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# Connectivity and Diameter in Distance Graphs

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**Abstract.** For  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ , the distance graph  $P_n^D$  has vertex set  $\{0, 1, \dots, n-1\}$  and edge set  $\{ij \mid 0 \leq i, j \leq n-1, |j-i| \in D\}$ . The class of distance graphs generalizes the important and very well-studied class of circulant graphs which have been proposed for numerous network applications. In view of fault tolerance and delay issues in these applications, the connectivity and diameter of circulant graphs have been studied in great detail.

Our main contributions are hardness results concerning computational problems related to the connectivity and diameter of distance graphs and a number-theoretic characterization of the connected distance graphs  $P_n^D$  for  $|D| = 2$ .

**Keywords.** Circulant graph; distance graph; multiple loop networks; connectivity; diameter

# 1 Introduction

Circulant graphs form an important and very well-studied class of graph [1, 9, 11, 12, 16, 18, 19]. They are Cayley graphs of cyclic groups and have been proposed for numerous network applications such as local area computer networks, large area communication networks, parallel processing architectures, distributed computing, and VLSI design. In view of fault tolerance and delay issues in these applications, the connectivity and diameter of circulant graphs have been studied in great detail [1, 2, 11, 12, 20, 24].

For  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ , the *circulant graph*  $C_n^D$  has vertex set  $[0, n-1] = \{0, 1, \dots, n-1\}$  and the neighbourhood  $N_{C_n^D}(i)$  of a vertex  $i \in [0, n-1]$  in  $C_n^D$  is given by

$$N_{C_n^D}(i) = \{(i+d) \bmod n \mid d \in D\} \cup \{(i-d) \bmod n \mid d \in D\}.$$

Clearly, we may assume  $\max(D) \leq \frac{n}{2}$  for every circulant graph  $C_n^D$ .

Our goal is to investigate how some of the fundamental results concerning circulant graphs generalize to the similarly defined yet more general class of distance graphs: For  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ , the *distance graph*  $P_n^D$  has vertex set  $[0, n-1]$  and

$$\begin{aligned} N_{P_n^D}(i) &= \{i+d \mid d \in D \text{ and } (i+d) \in [0, n-1]\} \\ &\cup \{i-d \mid d \in D \text{ and } (i-d) \in [0, n-1]\} \end{aligned}$$

for all  $i \in [0, n-1]$ . Clearly, we may assume  $\max(D) \leq n-1$  for every distance graph  $P_n^D$ .

Every distance graph  $P_n^D$  is an induced subgraph of the circulant graph  $C_{n+\max(D)}^D$ . More specifically, distance graphs are the subgraphs of sufficiently large circulant graphs induced by sets of consecutive vertices, i.e. they represent the structure of small segments of circulant networks. Conversely, the following simple observation shows that every circulant graph is in fact a distance graph.

**Proposition 1** *A graph is a circulant graph if and only if it is a regular distance graph.*

*Proof:* Clearly, every circulant graph  $C_n^D$  is regular and isomorphic to the distance graph  $P_n^{D'}$  for  $D' = D \cup \{n - d \mid d \in D\}$ .

Now let  $P_n^D$  be a regular distance graph. Let  $D = \{d_1, d_2, \dots, d_k\}$  with  $d_1 < d_2 < \dots < d_k \leq n - 1$ . Since the vertex 0 has exactly  $k$  neighbours  $D$ ,  $P_n^D$  is  $k$ -regular.

Let  $i \in [1, k]$ . The vertex  $d_i - 1$  has exactly  $i - 1$  neighbours  $j$  with  $j < d_i - 1$ . Hence  $d_i - 1$  has exactly  $k + 1 - i$  neighbours  $j$  with  $j > d_i - 1$  which implies  $(d_i - 1) + d_{k+1-i} \leq n - 1$ . The vertex  $d_i$  has exactly  $i$  neighbours  $j$  with  $j < d_i$ . Hence  $d_i$  has exactly  $k - i$  neighbours  $j$  with  $j > d_i$  which implies  $d_i + d_{k+1-i} > n - 1$ .

We obtain  $d_i + d_{k+1-i} = n$  for every  $i \in [1, k]$  which immediately implies that  $P_n^D$  is isomorphic to the circulant graph  $C_n^{D'}$  for  $D' = \{d \in D \mid d \leq \frac{n}{2}\}$ .  $\square$

Originally motivated by coloring problems for infinite distance graphs studied by Eggleton, Erdős, and Skilton [6, 7], most research on distance graphs focused on colorings [3–5, 13, 14, 22, 23].

Our main contributions in the present paper are hardness results concerning computational problems related to the connectivity and diameter of distance graphs and a number-theoretic characterization of the connected distance graphs  $P_n^D$  for  $|D| = 2$ .

## 2 Results

Boesch and Tindell [2] observed that a circulant graph  $C_n^D$  is connected if and only if the greatest common divisor  $\gcd(\{n\} \cup D)$  of the integers in  $\{n\} \cup D$  equals 1. In fact, since  $C_n^D$  is vertex-transitive, it is connected if and only if it contains a path from the vertex 0 to the vertex 1 which is equivalent to the existence of integers  $l$  and  $l_d$  for  $d \in D$  such that  $1 = ln + \sum_{d \in D} l_d d$ . It is a well-known consequence of the Euclidean algorithm that the

existence of such integers is equivalent to the above gcd-condition. Therefore, deciding the connectivity of a circulant graph requires a simple polynomial time gcd-computation.

The most fundamental connectivity problem for distance graphs is the following.

CONNECTIVITY OF  $P_n^D$

**Instance:**  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ .

**Question:** Is  $P_n^D$  connected?

We have not been able to determine the complexity of CONNECTIVITY OF  $P_n^D$  and pose the following conjecture.

**Conjecture 2** CONNECTIVITY OF  $P_n^D$  is NP-hard.

Clearly,  $P_n^D$  is connected if and only if for every  $i \in [0, n - 2]$ , there is a path in  $P_n^D$  from  $i$  to  $i + 1$ . Equivalently, for every  $i \in [0, n - 2]$ , there are integers  $x_1, x_2, \dots, x_l$  such that

$$|x_i| \in D \text{ for all } i \in [1, l], \quad (1)$$

$$1 = \sum_{j=1}^l x_j, \text{ and} \quad (2)$$

$$i + \sum_{j=1}^k x_j \in [0, n - 1] \text{ for all } k \in [0, l]. \quad (3)$$

As noted before, 1 is an integral linear combination of the elements of  $D$  if and only if  $\gcd(D) = 1$ . Hence the existence of integers  $x_i$  which satisfy (1) and (2) can be decided in polynomial time. Unfortunately, these integers are by far not unique. Furthermore, given integers  $x_i$  which satisfy (1) and (2), deciding the existence of an ordering of them which satisfies (3) is in general a hard problem as we show next.

BOUNDED PARTIAL SUMS

**Instance:**  $x_0, x_1, x_2, \dots, x_l \in \mathbb{Z}$  and  $n \in \mathbb{N}$ .

**Question:** Is there a permutation  $\pi \in S_l$  such that

$$x_0 + \sum_{j=1}^k x_{\pi(j)} \in [0, n-1]$$

for all  $k \in [0, l]$ ?

**Proposition 3** BOUNDED PARTIAL SUMS *is NP-complete.*

*Proof:* Clearly, BOUNDED PARTIAL SUMS is in NP. We will reduce the classical NP-complete problem PARTITION [10] to BOUNDED PARTIAL SUMS:

PARTITION

**Instance:**  $x_1, x_2, \dots, x_k \in \mathbb{N}$ .

**Question:** Is there a set  $I \subseteq [1, k]$  such that  $\sum_{i \in I} x_i = \sum_{i \in [1, k] \setminus I} x_i$ ?

In order to relate to the preceding discussion we will reduce to instances of BOUNDED PARTIAL SUMS which satisfy (2).

Let  $x_1, x_2, \dots, x_{l-2} \in \mathbb{N}$  be an instance of PARTITION. Let  $X = \sum_{i=1}^{l-2} x_i$ .

Let  $x_0 = 0$ ,  $x_{l-1} = -X$ ,  $x_l = -X + 1$ , and  $n = X + 1$ . It is easy to see that the instance  $x_1, x_2, \dots, x_{l-1}$  of PARTITION is “yes”-instance if and only if the instance of BOUNDED PARTIAL SUMS defined by

$$x_0, 2x_1, 2x_2, \dots, 2x_{l-2}, x_{l-1}, x_l$$

and  $n$  is a “yes”-instance. This completes the proof.  $\square$

Clearly, if  $|D| = 1$ , then  $P_n^D$  is connected if and only if  $D = \{1\}$ . Already for  $|D| = 2$ , the following number-theoretic characterization of the pairs  $(n, D)$  for which  $P_n^D$  is connected is not simple.

**Theorem 4** Let  $n, d_1, d_2 \in \mathbb{N}$  be such that  $d_1 < d_2$ . For  $i \in [0, d_1 - 1]$ , let  $r_i = (id_2) \bmod d_1$  and  $s_i = (n - 1 - r_i) \bmod d_1$ . Furthermore, for  $i \in [1, d_1 - 1]$ , let

$$d_i^+ = \max \{r_i \mid i \in [0, i^* - 1]\} \text{ and}$$

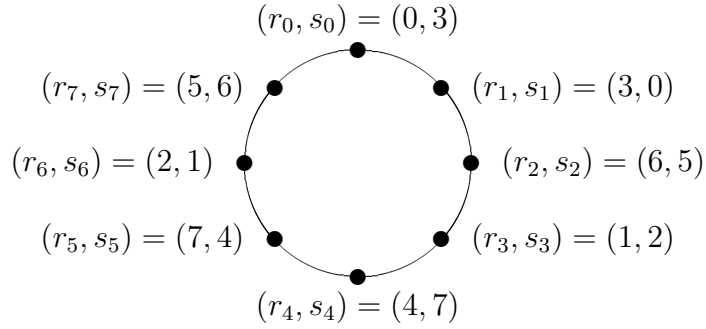
$$d_i^- = \max \left\{ s_{-i \bmod d_1} \mid i \in [0, d_1 - i^* - 1] \right\}.$$

Finally, let

$$d^* = \max_{i^* \in [1, d_1 - 1]} \min \{d_i^+, d_i^-\}.$$

(See Figure 1 for an example.)

$P_n^{\{d_1, d_2\}}$  is connected if and only if  $\gcd(\{d_1, d_2\}) = 1$  and  $d^* + d_2 \leq n - 1$ .



**Figure 1.**  $(d_1, d_2, n) = (8, 11, 20)$ ,  $(d_1^+, \dots, d_7^+) = (0, 3, 6, 6, 6, 7, 7)$ ,  
 $(d_1^-, \dots, d_7^-) = (7, 7, 7, 6, 6, 6, 0)$ , and  $d^* = 6$ .

*Proof:* “ $\Rightarrow$ ”: Let  $D = \{d_1, d_2\}$  and let  $P_n^D$  be connected. As we have already noted before, this implies that  $\gcd(D) = 1$ .

**Claim A** For every  $v \in [1, n - 1]$ , there is a path  $P : u_0 u_1 \dots u_l$  in  $P_n^D$  such that  $u_0 = 0$ ,  $u_l = v$ , and either

$$\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, d_2\}$$

or

$$\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, -d_2\}.$$

*Proof of Claim A:* Let  $P : u_0 u_1 \dots u_l$  be a path in  $P_n^D$  such that  $u_0 = 0$  and  $u_l = v$ . If  $Q : u_x u_{x+1} \dots u_{x+y}$  is a subpath of  $P$  with  $\{u_{x+1} - u_x, u_{x+y} - u_{x+y-1}\} = \{-d_2, d_2\}$  and  $u_{x+i} - u_{x+i-1} \in \{d_1, -d_1\}$  for  $i \in [2, y-1]$ , then  $(u_{x+y} - u_x) \bmod d_1 = 0$ , i.e.  $u_x$  and  $u_{x+y}$  differ by a multiple of  $d_1$ .

Therefore,  $Q$  can be replaced within  $P$  by a path  $Q' : v_0 v_1 \dots v_z$  in  $P_n^D$  with  $v_0 = u_x$ ,  $v_z = u_{x+y}$ , and either  $v_i - v_{i-1} = d_1$  for  $i \in [1, z]$  or  $v_i - v_{i-1} = -d_1$  for  $i \in [1, z]$ .

Note that replacing  $Q$  with  $Q'$  within  $P$  results in a walk  $P'$  which might not be a path. Since there is a path from  $u_0$  to  $u_l$  whose edge set is a subset of the edge set of  $P'$ , this easily implies the claim.  $\square$

Note that, by the definition of  $r_i$  and  $s_i$ ,  $r_i$  is the smallest integer at least 0 which has the same residue modulo  $d_1$  as  $id_2$  and  $(n-1-s_i)$  is the largest integer at most  $n-1$  which has the same residue modulo  $d_1$  as  $id_2$ .

Let  $i^* \in [1, d-1]$ .

First, let  $P : u_0 u_1 \dots u_l$  be a path in  $P_n^D$  such that  $u_0 = 0$ ,  $u_l \bmod d_1 = r_{i^*}$ , and  $\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, d_2\}$ . By the definition of the  $r_i$  and the choice of the path  $P$ , we obtain that for every  $i \in [1, i^*]$ , the path  $P$  contains an edge  $u_{j_{i-1}} u_{j_i}$  with  $u_{j_{i-1}} \bmod d_1 = r_{i-1}$ ,  $u_{j_i} \bmod d_1 = r_i$ , and  $u_{j_i} - u_{j_{i-1}} = d_2$ . This implies

$$n-1 \geq u_{j_{i-1}} + d_2 \geq r_{i-1} + d_2$$

for all  $i \in [1, i^*]$  and hence  $n-1 \geq d_{i^*}^+ + d_2$ .

Now, let  $P : u_0 u_1 \dots u_l$  be a path in  $P_n^D$  such that  $u_0 = 0$ ,  $u_l \bmod d_1 = r_{i^*}$ , and  $\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, -d_2\}$ . By the definition of the  $r_i$  and the choice of the



path  $P$ , we obtain that for every  $i \in [1, d_1 - i^*]$ , the path  $P$  contains an edge  $u_{j_{i-1}}u_{j_i}$  with  $u_{j_{i-1}} \bmod d_1 = r_{(d_1-i+1)} \bmod d_1$ ,  $u_{j_i} \bmod d_1 = r_{(d_1-i)} \bmod d_1$ , and  $u_{j_i} - u_{j_{i-1}} = -d_2$ . This implies

$$d_2 \leq u_{j_{i-1}} \leq n - 1 - s_{(d_1-i+1)} \bmod d_1$$

for all  $i \in [1, d_1 - i^*]$  and hence  $n - 1 \geq d_{i^*}^- + d_2$ .

Altogether, this easily implies that  $d^* + d_2 \leq n - 1$  which completes the proof of this implication.

“ $\Leftarrow$ ”: Let  $\gcd(D) = 1$  and  $d^* + d_2 \leq n - 1$ .

Since  $\gcd(D) = 1$ , we obtain that

$$\{r_i \mid i \in [0, d_1 - 1]\} = \{s_i \mid i \in [0, d_1 - 1]\} = [0, d_1 - 1],$$

i.e. the  $r_i$  and  $s_i$  represent all residues modulo  $d_1$ .

Therefore, it suffices to prove that for every  $i \in [0, d_1 - 1]$ , the graph  $P_n^D$  contains a path from 0 to some vertex  $v$  with  $v \bmod d_1 = r_i$ .

Let  $i^* \in [0, d_1 - 1]$ .

By the definition of  $d^*$  and by symmetry, we may assume that  $\max\{r_i \mid i \in [0, i^* - 1]\} \leq d^*$ . This implies that

$$\begin{aligned} & 0d_2(d_2 - d_1)(d_2 - 2d_1) \dots \\ & r_1(r_1 + d_2)(r_1 + d_2 - d_1)(r_1 + d_2 - 2d_1) \dots \\ & \dots \\ & r_{i^*-1}(r_{i^*-1} + d_2) \end{aligned}$$

is a path in  $P_n^D$  from 0 to some vertex  $v$  with  $v \bmod d_1 = r_{i^*}$ . This completes the proof.  $\square$

It is unclear whether Theorem 4 yields a polynomial time algorithm to check connectivity for  $P_n^D$  with  $|D| = 2$ . Furthermore, it would be interesting how the number-theoretic characterizations of the connectivity of circulant graphs given by Boesch and Tindell [2] and van Dorne [20] could be generalized to distance graphs.

While deciding connectivity is easy for circulant graphs, the exact calculation and minimization of the diameter of  $C_n^D$  are very difficult and well-studied problems even for the case  $|D| = 2$  [1, 11, 12, 25]. Many of the general upper and lower bounds on the diameter of circulant graphs easily generalize to distance graphs. The arguments used by Wong and Coppersmith [24] to obtain their classical estimates (cf. Theorems 4.6 and 4.7 in [11]) imply

$$\begin{aligned} \text{diam}(P_n^D) &\geq \frac{1}{2}(|D|!n)^{\frac{1}{|D|}} - |D| \text{ and} \\ \text{diam}(P_{d^k}^{\{1, d, \dots, d^{k-1}\}}) &\leq k(d-1). \end{aligned}$$

For our final hardness result, we consider the following decision problem which closely relates to the diameter of distance graphs.

SHORT PATH IN  $P_n^D$

**Instance:**  $n \in \mathbb{N}$ ,  $D \subseteq \mathbb{N}$  and  $l \in \mathbb{N}$ .

**Question:** Is there some  $u \in [0, n-2]$  such that  $P_n^D$  contains a path of length at most  $l$  between  $u$  and  $u+1$ ?

A natural certificate for a “yes”-instance of SHORT PATH IN  $P_n^D$  would be a path  $P$  of length at most  $l$  between two vertices  $u$  and  $u+1$  of  $P_n^D$ . The hardness of BOUNDED PARTIAL SUMS implies that an encoding of  $P$  which can be checked in polynomial time would most likely have to use at least  $\Omega(l)$  bits which would not be polynomially bounded in the encoding length of the triple  $(n, D, l)$ .

The construction in the following proof is inspired by van Emde Boas’s proof [21] that

WEAK PARTITION is NP-complete.

**Theorem 5** SHORT PATH IN  $P_n^D$  is NP-hard.

*Proof:* For an instance  $I$  of PARTITION, we will construct an instance  $I'$  of SHORT PATH IN  $P_n^D$  such that the encoding length of  $I'$  is polynomially bounded in the encoding length of  $I$ , and  $I$  is a “yes”-instance if and only if  $I'$  is a “yes”-instance.

Let  $x_1, x_2, \dots, x_k \in \mathbb{Z}$  be an instance of PARTITION.

Let

$$\begin{aligned} l &:= 2(k+1), \\ d &:= 2(k+1) \max(|x_1|, |x_2|, \dots, |x_k|) + 1, \text{ and} \\ n &:= 2d^{3k+4} + 1. \end{aligned}$$

For  $i \in [1, k]$  let

$$\begin{aligned} x_{i,1} &:= x_i d + d^{3i-1} + d^{3i} + 0 + 0, \\ x_{i,2} &:= 0 + 0 + d^{3i} + 0 + d^{3i+2}, \\ x_{i,3} &:= 0 + d^{3i-1} + 0 + d^{3i+1} + 0, \\ x_{i,4} &:= x_i d + 0 + 0 + d^{3i+1} + d^{3i+2}, \end{aligned}$$

and let

$$\begin{aligned} x_{k+1,1} &:= 1 + d^{3(k+1)-1} + d^{3(k+1)}, \\ x_{k+1,2} &:= 0 + 0 + d^{3(k+1)} + d^2, \end{aligned}$$

Writing the  $x_{i,j}$  as  $d$ -ary numbers, we obtain the following pattern.

	$d^0$	$d^1$	$d^2$	$d^3$	$d^4$	$d^5$	$d^6$	$d^7$	$d^8$	$\dots$	$d^{3k+2}$	$d^{3k+3}$
$x_{1,1}$		$x_1$	1	1								
$x_{1,2}$				1		1						
$x_{1,3}$			1		1							
$x_{1,4}$		$x_1$			1	1						
$x_{2,1}$		$x_2$				1	1					
$x_{2,2}$							1		1			
$x_{2,3}$					1			1				
$x_{2,4}$		$x_2$						1	1			
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$x_{k+1,1}$	1										1	1
$x_{k+1,2}$			1									1

Let

$$D := \{x_{i,j} \mid i \in [1, k], j \in [1, 4]\} \cup \{x_{k+1,1}, x_{k+1,2}\}.$$

The encoding length of the PARTITION instance is

$$\Omega \left( k + \sum_{i=1}^k \log(x_i) \right)$$

and the encoding length of  $(D, n, l)$  is

$$O \left( k^2 \log(k) \log(\max(|x_1|, |x_2|, \dots, |x_k|)) \right),$$

i.e. the latter is polynomially bounded in terms of the former.

**Claim A**  $(n, D, l)$  is a “yes”-instance of SHORT PATH IN  $P_n^D$  if and only if there are integers  $l_{i,j}$  such that

$$1 = l_{k+1,1}x_{k+1,1} + l_{k+1,2}x_{k+1,2} + \sum_{i=1}^k \sum_{j=1}^4 l_{i,j}x_{i,j} \text{ and} \quad (4)$$

$$l \geq |l_{k+1,1}| + |l_{k+1,2}| + \sum_{i=1}^k \sum_{j=1}^4 |l_{i,j}|. \quad (5)$$

*Proof of Claim A:* “ $\Rightarrow$ ” Let  $P$  be a path in  $P_n^D$  of length at most  $l$  between vertices  $u$  and  $u+1$ . If, for  $x_{i,j} \in D$ , going from  $u$  to  $u+1$  the path  $P$  uses  $l_{i,j}^+$  edges  $vw$  with  $w-v = x_{i,j}$  and  $l_{i,j}^-$  edges  $vw$  with  $v-w = x_{i,j}$ , then the integers  $l_{i,j} := l_{i,j}^+ - l_{i,j}^-$  satisfy (4) and (5).

“ $\Leftarrow$ ” Let the integers  $l_{i,j}$  be such that (4) and (5) hold. Consider the Algorithm 1 below.

```

r = 0;
u_r := 0;
while u_r ≠ 1 do
  if u_r ≤  $\frac{n-1}{2}$  and  $l_{i,j} > 0$  for some pair  $(i, j)$  then
    Choose a pair  $(i, j)$  such that  $l_{i,j} > 0$ ;
    u_{r+1} := u_r + x_{i,j};
    l_{i,j} := l_{i,j} - 1;
  else
    Choose a pair  $(i, j)$  such that  $l_{i,j} < 0$ ;
    u_{r+1} := u_r - x_{i,j};
    l_{i,j} := l_{i,j} + 1;
  end
end
end

```

### Algorithm 1

By the definition of  $n$  and  $D$ , we have  $\max(D) < \frac{n-1}{2}$ . Using this fact, it is straightforward to check that the sequence  $u_0, u_1, u_2, \dots$  produced by Algorithm 1 is a path of length at most  $l$  between the vertices 0 and 1, i.e.  $(n, D, l)$  is a “yes”-instance of SHORT PATH IN  $P_n^D$ .  $\square$

**Claim B**  $(n, D, l)$  is a “yes”-instance of SHORT PATH IN  $P_n^D$  if and only if  $x_1, x_2, \dots, x_k$

is a “yes”-instance of PARTITION.

*Proof:* “ $\Rightarrow$ ” If  $(n, D, l)$  is a “yes”-instance of SHORT PATH IN  $P_n^D$ , then Claim A implies the existence of integers  $l_{i,j}$  satisfying (4) and (5). For  $i \in [1, k]$ , let

$$\begin{aligned} l_i &= |l_{i,1}| + |l_{i,2}| + |l_{i,3}| + |l_{i,4}| \text{ and} \\ l_{k+1} &= |l_{k+1,1}| + |l_{k+1,2}|. \end{aligned}$$

By the definition of  $d$ , forming the sum in (4) and representing the involved numbers as  $d$ -ary numbers, there is never any carry contribution from one digit to the next. (In view of [21] we could say that there are no inheritance problems in this case.)

Considering the different digits of the sum in (4), this implies that

$$\begin{aligned} l_{k+1,1} &= 1, \\ l_{k+1,2} &= -1, \text{ and} \\ l_{1,1} + l_{1,3} &= 1. \end{aligned}$$

Furthermore, for  $i \in [1, k]$ ,

$$\begin{aligned} l_{i,2} &= -l_{i,1}, \\ l_{i,4} &= -l_{i,3}, \text{ and} \\ l_{i+1,1} + l_{i+1,3} &= -(l_{i,2} + l_{i,4}) \\ &= l_{i,1} + l_{i,3}. \end{aligned}$$

Therefore, for  $i \in [1, k]$ , if  $l_i > 0$ , then  $l_i \geq 2$ , and, if  $l_i = 2$ , then  $(l_{i,1}, l_{i,2}, l_{i,3}, l_{i,4})$  belongs

to

$$\{(1, -1, 0, 0), (-1, 1, 0, 0), (0, 0, 1, -1), (0, 0, -1, 1)\}. \quad (6)$$

Note that for each of these four possibilities, the contribution of  $\sum_{j=1}^4 l_{i,j} x_{i,j}$  to the second digit of (4) written as a  $d$ -ary number is either  $x_i$  or  $-x_i$ .

By a simple inductive argument, we obtain  $l_{i,1} + l_{i,3} = 1$  and  $l_i > 0$  for all  $i \in [1, k]$ . Since  $l_1 + l_2 + \dots + l_{k+1} \leq l = 2(k+1)$ , this implies  $l_i = 2$  for all  $i \in [1, k]$ . Therefore, since the second digit of the sum in (4) written as a  $d$ -ary number is 0, the  $l_{i,j}$  yield a solution for the PARTITION instance.

“ $\Leftarrow$ ” If the PARTITION instance is a “yes”-instance, then setting

$$(l_{k+1,1}, l_{k+1,2}) = (1, -1)$$

and suitably selecting  $(l_{i,1}, l_{i,2}, l_{i,3}, l_{i,4})$  from the four possibilities given in (6) according to some solution of the PARTITION instance, yields integers  $l_{i,j}$  which satisfy (4) and (5). In view of Claim A, this completes the proof of the claim.  $\square$

This completes the proof.  $\square$

Note that for the instances of SHORT PATH IN  $P_n^D$  constructed in the proof of Theorem 5, the length bound  $l$  is polynomially bounded in terms of the encoding length of  $D$ . Since a path in  $P_n^D$  of length at most  $l$  can be encoded using  $O(l \log(|D|))$  many bits, the restriction of SHORT PATH IN  $P_n^D$  to such instances actually yields an NP-complete problem.

As a final remark, we note that the existence of a monotonic path between two vertices of  $P_n^D$  is equivalent to the feasibility of an integer linear program in  $|D|$  dimensions which can be decided in polynomial time for bounded  $|D|$  [15].

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