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Packing Disjoint Cycles over Vertex Cuts

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Abstract

For a graph G, let $\nu(G)$ and $\nu'(G)$ denote the maximum cardinalities of packings of vertex-disjoint and edge-disjoint cycles of G, respectively. We study the interplay of these two parameters and vertex cuts in graphs. If G is a graph whose vertex set can be partitioned into three non-empty sets S, V_1, and V_2 such that there is no edge between V_1 and V_2, and $k = |S|$, then our results imply that $\nu(G)$ is uniquely determined by the values $\nu(H)$ for at most $2^{k+1}k!^2$ graphs H of order at most $\max\{|V_1|, |V_2|\} + k$, and $\nu'(G)$ is uniquely determined by the values $\nu'(H)$ for at most $2^{\binom{k}{2}+1}$ graphs H of order at most $\max\{|V_1|, |V_2|\} + k$.

Keywords: Connectivity; cycle; edge-disjoint cycles; graph; packings of cycles; vertex cut; vertex-disjoint cycles

1 Introduction

Packing vertex- or edge-disjoint cycles in graphs is a very well-studied and classical graph-theoretical problem. There is a vast amount of literature concerning conditions in terms of for instance order, size, vertex degrees, degree sums, independence number, chromatic number, and feedback vertex sets which are sufficient for the existence of some number of disjoint cycles which may additionally contain specified elements or satisfy certain length conditions. We refer the reader to [3,6–12,14,16,19,20,23–28] which is just a small selection. The algorithmic problems concerning cycle packings are typically hard [1, 2, 13, 15, 17, 18, 20, 22] and approximation algorithms were described [18, 22]. Several authors mention practical applications in computational biology such as reconstruction of evolutionary trees or genomic analysis.

The starting point for the research presented here is the simple observation that for graphs which contain cutvertices, the problems to find optimal packings of vertex- or edge-disjoint cycles essentially reduce to the blocks of the graph. Here we extend this observation and study the behaviour of these packing problems with respect to larger vertex cuts.

We consider finite and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$ which may contain parallel edges but no loops. The *neighbourhood* of a vertex u in a graph G is denoted by $N_G(u)$ and the *degree* of u in G is the number of incident edges. For a vertex x and a set of vertices Y in G, let $E_G(x,Y)$ denote the set of all edges of G between x and a vertex in Y. A *vertex cut* in G is a set of vertices S whose removal disconnects G. A *cycle* in G is a connected subgraph of G in which all vertices have degree 2. A *packing of vertex-disjoint/edge-disjoint cycles* in G is a set of cycles in G which are pairwise vertex-disjoint/edge-disjoint. Let $\nu(G)/\nu'(G)$ denote the maximum cardinality of a packing of vertex-disjoint/edge-disjoint cycles in G. While parallel edges occur naturally in our constructions and proofs, it is a reasonable restriction to consider graphs without loops, i.e. which do not contain cycles of length 1, because every loop is contained in every maximum packing of edge-disjoint cycles and in some maximum packing of vertex-disjoint cycles.

In Sections 2 and 3 we consider packings of vertex-disjoint/edge-disjoint cycles in graphs G which contain a vertex cut S. In both cases we prove that $\nu(G)$ and $\nu'(G)$ are uniquely determined by the values $\nu(H)$ and $\nu'(H)$ for graphs H which arise from G by some simple modifications and contain a vertex cut of cardinality strictly less than $|S|$. In Section 4 we discuss some algorithmical consequences of these results. In this context, we observe that for fixed l the graph properties "$\nu(G) \geq l$" and "$\nu'(G) \geq l$" can be decided in linear time for graphs of bounded clique-width.

2 Vertex-disjoint cycles

Throughout this section let G be a graph and let $S \cup V_1 \cup V_2$ be a partition of the vertex set of G into three non-empty sets such that there is no edge between V_1 and V_2, i.e. if G is connected, then S is a vertex cut of G.

For $x \in S$ and $i \in \{1,2\}$, let $G_{x,i}$ arise from G by deleting all edges in $E_G(x, V_{3-i})$.

For $x, y \in S$, let $G_{x,y}$ arise from G by

- deleting x and y,

- adding two new vertices z_1 and z_2,

- adding a new edge between z_1 and u for every edge between x or y and a vertex u in V_1, and

- adding a new edge between z_2 and v for every edge between x or y and a vertex v in V_2.

Note that there are parallel edges between z_i and the vertices in $N_G(x) \cup N_G(y) \cap V_i$ for $i \in \{1, 2\}$.

Clearly, for $x \in S$ and $i \in \{1, 2\}$, $G - x$ is a subgraph of $G_{x,i}$ and $G_{x,i}$ is a subgraph of G. Furthermore, deleting x from G, can only reduce $\nu(G)$ by 1. Therefore, for $x \in S$ and $i \in \{1, 2\}$,

$$\nu(G - x) \leq \nu(G_{x,i}) \leq \nu(G) \leq \nu(G - x) + 1. \tag{1}$$

We proceed to our main result in this section.

Theorem 1 *Let G, S, V_1, and V_2 be as above.*

(i) *If $\nu(G - x) < \nu(G_{x,i})$ for some $x \in S$ and $i \in \{1, 2\}$, then $\nu(G) = \nu(G - x) + 1$.*

(ii) *If $\nu(G - x) < \nu(G - y)$ for some x and y in S, then $\nu(G) = \nu(G - x) + 1$.*

(iii) *If $\nu(G - x) = \nu(G_{y,i})$ for all x and y in S and $i \in \{1, 2\}$, then $\nu(G) = \nu(G - x) + 1$ if and only if for every $x \in S$, there is some $y \in S$ such that x and y are non-adjacent and $\nu(G_{x,y}) \geq \nu(G - x) + 2$.*

Proof: (i) By (1),

$$\nu(G - x) + 1 \leq \nu(G_{x,i}) \leq \nu(G) \leq \nu(G - x) + 1$$

which implies $\nu(G) = \nu(G - x) + 1$.

(ii) By (1),
$$\nu(G - y) \leq \nu(G) \leq \nu(G - x) + 1 \leq \nu(G - y)$$
which implies $\nu(G) = \nu(G - x) + 1$.

(iii) Let $\nu(G - x) = \nu(G_{y,i})$ for all x and y in S and $i \in \{1, 2\}$. Let $\nu^- = \nu(G - x)$ for some $x \in S$. Note that ν^- is independent of the choice of x.

If $\nu(G) = \nu^- + 1$, then for every packing $\mathcal{C}$ of $\nu(G)$ vertex-disjoint cycles in G and for every $x \in S$, there is a unique cycle $C_x \in \mathcal{C}$ such that C_x contains an edge from $E_G(x, V_1)$ and an edge from $E_G(x, V_2)$. Let $x \in S$. Clearly, the cycle C_x contains at least two vertices from S. Let $y \in S$ be a vertex on C_x such that x and y are consecutive vertices from S on C_x. Note that $C_x = C_y$. If x and y are adjacent, then either $\nu(G_{x,1}) \geq \nu^- + 1$ or $\nu(G_{x,2}) \geq \nu^- + 1$ which is a contradiction. Hence x and y are non-adjacent. Since $G_{x,y}$ contains two vertex-disjoint cycles whose vertex set is contained in $\{z_1, z_2\} \cup V(C_x)$, we have $\nu(G_{x,y}) \geq \nu(G) + 1 = \nu^- + 2$.

Conversely, let $x, y \in S$ be such that x and y are non-adjacent and $\nu(G_{x,y}) \geq \nu^- + 2$. Let $\mathcal{C}_{x,y}$ be a packing of $\nu(G_{x,y})$ vertex-disjoint cycles in $G_{x,y}$. Since

$$\nu(G_{x,y} - \{z_1, z_2\}) \geq \nu(G_{x,y}) - 2 \geq \nu^-$$

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and
$$\nu(G_{x,y} - \{z_1, z_2\}) \le \nu(G - x) = \nu^-,$$
we have
$$\nu(G_{x,y} - \{z_1, z_2\}) = \nu(G_{x,y}) - 2 = \nu^-$$
and the vertices z_1 and z_2 are contained in distinct cycles C_{z_1} and C_{z_2} of $\mathcal{C}_{x,y}$, respectively. Clearly, the subgraph induced in G by the vertex set $(\{x, y\} \cup V(C_{z_1}) \cup V(C_{z_2})) \setminus \{z_1, z_2\}$ contains a subgraph H in which all vertices except for x and y are of degree 2 and $d_H(x) + d_H(y) = 4$. Since H contains at least one cycle, $\nu(G) \ge \nu(G_{x,y}) - 1 = \nu^- + 1$. By (1), $\nu(G) = \nu^- + 1$ which completes the proof. $\square$

3 Edge-disjoint cycles

Throughout this section let G be a graph and let $S \cup V_1 \cup V_2$ be a partition of the vertex set of G into three non-empty sets such that there is no edge between V_1 and V_2. Let $x \in S$ and let $E_1 \cup E_2$ be a partition of $E_G(x, S)$.

For a set $T \subseteq S \setminus \{x\}$, let $G(T)$ be the graph which arises from G by

- deleting x,

- adding two new vertices x_1 and x_2,

- adding a new edge between x_1 and y for all edges in $E_1 \cup E_G(x, V_1)$ between x and a vertex y,

- adding a new edge between x_2 and y for all edges in $E_2 \cup E_G(x, V_2)$ between x and a vertex y,

- and adding two new edges $e_{1,y} = x_1 y$ and $e_{2,y} = x_2 y$ for all $y \in T$.

Note that $d_{G(T)}(x_1) + d_{G(T)}(x_2) = d_G(x) + 2|T|$.

We proceed to our main result in this section.

Theorem 2 *If G, S, V_1, V_2, x, E_1, and E_2 are as above, then*
$$\nu'(G) = \max\left\{\nu'(G(T)) - |T| \mid T \subseteq S \setminus \{x\}\right\}.$$

Theorem 2 follows immediately from the next two lemmas.

Lemma 3 *If G, S, V_1, V_2, x, E_1, and E_2 are as above, then $\nu'(G) \ge \nu'(G(T)) - |T|$ for all sets $T \subseteq S \setminus \{x\}$.*

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Proof: We prove the result by induction on $|T|$. For $T = \emptyset$, every cycle in $G(\emptyset)$ corresponds to a subgraph of G of minimum degree at least 2. This immediately implies $\nu'(G) \geq \nu'(G(\emptyset))$. Now let $|T| \geq 1$. By induction, it suffices to determine a set $\Delta T \subseteq T$ with

$$\nu'(G(T \setminus \Delta T)) \geq \nu'(G(T)) - |\Delta T|. \tag{2}$$

Let $\mathcal{C}$ be a packing of $\nu'(G(T))$ edge-disjoint cycles in $G(T)$.

If there is some $y \in T$ such that $\mathcal{C}$ contains at most one cycle whose edge set intersects $\{e_{1,y}, e_{2,y}\}$, then $\Delta T = \{y\}$ satisfies (2). Hence we may assume that for every $y \in T$, the two edges $e_{1,y}$ and $e_{2,y}$ are contained in two different cycles in $\mathcal{C}$.

Next, we assume that there are two distinct vertices $y, z \in T$ and indices $i, j \in \{1, 2\}$ such that $e_{i,y}$ and $e_{j,z}$ are both contained in one cycle $C \in \mathcal{C}$. Let $e_{3-i,y}$ be contained in $C' \in \mathcal{C}$ and let $e_{3-j,z}$ be contained in $C'' \in \mathcal{C}$. If $|\{C, C', C''\}| = 2$, then $\Delta T = \{y, z\}$ clearly satisfies (2). Hence, we may assume that C, C' and C'' are three distinct cycles. Since $(E(C) \cup E(C') \cup E(C'')) \setminus \{e_{1,y}, e_{2,y}, e_{1,z}, e_{2,z}\}$ contains the edge set of a cycle, $\Delta T = \{y, z\}$ satisfies (2). Hence, we may assume that no cycle in $\mathcal{C}$ contains two of the edges in $E_T = \{e_{1,y}, e_{2,y} \mid y \in T\}$.

Now, for every $i \in \{1, 2\}$ and every $y \in T$, there is a cycle $C_{i,y}$ in $\mathcal{C}$ such that $\{e_{i,y}\} = E(C_{i,y}) \cap E_T$. The edge set in G corresponding to $(E(C_{1,y}) \cup E(C_{2,y})) \setminus \{e_{1,y}, e_{2,y}\}$ contains a cycle C_y for every $y \in T$. Furthermore, the edge set in G corresponding to $E(C)$ contains a cycle for every $C \in \mathcal{C} \setminus \{C_{i,y} \mid i \in \{1, 2\}, y \in T\}$. Altogether, this implies

$$\begin{aligned}
\nu'(G) &\geq |T| + |\mathcal{C} \setminus \{C_{i,y} \mid i \in \{1, 2\}, y \in T\}| \\
&= |T| + (\nu'(G(T)) - 2|T|) \\
&= \nu'(G(T)) - |T|.
\end{aligned}$$

This completes the proof. $\square$

Lemma 4 *If G, S, V_1, V_2, x, E_1, and E_2 are as above, then there is a set $T \subseteq S \setminus \{x\}$ such that $\nu'(G) = \nu'(G(T)) - |T|$.*

Proof: A cycle C in G is a crossing x-cycle, if it intersects $E_G(x, V_1) \cup E_1$ and $E_G(x, V_2) \cup E_2$. Let $\mathcal{C}$ be a packing of $\nu'(G)$ edge-disjoint cycles in G with the minimum possible number of crossing x-cycles. Since the union of two edge-disjoint crossing x-cycles which contain a common vertex apart from x contains two edge-disjoint cycles which are not crossing x-cycles, the choice of $\mathcal{C}$ implies that no two crossing x-cycles in $\mathcal{C}$ contain a common vertex apart from x.

For every crossing x-cycle $C \in \mathcal{C}$ choose a vertex $y_C \in S \setminus \{x\}$ such that C passes through y_C. Let $T = \{y_C \mid C \in \mathcal{C} \text{ is a crossing } x\text{-cycle}\}$.

Clearly, by the definition of y_C and T, there are two edge-disjoint cycles in $G(T)$ corresponding to each crossing x-cycle in $\mathcal{C}$. Furthermore, there is a cycle in $G(T)$ corresponding to each cycle in $\mathcal{C}$ which contains x but is not a crossing-x-cycle. Finally, there is a cycle in $G(T)$ corresponding to each cycle in $\mathcal{C}$ which does not contain x. Since all these cycles are edge-disjoint, we obtain $\nu'(G(T)) \geq \nu'(G) + |T|$. By Lemma 3, $\nu'(G) = \nu'(G(T)) - |T|$ which completes the proof. $\square$

4 Algorithmic consequences

Theorems 1 and 2 are clearly suitable for an inductive argument with respect to the cardinality of the vertex cut.

Theorem 5 *Let G be a graph and let $S \cup V_1 \cup V_2$ be a partition of the vertex set of G into three non-empty sets such that there is no edge between V_1 and V_2. Let $k = |S|$.*

(i) $\nu(G)$ is uniquely determined by the values $\nu(H)$ for at most $2^{k+1}k!^2$ graphs H of order at most $\max\{|V_1|, |V_2|\} + k$.

(ii) $\nu'(G)$ is uniquely determined by the values $\nu'(H)$ for at most $2^{\binom{k}{2}+1}$ graphs H of order at most $\max\{|V_1|, |V_2|\} + k$.

Proof: (i) We prove the statement by induction over k.

For $k = 1$, S contains exactly one vertex and no cycle of G intersects V_1 and V_2. Therefore, $\nu(G)$ equals $\max\{\nu(G[V_1 \cup S]) + \nu(G[V_2]), \nu(G[V_1]) + \nu(G[V_2 \cup S])\}$, i.e. $\nu(G)$ is uniquely determined by the values $\nu(H)$ for 4 graphs H of order at most $\max\{|V_1|, |V_2|\} + k$.

For $k \geq 2$, Theorem 1 implies that $\nu(G)$ is uniquely determined by the values $\nu(H)$ for $k + 2k + \binom{k}{2} \leq 2k^2$ graphs H whose vertex set $V(H)$ can be partitioned into three non-empty sets S', V_1', and V_2' such that there is no edge between V_1' and V_2', and $(|S'|, |V_1'|, |V_2'|) = (|S|, |V_1|, |V_2|) + (a, b, c)$ for some $(a, b, c) \in \{(-1, 0, 0), (-1, 1, 0), (-1, 0, 1), (-2, 1, 1)\}$. By induction, we obtain that $\nu(G)$ is uniquely determined by the values $\nu(H)$ for at most $2k^2 2^k (k-1)!^2 = 2^{k+1}k!^2$ graphs H of order at most $\max\{|V_1|, |V_2|\} + k$. This completes the proof of (i).

(ii) We prove the statement by induction over k.

For $k = 1$, S contains exactly one vertex and no cycle of G intersects V_1 and V_2. Therefore, $\nu'(G)$ equals $\nu'(G[V_1 \cup S]) + \nu'(G[V_2 \cup S])$, i.e. $\nu'(G)$ is uniquely determined by the values $\nu'(H)$ for 2 graphs H of order at most $\max\{|V_1|, |V_2|\} + k$.

For $k \geq 2$, Theorem 2 implies that $\nu'(G)$ is uniquely determined by the values $\nu'(H)$ for 2^{k-1} graphs H whose vertex set $V(H)$ can be partitioned into three non-empty sets S', V_1', and V_2' such that there is no edge between V_1' and V_2', and $(|S'|, |V_1'|, |V_2'|) = (|S|, |V_1|, |V_2|) + (-1, +1, +1)$. By induction, we obtain that $\nu'(G)$ is uniquely determined by the values $\nu'(H)$ for $2^{k-1}2^{\binom{k-1}{2}+1} = 2^{\binom{k}{2}+1}$ graphs H of order at most $\max\{|V_1|, |V_2|\} + k$. This completes the proof of (ii). $\square$

Considering the proofs of Theorems 1 and 2, it is not difficult to see that optimal packings of cycles in G can be derived efficiently from optimal packings of cycles in the graphs H from Theorem 5, too.

Since graphs of bounded tree-width and order n have vertex cuts of bounded order whose removal results in components of order at most $2n/3$ [21], such graphs seem to be a natural choice for an algorithmic application of Theorem 5. In view of [4, 5] our next and final result implies that for fixed $l \in \mathbb{N}$, the graph properties "$\nu(G) \geq l$" and "$\nu'(G) \geq l$"

can be decided in linear time for the even larger class of graphs of bounded clique-width. Furthermore, the corresponding cycle packings can be found efficiently, too.

Theorem 6 *For fixed $l \in \mathbb{N}$, the two graph properties "$\nu(G) \geq l$" and "$\nu'(G) \geq l$" can be expressed in monadic second order logic [4] avoiding quantification over sets of edges (MSO_1-logic).*

Proof: We only give details for the property "$\nu'(G) \geq l$" which is more difficult to express in monadic second order logic avoiding quantification over sets of edges.

Let G be a graph with $\nu'(G) \geq l$. If $\mathcal{C}$ is a packing of l edge-disjoint cycles of G with the smallest total size $\sum_{C \in \mathcal{C}} |E(C)|$, then every two cycles in $\mathcal{C}$ intersect in at most two vertices. This implies that every cycle in $\mathcal{C}$ contains at most $2(l-1)$ vertices which belong also to other cycles in $\mathcal{C}$. Therefore, it is easy to see that a graph G satisfies $\nu'(G) \geq l$ if and only if there are — not necessarily distinct — vertices v_j^i for $1 \leq i \leq l$ and $1 \leq j \leq 2(l-1)$, edges e_j^i for $1 \leq i \leq l$ and $1 \leq j \leq 4(l-1)$, and sets U_i of vertices for $1 \leq i \leq l$ such that

(i) $v_j^{i_1} \notin U_{i_2}$ for $1 \leq i_1, i_2 \leq l$ and $1 \leq j \leq 2(l-1)$,

(ii) $U_{i_1} \cap U_{i_2} = \emptyset$ for $1 \leq i_1, i_2 \leq l$ with $i_1 \neq i_2$,

(iii) e_{2j-1}^i and e_{2j}^i are distinct edges incident with v_j^i whose other endvertex lies in

$$V_i := U_i \cup \left\{ v_{j'}^i \mid 1 \leq j' \leq 2(k-1) \right\}$$

for $1 \leq i \leq l$ and $1 \leq j \leq 2(l-1)$,

(iv) every vertex in U_i has two distinct neighbours in V_i for $1 \leq i \leq l$, and

(v) $e_{j_1}^{i_1} \neq e_{j_2}^{i_2}$ for $1 \leq i_1, i_2 \leq l$ and $1 \leq j_1, j_2 \leq 4(l-1)$ with $i_1 \neq i_2$.

(*The vertices v_j^i correspond to the vertices of the i-th cycle C_i of a packing $\mathcal{C}$ of l edge-disjoint cycles which may belong to more than one cycle of $\mathcal{C}$. Note that the vertices v_j^i are allowed to coincide. Therefore, if C_i contains no vertex which belongs to another cycle of $\mathcal{C}$, then one can choose $v_1^i = v_2^i = \ldots = v_{2(l-1)}^i$ equal to an arbitrary vertex of C_i. The set U_i corresponds to the set of the remaining vertices of C_i. Since the sets U_i are disjoint, edges incident with vertices in distinct sets U_i are necessarily distinct.*)

Since the existence of the vertices v_j^i, the edges e_j^i, and the sets U_i and also the conditions (i)-(v) can clearly be expressed in monadic second order logic avoiding quantification over sets of edges, the proof is complete. $\square$

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