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Adaptive Controllers and Root Loci of Minimum-Phase Systems

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Abstract. We consider the class of multi-input, multi-output, finite-dimensional, minimum-phase systems $\dot{x} = Ax + Bu$, $y = Cx$, where it is required that $\det(CB) \neq 0$, but the state dimension is unknown. For this class, a simple universal adaptive high-gain controller—not based on any identification mechanism—is introduced that ensures exponential decay to zero of the solution of the closed-loop system. Moreover, the terminal system is "almost always" exponentially stable. The switching-type controller switches between constant gains at discrete points of time and is based on the simple Willems-Byrnes controller $u(t) = -k(t)y(t)$, $\dot{k}(t)^2$. The results are extended to solve the adaptive tracking problem for a certain class of reference signals.

1. Introduction

It is well known that for every single-input, single-output, minimum-phase systems belonging to the class

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), & x(0) &\in \mathbb{R}^n \\ y(t) &= cx(t), & cb &> 0 \end{aligned}\right\}, \tag{1}$$

the adaptive strategy introduced by Willems and Byrnes [1],

$$\left.\begin{aligned} u(t) &= -k(t)y(t) \\ \dot{k}(t) &= y(t)^2, \qquad k(0) \in \mathbb{R} \end{aligned}\right\} \tag{2}$$

leads to a closed-loop system with convergent finite-gain $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$ and asymptotically to zero-converging state $x(t)$ of

$$\dot{x}(t) = [A - k(t)bc]x(t), \qquad x(0) \in \mathbb{R}^n. \tag{3}$$

This approach uses neither any identification mechanism nor probing signals. In recent years, it has been extended and applied to various classes of minimum-phase systems, such as multivariable [2–5], higher relative degree [6, 7], infinite dimensional [8, 9], nonlinearly disturbed [10], and tracking [6, 11]. This list represents only a selection of references; see Ilchmann [12] for a survey and bibliography.

Almost all contributions have in common that $\lim_{t\to\infty} x(t) = 0$, but the problem of whether the decay is exponential is an open one. Moreover, the so-called *terminal system*

$$\dot{x}(t) = A_{k_\infty} x(t), \qquad \text{where} \qquad A_{k_\infty} := A - k_\infty bc,$$

can be unstable. However, computer simulations have shown, that "almost always" the terminal system has its eigenvalues in the open left-half plane only.

The purpose of this article is as follows: first, to introduce a simple modification of the feedback strategy (2) for a large class of multivariable minimum-phase systems having the benefit that the state $x(t)$ decays exponentially to zero; second, to prove topological properties of the terminal system, that is, that the terminal system is "almost always" exponentially stable; and third, to extend theses results to universal adaptive exponential tracking of certain signals.

The class of systems under consideration is the following class Σ of multi-input, multi-output, linear, minimum-phase systems

$$\left.\begin{array}{lll} \dot{x}(t) = Ax(t) + Bu(t), & y(t) = Cx(t), & x(0) \in \mathbb{R}^n \\ (A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}, & & n \text{ is arbitrary} \\ (A, B, C) \text{ is minimum phase}, & \det(CB) \neq 0 & \end{array}\right\} \tag{4}$$

where the linear system

$$\left.\begin{array}{ll} \dot{x}(t) = Ax(t) + Bu(t), & x(0) \in \mathbb{R}^n \\ y(t) = Cx(t) & \end{array}\right\} \tag{5}$$

associated with $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, is called *minimum phase* if it satisfies

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all} \quad s \in \bar{\mathbb{C}}_+. \tag{6}$$

Instead of continuous gain adaptation $K(t)$ in the feedback $u(t) = -K(t)y(t)$, we will consider piecewise constant gain implementation tuned by the function $\dot{s}(t) = y(t)^2$ and a prespecified *sequence of thresholds* $T_0 < T_1 < \ldots$, satisfying a certain growth condition. For example, for single-input, single-output systems such as system (1), with known sign of the high-frequency gain, $cb > 0$, the adaptive feedback strategy will be

$$\begin{array}{lll} u(t) = -k(t)y(t) & & \\ \dot{s}(t) = y(t)^2 & , \; s(0) = T_0 & \\ k(t) = T_i & , \; \text{if} & s(t) \in [T_{i-1}, T_i). \end{array}$$

At each time when the tuning function $s(t) = \int_0^t y(s)^2 ds + T_0$ reaches a threshold T_{i-1}, the feedback law will be changed to $u(t) = -T_i y(t)$. Eventually, the gain will be so large that the trajectory of the closed-loop subsystem $\dot{x}(t) = [A - k(t)bc]x(t)$ will decay exponentially (this is ensured by the minimum-phase assumption). Therefore, the integral

$\int_0^t y(s)^2 ds$ converges, and no more switchings will occur. The idea of using thresholds and piece-wise constant gain adaptation has been used, for single-input, single-output systems with positive high-frequency gain, by Ilchmann and Owens [13]; it is different from the so-called *piecewise smooth* approach, i.e., $u(t) = -k(t)K_{k(t)}y(t)$, where $\dot{k}(t) = y(t)^2$ is smooth and only K_k depends piecewise constantly on k; see, e.g., [2, 3]. This modification has the advantage that the closed-loop subsystem (3) is a piecewise constant system.

That such a result is, in principle, possible for a much larger class of systems has been proved by Mårtensson [14] and by Miller and Davison [15]. However, here we taylor an appropriate switching strategy for the specific class (4), which is simpler, and it is possible to show that "almost always" the terminal system is exponentially stable. This generic statement is given here in terms of the switching sequence, whereas Townley [16] has proved similar results in terms of the set of initial conditions $x_0 \in \mathbb{R}^n$.

This article is organized as follows. Basic properties of multivariable minimum-phase systems are collected in section 2. In section 3, the unstable root-loci of single-input, single-output, minimum-phse systems are studied in depth. These results have interest in their own right, and are also used in the following sections. In section 4, a modification of the Willems-Byrnes controller (2) is introduced that guarantees adaptive stabilization of systems belonging to Σ, and, as an improvement to other adaptive feedback strategies, yields exponential decay of the state, and a terminal system that is "almost always" (with respect to the sequences of thresholds) exponentially stable. The latter is shown in section 5. In section 6, it is proved that the stabilization result derived in section 4, in combination with an internal model, leads to an adaptive feedback strategy capable of exponentially tracking reference signals belonging to a certain class. Topological results are also valid in this case.

Nomenclature

$\|x\|_P$	$= \sqrt{< x, Px >}$ for $x \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$ positive definite
$\mathbb{C}_+(\mathbb{C}_-)$	Open right- (left-) half complex plane
$\sigma(A)$	The spectrum of the matrix $A \in \mathbb{C}^{n\times n}$
$GL_m(\mathbb{R})$	The set of all invertible matrices $M \in \mathbb{R}^{m\times m}$
$A_k := A - kBKC$	$(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, $K \in \mathbb{R}^{m\times m}$, $k \in \mathbb{R}$
$L_2(J)$	Vector space of measurable functions $f: J \to \mathbb{R}^n$, $J \subset \mathbb{R}$ some interval, such that $\int_J \|f(s)\|^2 ds < \infty$

2. Some properties of multivariate minimum-phase systems

In this section, some results on the system class (4) are collected that give a deeper insight into the system class and will be used in the following sections.

Remark 1. A multivariable system $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ is *minimum phase*, i.e., satisfies equation (6), if and only if it is stabilizable and detectable and the transfer function $C(sI_n - A)^{-1}B \in \mathbb{R}(s)^{m\times m}$ has no zeros in $\bar{\mathbb{C}}_+$. For a proof, see Ilchmann and Owens [17]. This shows that condition (6) is an extension of the well-known minimum-phase definition for single-input, single-output systems given usually in the frequency domain.

Lemma 1. *If $(A, B, C) \in \Sigma$ (see (4)), then a useful state-space description of the system can be achieved by the transformation*

$$\begin{pmatrix} y \\ z \end{pmatrix} = S^{-1}x,$$

where

$$S := [B(C\,B)^{-1}, V]$$

and $V \in \mathbb{R}^{n\times(n-m)}$ denotes a basis matrix of kerC. S has the inverse

$$S^{-1} = \begin{bmatrix} C \\ T \end{bmatrix},$$

where

$$T := (V^TV)^{-1}\, V^T[I_n - B(C\,B)^{-1}C].$$

The transformation $S^{-1}x$ converts equation (5) into

$$\left.\begin{aligned} \dot{y}(t) &= A_1y(t) + A_2z(t) + C\,Bu(t) \\ \dot{z}(t) &= A_3y(t) + A_4z(t) \end{aligned}\quad , \quad \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = S^{-1}x_0 \right\} \tag{7}$$

where $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times(n-m)}$, $A_3 \in \mathbb{R}^{(n-m)\times m}$, $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$, so that

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = S^{-1}AS.$$

If system (5) is miminum phase, then it follows from

$$\begin{vmatrix} sI_n - A & B \\ C & 0 \end{vmatrix} = \begin{vmatrix} sI_m - A_1 & -A_2 & CB \\ -A_3 & sI_{n-m} - A_4 & 0 \\ I_m & 0 & 0 \end{vmatrix} = |sI_{n-m} - A_4|\cdot|C\,B|$$

that $\sigma(A_4) \subset \mathbb{C}_-$.

Another important consequence of the minimum-phase assumption is that the present output is related to the past output and input data via the following inequality, where no information of the state variables is required.

Proposition 1. *Suppose $(A, B, C) \in \Sigma$, and $u(\cdot) : [0, \infty) \to \mathbb{R}^m$ is locally integrable. Then for every initial condition $x(0) = x_0 \in \mathbb{R}^n$ and positive definite matrix $P = P^T \in \mathbb{R}^{n \times n}$, there exists $M > 0$, such that for all $t > 0$ we have*

$$\frac{1}{2}\|y(t)\|_P^2 \leq M + M\int_0^t \|y(s)\|^2 ds + \int_0^t \langle y(s), PCBu(s)\rangle ds.$$

Although this inequality has been implicitly used in earlier works [18, 19], or in a more general framework, including nonlinear disturbances [17] and L_p-functions [5] for $p \geq 1$, we would like to give a straightforward proof in the present simple situation. The inequality is a basic tool for the proof of stability of the universal adaptive stabilizer presented in section 4.

Proof. Without restriction of generality, we may assume that system (5) is in the form of equations (7). Since A_4 is exponentially stable, there exist M_1, $\omega > 0$ such that

$$\|z(t)\| \leq M_1 e^{-\omega t} + M_1(Ly(\cdot))(t) \qquad \text{for all} \qquad t \geq 0 \tag{8}$$

where

$$(Ly(\cdot))(t) := \int_0^t e^{-\omega(t-s)}\|y(s)\|ds.$$

Integration of

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}(\|y(s)\|_P^2) &= \langle y(s), PA_1y(s) + PA_2z(s) + PCBu(s)\rangle \\ &\leq M_2\|y(s)\|^2 + M_2\|y(s)\| \, \|z(s)\| + \langle y(s), PCBu(s)\rangle\end{aligned}$$

where $M_2 := \|PA_1\| + \|PA_2\|$, yields

$$\begin{aligned}\frac{1}{2}\|y(t)\|_P^2 \leq \frac{1}{2}\|y(0)\|_P^2 &+ M_2\int_0^t \|y(s)\|^2 ds + M_2\int_0^t \|y(s)\| \cdot \|z(s)\| ds \\ &+ \int_0^t \langle y(s), PCBu(s)\rangle ds.\end{aligned} \tag{10}$$

Since L is a stable operator, we have for all $t \geq 0$,

$$\int_0^t (Ly(\cdot))(s)ds \leq \frac{1}{\omega^2}\int_0^t \|y(s)\|^2 ds. \tag{11}$$

For a proof, see, for example, Vidyasagar [20], p. 252. Using inequalities (8) and (11), and applying Hölder's inequality twice, yields for all $t \geq 0$,

$$\begin{aligned}\int_0^t \|y(s)\| \cdot \|z(s)\| ds &\leq M_1 \int_0^t e^{-\omega s}\|y(s)\| ds + M_1 \int_0^t \|y(s)\|(Ly(\cdot))(s) ds \\ &\leq M_1 \left[\int_0^t e^{-2\omega s} ds\right]^{1/2} \left[\int_0^t \|y(s)\|^2 ds\right]^{1/2} \\ &\quad + M_1 \left[\int_0^t \|y(s)\|^2 ds\right]^{1/2} \left[\int_0^t (Ly(\cdot))^2(s) ds\right]^{1/2} \\ &\leq M_1 (2\omega)^{-1/2} \left[\int_0^t \|y(s)\|^2 ds\right]^{1/2} + M_1 \frac{1}{\omega} \int_0^t \|y(s)\|^2 ds \\ &\leq \frac{M_1^2}{2\omega} + \int_0^t \|y(s)\|^2 ds + \frac{M_1}{\omega} \int_0^t \|y(s)\|^2 ds. \end{aligned} \tag{12}$$

Inserting inequality (12) into inequality (10) yields the desired inequality for

$$M := M_1^2/2\omega + 1/2\|y(0)\|_P^2 + 1 + M_1/\omega + M_2.$$ □

3. Root loci of minimum-phase systems

Suppose that for $k \in \mathbb{R}$ and $K \in \mathbb{R}^{m \times m}$, output feedback of the form

$$u(t) = -kKy(t) \tag{13}$$

is applied to system (5). Then the closed-loop system is given by

$$\dot{x}(t) = A_k x(t), \qquad x(0) = x_0,$$

where throughout this section we use the notation

$$A_k := A - kBKC. \tag{14}$$

We will study the unstable root loci of the parameterized matrix A_k, in particular, if (A, B, C) is single-input, single-output, and minimum phase. The results may be of interest in their own right and are basic to derive topological properties of the terminal system in section 5.

Remark 2. Basic properties of the linearly perturbed matrix A_k parameterized by $k \in \mathbb{R}$, with no extra assumptions on A, B, C, K, are derived by Kato [21], who shows that the number of eigenvalues of A_k is a constant l independent of k, with the exception of some *exceptional points*. Theses points originate from the algebraic singularities of the (branches of) the solutions of $\det(\lambda I_n - A_k) = 0$. In each compact set of $\mathbb{R}$, there is only a finite number of such exceptional points in k. Let $I \subset \mathbb{R}$ be an interval not containing any exceptional points. Then the eigenvalues $\lambda_i(k)$ of A_k depend analytically on $k \in I$. Moreover, the *total projection* $P_{\lambda_i}(k)$ on the *total eigenspace* associated with $\lambda_i(k)$ is analytic in $k \in I$, and so is the eigennilpotent $D_{\lambda_i}(k)$, which satisfies

$$D_{\lambda_i}(k) = (\lambda_i(k)I_n - A_k)\, P_{\lambda_i}(k).$$

Since $D_{\lambda_i}(k)$ is analytic in $k \in I$, there exists an analytic vector $w_{\lambda_i}(k) \not\equiv 0$ such that $w_{\lambda_i}(k) \in \ker D_{\lambda_i}(k)$ (see, e.g., Gohberg et al. [22], p. 388). Therefore,

$$v_{\lambda_i}(k) := P_{\lambda_i}(k) w_{\lambda_i}(k)$$

is an eigenvector of A_k belonging to $\lambda_i(k)$, and depending analytically on $k \in I$.

It is well known that single-input, single-output, minimum-phase systems are stabilizable by high-gain feedback $u = -ky$. This property carries over to multivariable systems in the following sense.

Remark 3. Suppose $(A, B, C) \in \Sigma$, $K \in \mathbb{R}^{m \times m}$, and $\sigma(CBK) \subset \mathbb{C}_+$. Then, if k tends to infinity, the eigenvalues of A_k are approaching the eigenvalues of $-kCBK$ and A_4, where A_4 is a stable matrix. This follows from the decomposition (7) together with Schur's formula (see, e.g., Gantmacher [23]), which yields

$$\det[\lambda I_n - (A - kBKC)] = \det \begin{bmatrix} \lambda I_m - A_1 + kCBK & A_2 \\ -A_3 & \lambda I_{n-m} - A_4 \end{bmatrix}$$

$$= \det(\lambda I_m - A_1 + kCBK) \cdot \det[(\lambda I_{n-m} - A_4) - A_3(\lambda I_m - A_1 + kCBK)^{-1} A_2].$$

Thus, in the limit we obtain

$$\lim_{k\to\infty} \sigma(A - kBCK) = \lim_{k\to\infty} \sigma(-kCBK) \cup \sigma(A_4).$$

Therefore, there exists a $k^* \geq 0$ such that $\sigma(A_k) \subset \mathbb{C}_-$ for all $k \geq k^*$. Note that the set $\{k \geq 0 \mid \sigma(A_k) \cap \bar{\mathbb{C}}_+ \neq \emptyset\}$ is not necessarily connected.

The following proposition shows, interestingly, that if k varies, then all unstable eigenvalues of A_k are moving.

Proposition 2. *Suppose $(A, B, C) \in \Sigma$ and $K \in \mathbb{R}^{m\times m}$. If $\lambda_k : I \to \bar{\mathbb{C}}_+$ denotes an analytic parameterization of any unstable eigenvalue of A_k on some open interval $I \subset \mathbb{R}$, then $\lambda_k \not\equiv$ constant on I.*

Proof. The decomposition of (A, B, C) given in equations (7) yields

$$\det[sI_n - A_k] = \det\begin{bmatrix} sI_m - A_1 + kCBK & A_2 \\ -A_3 & sI_{n-m} - A_4 \end{bmatrix}$$
$$= \det(sI_{n-m} - A_4)\cdot\det(sI_m - A_1 + kCBK - A_2(sI_{n-m} - A_4)^{-1}A_3).$$

Suppose, for some $\lambda \in \bar{\mathbb{C}}_+$, we have $\lambda_k = \lambda$ for all $k \in I$, i.e.,

$$\det(\lambda I_n - A_k) = 0 \qquad \text{for all} \qquad k \in I.$$

Since $\det(\lambda I_{n-m} - A_4) \neq 0$, it follows that for $M := \lambda I_m - A_1 - A_2(\lambda I_{n-m} - A_4)^{-1}A_3$, we have

$$\det(M + kCBK) = 0 \qquad \text{for all } k \in I,$$

or equivalently,

$$\det(\hat{M} + kI_m) = 0 \qquad \text{for all} \qquad k \in I,$$

where $\hat{M}$ is a Jordan form of $M(CBK)^{-1}$. Thus,

$$\det(\hat{M} + kI_m) = \prod_{i=1}^{m}(\mu_i + k) = 0 \qquad \text{for all} \qquad k \in I,$$

where $\mu_1, \ldots, \mu_m$ denote the eigenvalues of $\hat{M}$. Since the μ_i do not depend on k, this is a contradiction, and the proposition is proved. □

The following proposition shows in particular, that for single-input, single-output, minimum-phase systems with positive high-frequency gain $cb > 0$, the matrix A_k has, for all but finitely many k, only *distinct* unstable eigenvalues (see also Ilchmann and Owens [17]).

Proposition 3. *If $(A, b, c) \in \Sigma$ is single-input, single-output, then the set*

$$\mathcal{S} = \{k \in \mathbb{R} \mid A_k = A - kbc \quad \text{has eigenvalues in } \bar{\mathbb{C}}_+ \text{ of multiplicity } l \geq 2\}$$

is finite.

Proof. Choose coprime polynomials $\epsilon(\cdot), \psi(\cdot) \in \mathbb{R}[s]$ so that

$$c(sI_n - A)^{-1}b = \frac{\epsilon(s)}{\psi(s)}.$$

It can easily be seen that

$$c(sI_n - A_k)^{-1}b = \frac{\epsilon(s)}{\psi_k(s)},$$

where

$$\psi_k(s) := \psi(s) + k\epsilon(s).$$

Coppel ([24], theorem 10) has proved that, if (A, b, c) is detectable and stabilizable, then $s \in \bar{\mathbb{C}}_+$ is a zero of $\psi(\cdot)$ (including multiplicity) if and only if it is a zero of $\det(\cdot I_n - A)$. Since (A_k, b, c) is detectable and stabilizable (see remark 1), it follows that

$$\mathcal{S} = \{k \in \mathbb{R} \mid \psi_k(\cdot) \quad \text{has a zero in } \bar{\mathbb{C}}_+ \text{ of multiplicity } l \geq 2\}.$$

Next we will derive a necessary condition for $\psi_k(\cdot)$ having a repeated zero at $s \in \bar{\mathbb{C}}_+$, which is equivalent to

$$\psi_k(s) = \psi_k'(s) = 0,$$

respectively,

$$\psi(s) = -k\epsilon(s) \quad \wedge \quad \psi'(s) = -k\epsilon'(s). \tag{15}$$

Since $\epsilon(\cdot)$ and $\psi(\cdot)$ are coprime, this can hold only if $k = 0$ or

$$k \neq 0 \quad \wedge \quad \epsilon(s) \neq 0 \quad \wedge \quad \psi(s) \neq 0.$$

Now equations (15) yield

$$\frac{\psi(s)}{\epsilon(s)} = \frac{\psi'(s)}{\epsilon'(s)},$$

which is equivalent to

$$q(s) := \psi(s)\epsilon'(s) - \psi'(s)\epsilon(s) = 0.$$

But $\psi(s)/\epsilon(s)$ cannot equal $\psi'(s)/\epsilon'(s)$ for all $s \in \bar{\mathbb{C}}_+$, since the denominator polynomial of the latter is smaller than the former, so it must be that $q(s)$ is not the zero polynomial, which means that is has a finite number of zeros. Hence, equations (15) yield

$$\mathcal{S} \subset \{0\} \cup \left\{ k = -\frac{\psi(s)}{\epsilon(s)} = -\frac{\psi'(s)}{\epsilon'(s)} \;\middle|\; q(s) = 0,\, s \in \bar{\mathbb{C}}_+ \right\}.$$

Therefore, $\mathcal{S}$ is finite, and the proof is complete. □

Proposition 3 does not hold true for multivariable minimum phase systems. This can easily be seen from the example $A = B = C = I_n$, so that $A_k = (1 - k)I_n$.

Therefore, the important consequence of proposition 3 that the unstable subspace of A_k is spanned by piecewise analytic eigenvectors is no longer valid.

The following lemma shows that, for single-input, single-output systems, the projection of each fixed nonzero vector $\zeta \in \mathbb{R}^n$ onto the k-depending unstable subspace of A_k is nonzero, with the exception of discrete points $k \in \mathbb{R}$.

Lemma 2. *Suppose the system $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}$ is minimum phase and controllable. Let $I \subset \mathbb{R}$ be an open interval such that $\lambda_k : I \to \bar{\mathbb{C}}_+$ is an analytic parameterization of an unstable eigenvalue of $A_k = A - kbc$ with eigenvector v_k. Then, for $\zeta \in \mathbb{R}^n$, we have*

$$\langle v_k, \zeta \rangle = 0 \quad \text{for all} \quad k \in I \quad \Leftrightarrow \quad \zeta = 0.$$

Proof. Suppose $\langle v_k, \zeta \rangle = 0$ for all $k \in I$. Since

$$\begin{bmatrix} \lambda_k I_n - A_k & b \\ \zeta^T & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -kc & I_m \end{bmatrix} = \begin{bmatrix} \lambda_k I_n - A & b \\ \zeta^T & 0 \end{bmatrix},$$

it follows from the controllability assumption that

$$n \le rk \begin{bmatrix} \lambda_k I_n - A & b \\ \zeta^T & 0 \end{bmatrix} = rk \begin{bmatrix} \lambda_k I_n - A_k & b \\ \zeta^T & 0 \end{bmatrix}. \tag{16}$$

Since v_k is a right eigenvector, we have

$$\begin{bmatrix} \lambda_k I_n - A_k & b \\ \zeta^T & 0 \end{bmatrix} \begin{pmatrix} v_k \\ 0 \end{pmatrix} = 0,$$

and equation (16) yields

$$rk \begin{bmatrix} \lambda_k I_n - A & b \\ \zeta^T & 0 \end{bmatrix} = n \quad \text{for all} \quad k \in I. \tag{17}$$

Using the identity theorem of analytic functions and the fact that λ_k is not constant (see proposition 2), equation (17) yields

$$rk \begin{bmatrix} sI_n - A & b \\ \zeta^T & 0 \end{bmatrix} = n \quad \text{for all} \quad s \in \mathbb{C}, \tag{18}$$

and hence the $(n + 1) \times (n + 1)$ matrix in equation (18) is singular over the field $\mathbb{R}(s)$. Thus there exists a nonzero pair $(\varphi(\cdot), \alpha(\cdot)) \in \mathbb{R}[s]^n \times \mathbb{R}[s]$ such that

$$\varphi(s)^T(sI_n - A) = \alpha(s)\zeta^T \quad \text{and} \quad \varphi(s)^T b = 0 \quad \text{for all} \quad s \in \mathbb{C}. \tag{19}$$

If, for every nonzero pair $(\varphi(\cdot), \alpha(\cdot))$ satisfying equations (19), it holds that $\alpha(\cdot) \equiv 0$, then

$$\varphi(s)^T[sI_n - A, b] \equiv 0,$$

and, by right invertibility of $[sI_n - A, b]$, $\varphi(\cdot) \equiv 0$, which contradicts $(\varphi(\cdot), \alpha(\cdot)) \not\equiv 0$. Therefore, there exists a pair with $\alpha(\cdot) \not\equiv 0$. Considering $(sI_n - A)^{-1}$ as an element of $\mathbb{R}(s)^{n\times n}$, we obtain

$$0 = \alpha(s)\zeta^T(sI_n - A)^{-1}b,$$

and hence

$$0 = \zeta^T \sum_{i=0}^{\infty} s^{-(i+1)}A^i b,$$

which yields

$$0 = \zeta^T[b, Ab, \ldots, A^{n-1}b].$$

Since (A, b) is controllable, the controllability matrix is right invertible, which implies that $\zeta = 0$. This proves the lemma. □

The controllability assumption in lemma 2 is essential. Lemma 2 does not hold true for systems $(A, b, c) \in \Sigma$ that are not controllable. Consider, for example,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad c = (1, 0), \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad A_k = \begin{bmatrix} -k & 1 \\ 0 & -1 \end{bmatrix}.$$

For $k \in (-\infty, 0)$, the right eigenvector corresponding to $\lambda_k = -k$ is $v_k = (1, 0)^T$. However, for $\zeta = (0, 1)^T$, we obtain $\langle v_k, \zeta \rangle = 0$ for all $k \in (-\infty, 0)$.

Before the main result of this section is proved, a technical lemma is required.

Lemma 3. *If $A \in \mathbb{R}^{n\times n}$, $x_0 \in \mathbb{R}^n$, and $\lambda \in \bar{\mathbb{C}}_+$ so that*

$$Av = \lambda v \quad \text{and} \quad \langle v, x_0 \rangle \neq 0,$$

then the solution of the initial-value problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

is such that $e^{At}x_0 \not\to 0$ as $t \to \infty$.

Proof. The solution $x(t)$ can be written

$$x(t) = e^{At}x_0 = \alpha(x_0)e^{\lambda t}v + \tilde{x}(t),$$

where $\tilde{x}(t)$ is linearly independent of $e^{\lambda t}v$. $\alpha(x_0)$ denotes the coordinate of the projection of x_0 on $v\mathbb{R}$, which is given by

$$\alpha(x_0) = v(v^T v)^{-1} v^T x_0 = \frac{1}{\|v\|^2} v^T x_0.$$

Now, $\langle v, x_0 \rangle \neq 0$ yields $\alpha(x_0) \neq 0$, and the result follows, since λ is an unstable eigenvalue. □

We are now in a position to prove the main result of this section, which, in particular, says the following. Suppose (A, b, c) is a controllable minimum-phase system such that the closed-loop system $\dot{x}(t) = [A - kbc]x(t)$ is stable for k sufficiently large. Let $I \subset [0, \infty)$ be a finite union of closed intervals so that $A_k = A - kbc$ is exponentially stable whenever $k \in [0, \infty)\backslash I$. We obtain the result that, if $x_0 \in \mathbb{R}$ is fixed, then the solution of $\dot{x}(t) = A_k x(t)$, $x(0) = x_0$, is unstable for all but finitely many $k \in I$.

Theorem 1. *If the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \qquad x(0) = x_0 \\ y(t) &= cx(t), \end{aligned}$$

is controllable, minimum phase, and of relative degree 1, and $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, is fixed, then the solution of the initial value problem

$$\dot{x}(t) = [A - kbc]x(t), \qquad x(0) = x_0$$

satisfies the following:

(i) *The set*

$$\{k \in \mathbb{R} \mid \sigma(A - kbc) \cap \bar{\mathbb{C}}_+ \neq \emptyset \quad \text{and} \quad \lim_{t\to\infty} e^{[A-kbc]t} x_0 = 0\}$$

is discrete in $\mathbb{R}$.

(ii) *If $cb > 0$, then the set*

$$\{k \geq 0 \mid \sigma(A - kbc) \cap \bar{\mathbb{C}}_+ \neq \emptyset \quad \text{and} \quad \lim_{t\to\infty} e^{[A-kbc]t} x_0 = 0\}$$

is finite.

Proof. Let $\mathcal{E}$ denote the discrete set of exceptional points k defined in remark 2. (i) is proved if it can be shown that the set

$$\{k \in \mathbb{R} \setminus \mathcal{E} \mid \sigma(A_k) \cap \bar{\mathbb{C}}_+ \neq \emptyset \quad \text{and} \quad \lim_{t\to\infty} e^{A_k t} x_0 = 0\}$$

is discrete. Let $\mathbb{R} \setminus \mathcal{E} = \bigcup_{i \in \mathbb{N}} I_i$ be the countable union of disjoint open intervals. It remains to prove that for every $i \in \mathbb{N}$ the set

$$\{k \in I_i \mid \sigma(A_k) \cap \bar{\mathbb{C}}_+ \neq \emptyset \quad \text{and} \quad \lim_{t \to \infty} e^{A_k t} x_0 = 0\}$$

is discrete. Let $I := I_{i_0}$ for some $i_0 \in \mathbb{N}$, and $\lambda_k : I \to \bar{\mathbb{C}}_+$, $v_k : I \to \mathbb{C}^n$ denote analytic parameterizations of an unstable eigenvalue-eigenvector pair. It follows from lemma 2 that the analytic map $k \mapsto \langle v_k, x_0 \rangle$ is not identical zero on I. Thus, the set of zeros of $k \mapsto \langle v_k, x_0 \rangle$ is discrete in I. Now (i) is a consequence of lemma 3. (ii) follows from the fact that there exists a $k^* > 0$ such that $\sigma(A_k) \subseteq \bar{\mathbb{C}}_-$ for all $k \geq k^*$, and hence the set considered in (i) is bounded and therefore finite. □

Remark 4. Unfortunately, previous results cannot be extended to multivariable systems. Consider, for example, the controllable and observable minimum-phase system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = C^T = I_2.$$

$v_k = (1, 0)^T$ is an unstable eigenvector of $A - kBC = \operatorname{diag}\{1 - k, -1 - k\}$ with eigenvalue $(1 - k)$ for all $k \in [0, 1]$. Since $\langle v_k, e_2 \rangle = 0$ for all $k \in \mathbb{R}$, lemma 2 cannot be generalized. The same is true for theorem 1, since the set considered in (i) (respectively, (ii)) for $x_0 = (1, 0)^T$, is the continuum $(-\infty, 1]$ (respectively, $[0, 1]$).

4. Universal adaptive exponential stabilization

In this section we will show that a simple modification of the Willems–Byrnes controller (2) can be applied to a large class of systems to achieve exponential stabilization. In the most general case of the system class (4), where no special assumptions about the high-frequency gain are made apart from $\det(CB) \neq 0$, the switching strategy is based on the following result from linear algebra.

Lemma 4. *There exists a finite set*

$$\{K_1, \ldots, K_N\} \subseteq GL_m(\mathbb{R})$$

so that, for any $M \in GL_m(\mathbb{R})$ there exists $i \in \{1, \ldots, N\}$ such that

$$\sigma(MK_i) \subseteq \mathbb{C}_+.$$

Proof. See Mårtensson [3, 25]. □

The set given in lemma 4 is often called a *spectrum unmixing set* for $GL_m(\mathbb{R})$. Unfortunately, the cardinality of the unmixing sets constructed by Mårtensson [3, 25] is far too large to be convenient for applications. Little is know about the minimum cardinality of unmixing sets (see [26]). However, for $m = 1$, the set $\{1, -1\}$ is obviously unmixing, while for $m = 2$ there exists an unmixing set of cardinality 6. It has been shown by Zhu [27] that $GL_3(\mathbb{R})$ can be unmixed by a set having cardinality 32. We first introduce a universal adaptive stabilizer for the class (4), which ensures exponential decay to zero of the state.

Theorem 2. *Suppose* $\{K_1, \ldots, K_N\} \subseteq GL_m(\mathbb{R})$ *is a spectrum unmixing set and* $\{T_i\}_{i \geq 0} \subset \mathbb{R}^{\mathbb{N}}$ *is a sequence of thresholds satisfying*

$$T_i < T_{i+1}, \qquad \lim_{i \to \infty} \frac{T_i}{T_{i+1}} = 0. \tag{20}$$

If the adaptive control scheme

$$\left.\begin{aligned} u(t) &= -K(t)y(t) \\ \dot{s}(t) &= \|y(t)\|^2 \quad , \quad s(0) = T_0 \\ K(t) &= T_i \cdot K_{i \bmod N} \quad , \quad \textit{if} \quad s(t) \in [T_{i-1}, T_i),\ i = 1, 2, \ldots \end{aligned}\right\} \tag{21}$$

is applied to any system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{22}$$

belonging to the class (4), *with arbitrary initial condition* $x_0 \in \mathbb{R}^n$, *then the closed-loop system* (21), (22) *has a unique solution* $x(t)$, *and*

(i) $x(t)$ *decays exponentially to zero as* t *tends to* ∞,
(ii) *there exist* $i_0 \in \mathbb{N}$, $t^* \geq 0$ *such that*

$$[A - BK(t)C] = A - T_{i_0} BK_{i_0 \bmod N} C \quad \textit{for all} \quad t \geq t^*.$$

The intuition behind this control strategy is as follows. By the minimum-phase property, it follows from remark 3 and lemma 4 that there exists $i_0 \in \{1, \ldots, N\}$ such that

$$u(t) = -TK_{i_0 \bmod N} y(t)$$

yields an exponentially stable system

$$\dot{x}(t) = [A - TBK_{i_0 \bmod N} C]x(t),$$

provided T is sufficiently large. As long as the trajectory $x(t)$ is not decaying fast enough to zero, then $s(t)$ hits the next threshold, $K(t)$ in equations (21) travels through the spectrum unmixing set and increases the gain T_i, until finally the gain T_i is large enough and $A - T_i BK_{i \bmod N}C$ is exponentially stable. The growth condition (20) ensures that the system stays, for longer and longer periods, with an exponentially stable $A - T_i BK_{i \bmod N}C$ so that there exists a period that gives the system time to settle down. Exponential decay of the solution is then a simple consequence of the fact that $A - BK(t)C$ becomes finally a constant matrix. Therefore, asymptotic decay of the trajectory implies exponential decay.

Proof of theorem 2. (a) Discontinuities of the right-hand side of the closed-loop system occur whenever $s(t)$ leaves an interval $[T_{i-1}, T_i)$, which at times is

$$t_i := \min\{t \geq 0 \mid s(t) = T_i\}, \qquad i = 1, 2, \ldots$$

If the minimum does not exist for some T_j, then put $t_j = \infty$.

There exists a unique solution on the interval $[0, t_0)$, and, if $t_0 < \infty$, then the finite limit $x(t_0) := \lim_{t \to t_0} x(t)$ exists. Proceeding in this way for the next intervals, it follows that the solution $x(t)$ of the closed-loop system exists on $[0, t')$, where $\lim_{i\to\infty} t_i = t'$.

(b) We show that $s(\cdot) \in L_\infty(0, t')$. Suppose the contrary, i.e., infinitely many switches occur. By assumption, there exists $K_i \in \{K_1, \ldots, K_N\}$ such that $\sigma(CBK_i) \subseteq \mathbb{C}_+$. Let $P = P^T \in \mathbb{R}^{m\times m}$ be the unique positive-definite solution of

$$K_i^T(CB)^T P + PCBK_i = 2I_m. \tag{23}$$

Choose $\alpha > 0$ so that

$$-K_l^T(CB)^T P - PCBK_l \leq 2\alpha I_m \qquad \text{for all} \qquad l \in \{1, \ldots, N\}. \tag{24}$$

It then follows from proposition 1 that for all $l \in \mathbb{N}$ and for some $M > 0$, we have

$$\begin{aligned}\frac{1}{2}\|y(t)\|_P^2 &\leq M + M(T_{lN} - T_0) - \sum_{j=0}^{l-1} \int_{t_jN}^{t_{(j+1)N}} \langle y(\tau), PCBK(\tau)y(\tau)\rangle d\tau \\ &= M + (T_{lN} - T_0)\left[M - \frac{1}{T_{lN} - T_0}\sum_{j=0}^{l-1} \int_{t_jN}^{t_{(j+1)N}} \langle y(\tau), PCBK(\tau)y(\tau)\rangle d\tau\right].\end{aligned} \tag{25}$$

Without restriction of generality, one may assume that $i = N$; otherwise, start at a different time t_j with initial condition $x(t_j)$. Equations (23) and (24), together with the fact that $K(t)$ is constant on $[t_\lambda, t_{\lambda+1})$, yield

$$
\begin{aligned}
-\int_{t_{jN}}^{t_{(j+1)N}} & \langle y(\tau), PCBK(\tau)y(\tau)\rangle d\tau \\
& \leq \alpha T_{(j+1)N-1}(T_{(j+1)N-1} - T_{jN}) - T_{(j+1)N}(T_{(j+1)N} - T_{(j+1)N-1}) \\
& = T^2_{(j+1)N}\left[\alpha \frac{T^2_{(j+1)N-1} - T_{(j+1)N-1}T_{jN}}{T^2_{(j+1)N}} + \frac{T_{(j+1)N-1}}{T_{(j+1)N}} - 1\right]. \qquad (26)
\end{aligned}
$$

Since equation (26) tends to $-T^2_{(j+1)N}$ for $j \to \infty$, there exists a $M_1 > 0$ such that

$$
-\sum_{j=0}^{l-2}\int_{t_{jN}}^{t_{(j+1)N}} \langle y(\tau), PCBK(\tau)y(\tau)\rangle d\tau \leq M_1 \qquad \text{for all} \qquad l \in \mathbb{N}. \qquad (27)
$$

Equation (26) also yields

$$
\frac{-1}{T_{lN} - T_0}\int_{t_{(l-1)N}}^{t_{lN}} \langle y(\tau), PCBK(\tau)y(\tau)\rangle d\tau \leq \frac{T^2_{lN}}{T_{lN} - T_0}\left[\alpha \frac{T^2_{lN-1} - T_{lN-1}T_{jN}}{T^2_{lN}} + \frac{T_{lN-1}}{T_{lN}} - 1\right], \qquad (28)
$$

and the right-hand side of equation (28) tends to $-T_{lN}$ as $l \to \infty$.

Therefore, by inserting equations (27) and (28) into equation (25), the right-hand side of equation (25) tends to $-\infty$ as $l \to \infty$. This contradicts the positivity of the left-hand side of equation (25), and it is proved that $s(\cdot) \in L_\infty(0, t')$.

(c) It follows from (b) that $t' = \infty$, and part (ii) of theorem 2 is proved.

To prove part (i), let $t^* > 0$ such that

$$
K(t) = T_M K_{M \bmod N} \qquad \text{for all} \qquad t \geq t^*.
$$

By lemma 1, the terminal system

$$
\dot{x}(t) = [A - T_M B K_{M \bmod N} C]x(t), \qquad x(0) = x(t_{M-1}),
$$

can be expressed as

$$
\begin{aligned}
\dot{y}(t) &= (A_1 - T_M K_{M \bmod N} CB)y(t) + A_2 z(t) \\
\dot{z}(t) &= A_3 y(t) + A_4 z(t).
\end{aligned}
$$

Since $y(\cdot) \in L_2(0, \infty)$ and A_4 is stable, it follows that $z(\cdot) \in L_2(0, \infty)$. Furthermore, $\dot{y}(\cdot)$, $\dot{z}(\cdot) \in L_2(0, \infty)$. Therefore (see, e.g., Ilchmann and Owens [17]), $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$. Since the system becomes constant after finite time, asymptotic decay of $x(t)$ implies exponential decay, which proves part (i). This completes the proof of the theorem. □

Remark 5. If additional information on the spectrum of CB is available, then the output feedback and assumption (20) on the growth of the thresholds can be considerably simplified as follows:

(i) If it is assumed in theorem 2 that $\sigma(CB) \subseteq \mathbb{C}_+$, then the output feedback can be simplified to

$$u(t) = -K(t)y(t), \qquad K(t) = T_i \text{ , if } \quad s(t) \in [T_{i-1}, T_i),\ i = 1, 2, \ldots, \tag{29}$$

where the sequence of thresholds has to satisfy

$$T_0 < T_1 < \ldots \quad \text{and} \quad \lim_{i\to\infty} [T_i - T_{i-1}] = \infty. \tag{30}$$

To prove statements (i) and (ii) of theorem 2 for this case, only part (b) in the proof needs a change. By using the same argument, we can derive the inequality

$$\frac{1}{2}\|y(t_{i+1})\|_P^2 \le M + \int_0^{t_{i+1}} (M - K(\tau))\dot{s}(\tau)d\tau = M + \sum_{j=0}^{i} (M - T_j)(T_{j+1} - T_j),$$

where the right-hand side becomes negative for i sufficiently large. This contradiction proves $s(\cdot) \in L_\infty(0, t')$, and hence parts (i) and (ii) of theorem 2 follow as in the proof of theorem 2.

(ii) If it is assumed in theorem 2 that $\sigma(CB) \subset \mathbb{C}_+$ or $\sigma(CB) \subset \mathbb{C}_-$ (but which half plane actually contains the spectrum is unknown), then the output feedback can be simplified to

$$u(t) = -K(t)y(t), \quad K(t) = (-1)^i T_i \quad \text{if} \quad s(t) \in [T_{i-1}, T_i),\ i = 1, 2, \ldots, \tag{31}$$

where $T_0 < T_1 < \ldots$ is requierd to satisfy the following discrete version of a Nussbaum switching function (for the concept of Nussbaum switching functions, see Nussbaum [28] and Willems and Byrnes [1]):

$$\inf_{l\in\mathbb{N}} \frac{\Sigma_{i=1}^{l}(-1)^i T_i(T_i - T_{i-1})}{T_l - T_0} = -\infty, \qquad \sup_{l\in\mathbb{N}} \frac{\Sigma_{i=1}^{l}(-1)^i T_i(T_i - T_{i-1})}{T_l - T_0} = +\infty \tag{32}$$

Equation (32) is weaker than inequality (20). For example, it can be easily seen that $T_i := i^2$ satisfies equation (32) but does not satisfy inequality (20).

To prove theorem 2 in this case, only part (b) in the proof needs a change. By using the same argument, we can derive the inequality

$$\frac{1}{2}\|y(t_l)\|_P^2 \le M + (T_l - T_0) - \sum_{i=0}^{l} (-1)^i T_i (T_i - T_{i-1})$$

$$= M + (T_l - T_0)\left[M - \frac{\Sigma_{i=0}^{l}(-1)^i T_i (T_i - T_{i-1})}{T_l - T_0}\right].$$

Now equation (31) yields that the right-hand side of the above inequality takes arbitrary large negative values, thus contradicting the positivity of the left-hand side. This proves $s(\cdot) \in L_\infty(0, t')$, and statements (i) and (ii) of theorem 2 follow as in the proof of theorem 2.

5. Topological aspects of the terminal system

In theorem 2 it is proved that $x(t)$ decays exponentially to zero. This does not imply that the terminal system

$$\dot{x}(t) = [A - (-1)^M T_M B K_{M \bmod N} C]\, x(t)$$

is exponentially stable, where

$$M := \inf\{i \in \mathbb{N} \mid \lim_{t\to\infty} s(t) \le T_i\}.$$

In this section, we will consider, for fixed initial condition x_0, controllable single-input, single-output, minimum-phase systems of relative degree 1, that are systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \qquad x(0) = x_0 \\ y(t) &= cx(t), \end{aligned}\right\}, \tag{33}$$

with $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}$ and $cb \ne 0$.

We will show that the set of sequences of thresholds that lead to an exponentially stable terminal system is dense and that, at each time t_i where the gain T_i switches, the new trajectory

$$e^{[A-(-1)^{i+1}T_{i+1}bc](t-t_i)}x(t_i)$$

is not exponentially decaying with probability 1 whenever $[A - (-1)^{i+1}T_{i+1}bc]$ is not exponentially stable. To make this more precise, let

$$\mathfrak{J} := \{T = \{T_i\}_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid T \quad \text{satisfies} \quad (30)\},$$

and define a subspace of $\mathfrak{J}$ that is relevant for the switching strategy (31) as

$$\hat{\mathfrak{J}} := \{T \in \mathfrak{J} \mid T \quad \text{satisfies} \quad (32)\}.$$

For $\epsilon > 0$, we define the open ball with center $T = \{T_i\}_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ to be

$$\mathfrak{B}_\epsilon(T) := \{S = \{S_i\}_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid |S_i - T_i| < \epsilon \quad \text{for all} \quad i \in \mathbb{N}\}.$$

Using this terminology we have the following.

Consider a controllable, minimum-phase system of the form (33), with fixed $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$. Then the set of the sequences of thresholds $\mathfrak{J}_s$ (respectively, $\hat{\mathfrak{J}}_s$), i.e., all sequences $T \in \mathfrak{J}$ (respectively, $T \in \hat{\mathfrak{J}}$) so that T in combination with equation (29) (respectively, equation (31)) leads to an exponentially stable terminal system, is dense in $\mathfrak{J}$ (respectively, $\hat{\mathfrak{J}}$).

Theorem 3. *Suppose system* (33) *is a controllable, minimum-phase system with* $cb \neq 0$ *and fixed* $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$. *If* $T \in \mathfrak{J}$ *and equation* (29) (*respectively,* $T \in \hat{\mathfrak{J}}$ *and equation* (31)) *is applied to system* (33), *then for every* $\epsilon > 0$ *there exists* $\tilde{T} \in \mathfrak{B}_\epsilon(T)$ *such that* $\tilde{T}$, *instead of* T, *leads to an exponentially stable terminal system and* T *and* $\tilde{T}$ *differ in only finitely many points.*

Proof. We consider the switching strategy (31) only; (29) is simpler. Suppose the switching algorithm using the nominal sequence $\mathfrak{J}$ leads to a terminal system $[A - (-1)^M T_M bc]$, which is not exponentially stable. Then, in particular, $s(t) \in [T_{M-1}, T_M)$ for all $t \geq t_{M-1}$, the last switch occurs at time t_{M-1}, and

$$e^{[A-(-1)^M T_M bc](t-t_{M-1})} x(t_{M-1}) \to 0 \quad \text{as} \quad t \to \infty.$$

By theorem 1 (i), the set of $k \in \mathbb{R}$ so that

$$\sigma(A - kbc) \cap \bar{\mathbb{C}}_+ \neq \emptyset \quad \text{and} \quad \lim_{t \to \infty} e^{[A-(-1)^M T_M bc](t-t_{M-1})} x(t_{M-1}) = 0$$

is discrete. Thus we can choose $\epsilon' \in (0, \epsilon)$ and S_M instead of T_M so that

$$T_{M-1} < T_M - \epsilon' < S_M < T_M + \epsilon' < T_{M+1}.$$

and

$$e^{[A-(-1)^M S_M bc](t-t_{M-1})} x(t_{M-1}) \not\to 0 \quad \text{as} \quad t \to \infty.$$

Since (A, b, c) is detectable (see remark 1), $s(t)$ will leave the interval $[T_{M-1}, S_M)$. Proceeding in this way, and changing the nominal switching sequence at each switch, so that the projection of the state on the unstable subspace is nonzero, the switching strategy will, according to theorem 2, stop after finitely many switches. Therefore, the terminal system must be exponentially stable. This completes the proof. □

An immediate consequence of theorem 1 (i) is the following observation.

Remark 6. Suppose $T \in \mathfrak{T}$, and equation (29) (respectively, $T \in \hat{\mathfrak{T}}$ and equation (31)) is applied to a controllable minimum phase system (33) with $x_0 \neq 0$ and $cb > 0$ (respectively, $cb \neq 0$). Then at each time

$$t_i := \inf\{t \geq t_{i-1} \mid s(t) = T_i\}, \qquad t_0 := 0$$

when a new gain is chosen, the new trajectory

$$x(t;\, t_i) = e^{[A - T_{i+1}bc](t-t_i)}x(t_i)$$

(respectively,

$$\hat{x}(t;\, t_i) = e^{[A-(-1)^{i+1}T_{i+1}bc](t-t_i)}x(t_i))$$

satisfies

$$x(t;\, t_i) \not\to 0 \text{ (respectively, } x(t;\, t_i) \not\to 0)$$

as $t \to \infty$, with probability 1 with respect to $T_{i+1} \in \mathbb{R}$ if

$$\sigma(A - T_{i+1}bc) \cap \bar{\mathbb{C}}_+ \neq \emptyset$$

respectively,

$$\sigma(A - (-1)^{i+1}T_{i+1}bc) \cap \bar{\mathbb{C}}_+ \neq \emptyset).$$

6. Universal adaptive exponential tracking

In this section, we will show how to apply theorem 2 in order to obtain an adaptive tracking controller for the class of reference signals defined by

$$\mathcal{Y}_{\text{ref}} := \left\{ y_{\text{ref}}(\cdot) : \mathbb{R} \to \mathbb{R}^m \text{ a } \mathcal{C}^\infty\text{-function} \;\middle|\; \alpha\left(\frac{d}{dt}\right) y_{\text{ref}}(t) \equiv 0 \right\},$$

where $\alpha(s) \in \mathbb{R}[s]$ is a *known* monic polynomial with zeros in $\bar{\mathbb{C}}_+$ only.

The objective is to construct an adaptive control law, such that for any linear system belonging to class (4) and any reference signal $y_{\mathrm{ref}}(\cdot) \in \mathcal{Y}_{\mathrm{ref}}$, the closed-loop output response $y(t)$ generates an error $e(t)\ y(t) - y_{\mathrm{ref}}(t)$ that decays exponentially to zero.

The main idea, which, for single-input, single-output systems, goes back to Mareels [6] and Helmke et al. [11], is to use the knowledge of $\alpha(\cdot)$ to construct an internal model (that is a duplicated model of the dynamic reference signals) as part of a precompensator in the feedback loop. More precisely, let $\beta(\cdot) \in \mathbb{R}[s]$ be a monic Hurwitz polynomial of degree $p = \deg \alpha(\cdot)$, and choose a minimal realization of $\beta(s)/\alpha(s)$ denoted by $(\hat{A}, \hat{B}, \hat{C}, 1) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{p} \times \mathbb{R}^{1\times p} \times \mathbb{R}$. Then we will apply the precompensator

$$\dot{\xi}(t) = \hat{A}^*\xi(t) + \hat{B}^*v(t), \qquad u(t) = \hat{C}^*\xi(t) + I_m v(t), \qquad \xi(0) \in \mathbb{R}^{mp} \tag{34}$$

where

$$\hat{A}^* = \operatorname{diag}\{\hat{A}, \ldots, \hat{A}\} \in \mathbb{R}^{mp\times mp}, \qquad \hat{B}^* = \operatorname{diag}\{\hat{B}, \ldots, \hat{B}\} \in \mathbb{R}^{mp\times m},$$

$$\hat{C}^* = \operatorname{diag}\{\hat{C}, \ldots, \hat{C}\} \in \mathbb{R}^{m\times mp}.$$

The internal model (34) is connected in series with previous adaptive stabilizers to obtain the following adaptive tracking result.

Theorem 4. *Suppose* (A, B, C) *is a member of system* (4) *and* $\{K_1, \ldots, K_N\} \subset GL_m(\mathbb{R})$ *is a spectrum unmixing set. Let* $\{T_i\}_{i\geq 0} \in \mathbb{R}^{\mathbb{N}}$ *be a sequence of thresholds satisfying*

$$T_i < T_{i+1}, \qquad \lim_{i\to\infty} \frac{T_i}{T_{i+1}} = 0$$

and $(\hat{A}^*, \hat{A}^*, \hat{C}^*)$ *be defined as in precompensator* (34). *If* $\xi(0) \in \mathbb{R}^{mp}$, *an initial condition, and* $y_{\mathrm{ref}}(\cdot) \in \mathcal{Y}_{\mathrm{ref}}$, *a reference signal, are arbitrary, then the error feedback controller*

$$\left.\begin{aligned}
e(t) &= y(t) - y_{\mathrm{ref}}(t) \\
\dot{s}(t) &= \|y(t)\|^2 &&, s(0) = T_0 \\
v(t) &= -K(t)e(t) \\
K(t) &= T_i \cdot K_{i \bmod N} &&, \text{if } s(t) \in [T_{i-1}, T_i),\ i = 1, 2, \ldots \\
\dot{\xi}(t) &= \hat{A}^*\xi(t) + \hat{B}^*v(t) &&, \xi(0) \in \mathbb{R}^{mp} \\
u(t) &= \hat{C}^*\xi(t) + I_m v(t)
\end{aligned}\right\}, \tag{35}$$

applied to any system belonging to class (4), *with arbitrary initial condition* $x(0) \in \mathbb{R}^n$, *yields a unique solution*

$$(x(\cdot), \xi(\cdot), s(\cdot)) : [0, t') \to \mathbb{R}^{n+mp+1}$$

with the following properties:

(i) $t' = \infty$.
(ii) *$e(t)$ decays expoanentially as t tends to ∞.*
(iii) *There exists $i_0 \in \mathbb{N}$ such that $s(t) \le T_{i_0}$ for all $t \ge 0$, i.e., the switching mechanism stops after finitely many switches.*
(iv) *There exists a $c > 0$ such that for all $t \ge 0$,*

$$\|(\xi^T(t), x^T(t))^T\| \le c(1 + \max_{s\in[0,t]}\{\|y_{\mathrm{ref}}(s)\|\}).$$

Proof. The input–output behavior $v(\cdot) \mapsto y(\cdot)$ of the series interconnection formed by pre-compensator (34) and (A, B, C) is described by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}v(t), \qquad y(t) = \bar{C}\bar{x}(t), \qquad \bar{x}(0) \in \mathbb{R}^{m+n} \tag{36}$$

where

$$\bar{A} = \begin{bmatrix} A & B\hat{C}^* \\ 0 & \hat{A}^* \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} B \\ \hat{B}^* \end{bmatrix}, \qquad \bar{C} = [C, 0], \qquad \bar{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

It is easy to see that $(\bar{A}, \bar{B}, \bar{C})$ is minimum phase. The essential ingredient of the present proof is the following lemma due to Miller and Davison [29] (see also Townley and Owens [30]):

Lemma. *For every $y_{\mathrm{ref}}(\cdot) \in \mathcal{Y}_{\mathrm{ref}}$, there exists an $\tilde{x}_0 \in \mathbb{R}^{n+mp}$ such that*

$$y_{\mathrm{ref}}(t) = \bar{C}\tilde{x}(t), \qquad \dot{\tilde{x}}(t) = \bar{A}\tilde{x}(t), \qquad \tilde{x}(0) = \tilde{x}_0, \tag{37}$$

$$\|\tilde{x}(t)\| \le c\left(1 + \max_{s\in[0,t]}\{\| y_{\mathrm{ref}}(s)\|\}\right) \quad \text{for all } t \ge 0. \tag{38}$$

By equations (37) and (38), $x_e(t) := \bar{x}(t) - \tilde{x}(t)$ satisfies

$$\dot{\bar{x}}_e(t) = \bar{A}\bar{x}_e(t) + \bar{B}\bar{v}(t), \qquad \bar{x}_e(0) = \bar{x}(0) - \tilde{x}(0)$$

$$e(t) = \bar{C}x_e(t).$$

The problem has hence been converted into a standard stabilization problem for the error $e(t)$. By the Miller–Davison lemma above, and since $\bar{C}\bar{B} = CB$, theorem 2 can be applied. This proves (i)–(iii), (vi) is a consequence of inequality (38), and $\lim_{t\to\infty} x_e(t) = 0$. This completes the proof. □

Remark 7. The topological properties of the terminal system are valid in a similar manner for tracking of single-input, single-output systems. For brevity, the proof is omitted.

7. Concluding remarks

We have introduced a simple universal adaptive stabilizer for the class of multivariable minimum phase systems (A, B, C) with $\det(CB) \neq 0$. The switching strategy is piecewise constant between discrete points of time and is tuned by the function $\int_0^t \| y(s) \|^2 ds$ and a given sequence of thresholds. The feedback controller $u(t) = -T_i K_{i \bmod N} y(t)$ increases the gain T_i with each switch, and travels through a finite-spectrum unmixing set in order to find a stabilizing feedback matrix K_i. The minimum-phase assumption guarantees that, if necessary, finally the matrix $A - BT_i K_{i \bmod N} C$ is exponentially stable. Therefore, the switching algorithm stops after finitely many times, and the state of the terminal system $\dot{x}(t) = [A - T_M B K_{i \bmod N} C] x(t)$ decays exponentially. For single-input, single-output systems, the root loci of minimum-phase systems yield that the set of thresholds that produce an exponentially stable terminal system lies densely in the set of thresholds. Moreover, each time a new gain is implemented and the overall system is unstable, the probability is 1 that the new trajectory is unstable. In combination with an appropriate internal model, the previous results are applied to solve the adaptive tracking problem for a class of reference signals that are solutions of a known differential equation.

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