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Some Remarks on the Geodetic Number of a Graph

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Abstract. A set of vertices D of a graph G is geodetic if every vertex of G lies on a shortest path between two not necessarily distinct vertices in D. The geodetic number of G is the minimum cardinality of a geodetic set of G.

We prove that it is NP complete to decide for a given chordal or chordal bipartite graph G and a given integer k whether G has a geodetic set of cardinality at most k. Furthermore, we prove an upper bound on the geodetic number of graphs without short cycles and study the geodetic number of cographs, split graphs, and unit interval graphs.

Keywords. Cograph; convex hull; convex set; geodetic number; split graph; unit interval graph

1 Introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of a vertex u in G is denoted by $N_G(u)$. A set of pairwise non-adjacent vertices is called independent and a set of pairwise adjacent vertices is called a clique. A vertex is simplicial if its neighbourhood is a clique. The distance $d_G(u, v)$ between two vertices u and v in G is the length of a shortest path between u and v or ∞, if no such path exists. The diameter of G is the maximum distance between two vertices in G.

The *interval* $I[u, v]$ between two vertices u and v in G is the set of vertices of G which belong to a shortest path between u and v. Note that a vertex w belongs to $I[u, v]$ if and only if $d_G(u, v) = d_G(u, w) + d_G(w, v)$. For a set S of vertices, let the *interval* $I[S]$ of S be the union of the intervals $I[u, v]$ over all pairs of vertices u and v in S. A set of vertices S is called *geodetic* if $I[S]$ contains all vertices of G. Harary et al. [12] define the *geodetic number* $g(G)$ of a graph G as the minimum cardinality of a geodetic set. The calculation of the geodetic number is an NP-hard problem for general graphs [3] and [2, 7–10, 13] contain numerous results and references concerning geodetic sets and the geodetic number.

Our results are as follows. In Section 2 we simplify and refine the existing complexity result [3] by proving that the decision problem corresponding to the geodetic number remains NP-complete even when restricted to chordal or chordal bipartite graphs. In Section 3 we prove upper bounds on the geodetic number of graphs without short cycles and in particular for triangle-free graphs. Finally, in Section 4 we consider the geodetic number of cographs, split graphs and unit interval graphs.

2 Complexity results for chordal graphs

In this section we prove hardness results for the following decision problem.

GEODETIC SET
Instance: A graph G and an integer k.
Question: Does G have a geodetic set of cardinality at most k?

Our proofs will relate GEODETIC SET to the following well-known problem. Recall that a set of vertices D of a graph G is *dominating* if every vertex in $V(G) \setminus D$ has a neighbour in D.

DOMINATING SET
Instance: A graph G and an integer k.
Question: Does G have a dominating set of cardinality at most k?

A graph is *chordal* if it does not contain an induced cycle of length at least 4. Similarly, a bipartite graph is *chordal bipartite* if it does not contain an induced cycle of length at least 6. The problem DOMINATING SET is NP-complete for chordal graphs [5] and chordal bipartite graphs [16].

Theorem 1 GEODETIC SET *restricted to chordal graphs is NP-complete.*

Proof: Since the interval of a given set of vertices can be determined in polynomial time by shortest path methods, GEODETIC SET is in NP.

In order to prove NP-completeness, we describe a polynomial reduction of DOMINATING SET restricted to chordal graphs [5] to GEODETIC SET restricted to chordal graphs. Let (G, k) be an instance of DOMINATING SET such that G is chordal. Let the graph G' arise from G as follows: For every vertex $u \in V(G)$, add two new vertices x_u and y_u and add the new edges ux_u and x_uy_u. Furthermore add a new vertex z and new edges uz and x_uz for every $u \in V(G)$. Let $k' = k + |V(G)|$. Note that G' is chordal.

If G has a dominating set D with $|D| \leq k$, then let $D' = D \cup \{y_u \mid u \in V(G)\}$. Clearly, $\{x_u \mid u \in V(G)\} \cup \{z\} \subseteq I[\{y_u \mid u \in V(G)\}] \subseteq I[D']$. Furthermore, if $u \in V(G) \setminus D$, then there is a vertex $v \in D$ with $uv \in E(G)$. Since $d_{G'}(v, y_u) = 3$ and vux_uy_u is a path of length 3 in G', we have $u \in I[v, y_u] \subseteq I[D']$. Hence D' is a geodetic set of G' with $|D'| \leq k + |V(G)| = k'$.

Conversely, if G' has a geodetic set D' with $|D'| \leq k'$, then let $D = D' \cap V(G)$. Clearly, D is not empty. Since $\{y_u \mid u \in V(G)\} \subseteq D'$, we have $|D| \leq k' - |V(G)| = k$. If $u \in V(G) \setminus D$, then either there are two vertices $v, w \in D$ with $u \in I[v, w]$ or there are two vertices $v \in D$ and $w \in D' \setminus V(G)$ with $u \in I[v, w]$. In both cases, the distances within G' imply that v must be a neighbour of u. Hence D is a dominating set of G with $|D| \leq k$. $\Box$

Theorem 2 Geodetic Set *restricted to chordal bipartite graphs is NP-complete.*

Proof: In order to prove NP-completeness, we describe a polynomial reduction of Dominating Set restricted to chordal bipartite graphs [16] to Geodetic Set restricted to chordal bipartite graphs. Let (G, k) be an instance of Dominating Set such that G is chordal bipartite.

Let the graph G' arise from G as follows: Let A and B denote the partite sets of G. Add four new vertices a_1, a_2, b_1, b_2 and add new edges a_1b for all $b \in B \cup \{b_1, b_2\}$ and b_1a for all $a \in A \cup \{a_1, a_2\}$. Let $k' = k + 2$. Note that G' is chordal bipartite.

If G has a dominating set D with $|D| \leq k$, then let $D' = D \cup \{a_2, b_2\}$. Clearly, $a_1, b_1 \in I[a_2, b_2]$. Furthermore, if $a \in A \setminus D$, then there is a vertex $b \in D \cap B$ with $ab \in E(G)$. Since $d_{G'}(a_2, b) = 3$ and a_2b_1ab is a path of length 3 in G', we have $a \in I[a_2, b] \subseteq I[D']$. Hence, by symmetry, D' is a geodetic set of G' with $|D'| \leq k + 2 = k'$.

Conversely, if G' has a geodetic set D' with $|D'| \leq k'$, then let $D = D' \cap V(G)$. Clearly, $a_2, b_2 \in D'$ and D is not empty. If $a \in A \setminus D$, then either there are two vertices $b \in D \cap B$ and $v \in D$ with $a \in I[b, v]$ or there is a vertex vertex $b \in D \cap B$ with $a \in I[b, a_2]$. In both cases, the distances within G' imply that a must be a neighbour of b. Hence, by symmetry, D is a dominating set of G with $|D| \leq k$. $\Box$

3 Bounds for triangle-free graphs

In this section we prove upper bounds on the geodetic number for graphs without short cycles. The *girth* of a graph G is the length of a shortest cycle in G or ∞, if G has no cycles. Our first result is a probabilistic bound for graphs of large girth.

Theorem 3 *If G is a graph of order n, girth at least $4h$, and minimum degree at least δ, then*

(i)

$$g(G) \leq n \left(p + \delta(1-p)^{(\delta-1)\frac{(\delta-1)^h - 1}{\delta - 2} + 1} - (\delta - 1)(1-p)^{\delta \frac{(\delta-1)^h - 1}{\delta - 2} + 1} \right)$$

for every $p \in (0, 1)$ and

(ii)

$$g(G) \leq n \frac{\ln\left(\delta\left((\delta-1)\frac{(\delta-1)^h-1}{\delta-2}+1\right)\right)+1}{(\delta-1)\frac{(\delta-1)^h-1}{\delta-2}+1}.$$

Proof: For some $p \in (0,1)$, select every vertex of G independently at random with probability p and denote the set of selected vertices by A. If $B = V(G)\setminus I[A]$, then $I[A \cup B] = V(G)$ and hence $g(G) \leq |A \cup B|$.

Claim A $\mathbf{P}[u \in B] \leq \delta(1-p)^{(\delta-1)n_{\delta,h}+1} - (\delta-1)(1-p)^{\delta n_{\delta,h}+1}$ *for* $u \in V(G)$.

Proof of Claim A: Let $u \in V(G)$. Let d denote the degree of u and let $v_1, v_2, \ldots, v_d$ denote the neighbours of u. For $1 \leq i \leq d$, let V_i denote the set of vertices w with $d_G(u,w) = d_G(v_i,w)+1 \leq h$.

Since G has girth at least $4h$, there are no two distinct paths of length at most h between two vertices. Since the distance between u and a vertex w in V_i for $1 \leq i \leq d$ is at most h and $d_G(u,w) = d_G(v_i,w)+1$, the unique shortest path between u and w passes through v_i.

Let $n_i = |V_i|$ for $1 \leq i \leq d$. Let $\tilde{n} = n_1 + n_2 + \cdots + n_d$. Note that for $1 \leq i \leq d$,

$$n_i \geq n_{\delta,h} := \sum_{j=0}^{h-1}(\delta-1)^j = \frac{(\delta-1)^h-1}{\delta-2}.$$

If w_1 and w_2 belong to different sets among $V_1, V_2, \ldots, V_d$ and u does not belong to some shortest path between w_1 and w_2, then there is a path between w_1 and w_2 of length at most $2h$ which passes through u and a necessarily distinct shortest path between w_1 and w_2 of length strictly less than $2h$. The union of these two paths contains a cycle of length strictly less than $4h$, which is a contradiction. Hence the vertex u belongs to some shortest path between every two vertices from different sets among $V_1, V_2, \ldots, V_d$. Therefore, if the vertex u belongs to B, then $u \notin A$ and $A \cap V_i$ is non-empty for at most one index $1 \leq i \leq d$. We obtain $\mathbf{P}[u \in B] \leq (1-p)f_d(n_1, n_2, \ldots, n_d)$ for

$$\begin{aligned} f_d(n_1,n_2,\ldots,n_d) &= (1-p)^{\tilde{n}} + \sum_{i=1}^{d}\left(1-(1-p)^{n_i}\right)(1-p)^{\tilde{n}-n_i} \\ &= -(d-1)(1-p)^{\tilde{n}} + \sum_{i=1}^{d}(1-p)^{\tilde{n}-n_i} \end{aligned}$$

Since for $1 \leq i \leq d$,

$$\begin{aligned} \frac{\partial}{\partial n_i} f_d(n_1,n_2,\ldots,n_d) &= \ln(1-p)\left(-(d-1)(1-p)^{\tilde{n}} + \sum_{j\in\{1,2,\ldots,d\}\setminus\{i\}}(1-p)^{\tilde{n}-n_j}\right) \\ &= \sum_{j\in\{1,2,\ldots,d\}\setminus\{i\}} \ln(1-p)\left((1-p)^{\tilde{n}-n_j}-(1-p)^{\tilde{n}}\right) \\ &< 0, \end{aligned}$$

we obtain

$$\begin{aligned} f_d(n_1, n_2, \ldots, n_d) &\leq f_d(n_{\delta,h}, n_{\delta,h}, \ldots, n_{\delta,h}) \\ &= d(1-p)^{(d-1)n_{\delta,h}} - (d-1)(1-p)^{dn_{\delta,h}}. \end{aligned}$$

Since for $d \geq 1$ and $c \in (0,1)$

$$\begin{aligned} \frac{\partial}{\partial d}\left(dc^{(d-1)} - (d-1)c^d\right) &= c^{(d-1)} + \ln(c)dc^{(d-1)} - c^d - \ln(c)(d-1)c^d \\ &= c^{(d-1)}\left(1 - c + c\ln(c) + d\ln(c)(1-c)\right) \\ &\leq c^{(d-1)}\left(1 - c + c\ln(c) + \ln(c)(1-c)\right) \\ &= c^{(d-1)}\left(1 - c + \ln(c)\right) \\ &< 0, \end{aligned}$$

we obtain

$$\begin{aligned} \mathbf{P}[u \in B] &\leq (1-p)f_d(n_1, n_2, \ldots, n_d) \\ &\leq (1-p)f_d(n_{\delta,h}, n_{\delta,h}, \ldots, n_{\delta,h}) \\ &\leq (1-p)f_\delta(n_{\delta,h}, n_{\delta,h}, \ldots, n_{\delta,h}) \\ &= \delta(1-p)^{(\delta-1)n_{\delta,h}+1} - (\delta-1)(1-p)^{\delta n_{\delta,h}+1} \end{aligned}$$

which completes the proof of the claim. $\Box$

By Claim A, we obtain

$$\begin{aligned} \mathbf{E}[|A \cup B|] &\leq \sum_{u \in V(G)} \left(\mathbf{P}[u \in A] + \mathbf{P}[u \in B]\right) \\ &\leq n\left(p + \delta(1-p)^{(\delta-1)n_{\delta,h}+1} - (\delta-1)(1-p)^{\delta n_{\delta,h}+1}\right) \end{aligned}$$

which proves (i) by the first moment principle [1].

For $p = \frac{\ln\left(\delta\left((\delta-1)n_{\delta,h}+1\right)\right)}{(\delta-1)n_{\delta,h}+1}$, we obtain

$$\begin{aligned} \mathbf{E}[|A \cup B|] &\leq np + n\delta(1-p)^{(\delta-1)n_{\delta,h}+1} - n(\delta-1)(1-p)^{\delta n_{\delta,h}+1} \\ &\leq np + n\delta(1-p)^{(\delta-1)n_{\delta,h}+1} \\ &\leq np + n\delta e^{-p((\delta-1)n_{\delta,h}+1)} \\ &= n\frac{\ln\left(\delta\left((\delta-1)n_{\delta,h}+1\right)\right)+1}{(\delta-1)n_{\delta,h}+1} \end{aligned}$$

which proves (ii). $\Box$

For triangle-free graphs Theorem 3 immediately implies the following.

Corollary 4 *If G is a triangle-free graph of order n and minimum degree at least δ, then*

$$g(G) \leq n \left(\frac{2 \ln \delta}{\delta} + \left(1 + 2 \left(1 - \frac{1}{\delta} \right) \ln \delta \right) \frac{1}{\delta^2} \right).$$

Proof: This follows easily from Theorem 3 (i) for $h = 1$ and $p = \frac{2 \ln(\delta)}{\delta}$. $\Box$

We close this section with another simple bound for triangle-free graphs.

Proposition 5 *If G is a triangle-free graph of minimum degree at least 2 and M is a maximal matching in G, then $g(G) \leq 2|M|$.*

Proof: Let D denote the set of vertices of G which are incident with an edge in M. Since M is maximal, $V(G) \setminus D$ is an independent set. Hence every vertex $v \in V(G) \setminus D$ has two neighbours u and w in D. Since G is triangle-free, $v \in I[u, w] \subseteq I[D]$. $\Box$

4 Special graph classes

In this section we consider the geodetic number of *cographs*, *split graphs*, and *unit interval graphs*. We refer the reader to [6] for detailed definitions. Since the geodetic number of a disconnected graph equals the sum of the geodetic numbers of its components, we may restrict our attention to connected graphs.

Our first two results give exact values for the geodetic number of cographs and split graphs.

Theorem 6 *If G is connected cograph of order n and $G_1, \ldots, G_k, G_{k+1}, \ldots, G_t$ are the subgraphs of G induced by the vertex sets of the connected components of the complement of G where $|V(G_i)| \geq 2$ if and only if $1 \leq i \leq k$, then*

$$g(G) = \begin{cases} n & , \text{ if } k = 0, \\ g(G_1) & , \text{ if } k = 1, \text{ and} \\ \min \left\{ 4, \min\limits_{1 \leq i \leq k} g(G_i) \right\} & , \text{ if } k \geq 2. \end{cases}$$

Proof: First, let $k = 0$. Since G is complete, $g(G) = n$.

Next, let $k = 1$. If D is a geodetic set of G, then $D \cap V(G_1)$ is a geodetic set of G_1. Conversely, since G_1 is non-complete, every geodetic set of G_1 is also a geodetic set of G. Hence $g(G) = g(G_1)$.

Finally, let $k \geq 2$. Since two non-adjacent vertices from G_1 together with two non-adjacent vertices from G_2 form a geodetic set of G, we have $g(G) \leq 4$. Furthermore, since G_i is non-complete for $1 \leq i \leq k$, every geodetic set of G_i contains two non-adjacent vertices and hence it is also a geodetic set of G. Thus $g(G) \leq \min\limits_{1 \leq i \leq k} g(G_i)$.

If $g(G) = 2$, then a geodetic set of G with two elements consists of two non-adjacent vertices which must both belong to G_j for some $1 \leq j \leq k$. Hence $g(G_j) = \min_{1 \leq i \leq k} g(G_i) = 2$.

If $g(G) = 3$ and D is a geodetic set of G with three elements which do not all belong to G_j for some $1 \leq j \leq k$, then there are two distinct indices $1 \leq j_1, j_2 \leq k$ such that D contains exactly two non-adjacent vertices u and v from G_{j_1} and D contains exactly one vertex w from G_{j_2}. Since $w \in I[u, v]$, $I[u, w] = \{u, w\}$, and $I[v, w] = \{v, w\}$, we obtain that $\{u, v\}$ is a geodetic set of G contradicting $g(G) = 3$. Hence, if $g(G) = 3$, every geodetic set of G with three elements belongs to G_j for some $1 \leq j \leq k$ and $g(G_j) = \min_{1 \leq i \leq k} g(G_i) = 3$.

This completes the proof. $\Box$

Using Theorem 6 and modular decompositions [14, 15], the geodetic number of a cograph can be computed in linear time.

Theorem 7 *Let G be a connected split graph. Let $V_1 \cup V_2$ be a partition of $V(G)$ such that V_1 is a maximal independent set and V_2 is a clique. Let S denote the set of simplicial vertices of G. Let U denote the set of vertices $u \in V_2 \setminus S$ which have exactly one neighbour in V_1, say u', $V_2 \cap S \subseteq N_G(u')$ and $d_G(u', w) = 2$ for all $w \in V_1 \setminus \{u'\}$.*

(i) If $U = \emptyset$, then $g(G) = |S|$.

(ii) If $U \neq \emptyset$ and there is a vertex $v \in V_2 \setminus S$ such that

$$(N_G(v) \cap V_1) \cap \left(\bigcup_{u \in U \setminus \{v\}} (N_G(u) \cap V_1) \right) = \emptyset,$$

then $g(G) = |S| + 1$.

(iii) If $U \neq \emptyset$ and there is no vertex $v \in V_2 \setminus S$ as specified in (ii), then $g(G) = |S| + 2$.

Proof: Let $\tilde{U} = V(G) \setminus I[S]$. It is easy to see that $U \subseteq \tilde{U}$. Let $u \in \tilde{U}$. Since $V_1 \subseteq S$, we have $u \in V_2 \setminus S$ and u has at most one neighbour in V_1. Since V_1 is maximal independent, u has exactly one neighbour u' in V_1. If u' is non-adjacent to some vertex $v \in V_2 \cap S$, then $u \in I[u', v]$, which is a contradiction. Hence $V_2 \cap S \subseteq N_G(u')$. If there is a vertex $w \in V_1$ such that $d_G(u', w) \geq 3$, then w has a neighbour x in V_2 and $u'uxw$ is a shortest path between u' and w, which implies the contradiction $u \in I[u', w]$. Hence $\tilde{U} = U$. Since the vertices in S belong to every geodetic set of G, this implies (i).

If $g(G) = |S| + 1$, then $U \neq \emptyset$ and G has a geodetic set D with $D = S \cup \{v\}$ for some $v \in V_2 \setminus S$. Let $u \in U \setminus \{v\}$ and let u' denote the neighbour of u in V_1. Since D is a geodetic set, $u \in I[u', v]$ which implies that v is non-adjacent to u', i.e. v satisfies the condition specified in (ii). Conversely, if the hypothesis of (ii) is satisfied, then $S \cup \{v\}$ is a geodetic set of G which implies $g(G) = |S| + 1$. This implies (ii).

Furthermore, if the hypothesis of (iii) is satisfied, then $g(G) \geq |S| + 2$. Since the vertices in U are non-simplicial, every vertex u in U is adjacent to a vertex in V_2 which is non-adjacent to the unique neighbour of u in V_1. By the hypothesis of (iii), this implies that there are two vertices u_1 and u_2 in U such that their unique neighbours in V_1 are distinct. If $u \in U \setminus \{u_1, u_2\}$ and u' is the unique neighbour of u in V_1, then u' is non-adjacent either to u_1 or to u_2 and $u \in I[u', u_1] \cup I[u', u_2]$. Hence $g(G) \leq |S| + 2$ which implies (iii). $\Box$

Using Theorem 7, the geodetic number of a split graph can be computed in linear time.

Our final result is a best-possible upper bound on the geodetic number of unit interval graphs.

Theorem 8 *If G is a connected unit interval graph with s simplicial vertices and diameter d, then $g(G) \leq s + 2(d-1)$.*

Proof: Let $v_1, v_2, \ldots, v_n$ be a *canonical ordering* of the vertices of G [4,11,17], i.e. for every edge v_iv_j with $1 \leq i < j \leq n$ the vertices $v_i, v_{i+1}, \ldots, v_j$ form a clique. Let S denote the set of simplicial vertices. For $0 \leq i \leq d$, let

$$I_i = \{v \in V(G) \mid d_G(v_1, v) = i \text{ and } d_G(v, v_n) = d - i\}.$$

Clearly, $I := I_0 \cup I_1 \cup \ldots \cup I_d = I[v_1, v_n] \subseteq I[S]$. For $0 \leq i \leq d$, let

$$R_i = \{v \in V(G) \setminus (I \cup S) \mid d_G(v_1, v) = i\}.$$

Clearly, for $1 \leq i \leq d$ and $u \in R_i$, we have $I_{i-1} \cup R_i \cup I_i \subseteq N_G(u) \cup \{u\}$. If $I_{i-1} \cup R_i \cup I_i = N_G(u) \cup \{u\}$, then we call u *quasi-simplicial*.

For $0 \leq k \leq d$, let i_k and i'_k denote the first and last vertex in I_k according to the canonical ordering of the vertices. For $1 \leq k \leq d$ with $R_k \neq \emptyset$, let r_k and r'_k denote the first and last vertex in R_k according to the canonical ordering of the vertices.

Starting with the empty set, we construct a geodetic set D of G as follows: Add all vertices in S to D. If R_1 is non-empty, then, since no vertex in R_1 is simplicial, R_2 is non-empty and $R_1 \subseteq I[v_1, r_2]$. In this case add r_2 to D. Similarly, if R_d is non-empty, then, since no vertex in R_d is simplicial, R_{d-1} is non-empty and $R_d \subseteq I[v_n, r'_{d-1}]$. In this case add r'_{d-1} to D. If $2 \leq k \leq d-1$ and R_k contains quasi-simplicial vertices, then i_{k-1} and i'_k are non-adjacent and $R_k \subseteq I[i_{k-1}, i'_k]$. In this case add i_{k-1} and i'_k to D. If $2 \leq k \leq d-1$ and R_k is non-empty but contains no quasi-simplicial vertices, then every vertex in R_k is adjacent either to r'_{k-1} or to r_k, i.e. in particular at least one of these two vertices is well-defined. If r_k exists, then all vertices of R_k which are adjacent to r_k belong to $I[v_1, r_k]$. In this case add r_k to D. Similarly, if r'_{k-1} exists, then all vertices of R_k which are adjacent to r'_{k-1} belong to $I[v_n, r'_{k-1}]$. In this case add r'_{k-1} to D.

From the definition of D and the previous observations, it is clear that D is a geodetic set of G. Furthermore, D contains at most $g + 2(d-1)$ vertices which completes the proof. $\Box$

The graphs illustrated in Figure 1 show that Theorem 8 is best-possible. In this figure only the edges v_iv_j with $j = \max\{k \mid v_iv_k \in E(G)\}$ and $i = \min\{k \mid v_kv_j \in E(G)\}$ are shown. Note that $R_1 = \{r_1^1, r_1^2\}$ and $R_2 = \{r_2^1, r_2^2, r_2^3, r_2^4\}$.

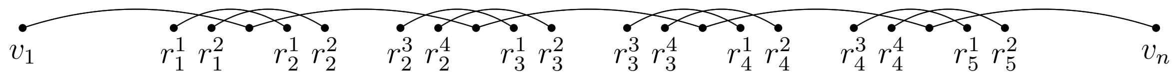

Figure 1 Extremal Graph for Theorem 8

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