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# TRACKING WITH PRESCRIBED TRANSIENT PERFORMANCE FOR HYSTERETIC SYSTEMS

ACHIM ILCHMANN*, HARTMUT LOGEMANN†, AND EUGENE P. RYAN†

**Abstract.** Tracking of reference signals (assumed bounded with essentially bounded derivative) is considered for a class of single-input, single-output, nonlinear systems, described by a functional differential equation with a hysteresis nonlinearity in the input channel. The first control objective is tracking, by the output, with prescribed accuracy: determine a feedback strategy which ensures that, for every reference signal and every system of the underlying class, the tracking error ultimately satisfies the prescribed accuracy requirements. The second objective is guaranteed output transient performance: the graph of the tracking error should be contained in a prescribed set (performance funnel). Under a weak sector boundedness assumption on the hysteresis operator, both objectives are achieved by a memoryless feedback which is universal for the underlying class of systems.

**Key words.** disturbance rejection, functional differential equations, hysteresis, nonlinear systems, tracking, transient behaviour.

**AMS subject classifications.** 93D15, 34K20, 34C55, 47J40.

**1. Introduction.** We consider a class $\mathcal{N}$ of nonlinear, single-input, single-output systems modelled by nonlinear functional differential equations of the form

$$\dot{y}(t) = f(p(t), T(y)(t)) + g\, v(t), \quad y|_{[-h,0]} = y^0 \in C[-h, 0], \tag{1.1}$$

with input $u$ and output $y$. We assume that the continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in its second argument, $p \in L^\infty(\mathbb{R}_+)$ ($\mathbb{R}_+ := [0, \infty)$) is a perturbation or disturbance, $T$ is a causal operator (of a class to be described in due course), $g \neq 0$ is a real parameter, and $h \geq 0$ quantifies the "memory" in the system. With reference to

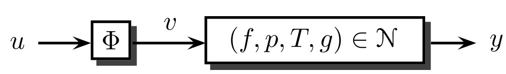


Fig. 1.1. *System with input hysteresis.*

Figure 1.1, the main concern is control of a cascade consisting of a hysteresis operator $\Phi$ (with properties to be defined in Section 2) and a nonlinear system $(f, p, T, g) \in \mathcal{N}$:

$$\dot{y}(t) = f(p(t), T(y)(t)) + g\, \Phi(u)(t), \quad y|_{[-h,0]} = y^0 \in C[-h, 0]\,. \tag{1.2}$$

We remark that, in a systems and control context, hysteretic effects have received increasing attention in recent years: applications include passivity-based control of hysteresis in smart actuators [5], inverse compensation of hysteresis [12, 19, 20], integral control in the presence of hysteretic actuators [15], stability of hysteretic feedback systems [16, 17], and positioning control problems using piezo electric actuators [4].

In the present paper, the primary control objective is tracking with prescribed accuracy: given $\lambda > 0$ (arbitrarily small), determine a single feedback strategy which

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ensures that, for every $(f, p, T, g) \in \mathcal{N}$, every admissible $\Phi$ and every reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$, the tracking error $e = y - r$ is ultimately bounded by $\lambda$ (that is, $|e(t)| < \lambda$ for all $t$ sufficiently large or, equivalently, $\limsup_{t\to\infty} |e(t)| < \lambda$). The second objective is guaranteed output transient performance: for some prescribed function $\beta : [0,\infty) \to [0,\infty)$, the tracking error $e$ is required to satisfy $\beta(t)|e(t)| < 1$ for all $t \geq 0$. Under mild assumptions on the operators $T$ and $\Phi$ (including in particular a weak sector boundedness condition for $\Phi$), both objectives are achieved by a memoryless feedback of the form $u(t) = \nu(k(t))e(t)$, with $k(t) = \alpha(\beta(t)|e(t)|)$ (for suitably chosen functions $\alpha$ and $\nu$), whilst maintaining boundedness of the control $u$ and of the "gain" function $k$. If the parameter $g$ in (1.2) is known to be positive, then the control may take the simplified form $u(t) = -k(t)e(t)$.

In an adaptive control context, the issue of tracking with prescribed transient behaviour dates back at least to the work of Miller & Davison [18]. The approach of the present paper is intrinsically different, the essence of which centres on the concept of a performance funnel, introduced in [8] (with extensions thereof in [9, 10, 11]),

$$F_\beta := \left\{ (t, e) \in \mathbb{R}_+ \times \mathbb{R} \;\middle|\; \beta(t)\,|e| < 1 \right\} \tag{1.3}$$

associated with the function $\beta : \mathbb{R}_+ \to \mathbb{R}$ (the reciprocal of which determines the funnel boundary). The memoryless feedback, alluded to above, ensures that, for every

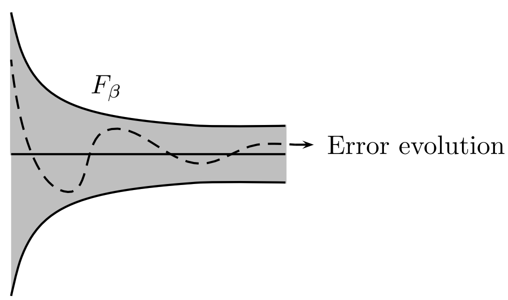


FIG. 1.2. *Performance funnel $F_\beta$.*

reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$, the tracking error $e = y - r$ evolves within the funnel $F_\beta$ and all signals are bounded. For example, if $\beta \in W^{1,\infty}(\mathbb{R}_+)$ is chosen so that $\liminf_{t\to\infty} \beta(t) \geq 1/\lambda > 0$, then evolution within the funnel ensures that the first control objective is achieved: other properties may be imposed on $\beta$ in order to "shape" the transient behaviour; for example, if $\beta$ is chosen as the function $t \mapsto \min\{t/\tau, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $\tau > 0$.

The paper is structured as follows. Section 2 first makes precise the class $\mathcal{N}$ of nonlinear systems and the class of admissible hysteresis operators which constitute the cascades of the form shown in Figure 1 underlying the paper: a prototype subclass of linear retarded systems illustrates the former system class; explicit constructions of backlash, Preisach and Prandtl operators serve to illustrate the latter hysteresis class. Then, we proceed to elucidate the concept of a performance funnel and to formulate the associated control problem. Section 2 terminates with a description of the proposed memoryless feedback control. Section 3 addresses the fundamental question of well posedness of the closed-loop system. This question is answered in the affirmative in Theorem 3.1, a proof of which is provided in the Appendix. Theorem 4.1 in Section 4 establishes the main result of the paper, namely, that the proposed feedback structure ensures attainment of the control objectives of asymptotic tracking with prescribed accuracy and transient behaviour: Corollary 4.1 identifies an additional assumption on the input hysteresis under which rejection of continuous and bounded input disturbances is achieved; Corollary 4.2 highlights a simplified control structure applicable to cases wherein the sign of the non-zero system parameter $g$ is known *a priori*. Finally, in Section 5, a problem of tracking with disturbance rejection is considered – in a context of second-order hysteretic systems and reference signals of class $W^{2,\infty}(\mathbb{R}_+)$ – and resolved via an application of Theorem 3.1 and Corollary 4.2.

**2. Formulation of the control problem.** The purpose of this section is to give a precise formulation of the problem. We first assemble some notation and terminology.

**2.1. Notation and terminology.** Set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{C}_+ := \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s \geq 0\}$ (the *closed* right-half real line and the *closed* right-half complex plane, respectively). Let $I \subset \mathbb{R}_+$ be an interval. We denote the space of continuous functions $I \to \mathbb{R}^n$ by $C(I,\mathbb{R}^n)$: if $I = [a,b]$ or $I = [a,b)$ and $n = 1$, then we simply write $C[a,b]$ or $C[a,b)$. Moreover, $BV[a,b]$ denotes the space of real-valued functions of bounded variation defined on $[a,b]$. For $h,t \in \mathbb{R}_+$, $w \in C[-h,t]$, $\tau > t$ and $\delta > 0$, define

$$\mathcal{C}(w;h,t,\tau,\delta) := \left\{ x \in C[-h,\tau] \,\middle|\, x|_{[-h,t]} = w,\ |x(s) - w(t)| \leq \delta \ \ \forall\ s \in [t,\tau] \right\}.$$

The space of essentially bounded (respectively, locally essentially bounded) measurable functions $I \to \mathbb{R}$ is denoted by $L^\infty(I)$ (respectively, $L^\infty_{\mathrm{loc}}(I)$). The space of locally absolutely continuous bounded functions $I \to \mathbb{R}$ with essentially bounded derivative is denoted by $W^{1,\infty}(I)$: the space of continuously differentiable bounded functions $I \to \mathbb{R}$ with locally absolutely continuous bounded first derivative and essentially bounded second derivative is denoted by $W^{2,\infty}(I)$. An operator $S : C[-h,\infty) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, $h \geq 0$, is *causal* if, and only if, for all $x, y \in C[-h,\infty)$ and all $\tau > 0$,

$$x|_{[-h,\tau]} = y|_{[-h,\tau]} \quad \Longrightarrow \quad S(x)(t) = S(y)(t) \ \text{ for a.a. } t \in [0,\tau].$$

We will have occasion to give meaning to $S(x)$, where $x \in C(I)$ and $I$ is a bounded interval of the form $[-h,a)$ or $[-h,a]$ with $0 < a < \infty$. This we do by showing that $S$ "localizes", in a natural way, to an operator $\tilde{S} : C(I) \to L^\infty_{\mathrm{loc}}(J)$, where $J := I \backslash [-h,0)$. For each $x \in C(I)$ and each $\sigma \in J$, define $x_\sigma \in C[-h,\infty)$ by

$$x_\sigma(t) := \begin{cases} x(t), & t \in [-h,\sigma] \\ x(\sigma), & t > \sigma\,. \end{cases}$$

By causality, we may define $\tilde{S}(x) \in L^\infty_{\mathrm{loc}}(J)$ by the property

$$\tilde{S}(x)|_{[0,\sigma]} = S(x_\sigma)|_{[0,\sigma]} \quad \forall\ \sigma \in J.$$

Henceforth, we will not distinguish notationally between an operator $S$ and its "localisation" $\tilde{S}$, the correct interpretation being clear from the context.

**2.2. Nonlinear system class.** With reference to (1.1), we first define the class of operators $\mathcal{O}_h$, parameterized by $h \geq 0$, to which $T$ belongs.

Definition 2.1. **(Operator class $\mathcal{O}_h$)**
An operator $T$ is deemed to be of class $\mathcal{O}_h$ if, and only if, the following hold.

(i) $T\colon C[-h,\infty) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$.

(ii) $T$ is a causal operator.

(iii) For all $t \geq 0$ and all $w \in C[-h,t]$, there exist $\tau > t$, $\delta > 0$ and $c_0 > 0$ such that

$$\text{ess-sup}_{s\in[t,\tau]}|T(x)(s) - T(y)(s)| \;\leq\; c_0 \sup_{s\in[t,\tau]} |x(s) - y(s)| \quad \forall\, x, y \in \mathcal{C}(w; h, t, \tau, \delta);$$

(iv) For all $c_1 > 0$, there exists $c_2 > 0$ such that, for all $y \in C[-h,\infty)$,

$$\sup_{t\in[-h,\infty)} |y(t)| \leq c_1 \quad \Longrightarrow \quad \text{ess-sup}_{t\in\mathbb{R}_+}|T(y)(t)| \leq c_2\,. \qquad \diamond$$

In interpreting property (iii) of the operator class $\mathcal{O}_h$, recourse should be made to the "localization" procedure outlined previously.

We are now in a position to define the class $\mathcal{N}$ of nonlinear systems.

Definition 2.2. **(System class $\mathcal{N}$)**
The class $\mathcal{N}$ is comprised of single-input, single-output, nonlinear systems $(f, p, T, g)$ of the form (1.1), satisfying the following assumptions:

(i) $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(z,\cdot)$ is locally Lipschitz for every $z \in \mathbb{R}$;

(ii) $g \in \mathbb{R}$ is non-zero;

(iii) $p \in L^\infty(\mathbb{R}_+)$;

(iv) $T \in \mathcal{O}_h$, where $h \geq 0$. $\diamond$

With reference to Figure 2.1, a system (1.1) of class $\mathcal{N}$ can be thought of as an interconnection of two (sub) systems. The dynamical system $\Lambda_1$, which can be influenced directly by the system input $v$, is also driven by the output $w$ from the system $\Lambda_2$, formulated as a causal operator mapping the system output $y$ to $w$ (an internal quantity, unavailable for feedback purposes); for example, $\Lambda_2$ can encompass infinite-dimensional processes (e.g. delays and diffusions) or nonlinear, input-to-state stable systems given by

$$\dot{z}(t) = a(z(t), y(t)), \quad w(t) = c(z(t)), \quad z(0) = z^0 \in \mathbb{R}^m,$$

with locally Lipschitz functions $a : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $c : \mathbb{R}^m \to \mathbb{R}$ (for details, see [8]).

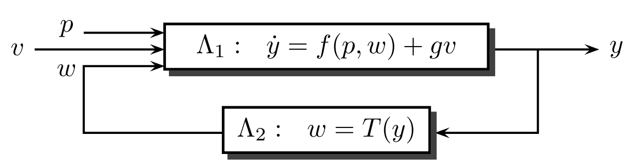


Fig. 2.1. *System of class $\mathcal{N}$.*

By way of illustration, in the following we consider a class of linear retarded systems and show that it is contained in $\mathcal{N}$.

Example 2.3. Let $h > 0$, let $A$ be an $n \times n$-matrix with entries in $BV[0,h]$ and let $b, c^T \in \mathbb{R}^n$. Consider the retarded system

$$\dot{x} = \mathrm{d}A * x + bv\,, \qquad x|_{[-h,0]} = x^0 \in C([-h,0], \mathbb{R}^n)\,, \tag{2.1a}$$

$$y = cx\,, \tag{2.1b}$$

where $(\mathrm{d}A * x)(t) := \int_0^h \mathrm{d}A(\tau)x(t-\tau)$ for all $t \in \mathbb{R}_+$. We assume that the system (2.1) satisfies the following two conditions:

- *minimum-phase* condition, i.e.,

$$\det \begin{pmatrix} sI - \hat{A}(s) & -b \\ c & 0 \end{pmatrix} \neq 0 \quad \forall\, s \in \mathbb{C}_+ \,,$$

  where $\hat{A}(s) := \int_0^h \exp(-s\tau)\mathrm{d}A(\tau)$.
- *relative degree one* condition, i.e., $cb \neq 0$.

It is well-known that, under these assumptions, there exists a similarity tranformation which takes the system into the form

$$\dot{y} = \mathrm{d}A_{11} * y + \mathrm{d}A_{12} * z + cbv \,, \quad y|_{[-h,0]} = y^0 \,, \tag{2.2a}$$

$$\dot{z} = \mathrm{d}A_{21} * y + \mathrm{d}A_{22} * z \,, \quad z|_{[-h,0]} = z^0 \,, \tag{2.2b}$$

where, by the minimum-phase condition, $A_{22}$ has the property that

$$\det(sI - \hat{A}_{22}(s)) \neq 0 \quad \forall\, s \in \mathbb{C}_+ \,, \tag{2.3}$$

see [7, 14] for details. For given $z^0 \in C([-h,0],\mathbb{R}^{n-1})$ and given $\xi \in C[-h,\infty)$, let $z(\cdot; z_0, \xi)$ denote the unique solution of the initial-value problem

$$\dot{z} = \mathrm{d}A_{22} * z + \mathrm{d}A_{21} * \xi \,, \qquad z|_{[-h,0]} = z^0 \,.$$

Setting

$$T(\xi) := \mathrm{d}A_{11} * \xi + \mathrm{d}A_{12} * z(\cdot\,; 0, \xi) \,, \quad p := \mathrm{d}A_{12} * z(\cdot\,; z_0, 0) \,,$$

equation (2.2a) can be expressed as

$$\dot{y} = p + T(y) + cbv \,, \qquad y^0 = cx^0 \,. \tag{2.4}$$

By a standard result from the theory of retarded functional differential equations (see [6, Corollary 6.1, p. 215]), (2.3) implies that the zero solution of the retarded equation $\dot{z} = \mathrm{d}A_{22} * z$ is exponentially stable, so that there exists $K > 0$ such that, for all $z^0 \in C([-h,0],\mathbb{R}^{n-1})$ and all $\xi \in C[-h,\infty)$,

$$\sup_{t\in[0,\infty)} |z(t; z_0, \xi)| \leq K\big( \sup_{t\in[-h,0]} |z^0(t)| + \sup_{t\in[-h,\infty)} |\xi(t)|\big) \,.$$

We conclude that $p$ is bounded and that $T \in \mathcal{O}_h$. Consequently, the system given by (2.4) is in the system class $\mathcal{N}$. $\diamond$

**2.3. Class of input nonlinearities.** Causal operators $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ satisfying some or all of the following conditions will be considered.

**H1.** There exists $c_0 > 0$ such that, for all $t \geq 0$ and all $w \in C[0,t]$, there exist $\tau > t$ and $\delta > 0$ such that

$$\sup_{s\in[t,\tau]} |\Phi(u_1)(s) - \Phi(u_2)(s)| \leq c_0 \sup_{s\in C[t,\tau]} |u_1(s) - u_2(s)| \quad \forall\, u_1, u_2 \in \mathcal{C}(w; 0, t, \tau, \delta) \,. \tag{2.5}$$

**H2.** For all $\omega > 0$ and all $u \in C[0,\omega)$, there exists $c_1 > 0$ such that

$$\sup_{s\in[0,t]} |\Phi(u)(s)| \leq c_1 \big(1 + \sup_{s\in[0,t]} |u(s)|\big) \quad \forall\, t \in [0,\omega) \,.$$

**H3.** There exist $c_2 > 0$ and $c_3 > 0$ such that, for all $u \in C(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$,

$$|u(t)| \geq c_2 \quad \Longrightarrow \quad c_3 u^2(t) \leq u(t)(\Phi(u)(t)).$$

**H4.** For each bounded $d \in C(\mathbb{R}_+)$, there exists $c_d > 0$ such that,

$$|\Phi(u+d)(t) - \Phi(u)(t)| \leq c_d \quad \forall u \in C(\mathbb{R}_+),\ \forall\, t \in \mathbb{R}_+\,.$$

Again, in interpreting H1 and H2, recourse should be made to the "localization" procedure outlined at the beginning of this section. A sufficient condition for H1, H2 and H4 to be satisfied is that $\Phi$ is *Lipschitz continuous* in the sense that there exists a Lipschitz constant $L \geq 0$ such that

$$\sup_{t\in\mathbb{R}_+} |\Phi(u_1)(t) - \Phi(u_2)(t)| \leq L \sup_{t\in\mathbb{R}_+} |u_1(t) - u_2(t)| \quad \forall\, u_1, u_2 \in C(\mathbb{R}_+).$$

We emphasize that many hysteresis operators satisfy conditions H1–H4, where we say that $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if, and only if, $\Phi$ is causal and rate independent. Here *rate independence* means that $\Phi(u \circ \zeta) = (\Phi u) \circ \zeta$ for every $u \in C(\mathbb{R}_+)$ and every time transformation $\zeta$, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *time transformation* if, and only if, it is continuous, non-decreasing and surjective.

We briefly digress to state the following lemma (which will play a role in Corollary 4.1 below). The proof is routine and is therefore omitted.

Lemma 2.4. *Let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be causal, let $d \in C(\mathbb{R}_+)$ be bounded and define the causal operator $\Phi_d : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by $\Phi_d(u) = \Phi(u+d)$ for all $u \in C(\mathbb{R}_+)$. Then the following statements hold:*

(i) *If $\Phi$ satisfies any of the assumptions* H1 *or* H2*, then so does $\Phi_d$.*

(ii) *If $\Phi$ satisfies* H3 *and* H4*, then* H3 *holds for $\Phi_d$.*

In the following, we give examples of hysteresis operators satisfying H1–H4.

**Backlash hysteresis.** A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references, see for example [2], [3], [13] and [15]. Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1, v_2) := \max\{v_1 - \sigma\,,\, \min\{v_1 + \sigma, v_2\}\} = \begin{cases} v_1 - \sigma, & \text{if } v_2 < v_1 - \sigma \\ v_2, & \text{if } v_2 \in [v_1 - \sigma, v_1 + \sigma] \\ v_1 + \sigma, & \text{if } v_2 > v_1 + \sigma\,. \end{cases}$$

Let $C_{\mathrm{pm}}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and all $\xi \in \mathbb{R}$, we define the operator $\mathcal{B}_{\sigma,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$\mathcal{B}_{\sigma,\xi}(u)(t) = \begin{cases} b_\sigma(u(0), \xi) & \text{for } t = 0\,, \\ b_\sigma(u(t), (\mathcal{B}_{\sigma,\xi}(u))(t_i)) & \text{for } t_i < t \leq t_{i+1},\ i = 0, 1, 2, \dots\,, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{n\to\infty} t_n = \infty$ and $u$ is monotone on each interval $[t_i, t_{i+1}]$. We remark that $\xi$ plays the role of an "initial state". It is not difficult to show that the definition is independent of the choice of the partition $(t_i)$. Figure 2.2 illustrates how $\mathcal{B}_{\sigma,\xi}$ acts. It is well-known that $\mathcal{B}_{\sigma,\xi}$ extends to a Lipschitz continuous operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator,

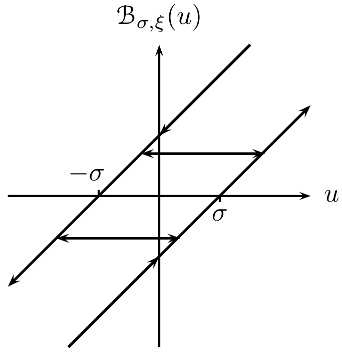


FIG. 2.2. *Backlash hysteresis*

which we shall denote by the same symbol $\mathcal{B}_{\sigma,\,\xi}$. It is well-known (and easy to check) that $\mathcal{B}_{\sigma,\,\xi}$ is a hysteresis operator. By Lipschitz continuity, $\mathcal{B}_{\sigma,\,\xi}$ satisfies H1, H2 and H4. It is trivial that $\mathcal{B}_{\sigma,\,\xi}$ also enjoys property H3. We also remark that the operator $\mathcal{B}_{\sigma,\,\xi}$ is in the class $\mathcal{O}_0$.

**Preisach and Prandtl hysteresis.** The Preisach operator described below encompasses both backlash and Prandtl operators. It can model complex hysteresis effects: for example, nested loops in input-output characteristics. Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let $\mu$ be a signed Borel measure on $\mathbb{R}_+$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset \mathbb{R}_+$, where $|\mu|$ denotes the total variation of $\mu$. Denoting the Lebesgue measure on $\mathbb{R}$ by $\mu_L$, let $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(\mathcal{P}_\xi(u))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(u))(t)} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + w_0\,, \qquad \forall\, u \in C(\mathbb{R}_+)\,,\ \ \forall\, t \in \mathbb{R}_+\,, \quad (2.6)$$

is called a *Preisach* operator, cf. [3, p. 55]. It is well-known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \geq 0$). Under the assumption that the measure $\mu$ is finite and $w$ is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $|\mu|(\mathbb{R}_0)\|w\|_\infty$, see [15], and thus, $\mathcal{P}_\xi$ satisfies H1, H2 and H4. If, in addition, $\mu$ and $w$ are non-negative and the following hold

$$\mu \neq 0, \quad \int_0^\infty \sigma\mu(\mathrm{d}\sigma) < \infty, \quad 0 < \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma)\,,$$

then H3 is also satisfied (a proof of this fact is provided in Appendix 1). As in the case of backlash, we remark that the operator $\mathcal{P}_\xi$ is also in the class $\mathcal{O}_0$.

Setting $w(\cdot,\cdot) = 1$ and $w_0 = 0$ in (2.6), we obtain the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$\mathcal{P}_\xi(u)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(u))(t)\mu(\mathrm{d}\sigma) \quad \forall\, u \in C(\mathbb{R}_+)\,,\ \ \forall\, t \in \mathbb{R}_+\,. \qquad (2.7)$$

For $\xi \equiv 0$ and $\mu$ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$ (where $\chi_{[0,5]}$ denotes the indicator function of the interval $[0,5]$), the Prandtl operator is illustrated in Figure 2.3.

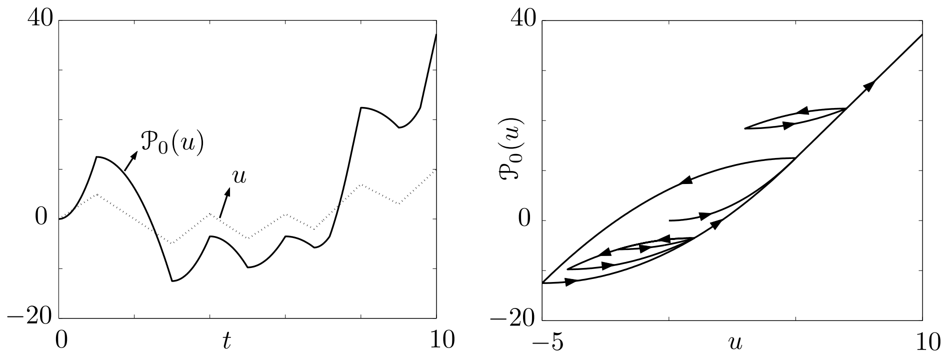


FIG. 2.3. *Example of Prandtl hysteresis*

**2.4. Control objectives and the performance funnel.** The first control objective is approximate tracking, by the output $y$ of system (1.2) (illustrated in Figure 1.1), of reference signals $r \in W^{1,\infty}(\mathbb{R}_+)$. In particular, for arbitrary $\gamma \geq 0$ and $\lambda > 0$, we seek an output feedback strategy which ensures that, for every $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h, 0]$ with $\gamma|y^0(0) - r(0)| < 1$, the unique solution of closed-loop system is bounded and the tracking error $e(t) = y(t) - r(t)$ is ultimately bounded by $\lambda$ (that is, $|e(t)| < \lambda$ for all $t$ sufficiently large). The second control objective is prescribed transient behaviour of the tracking error signal. We capture both objectives in the concept of a performance funnel, introduced in [8] and defined in (1.3), associated with a function $\beta : \mathbb{R}_+ \to \mathbb{R}$ (the reciprocal of which determines the funnel boundary) belonging to

$$W_{\gamma,\lambda} := \left\{ \beta \in W^{1,\infty}(\mathbb{R}_+) \middle| \ \beta(0) = \gamma,\ \beta(s) > 0 \ \ \forall\, s > 0,\ \liminf_{s\to\infty} \beta(s) \geq 1/\lambda \right\},$$

with $\gamma \geq 0$ and $\lambda > 0$. The aim is an output feedback strategy ensuring that, for every reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$ and every $y^0 \in C[-h, 0]$ with $\gamma|y^0(0) - r(0)| < 1$, the tracking error $e = y - r$ evolves within the funnel $F_\beta$ and all signals are bounded. For every $\gamma \geq 0$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$, evolution within the funnel ensures that the first control objective is achieved: moreover, $\beta$ can be chosen to influence the transient behaviour; for example, reiterating comments in the Introduction, if $\tau > 0$, $\gamma = 0$ and $\beta$ is chosen as the function $t \mapsto \min\{t/\tau, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $\tau > 0$ for all $y^0 \in C[-h, 0]$ and all $r \in W^{1,\infty}(\mathbb{R}_+)$.

REMARK 2.5. Some elucidation on the role of the parameter $\gamma \geq 0$ is warranted. In the absence of *a priori* information on the initial function $y^0 \in C[-h, 0]$, we simply set $\gamma = 0$. On the other hand, if sufficient *a priori* information is available to compute an upper bound $\delta > 0$ for the quantity $|y^0(0) - r(0)|$, then any $\gamma \in [0, 1/\delta)$ may be chosen: in particular, the choice $0 < \gamma < 1/\delta$, yields a uniform bound, viz. $\sup_{t\in\mathbb{R}_+} |y(t) - r(t)| \leq 1/\beta^*$, $\beta^* := \inf_{t\in\mathbb{R}_+} \beta(t) > 0$, on the tracking error associated with the solution $y$ corresponding to any initial function $y^0$ and reference signal $r$ with the property $\gamma|y^0(0) - r(0)| < 1$. This observation will play a role in Section 5 below. In many situations, a non-decreasing function $\beta$ is a natural choice, in which case $\beta^* = \gamma$.

**2.5. Output feedback.** Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and let $\alpha : [0, 1) \to \mathbb{R}_+$ be a locally Lipschitz unbounded injection (for example, $\alpha\colon s \mapsto 1/(1 - s)$). For

$r \in W^{1,\infty}(\mathbb{R}_+)$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$, consider the control strategy

$$u(t) = \nu(k(t))\,\big(y(t) - r(t)\big), \qquad k(t) = \alpha\big(\beta(t)\,|y(t) - r(t)|\big)\,. \tag{2.8}$$

The main contribution of the paper is to show that the feedback (2.8) applied to any cascade (as in Figure 1), given by (1.2), achieves the control objectives provided that the function $\nu$ has the following properties

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(k) = -\infty\,. \tag{2.9}$$

A simple example of a function satisfying (2.9) is $\nu : k \mapsto k\,\cos k$. In view of the nature of the function $\alpha$, care must be exercised in interpreting the closed-loop system. This we do in the next section, wherein we show that the closed-loop initial-value problem is well posed.

**3. The closed-loop system.** Let $(f, p, T, g) \in \mathcal{N}$ and let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be a causal operator satisfying H1. Let $r \in W^{1,\infty}(\mathbb{R}_+)$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and let $\alpha\colon [0, 1) \to \mathbb{R}_+$ be a locally Lipschitz unbounded injection. The conjunction of the system (1.2) and control (2.8) yields the closed-loop initial-value problem

$$\left.\begin{aligned} &\dot y(t) = f(p(t), T(y)(t)) + g\,\Phi(u)(t), \quad y|_{[-h,0]} = y^0 \in C[-h, 0],\\ &u(t) = \nu(k(t))\big(y(t) - r(t)\big),\\ &k(t) = \alpha\big(\beta(t)|y(t) - r(t)|\big). \end{aligned}\right\} \tag{3.1}$$

Writing

$$\mathcal{D} := \{(t, z) \in \mathbb{R}_+ \times \mathbb{R} \mid \beta(t)|z - r(t)| < 1\}, \tag{3.2}$$

then, by a *solution* of (3.1), we mean a continuous function $y : I \to \mathbb{R}$ on some interval $I$ of the form $[-h, \rho]$, with $0 < \rho < \infty$, or of the form $[-h, \omega)$, with $0 < \omega \le \infty$, such that (a) $y|_{[-h,0]} = y^0$ and (b) $y|_J$, $J := I \setminus [-h, 0)$, has graph in $\mathcal{D}$, is locally absolutely continuous and satisfies the differential equation in (3.1) almost everywhere on $J$. A solution is *maximal* if, and only if, it has no right extension that is also a solution.

THEOREM 3.1. *Let $(f, p, T, g) \in \mathcal{N}$ and let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be a causal operator satisfying* H1 *and* H2. *Let $r \in W^{1,\infty}(\mathbb{R}_+)$, $\gamma \ge 0$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and let $\alpha : [0, 1) \to \mathbb{R}_+$ be a locally Lipschitz unbounded injection. Then, for each $y^0 \in C[-h, 0]$ with $\gamma|y^0(0) - r(0)| < 1$, the initial-value problem* (3.1) *has a unique maximal solution $y \in C[-h, \omega)$. Moreover, if $\omega < \infty$, then* $\limsup_{t\uparrow\omega} \beta(t)|y(t) - r(t)| = 1$ *(or, equivalently,* $\limsup_{t\uparrow\omega} k(t) = \infty$*).*

A proof of this theorem is contained in the Appendix. We emphasize that, in Theorem 3.1, the causal operator $\Phi$ is required only to satisfy H1 and H2 and the function $\nu$ is assumed only to be locally Lipschitz. These assumptions are not sufficient to ensure that, for each $y^0 \in C[-h, 0]$, the unique maximal solution $y \in C[-h, \omega)$ is such that $\omega = \infty$; however, if $\Phi$ is such that H3 also holds and $\nu$ has properties (2.9), then $\omega = \infty$. The latter is the essence of Theorem 4.1 below.

**4. The main result.** We are now in a position to state and prove the main result of the paper, part (ii) of which asserts that the tracking error evolves within the performance funnel (and so the control objectives are achieved) and, moreover, is bounded away from the funnel boundary.

Theorem 4.1. *Let $(f,p,T,g) \in \mathcal{N}$ and let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be causal and such that* H1–H3 *are satisfied. Let $\gamma \geq 0$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function with properties* (2.9) *and let $\alpha : [0,1) \to \mathbb{R}_+$ be a locally Lipschitz unbounded injection. For each $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h,0]$ with $\gamma|y^0(0) - r(0)| < 1$, the unique maximal solution $y : [-h,\omega) \to \mathbb{R}$ of the closed-loop initial-value problem* (3.1) *is such that:*

(i) $\omega = \infty$*;*

(ii) *there exists $\varepsilon \in (0,1)$ such that $\beta(t)\,|y(t) - r(t)| \leq 1 - \varepsilon$ for all $t \in \mathbb{R}_+$;*

(iii) *the continuous functions $u$, $\Phi u : \mathbb{R}_+ \to \mathbb{R}$ and $k\colon \mathbb{R}_+ \to \mathbb{R}_+$ are bounded.*

**Proof.** Let $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h,0]$ be such that $\gamma|y^0(0) - r(0)| < 1$. An application of Theorem 3.1 establishes the existence of a unique maximal solution $y \in C[0,\omega)$ of (3.1), with $0 < \omega \leq \infty$. Since $(t, y(t)) \in \mathcal{D}$ for all $t \in [0,\omega)$ and $r$ is bounded, it follows from (3.2) that $y$ is bounded; moreover, by property (iv) of the operator class $\mathcal{O}_h$ (see Definition 2.1), the function $Ty$ is bounded. Writing

$$e(t) = y(t) - r(t), \quad k(t) = \alpha(\beta(t)|e(t)|), \quad u(t) = \nu(k(t))e(t) \quad \forall\, t \in [0,\omega),$$

we have $\beta(t)|e(t)| < 1$ for all $t \in [0,\omega)$ and, since $y$ and $r$ are bounded, the function $e = y - r$ is bounded. Moreover,

$$\dot e(t) = f(p(t), T(y)(t)) + g\,\Phi(u)(t) - \dot r(t) \quad \text{for a.a. } t \in [0,\omega).$$

By continuity of $f$, boundedness of $T(y)$ and $e$, and essential boundedness of $p$ and $\dot r$, there exists $c_0 > 0$ such that

$$e(t)\dot e(t) \leq c_0 + g\,e(t)\Phi(u)(t) \quad \text{for a.a. } t \in [0,\omega). \tag{4.1}$$

Observe that, by boundedness of $\beta$ and $e$, essential boundedness of $\dot\beta$ and inequality (4.1), there exists $c_1 > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\beta(t)e(t)\big)^2 &= 2\beta(t)\dot\beta(t)e^2(t) + 2\beta^2(t)e(t)\dot e(t)\\ &\leq c_1\big(1 + g\,e(t)\Phi(u)(t)\big) \quad \text{for a.a. } t \in [0,\omega).\end{aligned} \tag{4.2}$$

Next, we show that $k$ is bounded. By properties (2.9) of $\nu$, there exists a strictly increasing unbounded sequence $(k_n)_{n\in\mathbb{N}}$, with $k_n > \alpha(1/2)$ for all $n \in \mathbb{N}$, such that $\big(g\,\nu(k_n)\big)$ is a strictly decreasing unbounded sequence, with $g\,\nu(k_n) < 0$ for all $n \in \mathbb{N}$. Seeking a contradiction, suppose that $k$ is unbounded. For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0,\omega)|\ k(t) = k_{n+1}\}, \quad \sigma_n := \sup\{t \in [0,\tau_n]|\ \nu(k(t)) = \nu(k_n)\} < \tau_n\,,$$

wherein the latter inequality holds since $|\nu(k(\tau_n))| = |\nu(k_{n+1})| > |\nu(k_n)|$. Then, the following inequalities hold:

$$\left.\begin{aligned}&k_n \leq k(t) \quad \text{and} \quad |\nu(k_n)| \leq |\nu(k(t))|\\ &\beta(t)|e(t)| = \alpha^{-1}(k(t)) \geq \alpha^{-1}(k_n) > 1/2\\ &|e(t)| \geq 1/\big(2\sup_{s\geq 0}\beta(s)\big) =: c_2\end{aligned}\right\} \quad \forall\, t \in [\sigma_n,\tau_n],\ \forall\, n \in \mathbb{N} \tag{4.3}$$

wherein $\alpha^{-1}$ denotes the inverse of the bijection $\alpha : [0,1) \to [\alpha(0),\infty)$. By property H3 of $\Phi$, there exist $\Delta, \delta > 0$ such that

$$|u(t)| \geq \Delta \quad \Longrightarrow \quad \delta u^2(t) \leq u(t)\Phi(u)(t).$$

Choose $N \in \mathbb{N}$ sufficiently large so that $c_2|\nu(k_N)| \geq \Delta$. By (4.3), it follows that

$$|u(t)| = |\nu(k(t))e(t)| \geq c_2|\nu(k_N)| \geq \Delta \quad \forall\, t \in [\sigma_n, \tau_n],\ \forall\, n > N\,,$$

and so

$$\begin{aligned}\nu(k(t))e(t)\Phi(u)(t) &= u(t)\Phi(u)(t) \\ &\geq \delta u^2(t) = \delta(\nu(k(t))e(t))^2 \quad \forall\, t \in [\sigma_n, \tau_n],\ \forall\, n > N. \end{aligned} \tag{4.4}$$

Since $g\nu(k(t)) \leq g\nu(k_n) < 0$ for all $t \in [\sigma_n, \tau_n]$ and all $n \in \mathbb{N}$, we may conclude, from (4.3) and (4.4), that

$$\begin{aligned} g\, e(t)(\Phi u)(t) &\leq \delta g\, \nu(k(t))e^2(t) \\ &= -\delta|g\,\nu(k(t))|e^2(t) \leq -\delta c_2^2|g\,\nu(k_n)| \quad \forall\, t \in [\sigma_n, \tau_n],\ \forall\, n > N\,, \end{aligned}$$

which, in conjunction with (4.2), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\beta(t)e(t)\big)^2 \leq c_1\big(1 - \delta c_2^2|g\,\nu(k_n)|\big) \quad \forall\, t \in [\sigma_n, \tau_n],\ \forall\, n > N \tag{4.5}$$

Now fix $m > N$ sufficiently large so that

$$\delta\, c_2^2\, |g\,\nu(k_m)| > 1.$$

By (4.5), we have

$$\big(\beta(\tau_m)e(\tau_m)\big)^2 - \big(\beta(\sigma_m)e(\sigma_m)\big)^2 < 0,$$

and so $\beta(\tau_m)|e(\tau_m)| < \beta(\sigma_m)|e(\sigma_m)|$, whence the contradiction

$$0 > \alpha\big(\beta(\tau_m)|e(\tau_m)|\big) - \alpha\big(\beta(\sigma_m)|e(\sigma_m)|\big) = k(\tau_m) - k(\sigma_m) \geq 0.$$

This proves boundedness of $k$.

We may now infer that there exists $\varepsilon > 0$ such that $\beta(t)|e(t)| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$. By the second assertion of Theorem 3.1, it follows that $\omega = \infty$. Finally, boundedness of $e$ and $k$ implies boundedness of $u = \nu(k)e$ whence, by property H2 of $\Phi$, boundedness of $\Phi u$. This completes the proof. □

Finally, let us consider the closed-loop system (3.1) in the presence of a bounded continuous input disturbance $d$, that is, we replace (3.1) by

$$\left.\begin{aligned} &\dot{y}(t) = f(p(t), T(y)(t)) + g\,\Phi(u + d)(t), \quad y|_{[-h,0]} = y^0 \in C[-h, 0], \\ &u(t) = \nu(k(t))\big(y(t) - r(t)\big), \\ &k(t) = \alpha\big(\beta(t)|y(t) - r(t)|\big). \end{aligned}\right\} \tag{4.6}$$

The following result shows that, if $\Phi$ satisfies H1–H4, then the conclusions of Theorem 4.1 remain valid in the presence of bounded continuous input disturbances $d$.

Corollary 4.1. *Let $(f, p, T, g) \in \mathcal{N}$ and let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be causal and such that* H1–H4 *are satisfied. Let $\gamma \geq 0$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function with properties* (2.9) *and let $\alpha : [0, 1) \to \mathbb{R}_+$ be a locally Lipschitz unbounded injection. Then, for each bounded $d \in C(\mathbb{R}_+)$, and each $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h, 0]$ with $\gamma|y^0(0) - r(0)| < 1$, the unique maximal solution $y : [-h, \omega) \to \mathbb{R}$ of the closed-loop initial-value problem* (4.6) *is such that* *statements* (i)-(iii) *of* Theorem 4.1 *hold.*

The proof of Corollary 4.1 is a straightforward application of Lemma 2.4 and Theorem 4.1.

Inspection of the proof of Theorem 4.1 reveals that the role of properties (2.9) of $\nu$ is simply to ensure the existence of a strictly increasing unbounded sequence $(k_n)$, with $k_n > \alpha(1/2)$ for all $n$, such that $\big(g\,\nu(k_n)\big)$ is a strictly decreasing unbounded sequence with $g\,\nu(k_n) < 0$ for all $n$. If $(f,p,T,g) \in \mathcal{N}$ is such that the sign of $g$ is known *a priori*, then the latter property is assured if $\nu$ is chosen to be the linear function $k \mapsto -k\,\mathrm{sgn}(g)$. This observation leads immediately to the following result.

COROLLARY 4.2. *Let $(f,p,T,g) \in \mathcal{N}$ be such that $g > 0$. Let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be causal and such that* H1–H4 *are satisfied. Let $\gamma \geq 0$, $\lambda > 0$ and $\beta \in W_{\gamma,\lambda}$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$, $k \mapsto -k$ and let $\alpha : [0,1) \to \mathbb{R}_+$ be a locally Lipschitz unbounded injection. Then, for each bounded $d \in C(\mathbb{R}_+)$, and each $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h,0]$ with $\gamma|y^0(0) - r(0)| < 1$, the unique maximal solution $y : [-h,\omega) \to \mathbb{R}$ of the closed-loop initial-value problem* (4.6) *is such that statements* (i)-(iii) *of* Theorem 4.1 *hold.*

**5. Tracking and disturbance rejection for second-order hysteretic systems.** Consider the problem of tracking a reference signal $\rho \in W^{2,\infty}(\mathbb{R}_+)$ for single-input systems of the following form:

$$m\ddot{x} + c\dot{x} + \Psi(x) = \Phi(u+d) + q, \quad x(0) = x^0, \quad \dot{x}(0) = x^1, \quad m > 0, \tag{5.1}$$

with control input $t \mapsto u(t) \in \mathbb{R}$, bounded disturbances $d \in C(\mathbb{R}_+)$ and $q \in L^\infty(\mathbb{R}_+)$, and causal operators $\Psi$ and $\Phi$. In a mechanical context, $x(t)$ represents displacement at time $t \in \mathbb{R}_+$, and $m, c \in \mathbb{R}$ are the mass and damping constants. The operator $\Psi$ models a restoring force which may exhibit hysteresis phenomena, a particular example of which is the "hysteric spring" model discussed in, for example, [1]; the operator $\Phi$ may model hysteretic actuation as in, for example, micro-positioning control problems using piezo-electric actuators or smart actuators investigated in, *inter alia*, [4, 5, 12, 19, 20]. Without loss of generality, we may assume that $m = 1$. We also assume that both the displacement $x(t)$ and velocity $\dot{x}(t)$ are available for feedback purposes. Finally, we assume that the vector of initial data $(x^0, x^1)$ belongs to a known compactum and, moreover, the vector $(\rho(0), \dot{\rho}(0))$ also belongs to a known compactum, viz. there exist compact $X, Y \subset \mathbb{R}^2$ such that

$$(x^0, x^1) \in X, \quad (\rho(0), \dot{\rho}(0)) \in Y.$$

Fix $\lambda > 0$ and $\eta > 0$. The control objective is formulated as follows: determine a (time-dependent) feedback strategy which ensures the existence of a constant $M > 0$ such that, for every $\rho \in W^{2,\infty}(\mathbb{R}_+)$ with $(\rho(0), \dot{\rho}(0)) \in Y$, for all initial data $(x^0, x^1) \in X$ and all bounded disturbances $d \in C(\mathbb{R}_+)$ and $q \in L^\infty(\mathbb{R}_+)$, the closed-loop initial-value problem has unique solution $x$ on $\mathbb{R}_+$ and there exists $\delta \in (0,1)$ such that the tracking error $x - \rho$ approaches the interval $[-\delta\lambda\,,\,\delta\lambda]$ $\eta$-exponentially fast, in the following sense:

$$|x(t) - \rho(t)| \leq M e^{-\eta t} + \delta\lambda \ \ \forall\, t \in \mathbb{R}_+.$$

We proceed to construct a feedback which achieves this objective. Define

$$y^* := \max\big\{|x^0 - \rho^0 + (x^1 - \rho^1)/\eta| \;\big|\; (x^0, x^1) \in X,\ (\rho^0, \rho^1) \in Y\big\}. \tag{5.2}$$

Let $\gamma > 0$ be such that $\gamma < \min\{1/\lambda\,,\,1/y^*\}$. Let $\tau > 0$ be arbitrary and define $\beta \in W_{\gamma,\lambda}$ by

$$\beta(t) := \min\{\max\{\gamma\lambda\,,\,t/\tau\}\,,\,1\}/\lambda. \tag{5.3}$$

Observe that $\beta$ is non-decreasing with $\min_{t\in\mathbb{R}_+}\beta(t) = \beta(0) = \gamma$ and $\max_{t\in\mathbb{R}_+}\beta(t) = \beta(\tau) = 1/\lambda$. Let $\alpha\colon [0,1) \to \mathbb{R}$ be a locally Lipschitz unbounded injection. Introducing the feedback strategy

$$\begin{aligned} u(t) &= -k(t)\big(x(t) - \rho(t) + ((\dot x(t) - \dot\rho(t))/\eta)\big),\\ k(t) &= \alpha\big(\beta(t)|x(t) - \rho(t) + ((\dot x(t) - \dot\rho(t))/\eta)|\big), \end{aligned}$$

we arrive at the closed-loop initial-value problem

$$\left.\begin{aligned} &\ddot x(t) + c\dot x(t) + \Psi(x)(t) = \Phi(u+d)(t) + q(t), \quad c \in \mathbb{R}\,,\\ &(x(0), \dot x(0)) = (x^0, v^0) \in X,\\ &u(t) = -k(t)(y(t) - r(t)),\ k(t) = \alpha(\beta(t)|y(t) - r(t)|),\\ &y(t) = x(t) + (\dot x(t)/\eta), \quad r(t) := \rho(t) + (\dot\rho(t)/\eta,), \quad (\rho(0), \dot\rho(0)) \in Y. \end{aligned}\right\} \tag{5.4}$$

THEOREM 5.1. *Let $\Psi$ be a causal operator of class $\mathcal{O}_0$ and let $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be a causal operator satisfying* (H1)-(H4)*. Define*

$$M := (x^* + 1/\gamma)e^{\eta\tau}\,, \quad \textit{where} \;\; x^* := \max\big\{|x^0 - \rho^0|\;\big|\;\; (x^0, v^0) \in X,\ (\rho^0, \rho^1) \in Y\big\}.$$

*For every $\rho \in W^{2,\infty}(\mathbb{R}_+)$ with $(\rho(0), \dot\rho(0)) \in Y$, $(x^0, v^0) \in X$, $q \in L^\infty(\mathbb{R}_+)$ and bounded $d \in C(\mathbb{R}_+)$, the closed-loop initial-value problem* (5.4) *has a unique maximal solution $x\colon [0,\omega) \to \mathbb{R}$. Moreover,*

(i) $\omega = \infty$*;*

(ii) *there exists $\delta \in (0,1)$ such that $|x(t) - \rho(t)| \le Me^{-\eta t} + \delta\lambda$ for all $t \in \mathbb{R}_+$;*

(iii) *the continuous function $\dot x$ is bounded and $\limsup_{t\to\infty}|\dot x(t) - \dot\rho(t)| < 2\eta\lambda$;*

(iv) *the continuous functions $u$, $\Phi(u+d)$ and $k$ are bounded.*

**Proof.** Let $\rho \in W^{2,\infty}(\mathbb{R}_+)$ with $(\rho(0), \dot\rho(0)) = (\rho^0, \rho^1) \in Y$, let $(x^0, v^0) \in X$, $q \in L^\infty(\mathbb{R}_+)$ and let $d \in C(\mathbb{R}_+)$ be bounded. Define the causal operator $\tilde T\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$(\tilde T y)(t) := e^{-\eta t}x^0 + \eta\int_0^t e^{-\eta(t-s)}y(s)\mathrm{d}s, \quad \forall\, t \in \mathbb{R}_+, \ \ \forall\, y \in C(\mathbb{R}_+).$$

It is clear that $\tilde T$ is of class $\mathcal{O}_0$; moreover, since $\Psi \in \mathcal{O}_0$, the operator $T$ given by

$$(Ty)(t) := (\eta - c)\big(y(t) - (\tilde T y)(t)\big) - (1/\eta)(\Psi(\tilde T y))(t) \quad \forall\, t \in \mathbb{R}_+, \ \ \forall\, y \in C(\mathbb{R}_+),$$

is also of class $\mathcal{O}_0$. Defining

$$f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R},\ (w,z) \mapsto w + z, \quad p(\cdot) := \frac{q(\cdot)}{\eta}, \quad g := \frac{1}{\eta}\,,$$

(in which case $(f, p, T, g) \in \mathcal{N}$, with $g > 0$, and $r \in W^{1,\infty}(\mathbb{R}_+)$), consider the initial-value problem

$$\left.\begin{aligned} &\dot y(t) = f(p(t), (Ty)(t)) + g\,(\Phi(u+d))(t), \quad y(0) = y^0 := x^0 + (v^0/\eta)\\ &u(t) = -k(t)(y(t) - r(t)), \quad k(t) = \alpha(\beta(t)|y(t) - r(t)|) \end{aligned}\right\}. \tag{5.5}$$

Observe that $\gamma|y^0 - r(0)| \leq \gamma y^* < 1$ and, in the context of problem (5.5), all hypotheses of Theorem 3.1 and Corollary 4.2 are in place.

The initial-value problems (5.4) and (5.5) are equivalent in the sense that, if $y : [0,\omega) \to \mathbb{R}$ is a (maximal) solution of (5.5), then $x : [0,\omega) \to \mathbb{R}$, $t \mapsto (\tilde{T}y)(t)$, is a (maximal) solution of (5.4) and, conversely, if $x : [0,\omega) \to \mathbb{R}$ is a (maximal) solution of (5.4), then $y : [0,\omega) \to \mathbb{R}$, $t \mapsto x(t) + (\dot{x}(t)/\eta)$, is a (maximal) solution of (5.5).

By Theorem 3.1, (5.5) has unique maximal solution $y : [0,\omega) \to \mathbb{R}$, and so $x : [0,\omega) \to \mathbb{R}$, $t \mapsto (\tilde{T}y)(t)$ is the unique maximal solution of (5.4). By Corollary 4.2, $\omega = \infty$ and the functions $u$, $\Phi(u+d)$ and $k$ are bounded, thereby establishing assertions (i) and (iv). It remains only to prove assertions (ii) and (iii). By Corollary 4.2, there exists $\varepsilon > 0$ such that $\beta(t)|y(t) - r(t)| \leq 1 - \varepsilon =: \delta$ for all $t \in \mathbb{R}_+$. Recalling the definition of $\beta$, it follows that $\gamma|y(t) - r(t)| \leq \delta$ for all $t \in \mathbb{R}_+$. Since

$$\dot{x}(t) = -\eta x(t) + \eta y(t) \quad \text{and} \quad \dot{\rho}(t) = -\eta\rho(t) + \eta r(t) \quad \forall\, t \in \mathbb{R}_+,$$

we may infer that

$$|x(t) - \rho(t)| \leq e^{-\eta t}|x^0 - \rho(0)| + \frac{\delta\eta}{\gamma}\int_0^t e^{-\eta(t-s)}\mathrm{d}s < x^* + \frac{1}{\gamma} = Me^{-\eta\tau} \quad \forall\, t \in \mathbb{R}_+,$$

and so, *a fortiori*, $|x(t) - \rho(t)| \leq Me^{-\eta t}$ for all $t \in [0,\tau]$. Furthermore, since $|y(t) - r(t)| \leq \delta\lambda$ for all $t \geq \tau$, we conclude that

$$|x(t) - \rho(t)| \leq e^{-\eta(t-\tau)}|x(\tau) - \rho(\tau)| + \delta\eta\lambda\int_\tau^t e^{-\eta(t-s)}\mathrm{d}s \leq Me^{-\eta t} + \delta\lambda \;\forall\, t \geq \tau.$$

Assertion (ii) now follows. Finally, $|\dot{x}(t) - \dot{\rho}(t)| \leq \eta|x(t) - \rho(t)| + \eta|y(t) - r(t)| \leq \eta Me^{-\eta t} + 2\delta\eta\lambda$ for all $t \geq \tau$, whence assertion (iii). □

REMARK 5.1. The essence of the above proof is first to define the variable $y(t)$ as an appropriate linear combination, viz. $x(t) + \dot{x}(t)/\eta$, of the variables $x(t)$ and $\dot{x}(t)$ (assumed available for feedback) and then recast the closed-loop initial value problem in the form of (5.5) to which Theorem 3.1 and Corollary 4.2 may be applied. In particular, given $\rho \in W^{2,\infty}(\mathbb{R}_+)$ and defining $r := \rho + \dot{\rho}/\eta \in W^{1,\infty}$, the following relation holds: $y - r = H(D)(x - \rho)$, where $D$ is the differential operator and $H$ is the Hurwitz polynomial $s \mapsto 1 + s/\eta$. The approach extends to tracking, with disturbance rejection, of signals $\rho \in W^{n,\infty}(\mathbb{R}_+)$ for higher-order hysteretic systems in the obvious manner. Consider a generalization of (5.1) of the form $P(D)x + \Psi(x) = \Phi(u+d) + q$, where $P$ is a monic real polynomial of degree $n$ and $(x(0), \dot{x}(0), \ldots, x^{(n-1)}(0)) \in X \subset \mathbb{R}^n$. Assume that $x(t)$ and the derivatives $\dot{x}(t), \ldots, x^{(n-1)}(t)$ are available for feedback and define $y(t)$ as a linear combination, viz. $y(t) = x(t) + c_1\dot{x}(t) + \cdots + c_n x^{(n-1)}(t)$, with the property that $H : s \mapsto 1 + c_1 s + \cdots + c_n s^{n-1}$ is a Hurwitz polynomial of degree $n-1$. Given $\rho \in W^{n,\infty}(\mathbb{R}_+)$ with $(\rho(0), \dot{\rho}(0), \ldots, \rho^{(n-1)}(0)) \in Y \subset \mathbb{R}^n$ and defining $r := H(D)\rho$, we have the relation $y - r = H(D)(x - \rho)$. If $\eta > 0$ is such that every root of $H$ has real part less than $-\eta$, then the arguments used in establishing Theorem 5.1 apply *mutatis mutandis* to conclude that, under the feedback $u(t) = -k(t)(y(t) - r(t))$, $k(t) = \alpha(\beta(t)|y(t) - r(t)|)$ and for suitably defined $M > 0$, we achieve the performance objective $|x(t) - \rho(t)| \leq Me^{-\eta t} + \delta\lambda$ for all $t \in \mathbb{R}_+$ (whilst maintaining boundedness of all signals). The proof of this intuitively clear generalization is routine and is therefore omitted.

EXAMPLE 5.2. For purposes of illustration, consider system (5.4) with

$$m = 1, \quad c = 0, \quad d = \frac{\sin}{2}, \quad q = \frac{\cos}{2}, \quad \Psi = \mathcal{B}_{\frac{1}{2},0}, \quad \Phi = \mathcal{P}_0,$$

where $\mathcal{B}_{\frac{1}{2},0}$ is the backlash operator (with $\sigma = 1/2$ and $\xi = 0$) illustrated in Figure 2.2, and $\mathcal{P}_0$ is the Prandtl operator, given by (2.7) with $\xi = 0$ and $\mu(E) := \int_E \chi_{[0,5]}(\rho)\mathrm{d}\rho$, illustrated in Figure 2.3. Assume that $X = Y = [-1,1] \times [-1,1]$. For the function $\alpha$, we take $s \mapsto 1/(1-s)$. Adopting the performance parameter values $\lambda = 0.02$ and $\eta = 1$, we have $x^* = 2$ and $y^* = 4$. Choosing $\gamma = 1/4$, yields $M = 6e^{\tau}$ and so, by Theorem 5.1, for all $(x^0, v^0) \in X$ and $\rho \in W^{2,\infty}(\mathbb{R}_+)$ with $(\rho(0), \dot\rho(0)) \in Y$, the unique global solution of the closed-loop system has the property that $d_\lambda(x(t)-\rho(t)) \le Me^{-t}$. Figure 5.1 depicts the (MATLAB generated) solution $x$ (solid line) for $\tau = 1$, $\rho\colon t \mapsto 1 + \big(\sin(t/2)\big)/2$ (dashed line) and initial data $x^0 = 0 = v^0$.

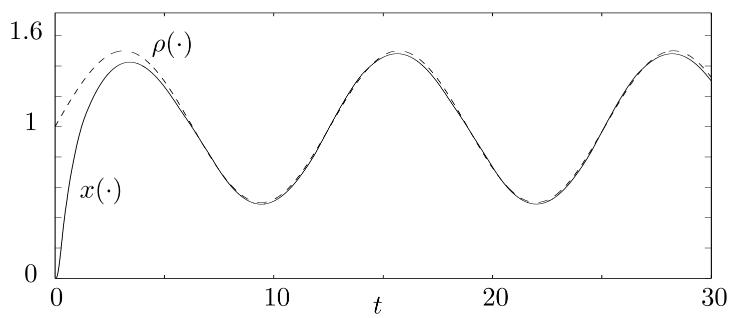


FIG. 5.1. *Example: solution $x$ (solid line) and reference $\rho$ (dashed line)*

## 6. Appendix 1: property H3 of the Preisach operator.

PROPOSITION 6.1. *Let $\mathcal{P}_\xi$ be the Preisach operator defined in* (2.6)*. Assume that the measure $\mu$ is non-negative, with*

$$0 < \alpha_1 := \mu(\mathbb{R}_+) < \infty \quad \textit{and} \quad 0 < \alpha_2 := \int_0^\infty \sigma\mu(\mathrm{d}\sigma) < \infty.$$

*Assume further that*

$$0 < \beta_1 := \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma) \quad \textit{and} \quad \beta_2 := \operatorname{ess\,sup}_{(s,\sigma)\in\mathbb{R}\times\mathbb{R}_+} w(s,\sigma) < \infty\,.$$

*Then, for every $0 < \varepsilon < \alpha_1\beta_1$,*

$$|u(t)| \ge (\alpha_2\beta_2)/\varepsilon \implies (\alpha_1\beta_1 - \varepsilon)u^2(t) \le (\mathcal{P}_\xi(u))(t)u(t)\,, \quad \forall\, u \in C(\mathbb{R}_+),\ \forall\, t \in \mathbb{R}_+\,. \tag{6.1}$$

**Proof.** Note initially that, by the definition of the backlash operator, we have

$$(\mathcal{B}_{\sigma,\,\xi(\sigma)}(u))(t) \in [u(t)-\sigma, u(t)+\sigma] \quad \forall\, u \in C(\mathbb{R}_+),\ \forall\, t \in \mathbb{R}_+,\ \forall\, \sigma \in \mathbb{R}_+\,.$$

Let $\varepsilon \in (0, \alpha_1\beta_1)$, $u \in C(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$.

*Case 1.* Assume that $u(t) \geq (\alpha_2\beta_2)/\varepsilon$. Writing $E_1 = [0, u(t)]$ and $E_2 = (u(t), \infty)$, it follows that

$$\begin{aligned}
(\mathcal{P}_\xi u)(t) &\geq \left(\int_{E_1} + \int_{E_2}\right) \int_0^{u(t)-\sigma} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) \\
&\geq \beta_1 \int_{E_1} (u(t)-\sigma)\,\mu(\mathrm{d}\sigma) + \beta_2 \int_{E_2} (u(t)-\sigma)\,\mu(\mathrm{d}\sigma) \\
&= \big(\beta_1\mu(E_1) + \beta_2\mu(E_2)\big)u(t) - \beta_1 \int_{E_1} \sigma\,\mu(\mathrm{d}\sigma) - \beta_2 \int_{E_2} \sigma\,\mu(\mathrm{d}\sigma) \\
&\geq \alpha_1\beta_1 u(t) - \alpha_2\beta_2\,,
\end{aligned}$$

and so we may conclude that

$$u(t) \geq (\alpha_2\beta_2)/\varepsilon \quad \Longrightarrow \quad (\mathcal{P}_\xi u)(t) \geq (\alpha_1\beta_1 - \varepsilon)u(t). \tag{6.2}$$

*Case 2.* Now assume that $u(t) \leq -(\alpha_2\beta_2)/\varepsilon$. Writing $E_1 = [0, -u(t)]$ and $E_2 = (-u(t), \infty)$, we have

$$\begin{aligned}
(\mathcal{P}_\xi u)(t) &\leq \left(\int_{E_1} + \int_{E_2}\right) \int_0^{u(t)+\sigma} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) \\
&\leq \beta_1 \int_{E_1} (u(t)+\sigma)\,\mu(\mathrm{d}\sigma) + \beta_2 \int_{E_2} (u(t)+\sigma)\,\mu(\mathrm{d}\sigma) \\
&\leq \alpha_1\beta_1 u(t) + \alpha_2\beta_2\,,
\end{aligned}$$

from which we may infer that

$$u(t) \leq -(\alpha_2\beta_2)/\varepsilon \quad \Longrightarrow \quad (\mathcal{P}_\xi u)(t) \leq (\alpha_1\beta_1 - \varepsilon)u(t). \tag{6.3}$$

Since $u \in C(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$ are arbitrary, the conjunction of (6.2) and (6.3) yields (6.1). □

**7. Appendix 2: proof of Theorem 3.1.** To facilitate the proof, we first consider, with notation and assumptions as in Section 3, the following family of initial-value problems, parameterized by $t_0 \in \mathbb{R}_+$,

$$\left.\begin{aligned}
&\dot y(t) = f(p(t), ((y)(t)) + g\,\Phi(u)(t), \quad y|_{[-h,0]} = y^0 \in C[-h, t_0], \\
&u(t) = \nu(k(t))\big(y(t) - r(t)\big), \\
&k(t) = \alpha\big(\beta(t)|y(t) - r(t)|\big).
\end{aligned}\right\} \tag{7.1}$$

We will prove the following theorem, of which Theorem 3.1 is a special case ($t_0 = 0$).

Theorem 7.1. *Under the assumptions of* Theorem 3.1*, for every $t_0 \in \mathbb{R}_+$ and every $y^0 \in C[-h, t_0]$ with $(t, y^0(t)) \in \mathcal{D}$ for all $t \in [0, t_0]$, the initial-value problem* (7.1) *has a unique maximal solution $y \in C[-h, \omega)$. Moreover, if $\omega < \infty$, then* $\limsup_{t\uparrow\omega} \beta(t)|y(t) - r(t)| = 1$ (*or, equivalently,* $\limsup_{t\uparrow\omega} k(t) = \infty$). By a solution of (7.1) we mean the obvious generalization of the earlier concept: a continuous function $y : I \to \mathbb{R}$ on an interval of the form $[-h, \rho]$, with $t_0 < \rho < \infty$, or of the form $[-h, \omega)$, with $t_0 < \omega \leq \infty$, such that (a) $y|_{[-h,t_0]} = y^0$ and (b) $y|_J$, $J := I \setminus [-h, t_0)$, is a locally absolutely continuous function, with graph in $\mathcal{D}$ and satisfying the differential equation in (7.1) almost everywhere on $J$.

**Proof of Theorem 7.1.** Let $t_0 \in \mathbb{R}_+$ and $y^0 \in C[-h, t_0]$ be such that $(t, y^0(t)) \in \mathcal{D}$ for all $t \in [0, t_0]$.

Step 1. First, we establish the existence of a unique solution on an interval $[-h, \rho]$ with $\rho > t_0$ sufficiently close to $t_0$. By property (iii) of the operator class $\mathcal{O}_h$, there exist $\tau_0 > t_0$, $\delta_0 > 0$ and $c_0 > 0$ such that

$$\begin{aligned}\text{ess-sup}_{t\in[t_0,\tau_0]}&|T(y_1)(t) - T(y_2)(t)| \\ &\le c_0 \max_{t\in[t_0,\tau_0]} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in \mathcal{C}(y^0; h, t_0, \tau_0, \delta_0).\end{aligned}$$

We may assume that $\delta_0 \in (0, 1)$ and $\tau_0 - t_0 > 0$ are sufficiently small so that

$$\mathcal{D}_0 := [t_0, \tau_0] \times [y^0(t_0) - \delta_0\, ,\, y^0(t_0) + \delta_0] \subset \mathcal{D}.$$

Next, consider the map

$$U : \mathcal{D} \to \mathbb{R}, \quad (t, z) \mapsto \nu(\alpha(\beta(t)|z - r(t)|))(z - r(t)).$$

Since $\alpha$ and $\nu$ are locally Lipschitz and $\beta$ and $r$ are bounded, it follows that there exists $c_1 > 0$ such that

$$|U(t, z_1) - U(t, z_2)| \le c_1|z_1 - z_2| \quad \forall\, (t, z_1), (t, z_2) \in \mathcal{D}_0.$$

For each $\rho \in (t_0, \tau_0]$, define $\mathcal{C}^0_\rho := \mathcal{C}(y^0; h, t_0, \rho, \delta_0)$. Observe that, if $y \in \mathcal{C}^0_\rho$, then $(t, y(t)) \in \mathcal{D}_0$ for all $t$ such that $t_0 \le t \le \rho \le \tau_0$. Therefore, for each $\rho \in [t_0, \tau_0]$, we may define an operator $\mathbf{U}_\rho : \mathcal{C}^0_\rho \to C[0, \rho]$ by

$$(\mathbf{U}_\rho y)(t) := U(t, y(t)) \quad \forall\, t \in [0, \rho],$$

and record the following fact:

$$|(\mathbf{U}_\rho y_1)(t) - (\mathbf{U}_\rho y_2)(t)| \le c_1|y_1(t) - y_2(t)| \quad \forall\, t \in [0, \rho],\ \forall\, y_1, y_2 \in \mathcal{C}^0_\rho\,. \tag{7.2}$$

Defining $w \in C[0, t_0]$ by $w(t) := U(t, y^0(t))$ for all $t \in [0, t_0]$, we have in particular,

$$(\mathbf{U}_\rho y)(t) = w(t) \quad \forall\, t \in [0, t_0],\ \forall\, y \in \mathcal{C}^0_\rho\,.$$

By hypothesis H1 on $\Phi$, there exist $\tau_1 \in (t_0, \tau_0]$, $\delta_1 \in (0, \delta_0]$ and $c_2 > 0$ such that

$$\begin{aligned}\max_{t\in[0,\tau_1]} &|\Phi(v_1)(t) - \Phi(v_2)(t)| \\ &\le c_2 \max_{t\in[0,\tau_1]} |v_1(t) - v_2(t)| \ \ \forall\, v_1, v_2 \in \mathcal{C}(w; 0, t_0, \tau_1, \delta_1)\,.\end{aligned} \tag{7.3}$$

Furthermore, by continuity of $U$, there exist $\tau_2 \in (t_0, \tau_1]$ and $\delta_2 \in (0, \delta_0]$ such that, if $\rho \in (t_0, \tau_2]$, then

$$\mathbf{U}_\rho y \in \mathcal{C}(w; 0, t_0, \rho, \delta_1) \quad \forall\, y \in \mathcal{C}(y^0; h, t_0, \rho, \delta_2) \subset \mathcal{C}^0_\rho\,. \tag{7.4}$$

For each $\rho \in (t_0, \tau_2]$, we define $\mathcal{C}_\rho := \mathcal{C}(y^0; h, t_0, \rho, \delta_2)$. Invoking (7.2)–(7.4), we may conclude that there exists $c_3 > 0$ such that, for every $\rho \in (t_0, \tau_2]$,

$$\max_{t\in[0,\rho]} |\Phi(\mathbf{U}_\rho y_1)(t) - \Phi(\mathbf{U}_\rho y_2)(t)| \le c_3 \max_{t\in[0,\rho]} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in \mathcal{C}_\rho\,. \tag{7.5}$$

Furthermore, as a consequence of (7.5), there exists $c_4 > 0$ such that, for every $\rho \in (t_0, \tau_2]$,

$$|\Phi(\mathbf{U}_\rho y)(t)| \le c_4 \quad \forall\, t \in [0,\rho],\ \forall\, y \in \mathcal{C}_\rho\,.$$

Equipped with the metric

$$(y_1, y_2) \mapsto \mu_\rho(y_1, y_2) := \max_{t\in[-h,\rho]} |y_1(t) - y_2(t)|\,,$$

$\mathcal{C}_\rho$ is a complete metric space. For each $\rho \in (t_0, \tau_2]$, define the operator $\mathbf{C}_\rho$ on $\mathcal{C}_\rho$ by

$$\mathbf{C}_\rho(y)(t) := \begin{cases} y^0(t), & t \in [-h, t_0] \\ y^0(t_0) + \int_{t_0}^t \big(f(p(s), T(y)(s)) + g\,\Phi(\mathbf{U}_\rho y)(s)\big)\,\mathrm{d}s, & t \in (t_0, \rho] \end{cases}$$

We proceed to show that there exists $\rho^* \in (t_0, \tau_2]$ such that, for all $\rho \in (t_0, \rho^*]$, $\mathbf{C}_\rho(\mathcal{C}_\rho) \subset \mathcal{C}_\rho$ and $\mathbf{C}_\rho$ is a contraction (and, consequently, for each such $\rho$, $\mathbf{C}_\rho$ has a unique fixed point). By property (iv) of the operator class $\mathcal{O}_h$, there exists $c_5 > 0$ such that, for every $\rho \in (t_0, \rho^*]$,

$$|T(y)(t)| \le c_5 \quad \text{for a.a. } t \in [t_0, \rho],\ \forall\, y \in \mathcal{C}_\rho\,.$$

By the local Lipschitz property of $f$, together with essential boundedness of $p$, there exists $c_6 > 0$ such that

$$|f(p(t), x_1) - f(p(t), x_2)| \le c_6 |x_1 - x_2| \\ \text{for a.a. } t \in [t_0, \tau_2] \text{ and all } x_1, x_2 \in \mathbb{R} \text{ with } |x_1|, |x_2| \le c_5.$$

Set $c_7 := \max\big\{|f(q,x)|\ \big|\ |q| \le \|p\|_{L^\infty},\ |x| \le c_5\big\}$ and fix $\rho^* \in (t_0, \tau_2]$ sufficiently close to $t_0$ so that

$$(\rho^* - t_0)\big(c_7 + c_0 c_6 + c_3|g| + c_4|g|\big) \le \delta_2.$$

Let $\rho \in (t_0, \rho^*]$ and $y \in \mathcal{C}_\rho$. By definition, $(\mathbf{C}_\rho y)|_{[-h,t_0]} = y^0$ and, moreover,

$$\begin{aligned} |\mathbf{C}_\rho(y)(t) - y^0(t_0)| &\le \int_{t_0}^{\rho} |f(p(s), T(y)(s)) + g\,\Phi(\mathbf{U}_\rho y)(s)|\mathrm{d}s \\ &\le (\rho - t_0)(c_7 + c_4|g|) \le \delta_2 \quad \forall\, t \in [t_0, \rho]\,. \end{aligned}$$

Therefore, $\mathbf{C}_\rho(y) \in \mathcal{C}_\rho$, establishing that $\mathbf{C}_\rho(\mathcal{C}_\rho) \subset \mathcal{C}_\rho$ for all $\rho \in (t_0, \rho^*]$. Furthermore, for $\rho \in (t_0, \rho^*]$ and $y_1, y_2 \in \mathcal{C}_\rho$,

$$\begin{aligned} \mu_\rho(\mathbf{C}_\rho(y_1)\,,\, \mathbf{C}_\rho(y_2)) &= \sup_{t\in[t_0,\rho]} \Big| \int_{t_0}^{t} \big(f(p(s), T(y_1)(s)) - f(p(s), T(y_2)(s)) \\ &\qquad\qquad + g\,\Phi(\mathbf{U}_\rho y_1)(s) - g\,\Phi(\mathbf{U}_\rho y_2)(s)\big)\,\mathrm{d}s\Big| \\ &\le (\rho - t_0)\Big( \sup_{t\in[t_0,\rho]} |f(p(t), T(y_1)(t)) - f(p(t), T(y_2)(t))| \\ &\qquad\qquad + |g| \sup_{t\in[t_0,\rho]} |\Phi(\mathbf{U}_\rho y_1)(t) - \Phi(\mathbf{U} y_2)(t)|\Big) \\ &\le (\rho - t_0)\,(c_0 c_6 + c_3|g|)\,\mu_\rho(y_1, y_2). \end{aligned}$$

Since $(\rho - t_0)(c_0c_6 + c_3|g|) \leq \delta_2 < 1$, it follows that $\mathbf{C}_\rho : \mathcal{C}_\rho \to \mathcal{C}_\rho$ is a contraction. Therefore, for each $\rho \in (0, \rho^*]$, $\mathbf{C}_\rho$ has a unique fixed point $y \in \mathcal{C}_\rho$ and so (7.1) has a unique solution in $\mathcal{C}_\rho$. We emphasize that the uniqueness property is specific to solutions of class $\mathcal{C}_\rho$: there may exist other solutions on $[-h, \rho]$ which are not of class $\mathcal{C}_\rho$. However, the following argument establishes the existence of precisely one solution for $\rho \in (t_0, \rho^*]$ with $\rho - t_0$ sufficiently small. Let $\tilde{y}$ be a solution on $[-h, \tilde{\rho}]$ (not necessarily of class $\mathcal{C}_{\tilde{\rho}}$) for some $\tilde{\rho} \in (0, \rho^*]$. Define

$$\Delta := \big\{ t \in [t_0, \tilde{\rho}] \;\big|\; |\tilde{y}(t) - y^0(t_0)| = \delta \big\}, \quad \rho := \begin{cases} \inf \Delta, & \Delta \neq \emptyset, \\ \tilde{\rho}, & \Delta = \emptyset. \end{cases}$$

Clearly, $\rho > t_0$ and $\tilde{y}|_{-h,\rho]}$ is in $\mathcal{C}_\rho$. Therefore, $\tilde{y}|_{-h,\rho]}$ is the unique solution of (7.1) on $[-h, \rho]$.

Step 2. Next, we show that any two solutions must coincide on the intersection of their domains. Let $y_1 \in C(I_1)$ and $y_2 \in C(I_2)$ be solutions of (7.1). For contradiction, suppose that there exists $t \in J := I_1 \cap I_2$ such that $y_1(t) \neq y_2(t)$. Let $t^* := \inf\{t \in J |\; y_1(t) \neq y_2(t)\}$. Evidently, $t^* < \sup J$. By the result in Step 1 above, $t^* > t_0$. An application of the result of Step 1 in the context of an initial-value problem of the form (7.1), with $t^*$ replacing $t_0$ and with the function $y_1|_{[-h,t^*]} \in C[-h, t^*]$ replacing $y^0$, yields the existence of a unique solution $y \in C[-h, \rho]$ for some $\rho > t^*$. It follows that $y_1(t) = y_2(t) = y(t)$ for all $t \in [-h, \rho] \cap J$ and so there exists $t > t^*$ with $y_1(t) = y_2(t)$. This contradicts the definition of $t^*$.

Step 3. We now establish the existence of a unique maximal solution. Let $R$ be the set of all $\rho > t_0$ such that there exists a solution $y_\rho \in C[-h, \rho]$ of (7.1). By Step 1, we know that $R \neq \emptyset$. Let $\omega := \sup R$ ($\omega = \infty$ is possible) and define $y \in C[-h, \omega)$ by the property

$$y|_{[-h,\rho]} = y_\rho \quad \text{for all} \quad \rho \in R.$$

The function $y$ is well defined since, by Step 2, we have

$$\big(\rho_1, \rho_2 \in R \;\wedge\; \rho_2 \leq \rho_1\big) \quad \Longrightarrow \quad y_{\rho_2} = y_{\rho_1}|_{[-h,\rho_2]}.$$

Clearly, $y$ is a maximal solution of (7.1) and uniqueness follows by Step 2.

Step 4. Assume that $\omega < \infty$. We have to show that $\limsup_{t\uparrow\omega} \beta(t)|y(t) - r(t)| = 1$. Seeking a contradiction, suppose that the latter does not hold, in which case $\limsup_{t\uparrow\omega} \beta(t)|y(t) - r(t)| < 1$. Then $k$ is bounded and therefore, since $y$ is bounded, the function $u$ is also bounded. By property (iv) of the operator class $\mathcal{O}_h$, $Ty$ is essentially bounded and, by property H2, $\Phi u$ is bounded. From the differential equation in (7.1), it now follows that $\dot{y}$ is essentially bounded on $[0, \omega)$. Therefore, $y$ is uniformly continuous on $[-h, \omega)$ and so extends to $y^* \in C[-h, \omega]$. Furthermore,

$$\beta(\omega)|y^*(\omega) - r(\omega)| = \lim_{t\uparrow\omega} \beta(t)|y^*(t) - r(t)| = \limsup_{t\uparrow\omega} \beta(t)|y(t) - r(t)| < 1$$

showing that $(\omega, y^*(\omega)) \in \mathcal{D}$. An application of the result in Step 1 in the context of an initial-value problem of the form (7.1), with $\omega$ replacing $t_0$ and $y^*$ replacing $y^0$, yields the existence of a unique solution $y^e \in C[-h, \rho]$ for some $\rho > \omega$, with $y^e|_{[-h,\omega)} = y$. This contradicts maximality of the solution $y$. □

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