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ADAPTIVE HIGH-GAIN FAST-SAMPLING P-CONTROL *

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Abstract

In this paper we show that well-known high-gain universal adaptive P-controllers can be implemented digitally, via adaptive sampling, provided that the length of the sampling interval increases sufficiently fast, as the proportional gain increases. Both stabilization and λ–tracking of arbitrary bounded and essentially smooth reference signals are considered.

1 Introduction

In this paper we will show that the ideas and techniques of high–gain adaptive output feedback control carry over to a more practically relevant situation where the output of the system is not available continuously, but is only available at sampled instants of time. This situation arises naturally in cases when digital computations of control inputs are used.

It is well-known (see Willems and Byrnes (1984)) that

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = y^2(t)$$

is a continuous-time, high-gain adaptive controller for a class of systems known as minimum-phase, positive high-frequency gain systems. This controller arose from the work of Nussbaum (1983) and Morse (1983) and has been developed by, for example, Mårtensson (1986) and Ilchmann (1993). All of these papers are similar in spirit in the sense that the adaptation of the controller gain is not based on any attempt to identify the parameters of the system. This paper continues in this spirit.

We focus on adaptive control of minimum-phase, multi–output systems with unknown dimension and matrix entries, with the spectrum of the high–frequency gain unmixed. More precisely, let the system to be controlled be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

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where $A \in \mathbb{R}^{n\times n}$, B, $C^T \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$ and n are unknown.

We assume that (1.1) satisfies certain qualitative structural assumptions, namely that (1.1) is *minimum phase*, i.e.

$$\det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \forall\, s \in \overline{\mathbb{C}_+}, \tag{1.2}$$

and has "high–frequency gain" CB satisfying either

$$\sigma(CB) \subset \mathbb{C}_+ \tag{1.3}$$

or, more generally,

$$\sigma(sCB) \subset \mathbb{C}_+ \text{ for some } s \in \{-1, 1\}, \tag{1.4}$$

i.e. the spectrum of CB is unmixed. These two qualitative properties of minimum-phase and unmixed high-frequency gain systems are well-known in the control engineering literature. Both properties can often be tested without using detailed knowledge of the system. This makes the design of controllers based only on this qualitative knowledge important and well-motivated.

The main *control objective* is to design a simple scalar adaptation law

$$k_{j+1} = f(k_j, y_j), \quad t_{j+1} = g(t_j, k_j), \tag{1.5}$$

so that the proportional sampled-data output feedback

$$u(t) = -k_j y_j, \quad t \in [t_j, t_{j+1}), \tag{1.6}$$

which uses sampled output information $y_j := y(t_j)$, when applied to a system (1.1) satisfying (1.2) and (1.3) or (1.4) yields a closed–loop system (1.1), (1.5), (1.6) with convergent gain adaptation, positive sampling interval length, and stabilized sampled output.

Further control objectives would be to ensure that the continuous-time output converges to zero, i.e. $\lim_{t\to\infty} y(t) = 0$, or to solve the λ–tracking problem, i.e. for a given bounded reference signal $y_{\text{ref}}(t)$, with bounded derivative, and a prespecified but arbitrary $\lambda > 0$, the system output $y(t)$ should track asymptotically towards the λ–ball around $y_{\text{ref}}(t)$ at sampling time instants, i.e.

$$\lim_{j\to\infty} dist\,(|y_j - y_{\text{ref}}(t_j)\|, [0, \lambda]) \;=\; 0.$$

The main novelty, distinguishing this problem from either continuous or discrete-time adaptive control, is the need to develop suitable mechanisms for adjusting the variable sampling rate. This issue arose in Owens (1996), to our knowledge the only existing paper on this topic. The λ–tracking concept is adopted from Ilchmann and Ryan (1994).

2 Sampling stabilization and λ–stabilization of first order systems

In this section we restrict attention to the simplest class of systems, i.e. scalar systems of the form

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t), \quad x(0) = x_0 \\ y(t) &= cx(t) \end{aligned} \tag{2.1}$$

with $a, b, c, x_0 \in \mathbb{R}$ and high frequency gain $cb > 0$, all unknown.

The basic ideas underlying the adaptation, especially the adaptation of the sampling rate, are already apparent in this simple situation.

Proposition 2.1 *The output feedback adaptive control law*

$$u(t) = -k_j y_j, \quad t \in [t_j, t_{j+1}) \tag{2.2}$$

where $y_j := y(t_j)$, and $\{k_j\}_{j\in\mathbb{N}_0}$ and $\{t_j\}_{j\in\mathbb{N}_0}$ are generated by the sampling–time and gain adaptation mechanism

$$\left.\begin{aligned} h_j &= \tfrac{1}{k_j \log k_j} && j \in \mathbb{N}_0 \\ t_{j+1} &= t_j + h_j && j \in \mathbb{N}_0 \\ k_{j+1} &= k_j + k_j h_j y_j^2 && j \in \mathbb{N}_0 \end{aligned}\right\} \tag{2.3}$$

with $t_0 = 0$, $k_0 > 1$, applied to any system given by (2.1) with $cb > 0$, yields a closed–loop system which admits a unique solution $x(\cdot)$ defined on the whole half–axis $[0, \infty)$ and satisfies

(i) $\lim_{j\to\infty} k_j = k_\infty \in \mathbb{R}$

(ii) $\lim_{j\to\infty} h_j = h_\infty > 0$

(iii) $\{y_j\}_{j\in\mathbb{N}_0} \in l^2$

(iv) $\lim_{t\to\infty} x(t) = 0$.

Remark 2.2 (i) It is well known, see e.g. Willems and Byrnes (1984), that the continuous-time adaptive control law

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = y^2(t) \tag{2.4}$$

will stabilize any system given by (2.1) with $cb > 0$. An Euler discretization of the k–dynamics in (2.4), with a step length η_j, is given by

$$\frac{k_{j+1} - k_j}{\eta_j} = y_j^2. \tag{2.5}$$

On the other hand, sampling the x–dynamics on a sampling interval of length h_j gives $x(t_j)$ determined approximately by an Euler discretization with step length h_j. Since the stiffness increases affinely with $k(t)$ one would to need to sample at a rate faster than $1/k(t)$. It is also natural to sample the x-dynamics (which are responding to changes in k) more rapidly than the numerical integration of the k-dynamics. Natural choices are $\eta_j = \frac{1}{\log k_j}$ and $h_j = t_{j+1} - t_j = [k_j \log k_j]^{-1} = o(\eta_j)$.
(ii) Note that our gain/sampling rate adaptation does not satisfy requirement (12) in Owens (1996) since

$$\lim_{h\to 0} hk(h) = \lim_{k\to\infty} \frac{1}{k \log k} k = 0.$$

More importantly, in the context of adaptive control without identification, we do not require the extra assumptions (12) and (22) which are imposed in Owens (1996).
(iii) We observe that (2.2) and (2.3) give a "high–gain, fast–sampling" adaptive controller.

Proof of Proposition 2.1: Applying Variation–of–Constants yields

$$x(t) = e^{a(t-t_j)} x_j - cb \int_{t_j}^{t} e^{a(t-\tau)} k_j x_j d\tau \tag{2.6}$$

for $t \in [t_j, t_{j+1})$, where $x_j := x(t_j)$. From (2.6), existence and uniqueness of the solution are immediate. To prove (i) – (iv) we proceed in a series of steps.
Step 1: To study the evolution of $x(\cdot)$ at sampling instances t_j, we observe that (2.6) gives

$$x_{j+1} = \mathbf{\Phi}_{h_j,k_j} x_j, \quad j \in \mathbb{N}_0, \tag{2.7}$$

where

$$\mathbf{\Phi}_{h,k} := \begin{cases} e^{ah} - \frac{cb}{a}[e^{ah} - 1]k, & \text{if } a \neq 0 \\ 1 - cbkh, & \text{if } a = 0. \end{cases}$$

Expanding the exponential in $\mathbf{\Phi}_{h,k}$ into a power series gives

$$\mathbf{\Phi}_{h,k} = 1 - \frac{cb}{\log k} + O(h). \tag{2.8}$$

Step 2: We will prove that $\{k_j\}_{j\in\mathbb{N}_0} \in l^\infty$. On the contrary, suppose $\lim_{j\to\infty} k_j = \infty$. We consider the Lyapunov–function candidate $V(y_j) := y_j^2$. Then

$$\begin{aligned} \Delta V(y_j) &:= V(y_{j+1}) - V(y_j) = \left[(\mathbf{\Phi}_{h_j k_j})^2 - 1\right] y_j^2 \\ &= \Big[-2\tfrac{cb}{\log k_j} + 2O(h_j) + \tfrac{(cb)^2}{(\log k_j)^2} - \tfrac{cb}{\log k_j} O(h_j) \\ &\quad + O(h_j)^2\Big] y_j^2. \end{aligned}$$

Hence, using the form of h_j in terms of k_j and by estimating all terms above except $-2cb/(\log k_j)$, there exists

j_0 sufficiently large (i.e. with $h_{j_0} = \frac{1}{k_{j_0} \log k_{j_0}}$ sufficiently small) so that

$$\Delta V(y_j) \leq -\frac{cb}{\log k_j} y_j^2 \quad \text{for all } j \geq j_0.$$

This is the crucial step. Hence, for all $N > j_0$,

$$\begin{aligned} y_N^2 - y_{j_0}^2 &= \sum_{j=j_0}^{N-1} \Delta V(y_j) \\ &\leq -\sum_{j=j_0}^{N-1} \frac{cb}{\log k_j} y_j^2. \end{aligned} \tag{2.9}$$

Using the formula for the adaptation of k_j in (2.9) gives

$$y_N^2 - y_{j_0}^2 \leq -cb[k_N - k_{j_0}] \tag{2.10}$$

for all $N > j_0$. Therefore

$$k_N \leq \frac{1}{cb}\left[-y_N^2 + y_{j_0}^2 + cbk_{j_0}\right] \leq \frac{y_{j_0}^2}{cb} + k_{j_0},$$

which clearly contradicts unboundedness of $\{k_j\}_{j \in \mathbb{N}_0}$. Hence $\{k_j\}_{j \in \mathbb{N}_0} \in l^\infty$.
Step 3: Boundedness of $\{k_j\}_{j \in \mathbb{N}_0}$ yields (i) and (ii). (iii) is a consequence of (2.3). It remains to prove (iv). Using the boundedness and monotonicity of k_j and h_j in (2.6) with $t \in [t_j, t_{j+1})$ gives

$$|x(t)| \leq \left[e^{|a|h_0} + cb\, e^{|a|h_0} k_\infty h_0\right] |x_j|. \tag{2.11}$$

Since $\{y_j\}_{j \in \mathbb{N}_0} \in l^2$ and $c \neq 0$ it follows that x_j tends to zero. Therefore (2.11) yields (iv).
This completes the proof. □

We end this section by considering λ–stabilization of scalar systems described by (2.1). This will provide the intuition for the general case which we consider in Section 5.

In the context of continuous adaptive feedback control and with the weakened control objective of ensuring that $y(t)$ should tend to $[-\lambda, \lambda]$ (a λ–strip), for some $\lambda > 0$ prespecified, the gain adaptation in (2.4) is modified by incorporating a "dead–zone":

$$\dot{k}(t) = \begin{cases} |y(t)|(|y(t)| - \lambda), & |y(t)| \geq \lambda \\ 0, & |y(t)| < \lambda. \end{cases}$$

This dead-zone idea has been used, in conjuction with suitable output feedback control laws, to extend applicability of the high-gain adaptive controllers to rejection of measurement noise and tracking of large classes of reference signals with guaranteed robustness in the presence of nonlinear disturbances, see Ilchmann and Ryan (1994), and for nonlinear systems in Allgöwer et al. (1995). The analogue for sampling stabilization of scalar systems is given as follows.

Proposition 2.3 *Let $\lambda > 0$. If instead of (2.3), k_j is adapted according to*

$$k_{j+1} = \begin{cases} k_j + k_j h_j\, y_j^2, & |y_j| \geq \lambda \\ k_j, & |y_j| < \lambda, \end{cases} \tag{2.12}$$

then the conclusions of Proposition 2.1 hold true but with (iii) and (iv) replaced by

(iii′) y_j converges to the closed interval $[-\lambda, \lambda]$ as $j \to \infty$.

(iv′) $x(\cdot)$ is bounded.

3 Sampling stabilization of multivariable systems

In Section 2 we considered the adaptive-sampling stabilization of scalar systems, where we could show that the continuous-time output $y(t)$ and state $x(t)$ both converged. In the multivariable case, we would not expect to obtain such strong results for the continuous-time output. In particular, whilst the sequence $y(t_j)$ converges to zero, the continuous-time output $y(t)$ need not converge to zero. The following example, due to Owens (1996), illustrates this for a controllable and observable, two dimensional, single–input single–output, minimum phase system.

Example 3.1 *The system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = (1, 0) \tag{3.1}$$

is a controllable, observable and minimum phase system with high–frequency gain $CB = 1$. Since

$$e^{A\delta} = \begin{bmatrix} \cos 2\pi\delta & \frac{1}{2\pi}\sin 2\pi\delta \\ -2\pi \sin 2\pi\delta & \cos 2\pi\delta \end{bmatrix},$$

it is clear that the sampled pair $(e^{A\delta}, C)$, corresponding to a sampling of (3.1) with sampling time δ, is observable ((3.1) is δ-sampled observable) if and only if $\sin 2\pi\delta \neq 0$, i.e. $\delta \notin \pi\mathbb{Z}$. In particular, (3.1) is not 1–sampled observable. It is not surprising therefore to find that, with initial data

$$x(0) = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}, \quad t_0 = 0, \quad k_0 > 1$$

$h_0 = \frac{1}{k_0 \log k_0} = 1$, the adaptive feedback law (2.2), (2.3) (where y_j^2 is replaced by $\|y_j\|^2$) applied to (3.1) yields $y_j = 0$, $h_j = 1$, $k_j = k_0$ for all $j \in \mathbb{N}_0$ and $u(\cdot) \equiv 0$, but $x(t) = (\sin 2\pi t, 2\pi \cos 2\pi t)$. Hence (iv) of Proposition 2.1 is not satisfied.

Thus, the best we can expect is the following result.

Theorem 3.2 *Suppose (1.1) satisfies (1.2) and (1.3), i.e. it is minimum phase and $\sigma(CB) \subset \mathbb{C}_+$. Then the adaptive-sampling output feedback law*

$$u(t) = -k_i y_i, \quad t \in [t_i, t_{i+1}), \tag{3.2}$$

where $y_i := y(t_i)$ and $\{k_j\}_{j\in\mathbb{N}_0}$ and $\{t_j\}_{j\in\mathbb{N}_0}$ are generated by the gain and sampling–time adaptation mechanism

$$\left.\begin{array}{rcll} h_i &=& \frac{1}{k_i \log k_i}, & i \in \mathbb{N}_0 \\ t_{i+1} &=& t_i + h_i, & i \in \mathbb{N}_0 \\ k_{i+1} &=& k_i + k_i h_i \|y_i\|^2, & i \in \mathbb{N}_0 \end{array}\right\} \tag{3.3}$$

with $t_0 = 0$ and $k_0 > 1$, applied to (1.1) yields a closed–loop system which admits a unique solution $x(\cdot)$ defined on the whole half–axis $[0, \infty)$. Here $x_i = x(t_i)$. Moreover

(i) $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$

(ii) $\lim_{i\to\infty} h_i = h_\infty > 0$

(iii) $\{y_i\}_{i\in\mathbb{N}_0} \in l^2$.

The main point to emphasize is the relationship between gain k and sampling rate h. Choosing $hk = (\log k)^{-1}$ has useful consequences summarised as follows.

Lemma 3.3 *Consider a system (1.1) satisfying (1.2) and (1.3) and let*

$$h = \frac{1}{k \log k}.$$

Then there exists $\overline{k} > 0$ sufficiently large such that for all $k \geq \overline{k}$, the feedback

$$\begin{array}{l} u(t) = -k y_i,\ t \in [t_i, t_{i+1}), \\ t_{i+1} = t_i + h,\ t_0 = 0, \end{array} \tag{3.4}$$

applied to (1.1) yields a closed–loop system

$$\dot{x}(t) = Ax(t) - kBCx_i, \quad t \in [t_i, t_{i+1}) \tag{3.5}$$

with a unique and exponentially decaying solution. Here $x_i = x(t_i)$. Moreover, the associated discrete time system

$$x_{i+1} = [I_n + h(A - kBC) + h^2 U_{h,k}] x_i \tag{3.6}$$

where

$$U_{h,k} = \left(\frac{1}{2!}A + \frac{1}{3!}hA^2 + \ldots\right)[A - kBC]$$

is power stable. More precisely, there exists some $M > 0$ independent of $\overline{k}$, so that, for all $i_0 \geq 0$,

$$\|x_{i+1}\| \leq M \zeta_h^{i+1-i_0} \|x_{i_0}\| \quad \forall i \ \geq \ i_0 \tag{3.7}$$

where, for some $R \in \mathbb{R}^{n\times n}$ independent of k,

$$\zeta_h := \left(1 - \frac{h}{4\|R\|}\right)^{\frac{1}{2}}.$$

If the sign of the high frequency gain CB is unknown then the adaptation feedback law has to learn the sign of the feedback. For continuous–time feedback one possibility is to set $u(t) = -k(t)\sin\sqrt{k(t)}y(t)$, where the basic idea is to keep the sign positive or negative for longer and longer periods until finally, if the sign is correct, the system is stabilized and $\sin\sqrt{k(t)}$ does not switch again. For adaptive-sampling feedback control, this idea is realized as follows:

For a monotone non–decreasing sequence $1 < k_0 \leq k_1 \leq \ldots$ define a *switching sequence* $\{S_i\}_{i\in\mathbb{N}_0} \subset \{-1, 1\}$ by

$$\chi_i := \begin{cases} 1, \text{ if } k_0 = \ldots = k_i \\ \frac{1}{k_i - k_0}\sum_{j=0}^{i-1}(k_{j+1} - k_j)S_j, \text{ otherwise} \end{cases} \tag{3.8}$$

and the algorithm:

$$\begin{array}{lll} & \texttt{set}\ L = 1, & \\ (*) & \texttt{while}\ -1 + \frac{1}{2L} \leq \chi_i & \texttt{set}\ S_i = 1, \\ & L = L + 1, & \\ (**) & \texttt{while}\ \chi_i \leq 1 - \frac{1}{2L} & \texttt{set}\ S_i = 1, \\ & L = L + 1,\ i = i + 1, & \\ & \texttt{go to (*).} & \end{array} \tag{3.9}$$

If $\{k_i\}_{i\in\mathbb{N}_0}$ diverges to infinity, then the switching algorithm (3.8) ensures that χ_i has the two accumulation points + 1 and -1. Thus S_i will stay at +1, respectively -1, for longer and longer intervals and it is then natural to choose the feedback

$$u(t) = -k_i S_i y_i, \quad t \in [t_i, t_{i+1}).$$

Theorem 3.4 *Suppose the system (1.1) satisfies (1.2) and (1.4). Let $S_0 = 1$ and $k_0 > 1$. Then the adaptive-sampling output feedback law*

$$u(t) = -k_i S_i y_i, \quad t \in [t_i, t_{i+1}),$$

where

$$y_i := y(t_i)$$

and the gain and sampling–time are adapted according to (3.3) in Theorem 3.2, with switching sequence $\{S_i\}_{i\in\mathbb{N}_0}$ defined by (3.8) and (3.9), applied to (1.1) yields a closed–loop system which admits a unique solution $x(\cdot)$ defined on the whole half–axis $[0, \infty)$. Moreover

(i) $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$

(ii) $\lim_{i\to\infty} h_i = h_\infty > 0$

(iii) $\lim_{i\to\infty} \chi_i = \chi_\infty \in (-1, 1)$

(iv) there exists some i_0 such that $S_i = S_{i_0}$ for all $i \geq i_0$

(v) $\{y_i\}_{i\in\mathbb{N}_0} \in l^2$.

Remark 3.5 *The switching procedure of (3.8), (3.9) is similar to the one used in Owens (1996).*

4 Stabilization of the state by sampling output feedback

We have seen in Example 3.1 that the adaptive algorithm (3.2), (3.3) does not, in general, guarantee that the continuous-time state $x(t)$, or even the output $y(t)$, converges to zero but only the sampled output $y(t_i)$ at sampling times t_i. However, Example 3.1 is "pathological" since the sampling times occur exactly where the output vanish. Since the continuous-time system (1.1) is detectable (this is a consequence of the minimum phase assumption, see Ilchmann (1993)), the aim is to choose the sampling periods in such a way that sampling preserves detectability.

It is well known that the sampled system (with constant sampling period $h > 0$) is detectable if, and only if,

$$\begin{array}{l} \frac{\lambda-\mu}{2\pi i} h \notin \mathbb{Z} \text{ for any } \lambda \neq \mu, \\ \lambda,\, \mu \in \sigma(A) \cup \{0\}. \end{array} \tag{4.1}$$

We shall modify the adaptive sampling time algorithm (3.3) under the additional assumption that (4.1) holds for some known h. If (4.1) holds for h then it holds for $\frac{h}{q}$ for any $q \in \mathbb{N}$. This is the key observation in the following result.

Theorem 4.1 *Suppose the system (1.1) satisfies (1.2) and (1.3). Let h be such that (4.1) holds. Then the adaptive sampling output feedback law*

$$u(t) = -k_i y_i, \quad t \in [t_i, t_{i+1}),$$

where $y_i := y(t_i)$, and $\{k_i\}_{i\in\mathbb{N}_0}$ and $\{t_i\}_{i\in\mathbb{N}_0}$ are generated by the gain and sampling–time adaptation mechanism

$$\left.\begin{array}{rcl} \widehat{h}_i &=& \frac{h}{j}, \text{ where } j \text{ is such that} \\ h_i &=& \frac{1}{k_i \log k_i} \in \left[\frac{1}{j+1}, \frac{1}{j}\right) \\ t_{i+1} &=& t_i + \widehat{h}_i \\ k_{i+1} &=& k_i + k_i \widehat{h}_i \|y_i\|^2 \end{array}\right\} \tag{4.2}$$

with $t_0 = 0$, $k_0 > 1$, applied to (1.1) yields a closed–loop system which admits a unique solution $x(\cdot)$ defined on the whole axis $[0,\infty)$. Moreover,

(i) $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$

(ii) there exist $i_0, j_\infty \in \mathbb{N}$ such that $\widehat{h}_i = \frac{1}{j_\infty} h$ for all $i \geq i_0$

(iii) $\{y_i\}_{i\in\mathbb{N}_0} \in l^2$

(iv) $\lim_{t\to\infty} x(t) = 0$.

Remark 4.2 (i) The detectability of the sampled system at some known sampling time h is used in Ortega and Kreisselmeier (1990).
(ii) If $M > 0$ is known so that $\|A\| \leq M$, then we can choose $h \in \left(0, \frac{3}{M}\right)$, since

$$0 < \left| h \frac{\lambda - \mu}{2\pi i} \right| \leq \frac{3}{M} \frac{2M}{2\pi} < 1.$$

(iii) If A is rational, then $\det(\lambda I_n - A) \in \mathbb{Q}[\lambda]$ and therefore real and imaginary parts of the eigenvalues of A are algebraic. Since the difference of any two algebraic numbers is algebraic, it follows that for any $h \in \mathbb{Q}$ we have that $h(\lambda - \mu) \notin 2\pi i\mathbb{Z}$ and therefore (4.1) holds for any $h \in \mathbb{Q}$.

5 Sampling λ–tracking

In Proposition 2.3 we have demonstrated how a dead–zone is incorporated into the gain adaptation for adaptive sampling λ–stabilization of the class of scalar systems. We will now extend Proposition 2.3 to the adaptive sampling λ–tracking control of multivariable systems.

We suppose that the output of the system $y(\cdot)$ is to track a signal $y_{\text{ref}}(\cdot)$ and that the output measurements are corrupted additively via a noise term $n(\cdot)$. Both the output corrupting noise $n(\cdot)$ and the reference signal $y_{\text{ref}}(\cdot)$ are assumed to belong to $W^{1,\infty}$. Here $W^{1,\infty}$ is the Sobolev space of bounded functions which are absolutely continuous on compact intervals and have essentially bounded derivatives. The control objective is now to track the reference signal at the variable sampling instants, within an arbitrary prespecified λ-neighbourhood for the reference signal, so that asymptotically we require

$$\lim_{i\to\infty} dist\,(\|y(t_i) - y_{\text{ref}}(t_i)\|, [0,\lambda]) \;=\; 0.$$

This will be achieved, analogously to the λ–stabilization in Proposition 2.3, by incorporating a dead–zone into the gain adaptation.

Theorem 5.1 *Let $\lambda, \gamma > 0$, $y_{\text{ref}}(\cdot) \in W^{1,\infty}$, $\overline{u} \in \mathbb{R}^m$. Suppose the system (1.1) is minimum phase, so that (1.2) holds, and the "sign" of the high–frequency gain is known, so that (1.3) holds. Let $e_i = y_i - y_{\text{ref}}(t_i)$. Then the adaptive sampling output feedback law*

$$u(t) = -k_i e_i + \overline{u},\; t \in\; [t_i,\; t_{i+1}) \tag{5.1}$$

where $\{k_i\}_{i\in\mathbb{N}_0}$ and $\{t_i\}_{i\in\mathbb{N}_0}$ are generated by the gain and sampling–time adaptation mechanism

$$\left.\begin{array}{rcl} h_i &=& \frac{1}{k_i \log k_i},\; i \in \mathbb{N}_0 \\ t_{i+1} &=& t_i + h_i,\; i \in \mathbb{N}_0 \\ k_{i+1} &=& k_i + \gamma\delta(e_i) k_i h_i \|e_i\|^2 \\ \delta(e_i) &:=& \begin{cases} 1, & \text{if } \|e_i\| \geq \lambda \\ 0, & \text{if } \|e_i\| < \lambda \end{cases} \end{array}\right\} \tag{5.2}$$

with $t_0 = 0$, $k_0 > 1$, applied to (1.1), results in a closed–loop system which admits a unique solution $x(\cdot)$ on $[0,\infty)$. Moreover, *(i)* $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$,

(ii) $\lim_{i\to\infty} h_i = h_\infty > 0$,

(iii) $\lim_{i\to\infty} dist\,(\|e_i\|, [0,\lambda]) \;=\; 0.$

Remark 5.2 $\overline{u} \in \mathbb{R}^m$ and $\gamma > 0$ in (5.1) and (5.2) are design parameters which might be used to improve the performance considerably if more information is known about the real process. We could have included γ in (3.3) as well. If the output is corrupted by noise $n(\cdot)$, then

$$e(t) = y(t) + n(t) - y_{\text{ref}}(t).$$

In this case $y_{\text{ref}}(t) - n(t)$ can be viewed as a different reference signal. If $n(\cdot)$ is large, the λ–tracking controller forces the output to track the noise. In this case, λ should not be chosen too small.

6 Numerical examples

We apply the controllers (3.2,3.3) (see Figure 1), and (5.1,5.2) (see Figure 2) to (1.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -16 & 16 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$C = [2\ 4\ 7\ 4\ 1]$, $x_0 = 0.1(1,1,1,1,1)^T$, and in the latter $y_{\text{ref}}(t) = |\sin(0.5t)|$ (and $\gamma = 1$ and $\overline{u} = 0$).

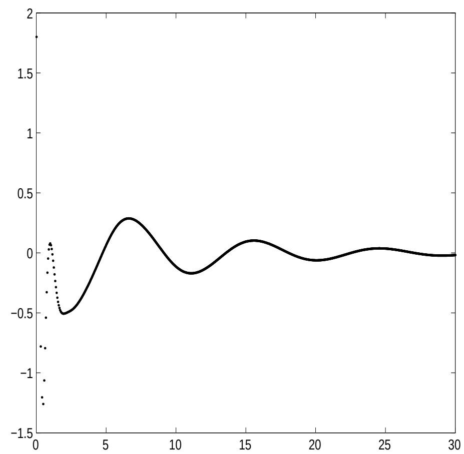

Figure 1: Output $y(t_i)$ with controller (3.2), (3.3) and $k_0 = 3$.

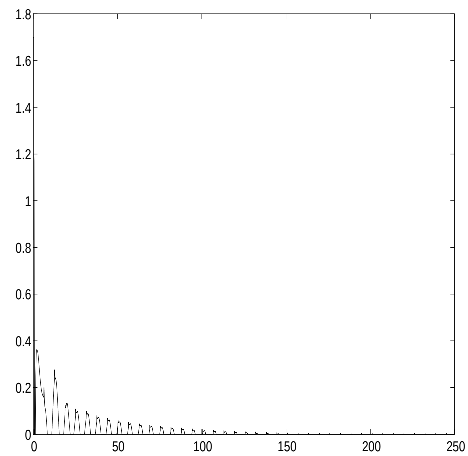

Figure 2: Distance error $dist(e(t_i), [-\lambda, \lambda])$ with tracking regulator (5.1), (5.2), $k_0 = 3$, $\lambda = 0.1$, $y_{ref}(t) = |\sin(0.5t)|$.

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