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Adaptive Multivariable pH Regulation of a Biogas Tower Reactor

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A simple adaptive high-gain regulator is designed for a non-linear multivariable biogas tower reactor. The controller achieves asymptotic tracking towards a pre-specified λ-neighbourhood of a constant reference signal within a pre-specified time T. *The adaptation strategy is robust against noise-corrupted output and changing system parameters. It is shown by experiments that it tolerates large disturbances. The results have been tested on an industrial pilot reactor.*

Keywords: Adaptive control; Biotechnology; Non-linear control; Process control; Robustness; λ-Tracking; Waste water treatment

1. Introduction

In this paper, we introduce an adaptive multivariable pH controller for a biogas tower reactor. The reactor is a new type for anaerobic treatment of waste water. It has been developed at the Department of Bioprocess and Biochemical Engineering at the Technical University of Hamburg Harburg. The adaptive controller was successfully tested over a period of two months at a biogas tower reactor in pilot scale located at the Deutsche Hefewerke (DHW) in Hamburg. The control objective is to keep the conversion rates of organic compounds into methane and carbon dioxide constant. This will be achieved by regulating the pH values in each reactor module.

There are numerous applications of control theory results to single-input single-output pH control of stirred tank reactors; see Goodwin et al. [5] or Flemming and Mogens [4], to name but a few. However, these results are not applicable to the biogas tower reactor, since a dominating feature of the new reactor principle is its modular reactor structure. Therefore, the plant consists of strongly coupled subsystems and the control problem is multi-input multi-output. Furthermore, the pH value has to be manipulated by the waste water influent rates and cannot be controlled by base or acid titration as in most pH control applications.

From the control engineering point of view, the first investigation of this new reactor principle is due to Pahl et al. [21] and Pahl and Lunze [20], where an overall model of order 36 was derived via theoretical process analysis and validated experimentally. This model is too complex for the controller design. It is used as a reference for reduced models. In Lunze and Pahl [16] a reduced four-dimensional model was used to analyse the dynamic couplings of the reactor modules in dependence on different operating conditions. It is also used for design, see Ilchmann et al. [9], where some experimental results of adaptive control of the biogas tower reactor are shown. Although some parameters, in particular those of the reactor rates, are uncertain, the reduced model preserves structural properties of the process since it obeys the mass balance equa-

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tions. It appears to us that under these circumstances an adaptive controller is appropriate.

However, adaptive control based on system identification cannot be applied. For this reason we apply an adaptive controller which relies on the structural properties of the system, such as minimum phase and relative degree one, and is moreover very simple in its design, it invokes no internal model and is applicable to multivariable systems of unknown state dimension. The adaptation is in the spirit of Ilchmann and Ryan [10]. It achieves tracking of the pH values within a pre-specified λ-neighbourhood of the constant reference signals and within a pre-specified time of 12 h. This is important since the biogas tower reactor is switched off over the weekend and at the beginning of each week the adaptive controller has to start up again and to find, due to changed system parameters, its appropriate gain.

The biogas tower reactor in pilot scale [17] is shown in Fig. 1. Waste water processed from a baker's yeast production contains sulphuric acid and organic compounds. This waste water is fed into the biogas tower reactor, where via microorganisms an anaerobic biochemical conversion of the organic influent compounds takes place. The total organic carbon is converted into carbon dioxide and methane. As a byproduct, hydrogen sulphide is produced which inhibits the activity of the methane-forming microorganisms. The overall biochemical reactions are described in detail by Friedmann and Märkl [6].

The reactor consists of four identical modules (see Fig. 1). At the top, a settler for effective biomass retention is integrated. For better mixing conditions, the waste water stream with flow rate f_{feed} can be split up into four influent streams with flow rates $f_{feed,i}$, where the ith stream is fed into the ith module. These inflow rates are the four manipulated variables u_i. To avoid gas accumulation in the upper zones of the reactor, gas-collecting devices are installed in each module. By these devices, the gas can be drawn off from the system. For better liquid–gas mass transfer, biogas is recirculated into the bottom of the reactor. To avoid accumulation of inhibiting hydrogen sulphide in the liquid phase, only methane and carbon dioxide are recirculated, whereas hydrogen sulphide is removed chemically. A single reactor module is shown in detail in Fig. 2.

Mixing behaviour: a baffle plate divides each module into two parts. Since the gas concentrations at both parts differ, a hydrostatic pressure causes a fluid circulation along the baffles similar to those found in airlift loop reactors [3]. The fluid circulation brings about a well-mixed system within one module. A part of the gas rises from one module to the next and causes an exchange mass flow in the liquid phase between the two modules that works in both directions. The volumetric flow rate f_{ex} of the exchange mass flow is controlled by the biogas flow rate drawn off [22].

Biochemical reactions: the microbacteria convert the educts of the influent into several products which are given in the liquid phase. By liquid–gas mass transfer these substances form biogas bubbles.

The global control objective is to keep the effluent concentrations of the waste water within legal limits. This means that the removal rates have to be stabilised. They depend on the non-measurable concentrations of weak acids inside the reactor and on several surrounding conditions given by certain pH values, temperature etc. It is known that there is a strong correlation between the concentrations of weak acids and the pH value [2]. Hence, removal rates are stabilised indirectly if pH values are constant in all modules.

Control variables are the pH values in each module. They are measurable on-line and are determined by the concentrations of weak acids, such as acetic acid and carbon dioxide. Changes in these educt and product concentrations are caused by biochemical reactions and liquid-gas mass transfer. The pH value in the coupled ith module is also influenced by the waste water flow rates which consist of the exchange flow between neighbouring modules, the accumulated inflow rates from the bottom modules, and the inflow rates fed directly into the module.

Manipulated variables are the inflow rates $f_{feed,i}$ for $i = 1,\ldots,4$. For the experiments (see Section 5) we applied the adaptive controller only to three instead of four modules. The reason is simply that the size of the actual settler was shown to be too small for sufficient biomass retention. This restriction will be avoided in future by reshaping the settler to an appropriate dimension.

The controller design has to obey the following *restrictions:*

- The main *time constants* of the process are about 6 h. Time changes concerning the growth of biomass are even more than one order of magnitude higher. Compared to these main time constants, the reactor is available for experiments only for a very short time. This limitation concerns the kind and number of experiments made. As the main consequence, well-known methods for system identification and adaptive control based on system identification cannot be applied.
- Since the process knowledge is incomplete, the process model has *uncertain parameters*, such as

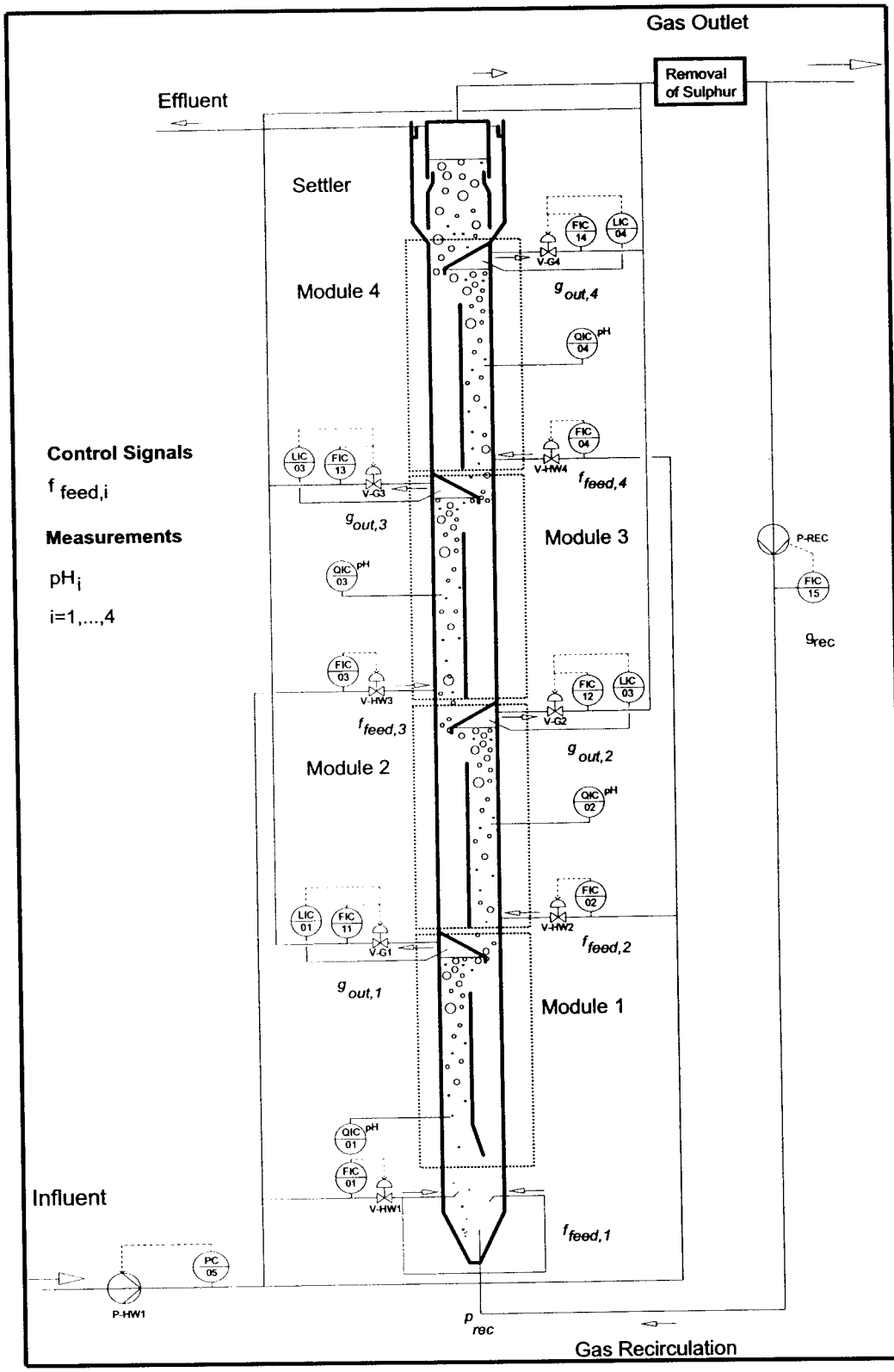

Fig. 1. Biogas tower reactor in pilot scale with its internal installations (height 20 m, diameter 1 m, volume 15 m^3).

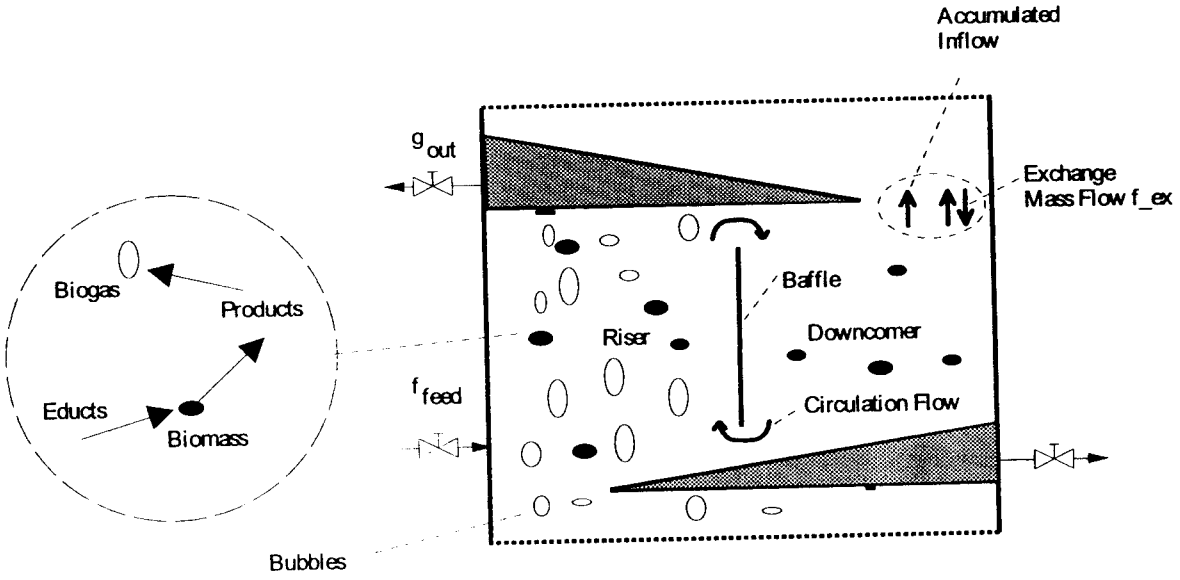

Fig. 2. Single module of the reactor.

the reaction rate parameters. Upper bounds of parameter errors are known but the uncertainty of some parameters covers orders of magnitude.

Control objectives:

1. The pH controller has to start up the reactor within 12 h at the beginning of each week in such a way that pH values reach the pre-specified operation area. The reason for this is that the reactor input is switched off over the weekend, which changes pH values considerably.
2. The pH controller has to attenuate the disturbances in order to keep all pH values within a pre-specified operating region. The controller should hold *all* pH values within an interval of ± 0.05 around the setpoints. This interval has been chosen with respect to the attainable measurement precision of the pH sensors.

The control objective has to be satisfied under the following *plant uncertainties and disturbances:*

- Plant parameters change considerably at varying exchange flow rates between neighbouring modules and the changing biomass concentration, which both influence the reaction rates. This leads to strong *perturbations* of the static reinforcement and the time constants of the plant.
- Stabilisation of the operating points has to be achieved in the presence of two main kinds of relevant disturbances. First, daily calibration of the pH electrodes causes stepwise output disturbances which have to be compensated by the controller. Furthermore, disturbance signals of unknown characteristics such as changing influent concentrations exist.

The *adaptive controller* introduced in the present paper to meet the above objectives consists of a non-linear state feedback law and a gain adaptation. Let $k(t)$ denote the time-varying gain which is tuned adaptively by the size of the error signal

$$\mathbf{e}(t) = (y_1(t) - w_1, \ldots, y_4(t) - w_4)^T$$

Here $y_i(t)$ denotes the pH value in the ith module and w_i the set point, $i = 1,\ldots,4$. The feedback law is designed in such a way that the strong relative degree one property of the system (which is crucial for the high-gain approach) is preserved after having closed the loop in the following sense: if $k(t)\mathbf{e}(t)$ is considered as input, then the system is of relative degree one. The second part of the adaptive controller is chosen so that it exploits the fact that the larger the gain $k(t)$ is, the smaller the error $e(t)$ becomes. However, we only want a gain as large as necessary to meet the control objectives. Therefore, we choose the *gain adaptation*

$$\dot{k}(t) = \begin{cases} \gamma(\|\mathbf{e}(t)\| - \lambda)\,\|\mathbf{e}(t)\|, & \|\mathbf{e}(t)\| \geq \lambda \\ 0, & \|\mathbf{e}(t)\| < \lambda \end{cases} \tag{1}$$

where $\lambda > 0$ is pre-specified. Note that $k(t)$ is strictly increasing as long as the error is outside the closed λ-ball; if the error enters the λ-ball, then it is kept constant. It will be shown that $k(t)$ converges to a finite limit whilst $e(t)$ approaches the λ-ball $\{e \in \mathbb{R}^4 | \|e\| < \lambda\}$ as t tends to ∞.

The paper is organised as follows. In Section 2

we derive and analyse the four-dimensional reduced model of the biogas tower reactor. In Section 3 the adaptation mechanism is introduced and convergence is shown. Some simulations which illustrate the effect of the adaptive regulator when applied to the model of the reactor are given in Section 4. In Section 5 we present and discuss our experimental results of the controller when tested at the real plant. The paper is finalised with some conclusions in Section 6 and the proofs of the main theorems are given in the Appendix.

2. Process Model

In this section, we derive a reduced model from the reference model given in Pahl and Lunze [20]. The non-linear process model consists of four subsystems, where each subsystem represents one module of the reactor (see Fig. 3). This model takes into account the biochemical reactions investigated by Friedmann and Märkl [6] and results in the mixing behaviour obtained by Reinhold [22]; it was validated by experiments; see Pahl and Lunze [20] for details. The model is based on a mass balance of compounds in the liquid and in the gas phase. Since within a single module the liquid phase is well mixed by the circulation flow, we assume *lumped* parameters. The overall model is then based on the mass balance of strongly coupled ideally stirred vessels. Four compounds in the liquid phase are considered. For a single module the mass balance of these liquid phase concentrations is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \frac{1}{V_{fl}}\left(\mathbf{C}_{in}(t)\mathbf{f}_{in}(t) - \mathbf{x}(t)\sum_{j=1}^{3} f_{in,j}(t)\right) + \begin{bmatrix} -r_{x_1}(t) \\ Kr_{x_1}(t) - \dfrac{m_{tr,x_2}(t)}{V_{fl}} \\ -r_{x_3}(t) \\ r_{x_3}(t) - \dfrac{m_{tr,x_4}(t)}{V_{fl}} \end{bmatrix} \tag{2}$$

where we use the following notation:

V_{fl}	liquid phase volume of each reactor module
$\mathbf{C}_{in}(t)$	matrix of inflow concentrations (feed concentrations and concentrations between neighbouring modules)
$\mathbf{f}_{in}(t)$	vector of volumetric inflow rates (feed flows and exchange flows between neighbouring modules)

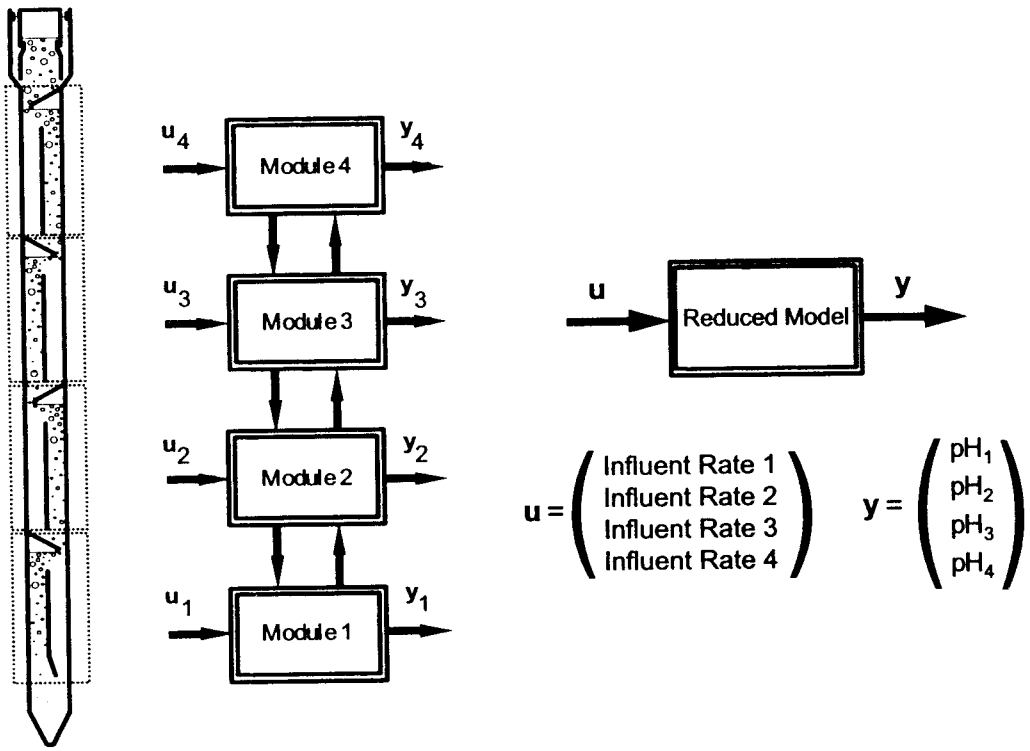

Fig. 3. Reduced model for pH controller design from the overall model of the Biogas tower reactor.

$f_{in,j}(t)$ volumetric inflow rate j
$r_*(t)$ reaction rate of compound $*$
$m_{tr,*}(t)$ liquid–gas mass transfer rate of compound $*$
K reaction rate parameter

and $\mathbf{x}(t) \in \mathbb{R}^4$ denotes the vector of the concentrations in the liquid phase. The first summand of the right-hand side of (2) describes the difference between mass inflows and mass outflows. Note that in addition to feed flows coupling mass flows exist. As explained in the Introduction, an exchange flow between neighbouring modules occurs in addition to the feed flows into each module. Hence, several mass flows have to be considered for the mass balance of a single module. All volumetric inflow rates and all influent concentrations are summarised in a vector $\mathbf{f}_{in}$ and a matrix $\mathbf{C}_{in}$. Since the liquid phase volume is constant, the volumetric outflow rate equals the sum of all influent rates $f_{in,j}(t)$ for $j = 1,2,3$. The right summand contains the source and sink terms. They depend non-linearly on the concentrations $x_1,\ldots,x_4$. For instance, for the two inner modules ($i = 2,3$) we have

$$\mathbf{C}_{in} = \begin{pmatrix} c_{feed,1} & x_{1,i-1} & x_{1,i} \\ c_{feed,2} & x_{2,i-1} & x_{2,i} \\ c_{feed,3} & x_{3,i-1} & x_{3,i} \\ c_{feed,4} & x_{4,i-1} & x_{4,i} \end{pmatrix},$$

$$\mathbf{f}_{in} = \begin{pmatrix} f_{feed,i} \\ f_{ex,i-1} + \sum_{j=1}^{i-1} f_{feed,j} \\ f_{ex,i} \end{pmatrix}$$

where $f_{ex,i}$ denotes the exchange flow rate between module i and the upper module $i + 1$.

The overall model consists of four modules, and is therefore given by four equations of the form (2) and some balance equations of the gas phase. The pH values depend non-linearly on the liquid phase concentrations $\mathbf{x}$.

Since the model is of high order and contains strong non-linearities, we were unable to use it directly for the controller design. Hence, a model of reduced order and complexity is derived from the overall model and used for the design of the pH controller. This will be described in the remainder of this section.

The educts and products of the biochemical reactions are mainly weak acids such as acetic acid, propionic acid and carbon dioxide. Their concentrations influence the pH value, which is defined by

$$pH = -\log_{10}\{\beta H^+\}$$

where H^+ denotes the hydronium concentration and β the activity coefficient, which can be assumed to be constant in our case.

Although in general the dissociation characteristics of the aqueous solutions are strongly non-linear and there is no one-to-one relationship between the respective concentrations, it is shown in Pahl and Lunze [20] that, within the relevant interval of the pH value [6.8, 7.5], a linear approximation describes the process sufficiently accurately with respect to controller design purposes:

$$pH(t) = pH_0 - m_1 x_1(t) - m_2 x_2(t) \tag{3}$$

where pH_0, m_1, m_2 are constant positive parameters and x_1, x_2 denote the acetic acid and the carbon dioxide concentration, respectively. Changes in the pH value are determined mainly by concentration changes of these compounds, whereas small concentration changes occurring in the other compounds showed only a weak influence on the pH value. We stress that a much more accurate model could be given. But in this case the reduction of the model complexity is an essential step in deriving a process model that is suitable for the controller design. Figure 4 depicts the pH value depending on the main concentrations c_{AC}, c_{CO_2}, where the concentrations of the other compounds, which vary only slightly in the region considered, are kept constant. This is explained and justified in Pahl and Lunze [20]. It can be seen that the quality of a linear approximation is sufficient in the relevant operation range $pH \in [6.8,\ldots,7.5]$, and $x_1 \in [0,\ldots,0.2]$ [mole/l], $x_2 \in [0,\ldots,0.3]$ [mole/l].

Now the reduced model is derived by combining (2) and (3) as follows. Considering (3) in the mass balances for x_1 and x_2 yields, for instance for $i = 2, 3$

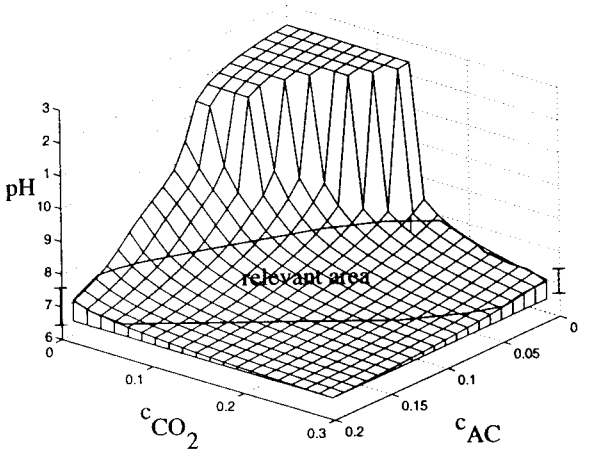

Fig. 4. Non-linear dependence of pH value on c_{AC} and c_{CO_2} for the relevant concentration intervals.

$$\frac{\mathrm{d}\, pH_i(t)}{\mathrm{d}t} = m_1\, \dot{x}_{1,i}(t) + m_2\, \dot{x}_{2,i}(t)$$

$$= \frac{1}{V_{fl}} [f_{feed,i}(pH_{feed,i} - pH_i) + f_{ex,i-1}(pH_{i-1} - pH_i) + f_{ex,i}(pH_{i+1} - pH_i)] + r_{x_1,i}(m_2 K - m_1) - \frac{m_2}{V_{fl}} m_{tr,x_2,i}$$

where the pH value of the inflow is calculated by

$$pH_{feed,i} = pH_0 + m_1\, x_{feed,1} + m_2\, x_{feed,2}$$

The sum of source and sink terms

$$r_i(t) := r_{x_1,i}\,(m_2 K - m_1) - \frac{m_2}{V_{fl}} m_{tr,x_2,i} \tag{4}$$

can be interpreted as a fictitious reaction rate which summarises pH changes caused by the biochemical reactions and the liquid–gas mass transfer of carbon dioxide. It can easily be shown that these rates satisfy the crude affine linear bound

$$0 < r_i(t) < \rho_0 + \rho_1\, pH_i(t)\,, \quad i = 1,\ldots,4 \tag{5}$$

for some positive ρ_0, $\rho_1 \in \mathbb{R}$, as long as the pH value is in the relevant region. In matrix notation we derive the following non-linear state space model

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{r}(\mathbf{y}(t)) - \mathbf{G}(\mathbf{y}(t))\, \mathbf{u}(t) \tag{6}$$

where

$$\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T = (pH_1(t), pH_2(t), pH_3(t), pH_4(t))^T \in \mathbb{R}^4$$

denote the state respective output signals,

$$\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T = \frac{1}{V_{fl}} (f_{feed,1}(t), f_{feed,2}(t), f_{feed,3}(t), f_{feed,4}(t))^T$$

are the influent rates, i.e. the input signals,

$$\mathbf{A} = \begin{bmatrix} -a_1 & a_1 & 0 & 0 \\ a_1 & -(a_1 + a_2) & a_2 & 0 \\ 0 & a_2 & -(a_2 + a_3) & a_3 \\ 0 & 0 & a_3 & -a_3 \end{bmatrix} \tag{7}$$

with $a_i = f_{ex,i} > 0$, $i = 1,\ldots,3$, stands for the exchange rates between the neighbouring modules,

$$\mathbf{r}(\mathbf{y}) = (r_1(y_1),\ldots,r_4(y_4))^T \tag{8}$$

is the vector of the source terms satisfying (5) and

$$\mathbf{G}(\mathbf{y}) = \begin{bmatrix} y_1 - y_{feed} & 0 & 0 & 0 \\ y_2 - y_1 & y_2 - y_{feed} & 0 & 0 \\ y_3 - y_2 & y_3 - y_2 & y_3 - y_{feed} & 0 \\ y_4 - y_3 & y_4 - y_3 & y_4 - y_3 & y_4 - y_{feed} \end{bmatrix},$$

$$y_{feed} > 0 \tag{9}$$

represents the effect of the inflows on the upper modules.

From a mathematical point of view, we observe the following properties of the model (6):

- For any initial condition $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^4_+$ and any locally integrable $\mathbf{u}(\cdot) : [0, \infty) \to \mathbb{R}^4$, there exists a unique solution $\mathbf{y}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^4_+$ of (6) as long as $\mathbf{y}(t)$ remains in $\mathbb{R}^4_+$.
- The non-linear system (6) is of strong relative degree one [14] as long as $y_i(t) \neq y_{feed}$ for all $i = 1,\ldots,4$. In Theorem 3.1 we will prove that this holds true for the initial conditions and setpoints relevant in this process.
- The matrix $\mathbf{A}$ is singular since $\mathbf{A}\,(1,\ldots,1)^T = 0$. Since rk $\mathbf{A} = 3$, the spectrum of $\mathbf{A}$ consists of 0 and some eigenvalues unequal 0. By Gershgorin's circle criterion we might easily conclude that all other eigenvalues are lying in the open left half complex plane; in fact they are negative since A is symmetric.

  Hence, if the reaction rates are zero, i.e. $\mathbf{r}(\cdot) \equiv 0$, and if $\mathbf{u}(\cdot) \equiv 0$, then for every initial condition $\mathbf{y}_0 \in \mathbb{R}^4$ the solution of (6) tends to the steady state of this compartimental model $c \cdot (1,\ldots,1)^T$, for some $c \in \mathbb{R}$. If $\mathbf{r}(\cdot) \not\equiv 0$ and $\mathbf{u}(\cdot) \equiv 0$, then by (5) the system becomes unstable. This is also intuitively clear: if the input is switched off or is relatively small compared to $r(\cdot)$, then the pH values will increase until all concentrations of weak acids are zero.
- If (6) is in equilibrium, i.e. for some $\bar{\mathbf{y}} = (\bar{y}_1,\ldots,\bar{y}_4)^T$ and $\bar{\mathbf{u}} = (\bar{u}_1,\ldots,\bar{u}_4)^T \in \mathbb{R}^4$ we have $0 = \mathbf{A}\bar{\mathbf{y}} + \mathbf{r}(\bar{\mathbf{y}}) - \mathbf{G}(\bar{\mathbf{y}})\bar{\mathbf{u}}$, and if $0 < y_{feed} < \bar{y}_1 < \ldots < \bar{y}_4$, then it is straightforward to show that $\bar{u} \in \mathbb{R}^4_+$.

We stress that the model (6) is very simple compared to the reference model by Pahl and Lunze [20]. Process information is lost due to the accumulation of several reaction rates into $\mathbf{r}(\mathbf{y})$. Thus, the model cannot be used for prediction purposes. However, it obeys the mass balance Eq. (1) and hence exhibits the essential structural properties. This will be important for the design of the adaptive controller in the following section.

3. Adaptive Controller

In this section, we will present the adaptive regulator which achieves, when applied to (6), tracking of the reference signals within a pre-specified λ-neighbourhood whilst preserving boundedness of the remaining variables.

3.1. The High-Gain Approach to Adaptive Control

The concept of high-gain stabilisation without identification was initiated by Willems and Byrnes [24] and Morse [18]. From then on, it became a rapidly growing field of interest for many different system classes; see Ilchmann [7] for a bibliography. The idea of incorporating a 'dead-zone' in the adaptation was first used by Miller and Davison [19] and the λ-tracking concept as used in the present paper is similar to that in Ilchmann and Ryan [10] and to Allgöwer et al. [1].

To provide the reader with an intuition of this concept, we consider the simplest example we can think of. Although the actual model, to which the adaptive regulator is applied, is non-linear, we will first study the effect of the adaptive controller on a linear first-order system of the form

$$\dot{y}(t) = a\, y(t) + g\, u(t), \quad y(0) = y_0 \tag{10}$$

for unknown y_0, $a \in \mathbb{R}$, $g > 0$. Consider the time-varying proportional error feedback

$$e(t) = y(t) - y_{ref}, \quad u(t) = -k(t)e(t) \tag{11}$$

where $y_{ref} \in \mathbb{R}$ denotes the setpoint to be tracked and the monotonically non-decreasing function $k(\cdot) : [0, \infty) \to [0, \infty)$ is determined by the gain adaptation (1) with $\lambda = 1$.

If (11) is applied to (10) (for simplicity we may assume that finite escape time of the nonlinear closed-loop system (10)–(11) does not occur), then the closed-loop system may be rewritten as (1) plus $\dot{e}(t) = [a - k(t)g]\ e(t) + a\, y_{ref}$, $e(0) = e_0 = y_0 - y_{ref}$ and an application of the variation-of-constants formula together with the monotonicity of $t \mapsto k(t)$ leads, for all $t \geq t_0$ and sufficiently large $t_0 \geq 0$, to

$$|e(t)| \leq e^{[a-k(t_0)g](t-t_0)}|e(t_0)| + \frac{|ay_{ref}|}{-[a - k(t_0)g]} \tag{12}$$

Suppose, for a moment, that $k(t)$ tends to ∞ as t tends to ∞. Then (12) yields $|e(t)| \to 0$ as $t \to \infty$. Therefore, there exists some $t_1 > 0$, such that $|e(t)| \leq \lambda$ for all $t \geq t_1$. However, this contradicts our assumption of unboundedness of $k(\cdot)$ and hence (1) implies boundedness of $k(\cdot)$. Roughly speaking, the gain increases as long as $y(t)$ is outside the interval $[y_{ref} - \lambda, y_{ref} + \lambda]$. If the gain is sufficiently large, then $|e(t)|$ tends towards $[0, \lambda]$.

A remarkable property of the gain adaptation (1) is that it tolerates output which is corrupted by additive noise. If an upper bound for the noise is known in advance and $\lambda > 0$ is chosen larger than this bound, then the error $e(t) := y(t) + n(t) - y_{ref}$, where $n(\cdot)$ denotes the noise–signal, is forced into the λ-strip. This property is due to the 'dead-zone' incorporated in the gain adaptation.

3.2. Adaptive Controller

We will prove that the same idea as in Section 3.1, i.e. high-gain to produce a stable output of the closed-loop system and converging gain if the error is sufficiently small, also works for our non-linear, multivariable model (6) of the biogas tower reactor. However, the application of the λ-tracking concept is not directly possible. This is due to the non-negligible non-linear couplings of the modules. However, we will introduce a feedback which decouples this structure whilst preserving the structural properties of strong relative degree one and minimum phase.

The control objective is to λ-track a constant reference signal

$$\mathbf{y}_{ref} = (w_1, \ldots, w_4)^T \quad \text{with} \quad y_{feed} < w_1 < \ldots < w_4$$

λ-Tracking means that we want the output $y_i(t)$ to reach asymptotically a λ-neighbourhood of the reference signal w_i. More precisely, we want

$$\mathbf{e}(t) = (y_1(t) - w_1, \ldots, y_4(t) - w_4)^T = \mathbf{y}(t) - \mathbf{y}_{ref}$$

to approach the λ-ball $\{\mathbf{e} \in \mathbb{R}^4 \mid \|\mathbf{e}\| < \lambda\}$ as t tends to ∞. This will be achieved as follows.

The *non-linear adaptive feedback* is defined by

$$\mathbf{u}(t) = \bar{\mathbf{u}} + \mathbf{v}(t) \tag{13}$$

where the components of $\mathbf{v}(t) = (v_1(t), \ldots, v_4(t))^T$ are given by

$$v_1(t) := \frac{k(t)}{y_1(t) - y_{feed}}(y_1(t) - w_1)$$

$$v_2(t) := \frac{k(t)}{y_2(t) - y_{feed}}(y_2(t) - w_2) - \frac{y_2(t) - y_1(t)}{y_2(t) - y_{feed}} u_1(t)$$

$$v_3(t) := \frac{k(t)}{y_3(t) - y_{feed}}(y_3(t) - w_3)$$

$$- \frac{y_3(t) - y_2(t)}{y_3(t) - y_{feed}} [u_1(t) + u_2(t)]$$

$$v_4(t) := \frac{k(t)}{y_4(t) - y_{feed}} (y_4(t) - w_4)$$

$$- \frac{y_4(t) - y_3(t)}{y_4(t) - y_{feed}} [u_1(t) + u_2(t) + u_3(t)]$$

$\bar{\mathbf{u}} \in \mathbb{R}^4$ may be arbitrary, but a sensible choice would be an approximation of the component of an equilibrium point $(\bar{\mathbf{y}}, \bar{\mathbf{u}})$, i.e.

$$\mathbf{A}\bar{\mathbf{y}} + \mathbf{r}(\bar{\mathbf{y}}) + \mathbf{G}(\bar{\mathbf{y}})\bar{\mathbf{u}} = \mathbf{0} \tag{14}$$

We stress that the u_i-terms in the v_is are chosen in such a way that, if (13) is applied to (6), then $k(t)e_i(t)$ does only influence $\dot{e}_i(t)$ directly. In fact, we will see in the Appendix (see (22)) that the closed-loop system may be written as

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) + \mathbf{f}(\mathbf{e}(t)) - k(t)\mathbf{e}(t)$$

where $\mathbf{f}$ is some affine linearly bounded function.

The *gain adaptation* is given, for pre-specified $\lambda > 0$, by (1). The design parameters $k(0) = k_0 \geq 0$ and $\gamma > 0$ influence the dynamics and size of the gain. The gain is strictly increasing as long as the error is outside the closed λ-ball; if the error enters the λ-ball, then it is kept constant.

3.3. The λ-Tracker

Our main result is convergence of the simple adaptive strategy (1),(13) if applied to the model (6). Certainly, it has to be shown that the feedback (13) is well defined, i.e. $y_i(t) > y_{feed}$ for all $t \geq 0$, $i = 1,\ldots,4$.

We suppose that the initial values for the pH values within the modules are strictly increasing from the lower to the upper modules. This is always the case since the sum of gas compounds solved in the liquid phase of the biogas tower reactor influences the pH value in dependence of the hydrostatic pressure. Products such as carbon dioxide are the more soluble the higher the hydrostatic pressure. We also suppose that the reference signals, which the pH values should converge to (within a neighbourhood), are ordered in size.

Theorem 3.1. Suppose

$$y_{feed}, \gamma, \lambda > 0, \quad k_0 \geq 0, \quad \bar{u} \in \mathbb{R}^4_+$$

$$\mathbf{y}_0 = (y_0^1,\ldots,y_0^4) \quad \text{with} \quad y_{feed} < y_0^1 < \ldots < y_0^4$$

and

$$\mathbf{y}_{ref} = (w_1,\ldots,w_4) \quad \text{with} \quad y_{feed} < w^1 < \ldots < w^4 \tag{15}$$

are given. The λ-tracker (1),(13) applied to any system (6) with initial conditions $\mathbf{y}(0) = \mathbf{y}_0$, $k(0) = k_0$, yields a closed-loop system which admits a unique solution

$$(\mathbf{y}(\cdot),k(\cdot)) : [0, \infty) \to \mathbb{R}^4_+ \times [0, \infty)$$

and satisfies

1. $y_i(t) > y_{feed}$ for all $t \geq 0$ and $i = 1,\ldots,4$, i.e. (13) is well defined;
2. $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$ exists;
3. $\|\mathbf{y}(t) - \mathbf{y}_{ref}\|$ approaches $[0, \lambda]$ as t tends to ∞.

Proof. See Appendix. □

Remark 3.2. In Theorem 3.1 we prove that for the adaptive feedback controller $y(t)$ remains larger than y_{feed}, i.e. $y_i(t) > y_{feed}$ for $i = 1,\ldots,4$, provided the components of the initial condition $y(0)$ satisfy this condition. Moreover, it also holds true that the ordering $y_{feed} < y_0^1 < \ldots < y_0^4$ is preserved for the components $y_i(t)$, provided the non-linearities $r_i(y_i)$, see (5), are sufficiently small in terms of $\bar{y}, y_{feed}$ and $\mathbf{A}$. We prove

$$y_{feed} < y_1(t) < \ldots < y_4(t) \quad \text{for all} \quad t \geq 0 \tag{16}$$

if $\mathbf{r}(\cdot) \equiv 0$. It is then easy to conclude that (16) remains valid if $\mathbf{r}(\cdot)$ is small.

For equilibrium (14) it is obvious from (23), note $k = 0$, that $\bar{u}_i > 0$ for $i = 1,\ldots,4$. Applying (23) once more, we obtain

$$\frac{d}{dt}(y_1(t) - y_2(t)) = -a_3[y_3(t) - y_2(t)] - (y_1(t) - y_{feed})\bar{u}_1 + (y_2(t) - y_{feed})\bar{u}_2 - k(t)[y_1(t) - w_1 - y_2(t) + w_2] \tag{17}$$

If (16) is false for the first time at $t = t'$, and we have $y_1(t') = y_2(t')$, then (17) yields

$$\frac{d}{dt}(y_1(t) - y_2(t))|_{t=t'} = -a_3[y_3(t') - y_2(t')] - (y_2(t') - y_{feed})[\bar{u}_1 + \bar{u}_2] - k(t)[w_2 - w_1] < 0$$

This contradicts $y_1(t') - y_2(t') = 0$. To prove the remaining inequalities in (16) proceed in a similar manner, this is omitted for brevity. Note that (16) is not crucial for the application of the λ-tracker (1),(13), it is only an additional information.

3.4. λ-Tracking Within Pre-specified Time

In Theorem 3.1 we have guaranteed that the error approaches the λ-ball asymptotically; nothing is said about the length of time it will take. By a simple modification of the gain adaptation (1) we can ensure that the error will enter the λ-ball within a pre-specified time T.

Consider, for pre-specified $\gamma_1,\gamma_2,\lambda,T>0$ and $k_0 \geq 0$

$$k(t) = k_0 + \gamma_1 \int_0^t d_\lambda(\mathbf{e}(s))\|\mathbf{e}(s)\|ds + \gamma_2 \begin{cases} \dfrac{\|\mathbf{e}(t)\|^2}{T-t}, & t \in [0, t^*) \\ k^*, & t \geq t^* \end{cases} \tag{18}$$

where

$$d_\lambda(\mathbf{e}) := \begin{cases} (\|\mathbf{e}\| - \lambda), & \|\mathbf{e}\| \geq \lambda \\ 0, & \|\mathbf{e}\| < \lambda \end{cases},$$

$$t^* := \min\left\{ t \in [0,T) \middle| \|\mathbf{e}(t)\| = \frac{3}{4}\lambda \right\},$$

$$k^* := \frac{\|\mathbf{e}(t^*)\|^2}{T - t^*}$$

Note that (18) is equivalent to (1) if $\gamma_2 = 0$. The intuition behind this gain adaptation is as follows: if t approaches T, then the third term on the right-hand side of (18) becomes very large, as long as $\|\mathbf{e}(t)\|$ is not very close to zero. Thus, the high gain forces the error to tend to zero until it hits $\frac{3}{4}\lambda$, then the error is within the λ-ball and from then on the adaption is not different from the previous one in (1).

Theorem 3.3. Suppose the assumptions of Theorem 3.1 and $\gamma_1,\gamma_2,T>0$. If the λ-tracker (13), (18) is applied to any system (6), then the closed-loop system possesses a unique solution

$$(\mathbf{y}(\cdot), k(\cdot)) : [0, \infty] \to \mathbb{R}_+^4 \times [0, \infty)$$

and satisfies

1. $y_i(t) > y_{feed}$ for all $t \geq 0$, $i = 1,\ldots,4$, i.e. (13) is well defined;
2. $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$ exists;
3. there exists some $t^* \in [0,T)$ such that $\|e(t^*)\| = \frac{3}{4}\lambda$;
4. $\|\mathbf{y}(t) - \mathbf{y}_{ref}\|$ approaches $[0, \lambda]$ as t tends to ∞.

Proof. See Appendix. □

Remark 3.4.

1. *Robustness* of the adaptive controllers (1),(13) and (18) holds with respect to the following perturbation and disturbance by the time-varying non-linearity $\boldsymbol{\delta} : \mathbb{R}_+ \times \mathbb{R}^4$ so that the affine linear bound

$$\|\mathbf{Ay} + \mathbf{r}(\mathbf{y}) + \boldsymbol{\delta}(t, \mathbf{y})\| \leq \alpha_1 + \alpha_2\|\mathbf{y}\| \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}^4, \quad t \in \mathbb{R}_+$$

still holds for some unknown constants $\alpha_1,\alpha_2 > 0$. For instance, the matrix $\mathbf{A}$ may be perturbed by a time-varying matrix $\tilde{\mathbf{A}}(t)$, or we have unmodelled non-linearities or the exchange flow rates may change during the operation. The proof of this is straightforward and omitted for brevity. Robustness against changing exchange flow rates is of practical importance because the exchange flow rates may be changed during the operation so as to control the distribution of the biomass.

As mentioned in Section 3.1, the adaptive controller can cope with output-corrupted noise. This is due to the dead-zone incorporated in the gain adaptation.

The right-hand side of (6) might also be modelled by some retarded or integrodifferential system as considered in Ilchmann and Logemann [8]. The proof that these perturbations can be considered for our particular application is also omitted for brevity.

We are also convinced that the results remain valid in the presence of model mismatch, such as fast actuator dynamics (provided the parasitic poles are sufficiently large), respectively if there are some unmodelled high-frequency parasitics which are fast enough.

2. *Input constraints* were implemented in the simulations and the adaptive controllers worked fine as long as the constraints were not too tight (see Section 4). More importantly, the real application was installed with input constraints and worked successfully for two months (see Section 5). However, we were unable to prove these results theoretically but we are hopeful that this will be possible. In an application of a similar control concept to anaerobic fermentation processes (which does not have the strong non-linear coupling as in (6)) we give bounds for the input constraints so that the controller achieves λ-tracking, see Ilchmann and Weirig [12].

4. Simulations

The application of the adaptive regulators (1),(13) and (18),(13) to (6) was simulated for representative plant parameter sets and controller design parameters. In all simulations the given objectives were met as stated in Theorems 3.1 and 3.3 with reasonably good transient performance as shown below.

As a typical example we choose the following system parameters for (6), for $i = 1,\ldots,4$:

$$a_i = 0.1[h^{-1}]\,,\ \rho_{1i} = 10^{-3}[h^{-1}]\,,$$
$$\rho_{0i} = 0.3 \cdot 10^{-3}[h^{-1}]\,,\ y_{feed} = 4.5 \qquad (19)$$

where ρ_{0i} and ρ_{1i} are the reaction rate parameters according to (4). On the time interval [0, 1.8], the input of the system is a constant $\mathbf{u}(\cdot) \equiv \bar{\mathbf{u}}$ and the system is in steady state. At $t = 1.8$[h], we will apply our control mechanisms to track two reference signals of different amplitude (Simulation 1 and Simulation 2). The effect of input constraints will also be demonstrated. Note that the initial time is $t = 1.8$[h] and not $t = 0$ as in Theorem 3.1.

Simulation 1. Set

$$\frac{1}{u_{max}}\,\mathbf{u}(t) = \frac{1}{u_{max}}\,\bar{\mathbf{u}} := (0.05, 0.025, 0.025, 0.01)^T$$
$$\text{for all } t \in [0, 1.8],\ u_{max} = 0.084[\mathrm{h}^{-1}]$$

In Fig. 5 it is shown that the system is at rest on this interval with

$$\mathbf{y}(t) = \bar{\mathbf{y}} = (7.3, 7.35, 7.4, 7.45)^T$$
$$\text{for all } t \in [0, 1.8]$$

At $t = 1.8$[h] switch on the adaptive regulator (1),(13) with design parameters

$$\gamma = 1.4[\mathrm{h}^{-1}]\,,\ \lambda = 0.05\,,\ k(1.8) = 0$$
$$\bar{\mathbf{u}} := (0.05, 0.025, 0.025, 0.01)^T \qquad (20)$$

to track the reference signal

$$\mathbf{y}_{ref} := (7.1, 7.15, 7.2, 7.25)^T$$

The transient behaviour is shown in Fig. 5. The norm of the error signals reaches the desired tolerance band in less than 3 h and stays within this band afterwards.

The gain settles at approximately 11. Note that there is no overshoot for $y(t)$ and oscillation does not occur. These simulations have a practical background. The biogas tower reactor might have been shut down for several days, e.g. over the weekend when yeast production rests. During this time pH values increase and the control objective is now to steer them down to some appropriate operating levels for high biogas production.

Simulation 2. A similar simulation as in Simulation 1 is shown in Fig. 6, where the only difference is that the given reference signal is further away from $\bar{\mathbf{y}}$, namely

$$\mathbf{y}_{ref} := (6.9, 6.94, 6.96, 6.98)^T$$

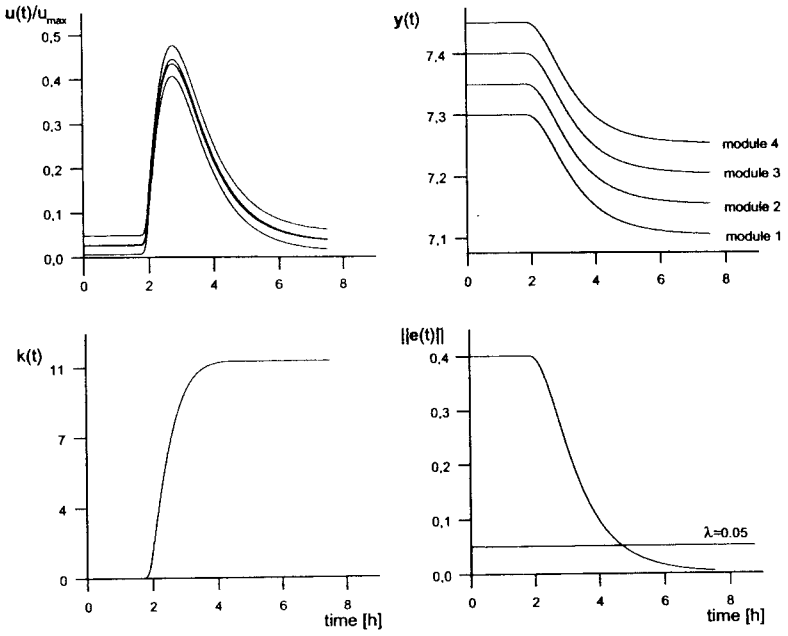

Fig. 5. Adaptive regulator (1),(13) with design parameters (20) applied to (6) with systems parameters (19).

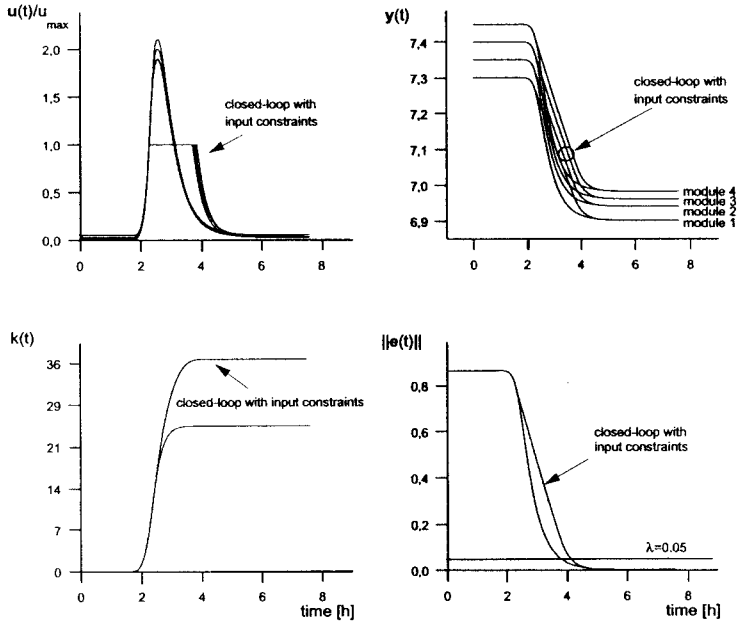

Fig. 6. Adaptive regulator (1),(13) (with and without input constraints) with parameters (20) applied to (6) with parameters (19). Reference step which causes saturation of control signals.

Qualitatively we obtain the same result. An interesting difference occurs if the input signals are subjected to the following saturations:

$$u_i(t) \in [0, u_{max}] \quad \text{for} \quad i = 1,\ldots,4$$
$$\text{and } u_{max} = 0.084[\text{h}^{-1}] \tag{21}$$

In the case of input saturation (21), the time until the norm of the error reaches the tolerance band is longer. Consequently, also the gain settles down at a higher value since the error $\|e(t)\|$ remains outside the λ-strip for a longer time. Note that in all three simulations the control objectives are met without any oscillations, which was also typical for all other simulations made. We would also like to stress that the term 'high-gain' might be misleading. Although theoretically our approach is high-gain, in practice the terminal gain is only slightly larger than necessary for achieving the given control objectives.

5. Experiments

In this section, experimental results are given which show that the controller (1),(13) and its modified version (18) worked successfully in the biogas tower reactor. Although the model was derived and analysed for four modules, the controller was tested only at three modules. The reason is simply that the size of the actual settler was shown to be too small for sufficient biomass retention and module four was also taken as a settler. This restriction will be avoided in the future by reshaping the settler to appropriate dimensions. It is easy to see that our previous analysis holds true also in the general case of i modules, where $i \geq 2$. The control law was implemented on the process control system by a discrete integration algorithm where a sampling time of 6 min was used. Note that although not theoretically proved for the present *non-linear* application, we proved in Ilchmann and Townley [11] that λ-tracking of linear systems is feasible if the output sampling period is chosen sufficiently small.

The controller (1),(13) was tested with the parameters (20) but $\bar{\mathbf{u}} = (0, 0, 0)^T$ and reference signal

$$\mathbf{y}_{ref} := (6.9, 6.975, 7.075)^T$$

where waste water was not fed into the fourth module and consequently we only have three reference signals. Note that the control loop contains the saturation of control signals because the control range of the valves used for feeding the waste water into the reactor is bounded. We put

$$u_i(t) \in [0, u_{max}] \quad \text{for} \quad u_{max} = 0.084[\text{h}^{-1}]$$

The experimental results are shown in Fig. 7. The command tracking is reached without any oscillation. The desired pH tolerance band is entered after 24 h. The gain is about $57[\text{h}^{-1}]$ at $t = 75[\text{h}]$ (with respect to a dimensionless input signal in Fig. 5), which is not too large. The stepwise disturb-

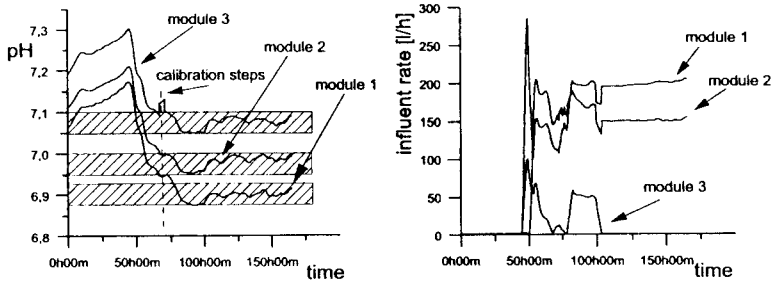

Fig. 7. Experimental results: adaptive regulation of the pH values with controller (1),(13).

ances are caused by the calibration of the sensors at $t = 70$[h].

In the second experiment the modified controller (1),(18) with parameters

$$\gamma_1 = 1.4[h^{-1}]\,,\ \gamma_2 = 7140[-]\,,\quad T = 12[h]\,,$$
$$\lambda = 0.05\,,\ k(0) = 0$$

is used. The results are depicted in Fig. 8. Note that the reactor was shut down over the weekend, which means that the influent rates were set almost to zero, similar to the first experiment. As shown in Fig. 8 the controller is started at pH values which are at the upper level of the possible range. After less than 6 h the pH values are inside the tolerance band. The pH value of module 3 leaves the tolerance band because of the saturation of the influent rate 3. The setpoint of pH_3, which is chosen too large for this experiment, cannot be realised within the possible range of input flow rates. For such a situation we implemented the strategy to switch off the gain adaptation as long as the input hits the input constraints; here $u_3(t)$ does it at $t = 52$[h], see Fig. 8. The other pH values reach the tolerance band and then remain inside it. The gain settles also below $60[h^{-1}]$ at $t = 52$[h] (with respect to a dimensionless input signal in Fig. 5), which is not too large. With the modified controller start-up is also performed without any oscillations.

From a practical point of view, it is sensible to reset the gain whenever the control engineer is sure that an increase of the gain was due to the reaction to unusual disturbances which have vanished.

6. Conclusions

We have designed a simple adaptive regulator for a four-dimensional non-linear model of a biogas tower reactor. A reactor of this type has been developed at the Department of Bioprocess and Biochemical Engineering at the Technical University of Hamburg Harburg. A pilot reactor of almost full scale (20 m high, 1 m in diameter) located at the Deutsche Hefewerke (DHW) in Hamburg was used for testing the controller. The regulator is not based on any identification mechanism but on a simple high-gain principle. It is proved that it achieves setpoint tracking within tolerance bands of pre-specified bounds. If tracking is required within a given time, it is proved that this can be achieved with a minor modification of the gain adaptation. The regulator is very simple in its design and robust with respect to parameter uncertainties. We have shown in simulations that the controller works well if applied to the model, the command tracking shows no overshoots and the gain does not become much larger than is necessary for tracking. Most importantly, the adaptive regulator has been successfully applied to the real biogas tower reactor for two months. In practice, the adaptation mechanism

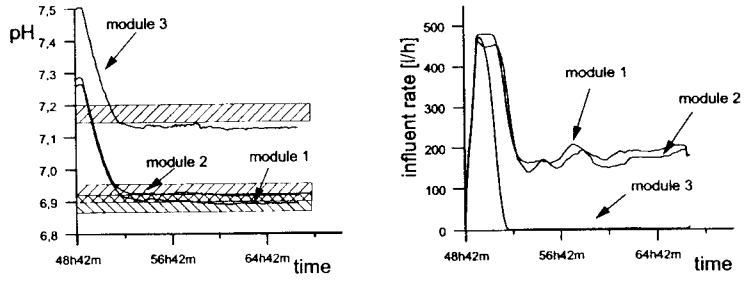

Fig. 8. Experimental results: start up with the modified controller (1),(18) after the reactor was shut down for several days.

was implemented with input saturation and the experiments were convincing. However, we could not prove mathematically that tracking can be achieved subject to input saturations. Another practically relevant, but theoretically not yet investigated, modification is gain resetment. Each experiment has been performed over a period of one week. The gain adaption was applied over this period and at each start-up (at the beginning of the week) the gain was reset. For longer operation periods, a reset should be done automatically if the gain reaches an upper bound due to strong disturbances.

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Appendix

Proof of Theorem 3.1. With the notation

$$\mathbf{e}(t) := \mathbf{y}(t) - \mathbf{y}_{ref},$$

$$d_\lambda(\mathbf{e}) := \begin{cases} (\|\mathbf{e}\| - \lambda), & \|\mathbf{e}\| \geq \lambda \\ 0, & \|\mathbf{e}\| < \lambda \end{cases},$$

$$\overline{\mathbf{U}} := \begin{bmatrix} \overline{u}_1 & & \\ & \ddots & \\ & & \overline{u}_4 \end{bmatrix}$$

the closed-loop system (6),(13),(1) may be written as

$$\begin{aligned}\dot{\mathbf{e}}(t) &= -(k(t)\mathbf{I}_4 - \mathbf{A} + \overline{\mathbf{U}})\,\mathbf{e}(t)\\ &\quad + (\mathbf{A} - \overline{\mathbf{U}})\mathbf{y}_{ref}\\ &\quad + \mathbf{r}(\mathbf{e}(t) + \mathbf{y}_{ref})\\ &\quad + y_{feed}\overline{\mathbf{U}}(1,\ldots,1)^T, \quad \mathbf{e}(0) = \mathbf{y}(0) - \mathbf{y}_{ref}\\ \dot{k}(t) &= \gamma d_\lambda(\mathbf{e}(t))\,\|\mathbf{e}(t)\|, \qquad k(0) = k_0\end{aligned} \tag{22}$$

By the classical theory of ordinary differential equations, (22) possesses a unique solution

$$(\mathbf{e}(\cdot),k(\cdot)) : [0,\tilde{t}) \to \mathbb{R}^5$$

maximally extended over $[0,\tilde{t}]$ for some $\tilde{t} \in (0,\infty]$.

We now proceed in several steps.

a. To prove 1. on $[0,\tilde{t})$, first rewrite (6),(13) in **y**-coordinates as follows (for brevity we omit the argument t)

$$\begin{aligned}\dot{y}_1 &= a_1(y_2 - y_1)\\ &\quad + r_1(y_1) - (y_1 - y_{feed})\overline{u}_1 - k(y_1 - w_1)\\ \dot{y}_2 &= a_1(y_1 - y_2) + a_2(y_3 - y_2)\\ &\quad + r_2(y_2) - (y_2 - y_{feed})\overline{u}_2 - k(y_2 - w_2)\\ \dot{y}_3 &= a_2(y_2 - y_3) + a_3(y_4 - y_3)\\ &\quad + r_3(y_3) - (y_3 - y_{feed})\overline{u}_3 - k(y_3 - w_3)\\ \dot{y}_4 &= a_3(y_3 - y_4)\\ &\quad + r_4(y_4) - (y_4 - y_{feed})\overline{u}_4 - k(y_4 - w_4)\end{aligned} \tag{23}$$

By assumption (15), the initial values $y_1(0),\ldots,y_4(0)$ are greater than y_{feed}. Seeking a contradiction to 1., suppose there exist some $t' \in (0,\tilde{t})$ such that

$$y_{i_0}(t') = y_{feed} \quad \text{and} \quad y_i(t') \geq y_{feed}$$
$$\text{for} \quad \text{all } i \in \{1,2,3,4\}\backslash\{i_0\}$$

If $i_0 = 1$, then by (23) we have

$$\begin{aligned}\dot{y}_1(t') &= a_1(y_2(t') - y_{feed}) + r_1(y_1(t'))\\ &\quad - k(t')(y_{feed} - w_1)\end{aligned}$$

whence, by (5) and (15), we may conclude $\dot{y}_1(t') > 0$. This contradicts the assumption $y_1(t') = y_{feed}$. The proof for $i_0 = 2, 3, 4$ is analogous and omitted for brevity. This proves the statement 1. for all $t \in [0,\tilde{t})$.

b. We shall prove that $k(\cdot)$ is bounded on $[0,\tilde{t})$. Differentiation of the Lyapunov-like candidate

$$V(\mathbf{e}) := \frac{1}{2}\,d_\lambda(\mathbf{e})^2$$

along (22) yields, for all $t \in [0,\tilde{t})$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{e}(t)) &= d_\lambda(\mathbf{e}(t))\,\frac{2}{\|\mathbf{e}(t)\|}\,\langle \mathbf{e}(t), \dot{\mathbf{e}}(t)\rangle\\ &= -d_\lambda(\mathbf{e}(t))\,\frac{2}{\|\mathbf{e}(t)\|}\,\langle \mathbf{e}(t), k(t)\mathbf{e}(t)\rangle\\ &\quad + d_\lambda(\mathbf{e}(t))\,\frac{2}{\|\mathbf{e}(t)\|}\\ &\quad \langle \mathbf{e}(t), (\mathbf{A} - \overline{U})[\mathbf{e}(t) + \mathbf{y}_{ref}]\\ &\quad + \mathbf{r}(\mathbf{e}(t) + \mathbf{y}_{ref}) + y_{feed}\overline{U}\cdot(1,\ldots,1)^T\rangle\end{aligned}$$

By assumption on $\mathbf{r}(\cdot)$ there exists some $M > 0$, so that for all $t \in [0,\tilde{t})$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{e}(t)) &\leq -\frac{2}{\gamma}\,k(t)\gamma d_\lambda(\mathbf{e}(t))\,\|\mathbf{e}(t)\|\\ &\quad + M d_\lambda(\mathbf{e}(t))\,(1 + \|\mathbf{e}(t)\|)\\ &\leq -\frac{2}{\gamma}\,k(t)\dot{k}(t) + M d_\lambda(\mathbf{e}(t))\\ &\quad \left(\frac{1}{\lambda} + 1\right)\|\mathbf{e}(t)\|\\ &= \left[-\frac{2}{\gamma}\,k(t) + \frac{M(1+\lambda)}{\lambda\gamma}\right]\dot{k}(t)\end{aligned} \tag{24}$$

and hence, by integration and the substitution of $\mu = k(\tau)$,

$$\begin{aligned}0 \leq V(\mathbf{e}(t)) &\leq V(\mathbf{e}(0))\\ &\quad + \int_0^t \left[-\frac{2}{\gamma}\,k(\tau)\right.\\ &\quad \left. + \frac{M(1+\lambda)}{\lambda\gamma}\right]\dot{k}(\tau)d\tau\\ &= V(\mathbf{e}(0)) + \int_{k(0)}^{k(t)}\left[-\frac{2}{\gamma}\,\mu + \frac{M(1+\lambda)}{\lambda\gamma}\right]d\mu\\ &= V(\mathbf{e}(0)) - \frac{k(t)^2 - k(0)^2}{\gamma}\\ &\quad + \frac{M(1+\lambda)}{\lambda\gamma}\,(k(t) - k(0))\end{aligned} \tag{25}$$

If $\lim_{t\to\tilde{t}} k(t) = \infty$, then the right-hand side of (25) tends to $-\infty$ as t tends to $\tilde{t}$, hence contradicting boundedness from below. This contradiction proves boundedness of $k(\cdot)$ on $[0,\tilde{t})$.

c. Boundedness of $k(\cdot)$ on $[0,\tilde{t})$ yields, by (25), boundedness of $e(\cdot)$ on $[0,\tilde{t})$. Hence, there does not exist a finite escape time, i.e. $\tilde{t} = \infty$, and therefore 1. and 2. are established.

d. By (24) and boundedness of $e(\cdot)$ and $k(\cdot)$ on $[0, \infty]$, there exists some $M_1 > 0$ such that, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{e}(t)) \leq M_1 \dot{k}(t)$$

The derivative of the sign-indefinite Lyapunov function

$$W(\mathbf{e},k) := V(\mathbf{e}) - (M_1 + 1)k$$

along (22) satisfies, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} W(\mathbf{e}(t),k(t)) = \frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{e}(t)) - (M_1 + 1)\dot{k}(t) \leq -\dot{k}(t) \qquad (26)$$

Since $\mathbf{e}(\cdot)$ and $k(\cdot)$ are bounded, we may apply LaSalle's invariance principle [15] and conclude from (26) that (3) holds true. This completes the proof. □

Proof of Theorem 3.3. The closed-loop system (6),(13),(18) may be written as, for all $t \in [0, T)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} \mathbf{e}(t) &= -(k(t)\mathbf{I}_4 - \mathbf{A} + \overline{U})\,\mathbf{e}(t) + (\mathbf{A} - \overline{U})\,\mathbf{y}_{ref} \\ &\quad + \mathbf{r}(\mathbf{e}(t) + \mathbf{y}_{ref}) + y_{feed}\overline{\mathbf{U}}(1,\ldots,1)^T \\ \dot{k}(t) &= \gamma_1 d_\lambda(\mathbf{e}(t))\,\|\mathbf{e}(t)\| \\ &\quad + \gamma_2 \begin{cases} 2\dfrac{\langle \mathbf{e}(t),\dot{\mathbf{e}}(t)\rangle}{T-t} + \dfrac{\|\mathbf{e}(t)\|^2}{(T-t)^2}, & t \in [0, t*) \\ 0, & t \geq t* \end{cases} \qquad (27)\end{aligned}$$

with initial conditions $\mathbf{e}(0) = \mathbf{y}(0) - \mathbf{y}_{ref}$, $k(0) = k_0$.

It is straightforward to check that the right-hand side of (27) is continuous and locally Lipschitz in $(\mathbf{e},k)$ for fixed $t \in [0, T)$, and locally integrable on $t \in [0, T)$ for fixed $(\mathbf{e},k)$. Hence, see for example, Sontag [23], there exists a unique solution $(\mathbf{e}(\cdot),k(\cdot))$: $[0, \tilde{t}) \to \mathbb{R}^5$ maximally extended over $[0, \tilde{t})$, $\tilde{t} < T$.

Note that $t \mapsto k(t)$ is no longer monotone as in Theorem 3.1.

We first show that $t* < \tilde{t}$. Differentiation of the Lyapunov-like candidate

$$W(\mathbf{e}) := \frac{1}{2}\|\mathbf{e}\|^2$$

along the solution component of (27) and using (5) yields, for some $M > 0$ and all $t \in [0, \tilde{t})$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W(\mathbf{e}(t)) = \langle \mathbf{e}(t),\dot{\mathbf{e}}(t)\rangle &\leq -[k(t) - M]\,\|\mathbf{e}(t)\|^2 \\ &\quad + M + M\,\|\mathbf{e}(t)\| \\ &\leq -[k(t) - M - 1]\,\|\mathbf{e}(t)\|^2 \\ &\quad + M + M^2 \qquad (28)\end{aligned}$$

With the notation $\tilde{k}(t) = k(t) - M - 1$ and $\tilde{M} = M + M^2$, an application of Lemma 3.2.4 in Ioannou and Sun [13] to (28) yields

$$W(\mathbf{e}(t)) \leq e^{-\int_0^t \tilde{k}(\tau)d\tau}\, W(\mathbf{e}(0)) + \int_0^t e^{-\int_s^t \tilde{k}(\tau)d\tau}\, \tilde{M}\, ds \qquad (29)$$

If $k(\cdot)$ is unbounded on $[0, \tilde{t})$, then (29) yields $\mathbf{e}(t) \to 0$ as $t \to \tilde{t}$, and therefore either $t* < \tilde{t}$ or $k(\cdot)$ is bounded on $[0, \tilde{t})$. If the latter holds true, it follows from (27) that $\mathbf{e}(\cdot)$ is bounded, which contradicts that finite escape time occurs at $\tilde{t}$. This proves $t* < \tilde{t}$.

The remainder of the proof follows from Theorem 3.1 since for $t \geq t*$ the gain adaptation (1) and (18) coincide. This completes the proof. □

Nomenclature

$\langle x,y\rangle$	Euclidean scalar product for $x,y \in \mathbb{R}^n$
a	exchange flow rate
β	activity coefficient
γ	design parameter of the controller
λ	size of the tolerance band
c	concentration of compound in liquid phase
f	volumetric liquid flow
H^+	hydronium concentration
i	module number
K	reaction weighting (without index)
m	mass flow
pH	pH value
r	fictitious reaction rate
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of positive real numbers
ρ	reaction rate parameter
t	time
u	input signal
V	volume
x	state variable
$\mathbf{x}$	state vector
y	output signal
$\mathbf{y}$	output vector

Indices

ex	exchange
feed	waste water inflow
fl	fluid
in	inflow
tr	transfer