Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 08/11

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2008

Impressum:
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http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

ilmedia

Cohabitation of Independent Sets and Dominating Sets in Trees

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Abstract. We give a constructive characterization of trees that have a maximum independent set and a minimum dominating set which are disjoint and show that the corresponding decision problem is NP-complete for general graphs.

Keywords. domination; independence; inverse domination
AMS subject classification. 05C69

1 Introduction

We consider finite, undirected and simple graphs $G = (V, E)$ with vertex set V and edge set E. A set $I \subseteq V$ of vertices is an *independent set* of G, if no two vertices from I are adjacent in G. The maximum cardinality of an independent set of G is the *independence number* $\alpha(G)$ of G. A set $D \subseteq V$ of vertices is a *dominating set* of G, if every vertex in $V \setminus D$ has a neighbour in D. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$ of G.

Minimum independent and maximum dominating sets are among the most fundamental and well-studied graph theoretic concepts [7]. As early as 1978 Bange, Barkauskas, and Slater [1] and Slater [10] characterized trees which have two disjoint minimum dominating sets and two disjoint maximum independent sets, respectively. In [2, 4, 6] the problem of finding two minimum dominating sets of minimum intersection is studied while in [8] trees with two disjoint minimum independent dominating sets are characterized. In [3, 5, 9] the minimum cardinality of a dominating set which lies in the complement of a minimum dominating set is studied.

Complementing this previous research we consider graphs $G = (V, E)$ that have a maximum independent set I and a minimum dominating set D which are disjoint. We call such a pair of sets (I, D) an (α, γ)-pair of G. Intuitively, two independent sets or two dominating sets compete for similar types of vertices while an independent set and a dominating set seem easiler to reconcile. After proving that the decision problem whether a given graph has an (α, γ)-pair is NP-complete, we give a constructive characterization of trees with an (α, γ)-pair.

Theorem 1 *The problem to decide whether an input graph has an (α, γ)-pair is NP-complete.*

Proof: For a 3SAT instance $\mathcal{C}$ with n variables and m clauses, we will describe a graph $G = (V, E)$ of order polynomial in n and m such that $\mathcal{C}$ is satisfiable if and only if G has an (α, γ)-pair.

For every boolean variable x of $\mathcal{C}$, the graph G contains a copy H_x of the gadget shown in Figure 1 with two specified vertices x and $\bar{x}$.

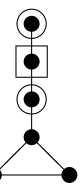

Figure 1: The gadget H_x.

For every clause C, the graph G contains $3n + 1$ disjoint paths of length three

$$P_1^C, P_2^C, \ldots, P_{3n+1}^C.$$

In each of these paths P_i^C we specify one endvertex x_i^C. If C contains the literal y, then G contains the edges yx_i^C for $1 \leq i \leq 3n + 1$. The graph G contains no further vertices or edges.

Clearly, every independent set of G contains at most three vertices from every of the gadgets H_x and at most two vertices from every of the paths P_i^C, i.e. $\alpha(G) \leq 3n+2m(3n+1)$. Since choosing three independent vertices from every of the gadgets H_x and the vertices at distance one and three from x_i^C from every of the paths P_i^C yields an independent set of order $3n + 2m(3n + 1)$, we have $\alpha(G) = 3n + 2m(3n + 1)$.

Clearly, every dominating set of G contains at least two vertices from every of the gadgets H_x and at least one vertex from every of the paths P_i^C. Hence $\gamma(G) \geq 2n + m(3n + 1)$. Furthermore, since choosing x, $\bar{x}$ and the neighbour of the endvertex from every of the gadgets H_x and the vertex at distance two from x_i^C from every of the paths P_i^C yields a dominating set of order $3n + m(3n + 1)$, we have $\gamma(G) \leq 3n + m(3n + 1)$.

If $\mathcal{C}$ has a satisfying truth assignment, then choosing three independent vertices containing the false literal among x and $\bar{x}$ from every of the gadgets H_x and the vertices at distance one and three from x_i^C from every of the paths P_i^C yields a maximum independent set I. Furthermore, choosing the true literal among x and $\bar{x}$ and the neighbour of the endvertex from every of the gadgets H_x and the vertex at distance two from x_i^C from every of the paths P_i^C yields an dominating set D of order $2n+m(3n+1)$. Hence (I, D) is an (α, γ)-pair of G.

Conversely, if G has an (α, γ)-pair (I, D), then we may assume that D contains exactly one of the two vertices x and $\bar{x}$ from every of the gadgets H_x. If one of the vertices x_i^C from some path P_i^C is not dominated by a vertex from one of the gadgets H_x, then D must contain at least two vertices from every of the $3n+1$ paths P_i^C and at least one vertex from every of the remaining paths. Hence $|D| \geq 3n + 1 + m(3n + 1)$ which is a contradiction. Therefore, all of the vertices x_i^C from every of the paths P_i^C are dominated by a vertex from one of the gadgets H_x. Hence the literals contained in D define a satisfying truth assignment for $\mathcal{C}$ and the proof is complete. $\Box$

2 Trees with an (α, γ)-pair

In this section we will describe a polynomial time procedure to decide whether a given tree has an (α, γ)-pair. We describe suitable reductions and explain how these reductions yield a constructive characterization of trees with an (α, γ)-pair.

The first lemma deals with some small trees.

Lemma 2 *(i) For $2 \leq n \leq 6$ the path $P_n : u_1 u_2 \ldots u_n$ has the following (α, γ)-pair (I_n, D_n):*

$$\begin{aligned}
(I_2, D_2) &= (\{u_1\}, \{u_2\}) \\
(I_3, D_3) &= (\{u_1, u_3\}, \{u_2\}) \\
(I_4, D_4) &= (\{u_1, u_4\}, \{u_2, u_3\}) \\
(I_5, D_5) &= (\{u_1, u_3, u_5\}, \{u_2, u_4\}) \\
(I_6, D_6) &= (\{u_1, u_3, u_6\}, \{u_2, u_5\}).
\end{aligned}$$

(ii) The tree $T^* = (V^*, E^*)$ with

$$\begin{aligned}
V^* &= \{u_0, u_1, v_0, v_1, v_2, w_0, w_1, w_2, w_3, x\} \\
E^* &= \{u_0u_1, u_1x, v_0v_1, v_1v_2, v_2x, w_0w_1, w_1w_2, w_2w_3, w_3x\}
\end{aligned}$$

has the (α, γ)-pair

$$(\{u_0, v_0, w_0, v_2, w_2\}, \{u_1, v_1, w_1, x\}).$$

Proof: It is very easy to check that the given sets are maximum independent sets and minimum dominating sets which are disjoint. $\Box$

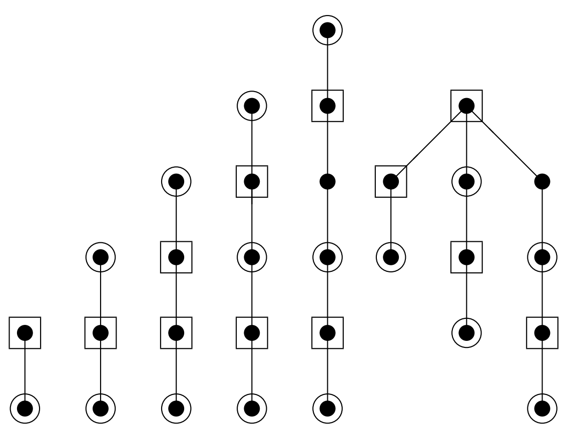

Figure 2 The trees $P_2, P_3, \ldots, P_6$ and T^*.

Lemma 3 *Let T contain a path $P : u_0u_1 \dots u_5$ such that $d_T(u_0) = 1$ and $d_T(u_1) = d_T(u_2) = d_T(u_3) = d_T(u_4) = 2$.*

(i) $\alpha(T') + 2 \le \alpha(T) \le \alpha(T') + 3$ for $T' = T - \{u_0, u_1, \dots, u_4\}$.

(ii) If $\alpha(T) = \alpha(T') + 3$, then T has an (α, γ)-pair if and only if $T'' = T - \{u_0, u_1\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T'') + 1$ and $\gamma(T) = \gamma(T'') + 1$.

(iii) If $\alpha(T) = \alpha(T') + 2$, then T has an (α, γ)-pair if and only if $T''' = T - \{u_0, u_1, u_2\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T''') + 1$ and $\gamma(T) = \gamma(T''') + 1$.

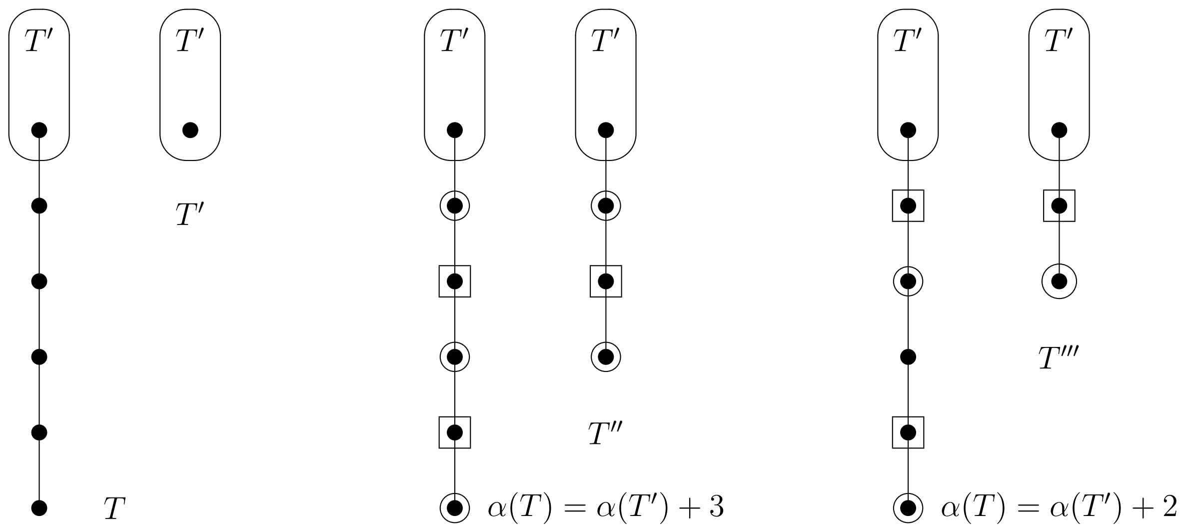

Figure 3 The trees T, T', T'' and T'''.

Proof: (i) The first inequality follows, since for every independent set I' of T' the set $I' \cup \{u_0, u_2\}$ is an independent set of T. The second inequality follows, since every independent set I of T contains at most three of the vertices in $\{u_0, u_1, \dots, u_4\}$ and $I \setminus \{u_0, u_1, \dots, u_4\}$ is an independent set of T'.

(ii) Let $\alpha(T) = \alpha(T') + 3$. Note that this implies that every maximum independent set of T contains u_0, u_2 and u_4. Therefore, if T has an (α, γ)-pair (I, D), then $u_0, u_2, u_4 \in I$ and hence $u_1, u_3 \in D$. Clearly, $\alpha(T'') \le \alpha(T') + 2$. Since $I \setminus \{u_0\}$ is an independent set in T'', we have $\alpha(T'') \ge \alpha(T) - 1 = \alpha(T') + 2$ and thus $\alpha(T) = \alpha(T') + 3 = \alpha(T'') + 1$. Clearly, $\gamma(T) \le \gamma(T'') + 1$. Since $D \setminus \{u_1\}$ is a dominating set in T'', we have $\gamma(T'') \le \gamma(T) - 1$ and thus $\gamma(T) = \gamma(T'') + 1$. Now $(I \setminus \{u_0\}, D \setminus \{u_1\})$ is an (α, γ)-pair of T''.

Conversely, if T'' has an (α, γ)-pair (I'', D''), $\alpha(T) = \alpha(T'') + 1$ and $\gamma(T) = \gamma(T'') + 1$, then $(I'' \cup \{u_0\}, D'' \cup \{u_1\})$ is an (α, γ)-pair of T.

(iii) Let $\alpha(T) = \alpha(T') + 2$. If T has an (α, γ)-pair (I, D), then we may assume without loss of generality that $u_0, u_3 \in I$ and $u_1, u_4 \in D$. Clearly, $\alpha(T''') \le \alpha(T') + 1$. Since $I \setminus \{u_0\}$ is an independent set in T''', we have $\alpha(T''') \ge \alpha(T) - 1 = \alpha(T') + 1$ and thus $\alpha(T) = \alpha(T') + 2 = \alpha(T''') + 1$. Clearly, $\gamma(T) \le \gamma(T''') + 1$. Since $D \setminus \{u_1\}$ is a dominating set in T''', we have $\gamma(T''') \leq \gamma(T) - 1$ and thus $\gamma(T) = \gamma(T''') + 1$. Now $(I \setminus \{u_0\}, D \setminus \{u_1\})$ is an (α, γ)-pair of T'''.

Conversely, if T''' has an (α, γ)-pair (I''', D'''), $\alpha(T) = \alpha(T''') + 1$ and $\gamma(T) = \gamma(T''') + 1$, then $(I''' \cup \{u_0\}, D''' \cup \{u_1\})$ is an (α, γ)-pair of T. $\Box$

Combining Lemma 2 (i) with Lemma 3 it is easy to check that the only paths P_n with an (α, γ)-pair satisfy $n \in \{2, 3, 4, 5, 6, 7, 8, 10\}$.

Lemma 4 *Let T contain a path $P : u_0 u_1 \ldots u_r w v_s v_{s-1} \ldots v_0$ with $r, s \geq 0$ such that $d_T(u_0) = d_T(v_0) = 1$, $d_T(u_i) = 2$ for $1 \leq i \leq r$ and $d_T(v_j) = 2$ for $1 \leq j \leq s$.*

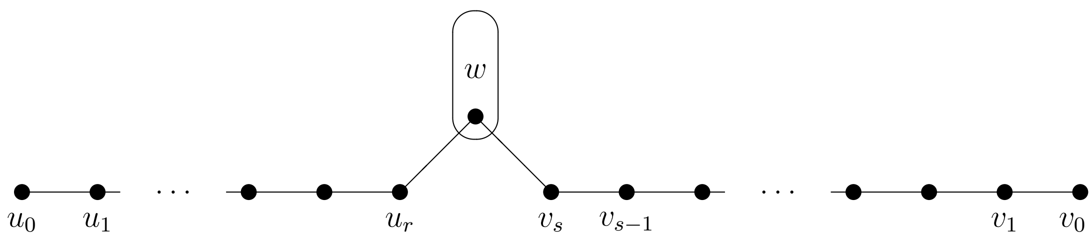

Figure 4 *The path $P : u_0 u_1 \ldots u_r w v_s v_{s-1} \ldots v_0$.*

(i) If $r = 2k$ and $s = 2l$ for some $0 \leq k, l \leq 1$ with $k \geq l$, then T has an (α, γ)-pair if and only if $T' = T - \{u_i \mid 0 \leq i \leq 2k\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + k$.

(ii) If $r = 2k + 1$ and $s = 0$ for some $0 \leq k \leq 1$, then T has an (α, γ)-pair if and only if $T' = T - \{u_i \mid 0 \leq i \leq 2k + 1\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + 1$.

(iii) If $r = s = 1$, then T has an (α, γ)-pair if and only if $T' = T - \{u_0, u_1\}$ has an (α, γ)-pair.

(iv) If $r = s = 3$, then T has an (α, γ)-pair if and only if $T' = T - \{u_0, u_1, u_2, v_0, v_1, v_2\}$ has an (α, γ)-pair and $\alpha(T) = \alpha(T') + 2$.

(v) If $r = 1$, $s = 2$ and $d_T(w) = 3$, then T has an (α, γ)-pair if and only if $T' = T - V(P)$ has an (α, γ)-pair.

(vi) If $r = 1$, $s = 3$ and $d_T(w) = 3$, then T has an (α, γ)-pair if and only if $T' = T - \{u_0, u_1\}$ has an (α, γ)-pair.

(vii) If $r = 2$, $s = 3$ and $d_T(w) = 3$, then T has an (α, γ)-pair if and only if $T' = T - \{u_0, u_1, v_0, v_1, v_2, v_3\}$ has an (α, γ)-pair.

Proof: (i) Note that every maximum independent set I of T satisfies $I \cap V(P) = \{u_{2i} \mid 0 \leq i \leq k\} \cup \{v_{2j} \mid 0 \leq j \leq l\}$.

Therefore, if T has an (α, γ)-pair (I, D), then $u_{2i} \in I$ for $0 \leq i \leq k$, $v_{2j} \in I$ for $0 \leq j \leq l$, $u_{2i+1} \in D$ for $0 \leq i \leq k - 1$ and $v_{2j+1} \in D$ for $0 \leq j \leq l - 1$. Clearly, $\alpha(T) \leq \alpha(T') + k + 1$.

Since $I \setminus \{u_{2i} \mid 0 \leq i \leq k\}$ is an independent set in T', we have $\alpha(T') \leq \alpha(T) - (k+1)$ and thus $\alpha(T) = \alpha(T') + k + 1$. Clearly, $\gamma(T) \leq \gamma(T') + k$ — note that $k = 0$ implies $l = 0$ and $w \in D$. Since $D \setminus \{u_{2i+1} \mid 0 \leq i \leq k-1\}$ is a dominating set in T', we have $\gamma(T') \leq \gamma(T) - k$ and thus $\gamma(T) = \gamma(T') + k$. Now $(I \setminus \{u_{2i} \mid 0 \leq i \leq k\}, D \setminus \{u_{2i+1} \mid 0 \leq i \leq k-1\})$ is an (α, γ)-pair of T'.

Conversely, if T' has an (α, γ)-pair (I', D'), $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + k$, then in view of $l \leq 1$ we may assume that $v_{2l} \in I'$. Hence $w \notin I'$ and $(I' \cup \{u_{2i} \mid 0 \leq i \leq k\}, D' \cup \{u_{2i+1} \mid 0 \leq i \leq k-1\})$ is an (α, γ)-pair of T.

(ii) If T has an (α, γ)-pair, then it has an (α, γ)-pair (I, D) such that $v_0 \in I$, $w \in D$, $|I \cap \{u_i \mid 0 \leq i \leq 2k+1\}| = k + 1$ and $|D \cap \{u_i \mid 0 \leq i \leq 2k+1\}| = 1$. Similarly, if T' has an (α, γ)-pair, then it has an (α, γ)-pair (I', D') such that $v_0 \in I$ and $w \in D$. This easily implies that $\alpha(T) = \alpha(T') + k + 1$, $\gamma(T) = \gamma(T') + 1$ and that T has an (α, γ)-pair if and only if T' has an (α, γ)-pair.

(iii) If T has an (α, γ)-pair, then it has an (α, γ)-pair (I, D) such that $v_0 \in I$ and $v_1 \in D$. Similarly, if T' has an (α, γ)-pair, then it has an (α, γ)-pair (I', D') such that $v_0 \in I$ and $v_1 \in D$. This easily implies that $\alpha(T) = \alpha(T') + 1$, $\gamma(T) = \gamma(T') + 1$ and that T has an (α, γ)-pair if and only if T' has an (α, γ)-pair.

(iv) Note that every minimum dominating set of T contains w, u_1 and v_1. Similarly every minimum dominating set of T' contains w. This easily implies that $\alpha(T) = \alpha(T') + 2$, $\gamma(T) = \gamma(T') + 2$ and that T has an (α, γ)-pair if and only if T' has an (α, γ)-pair.

(v) It is easy to see that $\alpha(T) = \alpha(T') + 3$ and $\gamma(T) = \gamma(T') + 2$. If T has an (α, γ)-pair, then it has an (α, γ)-pair (I, D) such that $u_0, v_0, v_2 \in I$ and $u_1, v_1 \in D$. This easily implies that T has an (α, γ)-pair if and only if T' has an (α, γ)-pair.

(vi) It is easy to see that $\alpha(T) = \alpha(T') + 1$. Similarly, since T' has a minimum dominating set containing w, we have $\gamma(T) = \gamma(T') + 1$ which again implies the desired result.

(vii) Note that T has a maximum independent set containing u_2 and a minimum dominating set containing w. This easily implies that $\alpha(T) = \alpha(T') + 3$ and $\gamma(T) = \gamma(T') + 2$ which again implies the desired result. □

Lemma 5 *Let T contain three internally vertex disjoint paths $P : u_0u_1x$, $Q : v_0v_1v_2x$ and $R : w_0w_1w_2w_3x$ such that $d_T(u_0) = d_T(v_0) = d_T(w_0) = 1$, $d_T(u_1) = d_T(v_1) = d_T(v_2) = d_T(w_1) = d_T(w_2) = d_T(w_3) = 2$ and $d_T(x) = 4$, then T has an (α, γ)-pair if and only if $T' = T - \{u_0, u_1, v_0, v_1, w_0, w_1, w_2, w_3\}$ has an (α, γ)-pair.*

Proof: Note that T has a maximum independent set I such that $I \cap (V(P) \cup V(Q) \cup V(R)) = \{u_0, v_0, w_0, v_2, w_2\}$ and a minimum dominating set D such that $D \cap (V(P) \cup V(Q) \cup V(R)) = \{u_1, v_1, w_1, x\}$. This easily implies that $\alpha(T) = \alpha(T') + 4$ and $\gamma(T) = \gamma(T') + 3$.

If T has an (α, γ)-pair, then T has an (α, γ)-pair (I, D) such that $I \cap (V(P) \cup V(Q) \cup V(R)) = \{u_0, v_0, w_0, v_2, w_2\}$ and $D \cap (V(P) \cup V(Q) \cup V(R)) = \{u_1, v_1, w_1, x\}$. In this case $(I \setminus \{u_0, v_0, w_0, w_2\}, D \setminus \{u_1, v_1, w_1\})$ is an (α, γ)-pair of T'. Conversely, if T' has an (α, γ)-pair, then T' has an (α, γ)-pair (I', D') such that $v_2 \in I'$ and $x \in D'$. In this case $(I' \cup \{u_0, v_0, w_0, w_2\}, D' \cup \{u_1, v_1, w_1\})$ is an (α, γ)-pair of T which completes the proof. □

For integers $k \geq 1$ and $d_1 \geq d_2 \geq \ldots \geq d_k \geq 1$ a tree T is said to have a $(d_1, d_2, \ldots, d_k)$-*tinsel* $(P_1, P_2, \ldots, P_k)$ *pending on* v if $P_1, P_2, \ldots, P_k$ are k internally vertex disjoint paths in T such that

$$P_i : u_{i,0}u_{i,1}\ldots u_{i,d_i-1}v,$$

$d_T(u_{i,0}) = 1$ and $d_T(u_{i,j}) = 2$ for $1 \leq i \leq k$ and $1 \leq j \leq d_i - 1$ and $d_T(v) = k+1$. For integers $\partial d_1, \partial d_2, \ldots, \partial d_k$ with $0 \leq \partial d_i \leq d_i$ for $1 \leq i \leq k$, the tree

$$T - \bigcup_{i=1}^{k} \bigcup_{j=0}^{\partial d_i - 1} \{u_{i,j}\}$$

is said to arise from the tree T by $(\partial d_1, \partial d_2, \ldots, \partial d_k)$-*cutting* the $(d_1, d_2, \ldots, d_k)$-tinsel $(P_1, P_2, \ldots, P_k)$. Note that a tree T which is not a path and is rooted at an endvertex of a longest path has a tinsel $(P_1, P_2, \ldots, P_k)$ pending on some vertex v such that $k \geq 2$ and all vertices of the paths P_i are either v or descendants of v.

The next result summazies the reductions captured by Lemmas 3 through 5 and yields a constructive characterization of trees having an (α, γ)-pair.

Theorem 6 *Let $T = (V, E)$ be a tree which is not a path and different from the tree T^*. Let $(P_1, P_2, \ldots, P_k)$ be a $(d_1, d_2, \ldots, d_k)$-tinsel pending on v with $k \geq 2$.*

The tree T has an (α, γ)-pair if and only if the tree T' which arises from the tree T by $(\partial d_1, \partial d_2, \ldots, \partial d_k)$-cutting the $(d_1, d_2, \ldots, d_k)$-tinsel $(P_1, P_2, \ldots, P_k)$ has an (α, γ)-pair and $(\alpha(T) - \alpha(T'), \gamma(T) - \gamma(T')) = (\partial\alpha, \partial\gamma)$ where

(i) if $d_1 \geq 5$ and $\alpha(T) = \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) + 3$, then $(\partial d_1, \partial d_2, \ldots, \partial d_k) = (2, 0, \ldots, 0)$ and $(\partial\alpha, \partial\gamma) = (1, 1)$.

(ii) if $d_1 \geq 5$ and $\alpha(T) = \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) + 2$, then $(\partial d_1, \partial d_2, \ldots, \partial d_k) = (3, 0, \ldots, 0)$ and $(\partial\alpha, \partial\gamma) = (1, 1)$.

(iii) if there are two indices $1 \leq i < j \leq k$ such that $d_i, d_j \in \{1, 3\}$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial\alpha, \partial\gamma) = \left(\frac{d_i+1}{2}, \frac{d_i-1}{2}\right)$.

(iv) if $d_k = 1$ and there is an index $1 \leq i < k$ such that $d_i \in \{2, 4\}$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial\alpha, \partial\gamma) = \left(\frac{d_i}{2}, 1\right)$.

(v) if there are two indices $1 \leq i < j \leq k$ such that $d_i = d_j = 2$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial\alpha, \partial\gamma) = (1, 1)$.

(vi) if there are two indices $1 \leq i < j \leq k$ such that $d_i = d_j = 4$, then $\partial d_i = \partial d_j = 3$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \notin \{i, j\}$ and $\partial\alpha = 2$.

(vii) if $k = 2$ and $(d_1, d_2) = (3, 2)$, then $T' = T - (V(P_1) \cup V(P_2))$.

(viii) if $k = 2$ and $(d_1, d_2) = (4, 2)$, then $(\partial d_1, \partial d_2) = (0, 2)$.

(ix) if $k = 2$ and $(d_1, d_2) = (4, 3)$, then $(\partial d_1, \partial d_2) = (4, 2)$.

(x) if $k = 3$ and $(d_1, d_2) = (4, 3, 2)$, then $(\partial d_1, \partial d_2, \partial d_3) = (4, 2, 2)$.

Furthermore, one of the cases (i)-(x) occurs.

Proof: If $d_1 \geq 5$, then, by Lemma 3 (i), $2 \leq \alpha(T) - \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) \leq 3$. Now, by Lemma 3 (ii) and (iii), either (i) or (ii) occurs. Hence we may assume that $d_1 \leq 4$, i.e. all d_i are at most 4. If there are two odd d_i's, then, by Lemma 4 (i), the case (iii) occurs. Hence we may assume that at most one of the d_i is odd. If $d_k = 1$, then, by Lemma 4 (ii), the case (iv) occurs. Hence we may assume that all d_i are either 2, 3 or 4. If there are two d_i's equal to 2, then, by Lemma 4 (iii), the case (v) occurs. Hence we may assume that at most one of the d_i is 2. If there are two d_i's equal to 4, then, by Lemma 4 (iv), the case (vi) occurs. Hence we may assume that at most one of the d_i is 4. If $k \geq 3$, then $k = 3$, $(d_1, d_2, d_3) = (4, 3, 2)$ and, by Lemma 5, the case (x) occurs. Hence we may assume $k = 2$ and, by Lemma 4 (v) through (vii), one of the cases (vii) through (ix) occurs. This completes the proof. □

Corollary 7 *It is possible to decide in polynomial time whether a given tree of order at least 2 has an (α, γ)-pair.*

Proof: If T is a path of order at most 6 or the tree T^*, then, by Lemma 2, T has an (α, γ)-pair. If T is a path of order at least 7, then Lemma 3 allows to reduce the decision problem to a smaller tree in polynomial time. If T is neigther a path not the tree T^*, then Theorem 6 allows to reduce the decision problem to a smaller tree in polynomial time. □

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