Ilchmann, Achim; Owens, David H. :

Robust universal exponential adaptive stabilization in the presence of nonlinearities

Zuerst erschienen in:
Proceedings of the First European Control Conference, ECC '91 : 2-5 July 1991, Grenoble, France / ed.: C. Commault. - Paris : Hermes, 1991, Bd. 3, S. 2580-2585,
ISBN 2-86601-282-8

ROBUST UNIVERSAL EXPONENTIAL ADAPTIVE STABILIZATION IN THE PRESENCE OF NONLINEARITIES

Achim Ilchmann and David H. Owens
School of Engineering
University of Exeter
North Park Road
Exeter, Devon EX4 4QF
United Kingdom
FAX No.: 0392-217965

Abstract: A unified approach to exponential adaptive stabilization via switching and gain adaptation is presented for a class of systems satisfying an integral inequality on input-output data. Two types of switching strategy are discussed. The approach leads naturally to a robust control in the presence of linear and nonlinear perturbations. Here, we only consider robustness with respect to sector bounded actuator dynamics.

Keywords: adaptive control, robustness, stability, nonlinear control

1 Introduction

In recent papers (see, e.g. [1] - [8]) attention has been focussed on the stabilization of multivariable systems in the state space form S(A,B,C) in $\mathbb{R}^n$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

(where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$) in the presence of almost complete ignorance of system structure. More precisely, for m-input/m-output systems, the state dimension n need not be known but it is assumed that it is known that CB is nonsingular and that the system is *minimum- phase*, i.e. the 'zero polynomial'

$$z(s) = \begin{vmatrix} sI - A & -B \\ C & 0 \end{vmatrix} \tag{1.2}$$

has roots only in the open left-half complex plane. Under these conditions it has been demonstrated that output feedback control laws of the form

$$u(t) = -N(t)k(t)y(t) \tag{1.3}$$

are capable of stabilizing all systems with the above (or similar) properties. Here $k(t)$ is an adaptive time-varying gain using output data to 'search for the correct gain level' whilst $N(t)$ adapts using data to 'obtain the correct sign of feedback'.

A wide range of results are now available demonstrating the feasibility of the problem and describing the separate effects of 'small' state nonlinearities [9], input/output nonlinearities [10], switching strategies [11] and preliminary results on exponential stabilization [12] in a number of cases.

The purpose of this paper is to provide a partial unification of the ideas by

a) expressing the class of systems in integral inequality form,

b) interpreting the inequalities in a form directly relevant to robustness studies,

c) including exponential weighting as a mechanism for ensuring exponential stabilization in a similar manner to [12],

d) demonstrating the possibility of including recently introduced [11] switching strategies within the general framework, and

e) extending the switching strategy to include robustness with respect to i/o nonlinearities.

One consequence of the approach is the possibility of including nonlinear and infinite-dimensional systems within the system class. See, for example, [7] and [10] for the details of related work.

In the following $L_p^q(a,b)$ denotes the linear space of p-integrable functions on the open internal $[a,b)$ with values in $\mathbb{R}^q$ (q defined by the context).

2 Properties of the System Class

For ease of presentation, the paper concentrates on single-input/single-output (SISO) systems with the property $CB \neq 0$ and minimum-phase structure. Under these conditions, the following result is well-known, see e.g. [9], and the proof is omitted.

Proposition 1: Suppose $(A,B,C) \in \Sigma$, then the system is similarly equivalent to

$$\begin{aligned} \dot{y}(t) &= A_{11}y(t) + A_{12}z(t) + CBu(t) \\ \dot{z}(t) &= A_{21}y(t) + A_{22}z(t) \end{aligned} \tag{2.1}$$

where $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is asymptotically stable and the state is $x(t) = S^{-1}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$,
$S \in \mathbb{R}^{n\times n}$ invertible. □

In order to include the possibility of exponential stabilization in the manner of [12] let $\omega(t)$ be a continuously differentiable, positive and non-increasing function with limit $\omega_\infty \in \mathbb{R}$ as $t \to +\infty$. Also use the notation

$$\hat{v}(t) := v(t)\exp\{\omega(t)t\} \tag{2.2}$$

where $v(\cdot): \mathbb{R} \to \mathbb{R}^r$ is absolutely continuous. Under this structure, the following proposition can be proved:

Proposition 2: Consider $(A, B, C) \in \Sigma$. Then, for any initial condition $x(t_0)$ and any $\omega(\cdot)$ with $\omega_\infty = 0$, there exists a constant $K \geq 0$ such that, for all measurable inputs $u(\cdot)$, the response $y(\cdot)$ satisfies, for arbitrary $t \geq t_0$, $t_0 \geq 0$.

$$\frac{1}{2}\hat{y}^2(t) \leq K + K\int_{t_0}^{t} \hat{y}^2(s)ds + CB\int_{t_0}^{t} \hat{u}(s)\hat{y}(s)ds$$

Proof: The exponentially weighted state space model has the form

$$\begin{aligned} d\hat{y}/dt &= (A_{11} + \omega + \dot{\omega}t)\hat{y} + A_{12}\hat{z} + CB\hat{u} \\ d\hat{z}/dt &= A_{21}\hat{y} + (A_{22} + (\omega + \dot{\omega}t)I)\hat{z} \end{aligned}$$

Note that, under the stated conditions $\limsup_{t\to+\infty} (\omega(t) + \dot{\omega}(t)t) = \omega_\infty = 0$ and hence the system $\frac{d\hat{z}}{dt} = (A_{22} + (\omega + \dot{\omega}t))\hat{z}$ is an exponentially stable linear time-varying system. A simple calculation yields

$$\frac{1}{2}\frac{d}{dt}(\hat{y}^2(t)) \leq (A_{11} + \omega(0))\hat{y}^2 + \hat{y}A_{12}\hat{z} + CB\hat{u}\hat{y}.$$

Integrating from t_0 to $t \geq t_0$ and noting that, for suitable choice of $K_1, K_2 > 0$

$$\int_{t_0}^{t} \hat{y}(s)A_{12}\hat{z}(s)ds \leq K_1 + K_2\int_{t_0}^{t} \hat{y}^2(s)ds,$$

the result is easily proved. The details are omitted for brevity. □

The introduction of exponential weighting is motivated by the following proposition relating properties of N, k, ω and $\hat{y}$ to the existence of exponential decay rates on the system state. The conditions of the result will become 'targets' for subsequent analysis and the underlying basis of the proofs.

Proposition 3: With the control of equation (1.3), suppose that $N(t)\,k(t)$ is bounded on $[0, \infty)$, that $\omega_\infty > 0$ and that $\hat{y}(\cdot) \in L_2(0, \infty)$. Then there exists real numbers $M \geq 0$ and $\lambda > 0$ such that

$$\|x(t)\| \leq Me^{-\lambda t}, \quad t \geq 0$$

Proof: Since $\omega_\infty > 0$ and $\hat{y}(\cdot) \in L_2(0, \infty)$, $y(\cdot)\,e^{\lambda\cdot} \in L_2(0, \infty)$ for all $\lambda \leq \omega_\infty$. Choose $\lambda \in (0, \omega_\infty)$ so that $A_{22} + \lambda I_{n-1}$ is asymptotically stable. A simple calculation (details omitted for brevity) then indicates that $z(\cdot)\,e^{\lambda\cdot} \in L_2^{n-1}(0, \infty)$ and hence $x(\cdot)e^{\lambda\cdot} \in L_2^n(0, \infty)$. Since

$$\frac{d}{dt}\left(x(t)e^{\lambda t}\right) = \left\{\lambda I_n + \begin{bmatrix} A_{11} - N(t)k(t)\cdot CB & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right\} x(t)\,e^{\lambda t}$$

it follows that $\frac{d}{dt}\left(x(t)e^{\lambda t}\right) \in L_2(0, \infty)$, and therefore $x(t)\,e^{\lambda t} \in L_\infty(0, \infty)$. The result now follows trivially. □

To conclude this section, Proposition 2 is used to extend the potential of the analysis. More precisely, throughout the remainder of the paper attention is focussed on the class of SISO systems with the property that, for any initial internal 'state' and any $\omega(\cdot)$ of the specified form with $\omega_\infty = 0$, there exists a constant K such that for all inputs $u(\cdot)$, the input response satisfies, for $t \geq t_0$, the inequality in Proposition 2 with $CB \neq 0$.

The use of the inequality permits nonlinear and infinite dimensional perturbations and hence includes a natural robustness into the control law performance and analysis.

In the following sections, consideration is given to (robust) adaptive controllers of the form of (1.3) capable of stabilizing all systems in the class defined above.

3 The Case of CB of Known Sign

In this case, let

$$N(\cdot) := -\operatorname{sgn} CB \tag{3.1}$$

so that we consider the exponentially weighted output $\hat{y}(\cdot) = cx(\cdot)\,e^{\omega\circ k}$ of the closed loop adaptive system

$$\dot{x}(t) = [A - \operatorname{sgn} CB \cdot k(t) \cdot BC]x(t) \tag{3.2}$$

Suppose that $k(\cdot) \in PC(0, \infty)$ is positive and nondecreasing.

Using the inequality of Proposition 2, we show in the following Lemma 1 that $\lim_{t\to\infty} k(t) = \infty$ and $\lim_{t\to\infty} \omega(t) = \omega_\infty = 0$ implies $\hat{y}(\cdot) \in L_2(0, \infty) \cap L_\infty(0, \infty)$. The intuition is, that for $\lim_{t\to\infty} k(t) = k_\infty$ finite but large enough and ω_∞ positive but small enough the result of Lemma 1 is still valid. In this case a suitable adaptive mechanism may be capable of ensuring that $x(t)$ is exponentially bounded in t.

To find k_∞ large enough and $\omega_\infty > 0$ small enough adaptively by using the information of the output $y(\cdot)$, we consider the following feedback and the adaptation law

$$\begin{aligned} u(t) &= -\operatorname{sgn}(CB) \cdot k(t) \cdot y(t) && (3.3a) \\ \dot{k}(t) &= |e^{\omega\circ k(t)t}\, y(t)|^2, \quad k(0) \in \mathbb{R}_+ && (3.3b) \\ \omega(t) &= [1 + t]^{-1} && (3.3c) \end{aligned}$$

This is only a prototype example of a fairly rich class of gains $k(\cdot)$ and weighting function $\omega(\cdot)$ which are related in the following way. $\omega(\cdot)$ at time t is calculated by using data on present and past values of the gain $k(\cdot)|_{[0,t]}$ so that (i) $\omega \circ k(t)$ is nonincreasing if $k(t)$ is nondecreasing, (ii) $\omega \circ k(t) \geq 0$ if $k(t) \geq 0$, and (iii) $\omega \circ k(t)$ goes to zero as $k(t)$ tends to infinity. $k(\cdot)$ is constructed from measurements $\hat{y}(\cdot)$ so that (i)

$k(t) \geq 0$ and nondecreasing, (ii) $k(\cdot)$ is bounded if $\hat{y}(\cdot) \in L_2 \cap L_\infty$, and (iii) $\hat{y}(\cdot) \in L_2$ if $k(\cdot)$ is bounded. A precise definition of these classes will be given in a future paper, here we only consider (3.3).

Now we are in a position to present our theorem on exponential stabilisation.

Theorem 1: Suppose $(A, B, C) \in \Sigma$. Then the feedback evolution (3.3) applied to the system (1.1) ensures that the solution of the adaptive closed-loop system (3.2) exists on the whole of $\mathbb{R}$ and is exponentially stable, i.e. there exists $\mu, \lambda \geq 0$ such that

$$\|x(t)\| \leq \mu e^{-\lambda t} \quad \text{for all} \quad t \geq 0 \tag{3.4}$$

Moreover

$$\begin{aligned} \lim_{t\to\infty} k(t) &= k_\infty < \infty & (3.5)\\ \lim_{t\to\infty} \omega \circ k(t) &= \omega_\infty > 0 & (3.6)\\ \hat{y}(\cdot),\ e^{\omega_\infty \cdot} y(\cdot) &\in L_2(0,\infty) & (3.7)\end{aligned}$$

For the proof of Theorem 1 the following Lemma is needed:

Lemma 1 If there exists a solution of (3.2) on $[0, t')$, $t' \leq \infty$, and $\lim_{t\to t'} k(t) = \infty$, and $\omega_\infty = 0$, then $\hat{y}(\cdot) \in L_2(0, t') \cap L_\infty(0, t')$.

Proof: As $\omega_\infty = 0$, Proposition 2 holds and substituting for $\hat{u}(t)$ gives

$$\frac{1}{2}\hat{y}^2(t) \leq K + K\int_0^t \hat{y}^2(s)ds - |CB| \int_0^t k(s)\hat{y}^2(s)ds.$$

If $\hat{y}(\cdot)$ is not an element of $L^2(0, t')$, a simple calculation yields

$$\lim_{t\to t'} \int_0^t k(s)\hat{y}^2(s)ds \left[\int_0^t \hat{y}^2(s)ds\right]^{-1} = +\infty.$$

For large enough t, this contradicts the necessary positivity of the right hand side of the inequality. Consequently $\hat{y}(\cdot)$ lies in $L_2(0, t')$ and inspection of the inequality then yields the consequence that $\hat{y}(t)^2$ is uniformly bounded on $[0, t')$. □

Proof of Theorem 1: We proceed in several steps.

(i) Suppose there exists a solution of (3.2) on $(-\infty, t')$ and $k(\cdot) \notin L_\infty(0, t')$. Then by (3.3c) $\lim_{t\to t'} \omega \circ k(t) = \omega'_\infty = 0$ and Lemma 1 implies $\hat{y}(\cdot) \in L_2(0, t') \cap L_\infty(0, t')$. Now (3.3b) gives $k(\cdot) \in L_\infty(0, t')$.

(ii) Since (i) gives $k(\cdot) \in L_\infty(0, t')$, (3.2) satisfies a global Lipschitz condition on $(0, t')$. Therefore (3.2) does not have a finite escape time on $[0, t')$.

(iii) Suppose $[0, t')$, $t' < \infty$, is the maximal interval of existence of $k(\cdot)$, i.e. $\lim_{t\to t'} k(t) = \infty$. Then by (ii), there exists a unique solution $x(\cdot) \in L_2(0, t') \cap L_\infty(0, t')$. This contradicts unboundedness of $k(\cdot)$ and proves $t' = \infty$.

(iv) Put $t' = \infty$ in the proof of (i). Since $k(t)$ is nondecreasing (i) proves (3.5), and (3.6) follows from (3.3c). (3.3) implies $\hat{y} \in L_2(0, \infty)$ and hence (3.7) is proved. Finally, (3.4) follows from Proposition 3. □

If the sign of CB in unknown, we could introduce a switching function and a Nussbaum-gain in the spirit of [1], and obtain generalizations of Lemma 1 and Theorem 1. For brevity we leave this for a future paper. Instead, a different switching concept to the Nussbaum-type setup will be investigated.

4 Exponential Stabilization using Alternative Switching Strategies

It is the purpose of this section to demonstrate that the new switching strategy introduced in [11] can be used in conjunction with the exponential weighting method of previous sections to produce exponential stabilization of the system state in the sense defined by the theorems of the previous section. For completeness the switching strategy is defined fully as follows: the switching function $N(t)$ takes only values ± 1 and changes sign at times $0 = t_0 < t_1 < \ldots$ Defining the *switching decision function* $\psi(\cdot)$ by

$$\psi(t) := \int_0^t N(\tau)k(\tau)\hat{y}^2(\tau)d\tau \left[\int_0^t \hat{y}^2(\tau)d\tau\right]^{-1} \tag{4.1}$$

with $k(\cdot)$ defined in (3.3b), let $\{\lambda_i\}_{i\in\mathbb{N}}$ be a strictly increasing, unbounded sequence of real, positive numbers or "thresholds". With this notation, $N(t)$ is defined by the following algorithm for the switching times $\{t_i\}$:

$$\begin{array}{ll} & i = 0 \\ & N(t_i) = 1 \\ (*) & t_{i+1} > t_i \text{ is defined by the property} \\ & \quad N(t_i)\psi(t) < \lambda_{i+1}k(t_0),\ t \in (t_i, t_{i+1}) \\ & \quad N(t_i)\psi(t_{i+1}) = \lambda_{i+1}k(t_0) \qquad (4.2) \\ & N(t) = N(t_i),\ t \in [t_i, t_{i+1}) \\ & N(t_{i+1}) = -N(t_i) \\ & i := i + 1 \\ & \text{go to } (*) \end{array}$$

which is well-defined as

a) $\psi(t)$ is monotonic on any interval $t > 0$ where $N(t)$ is constant

b) $\psi(t_0) = k(t_0)$ ensures correct initialization of the algorithm.

Using Proposition 2, if $\omega_\infty = 0$ then

$$\frac{1}{2}y^2(t) \leq K + \int_0^t \hat{y}^2(s)ds\,[K + CB\psi(t)] \tag{4.3}$$

which is fundamental to the proof of the following result generalizing Theorem 1 to the case of the above switching policy.

Theorem 2: Suppose $(A, B, C) \in \Sigma$. Then the feedback and adaption law

$$\begin{aligned} u(t) &= -N(t)k(t)y(t) && (4.4a)\\ \dot{k}(t) &= \left|e^{\omega o k(t)\cdot t}y(t)\right|^2, \quad k(0) \in \mathbb{R}_+ && (4.4b)\\ \omega(t) &= [1+t]^{-1} && (4.4c)\end{aligned}$$

where $N(\cdot)$ is produced by (4.2), applied to (1.1) yields a closed loop system whose solution exists on the whole of $\mathbb{R}$ and satisfies (3.4)–(3.7), and moreover

$$\psi(t) \quad \text{has a finite limit} \quad \psi_\infty \text{ as } t \to +\infty, \qquad (4.5)$$

the switching function $N(t)$ switches only a finite number of times $t_1, t_2, \ldots, t_M$ so that $N(t)$ is constant for $t \geq t_M$. (4.6)

Proof:

(i) We show that if there exists a solution $x(\cdot) \in L_2(0, t')$, $t' \leq \infty$, of the adaptive closed loop system $\dot{x} = [A + NkBC]x$ then $k(\cdot) \in L_\infty(0, t')$. Suppose $k(\cdot) \notin L_\infty(0, t')$. Then it follows from (4.4c) that $\lim_{t\to t'} \omega \circ k(t) = \omega'_\infty = 0$ and (4.3) is valid. Thus $\psi(\cdot) \in L_\infty(0, t')$ since otherwise $CB\psi(\cdot)$ would take arbitrarily positive and negative values and (4.3) would imply a contradiction. As $\psi(\cdot)$ is monotonic on each interval $[t_i, t_{i+1})$ it has a finite limit $\lim_{t\to t'} \psi(t) = \psi_\infty$ and hence a finite number of switching times $t_1, \ldots, t_M < t'$ occur. In particular, for $t \in [t_M, t'), N(t)$ is constant with value $+1$ or -1. If $N(t) = -\operatorname{sgn} CB$ for $t \in [t_M, t')$ then Lemma 1 gives $\hat{y}(\cdot) \in L_2(0, t') \cap L_\infty(0, t')$ and therefore, by using (4.4b), $k(\cdot) \in L_\infty(0, t')$. This contradicts the original assumption that $k(t)$ is unbounded. It remains to consider the case that $N(t) = \operatorname{sgn} CB$ for $t \in [t_M, t')$. Then the argument in part (i) of the proof of Proposition 3.5 in [11] yields that $\psi(t)$ is unbounded. This is a contradiction and the proof of (i) is complete.

(ii) In a similar manner to the proof of Theorem 1 it can be shown that the solution $x(\cdot)$ exists on the whole of $\mathbb{R}$ and is unique. We omit the proof for brevity.

(iii) For $k(\cdot) \in L_\infty(0, \infty)$ the arguments of part (iv) of the proof of Theorem 1 remain valid with $|\psi(t)| \leq k_\infty$ for all $t \geq 0$, which follows from (4.1). Using algorithm 4.2 and note (i) following it, the boundedness of ψ hence yields (4.5) and (4.6). The proof of the theorem is now complete. □

Clearly, the use of the switching algorithm (4.2) retains all of the global stabilization properties of the Nussbaum switching rule whilst avoiding the "growth" requirements of N. Adaptation of the gain $k(t)$ is the only growth mechanism in the control law and hence, intuitively, there is a possibility of achieving stabilization with reduced limit gains k_∞. The proof of only a finite number of switching times is an important aspect of the result, although the actual number in a given situation will depend upon $k(0), x(0)$ and the choice of thresholds $\{\lambda_i\}$. The latter problem will be the subject of further study.

5 Robustness of the Alternative Switching Strategy

The use of the switching decision function (4.1) includes a large degree of robustness of the control laws. The presence of input/output actuator and sensor nonlinearities is not included however. The purpose of this section is to indicate the robustness of the alternative switching strategy of Section 4 to this situation by consideration of input nonlinearities. The results are a parallel of those appearing in [11] for the Nussbaum switching case. They also indicate that the choice of thresholds $\{\lambda_i\}$ in Section 4 may be useful in ensuring robustness.

The actuator nonlinearity is assumed to be unknown and memoryless and possibly time-varying and is defined by the conditions

$$u(t) = \xi(v(t), t), \quad v(t) = N(t)k(t)y(t) \qquad (5.1)$$

where $\xi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be continuous and locally Lipschitz in v and measurable and locally integrable in t, moreover it is a sector linear bounded function, i.e. for some unknown real numbers $b \geq a > 0$ and all $t \geq 0$ it satisfies

$$\begin{array}{lllll} a\lambda & \leq & \xi(\lambda, t) & \leq & b\lambda, \quad \lambda \geq 0 \\ a\lambda & \geq & \xi(\lambda, t) & \geq & b\lambda, \quad \lambda \leq 0. \end{array} \qquad (5.2)$$

Without loss of generality, assume that $0 < a \leq 1 \leq b$ by absorbing, if necessary, a suitable scalar into the scalar factor CB. For notational simplicity, the dependence of ξ on t will be dropped in the following development.

An important function in the analysis is

$$\begin{aligned}\psi_0(t) &= \int_0^t \hat{u}(s)\hat{y}(s)ds \left[\int_0^t \hat{y}^2(s)ds\right]^{-1}\\ &= \int_0^t e^{\omega(s)s}\xi(N(s)k(s)y(s))\hat{y}(s)ds \left[\int_0^t \hat{y}^2(s)ds\right]^{-1}\end{aligned} \qquad (5.3)$$

Clearly ψ_0 cannot be computed since ξ is not known. Switching will hence be based on

$$\psi(t) := \frac{\int_0^t N(s)k(s)\hat{y}^2(s)ds}{\int_0^t \hat{y}^2(s)ds} \qquad (5.4)$$

with 'thresholds' related to the values of

$$\phi(t) := \int_0^t k(s)\hat{y}^2(s)ds \left[\int_0^t \hat{y}^2(s)ds\right]^{-1} \qquad (5.5)$$

Clearly $\phi(t)$ is positive and non-decreasing if $k(t)$ is positive and non-decreasing.

The proposed algorithm will require certain initialization procedures and hence, noting that $N(t) \in \{-1,+1\}$ implies that

$$|\psi(t)| \leq \phi(t), \quad t \geq 0, \tag{5.6}$$

the algorithm is assumed to be initialized by the choice of arbitrary switching $0 = t_0 < t_1 < t_2 < \ldots < t_A$ ($A \geq 2$) to assume that $|\psi(t_A)| < \phi(t_A)$. It is further assumed that the designer now specifies a sequence $\{\hat{\lambda}_i\}$ of strictly decreasing positive real numbers converging to zero where $\hat{\lambda}_0 \in (0,1)$ is such that $|\psi(t_A)| < (1-\hat{\lambda}_0)\phi(t_A)$. Note that, with the above definitions,

$$\eta(t) := \psi(t)\cdot\phi(t)^{-1} \in (-1,1), \quad t \geq t_A \tag{5.7}$$

and note that $\eta(t)$ is monotonic on any interval where $N(t)$ is constant with

$$N(t)\dot{\eta}(t) \geq 0 \tag{5.8}$$

The suggested switching algorithm is now as follows for $t \geq t_A$,

$$\begin{array}{ll} & i = A \\ & N(t_i) = -N(t_{i-1}) \\ (*) & t_{i+1} > t_i \text{ is defined by the property} \\ & \quad N(t_i)\eta(t) < 1-\hat{\lambda}_{i+1}, \ t \in (t_i, t_{i+1}) \\ & \quad N(t_i)\eta(t_{i+1}) = 1-\hat{\lambda}_{i+1} \\ & N(t) = N(t_i), \ t \in [t_i, t_{i+1}) \\ & N(t_{i+1}) = -N(t_i) \\ & i = i+1 \\ & \text{go to } \ (*) \end{array} \tag{5.9}$$

The following analysis arises to demonstrate that, with the gain adaptations of Section 4, the global adaptive stabilization is retained in the presence of any input nonlinearity of the specified form (5.1)–(5.2). Note that algorithm (5.9) is essentially that of (4.2) with adaptive thresholds

$$\lambda_i = (1-\hat{\lambda}_i)\phi(t_i)/k(0) \tag{5.10}$$

It is convenient to define

$$\begin{array}{rcl} \bar{s}(t) & = & \begin{cases} b, & N(t) = +1 \\ -a, & N(t) = -1 \end{cases} \\ \underline{s}(t) & = & \begin{cases} a, & N(t) = +1 \\ -b, & N(t) = -1 \end{cases} \end{array} \tag{5.11}$$

and note that, for all $t \geq 0$,

$$\underline{s}ky^2 \leq \xi(Nky)y \leq \bar{s}ky^2 \tag{5.12}$$

If $\underline{\psi}$ (resp $\overline{\psi}$) are functions obtained from ψ by replacing N in (5.4) by $\underline{s}$ (resp $\bar{s}$), then a simple calculation yields, for $t \geq 0$,

$$\underline{\psi}(t) \leq \psi_0(t) \leq \overline{\psi}(t) \tag{5.13}$$

and, with $a \leq 1 \leq b$,

$$\underline{\psi}(t) \leq \psi(t) \leq \overline{\psi}(t). \tag{5.14}$$

Note also that

$$\begin{array}{rcl} \overline{\psi}(t) & = & \frac{(b-a)}{2}\phi(t) + \frac{(b+a)}{2}\psi(t) \\ \underline{\psi}(t) & = & -\frac{(b-a)}{2}\phi(t) + \frac{(b+a)}{2}\psi(t) \end{array} \tag{5.15}$$

and hence for $\alpha := (b-a)/(b+a) \in [0,1)$

$$\begin{array}{rcl} \overline{\psi}(t) & = & \frac{1}{2}(b+a)\phi(t)[\alpha + \eta(t)] \\ \underline{\psi}(t) & = & \frac{1}{2}(b+a)\phi(t)[-\alpha + \eta(t)] \end{array} \tag{5.16}$$

Equations (5.13), (5.14) and (5.16) provide bounds on $\psi_0(t), \psi(t)$ and relate the bounds to the switching algorithm.

To begin the analysis, the following lemma is first proved.

Lemma 2: Suppose that $k(t) \to \infty (t \to t')$. Then $\phi(t), \underline{\psi}(t), \overline{\psi}(t), \psi_0(t)$ and $\psi(t)$ have finite limits as $t \to t'$.

Proof: If $\phi(t)$ is unbounded, suppose that the switching algorithm switches an infinite number of times $0 = t_0 < t_1 < t_2 \ldots < t'$. By construction this implies that

$$\limsup_{t\to t'} \eta(t) = +1 \ , \ \liminf_{t\to t'} \eta(t) = -1 \tag{5.17}$$

From (5.16) this gives

$$\limsup_{t\to t'} \underline{\psi}(t) = +\infty \ , \ \liminf_{t\to t'} \overline{\psi}(t) = -\infty \tag{5.18}$$

and hence, using (5.13),

$$\limsup_{t\to t'} \psi_0(t) = +\infty \ , \ \liminf_{t\to t'} \psi_0(t) = -\infty \tag{5.19}$$

As $k(t)$ is unbounded, then $\omega_\infty = 0$ and inequality (4.3) becomes

$$\frac{1}{2}\hat{y}^2(t) \leq K + \int_0^t \hat{y}^2(s)ds \cdot [K - CB\psi_0(t)] \tag{5.20}$$

and a contradiction is obtained as the left-hand-side is always positive. If $\phi(t)$ is unbounded it must hence be true that there are only a finite number M of switches. Consequently, $\eta(t)$ has a limit $\beta \in (-1,1)$. If $\phi(t)$ is unbounded, it follows that $\int_0^t k\hat{y}^2 ds$ is unbounded and hence

$$\begin{array}{rcl} \beta & = & \lim_{t\to t'} \eta(t) \\ & = & \lim_{t\to t'} \left[\dfrac{\int_0^{t_M} Nk\hat{y}^2 ds + N(t_M)\int_{t_M}^t k\hat{y}^2 ds}{\int_0^{t_M} k\hat{y}^2 ds + \int_{t_M}^t k\hat{y}^2 ds} \right] \\ & = & N(t_M) = \pm 1 \end{array} \tag{5.21}$$

which is impossible by construction. It follows that ϕ is bounded and the boundedness of $\underline{\psi}, \overline{\psi}, \psi_0$ and ψ follows from (5.16), (5.13) and (5.14). □

We also need the following lemma:

Lemma 3: If $k(t) \to +\infty (t \to t')$, then $\hat{y}(\cdot) \in L_2(0,t') \cap L_\infty(0,t')$.

Proof: From the previous lemma, ψ_0 and ϕ are bounded and hence, for all $T > 0, t \in (T, t')$

$$\phi(t) \geq \int_0^T k\hat{y}^2 ds + k(T) \int_T^t \hat{y}^2 ds \left[\int_0^T \hat{y}^2 ds + \int_T^t \hat{y}^2 ds \right]^{-1} \tag{5.22}$$

It follows that $\hat{y}(\cdot) \in L_2(0,t')$ otherwise $\lim_{t\to t'} \phi(t) \geq k(T)$ for all T which contradicts boundedness. As $k(\cdot)$ is unbounded, $\omega_\infty = 0$ and hence (5.20) holds proving that $\hat{y} \in L_\infty(0,t')$. This completes the proof of the lemma. □

The above two lemmas on the behaviour in the presence of gain divergence form the basis of the proof of the following main theorem of this section. The theorem proves the retention of all the global stabilization properties of the adaptive algorithm in the presence of input nonlinearities and the switching algorithm (5.9).

Theorem 3: Using the control law $u(t) = N(t)k(t)y(t)$ with $N(t)$ defined by the switching algorithm (5.9) and its initialization phase, suppose that $k(t)$ is given by (4.4b), (4.4c). Then the closed-loop adaptive control stabilizes the system in the presence of the input nonlinearity (5.1), (5.2) in the sense that there exists a unique solution on the whole of $\mathbb{R}$ satisfying

$$\lim_{t\to\infty} k(t) = k_\infty < +\infty, \quad \lim_{t\to\infty} \omega(t) = \omega_\infty > 0 \tag{5.23}$$

$$\lim_{t\to\infty} \psi(t) = \psi_\infty \in \mathbb{R}, \quad \lim_{t\to\infty} \phi(t) = \phi_\infty \in \mathbb{R} \tag{5.24}$$

and the switching algorithm switches only at a finite number of times $t_1, t_2, \ldots, t_{M-1}, t_M$. Moreover, $\hat{y}(\cdot) \in L_2(0,\infty)$ so that

$$y(\cdot)\exp(\omega_\infty \cdot) \in L_2(0,\infty) \tag{5.25}$$

and there exists real numbers $\overline{M}, \lambda > 0$ such that

$$\|x(t)\| \leq \overline{M}\, e^{-\lambda t}, \quad \text{for } t \geq 0 \tag{5.26}$$

Proof: The uniqueness of the solution and the absence of a finite escape time can be proved in a similar manner to that of Theorem 1. This is omitted for brevity, so it remains to show that $k(\cdot)$ is bounded and that this implies (5.23)-(5.26).
If $k(t) \to +\infty (t \to \infty)$ then, by Lemma 3, $\hat{y}(\cdot) \in L_2(0,\infty) \cap L_\infty(0,\infty)$ which using property (4.4b) indicates that $k(\cdot)$ is bounded which provides a contradiction. By monotonicity, $k(t)$ is bounded with limit k_∞ and hence $\omega(t)$ is bounded with limit $\omega_\infty > 0$ and $\hat{y}(\cdot) \in L_2(0,\infty)$ follows from (4.4b). The existence of limits for ψ and ϕ now follows trivially by definition as does the exponential bound for the state (using Proposition 3). It remains only to show that there are only a finite number of switching times. This follows from the fact that $\hat{y} \in L_2(0,\infty)$ and the relation

$$\lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} \left[\frac{\sum_{j\geq 0} (-1)^j \int_{t_j}^{t_{j+1}} k\hat{y}^2 ds}{\sum_{j\geq 0} \int_{t_j}^{t_{j+1}} k\hat{y}^2 ds} \right] \in (-1,1) \tag{5.27}$$

This completes the proof of the theorem. □

References

[1] J.C. Willems and C.I. Byrnes, *Global adaptive stabilization in the absence of information on the sign of the high frequency gain*, Lecture Notes in Control and Information Sciences No. 62, Springer, Berlin, 1984.

[2] B. Mårtensson, *Adaptive Stabilization*, Doctoral Dissertation, Lund Institute of Technology, 1986.

[3] D.H. Owens, D. Prätzel-Wolters and A. Ilchmann, *Positive-Real Structure and High-Gain Adaptive Stabilization*, IMA J. Math. Control Inform. 4, 167-181, 1984.

[4] H. Logemann and D.H. Owens, *Robust High-gain Feedback Control of Infinite-Dimensional Minimum-Phase Systems*, IMA J. Math. Control Inform. 4, 195-220, 1987.

[5] D.E. Miller and E.J. Davison, *The Self-Tuning Robust Servomechanism Problem*, IEEE Trans. Automat. Contr., AC-34, 511-523, 1989.

[6] D.E. Miller and E.J. Davison, *An Adaptive Controller Which Provides Lyapunov Stability*, IEEE Trans. Automat. Contr., AC-34, 599-609, 1989.

[7] H. Logemann, *Adaptive Exponential Stabilization for a Class of Nonlinear Retarded Processes*, Math. Contr., Sign., and Syst.3, 255-269, 1990.

[8] H. Logemann and B. Mårtensson, *Adaptive Stabilization of Infinite Dimensional Szstems*, Report, Institut für Dynamische Systeme, Bremen, 1990.

[9] A. Ilchmann, D.H. Owens and D. Prätzel-Wolters, *High gain robust adaptive controllers for multivariable systems*, Systems Control Lett.8, 397-404, 1987.

[10] H. Logemann and D.H. Owens, *Input-output theory of high-gain adaptive stabilization of infinite-dimensional systems with non-linearities*, Int. J. Adapt. Contr. Sign. Proc.2, 193-216, 1989.

[11] A. Ilchmann and D.H. Owens, *Exponential stabilization using non-differential gain adaptation*, Preprint 29, Institut für Angewandte Mathematik, Hamburg, to appear in IMA J. Math. Control Inform., 1991.

[12] A. Ilchmann and D.H. Owens, *Adaptive stabilization with exponential decay*, Systems Control Lett. 14, 437-443, 1990.

Acknowledgement:
This work was in part supported by the German Research Association (DFG), and by the EEC Science Programme.