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Spectralizable Operators

Vladimir A. Strauss and Carsten Trunk

Abstract. We introduce the notion of spectralizable operators. A closed operator A in a Hilbert space is called spectralizable if there exists a non-constant polynomial p such that the operator $p(A)$ is a scalar spectral operator in the sense of Dunford. We show that such operators belongs to the class of generalized spectral operators and give some examples where spectralizable operators occur naturally.

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1. Introduction

Spectral operators (see [11]) are operators which possess a spectral resolution with properties comparable to the spectral function of self-adjoint operators in Hilbert spaces. In particular, this resolution is bounded, that is, a spectral operator has no spectral singularities.

One of the important directions of the development of modern operator theory is related to find spectral resolutions for more general classes of operators. Operators with spectral singularities belong to the class of the so-called generalized spectral operators [9]. This class is actively investigated [2, 3, 4, 5, 13, 24, 25, 26]. It is well known that for many concrete operator classes these resolutions exist in a generalized sense only thanks to some spectral singularities. In the well-known monograph of Colojoară and Foiaş [9] it is shown (Chapter 5, Corollary 5.7) that J-unitary and J-self-adjoint operators in Pontryagin spaces are examples of generalized spectral operators. This is based on two following facts:

1. a π-self-adjoint J-non-negative operator represents a generalized scalar spectral operator with the unique singularity in zero;

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2. for every π-self-adjoint operator A there exists a non-constant polynomial p that maps A to a π-non-negative operator.

Let us note that this is also correct for so-called definitizable operators in Krein spaces. For the basic theory of definitizable operators we refer to [20].

Another class of operators with spectral singularities was considered by Naimark [27]. There non-self-adjoint differential operators of second order on the semi-axis are studied. These investigations were continued by Lyance in [21] and [22]. There is a large number of papers connected with these problems including different kinds of differential and difference operators from mathematical physics and other areas. We mention here only [8, 12, 19, 23, 28]

The aim of our work is the investigation of the class of spectralizable operators.

Definition 1.1. A bounded operator A in a Hilbert space is called a *spectralizable operator* if there exists a non-constant polynomial p such that the operator $p(A)$ is a scalar spectral operator. The polynomial p is then called a *spectralizing polynomial for A*.

Spectralizable operators arise in many different problems, see, e.g., [7, 14, 15, 16, 31, 32, 33]. We mention that the term spectralizable was used first in [30] in a very special setting, where, in particular, one can find an example of a non-self-adjoint difference operator with a self-adjoint square. For operators with an identity iteration, see, e.g., [18]. In [6] operators are studied which have the property that the closure of its square is similar to some self-adjoint operator.

In this work we study spectral properties of spectralizable operators. It is easy to see (by some simple examples) that a spectralizable operator has in general a spectral function with singularities. There is some kind of similarity between these two classes of spectralizable and definitizable operators. This gives the expectation that the theory of spectralizable operators can be developed in a similar direction as the theory of definitizable operators.

We proceed as follows. In Section 2 we provide the main definitions and in 3 we prove the main result of this paper, i.e. that spectralizable operators possesses an eigen spectral function. In Section 4 we give an example of a spectralizable operator with an unbounded spectral function and in Section 5 we give some more examples of spectralizable operator. Finally, in Section 6 we construct with the help of the results from Section 3 an eigen spectral function for bounded operators A in Krein spaces which have the property that $p(A)$ is J-non-negative. For this we do not assume that A is J-self-adjoint or J-unitary.

2. Main Definitions

Let A be a bounded operator in a Hilbert space $\mathcal{H}$. Denote, as usual, by $\mathcal{L}(H)$ the set of all bounded operators in $\mathcal{H}$.

Spectral operators (see [11]) possess a spectral function with properties comparable to the spectral function of self-adjoint operators in Hilbert spaces. We introduce now the notion of a spectral function with a set of peculiar points.

Definition 2.1. Let $\Lambda = \{\lambda_k\}_1^n$ be a finite set of complex numbers and let

$$\mathfrak{R}_\Lambda := \{X \subset \mathbb{C} : X \text{ is a Borel set with } \partial X \cap \Lambda = \emptyset\}, \tag{2.1}$$

where ∂X is the boundary of X in $\mathbb{C}$. Let

$$E : \mathfrak{R}_\Lambda \to \mathcal{L}(H)$$

be a countably additive (with respect to weak topology) function, that maps $\mathfrak{R}_\Lambda$ to a commutative algebra of projections with $E(\mathbb{C}) = I$. The function E is called *a spectral function on* $\mathbb{C}$ *with the set* Λ *of peculiar points.*

Definition 2.2. A spectral function E with the set Λ of peculiar points is called *the eigen spectral function* of an operator A if the following holds for all $X \in \mathfrak{R}_\Lambda$.

a) $E(X)A = AE(X)$, $\sigma(A|_{E(X)\mathfrak{H}}) \subset \overline{X}$;
b) if $X \cap \Lambda = \emptyset$ then the operator $AE(X)$ is a scalar spectral operator and

$$AE(X) = \int_X \xi E(d\xi);$$

c) if $X \cap \Lambda \neq \emptyset$ then $AE(X)$ is not a scalar spectral operator.

The following is a consequence of [13].

Theorem 2.3. *The eigen spectral function with the set* Λ *of peculiar points of an operator is unique.*

Later, in Section 4, we will give an example where the spectral function is unbounded in a neighbourhood of a peculiar point. For this situation we introduce the following notion.

Definition 2.4. Let E be a eigen spectral function of A with the set of peculiar points Λ. If $\lambda \in \Lambda$ then λ will be called a peculiarity or a peculiar point (of A). The peculiarity λ is called regular if for a fixed neighbourhood X, $X \in \mathfrak{R}_\Lambda$, the operator family $\{E(X \cap Y)\}_{Y \in \mathfrak{R}_\Lambda}$ is bounded. The peculiarity λ is called singular in the opposite case.

Let us note that the notion of regular and singular peculiarity does not depend on the choice of X.

3. Main Results

In this section we show that spectralizable operators possesses an eigen spectral function. It is the main result of our paper.

Theorem 3.1. *Let A be a spectralizable operator and let p be its spectralizing polynomial. Then A has the eigen spectral function E with a finite number of peculiarities, moreover the peculiar set Λ is a subset of the set of roots of p', where p' is the derivative of p.*

Proof. If p' equals a constant then A is a scalar spectral operator and Theorem 3.1 is proved. Therefore, we assume that p' is not equal to a constant. We divide the proof in several steps.

1. Let $\xi_0 \in \sigma(A)$ with $p'(\xi_0) \neq 0$. We show that there exists the neighborhood $U(\xi_0)$ of ξ_0 with the properties

$$\begin{array}{l} p'(\xi) \neq 0 \text{ for every } \xi \in \overline{U(\xi_0)}, \text{ where } \overline{U(\xi_0)} \text{ is the closure of } U(\xi_0) \\ \text{and there is a projection } E^A(U(\xi_0)) \text{ commuting with } A \text{ such that} \\ A|_{E^A(U(\xi_0)\mathcal{H}} \text{ is a scalar spectral operator with} \end{array} \tag{3.1}$$

$$\sigma(A|_{E^A(U(\xi_0))\mathcal{H}}) \subset \overline{U(\xi_0)} \quad \text{and} \quad \sigma(A|_{(I-E^A(U(\xi_0)))\mathcal{H}}) \subset \overline{\mathbb{C}\backslash U(\xi_0)}.$$

Let $V(\xi_0)$ be an open ball with centre ξ_0 and

$$p'|_{\overline{V(\xi_0)}} \neq 0.$$

Due to the inverse function theorem there are a neighborhood $W(p(\xi_0))$ of the point $p(\xi_0)$ and a continuous function $p^{\smile}$ inverse to p,

$$p^{\smile} : W(p(\xi_0)) \to V(\xi_0) \quad \text{with} \quad p^{\smile}(p(\xi_0)) = \xi_0.$$

We will denote this extension by $p^{\smile}$ also. As an abbreviation we will write in the following

$$W := W(p(\xi_0)).$$

The pre-image of the polynomial p, $p^{-1}(W) = \{x \in \mathbb{C} : p(x) \in W\}$, has at most finitely many connected components. Hence, by choosing $V(\xi_0)$ sufficiently small, it is no restriction to assume that the closure of $p^{\smile}(W)$ is isolated in the closure of $p^{-1}(W)$.

Let $E^{p(A)}$ be the spectral resolution of $p(A)$. Then (see [11], Corollary XV.3.7) A commutes with $E^{p(A)}$, therefore the subspace $E^{p(A)}(W)\mathcal{H}$ is invariant with respect to A. So, due to the theorem of spectral mapping we obtain

$$\sigma(A|_{E^{p(A)}(W)\mathcal{H}}) \subset p^{-1}\left(\sigma(p(A)|_{E^{p(A)}(W)\mathcal{H}})\right) \subset \overline{p^{-1}(W)}.$$

and

$$\sigma(A|_{(I-E^{p(A)}(W))\mathcal{H}}) = p^{-1}\left(\sigma(p(A)|_{(I-E^{p(A)}(W))\mathcal{H}})\right) \subset \overline{p^{-1}(\mathbb{C}\backslash W)}.$$

Denote by Q_w,

$$Q_w : E^{p(A)}(W)\mathcal{H} \to E^{p(A)}(W)\mathcal{H},$$

the Riesz-Dunford projection of the operator $A|_{E^{p(A)}(W)\mathcal{H}}$ in $E^{p(A)}(W)\mathcal{H}$ which corresponds to the spectral set

$$\sigma(A|_{E^{p(A)}(W)\mathcal{H}}) \cap \overline{p^{\smile}(W)}.$$

Now we define

$$U(\xi_0) := p^{\smile}(W) \quad \text{and} \quad E^A(U(\xi_0)) := Q_w E^{p(A)}(W).$$

By the construction of $E^A(U(\xi_0))$ the polynomial p is a one-to-one map of the set $\sigma(A|_{E^A(U(\xi_0))\mathcal{H}})$ onto the set $\sigma(p(A)|_{E^{p(A)}(W)\mathcal{H}})$. As $p^{\smile}$ is analytic on W we have

$$A|_{E^A(U(\xi_0))\mathcal{H}} = p^{\smile}(p(A)|_{E^A(U(\xi_0))\mathcal{H}}).$$

Since $p(A)|_{E^A(U(\xi_0))\mathcal{H}}$ is a spectral operator of type 0 (i.e., a scalar spectral operator, see [11], Section XV.5 for details), $A|_{E^A(U(\xi_0))\mathcal{H}}$ is also a scalar spectral operator (see [11, Corollary XV.5.7]) and (3.1) is proved.

2. The spectrum of A is a compact set, therefore it is sufficient to define the eigen spectral function E of A for bounded sets. We set

$$\Lambda_0 := \{\xi \in \mathbb{C} : p'(\xi) = 0\}$$

and define $\mathfrak{R}_{\Lambda_0}$ in the same way as in (2.1) where we replace Λ by Λ_0. Let $X \in \mathfrak{R}_{\Lambda_0}$ be a bounded set which does not contain any zeros of p'. Then its closure $\overline{X}$ has the same property and $\overline{X}$ is a compact set. Therefore, there exists finitely many points $\xi_1, \ldots, \xi_N$ for some $N \in \mathbb{N}$ and neighborhoods $U(\xi_j)$ of ξ_j, $j = 1, \ldots, N$, which satisfy (3.1) with

$$\overline{X} \subset \cup_{j=1}^N U(\xi_j).$$

As $A|_{E^A(U(\xi_j))\mathcal{H}}$, $1 \le j \le N$, is a spectral operator, the projection $E^A(U(\xi_j) \cap U(\xi_k))$ is defined for $1 \le k \le N$ and

$$E^A(U(\xi_j) \cup U(\xi_k)) := E^A(U(\xi_j)) + E^A(U(\xi_k)) - E^A(U(\xi_j) \cap U(\xi_k))$$

is defined as usual. In a similar way, we define $E^A(\cup_{j=1}^N U(\xi_j))$. Then the operator $A|_{E^A(\cup_{j=1}^N U(\xi_j))\mathcal{H}}$ is a spectral operator. Therefore $E^A(X)$ is defined for all $X \in \mathfrak{R}_{\Lambda_0}$ with $X \cap \Lambda_0 = \emptyset$.

3. Let $X \in \mathfrak{R}_{\Lambda_0}$ and let $\{\lambda_l\}_{l=1}^L$, $L > 0$, be the set of roots of p' in X, where all roots are different, i.e. we don't take into account the multiplicity of the roots. Let $Z(\lambda_j)$, $1 \le j \le L$, be a bounded neighborhood of the point λ_j such that every pair of the closure of these neighborhoods contains disjoint elements. Then

$$Z := X \setminus \cup_{j=1}^L Z(\lambda_j) \in \mathfrak{R}_{\Lambda_0} \quad \text{with} \quad Z \cap \Lambda_0 = \emptyset.$$

By the second step of this proof, $A|_{E^A(Z)\mathcal{H}}$ is a spectral operator. Denote by $E^A(\cup_{j=1}^L Z(\lambda_j))$ the Riesz-Dunford projector of the operator $A|_{(I-E^A(Z))\mathcal{H}}$ corresponding to the spectral set $\cup_{j=1}^L Z(\lambda_j)$. Then we define

$$E^A(X) := E^A(Z) + E^A(\cup_{j=1}^L Z(\lambda_j)).$$

4. By the first three steps, E^A is a spectral function with the set Λ_0 of peculiar points. Moreover, a), b) and c) of Definition 2.2 are satisfied for all $X \in \mathfrak{R}_{\Lambda_0}$. If for some $\lambda \in \Lambda_0$ there exists a neighborhood $U(\lambda)$ of λ such that there is a projection $E^A(U(\lambda))$ such that $A|_{E^A(U(\lambda)\mathcal{H}}$ is a scalar spectral operator with $\sigma(A|_{E^A(U(\lambda))\mathcal{H}}) \subset \overline{U(\lambda)}$, then, using the construction above, E can be extended to all Borel sets containing λ in their boundary in such a way that a), b) and c) of Definition 2.2 remains valid. Hence, we choose Λ to be the set of all $\xi \in \Lambda_0$ with the property that there exists no neighborhood $U(\xi)$ of ξ such that there

exists a projection $E^A(U(\lambda))$ with $A|_{E^A(U(\lambda)\mathcal{H}}$ is a scalar spectral operator with $\sigma(A|_{E^A(U(\lambda))\mathcal{H}}) \subset \overline{U(\lambda)}$. Theorem 3.1 is proved. $\square$

4. Singularities of the spectral function

It is shown in Theorem 3.1 that a spectralizable operator has a spectral function with peculiar points. In this section we give a simple example for a singular peculiar point.

Example. Let $\mathcal{H}$ be the Hilbert space of all square summable sequences, $\mathcal{H} = l_2(\mathbb{N})$. Define

$$C : \mathcal{H} \to \mathcal{H} \quad \text{by} \quad C(x_n)_{n\in\mathbb{N}} = (n^{-2}x_n)_{n\in\mathbb{N}}$$

and set

$$\mathbf{B} := \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}.$$

We have

$$\mathbf{B}^2 = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},$$

hence $\mathbf{B}^2$ is a self-adjoint operator and $\mathbf{B}$ is spectralizable with spectralizing polynomial $p(t) = t^2$. Therefore it follows from Theorem 3.1 that zero is the only possible peculiar point. We will show that the spectral function is unbonded near zero. Denote by e_n, $n \in \mathbb{N}$, the sequence $(\delta_k^n)_{k\in\mathbb{N}}$. A simple computation shows

$$\pm\frac{1}{n} \in \sigma_p(\mathbf{B}) \quad \text{with eigenvector } \frac{1}{\sqrt{1+\frac{1}{n^2}}} \begin{pmatrix} e_n \\ \pm\frac{1}{n}e_n \end{pmatrix}.$$

The spectral projection $E([m^{-1}, 1])$, $m \in \mathbb{N}$, of $\mathbf{B}$ corresponding to the interval $[m^{-1}, 1]$ is given by

$$E([m^{-1},1]) \begin{pmatrix} (x_n)_{n\in\mathbb{N}} \\ (y_n)_{n\in\mathbb{N}} \end{pmatrix} = \sum_{n=1}^{m} \begin{pmatrix} \frac{1}{2}(x_ne_n + ny_ne_n) \\ \frac{1}{2}(n^{-1}x_ne_n + y_ne_n) \end{pmatrix}.$$

From this it is easily seen that the norm of $E([m^{-1}, 1])$ tends to infinity as $m \to \infty$.

5. Examples

It turns out that a large class of operators is spectralizable. We illustrate this with some examples.

Example. Let A be a bounded self-adjoint operator in some Hilbert space $\mathcal{H}$. Let B and C be bounded operators in $\mathcal{H}$ which commutes with A such that the operators

$$A^2 + BC \quad \text{and } A^2 + CB$$

are scalar spectral operators. An easy calculation shows that

$$\begin{bmatrix} A & B \\ C & -A \end{bmatrix}^2 = \begin{bmatrix} A^2+BC & 0 \\ 0 & A^2+CB \end{bmatrix}$$

in the Hilbert space $\mathcal{H}\times\mathcal{H}$, that is,

$$\begin{bmatrix} A & B \\ C & -A \end{bmatrix} \quad \text{is a spectralizable operator.}$$

Example. Let $A_0 \in \mathcal{L}(\mathcal{H})$ be a uniformly positive self-adjoint operator in some Hilbert space $\mathcal{H}$. Define

$$\mathbf{A}_0 := \begin{bmatrix} 0 & I \\ -A_0 & -A_0^{\frac{1}{2}} \end{bmatrix}.$$

An easy calculation shows that

$$\mathbf{A}_0^3 = \begin{bmatrix} A_0A_0^{\frac{1}{2}} & 0 \\ 0 & A_0A_0^{\frac{1}{2}} \end{bmatrix}.$$

It is a self-adjoint operator in the Hilbert space $\mathcal{H}\times\mathcal{H}$ and, hence, the operator $\mathbf{A}_0$ is spectralizable. We mention that this operator is a special case of (in general unbounded) operators considered in [14], [15], [32] and [33].

Example. Let $A_1 \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator in some Hilbert space $\mathcal{H}$. Define

$$\mathbf{A}_1 := \begin{bmatrix} 0 & I \\ -A_1 & -I \end{bmatrix}.$$

An easy calculation shows that

$$\mathbf{A}_1^2 + \mathbf{A}_1 = \begin{bmatrix} -A_1 & 0 \\ 0 & -A_1 \end{bmatrix},$$

which is a self-adjoint operator in the Hilbert space $\mathcal{H}\times\mathcal{H}$. Hence, $\mathbf{A}_1$ is a spectralizable operator. If, in addition, A_1 is uniformly positive, then, as in the example above, the operator $\mathbf{A}_1$ again fulfills the asumptions of [14], [15], [32] and [33].

Example. Let $B \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator in some Hilbert space $\mathcal{H}$. Define

$$\mathbf{A} := \begin{bmatrix} 0 & i(B+1) \\ i(B+1) & B \end{bmatrix}.$$

An easy calculation shows that

$$\mathbf{A}^3 + 2\mathbf{A}^2 + \mathbf{A} = \begin{bmatrix} -(B+2)(B+1)^2 & 0 \\ 0 & -(B+2)(B+1)^2 \end{bmatrix},$$

which is a self-adjoint operator in the Hilbert space $\mathcal{H}\times\mathcal{H}$. Hence, $\mathbf{A}$ is a spectralizable operator.

6. Operators in Krein spaces

There is a well developed theory of J-self-adjoint and J-unitary operators in Krein spaces (see [1] and [20] for details). Recall that a bounded self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *definitizable* if there exists a non-zero polynomial p such that

$$[p(A)x, x] \geq 0 \quad \text{for all} \quad x \in \mathcal{H}. \tag{6.1}$$

Then the spectrum of A is real or its non-real part consists of a finite number of points, cf. [20]. A definitizable operator possesses a spectral function defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not belong to some finite set of so-called critical points (see [20]).

In what follows, we show that bounded operators in Krein spaces, which are not necessarily J-self-adjoint or J-unitary, satisfying (6.1) possess a eigen spectral function with a set of peculiar points in the sense of Definition 2.2.

Theorem 6.1. *Let A be a bounded operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that there is a polynomial p such that the operator $p(A)$ is J-non-negative. Then A has the eigen spectral function E with finite number of peculiarities. Moreover the peculiar set Λ is a subset of the union of the set of roots of p and the set of roots of p', where p' is the derivative of p.*

Proof. Let us consider the J-non-negative operator $p(A)$ and its spectral function $E^{p(A)}$ (see [20]). As it is well known, for every positive $\epsilon > 0$ the operator $p(A)|_{(I-E^{p(A)}([-\epsilon,\epsilon]))\mathcal{H}}$ is a scalar spectral operator, hence, we can apply to it Theorem 3.1. Thus, we need only to show how to construct projections corresponding to the pre-image of p of small neighborhoods of zero. Let us consider the set $U_\epsilon := \{\xi \in \mathbb{C} : |p(\xi)| < \epsilon\}$ with $\epsilon > 0$. For every ϵ this set is bounded, so its closure is compact. Moreover the number of roots (without multiplicity) does not exceed the degree of the polynomial p. There exists $\delta > 0$ such that for every $0 < \epsilon < \delta$ the set U_ϵ represents an union of disjoint neighborhoods of roots of p. Each neighborhood contains only one root. Then for each neighborhood we can define a Riesz-Dunford projector similar as in step 2 of the proof of Theorem 3.1. By a reasoning similar to the steps 3 and 4 of the proof of Theorem 3.1, we construct the eigen spectral function E. □

Corollary 6.2. *Let A be a bounded operator and p as in Theorem 6.1. Let E be the eigen spectral function of A with the set of peculiar points Λ. If the closed set $X \in \mathfrak{R}_\Lambda$ is such that $p(\xi) > 0$ for every $\xi \in X$ and $X \cap \Lambda = \emptyset$, then the subspace $E(X)\mathcal{H}$ is uniformly positive, that is*

$$(E(X)\mathcal{H}, [\cdot,\cdot]) \text{ is a Hilbert space..}$$

Proof. The properties of the spectral function $E^{p(A)}$ of the J-non-negative operator $p(A)$ implies that $(E^{p(A)}(p(X))\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space, cf. [20]. By construction, $E(X)\mathcal{H}$ is a subspace of $E^{p(A)}(p(X))\mathcal{H}$. □

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