Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **07/09**

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2007

Impressum:
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http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

ilmedia

Domination in Graphs of Minimum Degree at least Two and large Girth

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Abstract We prove that for graphs of order n, minimum degree $\delta \geq 2$ and girth $g \geq 5$ the domination number γ satisfies $\gamma \leq \left(\frac{1}{3}+\frac{2}{3g}\right)n$. As a corollary this implies that for cubic graphs of order n and girth $g \geq 5$ the domination number γ satisfies $\gamma \leq \left(\frac{44}{135}+\frac{82}{135g}\right)n$ which improves recent results due to Kostochka and Stodolsky (An upper bound on the domination number of n-vertex connected cubic graphs, *manuscript* (2005)) and Kawarabayashi, Plummer and Saito (Domination in a graph with a 2-factor, *J. Graph Theory* **52** (2006), 1-6) for large enough girth. Furthermore, it confirms a conjecture due to Reed about connected cubic graphs (Paths, stars and the number three, *Combin. Prob. Comput.* **5** (1996), 267-276) for girth at least 83.

Keywords domination number; minimum degree; girth; cubic graph

1 Introduction

The domination number $\gamma(G)$ of a (finite, undirected and simple) graph $G=(V,E)$ is the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D. This parameter is one of the most well-studied in graph theory and the two volume monograph [4, 5] provides an impressive account of the research related to this concept.

Fundamental results about the domination number $\gamma(G)$ are upper bounds in terms of the order n and the minimum degree δ of the graph G. Ore [10] proved that $\gamma(G) \leq \frac{n}{2}$ provided $\delta \geq 1$. For $\delta \geq 2$ and all but 7 exceptional graphs Blank [1] and McCuaig and Shepherd [9] proved $\gamma(G) \leq \frac{2n}{5}$. Equality in these two bounds is attained for infinitely many graphs which were characterized in [9, 11, 16].

In [13] Reed proved that $\gamma(G) \leq \frac{3}{8}n$ for every graph G of order n and minimum degree at least 3 and he conjectured that this bound could be improved to $\left\lceil \frac{n}{3} \right\rceil$ for connected cubic graphs. While Reed's conjecture was disproved by Kostochka and Stodolsky [7] who constructed a sequence $(G_k)_{k\in\mathbb{N}}$ of connected cubic graphs with

$$\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3}+\frac{1}{69},$$

Kostochka and Stodolsky [8] proved $\gamma(G) \leq \frac{4}{11}n$ for every connected cubic graph G of order $n > 8$ and

$$\gamma(G) \quad \leq \quad \left(\frac{1}{3} + \frac{8}{3g^2}\right) n \tag{1}$$

for every connected cubic graph G of order $n > 8$ and girth g where the girth is the length of a shortest cycle in G. The last result improved a recent result due to Kawarabayashi, Plummer and Saito [6] who proved that

$$\gamma(G) \quad \leq \quad \left(\frac{1}{3} + \frac{1}{9k+3}\right) n \tag{2}$$

for every 2-edge connected cubic graph G of order n and girth at least $3k$ for some $k \in \mathbb{N}$.

The first to use the girth g of a graph G next to its order n and minimum degree δ to bound the domination number γ were probably Brigham and Dutton [2] who proved

$$\gamma \leq \left\lceil \frac{n}{2} - \frac{g}{6} \right\rceil$$

provided that $\delta \geq 2$ and $g \geq 5$. In [14, 15] Volkmann determined finite set of graphs $\mathcal{G}_i$ for $i \in \{1, 2\}$ such that

$$\gamma \leq \left\lceil \frac{n}{2} - \frac{g}{6} - \frac{3i+3}{6} \right\rceil$$

unless G is a cycle or $G \in \mathcal{G}_i$. Motivated by these results Rautenbach [12] proved that for every $k \in \mathbb{N}$ there is a finite set $\mathcal{G}_k$ of graphs such that if G is a graph of order n, minimum degree $\delta \geq 2$, girth $g \geq 5$ and domination number γ that is not a cycle and does not belong to $\mathcal{G}_k$, then

$$\gamma \leq \frac{n}{2} - \frac{g}{6} - k.$$

In the present paper we prove a best-possible upper bound on the domination number of graphs of minimum degree at least 2 and girth at least 5 which allows to improve (1) and (2) for large enough girth. Furthermore, it confirms Reed's conjecture [13] for cubic graphs with girth at least 83.

2 Results

We immediately proceed to our main result.

Theorem 1 *If $G = (V, E)$ is a graph of order n, minimum degree $\delta \geq 2$, girth $g \geq 5$ and domination number γ, then*

$$\gamma \quad \leq \quad \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor \frac{g+1}{3} \right\rfloor + 1\right)}\right) n.$$

Proof: For contradiction, we assume that $G = (V, E)$ is a counterexample of minimum sum of order and size. Let n, g and γ be as in the statement of the theorem. Since n and γ are linear with respect to the components of G and $\frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}$ is non-decreasing in g, the graph G is connected. Furthermore, the set of vertices of degree at least 3 is independent. We prove several claims restricting the structure of G.

Claim 1. *G has a vertex of degree at least* 3.

Proof of Claim 1: For contradiction, we assume that G has no vertex of degree at least 3. In this case G is a cycle of order at least g and $\gamma = \left\lceil\frac{n}{3}\right\rceil$.
If $n = g$, then

$$\left\lceil\frac{n}{3}\right\rceil = \begin{cases} \frac{n}{3} & < & \left(\frac{1}{3}+\frac{2}{3(g+1)}\right)n & \text{, if } g \equiv 0 \pmod 3, \\ \frac{n+2}{3} & = & \left(\frac{1}{3}+\frac{2}{3g}\right)n & \text{, if } g \equiv 1 \pmod 3 \text{ and} \\ \frac{n+1}{3} & < & \left(\frac{1}{3}+\frac{2}{3(g+2)}\right)n & \text{, if } g \equiv 2 \pmod 3. \end{cases}$$

If $n = g + 1$, then

$$\left\lceil\frac{n}{3}\right\rceil = \begin{cases} \frac{n+2}{3} & = & \left(\frac{1}{3}+\frac{2}{3(g+1)}\right)n & \text{, if } g \equiv 0 \pmod 3, \\ \frac{n+1}{3} & < & \left(\frac{1}{3}+\frac{2}{3g}\right)n & \text{, if } g \equiv 1 \pmod 3 \text{ and} \\ \frac{n}{3} & < & \left(\frac{1}{3}+\frac{2}{3(g+2)}\right)n & \text{, if } g \equiv 2 \pmod 3. \end{cases}$$

Finally, if $n \geq g + 2$, then

$$\left\lceil\frac{n}{3}\right\rceil \leq \frac{n+2}{3} \leq \left(\frac{1}{3}+\frac{2}{3(g+2)}\right)n.$$

Since

$$3\left\lfloor\frac{g+1}{3}\right\rfloor+1 = \begin{cases} g+1 & \text{, if } g \equiv 0 \pmod 3, \\ g & \text{, if } g \equiv 1 \pmod 3 \text{ and} \\ g+2 & \text{, if } g \equiv 2 \pmod 3, \end{cases}$$

we obtain in all cases the contradiction $\gamma \leq \left(\frac{1}{3}+\frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)n$ and the proof of the claim is complete. □

A path P in G between vertices x and y of degree at least 3 whose internal vertices are all of degree 2 will be called 2*-path* and we set $p_P(x) := y$ and $p_P(y) := x$.

Claim 2. *G has no two vertices u and v of degree at least* 3 *that are joined by a* 2*-path P of length* 1 *(mod 3).*

Proof of Claim 2: For contradiction, we assume that such vertices u and v and such a path P exist.

If V' denotes the set of internal vertices of the path, then $G[V']$ is a path of order 0 (mod 3) which has a dominating set D' of cardinality $\frac{|V'|}{3}$. Since the graph $G[V \setminus V']$ satisfies the assumptions of the theorem, we obtain, by the choice of G, that $G[V \setminus V']$ has a dominating set D'' of cardinality at most $\left(\frac{1}{3} + \frac{2}{3(3\lfloor\frac{g+1}{3}\rfloor+1)}\right)(n - |V'|)$. Now, $D' \cup D''$ is a dominating set of G and we obtain

$$
\begin{aligned}
\gamma &\leq |D'| + |D''| \\
&\leq \frac{|V'|}{3} + \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)(n - |V'|) \\
&< \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)n,
\end{aligned}
$$

which implies a contradiction and the proof of the claim is complete. □

Claim 3. *G has no vertex u of degree at least 3 that lies on a cycle C of length 1 (mod 3) whose vertices different from u are all of degree 2.*

Proof of Claim 3: For contradiction, we assume that such a vertex u and such a cycle C exist.

Let V' denote a minimal set of vertices containing a neighbour of u on the cycle C such that $G[V \setminus V']$ has no vertex of degree less than 2.

If u is of degree at least 4, then the graph $G[V']$ is a path of order 0 (mod 3) and we obtain the same contradiction as in Claim 2.

Hence we can assume that u is of degree 3. In this case the graph $G[V']$ arises from C by attaching a path to u. Since $G[V']$ has a spanning subgraph which is a path, it has a dominating set D' of cardinality at most $\left\lceil\frac{|V'|}{3}\right\rceil$.

As before, $G[V \setminus V']$ has a dominating set D'' with $|D''| \leq \left(\frac{1}{3} + \frac{2}{3(3\lfloor\frac{g+1}{3}\rfloor+1)}\right)(n - |V'|)$. Now $D' \cup D''$ is a dominating set of G and using $|V'| \geq g$ we obtain

$$
\begin{aligned}
\gamma &\leq |D'| + |D''| \\
&\leq \left\lceil\frac{|V'|}{3}\right\rceil + \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)(n - |V'|) \\
&= \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)n + \left(\left\lceil\frac{|V'|}{3}\right\rceil - \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)|V'|\right).
\end{aligned}
$$

Considering the three cases $|V'| = g$, $|V'| = g + 1$ and $|V'| = g + 2$ as in the proof of Claim 1 implies the contradiction $\gamma \leq \left(\frac{1}{3} + \frac{2}{3(3\lfloor\frac{g+1}{3}\rfloor+1)}\right)n$ and the proof of the claim is complete. □

Claim 4. *G has no vertex u of degree at least 3 that lies on two cycles C_1 and C_2 of lengths 2 (mod 3) whose vertices different from u are all of degree 2.*

Proof of Claim 4: For contradiction, we assume that such a vertex u and such cycles C_1 and C_2 exist.

Let V' denote a minimal set of vertices containing a neighbour of u on the cycle C_1 and a neighbour of u on the cycle C_2 such that $G[V \setminus V']$ has no vertex of degree less than 2.

If u is of degree at least 6, then the graph $G[V']$ consists of two disjoint paths of order 1 (mod 3) whose endvertices are adjacent to u. This easily implies that there is a set $D' \subseteq \{u\} \cup V'$ containing u such that every vertex in $V' \setminus D'$ has a neighbour in D' and $|D'| = \left\lceil \frac{|V'|}{3} \right\rceil$. Since $|V'| \geq g$, we obtain a similar contradiction as in the proof of Claim 3.

Hence we can assume that u is of degree at most 5. In this case the graph $G[V']$ consists of C_1 and C_2 and possibly a path attached to u. Again, it is easy to see that $G[V']$ has a dominating set D' of cardinality at most $\left\lceil \frac{|V'|}{3} \right\rceil$. Since $|V'| \geq g$, we obtain a similar contradiction as in the proof of Claim 3 and the proof of the claim is complete. □

Claim 5. *G has no two distinct vertices u and v of degree at least 3 such that u lies on a cycle C of length 2 (mod 3) whose vertices different from u are all of degree 2, and u and v are joined by a 2-path P of length 2 (mod 3).*

Proof of Claim 5: For contradiction, we assume that such vertices u and v, such a cycle C and such a path P exist.

Let V' denote a minimal set of vertices containing a neighbour of u on the cycle C and a neighbour of u on the path P such that $G[V \setminus V']$ has no vertex of degree less than 2.

If u is of degree at least 5, then the graph $G[V']$ is the union of two paths of order 1 (mod 3) which both have an endvertex that is adjacent to u. Again, there is a set $D' \subseteq \{u\} \cup V'$ containing u such that every vertex in $V' \setminus D'$ has a neighbour in D' and $|D'| = \left\lceil \frac{|V'|}{3} \right\rceil$. Since $|V'| \geq g$, we obtain a similar contradiction as in the proof of Claim 3.

Hence we can assume that u is of degree at most 4. Let P' denote the 2-path starting at u that is internally disjoint from C and P. Let w denote the endvertex of P' different from u, i.e. $w = p_{P'}(u)$. If $v \neq w$ or $v = w$ and v is of degree at least 4, then the graph $G[V']$ arises from C, P and P' by deleting v and w. If $v = w$ and v is of degree 3, then let P'' denote the 2-path starting at v that is internally disjoint from P and P'. Now the graph $G[V']$ arises from C, P, P' and P'' by deleting the endvertex of P'' different from v. In both cases, by the parity conditions, the graph $G[V']$ has a dominating set D' of cardinality at most $\left\lceil \frac{|V'|}{3} \right\rceil$. Since $|V'| \geq g$, we obtain a similar contradiction as in the proof of Claim 3 and the proof of the claim is complete. □

Claim 6. *G has no vertex u that is joined to three vertices v_1, v_2 and v_3 of degree at least 3 by three distinct 2-paths of lengths 2 (mod 3).*

Proof of Claim 6: For contradiction, we assume that such vertices u, v_1, v_2 and v_3 and such paths exist. Let P_1, P_2 and P_3 denote the three 2-paths joining u to v_1, v_2 and v_3, respectively. Let V_0' denote the set of internal vertices of the three paths and let V' denote a minimal set of vertices containing V_0' such that $G[V \setminus V']$ has no vertex of degree less than 2. In order to complete the proof of Claim 6, we insert another claim about the structure of $G[V']$.

Claim 7. If u, v_1, v_2, v_3, P_1, P_2, P_3, V_0' and V' are as above, then

(i) either $u \not\in V'$ and $G[V']$ is the union of three paths of order 1 (mod 3) each of which has an endvertex that is adjacent to u,

(ii) or $G[V']$ has a spanning subgraph which arises by identifying an endvertex in each of three or four paths of which three are of order 2 (mod 3),

(iii) or $|V'| \geq g$ and $G[V']$ has a spanning subgraph which arises by identifying an endvertex in each of three or four paths of which two are of order 2 (mod 3),

(iv) or $u \not\in V'$, $|V'| \geq g$ and $G[V']$ has a spanning subgraph which is the union of three paths each of which has an endvertex that is adjacent to u and two of these three paths are of order 1 (mod 3).

Proof of Claim 7: If w is a vertex of degree at most 1 in $G[V \setminus V_0']$, then let $P(w)$ denote the 2-path starting in w that is internally disjoint from V_0'. Note that $P(w)$ has length 0 if w is an isolated vertex in $G[V \setminus V_0']$.

First, we assume that $|\{v_1, v_2, v_3\}| = 3$, i.e. the vertices v_1, v_2 and v_3 are all distinct.

If u is of degree 3, then $V' = \{u\} \cup V_0'$ and (ii) holds.

If u is of degree at least 5, then $V' = V_0'$ and (i) holds.

Hence we can assume that u is of degree 4.

If either $p_{P(u)}(u) \not\in \{v_1, v_2, v_3\}$ or $p_{P(u)}(u) \in \{v_1, v_2, v_3\}$, say $p(u) = v_1$, and v_1 is not of degree 3, then (ii) holds.

Hence we can assume that $p(u) = v_1$ is of degree 3. Let P' denote the 2-path starting in v_1 that is internally disjoint from V_0' and $P(u)$.

If either $p_{P'}(v_1) \not\in \{v_2, v_3\}$ or $p_{P'}(v_1) \in \{v_2, v_3\}$, say $p_{P'}(v_1) = v_2$, and v_2 is not of degree 3, then (ii) holds.

Hence we can assume that $p_{P'}(v_1) = v_2$ is of degree 3. Let P'' denote the 2-path starting in v_2 that is internally disjoint from V_0' and P'.

If either $p_{P''}(v_2) \neq v_3$ or $p_{P''}(v_2) = v_3$ and v_3 is not of degree 3, then (ii) holds.

Hence we can assume that $p_{P''}(v_2) = v_3$ is of degree 3. Let P''' denote the 2-path starting in v_3 that is internally disjoint from V_0' and P''. Clearly, $p_{P'''}(v_3) \not\in \{u, v_1, v_2\}$ and (ii) holds. (Note that we can delete the edges incident to v_i in P_i for $1 \leq i \leq 3$ in order to obtain the spanning subgraph mentioned in (ii).)

Next, we assume that $|\{v_1, v_2, v_3\}| = 1$. Note that the 2-paths between u and $v_1 = v_2 = v_3$ form cycles of length at least g.

If u and v_1 are both of degree at least 5, then $V' = V'_0$ and (i) holds.

If u is of degree at most 4 and v_1 is of degree at least 5, then (ii) holds. (Note that if $v_1 \in V'$, then we can delete the edges incident to v_1 in P_i for $1 \leq i \leq 3$ in order to obtain the spanning subgraph mentioned in (ii).)

If u is of degree at least 5 and v_1 is of degree at most 4, then (ii) holds. (Note that if $u \in V'$, then we can delete the edges incident to u in P_i for $1 \leq i \leq 3$ in order to obtain the spanning subgraph mentioned in (ii).)

If u and v_1 are both of degree at most 4, then either $P(u) = P(v_1)$ and (ii) holds or $P(u) \neq P(v_1)$ and (iii) holds. (Note that in the last case we can delete the edges incident to v_1 in P_1 and P_2 in order to obtain the spanning subgraph mentioned in (iii)).

Finally, we assume that $|\{v_1, v_2, v_3\}| = 2$, say $v_1 = v_3 \neq v_2$. Note that the 2-paths P_1 and P_3 between u and $v_1 = v_3$ form a cycle of length at least g.

If v_1 is of degree at least 4, then we can argue similarly as in the case $|\{v_1, v_2, v_3\}| = 3$.

Hence we can assume that v_1 is of degree 3.

If u and v_1 are joined by a 2-path Q different from P_1 and P_3, then (iii) or (iv) hold depending on the degree of u. (Note that, if u is of degree four for instance, then we can delete the edge incident to u in Q and the edge incident to v_1 in P_1 in order to obtain the spanning subgraph mentioned in (iii)).

Hence we can assume that u and v_1 are not joined by a 2-path different from P_1 and P_3.

If u is of degree 4 and u and v_2 are joined by a 2-path different from P_2, then (iii) holds.

Hence we can assume that either u is of degree at least 5 or u and v_2 are not joined by a 2-path different from P_2.

In the remaining cases (iii) or (iv) hold which completes the proof of the claim. $\square$

We return to the proof of Claim 6.

Note that in Cases (i) or (iv) of the Claim 7 there is a set $D' \subseteq \{u\} \cup V'$ containing u such that every vertex in $V' \setminus D'$ has a neighbour in D' and either $|D'| \leq \frac{|V'|}{3}$ (Case (i)) or $|D'| \leq \left\lceil \frac{|V'|}{3} \right\rceil$ and $|V'| \geq g$ (Case (iv)). Furthermore, by the parity conditions, in Cases (ii) and (iii) of Claim 7, the graph $G[V']$ has a dominating set D' such that either $|D'| \leq \frac{|V'|}{3}$ (Case (ii)) or $|D'| \leq \left\lceil \frac{|V'|}{3} \right\rceil$ and $|V'| \geq g$ (Case (iii)).

As before, $G[V \setminus V']$ has a dominating set D'' with $|D''| \leq \left(\frac{1}{3} + \frac{2}{3(3\lfloor \frac{g+1}{3} \rfloor + 1)} \right) (n - |V'|)$ and $D' \cup D''$ is a dominating set of G. If $|D'| \leq \frac{|V'|}{3}$, then we obtain a similar contradiction as in Claim 2 and if $|D'| \leq \left\lceil \frac{|V'|}{3} \right\rceil$ and $|V'| \geq g$, then we obtain a similar contradiction as in Claim 3. This completes the proof of the claim. $\square$

We have by now analysed the structure of G far enough in order to describe a sufficiently small dominating set leading to the final contradiction. Let $V_{\geq 3}$ denote the set of vertices of degree at least 3 and let $n_{\geq 3} = |V_{\geq 3}|$. The graph $G[V \setminus V_{\geq 3}]$ is a collection of paths of order either 1 (mod 3) or 2 (mod 3).

Let $P_1, P_2, ..., P_s$ denote the set of vertices of the paths of order 1 (mod 3) and let $Q_1, Q_2, ..., Q_t$ denote the set of vertices of the paths of order 2 (mod 3).

By the above claims,

$$s + t \geq \frac{3n_{\geq 3}}{2} \quad \text{and} \quad s \leq n_{\geq 3}$$

which implies

$$t \geq \frac{n_{\geq 3}}{2} \quad \text{and} \quad \left(n_{\geq 3} - \frac{s}{3} - \frac{2t}{3}\right) \leq \frac{n_{\geq 3}}{3}.$$

For $1 \leq i \leq s$, the path $G[P_i]$ without its one or two endvertices has a dominating set D_i^P of cardinality $\frac{|P_i|-1}{3}$. For $1 \leq j \leq t$, the path $G[Q_j]$ without its two endvertices has a dominating set D_j^Q of cardinality $\frac{|Q_j|-2}{3}$.

Now the set

$$V_{\geq 3} \cup \bigcup_{i=1}^{s} D_i^P \cup \bigcup_{j=1}^{t} D_j^Q$$

is a dominating set of G and we obtain,

$$\begin{aligned}
\gamma &\leq n_{\geq 3} + \sum_{i=1}^{s} \left|D_i^P\right| + \sum_{j=1}^{t} \left|D_j^Q\right| \\
&= n_{\geq 3} + \sum_{i=1}^{s} \frac{|P_i| - 1}{3} + \sum_{j=1}^{t} \frac{|Q_j| - 2}{3} \\
&= \left(n_{\geq 3} - \frac{s}{3} - \frac{2t}{3}\right) + \sum_{i=1}^{s} \frac{|P_i|}{3} + \sum_{j=1}^{t} \frac{|Q_j|}{3} \\
&\leq \frac{n_{\geq 3}}{3} + \sum_{i=1}^{s} \frac{|P_i|}{3} + \sum_{j=1}^{t} \frac{|Q_j|}{3} \\
&\leq \frac{n}{3}.
\end{aligned}$$

This final contradiction completes the proof. □

Note that Theorem 1 is best possible for the union of cycles $C_{3\left\lfloor \frac{g+1}{3} \right\rfloor + 1}$. We derive some consequences of Theorem 1 for graphs of minimum degree at least 3.

Corollary 2 *If $G = (V, E)$ is a graph of order n, minimum degree $\delta \geq 3$, girth $g \geq 5$ and domination number γ, then*

$$\gamma \leq \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor \frac{g+1}{3} \right\rfloor + 1\right)}\right)\left(n - 4\alpha\left(G^4\right)\right) + \alpha\left(G^4\right)$$

where $\alpha\left(G^4\right)$ denotes the independence number of G^4, i.e. the maximum cardinality of a set $I \subseteq V$ of vertices such that every two vertices in I are at distance at least 5.

Proof: Let $I \subseteq V$ be a set of vertices such that every two vertices in I are at distance at least 5 and $|I| = \alpha(G^4)$. If $V' = I \cup N_G(I)$, then $|V'| \geq 4|I|$.

We will prove that $G[V \setminus V']$ has minimum degree at least 2. Therefore, for contradiction, we assume that there is a vertex $u \in V \setminus V'$ which has 2 neighbours v_1 and v_2 in V'. Clearly, $v_1 \in N_G(w_1)$ and $v_2 \in N_G(w_2)$ for some $w_1, w_2 \in I$. If $w_1 = w_2$, then $uv_1w_1v_2u$ is a cycle of length 4 which is a contradiction. If $w_1 \neq w_2$, then $w_1v_1uv_2w_2$ is a path of length 4 between two vertices of I which is a contradiction to the choice of I.

Therefore, $G[V \setminus V']$ has minimum degree at least 2 and, by Theorem 1, it has a dominating set D'' with $|D''| \leq \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)(n - |V'|)$. Now $I \cup D''$ is a dominating set of G and we obtain

$$\begin{aligned}
\gamma(G) &\leq |I| + |D''| \\
&\leq \frac{1}{4}|V'| + \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)(n - |V'|) \\
&\leq \alpha\left(G^4\right) + \left(\frac{1}{3} + \frac{2}{3\left(3\left\lfloor\frac{g+1}{3}\right\rfloor+1\right)}\right)\left(n - 4\alpha\left(G^4\right)\right)
\end{aligned}$$

which completes the proof. □

Since $\alpha(G) \geq \frac{n}{\Delta+1}$ for every graph G of order n and maximum degree Δ and the maximum degree of G^4 is at most $\Delta^2\left(\Delta^2 - 2\Delta + 2\right)$, we obtain the following immediate corollaries.

Corollary 3 *If $G = (V, E)$ is a cubic graph of order n, girth $g \geq 5$ and domination number γ, then*

$$\gamma \leq \left(\frac{44}{135} + \frac{82}{135g}\right)n.$$

Proof: If $g \leq 12$, then $\frac{44}{135} + \frac{82}{135g} \geq \frac{3}{8}$ and Reed's bound [13] implies the desired result. If $g \geq 13$, then G^4 is neither complete nor an odd cycle and Brooks' theorem [3] implies that $\alpha\left(G^4\right) \geq \frac{n}{\Delta(G^4)} \geq \frac{n}{45}$ and the result follows from Corollary 2. □

Note that $\frac{44}{135} + \frac{82}{135g} < \frac{1}{3}$ for $g \geq 83$ and hence Corollary 3 improves the bounds (1) and (2) due to Kostochka and Stodolsky [8] and Kawarabayashi, Plummer and Saito [6] and also confirms Reed's conjecture [13] for large enough girth.

Corollary 4 *For every $\Delta \geq \delta \geq 3$ there are constants $\alpha_{\delta,\Delta} < \frac{1}{3}$ and $\beta_{\delta,\Delta}$ such that if $G = (V, E)$ is a graph of order n, minimum degree δ, maximum degree Δ, girth $g \geq 5$ and domination number γ, then*

$$\gamma \leq \left(\alpha_{\delta,\Delta} + \frac{\beta_{\delta,\Delta}}{g}\right)n.$$

Instead of giving exact expressions for $\alpha_{\delta,\Delta}$ and $\beta_{\delta,\Delta}$ in Corollary 4, we pose it as an open problem to determine the best-possible values for these coefficients.

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