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Robustness of Stability of Time-Varying Linear Systems

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This paper introduces the concept of stability radius for time-varying linear systems. Invariance properties of the stability radius are analysed for the group of Bohl transformations. We also explore the relationship between the stability radius, the norm of a certain perturbation operator, and the solvability of a nonstandard differential Riccati equation. As an application we construct robust Lyapunov functions and show how they can be used to analyze robustness with respect to nonlinear perturbations. © 1989 Academic Press, Inc.

NOMENCLATURE

$PC(\mathbb{R}_+, \mathbb{C}^{m\times p})$	set of all piecewise continuous complex $m\times p$ matrix functions on $\mathbb{R}_+ = [0, \infty)$
$PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p})$	the set of all bounded matrix functions belonging to $PC(\mathbb{R}_+, \mathbb{C}^{m\times p})$
$PC^1([t_0, t_1), GL_n(\mathbb{C}))$	the set of piecewise continuously differentiable $n\times n$ functions on $[t_0, t_1)$ which have nonsingular values
$L_q(t_0, \infty; \mathbb{C}^r)$	the set of functions $h\colon [t_0, \infty) \to \mathbb{C}^r$ such that $\int_{t_0}^{\infty} \|h(s)\|^q \, ds < \infty$, $q, r \in \mathbb{N}$
$L_\infty(t_0, \infty; \mathbb{C}^r)$	the set of functions $h\colon [t_0, \infty) \to \mathbb{C}^r$ such that $\sup_{t\geq t_0} \|h(t)\| < \infty$, $r \in \mathbb{N}$

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1. Introduction

After playing a minor role in the early development of the state space approach the *problem of model uncertainty* has recently regained a prominent position in systems theory. In this paper we propose a framework for the robustness analysis of *time-varying* linear systems. Although this subject is of interest in itself it is also important in other fields, e.g., in the area of adaptive control, where the stability analysis of time-varying systems plays a central role.

Most of the work on robustness of time-invariant linear systems—including the successful H^∞ approach [6]—is based on transform techniques. It is not clear how to extend these techniques to the time-varying case. Recently a *state space approach* to robustness has been proposed in [9, 10] which is based on the concept of "stability radius." The purpose of the present paper is to extend this approach to a time-varying setting.

We consider a nominal system of the form

$$\dot{x}(t) = A(t)\, x(t), \qquad t \geqslant 0, \tag{1.1}$$

where $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times n})$.[1] The corresponding transition matrix is denoted by $\Phi(t, s)$, $t, s \geqslant 0$. We suppose that the nominal system (1.1) is exponentially stable; i.e., there exist constants M, $\omega > 0$ such that

$$\| \Phi(t, s)\| \leqslant M e^{-\omega(t-s)}, \qquad t \geqslant s \geqslant 0. \tag{1.2}$$

The matrix $A(t)$ is subjected to additive structured perturbations, so that the perturbed system is

$$\dot{x}(t) = [A(t) + B(t)\, D(t)\, C(t)]\, x(t), \qquad t \geqslant 0, \tag{1.3}$$

where $D(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p})$ is an *unknown* bounded time-varying disturbance matrix and $B(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times m})$, $C(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{p\times n})$ are *given* "scaling matrices" defining the "structure" of the perturbation, $m, p \geqslant 1$, see [10].

Formally (1.3) may be interpreted as a closed loop system obtained by applying the time-varying feedback

$$u(t) = D(t)\, y(t) \tag{1.4}$$

to the time-varying linear system

$$\begin{aligned} \dot{x}(t) &= A(t)\, x(t) + B(t)\, u(t) \\ y(t) &= C(t)\, x(t). \end{aligned} \tag{1.5}$$

[1] Most of our results can be extended to systems (1.1) with locally integrable instead of piecewise-continuous generator $A(\cdot)$.

Note, however, that $B(t)$, $C(t)$ do not represent input, output matrices but describe the structure and scale of uncertainty of the system parameters. Hence controllability and observability assumptions cannot be justified in this setting.

In the literature a variety of sufficient conditions have been derived which ensure exponential stability of the perturbed system $\dot{x}(t) = [A(t) + \Delta(t)]\, x(t)$; see [2–4, 14]. These conditions are given in terms of bounds for $\|\Delta(\cdot)\|_{L_\infty}$ and are conservative.

Our problem is to determine a *sharp* upper bound. For structured perturbations of the form (1.3) this bound is

$$r_{\mathbb{C}}(A; B, C) = \inf\{\|D(\cdot)\|_{L_\infty}; D(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p}) \text{ and (1.3) is not exponentially stable}. \tag{1.6}$$

We call $r_{\mathbb{C}}(A; B, C)$ the (comlex)[2] *stability radius* of the nomial system (1.1) with respect to perturbations with structure (B, C). In the unstructured case ($m = p = n$, $B(\cdot) = C(\cdot) = I_n$) the stability radius is simply the distance of the system (1.1) from the set of not exponentially stable systems with respect to the L_∞ norm. Guided by the results for time-invariant linear systems [10] we will primarily investigate how the stability radius (1.6) is related to the *perturbation operator*

$$\begin{aligned} L_{t_0}\colon L_2(t_0, \infty; \mathbb{C}^m) &\to L_2(t_0, \infty; \mathbb{C}^p) \\ u(\cdot) &\mapsto \left(t \mapsto \int_{t_0}^{t} C(t)\, \Phi(t, s)\, B(s)\, u(s)\, ds\right) \end{aligned} \tag{1.7}$$

and the existence of bounded Hermitian solutions of a *parametrized differential Riccati equation*

$$\dot{P}(t) + A^*(t)\, P(t) + P(t)\, A(t) - \rho C^*(t)\, C(t) - P(t)\, B(t)\, B^*(t)\, P(t) = 0, \quad t \geq t_0 \geq 0 \tag{1.8}$$

(with parameter $\rho \in \mathbb{R}$). Unfortunately these relationships are not as simple as in the time-invariant case and we have only been able to extend some of the results to time-varying systems. This reflects the fact that perturbation theory for time-varying systems is far less developed and more complicated than that of time-invariant systems.

We will proceed as follows. In Section 2 we list some preliminary results on Bohl exponents and exponential stability of time-varying systems. We also introduce the group of Bohl transformations which contains the group

[2] The real stability radius is defined analogously; see [8]. However, here we concentrate on the complex stability radius.

of Lyapunov transformations as a subgroup. In Section 3 we discuss invariance properties of the stability radius and in Section 4 the perturbation operator is studied and its relation to the stability radius is partially clarified. In Section 5 we establish a connection between the norm of the perturbation operator and the solvability of the parametrized differential Riccati equation. Finally in Section 7 we show how to determine a Lyapunov function of "maximal" robustness.

2. Bohl Exponent and Bohl Transformations

Consider a differential equation of the form

$$\dot{x}(t) = A(t)\, x(t), \qquad t \geqslant 0, \tag{2.1}$$

where $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n})$ generates a transition operator $\Phi(t, s)$, $t, s \geqslant 0$. Throughout the paper $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{C}^n$, $\|\cdot\|$ the associated norm, and $\|D\|$ the induced norm for any bounded linear operator $D \in \mathscr{L}(\mathbb{C}^p, \mathbb{C}^m)$. For a characterization of the stability behaviour of (2.1) the following definition due to *Bohl* [2] is useful.

Definition 2.1 (Bohl exponent). The (upper) *Bohl exponent* $k_B(A)$ of the system (2.1) is given by

$$k_B(A) = \inf\{\omega \in \mathbb{R} \mid \exists M_\omega > 0 : t \geqslant t_0 \geqslant 0 \Rightarrow \|\Phi(t, t_0)\| \leqslant M_\omega e^{\omega(t - t_0)}\}.$$

It is possible that $k_B(A) = \pm\infty$. If (2.1) is time-invariant, i.e., $A(\cdot) \equiv A \in \mathbb{C}^{n \times n}$, then

$$k_B(A) = \max_{i \in \underline{n}} \operatorname{Re} \lambda_i(A),$$

where $\lambda_i(A)$, $i \in \underline{n}$, are the eigenvalues of A.

The following properties of the Bohl exponent can be found in [4].

Proposition 2.2. (i) *The Bohl exponent of the system* (2.1) *is finite if and only if*

$$\sup_{0 \leqslant |t-s| \leqslant 1} \|\Phi(t, s)\| < \infty. \tag{2.2}$$

In particular $k_B(A)$ *is finite if* $A(\cdot)$ *satisfies*

$$\sup_{t \in \mathbb{R}_+} \int_t^{t+1} \|A(s)\|\, ds < \infty.$$

(*In this case we say* $A(\cdot)$ *is* integrally bounded.)

(ii) *If* $k_B(A) < \infty$ *it can be determined via*

$$k_B(A) = \limsup_{s,\, t-s \to \infty} \frac{\log \|\Phi(t, s)\|}{t-s}. \tag{2.3}$$

For later use we need the following more restrictive definition.

DEFINITION 2.3. (Strict Bohl exponent). The Bohl exponent of the system (2.1) is said to be *strict* if it is finite and

$$k_B(A) = \lim_{s,\, t-s \to \infty} \frac{\log \|\Phi(t, s)\|}{t-s}.$$

The proof of the following lemma is straightforward.

LEMMA 2.4. *Suppose* $a(\cdot) \in PC(\mathbb{R}_+, \mathbb{C})$ *has a strict finite Bohl exponent and* $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n})$; *then*

(i) $k_B(-a) = -k_B(a)$

(ii) $k_B(aI_n + A) = k_B(a) + k_B(A)$ (*shift property*).

Better known in the literature is the (*upper*) *Lyapunov exponent*

$$k_L(A) = \inf\{\omega \in \mathbb{R} \mid \exists M_\omega > 0 : t \geqslant 0 \Rightarrow \|\Phi(t, 0)\| \leqslant M_\omega e^{\omega t}\}.$$

For time-invariant systems the Bohl and Lyapunov exponents coincide whereas in general

$$k_L(A) \leqslant k_B(A).$$

EXAMPLE 2.5. *Perron* [14] has shown that for the scalar system

$$\dot{x}(t) = [\sin \log t + \cos \log t]\, x(t), \qquad t \geqslant 0$$

the exponents are different; see also [4].

In this paper we will study the following stability concept for time-varying linear systems.

DEFINITION 2.6. The system (2.1) is said to be *exponentially stable* if there exist M, $\omega > 0$ such that

$$\|\Phi(t, t_0)\| \leqslant M e^{-\omega(t - t_0)} \qquad \text{for all} \quad t \geqslant t_0 \geqslant 0$$

("for all $t \geqslant t_0 \geqslant 0$" means "for all $t_0 \geqslant 0$ and all $t \geqslant t_0$").

Remark 2.7. (i) It can be shown (see [16]) that (2.1) is exponentially stable if and only if it is *uniformly asymptotically stable*, i.e., there exists $k > 0$ independent of t_0 such that

$$\|\Phi(t, t_0)\| \leqslant k \qquad \text{for all} \quad t \geqslant t_0 \geqslant 0$$

and

$$\lim_{t \to \infty} \|\Phi(t, t_0)\| = 0 \qquad \text{uniformly in } t_0 \in \mathbb{R}_+. \tag{2.4}$$

(ii) The system is *asymptotically stable* (i.e., the above relations hold but k may depend on t_0 and the convergence in (2.4) need not be uniform) if and only if $k_L(A) < 0$.

The following characterizations of exponential stability are proved in [4].

THEOREM 2.8. *Suppose $k_B(A) < \infty$ and $p \in (0, \infty)$; then the following statements are equivalent*:

(i) (2.1) *is exponentially stable*

(ii) $k_B(A) < 0$

(iii) *there exists a constant c_p, such that*

$$\int_{t_0}^{\infty} \|\Phi(t, t_0)\|^p \, dt \leqslant c_p \qquad \textit{for all} \quad t_0 \geqslant 0.$$

If in addition $A(\cdot)$ is integrally bounded, then (i), (ii), (iii) *are equivalent to*

(iv) *for every bounded $f(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^n)$, the solution of the initial value problem*

$$\dot{x}(t) = A(t)\, x(t) + f(t), \qquad t \geqslant 0, x(0) = 0$$

is bounded.

We now analyse the effect of time-varying linear coordinate transformations $z(t) = T(t)^{-1} x(t)$ on the system (2.1), where $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$. The associated similarity transformation converts the system (2.1) into

$$\dot{z}(t) = \hat{A}(t)\, z(t), \qquad t \geqslant 0, \tag{2.5}$$

where

$$\hat{A}(t) = T(t)^{-1} A(t)\, T(t) - T(t)^{-1} \dot{T}(t).$$

The transition matrix of the system (2.5) is

$$\hat{\Phi}(t, s) = T(t)^{-1}\, \Phi(t, s)\, T(s). \tag{2.6}$$

Since these transformations will not, in general, preserve stability properties, additional assumptions have to be imposed. If one requires $T(\cdot)$, $T(\cdot)^{-1}$, $\dot{T}(\cdot)$ are bounded one obtains the so-called *Lyapunov transformations* introduced by *Lyapunov* in his famous memoir [13]. This group of transformations preserves the properties of stability, instability, and asymptotic stability. The property of exponential stability is invariant with respect to a larger group of transformations.

DEFINITION 2.9 (Bohl transformation). $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$ is said to be a *Bohl transformation* if

$$\inf\{\varepsilon \in \mathbb{R} \mid \exists M_\varepsilon > 0\ \forall t, s \geqslant 0 \colon \|T(t)^{-1}\| \cdot \|T(s)\| \leqslant M_\varepsilon e^{\varepsilon |t-s|}\} = 0.$$

In the following example scalar Bohl transformations are characterized.

EXAMPLE 2.10. Suppose $\theta(\cdot) \in PC^1(\mathbb{R}_+, \mathbb{C}^*)$, and let $a(\cdot) = \dot{\theta}(\cdot)^{-1}$ so that

$$\dot{\theta}(t) = a(t)\,\theta(t) \qquad \text{and} \qquad (\theta(t)^{-1})^{\cdot} = -a(t)\,\theta(t)^{-1}.$$

The fundamental solutions of these differential equations are

$$\varphi(t, t_0) = \theta(t)\,\theta(t_0)^{-1} \qquad \text{and} \qquad \tilde{\varphi}(t, t_0) = \theta(t)^{-1}\,\theta(t_0).$$

By Definition 2.9 $\theta(\cdot)$ is a Bohl transformation if and only if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$M_\varepsilon^{-1} e^{-\varepsilon(t-s)} \leqslant \tilde{\varphi}(t, s)^{-1} = \varphi(t, s) \leqslant M_\varepsilon e^{\varepsilon(t-s)} \qquad \text{for all} \quad t \geqslant s \geqslant 0$$

and this condition holds if and only if $a(\cdot)$ has strict Bohl exponent 0.

The following proposition implies, in particular, that Bohl transformations preserve exponential stability (but not necessarily stability and asymptotic stability).

PROPOSITION 2.11. (i) *The Bohl transformations form a group with respect to (pointwise) multiplication.*

(ii) *The Bohl exponent is invariant with respect to Bohl transformations.*

Proof. (i) is an immediate consequence of Definition 2.9. To prove (ii), let $\dot{x}(t) = \hat{A}(t)\,x(t)$ be similar to (2.1) via the Bohl transformation $T(\cdot)$.

Since the transition matrix of $\dot{x}(t)=\hat{A}(t)\,x(t)$ is given by $\hat{\Phi}(t,s)=T(t)^{-1}\,\Phi(t,s)\,T(s)$, one obtains

$$k_B(A)\leqslant k_B(\hat{A}).$$

By (i), it follows that $k_B(A)=k_B(\hat{A})$. ∎

EXAMPLE 2.12. Consider a periodic system

$$\dot{x}(t)=A(t)\,x(t), \qquad t\geqslant 0, \tag{2.7}$$

where $A(\cdot)\in PC(\mathbb{R}_+, \mathbb{C}^{n\times n})$ is of period $\mu>0$. By Erugin's Theorem (see [7]) (2.7) can be transformed via Lyapunov transformations into a time-invariant system $\dot{x}(t)=\Lambda x(t)$, where Λ is a diagonal real $n\times n$ matrix whose diagonal entries are just the characteristic exponents $\lambda_1, ..., \lambda_n$ of (2.7). Hence, by Proposition 2.11,

$$k_B(A)=k_B(\Lambda)=\max\{\lambda_1, ..., \lambda_n\}.$$

In the scalar case $(n=1)$ we have

$$k_B(a)=\frac{1}{\mu}\int_0^{\mu} a(t)\,dt.$$

It is noteworthy that in the scalar case not only periodic but arbitrary time-varying systems can be reduced to a time-invariant one via Bohl transformations.

PROPOSITION 2.13. *Every scalar system*

$$\dot{x}(t)=a(t)\,x(t), \qquad t\geqslant 0$$

which has strict, finite Bohl exponent can be transformed via the Bohl transformation

$$\theta(t)=\exp\left(\int_0^t (a(\tau)-k_B(a))\,d\tau\right)$$

into the time-invariant linear system

$$\dot{z}(t)=k_B(a)\,z(t), \qquad t\geqslant 0.$$

Proof. Use Example 2.10 and Lemma 2.4. ∎

Remark 2.14. Example 2.5 together with the previous proposition implies that, in general, a Bohl transformation does not preserve the Lyapunov exponent.

For later use, we add some known perturbation results concerning the Bohl exponent of the system

$$\dot{x}(t) = [A(t) + \Delta(t)]\, x(t), \qquad t \geq 0, \tag{2.8}$$

where $\Delta(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n})$.

PROPOSITION 2.15. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\limsup_{s,\, t-s \to \infty} \frac{1}{t-s} \int_s^t \|\Delta(\tau)\|\, d\tau < \delta$$

implies

$$k_B(A + \Delta) \leqslant k_B(A) + \varepsilon.$$

The proof is straightforward and can be found in [4].

Systems (2.1), (2.8) are called *asymptotically equivalent* (resp. *integrally equivalent*) if

$$\lim_{t \to \infty} \|\Delta(t)\| = 0 \qquad \left(\text{resp.} \int_0^\infty \|\Delta(t)\|\, dt < \infty\right).$$

The above proposition shows that asymptotically or integrally equivalent systems have the same Bohl exponent.

3. The Structured Stability Radius

In this section it is assumed that the nominal system (2.1) is subjected to perturbations of the form $\Delta(t) = B(t)\, D(t)\, C(t)$, so that the perturbed system is

$$\dot{x}(t) = [A(t) + B(t)\, D(t)\, C(t)]\, x(t), \qquad t \geq 0, \tag{3.1}$$

where $D(\cdot)$ is an unknown, bounded, time-varying disturbance matrix ($D(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{m \times p})$) and $B(\cdot)$ and $C(\cdot)$ are known time-varying *scaling matrices* defining the structure of the perturbation. Throughout this section we assume the triple $\Sigma = (A, B, C)$ consists of matrix functions

$$A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n}), \qquad B(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times m}), \qquad C(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{p \times n}). \tag{3.2}$$

By Proposition 2.15 the set of exponentially stable systems is open in $PC(\mathbb{R}_+, \mathbb{C}^{n \times n})$ with respect to the L_∞-norm.[3] Its complement, which is

[3] This expression is used although $\|\cdot\|_{L_\infty}$ is only a pseudo-norm on $PC(\mathbb{R}_+, \mathbb{C}^{n \times n})$.

closed, will be denoted by $\mathscr{U}_n(\mathbb{R}_+, \mathbb{C})$. We will call the elements of $\mathscr{U}_n(\mathbb{R}_+, \mathbb{C})$ *unstable* (not exponentially stable). Note, however, that with respect to this shorthand terminology an unstable system may in fact be asymptotically stable. The following definition extends the concept of stability radius introduced in [9, 10] to time-varying systems.

DEFINITION 3.1 (Stability radius). Given $\Sigma = (A, B, C)$, the *(complex) stability radius* $r_{\mathbb{C}}(A; B, C)$ is defined by

$$r_{\mathbb{C}}(A; B, C) = \inf\{\|D\|_{L_\infty};\, D \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p}),\, A + BDC \in \mathscr{U}_n(\mathbb{R}_+, \mathbb{C})\}. \tag{3.3}$$

The *unstructured stability radius* of (3.1) is defined by

$$r_{\mathbb{C}}(A) = r_{\mathbb{C}}(A; I_n, I_n).$$

Note that $r_{\mathbb{C}}(A; B, C) = \inf \emptyset = \infty$ if there does not exist a perturbation matrix $D \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p})$ such that $A + BDC \in \mathscr{U}_n(\mathbb{R}_+, \mathbb{C})$.

Remark 3.2. (i) The unstructed stability radius $r_{\mathbb{C}}(A)$ measures the distance of $A(\cdot)$ from the set $\mathscr{U}_n(\mathbb{R}_+, \mathbb{C})$ of unstable matrices with respect to the L_∞-norm.

(ii) In the time-invariant case it is known (see [10]) that

$$r_{\mathbb{C}}(A; B, C) = \frac{1}{\max_{\omega\in\mathbb{R}} \|G(i\omega)\|},$$

where $G(iw) = C(i\omega I_n - A)^{-1} B$ (in particular $r_{\mathbb{C}}(A, B, C) = \infty$ if $G \equiv 0$).

(iii) If $\Sigma = (A, B, C)$ consists of *real* matrix functions the *real stability radius* $r_{\mathbb{R}}(A; B, C)$ is defined in an analogous fashion. $r_{\mathbb{R}}(A; B, C)$ is more difficult to analyze and even in the time-invariant case computable formulae are only available for the special cases $m = 1$ or $p = 1$; see [11]. Although the real stability radius is obviously of great importance for applications it should be observed that the complex stability radius offers some advantages in dealing with nonlinear perturbations. In fact it can be seen from the results in [10] that, e.g., a multivariable version of the Aizerman conjecture holds true over $\mathbb{C}$ whereas it is known to be false over $\mathbb{R}$ even in the scalar case.

The unstructed stability radius has the following properties.

LEMMA 3.3. (i) $r_{\mathbb{C}}(A) = 0 \Leftrightarrow A \in \mathscr{U}_n(\mathbb{R}_+, \mathbb{C})$.

(ii) $r_{\mathbb{C}}(\alpha A) = \alpha r_{\mathbb{C}}(A)$ *for all* $\alpha \geq 0$.

(iii) $A \mapsto r_{\mathbb{C}}(A)$ *is continuous on* $PC(\mathbb{R}_+, \mathbb{C}^{n\times n})$.

(iv) $r_{\mathbb{C}}(A+\Delta) \geqslant r_{\mathbb{C}}(A) - \|\Delta(\cdot)\|_{L_\infty}$ *for any* $\Delta \in PC_b(\mathbb{R}_+, \mathbb{C}^{n\times n})$.

(v) $0 < r_{\mathbb{C}}(A) \leqslant -k_B(A)$ *if* $A(\cdot)$ *is exponentially stable.*

(vi) *If* $\Sigma = (A, B, C)$ *and* $k_B(A) < 0$, *then*

$$r_{\mathbb{C}}(A) \leqslant \| B(\cdot)_{|[t_0,\infty)} \|_{L_\infty} \cdot \| C(\cdot)_{|[t_0,\infty)} \|_{L_\infty} \cdot r_{\mathbb{C}}(A; B, C) \qquad \textit{for all} \quad t_0 \geqslant 0$$

(*where we define* $0 \cdot \infty = \infty$).

Proof. (i)–(iv) and (vi) follow directly from the definition. (i) yields the first inequality in (v) and the second is a consequence of $A - k_B(A) I_n \in \mathscr{U}_n(\mathbb{R}_+, \mathbb{C})$, since $k_B(A - k_B(A) I) = 0$ by Lemma 2.4(ii). ∎

The following proposition summarizes some elementary *invariance properties* of $r_{\mathbb{C}}(A; B, C)$.

PROPOSITION 3.4. *Let* $\Sigma = (A, B, C)$ *be given. Then*

(i) *If* $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$ *defines a Bohl transformation then*

$$r_{\mathbb{C}}(T^{-1}AT - T^{-1}\dot{T}; T^{-1}B, CT) = r_{\mathbb{C}}(A; B, C).$$

(ii) *If* $\theta(\cdot) \in PC^1(\mathbb{R}_+, \mathbb{C})$ *is a scalar Bohl transformation then*

$$r_{\mathbb{C}}(A - \theta^{-1}\dot{\theta} I_n; B, C) = r_{\mathbb{C}}(A; B, C).$$

(iii) *If* $\dot{x}(t) = A(t)\,x(t)$ *and* $\dot{x}(t) = \hat{A}(t)\,x(t)$ *are asymptotically or integrally equivalent then*

$$r_{\mathbb{C}}(\hat{A}; B, C) = r_{\mathbb{C}}(\hat{A}; B, C) = r_{\mathbb{C}}(A; B, C).$$

Proof. By Proposition 2.11(ii)

$$k_B(T^{-1}AT - T^{-1}\dot{T} + T^{-1}BDCT) = k_B(A + BDC)$$

for every $D(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p})$. Hence (i) and (ii) follow. (iii) is an immediate consequence of the equality

$$k_B(A + BDC) = k_B(\hat{A} + BDC)$$

resulting from Proposition 2.15. ∎

In contrast with the Bohl exponent the unstructured stability radius is not invariant with respect to Bohl transformations. In fact any exponentially stable time-invariant system $\dot{x}(t) = Ax(t)$ can be brought arbitrary close to an unstable system by constant similarity transformations.

The following example illustrates that there exist sequences of time-invariant systems such that $k_B(A_k) \to -\infty$, $r_{\mathbb{C}}(A_k) \to 0$ as $k \to \infty$.

EXAMPLE 3.5. Set

$$A_k = -\begin{bmatrix} k & k^3 \\ 0 & k \end{bmatrix}, \qquad D_k = k^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \text{for} \quad k \in \mathbb{N}.$$

Then $\lim_{k\to\infty} k_B(A_k) = -\infty$. However, $\sigma(A_k + D_k) = \{1/k, 1/k - 2k\}$ although $\lim_{k\to\infty} \|D_k\| \to 0$. Thus $\lim_{k\to\infty} r_{\mathbb{C}}(A_k) = 0$.

Remark 3.6. Suppose that $\dot{x}(t) = A(t)\, x(t)$ is periodic. By Proposition 2.11 and Lemma 3.3(v) no Bohl-equivalent system $\dot{x}(t) = \hat{A}(t)\, x(t)$ can have an unstructured stability radius larger than $-k_B(A)$. Example 2.12 shows that there always is a Bohl-equivalent system with stability radius equal to $-k_B(A)$. On the other hand it follows from results in [10] that for any $\varepsilon > 0$ there exists a Bohl-equivalent system $\dot{x}(t) = \tilde{A}(t)\, x(t)$ with $r_{\mathbb{C}}(\tilde{A}) < \varepsilon$. It is not clear whether analogous results hold for general time-varying systems.

For exponentially stable scalar systems the unstructured stability radius always coincides with the negative of the Bohl exponent. This is a direct consequence of the previous proposition and Proposition 2.13 for the case where the scalar system has a strict finite Bohl exponent. However the same result holds without this assumption.

PROPOSITION 3.7. *Suppose $a(\cdot) \in PC(\mathbb{R}_+, \mathbb{C})$ and the scalar system $\dot{x}(t) = a(t)\, x(t)$, $t \geq 0$ is exponentially stable; then*

$$r_{\mathbb{C}}(a) = -k_B(a). \tag{3.4}$$

We omit the proof which is straightforward.

Note that the proof of Lemma 3.3 shows that for time-varying scalar systems the *constant* disturbance $d(\cdot) \equiv r_{\mathbb{C}}(A)$ destabilizes the nominal system.

The following remark illustrates that there are essential differences in the properties of the stability radius for time-invariant and time-varying systems.

Remark 3.8. Suppose that $\dot{x}_i(t) = A_i x_i(t)$, $i = 1, 2$ are two exponentially stable *time-invariant* linear systems; then

$$r_{\mathbb{C}}(A_1 \oplus A_2) = \min\{r_{\mathbb{C}}(A_1), r_{\mathbb{C}}(A_2)\}. \tag{3.5}$$

This basic decomposition property of the stability radius is no longer true for time-varying systems. In fact one can construct, for any $\varepsilon > 0$, periodic functions $a_i(t)$, $i = 1, 2$ of the same period such that both scalar systems $\dot{x}_i(t) = a_i(t)\, x_i(t)$ are exponentially stable with Bohl exponent -1 (so that $r_{\mathbb{C}}(a_1) = r_{\mathbb{C}}(a_2) = 1$) whereas $r_{\mathbb{C}}(\mathrm{diag}(a_1, a_2)) < \varepsilon$.

4. The Perturbation Operator

In the time-invariant case (see [10]), the stability radius can be characterized as the inverse of the norm of the convolution operator

$$\begin{aligned} L_0\colon L_2(0, \infty; \mathbb{C}^m) &\to L_2(0, \infty; \mathbb{C}^p) \\ u(\cdot) &\mapsto \left(t \mapsto \int_0^t C e^{A(t-s)} B u(s)\, ds \right). \end{aligned} \tag{4.1}$$

In order to explore the possibility of obtaining similar results for time-varying systems we assume, throughout this section,

$$\begin{aligned} &A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times n}), \qquad B(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{n\times m}) \\ &C(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{p\times n}), \qquad k_B(A) < 0 \end{aligned} \tag{4.2}$$

With any such triple $\Sigma = (A, B, C)$ we associate a parametrized family of *perturbation operators* $(L_{t_0}^{\Sigma})_{t_0 \in \mathbb{R}_+}$ defined by

$$\begin{aligned} L_{t_0}^{\Sigma}\colon L_2(t_0, \infty; \mathbb{C}^m) &\to L_2(t_0, \infty; \mathbb{C}^p), \qquad t_0 \geqslant 0 \\ u(\cdot) &\mapsto \left(t \mapsto \int_{t_0}^t C(t)\, \Phi(t, s)\, B(s)\, u(s)\, ds \right). \end{aligned} \tag{4.3}$$

In the following proposition we will show that these maps are well-defined. Note that in the time-invariant case $\|L_{t_0}^{\Sigma}\| = \|L_0\|$ for all $t_0 \geqslant 0$.

Proposition 4.1. *Suppose* (4.2) *and let* $\Sigma = (A, B, C)$. *Then*

(i) $L_{t_0}^{\Sigma}$ *is a bounded operator.*

(ii) $t_0 \mapsto \|L_{t_0}^{\Sigma}\|$ *is monotonically decreasing on* $\mathbb{R}_+$.

(iii) $\|L_{t_0}^{\Sigma}\|^{-1} \leqslant r_{\mathbb{C}}(A; B, C)$ *for all* $t_0 \geqslant 0$.

(iv) *If* A, B, C *are periodic with some common period, then*

$$\|L_{t_0}^{\Sigma}\| = \|L_{t_1}^{\Sigma}\| \qquad \text{for all } t_0, t_1 \in \mathbb{R}_+.$$

(v) *In the unstructured case, i.e.,* $B(\cdot) \equiv C(\cdot) \equiv I_n$, *if*

$$\|\Phi(t, s)\| \leqslant M e^{-\omega(t-s)} \qquad \text{for all} \quad t \geqslant s \geqslant t_0 \text{ and some } M, \omega > 0$$

then

$$\frac{\omega}{M} \leqslant \|L_{t_0}^{\Sigma}\|^{-1} \leqslant \lim_{t\to\infty} \|L_t^{\Sigma}\|^{-1} \leqslant r_{\mathbb{C}}(A) \leqslant -k_B(A). \tag{4.4}$$

Proof. We write as shorthand notations L_{t_0} instead of $L^{\Sigma}_{t_0}$ and $L_2(t_0, r)$ instead of $L_2(t_0, \infty; \mathbb{C}^r)$, $r = m, p$.

(i) Let $u(\cdot) \in L_2(t_0, m)$; then by changing variables and using the inequality

$$\| f*v \|_{L_2} \leqslant \| f \|_{L_1} \cdot \| v \|_{L_2} \qquad \text{for} \quad f \in L_1, v \in L_2,$$

we obtain

$$\| L_{t_0} u \|^2_{L_2(t_0, p)} \leqslant (\| C \|_{L_\infty} \| B \|_{L_\infty} M)^2 \int_{t_0}^{\infty} \left[\int_{t_0}^{t} e^{-\omega(t-s)} \| u(s) \| \, ds \right]^2 dt$$

$$= (\| C \|_{L_\infty} \| B \|_{L_\infty} M)^2 \int_0^{\infty} \left| \int_0^{\tau} e^{-\omega(\tau - \sigma)} \| u(\sigma + t_0) \| \, d\sigma \right|^2 d\tau$$

$$\leqslant (\| C \|_{L_\infty} \| B \|_{L_\infty} M)^2 \, \| e^{-\omega \cdot} \|^2_{L_1(0,1)} \cdot \| u(\cdot + t_0) \|^2_{L_2(0, m)}$$

$$\leqslant [(\| C \|_{L_\infty} \| B \|_{\infty} M)^2 / \omega^2] \, \| u \|^2_{L_2(t_0, m)}.$$

This shows that L_{t_0} is bounded and the first inequality holds in (4.4).

(ii) Suppose $0 \leqslant t_0 < t_1$ and $u(\cdot) \in L_2(t_1, m)$, $\| u(\cdot) \| = 1$. Extending $u(\cdot)$ to $\bar{u}(\cdot)$ by $u(t) = 0$ for $t \in [t_0, t_1)$ yields $\bar{u}(\cdot) \in L_2(t_0, m)$ with $\| \bar{u}(\cdot) \| = 1$. Now

$$\| L_{t_1} u \|^2_{L_2(t_1, p)} = \| L_{t_0} \bar{u} \|^2_{L_2(t_0, p)},$$

from which (ii) follows.

(iii) Let $D(\cdot) \in PC_b(t_0, \infty; \mathbb{C}^{m \times p})$ be such that

$$\| D \|_{L_\infty} < \| L_{t_0} \|^{-1}; \tag{4.5}$$

then we have to show that the perturbed system

$$\dot{x}(t) = [A(t) + B(t) \, D(t) \, C(t)] \, x(t), \qquad t \geqslant t_0, \tag{4.6}$$

is exponentially stable. By Theorem 2.8 and Proposition 2.2 it is sufficient to prove that the solutions $x(\cdot) = x(\cdot; t'_0, x_0)$ of (4.6) (with $t'_0 \geqslant t_0$) satisfy for some $k > 0$

$$\sup_{t'_0 \geqslant t_0} \| x(\cdot; t'_0, x_0) \|_{L_2(t'_0, n)} \leqslant k \, \| x_0 \| \qquad \text{for all} \quad x_0 \in \mathbb{C}^n \tag{4.7}$$

$$\sup_{0 \leqslant |t - t'_0| \leqslant 1} \| x(t; t'_0, x_0) \| \leqslant k \, \| x_0 \| \qquad \text{for all} \quad x_0 \in \mathbb{C}^n. \tag{4.8}$$

Now, by variations of constants, for $t \geqslant t'_0$

$$x(t; t'_0, x_0) = \Phi(t, t'_0) \, x_0 + \int_{t'_0}^{t} \Phi(t, s) \, B(s) \, D(s) \, C(s) \, x(s; t'_0, x_0) \, ds \tag{4.9}$$

and hence for $y(t) := C(t)\,x(t)$, $y_0(t) := C(t)\,\Phi(t, t'_0)\,x_0 \in L_2(t'_0, p)$,

$$y(t) = y_0(t) + (L_{t'_0} D y)(t).$$

By the contraction principle and (4.5) this equation has a unique solution in $L_2(t'_0, p)$ and

$$\begin{aligned}\| y \|_{L_2(t'_0, p)} &\leqslant \|(I - L_{t'_0} D)^{-1}\| \cdot \| y_0 \|_{L_2(t'_0, p)} \\ &\leqslant (1 - \| L_{t'_0} D \|)^{-1} \| y_0 \|_{L_2(t'_0, p)} \\ &\leqslant (1 - \| L_{t'_0} \| \, \| D \|)^{-1} \| y_0 \|_{L_2(t'_0, p)}.\end{aligned}$$

So the norm is uniformly bounded in $t'_0 \geqslant t_0$.

Replacing $C(s)\,x(s; t'_0, x_0)$ by $y(s)$ in (4.9) yields

$$x(t; t'_0, x_0) = \Phi(t, t'_0)\,x_0 + \int_{t'_0}^{t} \Phi(t, s)\,B(s)\,D(s)\,y(s)\,ds.$$

Similar estimates to those used in (i) show that the input-to-state map

$$\begin{aligned}&M_{t'_0}\colon L_2(t'_0, m) \to L_2(t'_0, n) \\ &u(\cdot) \mapsto \left(t \mapsto \int_{t'_0}^{t} \Phi(t, s)\,B(s)\,u(s)\,ds \right)\end{aligned} \tag{4.10}$$

is uniformly bounded in $t'_0 \geqslant t_0$. Hence (4.7) is satisfied and a similar estimate to that in (i), applied to (4.9), yields (4.8).

(iv) Let $\mu > 0$ be the common period of A, B, C. The right shift

$$\begin{aligned}&S_\mu\colon L_2(t_0, r) \to L_2(t_0 + \mu, r) \\ &v(t) \mapsto v(t - \mu)\end{aligned}$$

is an isometry. Now $\Phi(t + \mu, s + \mu) = \Phi(t, s)$, hence

$$\begin{aligned}(S_\mu \circ L_{t_0} u)(t) &= \int_{t_0}^{t-\mu} C(t - \mu)\,\Phi(t - \mu, s)\,B(s)\,u(s)\,ds \\ &= \int_{t_0}^{t-\mu} C(t)\,\Phi(t, s + \mu)\,B(s)\,u(s)\,ds \\ &= \int_{t_0+\mu}^{t} C(t)\,\Phi(t, \tau)\,B(\tau - \mu)\,u(\tau - \mu)\,d\tau \\ &= \int_{t_0+\mu}^{t} C(t)\,\Phi(t, \tau)\,B(\tau)\,S_\mu u(\tau)\,d\tau \\ &= (L_{t_0+\mu} \circ S_\mu u)(t).\end{aligned}$$

This proves $\|L_{t_0}\| = \|L_{t_0+\mu}\|$ and the result follows since $t_0 \mapsto \|L_{t_0}\|$ is decreasing.

(v) The second and third inequalities in (4.4) follow from (ii) and (iii) and the last is a consequence of Lemma 3.3(v). ∎

Remark 4.2. From a control theoretic viewpoint $L^{\Sigma}_{t_0}$ may be thought of as the *input–output operator* of the system

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t) \qquad & x(t_0) &= 0 \\ y(t) &= C(t)\,x(t), & t &\geqslant t_0. \end{aligned} \tag{4.11}$$

If the triple $\Sigma = (A, B, C)$ is such that $k_B(A) < 0$ (*internal stability*) then by Proposition 4.1(i) the input–output operator $L^{\Sigma}_{t_0}$ is bounded (*external stability*). Under the additional assumption that the system (4.11) is bounded and uniformly controllable and uniformly observable the converse holds true; i.e., boundedness of $L^{\Sigma}_{t_0}$ implies $k_B(A) < 0$. This is proved in [1].

Throughout the remainder of this paper we use the notation

$$l(A; B, C) := \lim_{t_0 \to \infty} \|L^{\Sigma}_{t_0}\|^{-1}. \tag{4.12}$$

As a consequence of Proposition 4.1(iii) we obtain the following robustness result.

COROLLARY 4.3. *Suppose* $\Sigma = (A, B, C)$ *and* (4.2). *If* $D(\cdot) \in PC_b(0, \infty; \mathbb{C}^{m \times p})$ *satisfies*

$$\lim_{t_0 \to \infty} \|D(\cdot)|_{[t_0, \infty)}\|_{L_\infty} < l(A; B, C) \tag{4.13}$$

then the perturbed system (4.6) *is exponentially stable.*

In the unstructed case it is known that perturbations $D \in PC_b(\mathbb{R}_+, \mathbb{C}^{m \times p})$ of norm $\|D(\cdot)\|_{L_\infty} < \omega/M$ (ω, M as in Proposition 4.1(v)) do not destroy the exponential stability of the system (see [3]). In view of (4.4), Condition (4.13) is less conservative.

In contrast with time-invariant systems the following example shows that the inequality

$$l(A; B, C) \leqslant r_{\mathbb{C}}(A; B, C) \tag{4.14}$$

is in general strict.

EXAMPLE 4.4. Consider the scalar system

$$\dot{x}(t) = a(t)\,x(t), \qquad t \geqslant 0,$$

where $a(t) = -1 + k\alpha(t)$, $k \in \mathbb{R}$, $\alpha(\cdot) \in PC(\mathbb{R}_+, \mathbb{C})$ is periodic with period $3T$, $T = \ln 2$, given by

$$\alpha(t) = \begin{cases} 0 & t \in [3iT, (3i+1)\,T) \\ 1 & t \in [(3i+1)\,T, (3i+2)\,T), \qquad i \in \mathbb{N}_0. \\ -1 & t \in [(3i+2)\,T, 3(i+1)\,T) \end{cases}$$

Let $\Sigma = (a, 1, 1)$; then in view of Example 2.12, Proposition 3.7, and Proposition 4.1(iv) we have

$$-k_B(a) = r_{\mathbb{C}}(a) = 1 \qquad \text{and} \qquad l(A; 1, 1) = \| L_0^{\Sigma} \|^{-1}.$$

We will show that $\| L_0^{\Sigma} \|^{-1} < 1$.

Let $\beta(t) := k \int_0^t \alpha(\tau)\, d\tau$ and $u(t) = e^{\beta(t) - 2t}$. A straightforward calculation shows that

$$\| L_0^{\Sigma} u \|^2 - \| u \|^2 = \int_0^T e^{-2t}(1 - 2e^{-t})\, dt + \int_T^{\infty} e^{2\beta(t) - 2t}(1 - 2e^{-t})\, dt.$$

Since $1 - 2e^{-t} > 0$ for $t > T$ one can choose k so that the right hand side is positive.

Equality holds in (4.14) if the system Σ is asymptotically or integrally equivalent to a time-invariant system. To prove this we need the following

PROPOSITION 4.5. *Suppose that $\Sigma = (A, B, C)$ satisfies* (4.2) *and let $\dot{x}(t) = \hat{A}(t)\, x(t)$ be asymptotically or integrally equivalent to $\dot{x}(t) = A(t)\, x(t)$. Then for $\hat{\Sigma} = (\hat{A}, B, C)$*

$$\lim_{t_0 \to \infty} \| L_{t_0}^{\Sigma} - L_{t_0}^{\hat{\Sigma}} \| = 0. \tag{4.15}$$

In particular

$$l(A; B, C) = l(\hat{A}; B, C). \tag{4.16}$$

The proof is straightforward; see [8].

By Proposition 3.4(iii) and Remark 3.2(ii) we get

COROLLARY 4.6. *Suppose $\Sigma = (A, B, C)$ satisfies* (4.2) *and B, C are constant matrices. If $\dot{x}(t) = A(t)\, x(t)$ is asymptotically or integrally equivalent to a time-invariant $\dot{x}(t) = A_0 x(t)$, then*

$$r_{\mathbb{C}}(A; B, C) = l(A; B, C) = r_{\mathbb{C}}(A_0; B, C) = [\max_{\omega \in \mathbb{R}} \| C(i\omega I - A_0)^{-1} B \|]^{-1}.$$

It is clear from the definition of $L_{t_0}^{\Sigma}$ that this operator is invariant with respect to Bohl transformations if the transformation is applied not only to $A(\cdot)$ but also to $B(\cdot)$ and $C(\cdot)$:

$$L_{t_0}^{\Sigma} = L_{t_0}^{\Sigma_T}, \qquad t_0 \geq 0 \qquad \text{for} \quad \Sigma_T = (T^{-1}AT - T^{-1}\dot{T}, T^{-1}B, CT).$$

However, contrary to the Bohl exponent and the stability radius, $l(A; B, C)$ is not invariant when scalar Bohl transformations are applied to $A(\cdot)$ alone. In fact, if we apply Proposition 2.13 this is demonstrated by Example 4.4.

In order to fill the gap between $l(A; B, C)$ and $r_{\mathbb{C}}(A; B, C)$ one might try to use the scalar Bohl transformation θ and consider $\Sigma_\theta = (A - \theta^{-1}\dot{\theta}I_n, B, C)$. Then $r_{\mathbb{C}}(A; B, C) = r_{\mathbb{C}}(A - \theta^{-1}\dot{\theta}I_n; B, C)$ and it is easy to see that $L_{t_0}^{\Sigma_\theta} = \theta^{-1}L_{t_0}^{\Sigma}\theta$. Unfortunately we have not been able to prove or disprove the following

Conjecture 4.7. Suppose (4.2) and $\Sigma = (A, B, C)$; then

$$r_{\mathbb{C}}(A; B, C) = \sup\{l(A - \theta^{-1}\dot{\theta}I_n; B, C);\ \theta \text{ a scalar Bohl transformation}\}.$$

By Proposition 3.7 the conjecture holds true for scalar systems.

5. The Associated Parametrized Differential Riccati Equation

In this section we examine the parametrized *differential Riccati equation* $(\mathrm{DRE})_\rho$

$$\dot{P}(t) + A^*(t)\,P(t) + P(t)\,A(t) - \rho C^*(t)\,C(t) - P(t)\,B(t)\,B^*(t)\,P(t) = 0,$$

$$t \geq t_0,\ \rho \in \mathbb{R}$$

associated with the system

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), \qquad x(t_0) = x_0 \in \mathbb{C}^n \\ y(t) &= C(t)\,x(t). \end{aligned} \tag{5.1}$$

Throughout this section we assume (4.2).

For time-invariant $\Sigma = (A, B, C)$ it has been shown in [10] that the *algebraic Riccati equation* $(\mathrm{ARE})_\rho$

$$A^*P + PA - \rho C^*C - PBB^*P = 0$$

admits a Hermitian solution P if and only if $\rho \leq r_{\mathbb{C}}^2(A; B, C)$. Guided by this result we wish to determine the maximal ρ for which there exist bounded Hermitian solutions of $(\mathrm{DRE})_\rho$ on $[t_0, \infty)$. Various authors (see e.g.

[12, 15]) have studied differential Riccati equations with time-varying coefficients; however, their results cannot be applied to $(\text{DRE})_\rho$ if $\rho > 0$.

We will proceed via the *optimal control problem* $(\text{OCP})_\rho$

Minimize the cost functional

$$J_\rho(x_0, [t_0, t_1), u(\cdot)) := \int_{t_0}^{t_1} [\|u(s)\|^2 - \rho\, \|y(s)\|^2]\, ds$$

$$\text{for} \quad u(\cdot) \in L_2(t_0, t_1; \mathbb{C}^m) \qquad \text{subject to (5.1)}.$$

where $0 \leqslant t_0 \leqslant t_1 \leqslant \infty$, $x_0 \in \mathbb{C}^n$ and $\rho \in \mathbb{R}$. We begin by examining the *finite time* problem, where $t_1 < \infty$. Since the optimal control is expected to be feedback we start with some lemmata on the cost of feedback controls $u(t) = -F(t)\, x(t)$. To describe these costs we need the following well-known lemma about differential Lyapunov equations.

LEMMA 5.1. *Let* $\tilde{A}(\cdot)$, $R(\cdot) \in PC([t_0, \infty); \mathbb{C}^{n \times n})$ *and* $\tilde{\Phi}(\cdot, \cdot)$ *be the transition matrix of* $\dot{x}(t) = \tilde{A}(t)\, x(t)$.

(i) *The unique solution of the differential Lyapunov equation*

$$\dot{P}(t) + \tilde{A}^*(t)\, P(t) + P(t)\, \tilde{A}(t) + R(t) = 0, \qquad t \in [t_0, t_1] \tag{5.2}$$

with final value $P(t_1) = 0$ *is given by*

$$P(t) = \int_t^{t_1} \tilde{\Phi}^*(s, t)\, R(s)\, \tilde{\Phi}(s, t)\, ds, \qquad t \in [t_0, t_1].$$

(ii) *If* $\dot{x}(t) = \tilde{A}(t)\, x(t)$ *is exponentially stable and* $R(\cdot)$ *is bounded, then*

$$P(t) = \int_t^{\infty} \tilde{\Phi}^*(s, t)\, R(s)\, \tilde{\Phi}(s, t)\, ds$$

is the unique bounded solution of (5.2) *on* $[t_0, \infty)$.

LEMMA 5.2. *Suppose* $F(\cdot) \in PC([t_0, t_1]; \mathbb{C}^{m \times n})$, $t_1 < \infty$, $A_F(t) = A(t) - B(t)\, F(t)$ *with transition matrix* $\Phi_F(\cdot, \cdot)$, *and let*

$$u_F(t) = -F(t)\, x(t), \qquad t \in [t_0, t_1],$$

where $x(\cdot)$ *satisfies*

$$\dot{x}(t) = A_F(t)\, x(t), \qquad t \in [t_0, t_1],\ x(t_0) = x_0.$$

Then

$$J_\rho(x_0, [t_0, t_1), u_F(\cdot)) = \langle x_0, P_F(t_0)\, x_0 \rangle, \tag{5.3}$$

where

$$P_F(t)=\int_t^{t_1} \Phi_F^*(s,t)[F^*(s)F(s)-\rho C^*(s)C(s)]\,\Phi_F(s,t)\,ds, \qquad t\in[t_0,t_1] \tag{5.4}$$

is the solution of the differential Lyapunov equation $(\mathrm{DLE})_\rho$

$$\dot P(t)+A_F^*(t)P(t)+P(t)A_F(t) -\rho C^*(t)C(t)+F^*(t)F(t)=0, \qquad t\in[t_0,t_1]$$

with final value $P(t_1)=0$.

Proof. By (5.4) and the definition of J_ρ we obtain

$$\begin{aligned}\langle x_0, P_F(t_0)x_0\rangle &= \int_{t_0}^{t_1} [\|F(s)\Phi_F(s,t_0)x_0\|^2-\rho\,\|C(s)\Phi_F(s,t_0)x_0\|^2]\,ds\\ &= \int_{t_0}^{t_1} [\|u_F(s)\|^2-\rho\,\|C(s)\Phi_F(s,t_0)x_0\|^2]\,ds\\ &= J_\rho(x_0,[t_0,t_1),u_F(\cdot)).\end{aligned}$$

That P_F solves $(\mathrm{DLE})_\rho$ follows from Lemma 5.1(i) if we set $\tilde A(t)=A_F(t)$ and $R(t)=-\rho C^*(t)C(t)+F^*(t)F(t)$. ∎

Note the following relationship between the differential Riccati equation $(\mathrm{DRE})_\rho$ and the differential Lyapunov equation $(\mathrm{DLE})_\rho$.

Remark 5.3. $P(\cdot)$ is a solution of $(\mathrm{DRE})_\rho$ on $[t_0,t_1]$ if and only if $P(\cdot)$ is a solution of $(\mathrm{DLE})_\rho$ on $[t_0,t_1]$ with $F(t)=B^*(t)P(t)$.

Our construction procedure for solutions of $(\mathrm{DRE})_\rho$ (cf. proof of Theorem 5.7) is based on this simple observation.

LEMMA 5.4. *Let* $F(\cdot)\in PC([t_0,t_1],\mathbb{C}^{m\times n})$, $\bar u(\cdot)\in L_2(t_0,t_1;\mathbb{C}^m)$, $u_F(t)=-F(t)x(t)$, $t\in[t_0,t_1]$, *where now*

$$\begin{aligned}\dot x(t)&=A(t)x(t)+B(t)(u_F(t)+\bar u(t)),\\ &=A_F(t)x(t)+B(t)\bar u(t), \qquad t\in[t_0,t_1],\ x(t_0)=x_0.\end{aligned} \tag{5.5}$$

If $u(t)=u_F(t)+\bar u(t)$, $t\in[t_0,t_1]$, *then*

$$\begin{aligned}J_\rho(x_0,[t_0,t_1),u(\cdot))=\langle x_0,P_F(t_0)x_0\rangle &+\int_{t_0}^{t_1}\|u(s)+B^*(s)P_F(s)x(s)\|^2\,ds\\ &-\int_{t_0}^{t_1}\|[F(s)-B^*(s)P_F(s)]\,x(s)\|^2\,ds,\end{aligned}$$

where $P_F(\cdot)$ *is defined by* (5.4).

Proof. Differentiation of $V(t) := \langle x(t), P_F(t)\, x(t)\rangle$, $t \in [t_0, t_1]$, along the solution $x(\cdot)$ of (5.5) gives (we leave out the argument t)

$$\begin{aligned}\dot V &= \langle A_F x + B\bar u, P_F x\rangle + \langle x, \dot P_F x\rangle + \langle x, P_F(A_F x + B\bar u)\rangle \\ &= \langle B\bar u, P_F x\rangle + \langle x, P_F B\bar u\rangle + \langle x, (\rho C^*C - F^*F)\, x\rangle \\ &= -\|u_F\|^2 + \rho\,\|Cx\|^2 + 2\,\mathrm{Re}\,\langle \bar u, B^*P_F x\rangle \\ &= -\|u\|^2 + \rho\,\|Cx\|^2 + \|u + B^*P_F x\|^2 - \|(B^*P_F - F)\,x\|^2.\end{aligned}$$

Integrating on $[t_0, t_1]$ and using $P_F(t_1) = 0$ yields the result. ∎

If $\rho \geqslant 0$ and $0 \leqslant t_0 < t_1 < t_2 \leqslant \infty$, then

$$\begin{aligned} 0 &\geqslant \inf_{u \in L_2(t_0, t_1; \mathbb{C}^m)} J_\rho(x_0, [t_0, t_1), u(\cdot)) \\ &\geqslant \inf_{u \in L_2(t_0, t_2; \mathbb{C}^m)} J_\rho(x_0, [t_0, t_2), u(\cdot)), \end{aligned} \tag{5.6}$$

whereas the converse inequalities hold if $\rho \leqslant 0$. These inequalities show that the minimal costs are finite over an arbitrary interval if they are finite over $[0, \infty)$.

LEMMA 5.5. (i) $\inf_{u \in L_2(t_0, m)} J_\rho(0, \infty), u(\cdot)) = 0 \Leftrightarrow \rho \leqslant \|L^\Sigma_{t_0}\|^{-2}$ (*here by definition* $\|L^\Sigma_{t_0}\|^{-2} = \infty$ *if* $\|L^\Sigma_{t_0}\| = 0$).

(ii) *For every* $\rho \in (-\infty, \|L^\Sigma_{t_0}\|^{-2})$ *there exists a constant* $c_\rho > 0$ *such that*

$$\inf_{u \in L_2(t, m)} J_\rho(x_0, [t, \infty), u(\cdot)) \geqslant -c_\rho\,\|x_0\|^2 \qquad \textit{for all} \quad t \geqslant t_0,\ x_0 \in \mathbb{R}^n. \tag{5.7}$$

Proof. Statement (i) follows from the equivalence

$$\inf_{u \in L_2(t_0, m)} J_\rho(0, [t_0, \infty), u(\cdot)) = 0 \Leftrightarrow [\|u\|^2 - \rho\,\|L^\Sigma_{t_0} u\|^2] \geqslant 0 \quad \text{for all} \quad u \in L_2(t_0, m).$$

To prove (ii) we need only consider the case $\rho \in (0, \|L^\Sigma_{t_0}\|^{-2})$. Since

$$2\,\mathrm{Re}\langle a, b\rangle \leqslant \alpha\,\|a\|^2 + \alpha^{-1}\,\|b\|^2 \qquad \text{for all} \quad \alpha > 0,\ a, b \in L_2(t_0, p),$$

we have

$$\begin{aligned} &J_\rho(x_0, [t_0, \infty), u(\cdot)) \\ &\quad = \|u(\cdot)\|^2 - \rho\,\|(L^\Sigma_{t_0} u)(\cdot) + C(\cdot)\,\Phi(\cdot, t_0)\,x_0\|^2 \\ &\quad = \|u(\cdot)\|^2\ \|(L^\Sigma_{t_0} u)(\cdot)\|^2 - \rho\,\|C(\cdot)\,\Phi(\cdot, t_0)\,x_0\|^2 \\ &\qquad - 2\rho\,\mathrm{Re}\langle (L^\Sigma_{t_0} u)(\cdot), C(\cdot)\,\Phi(\cdot, t_0)\,x_0\rangle \\ &\quad \geqslant \|u(\cdot)\|^2 - \rho(1+\alpha)\|(L^\Sigma_{t_0} u)(\cdot)\|^2 - \rho(1+\alpha^{-1})\|C(\cdot)\,\Phi(\cdot, t_0)\,x_0\|^2. \end{aligned}$$

For sufficiently small α

$$J_\rho(x_0, [t_0, \infty), u(\cdot)) \geq -\rho(1+\alpha^{-1}) \| C(\cdot)\, \Phi(\cdot, t_0)\, x_0 \|^2.$$

Since $\dot{x}(t) = A(t)\, x(t)$ is exponentially stable, there exists $c > 0$ such that

$$\| C(\cdot)\, \Phi(\cdot, t_0)\, x_0 \|^2 \leq c \, \| x_0 \|^2 \qquad \text{for all} \quad t_0 \geq 0.$$

So we may take $c_\rho = \rho(1+\alpha^{-1})\, c$ to ensure (5.7) for t_0. The result follows for any $t \geq t_0$ since the left hand side of (5.7) is increasing in t. ∎

LEMMA 5.6. *Suppose* $A_k(\cdot) \in PC(t_0, t_1; \mathbb{C}^{n \times n})$, $k \in \mathbb{N}$, $t_1 < \infty$ *converges pointwise* to $\tilde{A}(\cdot) \in PC(t_0, t_1; \mathbb{C}^{n \times n})$ *on* $[t_0, t_1]$, i.e.,

$$\lim_{k \to \infty} \| A_k(t) - \tilde{A}(t) \| = 0 \qquad \textit{for all} \quad t \in [t_0, t_1]$$

and $\| A_k(t) \| < c$ *for all* $t \in [t_0, t_1]$, $k \in \mathbb{N}$. If $A_k(\cdot)$ *generates* $\Phi_k(\cdot, \cdot)$ *and* $\tilde{A}(\cdot)$ *generates* $\tilde{\Phi}(\cdot, \cdot)$, *then for every* $\varepsilon > 0$, *there exists* $k_0 \in \mathbb{N}$, *such that*

$$\| \Phi_k(t, s) - \tilde{\Phi}(t, s) \| < \varepsilon \qquad \textit{for all} \quad k \geq k_0,\ t_0 \leq s \leq t \leq t_1. \tag{5.8}$$

The proof is straightforward; see [8].

We are now in a position to solve the optimal control problem $(\text{OCP})_\rho$ on finite intervals, a main result of this section.

THEOREM 5.7. *Suppose* $\rho < \| L_{t_0}^{\Sigma} \|^{-2}$, $0 \leq t_0 < t_1 < \infty$. *Then*

(i) *There exists a (unique) Hermitian solution* $P^{t_1}(\cdot) \in PC^1(t_0, t_1; \mathbb{C}^{n \times n})$ *of* $(\text{DRE})_\rho$ *with* $P^{t_1}(t_1) = 0$.

(ii) *If* $\rho \geq 0$ $(\rho \leq 0)$ *then* $P^{t_1}(t)$ *is nonpositive (nonnegative) for all* $t \in [t_0, t_1]$.

(iii) *The minimal cost of* $(\text{OCP})_\rho$ *is*

$$\inf_{u \in L_2(t_0, t_1; \mathbb{C}^m)} J_\rho(x_0, [t_0, t_1), u(\cdot)) = \langle x_0, P^{t_1}(t_0)\, x_0 \rangle. \tag{5.9}$$

(iv) *The optimal control is given by*

$$u^*(t) = -B^*(t)\, P^{t_1}(t)\, x(t).$$

Proof. Starting with $P_0(\cdot) \equiv 0$ we recursively define a sequence $P_k(\cdot) \in PC^1(t_0, t_1; \mathbb{C}^{n \times n})$, $k \in \mathbb{N}$ by the sequence of differential Lyapunov equations

$$\begin{aligned} \dot{P}_k(t) + A_{k-1}^*(t)\, P_k(t) + P_k(t)\, A_{k-1}(t) - \rho C^*(t)\, C(t) & \\ + P_{k-1}(t)\, B(t)\, B^*(t)\, P_{k-1}(t) = 0, &\qquad t \in [t_0, t_1], \\ P_k(t_1) = 0, & \end{aligned} \tag{5.10}$$

where

$$A_{k-1}(t) = A(t) - B(t)\,B^*(t)\,P_{k-1}(t), \qquad t \in [t_0, t_1], k \geq 1.$$

We will show:

(a) $P^{t_1}(t) = \lim_{k \to \infty} P_k(t)$ exists for all $t \in [t_0, t_1]$

(b) $P^{t_1}(\cdot)$ is the unique Hermitian solution of $(\text{DRF})_\rho$ on $[t_0, t_1]$ with $P^{t_1}(t_1) = 0$.

After establishing (a), (b) we have, by Lemma 5.2 and Remark 5.3,

$$J_\rho(x_0, [t_0, t_1), -B^*(\cdot)\,P^{t_1}(\cdot)\,x(\cdot)) = \langle x_0, P^{t_1}(t_0)\,x_0 \rangle,$$

and applying Lemma 5.4 with $F(t) = B^*(t)\,P^{t_1}(t)$ yields

$$J_\rho(x_0, [t_0, t_1), u(\cdot)) = \langle x_0, P^{t_1}(t_0)\,x_0 \rangle + \int_{t_0}^{t_1} \|u(s) - u^*(s)\|^2\,ds.$$

This shows (iii) and (iv) so it remains to prove (a), (b), and (ii). Note that by (5.10), $P_k(t) = P_F(t)$, where $F(t) = B^*(t)\,P_{k-1}(t)$. Set

$$u_k(t) = -B^*(t)\,P_k(t)\,x_k(t), \qquad \bar{u}_k(t) = u_k(t) - u_{k-1}(t), \qquad t \in [t_0, t_1], k \geq 1,$$

where $x_k(\cdot)$ solves $\dot{x}_k(t) = A_k(t)\,x_k(t)$, $x_k(t_0') = x_0$ and $t_0' \in [t_0, t_1]$ is arbitrary. By Lemma 5.2 and Lemma 5.4

$$\begin{aligned} &\langle x_0, P_{k+1}(t_0')\,x_0 \rangle - \langle x_0, P_k(t_0')\,x_0 \rangle \\ &\quad = J_\rho(x_0, [t_0', t_1), u_k(\cdot)) - \langle x_0, P_k(t_0')\,x_0 \rangle \\ &\quad = -\int_{t_0'}^{t_1} \|[B^*(s)\,P_{k-1}(s) - B^*(s)\,P_k(s)]\,x_k(s)\|^2\,ds \leq 0 \end{aligned}$$

for all $k \geq 1$, $t_0' \in [t_0, t_1]$. But by Lemma 5.5(ii)

$$\langle x_0, P_k(t_0')\,x_0 \rangle \geq \inf_{u \in L_2(t_0', t_1; \mathbb{C}^m)} J_\rho(x_0, [t_0', t_1), u(\cdot)) \geq -c_\rho\,\|x_0\|^2.$$

So $(P_k(t_0'))_{k \geq 1}$ is a decreasing sequence uniformly bounded from below and the limit

$$\lim_{k \to \infty} P_k(t_0') = P^{t_1}(t_0') = (P^{t_1}(t_0'))^*$$

exists for every $t_0' \in [t_0, t_1]$. This proves (a). Moreover

$$\lim_{k \to \infty} A_k(t) = A(t) - B(t)\,B^*(t)\,P^{t_1}(t) \qquad \text{for all} \quad t \in [t_0, t_1] \tag{5.11}$$

and since $P_k(t)$, $k \geqslant 1$ is monotonically decreasing and bounded from below we see that $\| A_k(t) \|$ is uniformly bounded on $[t_0, t_1]$. Thus by Lemma 5.6 $\Phi_k(\cdot, \cdot)$ converges uniformly on $[t_0, t_1]$ to $\Phi^{t_1}(\cdot, \cdot)$ the evolution operator generated by $A(\cdot) - B(\cdot) B^*(\cdot) P^{t_1}(\cdot)$. Next we apply Lebesgue's dominated convergence theorem to the sequence

$$P_k(t) = -\int_{t_0}^{t_1} \Phi_{k-1}^*(s, t)[\rho C^*(s) C(s) - P_{k-1}(s) B(s) B^*(s) P_{k-1}(s)] \times \Phi_{k-1}(s, t)\, ds$$

to obtain

$$P^{t_1}(t) = -\int_t^{t_1} \Phi^{t_1 *}(s, t)[\rho C^*(s) C(s) - P^{t_1}(s) B(s) B^*(s) P^{t_1}(s)]\, \Phi^{t_1}(s, t)\, ds$$

Thus $P^{t_1}(\cdot)$ satisfies $(\mathrm{DRE})_\rho$ on $[t_0, t_1]$ and $P^{t_1}(t_1) = 0$. The uniqueness of the solution $P^{t_1}(\cdot)$ of $(\mathrm{DRE})_\rho$ with $P^{t_1}(t_1) = 0$ follows from general theorems. This proves (a), (b), and (ii) for $\rho \leqslant 0$.

Applying Lemma 5.2 and Remark 5.3 to the above equation yields

$$P^{t_1}(t) = -\int_t^{t_1} \Phi^*(s, t)[\rho C^*(s) C(s) + P^{t_1}(s) B(s) B^*(s) P^{t_1}(s)]\, \Phi(s, t)\, ds,$$

which proves (ii) for $\rho \geqslant 0$. (Note that $P_{k+1}^{t_1}(t) \geqslant P_k^{t_1}(t)$ holds for $k \geqslant 1$ and not for $k = 0$ if $\rho < 0$.) This shows (ii) and completes the proof. ∎

Using (5.6) we have

COROLLARY 5.8. *Suppose* $\rho < \| L_{t_0}^{\Sigma} \|^{-2}$, $0 \leqslant t_0 < t_1 < t_2 < \infty$. *Then*

$$P^{t_2}(t) \leqslant P^{t_1}(t) \quad \text{for all} \quad t \in [t_0, t_1] \quad \text{if} \quad \rho \geqslant 0$$
$$P^{t_2}(t) \geqslant P^{t_1}(t) \quad \text{for all} \quad t \in [t_0, t_1] \quad \text{if} \quad \rho < 0.$$

We now proceed to examine solutions of $(\mathrm{DRE})_\rho$ on *infinite* intervals and relate them to the *infinite time* optimal control problem $(\mathrm{OCP})_\rho$, $t_1 = \infty$. The following lemma plays a key role.

LEMMA 5.9. *Suppose* $t_0 \geqslant 0$, $\rho \in \mathbb{R}$, $u(\cdot) \in L_2(t_0, m)$, *and* $Q(\cdot) \in PC^1(t_0, \infty; \mathbb{C}^{n \times n})$ *is a bounded Hermitian solution of* $(\mathrm{DRE})_\rho$. *If* $x(\cdot)$ *solves*

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t), \qquad t \geqslant t_0,\ x(t_0) = x_0, \tag{5.12}$$

then

$$J_\rho(x_0, [t_0, \infty), u(\cdot)) = \int_{t_0}^{\infty} \| u(s) + B^*(s) Q(s) x(s) \|^2\, ds + \langle x_0, Q(t_0) x_0 \rangle. \tag{5.13}$$

In particular,

$$\langle x_0, Q(t_0)\, x_0\rangle \leqslant \inf_{u \in L_2(t_0, m)} J_\rho(x_0, [t_0, \infty), u(\cdot)), \qquad x_0 \in \mathbb{C}^n. \tag{5.14}$$

Proof. Since $k_B(A) < 0$ we have $x(\cdot) \in L_2(t_0, n)$ and thus $x(t) \to 0$ as $t \to \infty$. Now

$$\frac{d}{dt}\langle x(t), Q(t)\, x(t)\rangle$$
$$= \rho\, \| C(t)\, x(t)\|^2 + \| B^*(t)\, Q(t)\, x(t)\|^2 + 2\,\mathrm{Re}\langle B(t)\, u(t), Q(t)\, x(t)\rangle$$
$$= \rho\, \| C(t)\, x(t)\|^2 + \| u(t) + B^*(t)\, Q(t)\, x(t)\|^2 - \| u(t)\|^2.$$

Integrating over $[t_0, t_1]$ and taking limits as $t_1 \to \infty$ yields (5.13). Since (5.13) holds for all $u(\cdot) \in L_2(t_0, m)$, (5.14) follows. ▮

The above lemma yields immediately the following necessary condition for the existence of bounded Hermitian solutions of $(\mathrm{DRE})_\rho$.

PROPOSITION 5.10. *Suppose* (4.2) *and* $t_0 \geqslant 0$. *If* $Q(\cdot) \in PC^1(t_0, \infty; \mathbb{C}^{n\times n})$ *is a bounded Hermitian solution of* $(\mathrm{DRE})_\rho$ *on* $[t_0, \infty)$ *then*

$$\rho \leqslant \| L_{t_0}^{\Sigma}\|^{-2}. \tag{5.15}$$

Proof. By (5.14), $0 \leqslant J_\rho(0, [t_0, \infty), u(\cdot))$ for all $u \in L_2(t_0, m)$. This implies (5.15) by Lemma 5.5(i). ▮

The following converse result is the main theorem of this section.

THEOREM 5.11. *Suppose* (4.2), $\Sigma = (A, B, C)$, *and* $\rho < \| L_{t_0}^{\Sigma}\|^{-2}$, $t_0 \geqslant 0$. *Then we have*

(i) *There exists a unique stabilizing* (*i.e.*, $A - BB^*P^+$ *is exponentially stable*) *bounded Hermitian solution* $P^+(\cdot) \in PC^1(t_0, \infty; \mathbb{C}^{n\times n})$ *of* $(\mathrm{DRE})_\rho$ *on* $[t_0, \infty)$.

(ii) P^+ *is maximal in the sense that, for any bounded Hermitian solution*
$Q(\cdot) \in PC^1(t_0', \infty; \mathbb{C}^{n\times n})$ on $[t_0', \infty)$, $t_0' \geqslant t_0$,

$$Q(t) \leqslant P^+(t) \qquad \textit{for all} \quad t \geqslant t_0'.$$

(iii) *The minimal cost is*

$$\inf_{u \in L_2(t_0, m)} J_\rho(x_0, [t_0, \infty), u(\cdot)) = \langle x_0, P^+(t_0)\, x_0\rangle \tag{5.16}$$

and the optimal control is given by

$$u(t) = -B^*(t)\, P^+(t)\, x(t), \qquad t \geq t_0, \tag{5.17}$$

where $x(\cdot)$ *solves*

$$\dot{x}(t) = [A(t) - B(t)\, B^*(t)\, P^+(t)]\, x(t), \qquad t \geq t_0,\ x(t_0) = x_0. \tag{5.18}$$

Proof. First, let $\rho \geq 0$. By Lemma 5.5 and Theorem 5.7 there exists $c_\rho > 0$ such that for all $t_1 > t_0$, $t \in [t_0, t_1]$

$$\begin{aligned} -c_\rho \|x_0\|^2 &\leq \inf_{u \in L_2(t, m)} J_\rho(x_0, [t, \infty), u(\cdot)) \\ &\leq \inf_{u \in L_2(t, t_1; \mathbb{C}^m)} J_\rho(x_0, [t, t_1), u(\cdot)) \qquad (5.19)\\ &= \langle x_0, P^{t_1}(t)\, x_0 \rangle. \end{aligned}$$

Thus $P^{t_1}(t)$ is bounded below and since by Corollary 5.8 it is monotonically decreasing we have that

$$P^+(t) = \lim_{t_1 \to \infty} P^{t_1}(t) \tag{5.20}$$

exists for all $t \in [t_0, \infty)$. In an analogous way, existence of the limit (5.20) can be proved for the case $\rho < 0$.

In both cases, $P^{t_1}(\cdot)$ satisfies

$$P^{t_1}(t) = P^{t_1}(t_0) - \int_{t_0}^{t} [A^*(s)\, P^{t_1}(s) + P^{t_1}(s)\, A(s) - \rho C^*(s)\, C(s) - P^{t_1}(s)\, B(s)\, B^*(s)\, P^{t_1}(s)]\, ds.$$

Taking limits (as $t_1 \to \infty$) yields

$$P^+(t) = P^+(t_0) - \int_{t_0}^{t} [A^*(s)\, P^+(s) + P^+(s)\, A(s) - \rho C^*(s)\, C(s) - P^+(s)\, B(s)\, B^*(s)\, P^+(s)]\, ds$$

and differentiation shows that $P^+(\cdot) \in PC^1(t_0, \infty; \mathbb{C}^{n \times n})$ is a bounded Hermitian solution of $(\mathrm{DRE})_\rho$ on $[t_0, \infty)$.

Before showing that $P^+(\cdot)$ is stabilizing we prove (iii).

If $Q(\cdot) \in PC^1(t_0, \infty; \mathbb{C}^{n \times n})$ is a bounded Hermitian solution of $(\mathrm{DRE})_\rho$ and $A(\cdot) - B(\cdot)\, B^*(\cdot)\, Q(\cdot)$ is the generator of $\Phi_Q(\cdot, \cdot)$, then

$$\begin{aligned} &\frac{d}{ds} [\Phi_Q^*(s, t_0)\, Q(s)\, \Phi_Q(s, t_0)] \\ &\quad = \Phi_Q^*(s, t_0)[\rho C^*(s)\, C(s) - Q(s)\, B(s)\, B^*(s)\, Q(s)]\, \Phi_Q(s, t_0). \end{aligned}$$

Hence

$$\begin{aligned}&\langle x_0, Q(t_0)\, x_0\rangle \\ &\quad = \langle \Phi_Q(t, t_0)\, x_0, Q(t)\, \Phi_Q(t, t_0)\, x_0\rangle + \int_{t_0}^{t} \langle \Phi_Q(s, t_0)\, x_0, \\ &\qquad [Q(s)\, B(s)\, B^*(s)\, Q(s) - \rho C^*(s)\, C(s)]\, \Phi_Q(s, t_0)\, x_0 > ds \qquad (5.21)\end{aligned}$$

First we consider the case $\rho \leqslant 0$ for which $P^+(t) \geqslant 0$, $t \geqslant t_0$. The above equality with $Q(\cdot) = P^+(\cdot)$ yields

$$\langle x_0, P^+(t_0)\, x_0\rangle \geqslant J_\rho\, (x_0, [t_0, \infty), -B^*(\cdot)\, P^+(\cdot)\, \Phi_{P^+}(\cdot, t_0)\, x_0).$$

In particular $\hat{u}(\cdot) := -B^*(\cdot)\, P^+(\cdot)\, \Phi_{P^+}(\cdot, t_0)\, x_0 \in L_2(t_0, m)$ and applying (5.14) with $Q(\cdot) = P^+(\cdot)$ we find

$$\inf_{u \in L_2(t_0, m)} J_\rho(x_0, [t_0, \infty), u(\cdot)) = J_\rho(x_0, [t_0, \infty), \hat{u}(\cdot)) = \langle x_0, P^+(t_0)\, x_0\rangle.$$

The case $\rho > 0$ is more difficult. To do this we extend the finite time optimal control by 0 to $[t_0, \infty)$ and define $u_{t_1}(\cdot) \in L_2(t_0, m)$ by

$$u_{t_1}(t) = \begin{cases} -B^*(t)\, P^{t_1} x_{t_1}(t) & \text{for} \quad t_0 \leqslant t \leqslant t_1 \\ 0 & \text{for} \quad t_1 < t, \end{cases}$$

where $x_{t_1}(\cdot)$ solves

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u_{t_1}(t), \qquad t \geqslant t_0,\ x(t_0) = x_0.$$

Then by Theorem 5.7

$$\begin{aligned}&J_\rho(x_0, [t_0, \infty), u_{t_1}(\cdot)) \\ &\quad = \int_{t_0}^{t_1} [\| u_{t_1}(s)\|^2 - \rho\, \| C(s)\, x_{t_1}(s)\|^2]\, ds - \int_{t_1}^{\infty} \rho\, \| C(s)\, x_{t_1}(s)\|^2\, ds \\ &\quad = \langle x_0, P^{t_1}(t_0)\, x_0\rangle - \rho \int_{t_1}^{\infty} \| C(s)\, x_{t_1}(s)\|^2\, ds.\end{aligned}$$

By applying (5.14) to $P^+(\cdot)$ we obtain

$$J_\rho(x_0, [t_0, \infty), u(\cdot)) \geqslant \langle x_0, P^+(t_0)\, x_0\rangle \qquad \text{for all} \quad u \in L_2(t_0, m) \qquad (5.23)$$

and so

$$\begin{aligned}&\lim_{t_1 \to \infty} \int_{t_1}^{\infty} \| C(s)\, x_{t_1}(s)\|^2\, ds = 0, \\ &\lim_{t_1 \to \infty} J_\rho(x_0, [t_0, \infty), u_{t_1}(\cdot)) = \langle x_0, P^+(t_0)\, x_0\rangle.\end{aligned} \qquad (5.24)$$

Now from (5.22) we have for every $\alpha > 0$

$$\begin{aligned} 0 &\geqslant \langle x_0, P^{t_1}(t_0)\, x_0 \rangle \\ &\geqslant J_\rho(x_0, [t_0, \infty), u_{t_1}(\cdot)) \\ &= \int_{t_0}^{\infty} [\| u_{t_1}(s)\|^2 - \rho \, \| C(s)\, \Phi(s, t_0)\, x_0 + (L_{t_0}^{\Sigma} u_{t_1})(s)\|^2]\, ds \\ &\geqslant (1 - \rho(1+\alpha)\| L_{t_0}^{\Sigma}\|^2) \cdot \| u_{t_1}\|^2_{L_2(t_0, m)} - \rho(1+\alpha^{-1})\| C(\cdot)\, \Phi(\cdot, t_0)\, x_0\|^2_{L_2(t_0, p)} \end{aligned} \tag{5.25}$$

by the same estimate as that used in establishing Lemma 5.5. Choosing $\alpha > 0$ small enough we see there exists a constant K independent of t_0, so that for all $t_0 \geqslant 0$

$$\| u_{t_1}\|^2_{L_2(t_0, m)} \leqslant K \| x_0 \|^2. \tag{5.26}$$

Hence $\{u_{t_1}, t_1 \geqslant t_0\}$ is bounded in $L_2(t_0, m)$, so there exists a sequence $(u_{t_k})_{k \in \mathbb{N}}$, $t_k \to \infty$ which converges weakly to some $\hat{u}(\cdot) \in L_2(t_0, m)$. By (5.23) and (5.24), (u_{t_k}) is a minimizing sequence. It is easy to see that J_ρ is strictly convex. Moreover it follows from the last inequality in (5.25)—which holds for arbitrary $u \in L_2(t_0, m)$ instead of u_{t_1}—that $u \mapsto J_\rho(x_0, [t_0, \infty), u(\cdot))$ is coercive. Hence, by [5, p. 35], $\hat{u}(\cdot)$ is the unique optimal control and the minimum cost is

$$J_\rho(x_0, [t_0, \infty), \hat{u}(\cdot)) = \langle x_0, P^+(t_0)\, x_0 \rangle.$$

Lemma 5.9, implies that for $Q(\cdot) = P^+(\cdot)$

$$J_\rho(x_0, [t_0, \infty), \hat{u}(\cdot)) = \int_{t_0}^{\infty} \| \hat{u}(s) + B^*(s)\, P^+(s)\, x(s)\|^2\, ds + \langle x_0, P^+(t_0)\, x_0 \rangle$$

and so

$$\hat{u}(t) = -B^*(t)\, P^+(t)\, x(t), \qquad t \geqslant t_0. \tag{5.27}$$

To prove uniqueness and maximality, assume that $Q(\cdot)$ is a bounded Hermitian solution of $(\mathrm{DRE})_\rho$ on $[t_0', \infty)$. Using Lemma 5.9 and (5.16) we obtain

$$\langle x_0, Q(t)\, x_0 \rangle \leqslant \inf_{u \in L_2(t, m)} J_\rho(x_0, [t, \infty), u(\cdot)) = \langle x_0, P^+(t)\, x_0 \rangle$$

for all $t \geqslant t_0'$ and all $x_0 \in \mathbb{C}^n$. Hence the maximality of $P^+(\cdot)$. Now assume that $Q(\cdot)$ is stabilizing; then for every $t \geqslant t_0'$ the feedback control $u(s) = -B^*(s)\, Q(s)\, x(s)$, $s \geqslant t$ is in $L_2(t, \infty; \mathbb{C}^m)$, and so by Lemma 5.9

$$J_\rho(x_0, [t, \infty), u(\cdot)) = \langle x_0, Q(t)\, x_0 \rangle \leqslant \langle x_0, P^+(t)\, x_0 \rangle.$$

Hence by (5.16) uniqueness holds.

To prove that the feedback system (5.18) is exponentially stable we note that by (5.16) and (5.21)

$$\|\hat{u}\|^2_{L_2(t_0,\, m)} \leqslant K \|x_0\|^2$$

for some constant K independent of $t_0 \geqslant 0$. Hence the solution $x(\cdot)$ of (5.18) satisfies $\|x(\cdot)\|^2_{L_2(t_0,\, n)} < \bar{K} \|x_0\|^2$, with $\bar{K}$ independent of t_0. The exponential stabilization then follows by Theorem 2.8. ▮

Remark 5.12. If the system Σ is uniformly observable and $\rho > 0$ ($\rho < 0$), then

$$P^+(t) < -\gamma I_n \qquad (P^+(t) > \gamma I_n) \tag{5.28}$$

for some $\gamma > 0$ and all $t \geqslant t_0$; see [8].

Proposition 5.10 and Theorem 5.11, together, imply the following characterization of $\|L^\Sigma_{t_0}\|$ in terms of the solvability of $(\text{DRE})_\rho$:

$$\|L^\Sigma_{t_0}\|^{-2} = \sup\{\rho \in \mathbb{R};\ (\text{DRE})_\rho \text{ has a bounded Hermitian solution on } [t, \infty)\}. \tag{5.29}$$

More precisely, if $\rho < \|L^\Sigma_{t_0}\|^{-2}$, then $(\text{DRE})_\rho$ possesses a bounded Hermitian solution on $[t_0, \infty)$, whereas for $\rho > \|L^\Sigma_{t_0}\|^{-2}$ there does not exist such a solution. However, there may exist solutions on some smaller interval $[t_0', \infty)$, $t_0' > t_0$.

COROLLARY 5.13. *Suppose* (4.2); *then* $l(A; B, C)^2$ *is the supremum of all* $\rho \in \mathbb{R}$ *for which there exists a bounded Hermitian solution of* $(\text{DRE})_\rho$ *on some interval* $[t_0, \infty)$, $t_0 > 0$.

Remark 5.14. The above results are not applicable to the limiting parameter value $\rho^* = \|L^\Sigma_{t_0}\|^{-2}$ (resp. $\rho^* = l(A; B, C)^2$). In the time-invariant case it is known that $(\text{ARE})_\rho$ has a Hermitian solution for $\rho^* = \|L_0\|^{-2}$ but the corresponding closed loop system is no longer exponentially stable and there may not be a solution of the corresponding optimal control problem $(\text{OCP})_{\rho^*}$; see [10]. So the differential Riccati equation $(\text{DRE})_\rho$ and the optimal control problem $(\text{OCP})_\rho$ are decoupled at the parameter value $\rho^* = \|L_0\|^{-2}$.

Remark 5.15. In [8] we have shown that if Σ is uniformly controllable and the conditions of Theorem 5.11 are satisfied, then there exists a solution $P^-(\cdot)$ of $(\text{DRE})_\rho$ on $[t_0 + \sigma, \infty)$ for some $\sigma > 0$ such that the closed loop system $\dot{x}(t) = [A(t) - B(t)\, B^*(t)\, P^-(t)]\, x(t)$ is *completely unstable* (i.e., the adjoint system $\dot{x}(t) = -[A(t) - B(t)\, B^*(t)\, P^-(t)]^*\ x(t)$ is exponentially stable). However, in contrast to the time-invariant case, $P^-(\cdot)$ will *not* in general be a minimal solution of $(\text{DRE})_\rho$ on $[t_0 + \sigma, \infty)$.

Remark 5.16. If the assumptions of Theorem 5.11 are fulfilled then for each $t \geqslant t_0$ the map

$$\varphi(t, \cdot): (-\infty, \| L_{t_0} \|^{-2}) \to \mathbb{C}^{n \times n}$$
$$\rho \mapsto P_\rho^+(t)$$

is differentiable with respect to ρ and monotonically decreasing in ρ. Moreover, if Σ_ρ denotes the closed loop system obtained by applying the optimal feedback (5.17) then

$$\| L_{t_0}^{\Sigma_\rho} \|^{-2} = \| L_{t_0}^{\Sigma} \|^{-2} - \rho, \qquad \rho < \| L_{t_0}^{\Sigma} \|^{-2}.$$

This is proved in [8].

6. Nonlinear Perturbations and Robust Lyapunov Functions

In this section we briefly outline some consequences of the previous results for nonlinear perturbations of the form $\Delta(t) = B(t)\, N(C(t)\, x(t), t)$ so that the perturbed system is

$$\sum_N : \dot{x}(t) = A(t)\, x(t) + B(t)\, N(C(t)\, x(t), t), \qquad t \geqslant t_0,\ x(t_0) = x_0, \quad (6.1)$$

where (A, B, C) satisfies (4.2) and $N: \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}^m$ is continuously differentiable. We assume that $N(0, t) = 0$ so that 0 is an equilibrium state of (6.1). Our aim is to determine conditions on the "norm" of the nonlinear perturbation such that exponential stability of (6.1) is preserved.

For this, we have to consider the ε-modification of $(\mathrm{DRE})_\rho$

$$\dot{P}(t) + [A(t) + \varepsilon I_n]^*\, P(t) + P(t)[A(t) + \varepsilon I_n] - \rho C^*(t)\, C(t) - P(t)\, B(t)\, B^*(t)\, P(t) = 0. \quad (6.2)$$

Theorem 6.1. *Suppose* (4.2), $t_0 \geqslant 0$, *and*

$$\| N(y, t) \| \leqslant \gamma \| y \| \qquad \textit{for all} \quad t \geqslant t_0, y \in \mathbb{C}^p, \quad (6.3)$$

where $\gamma < \| L_{t_0}^{\Sigma} \|^{-1}$, $\Sigma = (A, B, C)$. *Then the origin is globally exponentially stable for the perturbed system* (6.1).

Proof. Chose $\rho \in (\gamma^2, \| L_{t_0}^{\Sigma} \|^{-2})$. One can show that for $\varepsilon > 0$ sufficiently small there exists a maximal bounded Hermitian solution of the differential Riccati equation (6.2) associated with $\Sigma^\varepsilon = (A + \varepsilon I_n, B, C)$. Consider the functional

$$V(t, x) = -\langle x, P_\rho^\varepsilon(t)\, x \rangle, \qquad t \geqslant t_0, x \in \mathbb{C}^n.$$

By Assumption (6.3), the solutions of (6.1) exist on $[t_0, \infty)$. The derivative of V along any solution $x(\cdot)$ of (6.1) is

$$\begin{aligned}\dot{V}(t, x(t)) &= -2\varepsilon V(t, x(t)) - \rho \, \|C(t)\, x(t)\|^2 - \|B^*(t)\, P^\varepsilon_\rho(t)\, x(t)\|^2 \\ &\quad - 2\,\mathrm{Re}\langle P^\varepsilon_\rho(t)\, x(t), B(t)\, N(C(t)\, x(t), t)\rangle \\ &= -2\varepsilon V(t, x(t)) - \|B^*(t)\, P^\varepsilon_\rho(t)\, x(t) + N(C(t)\, x(t), t)\|^2 \\ &\quad - [\rho \|C(t)\, x(t)\|^2 - \|N(C(t)\, x(t), t)\|^2].\end{aligned}$$

Hence

$$\dot{V}(t, x(t)) \leqslant -2\varepsilon V(t, x(t)) - \delta \, \|C(t)\, x(t)\|^2, \qquad t \geqslant t_0,$$

where $\delta = \rho - \gamma^2$. Integrating yields

$$V(t_1, x(t_1))\, e^{2\varepsilon t_1} - V(t_0, x(t_0))\, e^{2\varepsilon t_0} \leqslant -\delta \int_{t_0}^{t_1} e^{2\varepsilon t} \, \|C(t)\, x(t)\|^2 \, dt$$

for all $t_1 > t_0$ and since $V(t_1, x(t_1)) \geqslant 0$,

$$\int_{t_0}^{\infty} \varepsilon^{2\varepsilon(t - t_0)} \, \|C(t)\, x(t)\|^2 \, dt \leqslant -\delta^{-1} \, \langle x_0, P^\varepsilon_\rho(t_0)\, x_0\rangle. \tag{6.4}$$

Now if $A(\cdot)$ generates $\Phi(\cdot, \cdot)$,

$$\|x(t)\| \leqslant \|\Phi(t, t_0)\, x_0\| + \int_{t_0}^{t} \|\Phi(t, s)\, B(s)\, N(C(s)\, x(s), s)\| \, ds.$$

But there exist $M, \omega > 0$ such that $\|\Phi(t, s)\| \leqslant Me^{-\omega(t-s)}$, $t \geqslant s$. Hence

$$\begin{aligned}e^{\varepsilon(t - t_0)} \, \|x(t)\| &\leqslant Me^{-(\omega - \varepsilon)(t - t_0)} \, \|x_0\| \\ &\quad + \gamma \int_{t_0}^{t} M \, \|B\|_{L_\infty} \, e^{-(\omega - \varepsilon)(t - s)} \, e^{\varepsilon(s - t_0)} \, \|C(s)\, x(s)\| \, ds \\ &\leqslant Me^{-(\omega - \varepsilon)(t - t_0)} \, \|x_0\| + \gamma M \, \|B\|_{L_\infty} \\ &\quad \times \left[\int_{t_0}^{t} e^{-2(\omega - \varepsilon)(t - s)} \, ds\right]^{1/2} \left[\int_{t_0}^{t} e^{2\varepsilon(s - t_0)} \, \|C(s)\, x(s)\|^2 \, ds\right]^{1/2}.\end{aligned}$$

So, by (6.4), there exists a constant $K > 0$ such that

$$\|x(t)\| \leqslant Ke^{-\varepsilon(t - t_0)} \, \|x(t_0)\| \qquad \text{for all} \quad t \geqslant t_0 \geqslant 0.$$

This concludes the proof. ∎

The proof shows that $V(t, x) = -\langle x, P_\rho^\varepsilon(t)\, x\rangle$ is a *joint* Lyapunov function for all systems (6.1) satisfying (6.2) with $\gamma < \| L_{t0}^\Sigma \|^{-1}$. In the linear case one can choose $\varepsilon = 0$, i.e., $V(t, x) = -\langle x, P_\rho(t)\, x\rangle$.

A Lyapunov function could be called *of maximal robustness* with respect to perturbations of the structure $\Delta(t) = B(t)\, D(t)\, C(t)$ if it guarantees the exponential stability of *all* the perturbed systems $\sum_D$ with $\| D \|_{L_\infty} < r_{\mathbb{C}}(A; B, C)$. In the time-invariant case a Lyapunov function of maximal robustness can in fact be constructed using the maximal solution of the $(\text{ARE})_\rho$ with $\rho = r_{\mathbb{C}}^2(A; B, C)$; see [10]. The time-varying case is more complicated since $\| L_{t_0}^\Sigma \|^{-1}$ does not necessarily equal $r_{\mathbb{C}}(A; B, C)$.

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