11) is applied. From a mathematical point of view, the main difference between the two gain adaptations is that $k(t)$ in (11) is no longer monotone.

Theorem 2: *Consider the model* (4) *of an activated sludge process and suppose* (8) *and* (9) *are satisfied. Suppose furthermore that there exists some* $\delta \in (0, \lambda)$ *such that*

$$\alpha_1 < \frac{r(t) - 1}{V} \min\{k(0), k^*\}[\lambda - \delta] \tag{16}$$

In a reference signal $X_{\mathrm{ref}}(\cdot)$ *is proportional to the influent flow rate, i.e.* $X_{\mathrm{ref}}(\cdot) \equiv c_{\mathrm{ref}} F_{\mathrm{in}}(\cdot)$, *for some* $c_{\mathrm{ref}} > 0$, $\lambda \in (0, X_{\mathrm{ref}}(t))$ *for all* $t \geq 0$, *and if the derivative satisfies* (13), *then the proportional error feedback* (6) *with gain adaptation* (11) *for suffciently large* k^* *applied to the waste water process* (4), *for arbitrary initial condition* $(X_R(0), S(0), k(0)) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, *yields a closed-loop system which has a unique solution*

$$(X_R(\cdot), S(\cdot), k(\cdot)) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$$

on the whole of $\mathbb{R}_{\geq 0}$ and, moreover

(i) $\lim_{t\to\infty} k(t) = k^*$,

(ii) $X_R(\cdot)$ and $S(\cdot)$ are bounded,

(iii) $\lim_{t\to\infty} \operatorname{dist}(|X_{\mathrm{ref}}(t) - X_R(t)|, [0, \lambda]) = 0$.

Proof: Existence and uniqueness of the solution follows from the theory of ordinary differential equations, and it is easy to see that finite escape time cannot occur.

Positivity of $X_R(\cdot)$ and $S(\cdot)$ follows similarly as the Step 1 of the proof of Theorem 1.

By Remark 1 we might choose k^* sufficiently large so that (14) holds. Now boundedness of $k(\cdot)$ and $X_R(\cdot)$, $S(\cdot)$ follows as in Steps 2–4 of the proof of Theorem 1. This proves (ii).

To prove (i) and (iii) first note that, as opposed to Theorem 1, boundedness of $k(\cdot)$ only yields that the convolution of $\mathrm{e}^{-\sigma t}$ and $d_\lambda(e(t))^r$ is bounded on $[0, \infty)$. However, since k^* may be chosen sufficiently large, Claim 3 in Step 2 of the proof of Theorem 1 holds. This proves (i) and (iii) and therefore the proof is complete. □

5. Simulations of λ-tracking controllers applied to an activated sludge process

The simulations presented in the present section can be divided into three groups:

(i) simulations to show that Theorems 1 and 2 work successfully;

(ii) simulations if the adaptive gain $k(t)$ of the controller is replaced by a constant gain k;

(iii) simulations for the adaptive controller if applied to the model with additional sensor dynamics and noise.

Case (ii) has not been proved explicitly but it can easily be seen that all results hold true if the adaptive gain parameter $k(t)$ is replaced by a sufficiently large constant gain k. It is interesting to compare the non-adaptive results with the adaptive ones and to see that the high-gain controller is not 'very high'. More precisely, our experience of this problem is that in almost all cases the limit of the adaptively determined gain is only slightly larger than the constant gain necessary to achieve tracking. The third class of simulations comprises the realistic case that the model (4) is interconnected with sensor dynamics and also noise corrupting the output. This case goes beyond the theoretical results proved so far. However, we plot these results in the same figures and it might be interesting to demonstrate that the λ-tracker achieves satisfactory results even in this case.

The sensor dynamics are modelled by

$$T_m \dot{X}_m(t) = -X_m(t) + X_R(t) + n(t) \tag{17}$$

and additionally the dynamics are corrupted by some 'white Gaussian noise' $n(\cdot): \mathbb{R}_{\geq 0} \to \mathbb{R}$ with variance 1200, which is commonly used to simulate noisy measurements. The variance we choose corresponds to 6% noise on the data. Equation (17) models that the biomass concentration in the recycle stream is inferred by measuring the concentration of carbon dioxide and oxygen in the reactor (see e.g. Tenno *et al.* (1989)) or approximated on-line by turbidimetric measurements (see e.g. Verstraete and van Vaerenbergh (1986)).

The specific model we consider for the simulations is

$$\left.\begin{aligned} \dot{X}_R(t) &= \left[\frac{\dot{r}(t)}{r(t)} + \mu(t, S(t)) - \frac{F_{in}(t)}{V} \right. \\ &\quad \left. - c_d(t) + \frac{r(t)-1}{V} F_R(t)\right] X_R(t) \\ \dot{S}(t) &= -\frac{1}{Y(t)} \mu(t, S(t)) \frac{1}{r(t)} X_R(t) \\ &\quad - \frac{F_{in}(t) + F_R(t)}{V} S(t) + \frac{F_{in}(t)}{V} S_{in} \\ \tfrac{1}{12}\dot{X}_m(t) &= -X_m(t) + X_R(t) + n(t) \end{aligned}\right\} \tag{18}$$

with constant process parameters

$$V = 1.5 \times 10^7 \text{ (l)}, \qquad S_{in} = 300 \text{ (mgCOD/l)}$$

and uncertainties are modelled by time-varying parameters

$$\begin{aligned} \mu(t, S) &= \mu_m(t) \frac{S}{S + K_m(t)} \\ \mu_m(t) &= 0.2 + 0.1 \sin\left(\frac{2\pi t}{3} + \frac{4\pi}{3}\right) \\ K_m(t) &= 90 + 30 \sin\frac{\pi t}{2} \\ Y(t) &= 0.6 + 0.1 \sin\left(\frac{\pi t}{3} + \frac{\pi}{3}\right) \\ c_d(t) &= 10^{-4}\left(25 + 5 \sin\frac{\pi t}{12}\right) \\ r(t) &= 4 + \sin\frac{\pi t}{6} \text{ (l/h)} \end{aligned}$$

Our control objective is to track the reference signal

$$\left.\begin{aligned} X_{ref}(t) &= c_{ref} F_{in}(t) \\ c_{ref} &= 3.8 \times 10^{-3} \\ F_{in}(t) &= 3 \times 10^6 \left(1 + 0.25 \sin\frac{\pi t}{12}\right) \end{aligned}\right\} \tag{19}$$

within a tolerated error of 2–4%, so that we set $\lambda = 300$.

The above data coincide with the typical range for domestic waste water and are chosen identical to those in Schaper *et al.* (1990). We keep these data for all of the following simulations and the initial conditions are always set to

$$\begin{aligned} S(0) &= 8 \text{ (mgCOD/l)} \\ X_R(0) &= 11.4 \times 10^3 \text{ (mg/l)} \\ X_m(0) &= 0 \text{ (mg/l)} \end{aligned}$$

All simulations are done by MATLAB/SIMULINK using a 4th order Runge–Kutta method with constant integration step of 5 min corresponding to the time constant of the sensor.

5.1. *Non-adaptive control*

We like to show that the adaptively determined gain $k(t)$ is only slightly larger than the constant gain necessary to achieve the same control objective. For this reason we first apply a *non-adaptive* feedback

$$F_R(t) = \mathrm{sat}_{[0,10^6]}(k' e(t)) \tag{20}$$

where

$$e(t) = X_{ref}(t) - X_m(t), \qquad X_{ref}(\cdot) \quad \text{as in (19)}$$

to the process (18). Note that the actuator saturation bounds are 0 respectively 10^6 (l/h). The upper bound captures the physical limitation for the capacity of the valve. The particular process considered represents a real urban waste water, where the recirculation is carried out by electro-pump with a standard flow operating condition 300 (l/s).

The error with respect to different gains

$$k' = 10^3, \quad 2 \times 10^3, \quad 5 \times 10^3, \quad 7 \times 10^3$$

is depicted in figure 2.

The process is simulated over 24 h which captures a typical period of the process behaviour. Note the severe realistic conditions such as load disturbances and input constraints. We start at $t = 0$ with $e(0) = X_{ref}(0) - 0 = 11.4 \times 10^3$. Tracking is not achieved for $k' = 1000$, but $k' = 5000$ yields a satisfactory result. The higher k' the better the tracking becomes. However, the drawback of large k' is a considerable overshoot, which becomes apparent in the initial phase. This overshoot is due to the fact that the larger k', the faster the decay of the error to zero and hence the steeper $e(t)$ goes into zero; when the error hits $-\lambda$ at $t \approx 0.1$ (h), then k' has no influence and the system 'itself' ($F_R(t) = 0$) drives the error back into the λ-neighbourhood.

So there is a tradeoff between small k' to prevent a too big overshoot, and large enough k' to ensure λ-tracking.

Note the simulations without sensor dynamics for the constant gain $k' = 5000$, have a bigger overshoot

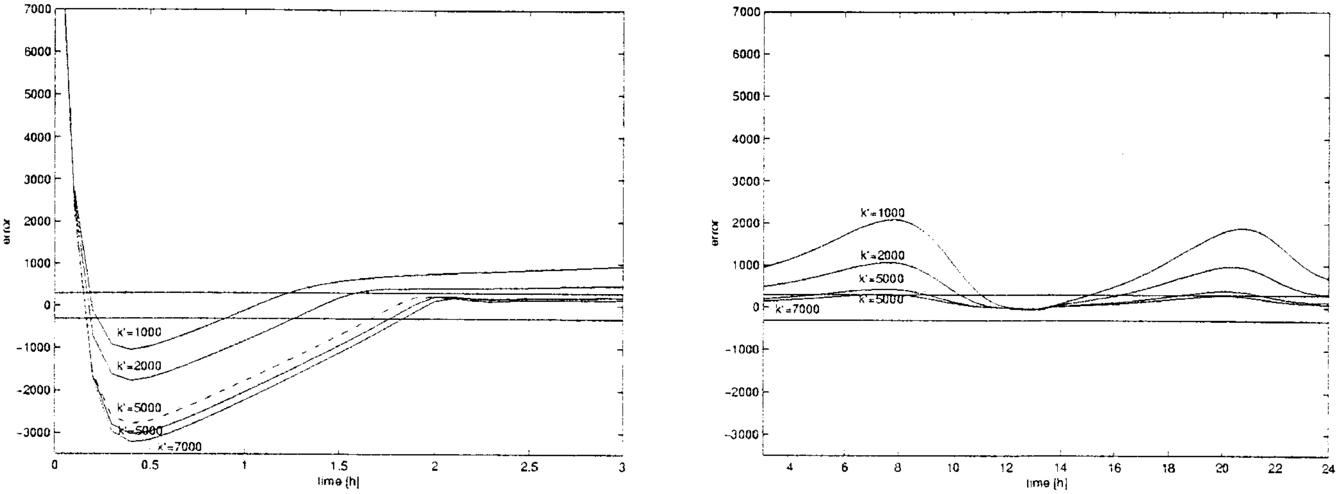

Figure 2. Error $e(t)$ for constant output feedback (20) applied to (18) for $k' = 10^3, 2 \times 10^3, 5 \times 10^3, 7 \times 10^3$ over time periods $[0, 3]$ and $[3, 24]$ and without noise. The dashed line shows no sensor dynamics, i.e. $X_m = X_R$, no noise, i.e. $n(\cdot) = 0$ and $k' = 5 \times 10^3$.

but are very similar in nature and achieve satisfactory λ-tracking, too.

5.2. *Gain adaptation with λ-tracker* (10)

In order to 'learn' adaptively a suitable gain we apply the gain adaptation (10) to (18) where $k(t)$ increases monotonically until the error stays within the λ-strip. First we consider an ideal case where there is not any noise corrupting the measurement, i.e. $n(\cdot) \equiv 0$.

We use the output feedback

$$F_R(t) = \text{sat}_{[0,10^6]}(k(t)e(t)) \tag{21}$$

where

$$e(t) = X_{\text{ref}}(t) - X_m(t), \qquad X_{\text{ref}}(\cdot) \quad \text{as in (19)}$$

and the gain adaptation

$$\dot{k}(t) := \gamma \begin{cases} |e(t)| - 300, & |e(t)| \geq 300 \\ 0, & |e(t)| < 300 \end{cases} \tag{22}$$

where $k(0) = 0$ and $\gamma = 1$ or 7.

In figure 3 the simulations are depicted over a period of 48 h. The larger γ, the faster the error tends into the λ-strip; but the price to pay is a worse transient behaviour (the overshoot of the error is large in the initial phase), and the limiting gain $k_\infty = \lim_{t \to \infty} k(t)$ becomes larger. The reason for this is similar to that in (20). Note also that for $\gamma = 1$ the limit gain is $k_\infty \approx 5000$, which is successful in the non-adaptive case, see figure 2. If there are no sensor dynamics, i.e. $X_m = X_R$, and no noise, then figure 3 depicts a bigger

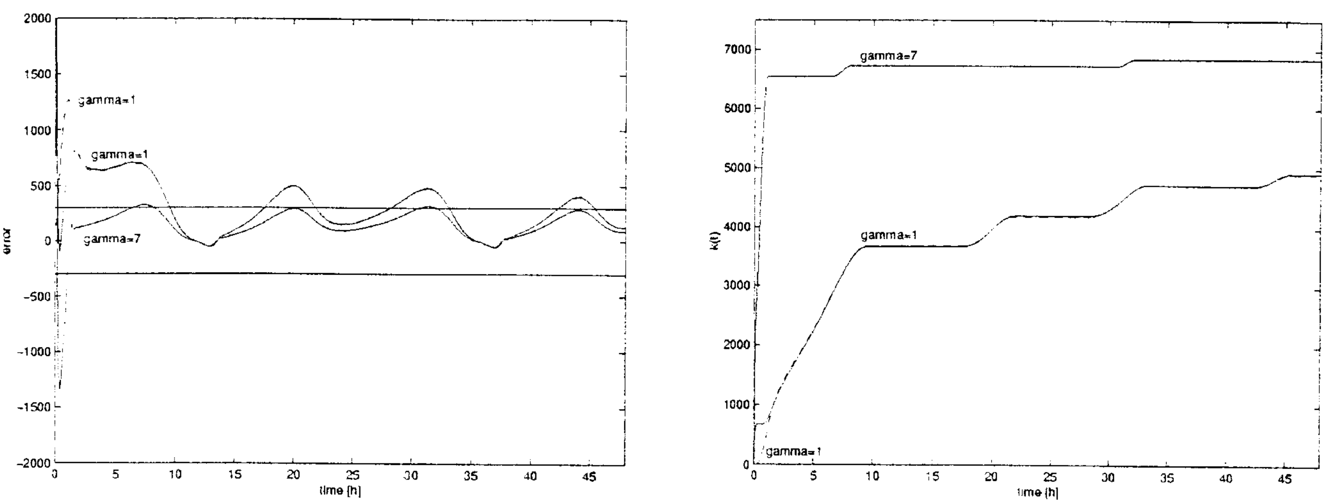

Figure 3. Error $e(t)$ and gain adaptation $k(t)$ for (21), (22) applied to (18) without any noise, $\gamma = 1, 7$. The dashed line shows no sensor dynamics, i.e. $X_m = X_R$, no noise, i.e. $n(\cdot) = 0$ and $\gamma = 1$.

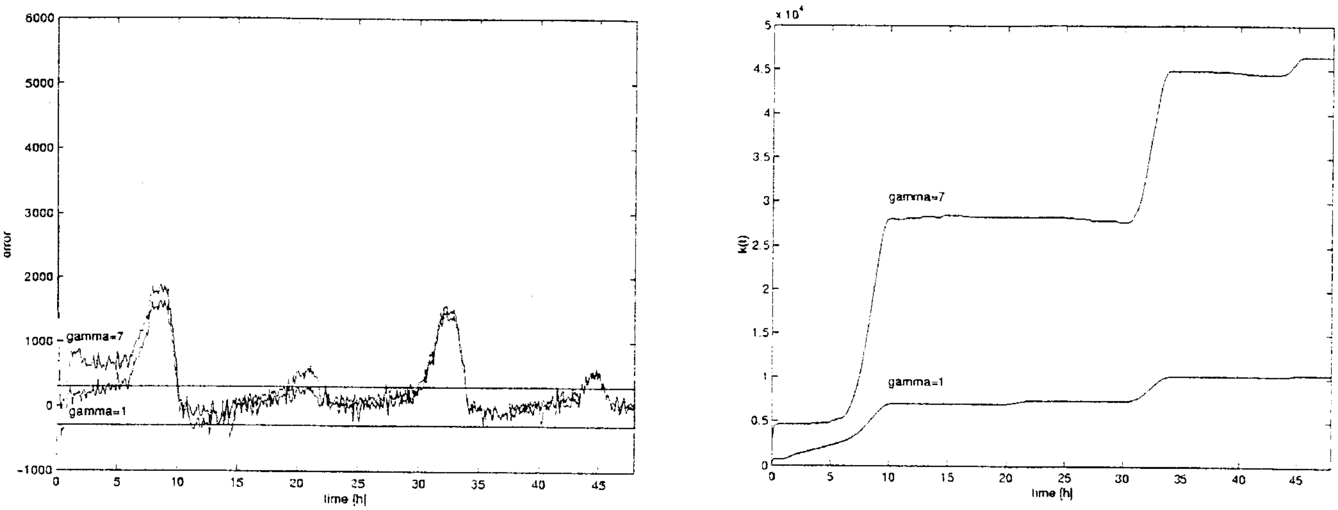

Figure 4. Error $e(t)$ and gain adaptation $k(t)$ for (21), (22) applied to (18), $\gamma = 1, 7$ in the presence of noise.

overshoot in the initial phase of 3 h, after that the difference is negligible.

The drawback of the gain adaptation (10) respectively (22) is that it cannot cope with too large 'white noise' corrupting the measurement or large noise occurring periodically. This is depicted in figure 4 where we apply (22) to (18), but this time with 'white noise' $n(\cdot)$ of variance 1500. This noise forces the error outside the λ-strip and causes the gain $k(t)$ to increase to infinity. Simulations over a period of 480 h for $\gamma = 1, 7$ show that the limiting gain become $k(480) = 5.5 \times 10^4$, 3.9×10^5, respectively.

We also simulated the λ-tracker (21) and (22) applied to (18) but with no sensor dynamics, i.e. $X_m = X_R$. In this case the simulation results are worse, the noise makes the limiting gain k_∞ larger. The reason is that the sensor dynamics are filtering the noise.

There is also a different realistic scenario which causes the gain to tend to infinity: Disturbances over finite time intervals $[t_n, t_{n+1}]$ might act on the system, and during the subsequent time inteval $[t_{n+1}, t_{n+2}]$ the period might be noise free. In the following section we introduce a modification of (22) which leads to an increase of the gain in noisy periods so that the error is forced back into the λ-strip, and to a decrease of the gain to a reference level in not so noisy periods.

5.3. *Gain adaptation with reference gain*

In order to overcome the unbounded gain adaptation described in the second part of § 5.2, we will use the modified gain adaptation (11) with reference value k^*. This will lead to satisfactory practical results. We choose

$$\dot{k}(t) = -\sigma[k(t) - k^*] + \gamma \begin{cases} (|e(t)| - 300)^\beta, & |e(t)| \geq 300 \\ 0, & |e(t)| < 300 \end{cases} \tag{23}$$

where the remaining constants $k^*, \sigma, \gamma, \beta$ are varied in the following. These design parameters have a substantial influence on the process dynamics and their choice is of crucial importance for obtaining good tracking.

5.3.1. *Variation of* k^*: A sensible choice of k^* would be $k(0) = k^*$, where k^* would be the first flat level when we applied (22) to (18), i.e. $k^* = k(t) \approx 3700$ for $t \in [8, 17]$ in figure 3, $\gamma = 1$.

If the reference gain is not large enought (for example $k^* = 10^3$ in figure 5), then tracking is not achieved. Increasing k^* improves the tracking and for $k^* = 5 \times 10^3$ the output follows the reference signal within the pre-defined error. Again there is a tradeoff: k^* needs to be large enough to achieve tracking when there is no noise, the k^* needs to be small to keep the sensitivity to noise small and hence the overshoot small.

From a dynamical point of view, $k^* = 10^3$ in figure 5 is the most intersting one. For $k(t) = k^*$ not sufficiently large tracking is not achieved, so that the error is outside the λ-strip and $k(t)$ will increase until $t \approx 8$ (h), where it infers the error into the λ-strip. But '$-\sigma[k(t) - k^*]$' on the right-hand side of (23) leads to a decrease of the gain, which results in an oscillation. For large k^*, see $k^* = 5000$ in figure 5, the oscillation is flattened and the situation is very close to the non-adaptive case. The gain is large enough to achieve λ-tracking, and then '$-\sigma[k(t) - k^*]$' does not decrease $k(t)$ substantially. However, when there are large disturbances corrupting

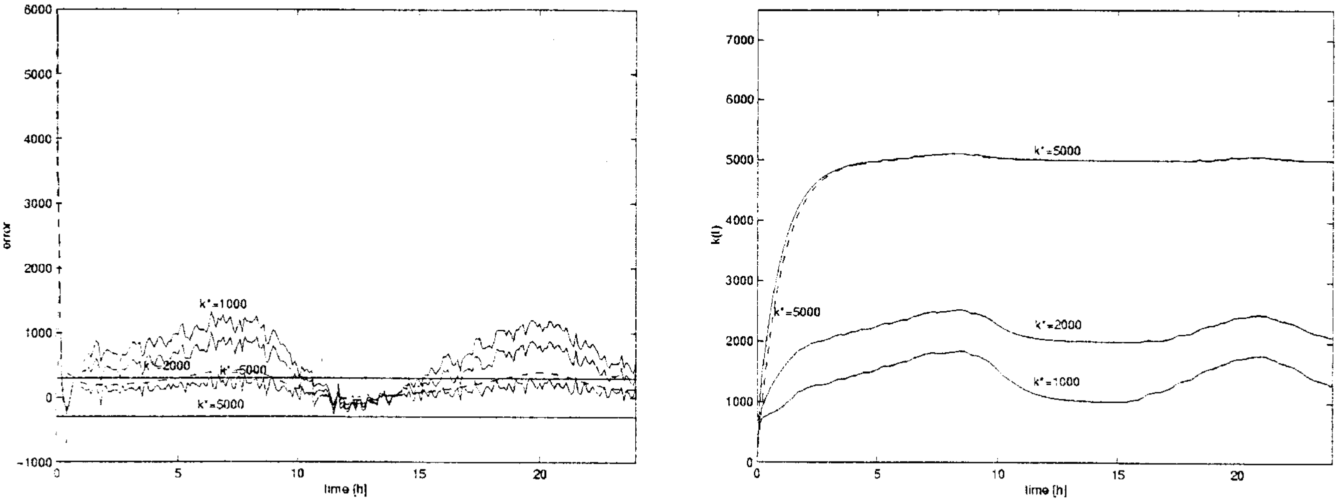

Figure 5. Error $e(t)$ and gain adaptation $k(t)$ for (21), (23) applied to (18) in the presence of noise, $k^* = 1000$, 2000, 5000 and $\sigma = 1$, $\gamma = 1$, $\beta = 1$. The dashed line shows no sensor dynamics, i.e. $X_{\rm m} = X_{\rm R}$, no noise, i.e. $n(\cdot) = 0$ and $k^* = 5000$.

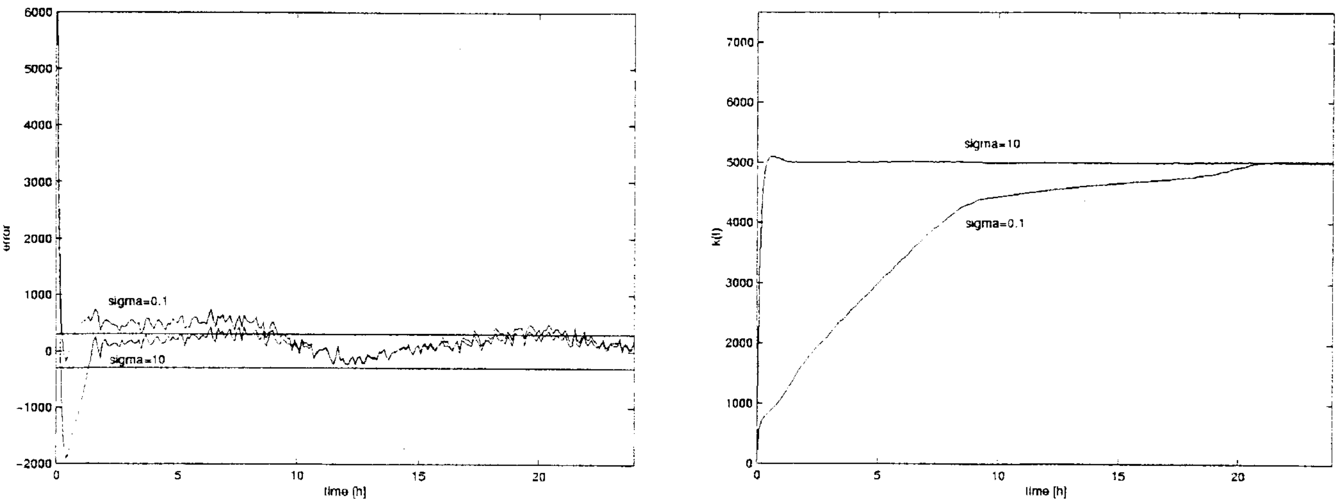

Figure 6. Error $e(t)$ and gain adaptation $k(t)$ for (21), (23) applied to (18) in the presence of noice, $k^* = 5000$, $\gamma = 1$, $\beta = 1$, and $\sigma = 0.1$, 10.

the system, then $k(t)$ will increase and so the error is forced back into the λ-strip and afterwards $k(t)$ will decrease towards k^*.

Note also that the controller works successfully in case of no sensor dynamics, i.e. $X_m = X_{\rm R}$ and no noise, i.e. $n(\cdot) = 0$. See dashed line in figure 5.

5.3.2. *Variation of* σ: Decreasing σ leads to slower gain dynamics. This avoids input saturation problems but the tracking performance is worse and for $\sigma = 0.1$ the output reaches the λ-strip only after 10 h, see figure 6. Certainly, for very small σ and error outside the λ-strip, we have $\dot{k}(t) \approx (|e(t)| - \lambda)^\beta$. Compare figure 6 for $\sigma = 0.1$ with figure 4 for $\gamma = 1$.

5.3.3. *Variation of* γ *and* β: The larger we choose γ and β the faster is the gain dynamic and the better is the tracking, but this causes a large $k(t)$. This leads to input saturations and high initial overshoot and whenever $e(t)$ leaves the λ-strip, the error is amplified by a large γ or by the exponent β and hence $k(t)$ increases. This becomes apparent when we set $\gamma = 10$ in figure 7 or even when $\beta = 2$ in figures 8 and 9. The actual tracking in case of $\beta = 1$ or $\beta = 2$ does not differ very much after the initial period $[0, 2]$ as can be seen in figure 8, but during the initial phase the difference is dramatic. Due to the large gain (see figures 8 and 9 for $\beta = 2$) the input action hits the saturation bounds and even if the error is already small, i.e. $t = 2.3$, the input action for $\beta = 2$ is almost a bang–bang control.

All previous simulations indicate that if more information about the system is available, this should be used

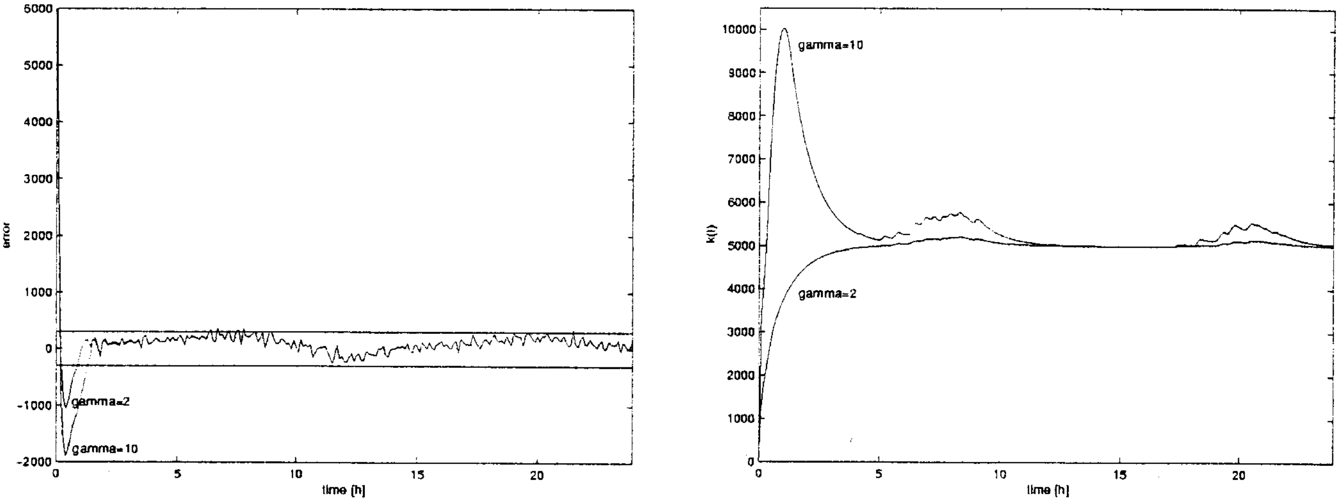

Figure 7. Error $e(t)$ and gain adaptation $k(t)$ for (21), (23) applied to (18) in the presence of noise, $k^* = 5000$, $\beta = 1$, $\sigma = 1$, $\gamma = 2, 10$.

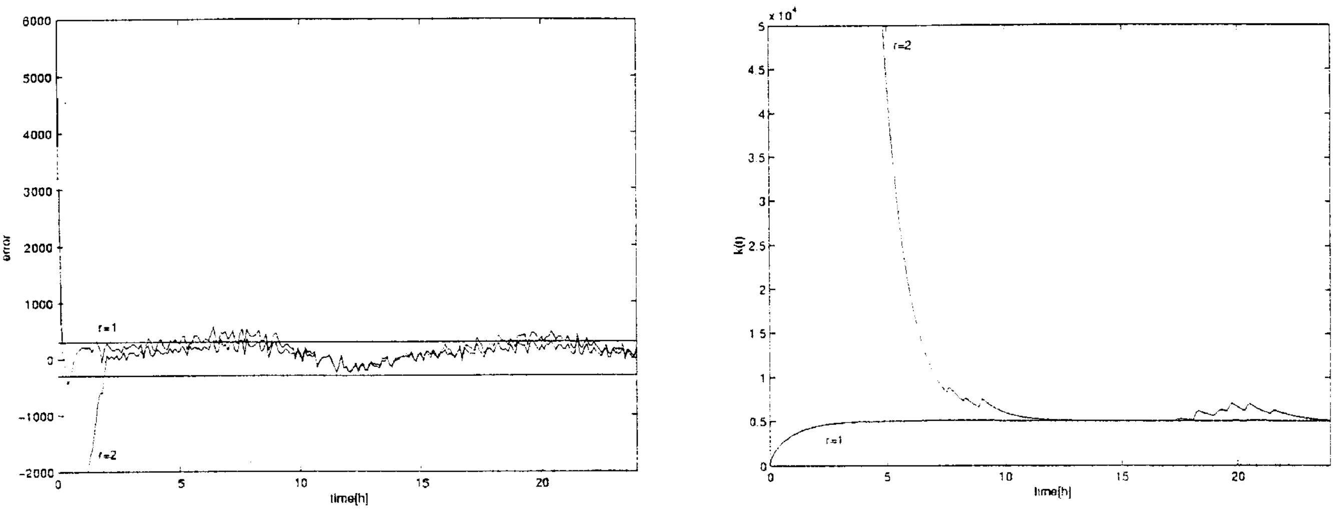

Figure 8. Error $e(t)$ and gain adaptation $k(t)$ for (21), (23) applied to (18) in the presence of noise, $k^* = 5000$, $\gamma = 1$, $\sigma = 1$, and $\beta = 1, 2$.

for the choice of the design parameters $k^*, \sigma, \gamma, \beta$ so that the transient behaviour can be improved.

6. Conclusions

Most widely used strategies for controlling biological waste water treatment are either based on real-time parameter identification and reconstruction of unavailable process variables (see e.g. Bastin and Dochain (1990) for linearizing controllers or Marsili-Libelli (1989) for self-tuning controllers or optimal control), or on linearization of the model around an operating point (or region) and to consider a linear process with non-stationary parameters which vary in limited ranges (Schaper *et al.* 1990). The latter approach leads to robust linear controllers (PI, PID, H_2, H_∞) which are designed with performance specifications defined in the frequency domain. Though very elegant solutions have been obtained, the complexity of the control law makes the implementation of these methods very difficult.

The λ-tracker introduced in the present paper achieves the same control objectives with much less effort and allows for a much larger class of systems than the aforementioned controllers.

The λ-tracker relies on structural properties of the process only and allows for unknown reaction kinetics and unknown time-varying process parameters. Only weak feasibility assumptions are assumed and estimation or identification mechanism or probing signals are not invoked. This has been proved mathematically and the simulations show that tracking is also ensured in situations which go beyong the theory. With little design effort good command tracking is achieved. The only

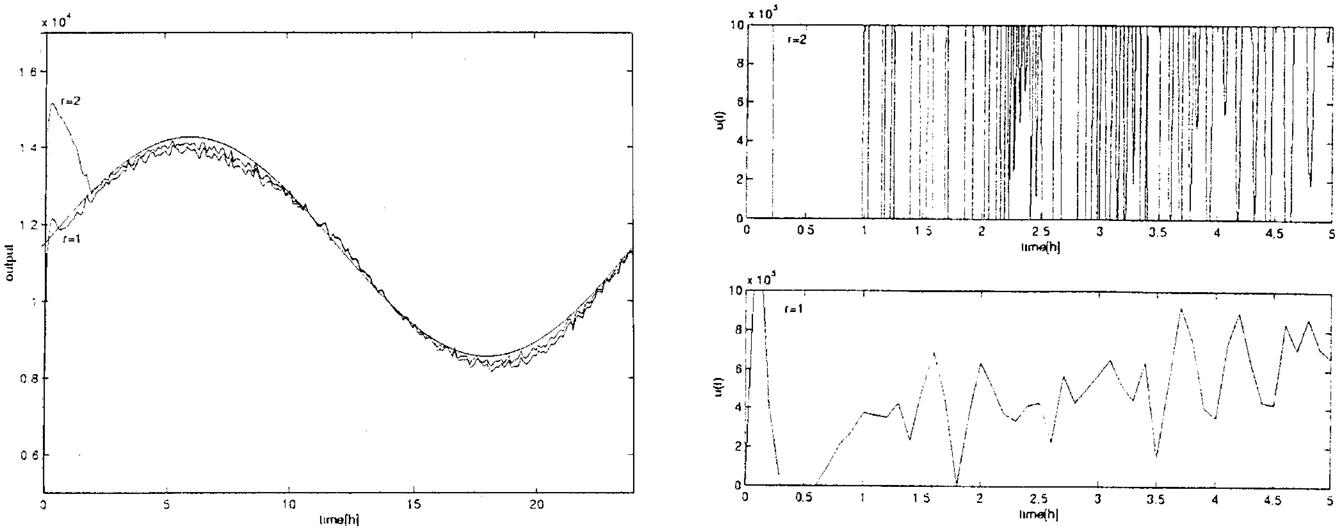

Figure 9. Output $y(t)$ and input $u(t)$ for (21), (23) applied to (18) in the presence of noise, $k^* = 5000$, $\gamma = 1$, $\sigma = 1$, and $\beta = 1, 2$.

price being paid is that a small prespecified tracking error is tolerated. This corresponds to a realistic tracking tollerance of the process.

It is also shown that rough process knowledge can immediately be used to tune the design parameters of the controller appropriately.

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