Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 07/21

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2007

Impressum:
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http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

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On Domains of Powers of Linear Operators and Finite Rank Perturbations

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Abstract. Let S and T be linear operators in a linear space such that $S \subset T$. In this note an estimate for the codimension of $\operatorname{dom} S^n$ in $\operatorname{dom} T^n$ in terms of the codimension of $\operatorname{dom} S$ in $\operatorname{dom} T$ is obtained. An immediate consequence is that for any polynomial p the operator $p(S)$ is a finite-dimensional restriction of the operator $p(T)$ whenever S is a finite-dimensional restriction of T. The general results are applied to a perturbation problem of selfadjoint definitizable operators in Krein spaces.

Mathematics Subject Classification (2000). Primary 47A05; Secondary 47B50.

Keywords. Powers of operators, Krein spaces, symmetric and selfadjoint operators, finite rank perturbations, definitizable operators.

1. Introduction

Let S and T be linear operators in a vector space and assume that S is a restriction of T, i.e., the domain $\operatorname{dom} S$ of S is a subset of the domain $\operatorname{dom} T$ of T and $Sx = Tx$ holds for all $x \in \operatorname{dom} S$. Clearly, for any $n \in \mathbb{N}$ also the n-th power S^n of S is a restriction of the n-th power T^n of T. In this note we verify the useful formula

$$\dim \left(\frac{\operatorname{dom} T^n}{\operatorname{dom} S^n} \right) \leq n \cdot \dim \left(\frac{\operatorname{dom} T}{\operatorname{dom} S} \right), \qquad n \in \mathbb{N}, \tag{1.1}$$

which relates the codimension of $\operatorname{dom} S^n$ in $\operatorname{dom} T^n$ with the codimension of $\operatorname{dom} S$ in $\operatorname{dom} T$. Note, that if S is a finite-dimensional restriction of T and p is a polynomial, then (1.1) immediately implies that also $p(S)$ is a finite-dimensional restriction of $p(T)$. This observation will be used to improve a classical result from

The work of T.Ya. Azizov was supported by the Russian Foundation for Basic Research (RFBR), Grant 05-01-00203-a.

P. Jonas and H. Langer on finite rank perturbations of definitizable selfadjoint operators in Krein spaces, cf., [JL, Theorem 1] and [ABT, Theorem 2.2].

2. Domains of Powers of Linear Operators

Let $\mathcal{X}$ be a vector space and let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathcal{X}$. Then it is well known (cf., e.g., [K, §7.6]) that there exists a subspace $\mathcal{G} \subset \mathcal{X}$ with

$$\mathcal{X} = \mathcal{M} \dotplus \mathcal{G} \quad \text{and} \quad \mathcal{N} = (\mathcal{N} \cap \mathcal{M}) \dotplus (\mathcal{N} \cap \mathcal{G}), \tag{2.1}$$

where $\dotplus$ denotes the direct sum of two subspaces. The subspaces $\mathcal{M}$ and $\mathcal{N}$ will be called isomorphic (in signs: $\mathcal{M} \cong \mathcal{N}$) if there exists a bijective linear mapping $F : \mathcal{M} \to \mathcal{N}$.

Let S and T be linear operators in $\mathcal{X}$ defined on the subspaces $\operatorname{dom} S$ and $\operatorname{dom} T$, respectively, and assume that $\operatorname{dom} S \subset \operatorname{dom} T$ and $Sx = Tx$, $x \in \operatorname{dom} S$, holds, i.e., S is a restriction of T. We shall also write $S \subset T$ in the sense of graphs. If $\mathcal{X}$ is a normed vector space the operators S and T may be unbounded and nonclosed. In the following, a relation between the codimensions of $\operatorname{dom} S^n$ in $\operatorname{dom} T^n$, $n \in \mathbb{N}$, and the codimension of $\operatorname{dom} S$ in $\operatorname{dom} T$ will be established, cf. (1.1) and Proposition 2.3. First two auxiliary statements will be proved, the following one of which may be of independent interest.

Proposition 2.1. *Let S and T be linear operators in $\mathcal{X}$ and assume that S is a restriction of T. Then for every $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$ we have*

$$\frac{\operatorname{dom} T^n}{\operatorname{dom} S^n} \cong \frac{\operatorname{dom} T^k \cap \operatorname{ran} T^{n-k}}{\operatorname{dom} S^k \cap \operatorname{ran} S^{n-k}} \times \frac{\ker T^{n-k}}{\ker S^{n-k}}.$$

Proof. Let $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$, set

$$\mathcal{Y} := \frac{\operatorname{dom} T^k \cap \operatorname{ran} T^{n-k}}{\operatorname{dom} S^k \cap \operatorname{ran} S^{n-k}} \quad \text{and} \quad \mathcal{Z} := \frac{\ker T^{n-k}}{\ker S^{n-k}},$$

and denote the cosets in $\mathcal{Y}$ and $\mathcal{Z}$ by $[\,\cdot\,]_{\mathcal{Y}}$ and $[\,\cdot\,]_{\mathcal{Z}}$, respectively. Since

$$\ker S^{n-k} = \ker T^{n-k} \cap \operatorname{dom} S^n$$

by (2.1) there exists a subspace $\mathcal{G} \subset \operatorname{dom} T^n$ with

$$\operatorname{dom} T^n = \operatorname{dom} S^n \dotplus \mathcal{G} \tag{2.2}$$

$$\ker T^{n-k} = \ker S^{n-k} \dotplus (\ker T^{n-k} \cap \mathcal{G}). \tag{2.3}$$

By Q we denote the projection in $\operatorname{dom} T^n$ onto $\operatorname{dom} S^n$ with respect to the decomposition (2.2). Now, we choose projections P_S in $\operatorname{dom} S^n$ onto $\ker S^{n-k}$ and $P_{\mathcal{G}}$ in $\mathcal{G}$ onto $\ker T^{n-k} \cap \mathcal{G}$ and define $P : \operatorname{dom} T^n \to \operatorname{dom} T^n$ by

$$Px := P_S Qx + P_{\mathcal{G}}(x - Qx), \quad x \in \operatorname{dom} T^n.$$

Then P is a projection in $\operatorname{dom} T^n$ onto $\ker T^{n-k}$ with

$$P \operatorname{dom} S^n = \ker S^{n-k}. \tag{2.4}$$

Let the linear mapping $F : \operatorname{dom} T^n \to \mathcal{Y} \times \mathcal{Z}$ be defined by

$$Fx := \big\{[T^{n-k}x]_{\mathcal{Y}}, [Px]_{\mathcal{Z}}\big\}, \quad x \in \operatorname{dom} T^n.$$

In the following we will show

$$\ker F = \operatorname{dom} S^n \quad \text{and} \quad \operatorname{ran} F = \mathcal{Y} \times \mathcal{Z}. \tag{2.5}$$

Let $x \in \operatorname{dom} T^n$ such that $Fx = 0$. Then we have $T^{n-k}x \in \operatorname{dom} S^k \cap \operatorname{ran} S^{n-k}$ and $Px \in \ker S^{n-k}$. Thus, there exists $u \in \operatorname{dom} S^n$ such that $T^{n-k}x = S^{n-k}u$ which implies $y := x - u \in \ker T^{n-k}$. Hence by (2.4)

$$y = Py = Px - Pu \in \ker S^{n-k}$$

and $x = u + y \in \operatorname{dom} S^n$ follows. Conversely, if $x \in \operatorname{dom} S^n$, then

$$T^{n-k}x = S^{n-k}x \in \operatorname{dom} S^k \cap \operatorname{ran} S^{n-k}$$

and $Px \in \ker S^{n-k}$ (see (2.4)), i.e. $Fx = 0$. In order to see that F is surjective, let $y \in \operatorname{dom} T^k \cap \operatorname{ran} T^{n-k}$ and $z \in \ker T^{n-k}$. Then there exists $x' \in \operatorname{dom} T^n$ with $T^{n-k}x' = y$ and for the vector $x := x' - Px' + z$ we have $T^{n-k}x = y$ and $Px = z$ and thus $Fx = \{[y]_{\mathcal{Y}}, [z]_{\mathcal{Z}}\}$. This establishes (2.5) which gives

$$\frac{\operatorname{dom} T^n}{\operatorname{dom} S^n} = \frac{\operatorname{dom} F}{\ker F} \cong \operatorname{ran} F = \mathcal{Y} \times \mathcal{Z}.$$

□

Lemma 2.2. *Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{X}$ be subspaces of $\mathcal{X}$ such that $\mathcal{M}_0 \subset \mathcal{M}_1$ and $\mathcal{N}_0 \subset \mathcal{N}_1$. Then we have*

$$\dim \frac{\mathcal{M}_1 \cap \mathcal{N}_1}{\mathcal{M}_0 \cap \mathcal{N}_0} \le \dim \frac{\mathcal{M}_1}{\mathcal{M}_0} + \dim \frac{\mathcal{N}_1}{\mathcal{N}_0}.$$

Proof. Denote the cosets in $(\mathcal{M}_1 \cap \mathcal{N}_1)/(\mathcal{M}_0 \cap \mathcal{N}_0)$ (resp. $\mathcal{M}_1/\mathcal{M}_0$, $\mathcal{N}_1/\mathcal{N}_0$) by $[\,\cdot\,]_{\mathcal{M}\cap\mathcal{N}}$ (resp. $[\,\cdot\,]_{\mathcal{M}}$, $[\,\cdot\,]_{\mathcal{N}}$). Then the mapping

$$F : \frac{\mathcal{M}_1 \cap \mathcal{N}_1}{\mathcal{M}_0 \cap \mathcal{N}_0} \to \frac{\mathcal{M}_1}{\mathcal{M}_0} \times \frac{\mathcal{N}_1}{\mathcal{N}_0}, \quad F[x]_{\mathcal{M}\cap\mathcal{N}} := ([x]_{\mathcal{M}}, [x]_{\mathcal{N}})$$

is a well-defined linear injection. □

The following proposition is the main result of this section.

Proposition 2.3. *Let S and T be linear operators in $\mathcal{X}$ and assume that S is a restriction of T. Then for every $n \in \mathbb{N}$ we have*

$$\dim \left(\frac{\operatorname{dom} T^n}{\operatorname{dom} S^n}\right) \le n \cdot \dim \left(\frac{\operatorname{dom} T}{\operatorname{dom} S}\right). \tag{2.6}$$

Corollary 2.4. *Let S and T be linear operators in $\mathcal{X}$ and assume that S is a finite-dimensional restriction of T, i.e., $\dim(\operatorname{dom} T/\operatorname{dom} S) < \infty$. Then for any polynomial p the operator $p(S)$ is a finite-dimensional restriction of the operator $p(T)$.*

Proof of Proposition 2.3. The assertion will be proved by induction. Obviously (2.6) is true for $n = 1$. Suppose that (2.6) holds for some $n \in \mathbb{N}$. Then with the help of Proposition 2.1 and Lemma 2.2 we obtain

$$\begin{aligned}\dim\left(\frac{\operatorname{dom} T^{n+1}}{\operatorname{dom} S^{n+1}}\right) &= \dim\left(\frac{\operatorname{dom} T^{n} \cap \operatorname{ran} T}{\operatorname{dom} S^{n} \cap \operatorname{ran} S}\right) + \dim\left(\frac{\ker T}{\ker S}\right)\\ &\le \dim\left(\frac{\operatorname{dom} T^{n}}{\operatorname{dom} S^{n}}\right) + \left[\dim\left(\frac{\operatorname{ran} T}{\operatorname{ran} S}\right) + \dim\left(\frac{\ker T}{\ker S}\right)\right].\end{aligned}$$

Now, an application of Proposition 2.1 for the case $k = 0$ and $n = 1$ implies the assertion of Proposition 2.3. □

We note that relation (2.6) is in general not an equality. As an example, consider the case $T = I$ and $S = T \upharpoonright M$ where $\operatorname{codim} M = 1$.

3. Finite Rank Perturbations of Definitizable Operators

In this section we apply the results of the previous section to symmetric operators of finite defect in Krein spaces. For the basic theory of Krein spaces and linear operators acting therein we refer to the monographs [AI] and [B].

A (possibly nondensely defined) operator S in a Krein space $(\mathcal{K}, [\cdot\,,\cdot])$ is called *symmetric* if $[Sx, x]$ is real for all $x \in \operatorname{dom} S$. Recall also that a closed symmetric operator S in $\mathcal{K}$ is said to be of *defect* $m \in \mathbb{N}$ if there exists a selfadjoint extension A of S in $\mathcal{K}$, i.e. $S \subset A = A^+$ (where A^+ denotes the adjoint of A with respect to $[\cdot\,,\cdot]$), such that $\dim(A/S) = m$. Observe that A can be multivalued if $\operatorname{dom} S$ is not dense in $\mathcal{K}$. However, it is always possible to choose a selfadjoint extension A of S which is an operator; then the defect of S coincides with $\dim(\operatorname{dom} A/\operatorname{dom} S)$.

A point $\lambda \in \mathbb{C}$ is said to be a *point of regular type* of a closed operator T in the Krein space $\mathcal{K}$ if $\ker(T - \lambda) = \{0\}$ and $\operatorname{ran}(T - \lambda)$ is closed. The set of points of regular type of T will be denoted by $r(T)$.

A selfadjoint operator A in $\mathcal{K}$ is said to be *definitizable* if its resolvent set $\rho(A)$ is nonempty and there exists a real polynomial q, $q \neq 0$, such that

$$[q(A)x, x] \ge 0 \quad \text{for all} \quad x \in \operatorname{dom} q(A).$$

We refer to [L] for a detailed study of the spectral properties of definitizable operators in Krein spaces. It was shown by P. Jonas and H. Langer in [JL, Theorem 1] that a definitizable operator remains definitizable under finite rank perturbations in resolvent sense if the perturbed operator is selfadjoint and has a nonempty resolvent set. This result was recently improved in [ABT, Theorem 2.2] where the assumption on the nonemptiness of the resolvent set of the perturbed operator was dropped. In the following we give a new simple proof of [ABT, Theorem 2.2] which makes use of the results in Section 2.

Theorem 3.1. *Let A and B be selfadjoint operators in the Krein space $\mathcal{K}$ and assume that the symmetric operator*

$$S := A \upharpoonright \operatorname{dom} S = B \upharpoonright \operatorname{dom} S, \quad \operatorname{dom} S := \big\{x \in \operatorname{dom} A \cap \operatorname{dom} B : Ax = Bx\big\},$$

is of finite defect. Then A is definitizable if and only if B is definitizable.

Proof. Assume that A is definitizable and let $q \neq 0$ be a real definitizing polynomial for A. We have to verify that $\rho(B)$ is nonempty. Then the assumption that S is of finite defect implies that

$$(A-\lambda)^{-1} - (B-\lambda)^{-1}, \qquad \lambda \in \rho(A) \cap \rho(B),$$

is a finite rank operator and therefore the statement follows from [JL, Theorem 1].

By [L] the set $\mathbb{C}\setminus\mathbb{R}$ with the exception of at most finitely many points belongs to $\rho(A)$ and it follows from $\sigma(q(A)) = q(\sigma(A))$ (cf. [DS, VII.9, Theorem 10]) that $\rho(q(A)) \cap (\mathbb{C}\setminus\mathbb{R}) \neq \varnothing$. Therefore $q(A)$ is a selfadjoint nonnegative operator in the Krein space $\mathcal{K}$ and hence

$$\mathbb{C}\setminus\mathbb{R} \subset \rho(q(A)) \tag{3.1}$$

holds. As $\rho(A) \subset r(S)$, and thus $r(S) \neq \varnothing$, a slight variation of the proof of [DS, VII.9, Theorem 7] shows that $q(S)$ is a closed operator. By Corollary 2.4 $q(S)$ has finite defect, and therefore also the symmetric extension $q(B)$ of $q(S)$ has finite defect and is closed. Thus, there exists a selfadjoint extension T of $q(B)$ which is an operator.

Next we show in the same way as in the proof of [CL, Proposition 1.1] that T has nonempty resolvent set. Since the symmetric form $[q(S)\cdot,\cdot]$ on $\operatorname{dom} q(S)$ is nonnegative, the form $[T\cdot,\cdot]$ on $\operatorname{dom} T$ has at most $n := \dim(\operatorname{dom} T/\operatorname{dom}(q(S)))$ negative squares. Moreover, (3.1) implies $\mathbb{C}\setminus\mathbb{R} \subset r(q(S))$ and thus $\operatorname{ran}(T-\lambda)$ is closed for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Assume now that there are $n+1$ different eigenvalues $\lambda_1,\ldots,\lambda_{n+1}$ in one of the open halfplanes with corresponding eigenvectors $x_1,\ldots,x_{n+1}$. As $\operatorname{dom} T$ is dense [B, Lemma I.10.4] implies that there exist vectors $z_1,\ldots,z_{n+1} \in \operatorname{dom} T$ such that $[x_i,z_j] = \delta_{ij}$, $i,j = 1,\ldots,n+1$. By setting $y_j := \overline{\lambda_j^{-1}} z_j \in \operatorname{dom} T$ we have $[Tx_i,y_j] = \delta_{ij}$, $i,j = 1,\ldots,n+1$, and it follows that

$$\mathcal{L} := \Big(\operatorname{span}\{x_1,\ldots,x_{n+1},y_1,\ldots,y_{n+1}\}, [T\cdot,\cdot]\Big)$$

is a Krein space which contains the neutral subspace $\operatorname{span}\{x_1,\ldots,x_{n+1}\}$. Hence $\mathcal{L}$ also contains an $(n+1)$-dimensional negative subspace, which is impossible as $[T\cdot,\cdot]$ has at most n negative squares. This shows that there exists a pair $\mu,\bar\mu \in r(T)$ which by the selfadjointness of T implies $\mu,\bar\mu \in \rho(T)$. The operator T is thus definitizable, see, e.g., [L, I.3, Example (c)] and, in particular, the set $\mathbb{C}\setminus\mathbb{R}$ with the possible exception of at most finitely many points belongs to $\rho(T)$.

For all $\lambda \in \rho(T)\cap\rho(A)$ we have $\ker(q(B)-\lambda) = \{0\}$ and both $\operatorname{ran}(q(B)-\lambda)$ and $\operatorname{ran}(B-\lambda)$ are closed. This together with $\sigma_p(q(B)) = q(\sigma_p(B))$ implies that there exists a point $\mu \in \mathbb{C}$ such that $\mu,\bar\mu \in r(B)$. But then the selfadjointness of B yields $\mu,\bar\mu \in \rho(B)$ and, in particular, $\rho(B) \neq \varnothing$. □

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