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On Packing Shortest Cycles in Graphs

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Abstract. We study the problems to find a maximum packing of shortest edge-disjoint cycles in a graph of given girth g (g-ESCP) and its vertex-disjoint analogue g-VSCP. In the case $g = 3$, Caprara and Rizzi (2001) have shown that g-ESCP can be solved in polynomial time for graphs with maximum degree 4, but is APX-hard for graphs with maximum degree 5, while g-VSCP can be solved in polynomial time for graphs with maximum degree 3, but is APX-hard for graphs with maximum degree 4. For $g \in \{4, 5\}$, we show that both problems allow polynomial time algorithms for instances with maximum degree 3, but are APX-hard for instances with maximum degree 4. For each $g \geq 6$, both problems are APX-hard already for graphs with maximum degree 3.

Keywords. algorithms; approximation algorithms; combinatorial problems; graph algorithms; shortest cycles; packing; complexity

1 Introduction

In [4], Garey and Johnson show that the problem to find the maximum number of vertex-disjoint triangles in a graph is NP-hard. In [6], Holyer proves that the problem to find the maximum number of edge-disjoint triangles in a graph is NP-hard. Both sources actually consider arbitrarily large cliques instead of triangles. In [2], Caprara and Rizzi study the approximability of both triangle packing problems. Here we consider the generalisations of these problems to shortest cycles instead of triangles. For graphs whose shortest cycles have length g, i.e. which are of girth g, we consider the problems of finding the maximum number of edge-disjoint g-cycles (g-ESCP) and of finding the maximum number of vertex-disjoint g-cycles (g-VSCP). Note that for each value of g, the restrictions of g-ESCP and of g-VSCP to graphs with maximum degree 3 coincide.

Caprara and Rizzi [2] give polynomial time algorithms for the restriction of 3-ESCP to graphs with maximum degree 4 and for the restriction of 3-VSCP to graphs with maximum degree 3, but they show that under all weaker maximum degree restrictions the problems are APX-hard.

Theorem 1 (Caprara and Rizzi [2]) *The restriction of* 3*-ESCP to graphs with maximum degree* 5 *and the restriction of* 3*-VSCP to graphs with maximum degree* 4 *are APX-hard.*

By considering uniform subdivisions of instances for the 3-ESCP and 3-VSCP problems, their result immediately implies that for each $k \in \mathbb{N}$, the restriction of $3k$-ESCP to graphs with maximum degree 5 and the restriction of $3k$-VSCP to graphs with maximum degree 4 are APX-hard.

In the present paper, we show that for any $g \geq 6$, g-ESCP — and thus g-VSCP — is APX-hard, even if restricted to graphs with maximum degree 3. For $g \in \{4, 5\}$, we give polynomial time algorithms to solve g-ESCP instances — and thus g-VSCP instances — with maximum degree 3, but we show that both problems are already APX-hard if restricted to graphs with maximum degree 4. Altogether we determine the exact maximum degree thresholds for the polynomial time solvability of these packing problems.

2 Exact Algorithms

As in [2], we define

Definition 1 *For a given graph G we call the graph H whose vertices correspond to the shortest cycles of G and for which two vertices are adjacent if the corresponding cycles of G share an edge the **auxiliary graph** of G.*

Clearly, for any fixed g and a given graph G of girth g the problem g-ESCP can be polynomially reduced to determining the independence number $\alpha(H)$ of the auxiliary graph H of G.

Theorem 2 *4-ESCP and 5-ESCP can be solved in polynomial time for graphs with maximum degree $\Delta \leq 3$.*

Proof: It is sufficient to show that the auxiliary graph of G is claw-free, because then its independence number can be determined in polynomial time [8]. Indeed, if G contains a cycle C that intersects three other cycles, each of these cycles uses two of the at most five edges between $V(C)$ and $V(G) \setminus V(C)$, so they cannot all be edge-disjoint. □

By the same argument, auxiliary graphs of graphs with girth 4 are quasi-line graphs, i.e. the neighbourhood of each of their vertices can be partitioned into two cliques. Indeed, only few graphs of girth 4 do not allow trivial reductions, so we can give a proof that does not rely on the results by Minty [8] in this case. Since the structure of graphs of girth 5 and their auxiliary graphs is much richer than in the case of girth 4, such a very simple and direct argument does not seem to exist in this case.

Alternative Proof for 4-ESCP: We can detect and remove all edges of a given graph G that are not contained in a 4-cycle and construct the auxiliary graph in polynomial time. We may assume that H is connected (otherwise, we would split the edge set of G into smaller instances) and contains no vertex of degree 1 (otherwise, we would include the corresponding 4-cycle in the packing and discard the edges that are only contained in the intersecting 4-cycles). We may further assume that H is not a cycle, since we can compute $\alpha(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$. Therefore G contains a 4-cycle that intersects at least three further 4-cycles. We finish the proof by showing that under these conditions, G is one of the four graphs shown in Figure 1.

First we assume that G contains a $K_{2,3}$ subgraph induced by the union of the independent sets $\{v_1, v_2\}$ and $\{w_1, w_2, w_3\}$. Since this subgraph contains only three 4-cycles, we may assume that w_1 has another neighbour x. In order for the edge w_1x to be contained in a 4-cycle, x must be adjacent to one of the vertices w_2 and w_3, w.l.o.g. it is adjacent to w_2. Therefore, G contains $K_{3,3} - e$ as a subgraph. Since the vertices x and w_3 have distance 3 in the $K_{3,3} - e$ subgraph and every edge of G is contained in a 4-cycle, any further edge that contains one of them must contain them both. Therefore, in this subcase either $G = K_{3,3} - e$ or $G = K_{3,3}$.

Now we assume that G contains no $K_{2,3}$ subgraph, i.e. each pair of different 4-cycles in G shares at most one edge. Since H is claw-free but contains a vertex of degree at least 3,

G contains three 4-cycles each pair of which shares exactly an edge. These three 4-cycles form a subgraph that consists of an induced 6-cycle $v_1w_1v_2w_2v_3w_3v_1$ and a vertex y with $N(y) = \{v_1, v_2, v_3\}$. As in the above paragraph, we may assume that G contains a path w_1xw_2 for a new vertex x, and either G contains no further edge or another edge xw_3. In the first case, $G = K_2 \times P_4 - e$, and in the second case, $G = K_2 \times C_4$. $\square$

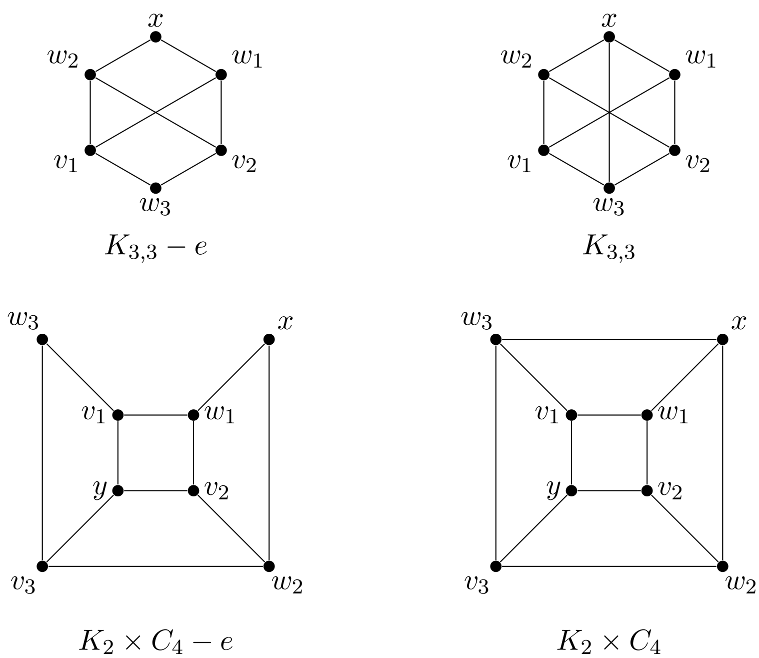

Figure 1: Graphs for Theorem 2

3 Hardness of Approximation

A MAX-SAT instance consists of a set X of some s Boolean variables $x_1, \ldots, x_s$ and of a set $\mathcal{Z}$ of some t clauses, which are subsets of the set L of literals, where L is the disjoint union of X and the set $\{\overline{x_1}, \ldots, \overline{x_s}\}$ of negations of the variables. We say that a truth assignment $X \to \{true, false\}$ satisfies a clause C, if C contains a Boolean variable set to true or the negation of a Boolean variable set to false. The maximum satisfiability problem asks for the maximum number of clauses that can be satisfied by a truth assignment.

Our proofs of the hardness of shortest cycle packings rely on a result of Berman and Karpinski on the 3-OCC-MAX 2SAT problem. This problem is the restriction of the maximum satisfiability problem to instances for which each clause contains at most two variables and each variable x occurs in at most three clauses, i.e. at most three clauses contain one of the literals x and $\overline{x}$.

Theorem 3 (Berman and Karpinski [1]) *For every $\varepsilon > 0$, it is NP-hard to approximate 3-OCC-MAX 2SAT within a factor of $\frac{2012}{2011} - \varepsilon$.*

Definition 2 *We call a 3-OCC-MAX 2SAT instance* ***reduced****, if for every literal $l_1 \in L$ and for every literal $l_2 \in L \setminus \{l_1\}$,*

1. *none of its clauses is of the form* $\{l_1, \overline{l_1}\}$*,*

2. *the instance contains a clause which contains* l_1 *and a clause which contains* $\overline{l_1}$*,*

3. *the instance does not contain both the clauses* $\{l_1, l_2\}$ *and* $\{\overline{l_1}, \overline{l_2}\}$*,*

4. *the instance does not contain two clauses* $\{l_1, \overline{l_2}\}$ *and* $\{l_1, l_2\}$ *and a third clause that contains the literal* $\overline{l_1}$*.*

Lemma 1 *For every* $\varepsilon > 0$*, it is NP-hard to approximate the restriction of 3-OCC-MAX 2SAT to reduced instances within a factor of* $\frac{2012}{2011} - \varepsilon$*.*

Proof: By Theorem 3, it is sufficient to show that for every unreduced instance I of the 3-OCC-MAX 2SAT problem, we can compute an integer d and a smaller instance I' with $OPT(I) = OPT(I') + d$ in polynomial time.

If the first condition on reduced instances is violated, we construct I' from I by removing the clause $\{l_1, \overline{l_1}\}$ and setting $d := 1$.

If the second condition is violated, then let d denote the number of clauses which contain l_1 or $\overline{l_1}$. We construct I' from I by removing these d clauses and the variable corresponding to l_1.

If the third condition is violated, we have two clauses $\{l_1, l_2\}$ and $\{\overline{l_1}, \overline{l_2}\}$. Let x_1 and x_2 be the two variables corresponding to the literals l_1 and l_2. If there exists a partial truth assignment $\{x_1, x_2\} \to \{true, false\}$ that satisfies all clauses in which the literals x_1, $\overline{x_1}$, x_2 and $\overline{x_2}$ occur, we set d to be the number of these clauses and construct I' from I by removing them. Otherwise, there are w.l.o.g. two clauses $\{l_1, l_3\}$ and $\{l_2, l_4\}$ for literals l_3, l_4 corresponding to two further variables. Any truth assignment that assigns the same value to l_1 and l_2 satisfies *at most* three of the four clauses in which l_1 and l_2 occur, while a truth assignment that assigns different values to these literals satisfies *at least* three of the four clauses. Therefore, we can set $d := 2$ and construct I' from I by removing the clauses $\{l_1, l_2\}$ and $\{\overline{l_1}, \overline{l_2}\}$ and replacing the literal l_1 by the literal $\overline{l_2}$.

If the fourth condition is violated, at most one of the literals l_2 and $\overline{l_2}$ occurs in a fourth clause; w.l.o.g. there is no further occurrence of the literal $\overline{l_2}$. Then any optimal truth setting remains optimal after the value of the variable corresponding to l_2 is adjusted such that l_2 is *true*. Therefore, we can set d to be the number of occurrences of the literal l_2 in I and construct I' from I by removing the clauses in which l_2 occurs and replacing the clause $\{l_1, \overline{l_2}\}$ by $\{l_1\}$. □

We can associate graphs to MAX-SAT instances via a construction from Karp's proof of the NP-completeness of STABLE SET [7].

Definition 3 *For a given MAX-SAT instance the vertices of the* ***SAT graph*** *correspond to the pairs* $(l, C) \in L \times \mathcal{Z}$ *with* $l \in C$*. Its edge set is a union* $E_C \cup E_V$ *of the set* E_C *of* ***clause edges*** *between each pair of vertices* (l_1, C) *and* (l_2, C) *that belong to the same clause, and of the set* E_V *of* ***variable edges*** *between each pair* (x, C_1) *and* $(\overline{x}, C_2)$ *of vertices.*

Obviously, the solution of the MAX-SAT problem corresponds the size of a maximum independent set in its SAT graph. For reduced 3-OCC MAX 2-SAT instances the SAT graphs have some properties summarized in the following lemma.

$(x_1 \vee x_2), (\overline{x_1}), (x_1 \vee \overline{x_2})$ $\qquad$ $(\overline{x_2} \vee \overline{x_3}), (x_1 \vee x_2), (\overline{x_1} \vee \overline{x_2}), (x_1 \vee x_3)$

Figure 2: SAT graphs to two *unreduced* 3-OCC MAX 2-SAT instances

Lemma 2 *The SAT graph corresponding to a reduced 3-OCC-MAX 2SAT instance is a simple graph, its vertex degrees range from 2 to 3, and its girth is at least 6.*

Proof: Let H be the SAT graph for a reduced 3-OCC-MAX 2SAT instance. Parallel edges in SAT graphs arise only if a clause contains two literals corresponding to the same variable, and this case is excluded for reduced graphs. The degree of any vertex in H cannot be larger than three because it is incident to at most one clause edge and at most two variable edges. As both literals corresponding to each variable occur in different clauses, there is no vertex of degree one.

Let us assume that H contains a 3-cycle T. Since the clause edges are a matching, T contains at least two variable edges. Therefore, all three vertices in T correspond to the same variable, so none of the three edges is a clause edge. This is impossible because (V, E_V) is a disjoint union of paths of length one and two.

Let us assume that H contains a 4-cycle Q. Since the clause edges are a matching, and every path with three edges in H contains at least one clause edge, clause edges and variable edges alternate on Q. Therefore, the vertices of C correspond to two clauses of the form $\{l_1, l_2\}$ and $\{\overline{l_1}, \overline{l_2}\}$, but this is the third excluded case for reduced instances.

Finally, we assume that H contains a 5-cycle P. As the clause edges are a matching, P contains at most two of them, so the vertices of P correspond to only two variables, and the 5-cycle corresponds to clauses $\{l_1, \overline{l_2}\}$, $\{l_1, l_2\}$ and $\{\overline{l_1}\}$, where the third clause may contain another literal. This is the fourth excluded case for reduced instances. □

To prove the hardness results, we use a construction which, under suitable conditions, provides us with graphs with given girth and given auxiliary graph.

Definition 4 *For any triangle-free graph H of maximum degree Δ and any integer $g \geq \max\{3, \Delta\}$, we call a graph G a $C(g, H)$**-graph**, if it is obtained from the disjoint union G' of g-cycles C_v for each vertex $v \in V(H)$ by the following identification process: For each $v \in V(H)$, we select $\deg_H(v)$ different **glueing edges** $\{e_{v,w}\}_{w \in N(v)} \subseteq E(C_v)$ such that for each pair $w \neq w'$ of neighbours of v, the distance between the vertex sets $e_{v,w}$ and $e_{v,w'}$ is at least $\left\lfloor \frac{g - \deg_H(v)}{\deg_H(v)} \right\rfloor = \left\lfloor \frac{g}{\deg_H(v)} - 1 \right\rfloor$. For each edge $\{v, w\} \in E(H)$ with $e_{v,w} = \{a_1, b_1\}$ and $e_{w,v} = \{a_2, b_2\}$, we define two **identification sets** $\{a_1, a_2\}$ and $\{b_1, b_2\}$. (Note that there are two possible choices.) We obtain G from G' as follows: For every edge $\{v, w\} \in E(H)$ with $e_{v,w} = \{a_1, b_1\}$ and $e_{w,v} = \{a_2, b_2\}$ and identification sets $\{a_1, a_2\}$ and $\{b_1, b_2\}$, we remove one of the two glueing edges $e_{v,w}$ and $e_{w,v}$ and identify a_1 with a_2 and b_1 with b_2.*

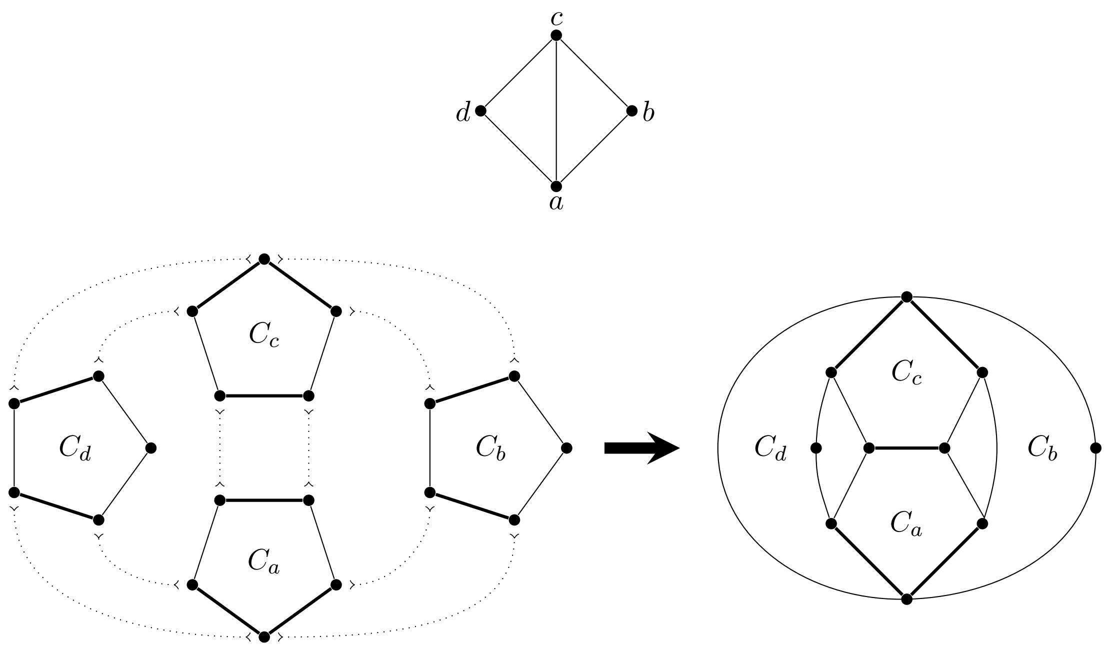

Figure 3: Construction of a $C(g, H)$-graph for $g = 5$ and $H = K_4 - e$

This construction does not yield a unique C_g-graph, as it depends upon the choice of glueing edges and identification sets. For fixed girth, it can be performed in polynomial time. Any $C(g, H)$-graph contains the g-cycles C_v for each $v \in V(H)$ as induced subgraphs, and for each edge $\{v, w\} \in H$, it contains an edge $e_{\{v,w\}}$ representing the pair of glueing edges $e_{v,w}$ and $e_{w,v}$. Figure 3 shows a $C(5, K_4 - e)$-graph that, besides the 5-cycles C_a, C_b, C_c, and C_d, contains two 3-cycles and a 4-cycle, so its auxiliary graph is not $K_4 - e$. However the following definition and lemma describe a condition on H that guarantees that the auxiliary graph of any $C(g, H)$-graph of H is H itself.

Definition 5 *The g-**weight** of a cycle C_H in a graph H is*

$$w_g(C_H) := \sum_{v \in V(C_H)} \left\lfloor \frac{g}{\deg_H(v)} - 1 \right\rfloor .$$

Lemma 3 *If a graph H contains no cycle C_H of g-weight $w_g(C_H) \leq g$, then the girth of any $C(g, H)$-graph G is g, and H is its auxiliary graph.*

Proof: It suffices to show that G contains no cycle $C \notin \{C_v : v \in H\}$ of length less than or equal to g. Let us assume that C is such a cycle. Then we can split the sequence of the edges of C into a sequence $P_1, P_2, \ldots, P_l$ of consecutive paths such that for $1 \leq i \leq l$ the path P_i is entirely contained in some g-cycle C_{v_i}. Furthermore, allowing paths of length 0, it is possible to choose these paths such that $\forall i \in \{1, \ldots, l\} : v_{i-1}v_i \in E(H)$ with $v_0 := v_l$. Clearly, the length of P_i is at least $\left\lfloor \frac{g}{\deg_H(v_i)} - 1 \right\rfloor$. Since the length of C is at most g, we obtain that $g \geq \sum_{i=1}^{l} \left\lfloor \frac{g}{\deg_H(v_i)} - 1 \right\rfloor$ and that $\forall i \in \{0, \ldots, l-2\} : v_i \neq v_{i+2}$. As $l > 1$, this implies that the sequence $v_0, v_1, v_2, \ldots, v_l$ contains a cycle C_H of H with $w_g(C_H) \leq g$. $\Box$

Theorem 4 *For every $\varepsilon > 0$ and every $g \geq 6$, it is NP-hard to approximate the restriction of g-ESCP to graphs with maximum degree at most 3 within a factor of $\frac{2012}{2011} - \varepsilon$.*

Proof: Let H be the SAT graph for a given reduced instance of the 3-OCC-MAX 2SAT problem. By Lemma 3, the theorem can be proved by a polynomial time construction of a graph G with girth g and maximum degree 3, such that H is the auxiliary graph of G.

We choose G as an arbitrary $C(g, H)$-graph. Since $g \geq 2\Delta(H)$, the glueing edges in any $C(g, H)$-graph G are disjoint, so $\Delta(G) \leq 3$. By Lemma 3, G has girth g, and H is its auxiliary graph, provided that the g-weight of any cycle C_H in H is greater than g.

As the clause edges on C_H are disjoint, each vertex of C_H is contained in a path of length one or two that contains only variable edges. Now each path of variable edges contains at most one vertex of degree three, so at least half of the vertices of C_H have degree two. Therefore, $w_g(C_H) = \sum_{v \in V(C_H)} \left\lfloor \frac{g}{\deg_H(v)} - 1 \right\rfloor \geq 3 \cdot \left(\left\lfloor \frac{g}{2} \right\rfloor + \left\lfloor \frac{g}{3} \right\rfloor - 2 \right)$. Since $g \geq 6$, this implies that the g-weight of C_H is strictly greater than g. $\square$

Theorem 5 *For every $\varepsilon > 0$ and each $g \in \{4, 5\}$, it is NP-hard to approximate the restriction of g-ESCP to graphs with maximum degree at most 4 within a factor of $\frac{6036}{6035} - \varepsilon$.*

Proof: Let H' be the SAT graph to an arbitrary reduced instance of the 3-OCC-MAX 2SAT problem. By the restrictions of the 3-OCC-MAX 2SAT instance, the vertices of degree 3 induce a subgraph of H' of maximum degree 1.

In polynomial time we can determine a set A of vertices of degree 2 of H' such that every cycle of H' contains a vertex of A and subject to this property the set A is minimal with respect to inclusion. Since A is independent, $|A| \leq \alpha(H')$.

Let H be the graph that we obtain by replacing each vertex $v \in A$ with neighbours a and b by five vertices v_i of degree 2, such that $av_1v_2v_3v_4v_5b$ is a path in H. It is easy to see that $\alpha(H) = \alpha(H') + 2|A|$ and that every independent set I of H efficiently yields an independent set I' of H' with $|I'| \geq |I| - 2|A|$. This implies that every independent set I of H for which $\frac{\alpha(H)}{|I|} \leq 1 + \delta$ would efficiently yield an independent set I' of H' for which $\frac{\alpha(H')}{|I'|} \leq \frac{1+\delta}{1-2\delta}$.

Since each cycle in H' contains at least two vertices of degree 2, each cycle C_H in H contains at least six vertices of degree 2, so $w_g(C_H) \geq 6 > g$, and by Lemma 3, any $C(g, H)$-graph has girth g, and its auxiliary graph is H.

The vertices of degree 3 in H induce a subgraph of maximum degree 1 as they do in H', i.e. a collection of isolated vertices and disjoint edges. We shall now argue why this allows us to use the freedom of choosing the glueing edges and the identification sets in such a way that we obtain a $C(g, H)$-graph G of maximum degree at most 4. For vertices v of H of degree 3 all neighbours of which are of degree 2, the identification processes involving the edges of C_v cannot create vertices of degree more than 4. If vw is an edge of H between two vertices of degree 3, then we choose the glueing edge $e_{v,w} = xy$ in C_v such that y is not contained in another glueing edge of C_v and the glueing edge $e_{w,v} = x'y'$ in C_w such that x' is not contained in another glueing edge of C_w. Furthermore, for the edge vw we choose the identification sets $\{x, x'\}$ and $\{y, y'\}$. By these choices, the identification processes involving the edges of C_v and C_w cannot create vertices of degree more than 4. Hence G is of maximum degree at most 4.

By Lemma 1, we can finish the proof by showing that the modified graph H is still the SAT graph of a reduced instance of the 3-OCC-MAX 2SAT problem. It suffices to show that for each vertex v of degree 2 in the SAT graph of a reduced instance J of the 3-OCC-MAX

2SAT problem, we can construct an instance J' whose SAT graph is the graph obtained by replacing v with a path $v_1v_2v_3v_4v_5$ as above.

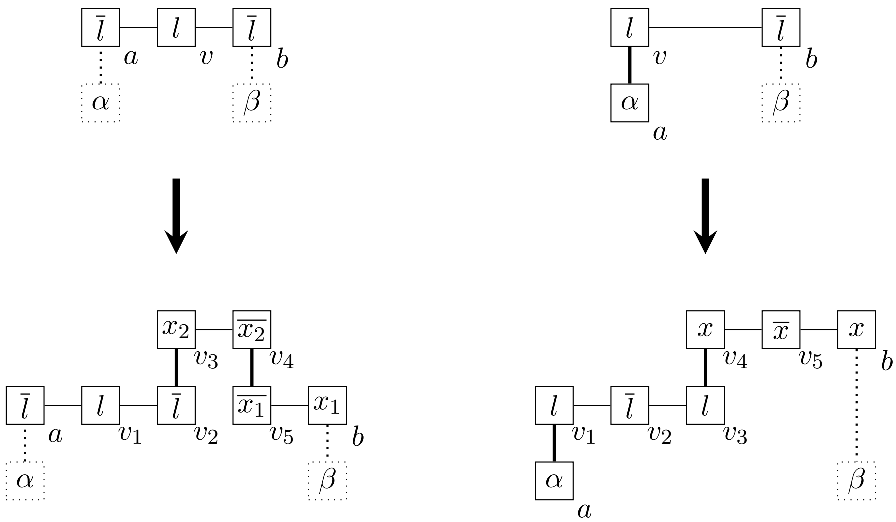

Figure 4: Replacement of a vertex v of degree two in Theorem 5

If v corresponds to the only literal l in a clause $\{l\}$, then there exist two further clauses C_1 and C_2 containing the literal $\bar{l}$. In this case, we construct J from J' by replacing the occurrence of $\bar{l}$ in C_2 with a new variable x_1 and adding two new clauses $\{\bar{l}, x_2\}$ and $\{\overline{x_2}, \overline{x_1}\}$, where x_2 is a second new variable. Otherwise, v corresponds to a literal l in a clause $\{l, \alpha\}$ that contains another literal α, and the literal $\bar{l}$ occurs in precisely one clause. In this case, we construct J from J' by replacing the occurrence of $\bar{l}$ by a new variable x and adding three new clauses $\{\bar{l}\}$, $\{l, x\}$, and $\{\overline{x}\}$. $\square$

Finally, we show the APX-hardness of g-VSCP for $g < 6$ by a similar construction in which each two g-cycles intersect in at most an edge.

Theorem 6 *For every $\varepsilon > 0$ and every $g \in \{3, 4, 5\}$, it is NP-hard to approximate the restriction of g-VSCP to graphs with maximum degree at most 4 within a factor of $\frac{2012}{2011} - \varepsilon$.*

Proof: Let H be the SAT graph for a given reduced instance of the 3-OCC-MAX 2SAT problem. We are going give a polynomial time construction of a graph G of girth g and maximum degree at most 4 whose shortest cycles are a set $\{C_v\}_{v\in V(H)}$, such that two cycles C_v and C_w are vertex-disjoint if and only if $\{v, w\} \notin E(H)$. Since vertex-disjoint packings of shortest cycles in G correspond to stable sets in H, Lemma 1 then implies the statement.

Let G' be the disjoint union of $|H|$ g-cycles C_v. Since the maximum degree of H is at most 3, we can select vertices $x_{v,w} \in C_v$ and $x_{w,v} \in C_w$ for each edge $\{v, w\} \in E(H)$, such that all $2|E(H)|$ selected vertices are pairwise different. We construct G by identifying $x_{v,w}$ with $x_{w,v}$ for each $\{v, w\} \in E(H)$. Then each vertex of G is contained in at most two g-cycles C_v, so the maximum degree of G is at most 4. It remains to show that the length of any cycle $C_G \notin \{C_v\}_{v\in H}$ is greater than g.

Indeed, the edge sequence C_G can be uniquely decomposed into maximal non-empty subpaths $P_1, P_2, \ldots, P_l$, such that $l > 2$, and for each $1 \le i \le l$ the edges of the path P_i are contained in some cycle C_{v_i} for $v_i \in H$. Then the edges $v_1 v_l$ and $v_i v_{i+1}$ for $i \in \{1, \ldots, l-1\}$ are contained in H, and since $\forall i \in \{1, \ldots, l-2\}$: $v_i \neq v_{i+2}$, the vertex sequence contains a cycle of H. Since the girth of H is at least 6 by Lemma 2, we have $|E(C_G)| \geq l \geq 6 > g$. $\square$

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