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A SIMPLE ADAPTIVE OBSERVER FOR NONLINEAR SYSTEMS[1]

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Abstract: In this note we present a high-gain observer for nonlinear uniformly observable SISO systems for which the high-gain parameter is determined adaptively on-line. The adaptation scheme is simple and universal in the sense that it is independent of the system the observer is designed for. Unlike in an earlier approach, the gain is adapted continuously in the present paper. This further simplifies the adaptation law and also leads to lower values of the high-gain parameter. We prove that the observer output error becomes smaller than a user specified bound for large times and that the adaptation converges.

Keywords: nonlinear observers, adaptive, high-gain, robustness, bioreactors

1. INTRODUCTION

For nonlinear systems that are uniformly observable for any $u(\cdot)$ (i.e. the states of the system can be determined from the output of the system and its derivatives, independently of the input)(Gauthier and Bornard, 1981), a high-gain observer has been suggested in (Tornambè, 1992). One of the advantages of this observer are its excellent robustness properties (Tornambè, 1992). By choosing the observer gain k large enough (therefore the name "high-gain") the observer error can be made arbitrarily small. The difficulty in practical applications is, however, the determination of an appropriate value for the observer gain. For values too low, the desired bounds on the observer error cannot be achieved. For values unnecessarily high, the sensitivity to noise increases, thus limiting the practical use.

In this paper we propose an adaptation scheme for the observer gain of the high-gain observer in (Tornambè, 1992) such that its advantages are retained and the observer gain is adjusted automatically until the observer output error becomes smaller than a desired target value.

The note is organized as follows: In Section 2 we recall the main result on the non-adaptive high-gain observer in (Tornambè, 1992). In Section 3 we present the adaptation scheme and prove (i) convergence of the observer output error towards an arbitrary small but prespecified λ-ball around zero, (ii) the boundedness of the observer error, and (iii) convergence of the adaptation scheme. Finally in Section 4 the usefulness of the proposed adaptive observer is illustrated by applying it to a simple example of a bioreactor in simulation.

2. HIGH-GAIN OBSERVER

The theory of non-adaptive high-gain observers as in (Tornambè, 1992) assumes that the system is given in observability normal form (Zeitz, 1989), also called the generalized controller canonical

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form (Fliess, 1990). In principle, every uniformly observable, suffienciently smooth SISO-system with input u and output y can be transformed into this normal form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \phi(\mathbf{x}, \mathbf{u}) \\ y &= x_1, \end{aligned} \tag{1}$$

with $\mathbf{x} = [x_1 \cdots x_n]^T$, $\mathbf{u} = [u\ \dot{u}\ u^{(2)} \cdots u^{(n)}]^T$ and $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes some continuous function.

A possibility to estimate the states of such systems (1) is the high-gain observer (Tornambè, 1992). The structure of the high-gain observer is a simple chain of integrators, each "corrected" by the injection of the output error $(y - \hat{y})$ multiplied by a factor depending on the constant observer gain k:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + p_1\, k\, (y - \hat{y}) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + p_{n-1}\, k^{n-1}\, (y - \hat{y}) \\ \dot{\hat{x}}_n &= \phantom{\hat{x}_n +{}} p_n\, k^n\, (y - \hat{y}) \\ \hat{y} &= \hat{x}_1, \end{aligned} \tag{2}$$

where $\hat{\mathbf{x}} = [\hat{x}_1 \cdots \hat{x}_n]^T$ denotes the estimate of the state $\mathbf{x}$ and $\hat{y}$ the estimate of the system's output (cf. Figure 1).

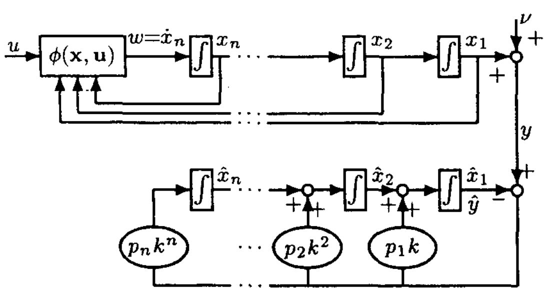


Fig. 1. High-gain observer for a system in observability normal form.

In contrast to the classical Luenberger observer (Luenberger, 1966), the high-gain observer does not consist of a replica of the system (1) plus correction terms as the nonlinearity $\phi(\mathbf{x}, \mathbf{u})$ is not modelled. The observer error will be denoted by

$$\begin{aligned} \mathbf{e}(t) &= [e_1(t), \dots, e_n(t)]^T \\ &= [x_1(t) - \hat{x}_1(t), \dots, x_n(t) - \hat{x}_n(t)]\,. \end{aligned}$$

The following theorem is proven in (Tornambè, 1992).

Theorem 1. (High-Gain Observer). *Assume that*

A1) the system (1) *exhibits no finite escape time and*

A2) the nonlinearity ϕ *in* (1) *is uniformly bounded, i.e. there exists some* $\mu > 0$ *such that* $||\phi(\mathbf{x}, \mathbf{u})|| \le \mu$ *for all* $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$.

If the coefficients $p_1, \dots, p_n \in \mathbb{R}$ *are such that* $s^n + \sum_{j=1}^{n} p_j s^{n-j}$ *is a Hurwitz polynomial with distinct roots, then for every* $d > 0$ *and every time* $\bar{t} > 0$ *there exists a finite observer gain* $\bar{k} = \bar{k}(d, \bar{t})$ *such that for all constant* $k \ge \bar{k}$ *the observer error satisfies:*

$$||\mathbf{e}(t)|| \le d \quad \textit{for all } t \ge \bar{t}.$$

This means that the observer error $\mathbf{e}$ can be made arbitrarily small in an arbitrarily short time by an appropriate choice (i.e. large enough) of the observer gain k.

In the Laplace domain, the relationship between $w(s) = \mathcal{L}\{\dot{x}_n(t)\}$, an additive output noise $\nu(s)$ (see Fig. 1), and the observer output error $e_1(s) = y(s) - \hat{y}(s)$ is given by:

$$e_1(s) = \frac{1}{(s+k)^n} w(s) + \frac{s^n}{(s+k)^n} \nu(s), \tag{3}$$

where, for simplicity, $p_i = \binom{n}{i}$ for $k = 1, \dots, n$. From (3) it is obvious that the larger the observer gain k is chosen, the smaller the influence w and thus of the nonlinearity ϕ (and therefore of u) on the observer output error. For a large observer gain, the observer is thus very robust, provided the dimension of the state-space is known (Tornambè, 1992). However, for large values of k the additive output noise is damped in the observer output error, and therefore undamped in the observer output and thus the observer is potemtially sensitive to noise.

3. ADAPTIVE HIGH-GAIN OBSERVER

To overcome the difficulty of having to choose the observer gain k, we propose the following simple adaptation law for k, where $e_1(t) = y(t) - \hat{y}(t)$:

$$\frac{d}{dt} k(t) = \gamma \cdot d^2_{\lambda,\hat{\lambda}}(e_1(t)), \qquad k(0) = k_0 \tag{4}$$

with

$$d_{\lambda,\hat{\lambda}}(e_1) = \begin{cases} \hat{\lambda} - \lambda & \text{for } |e_1| \ge \hat{\lambda} \\ |e_1| - \lambda & \text{for } \lambda \le |e_1| \le \hat{\lambda} \\ 0 & \text{for } |e_1| \le \lambda, \end{cases} \tag{5}$$

where $\hat{\lambda} > \lambda > 0$, $\gamma > 0$, are preassigned design parameters.

The idea behind this adaptation law is that the observer gain $k(t)$ is monotonically increasing (with a bounded derivative) as long as $|y(t) - \hat{y}(t)|$ lies outside the λ-strip $[0, \lambda]$ and $k(t)$ stops increasing as soon as $|y(t) - \hat{y}(t)|$ enters the λ-strip. The concept of introducing a dead-zone in the gain adaptation as in (4) and (5) is due to (Ilchmann and Ryan, 1994) where it was used for tracking.

In (Bullinger and Allgöwer, 1997), an adaptation law with amplitude discrete observer parameter was used instead of a continuous one. The advantage of the present continuous case is that there is no step size which has to increase quite rapidly (see also the comments in Section 4) and that the adaptation law is simpler.

We now prove that the high-gain observer (2) with a time-varying observer gain k determined by adaptation law (4), (5) guarantees convergence of the adaptation and boundedness of the observer error.

Theorem 2. Suppose that for any sufficiently smooth $u(\cdot)$ the function $\phi(\mathbf{x},\mathbf{u})$ in (1) is essentially bounded. Then the high-gain observer (2) with the adaptation law (4), (5) achieves, for any initial condition $\mathbf{x}(0) \in \mathbb{R}^n$, any coefficients $p_1, \dots, p_n \in \mathbb{R}$ such that $s^n + \sum_{j=1}^n p_j s^{n-j}$ is a Hurwitz polynomial, and with arbitrary design parameters $\hat{\lambda} > \lambda > 0$, $\gamma > 0$, $k(0) > 0$, $\hat{\mathbf{x}}(0) \in \mathbb{R}^n$ a nonlinear system (1), (2), (4), (5) where every solution exists on the whole of $[0,\infty)$ and

a) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$,

b) $\mathbf{e}(\cdot) \in L_\infty(0,\infty)$,

c) $\lim_{t\to\infty} \operatorname{dist}(e_1(t), [-\lambda,\lambda]) = 0$.

Theorem 2 states that the observer parameter $k(t)$ converges to a finite value while the observer output error $e_1(t)$ approaches the λ-strip $[-\lambda,\lambda]$ asymptotically.

Proof of a) The error differential equations are

$$\begin{aligned} \dot e_1 &= e_2 && - p_1 k\, e_1 \\ &\vdots \\ \dot e_{n-1} &= e_n && - p_{n-1}k^{n-1}\, e_1 \\ \dot e_n &= \phi(\mathbf{x},\mathbf{u}) && - p_n k^n\, e_1 . \end{aligned}$$

respectively

$$\begin{aligned} \dot{\mathbf{e}}(t) &= A_{k(t)}\mathbf{e}(t) + B\phi(\mathbf{x}(t),\mathbf{u}(t)), \\ \mathbf{e}(0) &= \mathbf{x}(0) - \hat{\mathbf{x}}(0) \end{aligned} \tag{6}$$

where

$$A_{k(t)} = k(t)\cdot K(t)\bar{A}K(t)^{-1}$$

$$\bar{A} = \begin{bmatrix} -p_1 & 1 & 0 & & \\ -p_2 & 0 & 1 & & \\ \vdots & & & \ddots & \\ -p_{n-1} & 0 & & & 1 \\ -p_n & 0 & & & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$K(t) = \operatorname{diag}\left\{1 \; k(t) \; \dots \; k(t)^{n-1}\right\}.$$

It follows from the theory of ordinary differential equations, that the initial value problem (4), (5), (6) possesses an absolutely continuous solution $(\mathbf{e}(\cdot), k(\cdot))\colon [0,\omega) \to \mathbb{R}^{n+1}$, maximally extended over $[0,\omega)$ for some $\omega \in (0,\infty]$. Seeking a contradiction to a), we suppose

$$\lim_{t\to\omega} k(t) = \infty.$$

Since $s^n + \sum_{j=1}^n p_j s^{n-j}$ is a Hurwitz polynomial, $\bar{A}$ is a Hurwitz matrix, and hence there exists a symmetric, positive definite matrix $\bar{P} \in \mathbb{R}^{n\times n}$ such that

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} = -I. \tag{7}$$

Set

$$P(t) = K(t)^{-1}\bar{P}K(t)^{-1}$$

which is symmetric and non-singular for every $k(t)$, and hence:

$$\begin{aligned} A_k^T P + PA_k &= \left(k\cdot K^{-1}\bar{A}^T K\right)\left(K^{-1}\bar{P}K^{-1}\right) \\ &\quad + \left(K^{-1}\bar{P}K^{-1}\right)\left(k\cdot K\bar{A}K^{-1}\right) \\ &= k\cdot K^{-1}\left(\bar{A}^T\bar{P} + \bar{P}\bar{A}\right)K^{-1} \\ &= -k\cdot K^{-2}, \end{aligned}$$

where, as in the following, the argument t is omitted.

Differentiating the Lyapunov function candidate

$$V(t,\mathbf{e}) = D^2(t,\mathbf{e}),$$

where

$$D(t,\mathbf{e}) = \begin{cases} \|\mathbf{e}\|_{P(t)} - \rho, & \text{for } \|\mathbf{e}\|_{P(t)} \geq \rho \\ 0, & \text{for } \|\mathbf{e}\|_{P(t)} \leq \rho, \end{cases}$$

$$\rho = \frac{\lambda}{2}\sqrt{\|\bar{P}^{-1}\|}, \text{ and } \|\mathbf{e}\|_{P(t)} = \sqrt{\mathbf{e}^T P(t)\mathbf{e}}$$

along the trajectory of (3) yields, for all $t \in [0,\omega)$,

$$\begin{aligned} \frac{d}{dt}V &= 2D(\mathbf{e})\cdot\frac{d}{dt}D(\mathbf{e}) = 2D(\mathbf{e})\frac{1}{2}\frac{\frac{d}{dt}\left(\mathbf{e}^T P\mathbf{e}\right)}{\|\mathbf{e}\|_{P(t)}} \\ &= \frac{D(\mathbf{e})}{\|\mathbf{e}\|_{P(t)}}\left(\frac{d}{dt}\mathbf{e}^T P\mathbf{e} + \mathbf{e}^T P\frac{d}{dt}\mathbf{e} + \mathbf{e}^T\frac{d}{dt}P\mathbf{e}\right) \\ &= \frac{D(\mathbf{e})}{\|\mathbf{e}\|_{P(t)}}\Bigg(\mathbf{e}^T\left(A_k^T P + PA_k\right)\mathbf{e} \\ &\quad + 2\mathbf{e}^T K^{-1}\bar{P}K^{-1}B\phi(\mathbf{x},\mathbf{u}) \\ &\quad + 2\mathbf{e}^T\left(\frac{d}{dt}K^{-1}\right)\bar{P}K^{-1}\mathbf{e}\Bigg) \\ &\leq \frac{D(\mathbf{e})}{\|\mathbf{e}\|_{P(t)}}\Bigg(-k\,\|\eta\|^2 + 2\,\|\tilde{K}\|\,\|\bar{P}\|\,\|\eta\|^2 \\ &\quad + 2\,\|\bar{P}\|\,\|K^{-1}B\|\,|\phi(\mathbf{x},\mathbf{u})|\,\|\eta\|\Bigg), \end{aligned} \tag{8}$$

where $\eta := K^{-1}\mathbf{e}$ and

$$\tilde{K} := K\frac{d}{dt}\left(K^{-1}\right) = \frac{\dot{k}}{k}\cdot\operatorname{diag}\{0,1,\dots,n-1\}.$$

By assumption there exists some $\mu > 0$ such that $\|\phi(\mathbf{x}(t),\mathbf{u}(t))\|_{L_\infty} \leq \mu$ for almost all $t \in [0,\omega)$, then substituting $\|\tilde{K}\| \leq n\frac{\dot{k}}{k}$ into (8) yields,

$$\begin{aligned} \frac{d}{dt}V \leq \frac{D(\mathbf{e})}{\|\mathbf{e}\|_{P(t)}}\Bigg(\left(-k + 2n\,\frac{\dot{k}}{k}\,\|\bar{P}\|\right)\|\eta\|^2 \\ + \frac{2}{k^{n-1}}\,\mu\,\|\bar{P}\|\,\|\eta\|\Bigg). \end{aligned} \tag{9}$$

By (4) and (5) $\dot{k}$ is bounded. Choosing $t_1 \in [0, \omega)$ such that

$$-k(t_1) + 2n\, \frac{\gamma(\hat{\lambda} - \lambda)^2}{k(t_1)}\, \|\bar{P}\| = -\frac{k(t_1)}{2}$$

yields by monotonicity of $t \mapsto k(t)$, for almost all $t \in [t_1, \omega)$,

$$\frac{d}{dt}V \le \frac{D(\mathbf{e})}{\|\mathbf{e}\|_{P(t)}} \Bigg(-\frac{k}{2}\|\eta\|^2 + \frac{2\,\mu}{k^{n-1}}\|\eta\|\,\|\bar{P}\| + \left(\|\eta\| - \frac{\mu}{k^{n-1}}\right)^2 \|\bar{P}\| \Bigg).$$

Choosing $t_2 \in [t_1, \omega)$ such that

$$\|\eta(t_2)\|^2\, \|\bar{P}\| = \frac{k(t_2)}{4}\|\eta(t_2)\|^2$$

yields for almost all $t \in [t_2, \omega)$,

$$\frac{d}{dt}V \le -\frac{D(\mathbf{e})}{\|\mathbf{e}\|_{P(t)}} \left(\frac{k}{4}\|\eta\|^2 - \frac{\mu^2\, \|\bar{P}\|}{k(t_2)^{2n-2}} \right). \quad (10)$$

As $\|\mathbf{e}\|^2_{P(t)} = \left(K^{-1}\mathbf{e}\right)^T \bar{P} K^{-1}\mathbf{e}$,

$$\|\bar{P}^{-1}\|^{-1}\, \|\eta\|^2 \le \|\mathbf{e}\|^2_{P(t)} \le \|\bar{P}\|\, \|\eta\|^2. \quad (11)$$

Substituting (11) into (10) yields

$$\frac{d}{dt}V \le -\frac{kV}{4\,\|\bar{P}\|} + \frac{\mu^2\, \|\bar{P}\|}{k(t_2)^{2n-2}}.$$

Choosing $t_3 \in [t_2, \omega)$ such that

$$k(t_3) = \frac{4\mu^2 \|\bar{P}\|^2 \|\bar{P}^{-1}\|}{\left(\frac{3}{4}\lambda\right)^2 k(t_2)^{2n-2}} \quad (12)$$

yields for almost all $t \in [t_3, \omega)$

$$\frac{d}{dt}V \le -\frac{k(t_3)V}{4\,\|\bar{P}\|} + \frac{\mu^2\, \|\bar{P}\|}{k(t_2)^{2n-2}}.$$

Therefore, for all $t \in [t_3, \omega)$,

$$V(t, \mathbf{e}(t)) \le e^{-\frac{k(t_3)}{4\,\|\bar{P}\|}(t - t_3)} V(t_3, \mathbf{e}(t_3)) + \frac{4\|\bar{P}\|^2 \mu^2}{k(t_3) k(t_2)^{2n-2}}. \quad (13)$$

If $\omega < \infty$, then (13) yields $\mathbf{e}(\cdot) \in L_\infty(0, \omega)$ and hence by (4) and (5), $k(\cdot) \in L_\infty(0, \omega)$. If $\omega = \infty$, then by (13),

$$\lim_{t \to \infty} \operatorname{dist}\left(V, \left[0, \left(\frac{3}{4}\lambda\right)^2 \|\bar{P}^{-1}\|^{-1}\right] \right) = 0.$$

$V \le \left(\frac{3}{4}\lambda\right)^2 \|\bar{P}^{-1}\|^{-1}$ implies by (11) that $\|\eta\| \le \frac{3}{4}\lambda$. Whence since $|e_1(t)| \le \|\eta(t)\|$ the dead-zone in the gain adaptation yields that $k(\cdot) \in L_\infty(0, \infty)$. This contradicts by (4) and (5) unboundedness of $k(\cdot)$, thus proving a).

Proof of b) We show that $\mathbf{e}(\cdot) \in L_\infty$. Recall that $\bar{A}$ was chosen to be Hurwitz and hence in particular $A_k = kK\bar{A}K^{-1}$ for $k = k_\infty$ is Hurwitz. With

$$\Delta(t) := A_{k(t)} - A_{k_\infty}$$

we might rewrite (6) as

$$\frac{d}{dt}\mathbf{e}(t) = \left(A_{k_\infty} + \Delta(t)\right) \mathbf{e}(t) + B\phi(\mathbf{x(t)}, \mathbf{u}(t)).$$

Since A_{k_∞} is Hurwitz and $\lim_{t \to \infty} \Delta(t) = 0$, it follows that $\dot{\eta} = [A_{k_\infty} + \Delta(t)]\,\eta$ is exponentially stable (see e.g. Theorem 8.6. in (Rugh, 1993)) and hence uniform boundedness of $B\phi(\mathbf{x}, \mathbf{u})$ yields boundedness of e (see e.g. Corollary 6.1 in (Khalil, 1996)).

As $k(\cdot)$ and $\mathbf{e}(\cdot)$ are both bounded, there does not exist a finite escape time, i.e. $\omega = \infty$, and hence b) is established.

Proof of c) We first prove that $t \mapsto \frac{d}{dt} d_{\lambda, \hat{\lambda}}(\mathbf{e}(t))$ is absolutely continuous on any compact interval. Since $t \mapsto \mathbf{e}(t)$ as the solution of (6) and $\mathbf{e} \mapsto d_{\lambda,\hat{\lambda}}(\mathbf{e})$ are trivially absolutely continuous, the composition $t \mapsto d_{\lambda,\hat{\lambda}}(\mathbf{e}(t))$ is absolutely continuous if it is of bounded variation (see e.g. (Hewitt and Stromberg, 1965) p. 297). The latter is easy to see and a proof is omitted. Therefore, $d_{\lambda,\hat{\lambda}}(\mathbf{e}(\cdot))$ is differentiable almost everywhere and we obtain, for almost all $t \ge 0$,

$$\frac{d}{dt} d_{\lambda,\hat{\lambda}}(\mathbf{e}(t)) \le |\dot{e}_1(t)|.$$

Since the right hand side of (6) is bounded, we may conclude that $\frac{d}{dt} d_{\lambda,\hat{\lambda}}(\mathbf{e}) \in L_\infty(0, \infty)$ and since $k \in L_\infty(0, \infty)$ is equivalent to $d_{\lambda,\hat{\lambda}}(\mathbf{e}) \in L_2(0, \infty)$ it follows from Lemma 2.1.7 in (Ilchmann, 1993) that $\lim_{t \to \infty} d_{\lambda,\hat{\lambda}}(t) = 0$ and hence

$$\lim_{t \to \infty} \operatorname{dist}\left(|y(t) - \hat{y}(t)|, [0, \lambda]\right) = 0.$$

This completes the proof. □

In Theorem 2 we were mainly interested in the observer *output* error approaching the λ-strip. This is the essential property needed for many observer-based control schemes, as for example the one in (Groebel *et al.*, 1995). We now derive an additional bound on the observer (state) error $\mathbf{e}(\cdot)$.

Theorem 3. Under the same assumptions as in Theorem 2 and $\mu > 0$ so that $\|\phi(\mathbf{x}(t), \mathbf{u}(t))\| \le \mu$ for all $t \ge 0$, the observer errors satisfy, for $i = 1, \dots, n$,

$$\limsup_{t \to \infty} |e_i(t)| \le \frac{2\mu \|\bar{P}\|}{k^{n+1-i}(0)}, \quad (14)$$

with $\|\bar{P}\|$ as in (7).

Sketch of the proof for Theorem 3: We use the same notation as in the proof of Theorem 2.

Since $k(\cdot)$ is monotone and converges, for arbitrary small but fixed $\epsilon \in (0,1)$ there exists some $\bar{t} \geq 0$ such that

$$2n\|\bar{P}\|\dot{k}(t) < \epsilon, \quad \text{for all } t \geq \bar{t}.$$

For arbitrary $\rho > 0$ (9) yields, for all $t \geq \bar{t}$,

$$\frac{d}{dt}V \leq -\frac{D(\mathrm{e})}{\|\mathrm{e}\|_{P(t)}}\left[\left(k - \frac{\epsilon}{k}\right)\|\eta\| - \frac{2\mu}{k^{n-1}}\|\bar{P}\|\right]\|\eta\|,$$

and therefore $\frac{d}{dt}V(t, \mathrm{e}(t)) < 0$ for $t \geq \bar{t}$, if

$$\left(k(t) - \frac{\epsilon}{k(t)}\right)\|\eta(t)\| > \hat{\epsilon} := \frac{2\mu\|\bar{P}\|}{k^{n-1}(0)}, \|\eta(t)\|_{\bar{P}} \geq \rho.$$

The latter is satisfied for $t \geq \bar{t}$, if

$$\|\eta(t)\|_{\bar{P}} \geq \|\eta(t)\|\sqrt{\|\bar{P}^{-1}\|} > \max\left\{\frac{\hat{\epsilon}\sqrt{\|\bar{P}^{-1}\|}}{k(t) - \frac{\epsilon}{k(t)}}, \rho\right\}.$$

Thus,

$$\limsup_{t\to\infty}\|\eta(t)\|_{\bar{P}} \leq \sqrt{\|\bar{P}^{-1}\|}\max\left\{\frac{\hat{\epsilon}}{k(0)}, \rho\right\}.$$

Using that $\rho > 0$ is arbitrary, (14) follows. □

Remark 4. The idea of the proof of Theorem 2 Part a) can be used to generalize the results in (Tornambè, 1992) to the case of non distinct roots. Furthermore, the same arguments as above can be used to show that no finite escape time can occur in the *non-adaptive* high-gain case either, and thus Assumption (A1) in Thm. 1 (Tornambè, 1992) can be removed.

The proposed adaptive high-gain observer is easy to implement (as only the state-space dimension of the system has to be known) and retains the advantages of the non-adaptive high-gain observer. Robustness is improved by the adaptation law as it enables the user to start with a small observer gain that is increased only as needed. In a non-adaptive scheme the observer gain is usually chosen in a conservative way, which causes the high-gain observer to be less performant in the presence of output measurement noise than the adaptive high-gain observer.

4. EXAMPLE

To demonstrate the adaptive high-gain observer, the proposed method is applied to the following generic bioreactor as given in (Bastin and Dochain, 1990) with parameters as in (Gauthier *et al.*, 1992):

$$\begin{aligned} \dot{m} &= \frac{a_1 ms}{a_2 m + s} - u\, m \\ \dot{s} &= -\frac{a_3 a_1 ms}{a_2 m + s} - u\, s + u\, a_4 \\ y &= m, \end{aligned} \tag{15}$$

where m and s denote the concentrations of the microorganism and the substrate respectively, u is the substrate inflow rate which is considered as input. All state variables are strictly positive and the parameters are $a_1 = a_2 = a_3 = 1$, $a_4 = 0.1$, $m(0) = 0.075$, $s(0) = 0.03$.

The system (15) can easily be transformed into the observability normal form by defining the new state variables x_1, x_2 as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Phi(m, s, u, \dot{u}) = \begin{bmatrix} m \\ \dot{m} \end{bmatrix}. \tag{16}$$

In (Gauthier *et al.*, 1992) it has been shown that assumptions A1) and A2) are satisfied for $u \in (0, a_1)$. For the observer, the following values are used: $\lambda = 0.02$, $\hat{\lambda} = 0.5$, $\gamma = 100$, $p_1 = 1$, $p_2 = 0.2$, $k_0 = 0.1$. The following substrate input flow profile is used for the simulations:

$$u(t) = \begin{cases} 0.08\,h^{-1} & t \in [0, 30)h \\ 0.02\,h^{-1} & t \in [30, 50)h \\ 0.08\,h^{-1} & t \geq 50h. \end{cases}$$

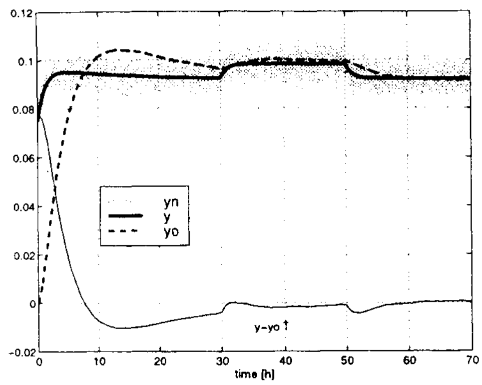
y is the plant output without, yn with noise, yo is the observer output, so is the estimated value for s, calculated via $\Phi^{-1}(x_{o_1}, x_{o_2}, u, \dot{u})$, where x_{o_1}, x_{o_2} are the states of the observer.

For the simulation (Figure 2) band-limited gaussian white noise with a rather high power spectral density of $0.25 \cdot 10^{-6}$ and a sampling time of 0.01 h is used. In Figure 2.a the bioreactor output y (the concentration of the microorganism) and its estimate y_o can be seen. After the transient phase due to different initial conditions in system and observer, the observer output follows very well the plant output, even though the noise level is quite high and the fact that there are changes in u. Exept in the first few hours, the output error $y - y_o$ stays within the λ-strip. In Figure 2.b the concentration of the substrate s and its observer estimate s_o are shown. Also here the observer error is rather small while the discontinuities in s_o are due to the fact that s_o is a function of the substrate input flow u. Figure 2.c shows that k increases rapidly (because of the large value γ) and then stays constant at a value that is not "high" as the name of the observer implies.

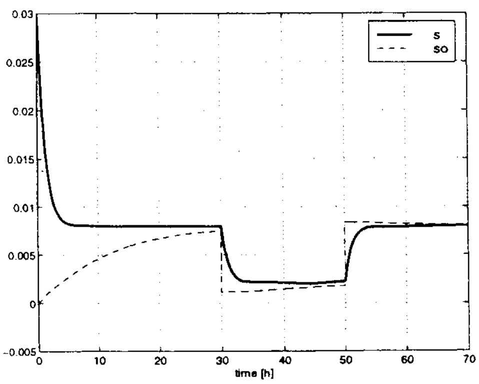
Note that the observer gain with the amplitude continuous adaptation is about a factor two lower than with the amplitude discrete adaptation as in (Bullinger and Allgöwer, 1997), where the final value for the observer gain was about 0.55 with the same setup, only the adaptation law being different.

CONCLUSIONS

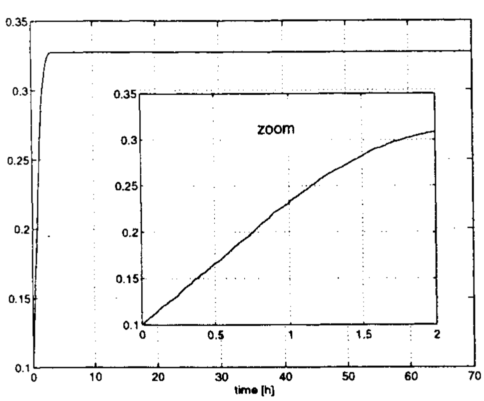
In this paper we introduce an adaptive extension to the high-gain observer originally proposed in (Nicosia and Tornambè, 1989). The adaptation of the observer gain is done via a very simple



(a) "True" (y), "true" with additive noise (yn), and estimated (yo) concentration of microorganism and observer output error $y - y_o$.



(b) Estimated (so) and "true" (s) concentration of substrate.



(c) Observer gain k.

Fig. 2. State-estimation of the bioreactor with noise.

adaptation law that is universal in the sense that it is independent of the system to be observed. Thus, this adaptation fits nicely into the concept of high-gain observation. We have proved that with the adaptive high-gain observer the observer output error becomes smaller than an arbitrary user-specified bound for large times and that the adaptation converges. The assumptions needed are the same as required for the non-adaptive high-gain observer.

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