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Minimum Degree and Density of Binary Sequences

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Abstract

For $d, k \in \mathbb{N}$ with $k \leq 2d$, let $g(d, k)$ denote the infimum density of binary sequences $(x_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ which satisfy the minimum degree condition $\sum\limits_{j=1}^{d} (x_{i+j} + x_{i-j}) \geq k$ for all $i \in \mathbb{Z}$ with $x_i = 1$. We reduce the problem to determine $g(d, k)$ to a combinatorial problem related to the generalized k-girth of a graph G which is defined as the minimum order of an induced subgraph of G of minimum degree at least k. Extending results of Kézdy and Markert, and of Bermond and Peyrat, we present a minimum mean cycle formulation which allows to determine $g(d, k)$ for small values of d and k. For odd values of k with $d + 1 \leq k \leq 2d$, we conjecture $g(d, k) = \frac{k^2-1}{2(dk-1)}$ and show that this holds for $k \geq 2d - 3$.

Keywords: Minimum degree; density; binary sequence; girth; generalized girth; power of cycle

Proposed running head: “Degree and Density of Sequences”

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1 Introduction

Let $d \in \mathbb{N}$ be fixed. For a two-way infinite binary sequence

$$X = (x_i)_{i\in\mathbb{Z}} = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \{0,1\}^{\mathbb{Z}},$$

we define *the minimum degree $\delta(X)$ of X* as

$$\delta(X) = \min\left\{\sum_{j=1}^{d}(x_{i+j} + x_{i-j}) \mid i \in \mathbb{Z}, x_i = 1\right\}.$$

If $x_i = 0$ for all $i \in \mathbb{Z}$, then we write $X = 0$ and call X *trivial*.

For $k \in \mathbb{N}$ with $k \leq 2d$, we consider the infimum density $g(d,k)$ of non-trivial binary sequences subject to a minimum degree condition defined as

$$g(d,k) = \inf\left\{\liminf_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} x_i \mid X = (x_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}, X \neq 0, \delta(X) \geq k\right\}.$$

Considering the binary sequence $(x_i)_{i\in\mathbb{Z}}$ with $x_i = 1$ if and only if $1 \leq i \leq k+1$, it follows that $g(d,k) = 0$ for $k \leq d$. While for such values of k, the calculation of $g(d,k)$ is trivial, for $k \geq d+1$, the calculation of $g(d,k)$ leads to an interesting combinatorial problem.

We prove as our first result that we can restrict ourselves to periodic sequences whose period is bounded in terms of d. Note that $g(d,2d) = 1$ for all $d \in \mathbb{N}$.

Theorem 1 *Let $d, k \in \mathbb{N}$ with $d \geq 2$ and $d+1 \leq k \leq 2d$. There is a non-trivial periodic binary sequence $X = (x_i)_{i\in\mathbb{Z}}$ whose period p is at most $d2^{2d+1}$ such that $\delta(X) \geq k$ and*

$$g(d,k) = \frac{1}{p}\sum_{j=1}^{p} x_j.$$

Proof: Let $0 < \epsilon < \frac{1}{3}$. Let $X = (x_i)_{i\in\mathbb{Z}}$ be a non-trivial binary sequence such that $\delta(X) \geq k$ and $\liminf\limits_{n\to\infty} \frac{1}{2n+1}\sum\limits_{j=-n}^{n} x_j \leq g(d,k) + \epsilon$. Since $x_i = 1$ for infinitely many $i \in \mathbb{Z}$, we have

$$\liminf_{n\to\infty} \frac{1}{2n+1}\sum_{j=-n}^{n} x_j \geq \frac{1}{2}\left(\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} x_j + \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} x_{-j}\right).$$

By symmetry, we may assume that $\liminf\limits_{n\to\infty} \frac{1}{n}\sum\limits_{j=1}^{n} x_j \leq g(d,k) + \epsilon$.

Note that $\delta(X) \geq k \geq d+1$ implies that X does not contain d consecutive 0-entries.

We call some $n \in \mathbb{N}$ *good* if

- $\frac{1}{n}\sum\limits_{j=1}^{n} x_j \leq g(d,k) + 2\epsilon$ and

- $(x_{j_1}, x_{j_1+1}, \ldots, x_{j_1+2d-1}) = (x_{j_2}, x_{j_2+1}, \ldots, x_{j_2+2d-1})$ for some $1 \leq j_1 \leq \lfloor \epsilon n \rfloor - 2d + 1$ and $n - \lfloor \epsilon n \rfloor + 1 \leq j_2 \leq n - 2d + 1$.

Claim *There are infinitely many good $n \in \mathbb{N}$.*

Proof of the claim: Let $n_1, n_2, \ldots, n_{2^{2d}} \in \mathbb{N}$ be such that $\frac{1}{n_i} \sum\limits_{j=1}^{n_i} x_j \leq g(d,k) + 2\epsilon$ for $1 \leq i \leq 2^{2d}$, $2d \leq \lfloor \epsilon n_1 \rfloor$, and $n_i \leq \lfloor \epsilon n_{i+1} \rfloor$ for $1 \leq i \leq 2^{2d} - 1$. Clearly, it suffices to prove that one of the n_i's is good. For contradiction, we assume that all n_i's are bad. Inductively, this implies that for $1 \leq i \leq 2^{2d}$, the sequence $(x_j)_{j \in \{1,2,\ldots,\lfloor \epsilon n_i \rfloor\}}$ contains i distinct subsequences of the form $(x_j, x_{j+1}, \ldots, x_{j+2d-1})$ with $1 \leq j \leq \lfloor \epsilon n_i \rfloor - 2d + 1$ which are different from all subsequences of the form $(x_j, x_{j+1}, \ldots, x_{j+2d-1})$ with $n_i - \lfloor \epsilon n_i \rfloor + 1 \leq j \leq n_i - 2d + 1$. Since there are exactly 2^{2d} distinct binary sequences of length d, this is impossible for $i = 2^{2d}$, which completes the proof of the claim. □

Let $n \in \mathbb{N}$ be good. Let $(x_{j_1}, x_{j_1+1}, \ldots, x_{j_1+2d-1}) = (x_{j_2}, x_{j_2+1}, \ldots, x_{j_2+2d-1})$ for $1 \leq j_1 \leq \lfloor \epsilon n \rfloor - 2d + 1$ and $n - \lfloor \epsilon n \rfloor + 1 \leq j_2 \leq n - 2d + 1$.

The non-trivial periodic binary sequence $X' = (x'_i)_{i \in \mathbb{Z}}$ with $x'_i = x_i$ for $j_1 + 2d \leq i \leq j_2 + 2d - 1$ of period $p' = j_2 - j_1$ satisfies $\delta(X') \geq k$ and

$$\frac{1}{p'} \sum_{j=1}^{p'} x'_j \leq \frac{1}{1 - 2\epsilon} \left(g(d,k) + 2\epsilon \right).$$

If $p' > 2d2^{2d}$, then the pigeonhole principle implies the existence of indices $1 \leq j_1, j_2 \leq p'$ with $(x'_{j_1}, x'_{j_1+1}, \ldots, x'_{j_1+2d-1}) = (x'_{j_2}, x'_{j_2+1}, \ldots, x'_{j_2+2d-1})$ and $j_1 + 2d \leq j_2 \leq j_1 + p' - 2d$. Let $X'' = (x''_i)_{i \in \mathbb{Z}}$ be the non-trivial p''-periodic binary sequence with $x''_i = x'_i$ for $j_1 + 2d \leq i \leq j_2 + 2d - 1$ with $p'' = j_2 - j_1$. Similarly, let $X''' = (x'''_i)_{i \in \mathbb{Z}}$ be the non-trivial p'''-periodic binary sequence with $x'''_i = x'_i$ for $j_2 + 2d \leq i \leq j_1 + p' + 2d - 1$ with $p''' = j_1 + p' - j_2$. Clearly, $p'', p''' < p'$, $\delta(X''), \delta(X''') \geq k$, and either $\frac{1}{p''} \sum\limits_{j=1}^{p''} x''_j \leq \frac{1}{1-2\epsilon} \left(g(d,k) + 2\epsilon \right)$ or $\frac{1}{p'''} \sum\limits_{j=1}^{p'''} x'''_j \leq \frac{1}{1-2\epsilon} \left(g(d,k) + 2\epsilon \right)$. This implies that for every $0 < \epsilon < \frac{1}{3}$, there is a non-trivial periodic binary sequence $X = (x_i)_{i \in \mathbb{Z}}$ whose period p is at most $d2^{2d+1}$ such that $\delta(X) \geq k$ and $\frac{1}{p} \sum\limits_{j=1}^{p} x_j \leq \frac{1}{1-2\epsilon} \left(g(d,k) + 2\epsilon \right)$. Since for every such sequence X, the quantity $\frac{1}{p} \sum\limits_{j=1}^{p} x_j$ is a rational number whose denominator is bounded by $d2^{2d+1}$, the desired result follows. □

For the further investigations, it is more convenient to consider a cyclic binary sequence

$$X = (x_0, x_1, \ldots, x_{p-1}) = x_0 x_1 \ldots x_{p-1}$$

of length p instead of a periodic binary sequence $(x_i)_{i \in \mathbb{Z}}$ with period p. As usual, we will consider indices modulo the length p. We say that an entry x_i of X *sees* another entry x_j of X if the cyclic distance of x_i and x_j is at least 1 and at most d. To avoid double-counting, we define *the minimum degree* $\delta(X)$ *of* X as the minimum number of distinct 1-entries of X seen by a 1-entry of X. Furthermore, we define *the density* $\mu(X)$ *of* X as

$$\mu(X) = \frac{1}{p}\sum_{j=1}^{p} x_j.$$

With these notions, Theorem 1 implies that $g(d,k)$ equals the minimum density of a non-trivial cyclic binary sequence X of length at most $d2^{2d+1}$ and minimum degree $\delta(X) \geq k$.

Our original motivation to study $g(d,k)$ comes from graph theory: For a finite, simple and undirected graph $G = (V,E)$ and $k \in \mathbb{N}$, the *k-girth $g_k(G)$ of G* is the minimum order of an induced subgraph of G of minimum degree at least k. The notion of k-girth was proposed and studied by Erdős et al. [3–5] and Bollobás and Brightwell [2]. It generalizes the usual girth, the length of a shortest cycle, which coincides with the 2-girth.

Kézdy and Markert studied bounds on this generalized girth [7, 8]. They conjectured that the d-th power of the cycle of length $n \geq 2d+1$, denoted by C_n^d, is the $2d$-regular graph with largest $(d+1)$-girth [8] (see also Chapter 5 of [7]). During the 1988 SIAM conference, Kézdy [6] posed the problem to determine the exact value of the $(d+1)$-girth of C_n^d. For odd values of d, this problem was solved by Bermond and Peyrat [1] who proved that for $d+1 \leq k \leq 2d$, the k-girth of C_n^d satisfies

$$\frac{g_k\left(C_n^d\right)}{n} \geq \frac{k}{2d}. \tag{1}$$

The bound (1) is best-possible whenever k is even in view of the induced subgraph of C_n^d where n is a multiple of d and which alternately contains $\frac{k}{2}$ consecutive vertices of C_n^d and does not contain the next $d - \frac{k}{2}$ consecutive vertices of C_n^d. For odd values of k, Bermond and Peyrat mentioned results for some small values of d and k, and proved the best-possible estimate $\frac{g_{2d-1}(C_n^d)}{n} \geq \frac{2d}{2d+1}$.

An induced subgraph G of C_n^d can be conveniently identified with a cyclic binary sequence $X = (x_0, x_1, \ldots, x_{n-1})$ of length n where 1-entries correspond to vertices of C_n^d which belong to G and 0-entries correspond to vertices of C_n^d which do not belong to G. This correspondence implies that $g(d,k)$ equals the minimum k-girth of the d-th power of cycles, i.e.

$$g(d,k) = \min\left\{\frac{g_k(C_n^d)}{n} \mid n \geq 3\right\}.$$

The above-mentioned results of Bermond and Peyrat imply that $g(d,k) = \frac{k}{2d}$ for $d+1 \leq k \leq 2d$ with even k and that $g(d, 2d-1) = \frac{2d}{2d+1}$. Kézdy and Markert determined $g(4,5) = \frac{12}{19}$ and $g(6,7) = \frac{24}{41}$ with the help of a computer. Bermond and Peyrat [1] claimed that $g(5,7) = \frac{5}{7}$ which is not correct (see Section 2). Furthermore, they conjectured that

$$g(d,k) = \frac{d(2d+3-k)}{2(d^2-(k-d-2)d-(k-d))}$$

for $d+1 \le k \le 2d$ with k odd. Since this expression is less than $\frac{k}{2d}$ if and only if $\left|k - \frac{3d}{2}\right| < \frac{d}{2}\sqrt{1-\frac{4}{d+1}}$, this conjecture is obviously not correct in view of (1).

Our results are as follows. In Section 2, we explain how for fixed values of d and k, the problem to determine $g(d,k)$ can be reduced to a minimum mean cycle problem on a suitably defined directed graph with arc costs. This allows to determine $g(d,k)$ and also the structure of optimal subgraphs of C_n^d for many small values of d and k and motivates a corresponding conjecture explained in Section 3. Moreover, in Section 4, we prove as our main result that our conjecture is true for $k = 2d-3$, i.e. we determine $g(d, 2d-3)$.

2 Minimum Mean Cycle Formulation

Given a directed graph $D = (V, A)$ and a cost function $c : A \to \mathbb{R}$, a *minimum mean cycle* is a directed cycle

$$C : v_1 v_2 \dots v_n v_1$$

in D for which

$$\overline{c}(A(C)) = \frac{1}{n} \sum_{a \in A(C)} c(a)$$

is minimum. Karp [9] observed that a minimum mean cycle can be found efficiently using shortest path methods.

For $d \in \mathbb{N}$ and $d+1 \le k \le 2d$, let $D = (V, A)$ be the directed graph whose vertex set V consists of all binary sequences

$$(x_{-d}, \dots, x_{-1}, x_0, x_1, \dots, x_d)$$

of length $2d+1$ with $x_0 = 1$ and $\sum\limits_{i=1}^{d}(x_i + x_{-i}) \ge k$ and which contains a directed arc (x, y) of cost $c((x,y)) = -i^*$ from a vertex $x = (x_{-d}, \dots, x_d)$ to a vertex $y = (y_{-d}, \dots, y_d)$ exactly if

$$(x_{i^*-d}, \dots, x_0, \dots, x_{i^*}, \dots, x_d) = (y_{-d}, \dots, y_{-i^*}, \dots, y_0, \dots, y_{d-i^*})$$

for $i^* = \min\{i \mid 1 \le i \le d, x_i = 1\}$. Note that i^* is well-defined and that the last condition implies that x and y can be suitably overlayed, i.e. there is a binary sequence z of length $2d+1+i^*$ such that x corresponds to the first $2d+1$ entries of z and y corresponds to the last $2d+1$ entries of z. See Figure 1 for an illustration.

Theorem 2 *If D and c are as above and C is a minimum mean cycle of D, then*

$$g(d,k) = -\frac{1}{\overline{c}(A(C))}.$$

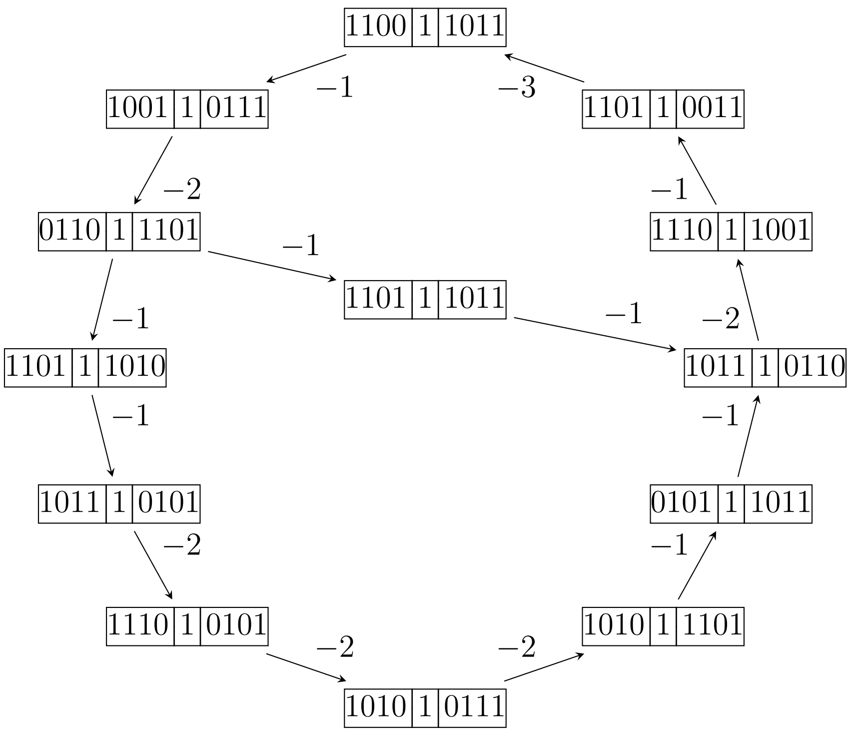

Figure 1: Induced subgraph of the directed graph D for $d = 4$ and $k = 5$.

Proof: Clearly, for every directed cycle $C : v_1v_2 \dots v_nv_1$ in D, suitably overlaying the sequences $v_1, v_2, \dots, v_n$ — as x and y above — results in a cyclic binary sequence X with $\delta(X) \geq k$. Since the number of 1-entries of X equals n and the length of X equals $-\sum\limits_{a\in A(C)} c(a)$, we obtain $\mu(X) = -\frac{1}{\bar{c}(A(C))}$.

Conversely, if X is a cyclic binary sequence with $\delta(X) \geq k$, the sequences of length $2d+1$ centered at the consecutive 1-entries of X define a directed closed walk W in D. By Euler's theorem, W contains a directed cycle C with $\bar{c}(A(C)) \leq \bar{c}(A(W))$. Since the length of W equals the number of 1-entries of X and the length of X is $-\sum\limits_{a\in A(C)} c(a)$, we obtain $\bar{c}(A(C)) \leq \bar{c}(A(W)) = -\frac{1}{\mu(X)}$.

These two observations clearly imply the desired result. $\square$

Table 1 summarizes some explicit values of $g(d, k)$ obtained by this approach together with realizing cyclic binary sequences. In fact, we determined optimal sequences for all values of d and $d + 1 \leq k \leq 2d$ with $d \leq 13$, $k \geq 2d - 7$, and k odd. For $(d, k) = (5, 7)$ for example, we obtained $g(5, 7) = \frac{24}{34}$, and a realizing cyclic binary sequence is

$$(1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0),$$

which we write shortly as $1^3 01^4 0101^2 01^4 01^2 0101^4 01^3 0^2$.

$2d$-k	(d,k)	$g(d,k)$	Optimal cyclic sequences, candidates for $\mathbf{u}$ highlighted
3	(4, 5)	12/19	$\mathbf{1^201}\ 1^201\ 0\ \mathbf{101^2}\ 101^2\ 0^2$
	(5, 7)	24/34	$\mathbf{1^301}\ 1^301\ 0\ \mathbf{1^201^2}\ 1^201^2\ 0\ \mathbf{101^3}\ 101^3\ 0^2$
	(6, 9)	40/53	$\mathbf{1^401}\ 1^401\ 0\ \mathbf{1^301^2}\ 1^301^2\ 0\ \mathbf{1^201^3}\ 1^201^3\ 0\ \mathbf{101^4}\ 101^4\ 0^2$
	(7, 11)	60/76	$\mathbf{1^501}\ 1^501\ 0\ \mathbf{1^401^2}\ 1^401^2\ 0\ \mathbf{1^301^3}\ 1^301^3\ 0\ \mathbf{1^201^4}\ 1^201^4\ 0\ \mathbf{101^5}\ 101^5\ 0^2$
5	(6, 7)	24/41	$\mathbf{10^21^3}\ 10^21^3\ 0^3\ \mathbf{1^30^21}\ 1^30^21\ 0\ \mathbf{1^20^21^2}\ 1^20^21^2\ 0$
			$\mathbf{101^201}\ 101^201\ 0^2\ \mathbf{1^20101}\ 1^20101\ 0\ \mathbf{10101^2}\ 10101^2\ 0^2$
	(7, 9)	40/62	$\mathbf{1^40^21}\ 1^40^21\ 0\ \mathbf{1^30^21^2}\ 1^30^21^2\ 0\ \mathbf{1^20^21^3}\ 1^20^21^3\ 0\ \mathbf{10^21^4}\ 10^21^4\ 0^3$
			$\mathbf{1^30101}\ 1^30101\ 0\ \mathbf{1^20101^2}1^20101^2\ 0\ \mathbf{10101^3}\ 10101^3\ 0^2\ \mathbf{101^301}\ 101^301\ 0^2$
		20/31	$\mathbf{101^201^2}\ 101^201^2\ 0^2\ \mathbf{1^201^201}\ 1^201^201\ 0$
7	(8, 9)	40/71	$\mathbf{1^40^31}\ 1^40^31\ 0\ \mathbf{1^30^31^2}\ 1^30^31^2\ 0\ \mathbf{1^20^31^3}\ 1^20^31^3\ 0\ \mathbf{10^31^4}\ 10^31^4\ 0^4$
			$\mathbf{1^20^21^201}\ 1^20^21^201\ 0\ \mathbf{10^21^201^2}\ 10^21^201^2\ 0^3\ \mathbf{1^201^20^21}\ 1^201^20^21\ 0\ldots$
			$\mathbf{101^30^21}\ 101^30^21\ 0^2\ \mathbf{1^30^2101}\ 1^30^2101\ 0\ \mathbf{1^20^2101^2}\ 1^20^2101^2\ 0\ldots$
			$\mathbf{101^20101}\ 101^20101\ 0^2\ \mathbf{1^2010101}\ 1^2010101\ 0\ \mathbf{1010101^2}\ 1010101^2\ 0^2\ldots$

Table 1

3 A Conjecture for $g(d,k)$

We have observed that all optimal sequences that we have computed can be obtained by applying a uniform construction rule.

Let U be the set of finite binary sequences starting and ending with a 1. For $\mathbf{u} \in U$ with $\mathbf{u} = 10^a\mathbf{v}$ for some $\mathbf{v} \in U$, the *shift operation* s applied to $\mathbf{u}$ results in $s(\mathbf{u}) = \mathbf{v}0^a1$, i.e. it removes all entries of $\mathbf{u}$ before the second 1 and appends them at the end in reverse order. For $\mathbf{u} = 11101$, for example, we obtain

$$s(\mathbf{u}) = 11011,\ \ s^2(\mathbf{u}) = s(s(\mathbf{u})) = 10111, \ \text{ and } \ s^3(\mathbf{u}) = 11101 = \mathbf{u}.$$

For $d,k \in \mathbb{N}$ with $d+1 \leq k \leq 2d$ and k odd, let U_k^d be the set of those sequences in U with length d and exactly $l = \frac{k+1}{2}$ many 1-entries.

Note that for $\mathbf{u} \in U_k^d$, we have $s^{l-1}(\mathbf{u}) = \mathbf{u}$.

The *shifted sequence* for $\mathbf{u}$ is the concatenation

$$X(\mathbf{u}) = \mathbf{u}\mathbf{u}0^{a_1+1}s(\mathbf{u})s(\mathbf{u})0^{a_2+1}\ldots0^{a_{l-2}+1}s^{l-2}(\mathbf{u})s^{l-2}(\mathbf{u})0^{a_{l-1}+1},$$

where a_i is the number of 0s between the i-th and $(i+1)$-st 1-entry of $\mathbf{u}$, i.e. $\mathbf{u} = 10^{a_1}10^{a_2}1\ldots10^{a_{l-1}}1$. For $\mathbf{u} = 11011 \in U_7^5$, we have

$$X(\mathbf{u}) = 11011\,11011\,0\,10111\,10111\,00\,11101\,11101\,0$$

which is a cyclic shift of the sequence for $(5,7)$ in Table 1.

A subsequence of consecutive entries of a cyclic binary sequence is called an *interval*.

Lemma 3 *Let $d,k \in \mathbb{N}$ be such that $d+1 \leq k \leq 2d$ and k is odd. Let $\mathbf{u} \in U_k^d$.*

(i) $X(\mathbf{u})$ has length $dk-1$,

(ii) $\mu(X(\mathbf{u})) = \frac{k^2-1}{2(dk-1)}$,

(iii) $\delta(X(\mathbf{u})) = k$, *and*

(iv) $g(d,k) \leq \mu(X(\mathbf{u}))$.

Proof: Let $\mathbf{u} = 10^{a_1}10^{a_2}1\dots10^{a_{l-1}}1$. The length of $X(\mathbf{u})$ equals

$$(l-1)2d + \sum_{i=1}^{l-1}(a_i+1) = (k-1)d + (d-1) = dk-1.$$

Furthermore, $X(\mathbf{u})$ contains $(l-1)2l = \frac{k^2-1}{2}$ many 1-entries. This implies (i) and (ii).

Note that the shifted sequences for $\mathbf{u}$ and for $s(\mathbf{u})$ are cyclic translates of each other. Furthermore, note that the reverse of a shifted sequence is also the cyclic translate of a shifted sequence. Therefore, in order to prove (iii), it suffices to consider the 1-entries within the first copy of $s(\mathbf{u})$ in $X(\mathbf{u})$.

By definition, the interval of $X(\mathbf{u})$ of length $2d+1$ centered at the first 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals (the central entry is highlighted)

$$10^{a_2}1\dots10^{a_{l-2}}10^{a_{l-1}}10^{a_1+1}\mathbf{1}0^{a_2}10^{a_3}1\dots10^{a_{l-1}}10^{a_1}11.$$

Hence this 1-entry sees $(l-1)$ 1-entries to the left and l 1-entries to the right, i.e. it sees $2l-1=k$ 1-entries.

For $2 \leq i \leq l-2$, the interval of $X(\mathbf{u})$ of length $2d+1$ centered at the i-th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals

$$10^{a_{i+1}}1\dots10^{a_{l-1}}10^{a_1+1}10^{a_2}1\dots10^{a_i}\mathbf{1}0^{a_{i+1}}1\dots10^{a_{l-1}}10^{a_1}110^{a_2}1\dots10^{a_i}1.$$

Again this 1-entry sees $2l-1=k$ 1-entries.

The interval of $X(\mathbf{u})$ of length $2d+1$ centered at the $(l-1)$-th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals $10^{a_1+1}10^{a_2}1\dots10^{a_{l-1}}\mathbf{1}0^{a_1}110^{a_2}1\dots10^{a_{l-1}}1$. Again this 1-entry sees $2l-1=k$ 1-entries.

The interval of $X(\mathbf{u})$ of length $2d+1$ centered at the l-th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals $010^{a_2}1\dots10^{a_{l-1}}10^{a_1}\mathbf{1}s(\mathbf{u})$. Again this 1-entry sees $2l-1=k$ 1-entries.

(iv) follows immediately from (ii) and (iii). □

We pose the following conjecture.

Conjecture 4 *If $d \in \mathbb{N}$ and $d+1 \leq k \leq 2d$ are such that k is odd, then*

$$g(d,k) = \frac{k^2-1}{2(dk-1)}.$$

Furthermore, a cyclic binary sequence X with $\delta(X) \geq k$ has density $g(d,k)$ if and only if X is the concatenation of copies of a shifted sequence $X(\mathbf{u})$ for some $\mathbf{u} \in U_k^d$.

The case $k = 2d-1$ of Conjecture 4 follows from the results and arguments in [1]. In this case U^d_{2d-1} contains only the element $\mathbf{u} = 1^d$ and $X(\mathbf{u}) = 1^{2d}01^{2d}0\dots1^{2d}0$.

Since we will prove Conjecture 4 for $k = 2d-3$, it is useful to consider the structure of $X(\mathbf{u})$ for $\mathbf{u} \in U^d_{2d-3}$. In this case, $\mathbf{u}$ is a sequence of length d containing $(d-1)$ 1-entries. If $\mathbf{u}^* = 10^{a_1}10^{a_2}1\dots10^{a_{l-1}}1$ with $a_1 = \dots = a_{l-2} = 0$ and $a_{l-1} = 1$, then $\mathbf{u}^* = 1^{d-2}01$ and

$$\begin{aligned} X(\mathbf{u}^*) &= 1^{d-2}01\,1^{d-2}01\,0\,1^{d-3}01^2\,1^{d-3}01^2\,0\,\dots101^{d-2}\,101^{d-2}\,0^2 \\ &= 1^{d-2}01^{d-1}0101^{d-3}01^{d-1}01^201^{d-4}01^{d-1}01^30\dots101^{d-1}01^{d-2}0^2 \end{aligned}$$

Since for every $\mathbf{u} \in U^d_{2d-3}$, there is some i with $s^i(\mathbf{u}^*) = \mathbf{u}$, every shifted sequence $X(\mathbf{u})$ for $\mathbf{u} \in U^d_{2d-3}$ arises from $X(\mathbf{u}^*)$ by a cyclic shift. In this sense, the conjectured extremal sequences are unique.

4 The Value of $g(d, 2d-3)$

Throughout this section let $d \geq 4$ and let $\mathcal{X}$ be the set of cyclic binary sequences X with $\delta(X) \geq 2d-3$. This section is devoted to the proof of Conjecture 4 for $k = 2d-3$, i.e. we will prove the following result.

Theorem 5 *Every $X \in \mathcal{X}$ satisfies $\mu(X) \geq \frac{(2d-3)^2-1}{2((2d-3)d-1)}$. Equality holds if and only if X is the concatenation of shifted sequences $X(\mathbf{u}^*)$ with $\mathbf{u}^* = 1^{d-2}01$.*

Before proving Theorem 5, we investigate structural properties of sequences in $\mathcal{X}$. Let

$$X = (x_0, x_1, \dots, x_{n-1}) = x_0x_1\dots x_{n-1} \in \mathcal{X} \text{ with } n \geq 2d+1.$$

Recall that an entry x_i of X sees another entry x_j of X, if x_j is in one of the intervals $x_{i-d}x_{i-d+1}\dots x_{i-1}$ or $x_{i+1}x_{i+2}\dots x_{i+d}$. We call x_i *regular* if it sees exactly $(2d-3)$ 1-entries and hence exactly three 0-entries. We first show that all irregular entries see more than $(2d-3)$ 1-entries and describe the local structure around regular 0-entries.

Lemma 6

(i) All entries of X see at most three 0-entries.

(ii) For every regular 0-entry x_i, either $x_{i+1} = x_{i+d} = 0$, or $x_{i-1} = x_{i-d} = 0$, or $x_{i-d} = x_{i+d} = 0$.

Proof: (i): By assumption, all 1-entries of X see at most three 0-entries. For contradiction, we assume that some 0-entry of X sees more than three 0-entries. This implies that X has an interval $X' = 10^a1$ such that some 0-entry of X' sees at least four 0-entries. Since $d \geq 4$ and each of the two 1-entries of X' see at most three 0-entries, we obtain $a \leq 3$. Moreover, the two 1-entries of X' together see at most $(6-a)$ distinct 0-entries. If $a \geq 2$, then every 0-entry of X' sees at most three 0-entries, a contradiction. Hence $a = 1$. If x_i

is the 0-entry in X', then each 1-entry of X' sees all but one entry seen by x_i. Thus it sees at least three 0-entries seen by x_i and the 0-entry x_i which is the final contradiction.

(ii): Again, the interval X' of the form 10^a1 of X containing the regular 0-entry x_i satisfies $a \leq 3$. If $a = 3$, then one of the two 1-entries of X' sees x_i and all three 0-entries seen by x_i which is a contradiction. If $a = 2$, then, by symmetry, we may assume that x_i is the first 0-entry of X'. Since the 1-entry x_{i-1} does not see one of the 0-entries seen by x_i, we have $x_{i+1} = x_{i+d} = 0$. Finally, if $a = 1$, then each of the 1-entries x_{i-1} and x_{i+1} does not see one of the 0-entries seen by x_i which implies $x_{i+d} = x_{i-d} = 0$ and completes the proof of (ii). $\square$

Let n_1 denote the number of 1-entries of X. Moreover, let n^+ denote the number of irregular entries of X.

We can relate the density of X to the number of irregular entries of X.

Lemma 7

$$\mu(X) = \frac{n_1}{n} \geq \frac{2d-3}{2d} + \frac{n^+}{2dn}.$$

Proof: By Lemma 6 (i), double-counting the pairs (x_i, x_j) where $x_i = 1$ and x_i sees x_j yields $(2d-3)(n-n^+) + (2d-2)n^+ \leq 2dn_1$ which implies $\mu(X) = \frac{n_1}{n} \geq \frac{2d-3}{2d} + \frac{n^+}{2d}$. $\square$

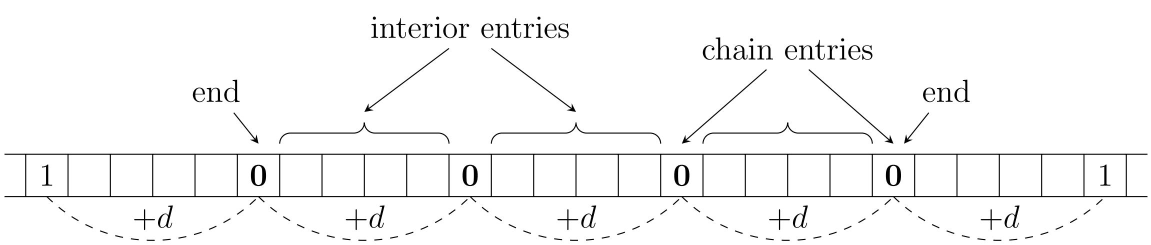

Figure 2: A chain of length 4 for $d = 5$.

A *chain* of X is a maximal subsequence

$$C = (x_i, x_{i+d}, \ldots, x_{i+kd})$$

of distinct 0-entries of X such that $k \geq 1$. A chain may be *cyclic* in which case $i \equiv i+(k+1)d \pmod{n}$. Otherwise C has two distinct *ends* x_i and x_{i+kd} where $x_{i-d} = 1 = x_{i+(k+1)d}$. Associated with the chain C are the *interior* entries of C, which are those entries that belong to one of the intervals $x_{i+jd+1}x_{i+jd+2}\ldots x_{i+jd+d-1}$, $0 \leq j \leq k-1$, between consecutive chain entries x_{i+jd} and $x_{i+(j+1)d}$ of C. We say that two chains *overlap*, if a chain entry of one chain is an interior entry of the second chain. Clearly, in this case, also a chain entry of the second chain is an interior entry of the first chain. Note that a chain may overlap itself.

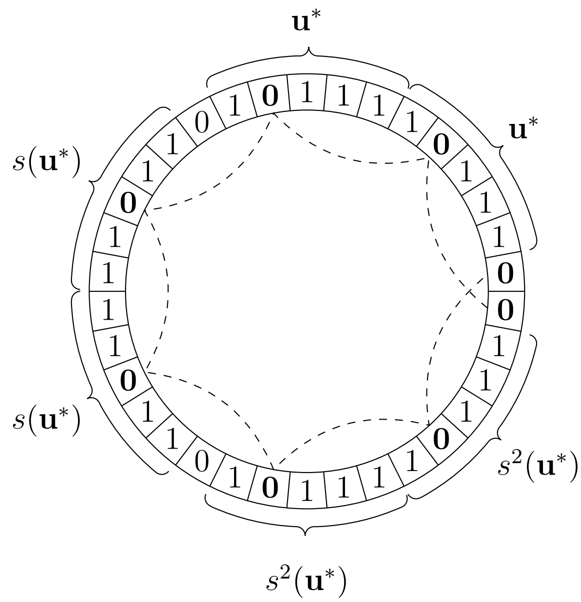

Figure 3: The example $X'(\mathbf{u}^*)$ for $d = 5$, i.e. with $\mathbf{u}^* = 11101$.

For example, the sequence $X'(\mathbf{u}^*) = x'_0 x'_1 \dots x'_{n-1}$ which arises from the shifted sequence $X(\mathbf{u}^*)$ for $\mathbf{u}^* = 1^{d-2}01$ by moving the final 0-entry to the beginning

$$\begin{aligned} X'(\mathbf{u}^*) &= x'_0 x'_1 \dots x'_{n-1} \\ &= \mathbf{0}1^{d-2}\mathbf{0}1^{d-1}\mathbf{0}101^{d-3}\mathbf{0}1^{d-1}\mathbf{0}1^2 01^{d-4}\mathbf{0}1^{d-1}\mathbf{0}1^3 0 \dots 1\mathbf{0}1^{d-1}\mathbf{0}1^{d-2}\mathbf{0} \end{aligned} \qquad (2)$$

has the single chain $C = (x'_{n-1}, x'_{d-1}, x'_{2d-1}, \dots, x'_{n-d}, x'_0)$ whose ends x'_{n-1} and x'_0 are both interior entries as well as chain entries of C. See Figures 2 and 3 for an illustration.

We will show that chains may overlap only in their respective ends. More precisely, in Lemma 8 (ii) below we show that if x_i is a chain entry of C which is an interior entry of chain C' and x_{i-d} is another chain entry of C, then x_i is an end of C and x_{i-1} is an end of $C' = (x_{i-1}, x_{i+d-1}, \dots)$. If this occurs, we call the interval $x_{i-1}x_i = 0^2$ a pair of *overlapping chain ends.*

Lemma 8

(i) Every regular 0*-entry of* X *belongs to some chain of* X*.*

(ii) If a chain entry of C *is an interior entry of the (not necessarily distinct) chain* C'*, then it belongs to a pair of overlapping chain ends.*

(iii) Let $x_{i-1}x_i$ *be a pair of overlapping chain ends. The intervals of length* $2d$ *ending and starting in* $x_{i-1}x_i = 0^2$ *have the form* $1^{d-1}01^{d-2}0^2$ *and* $0^2 1^{d-2}01^{d-1}$*, respectively.*

(iv) An end of a chain is regular in X *if and only if it belongs to a pair of overlapping chain ends.*

Proof: (i): This follows immediately from Lemma 6 (ii).

(ii): Let x_i be a chain entry of C which is an interior entry of C'. Then there must be chain entries x_j, x_{j+d} with $i-d < j < i$ of C'. By symmetry, we may assume that x_{i-d} is another chain entry of C. If $j < i-1$, then x_{i-1} sees at least four 0-entries, a contradiction. So $j = i-1$. Moreover, $x_{j-d} = 1 = x_{i+d}$, otherwise x_{i-2} or x_{i+1} sees four 0-entries. So x_i is an end of C and x_{i-1} is an end of C'.

(iii): Since both x_{i-1} and x_i already see three of the four 0-entries $x_{i-d}, x_{i-1}, x_i, x_{i+d-1}$, we obtain that $x_{i-d+1} = 1 = x_{i+d-2}$. Since each of these two entries sees three of the four 0-entries, too, all other entries seen by them must be 1, and the two intervals of X ending and starting in $x_{i-1}x_i$ have the required form.

(iv): It follows from (iii) that overlapping ends of chains are regular. Conversely, we assume that x_i is an end of a chain which is not an interior entry of any chain. By symmetry, we may assume that $x_{i-d} = 0$ and $x_{i+d} = 1$. If x_i is regular, then $x_{i-1} = 0$, otherwise x_{i-1} sees x_i and all the three 0-entries seen by x_i, a contradiction to Lemma 6 (i). But since x_{i-1} does not belong to a chain, it must be irregular by (i) and thus x_{i-1} sees only the two 0-entries x_i and x_{i-d}. So x_i must be irregular as well. □

Lemma 9 *Let $I = x_{j-d}x_{j-d+1}\dots x_{j+d}$ be an interval of $2d+1$ entries of X.*

(i) If I contains no irregular entry, then I contains a regular end of a chain.

(ii) If I does not contain a regular chain end but contains an irregular chain end, then it contains at least two irregular entries.

Proof: (i): Since the center x_j of I is regular, it sees exactly three 0-entries, all of which are regular. By the length of I, only two of them can belong to the same chain. So, by Lemma 8 (iv), the third must be a regular chain end belonging to a pair of overlapping chain ends.

(ii): For contradiction, we assume that I contains exactly one irregular entry, an irregular chain end. If the center x_j is not the irregular chain end itself, then it is regular. So it sees two further 0-entries apart from the irregular chain end. Since these are regular, they all belong to chains. Hence, by Lemma 8 (ii), one of them is a regular chain end, a contradiction. So let x_j be the irregular chain end. We may assume that $x_{j-d} = 0$. If x_j sees another 0-entry apart from x_{j-d}, then, by Lemma 8 (i) and (iv), this 0-entry is irregular. Otherwise, x_{j+1} is irregular, a contradiction. □

Lemma 10 *If X has a single chain whose ends overlap, then X has at least $d-3$ irregular entries.*

Proof: Let $(x_0, x_d, x_{2d}, \dots, x_{n-d+1}, x_1)$ be the chain and let $2 \le r \le d-2$. We prove that there is some irregular entry x_j with $2 \le j \le n-2$ and $j \equiv r \bmod d$.

If an entry at such a position satisfies $x_j = 0$, then, by Lemma 8 (i) and (ii), x_j is irregular. Hence, we may assume that $x_j = 1$ for all $2 \leq j \leq n-2$ with $j \equiv r \bmod d$. We choose a largest $s < r$ such that X has an entry $x_k = 0$ with $k \equiv s \bmod d$. Note that $x_1 = 0$ implies that s is well-defined and that $1 \leq s < r$. We claim that $x_{k-s+d+r}$ is irregular.

Note that every 1-entry in the interval $x_{k-s}x_{k-s+1}\dots x_{k-s+d}$ sees the three 0-entries x_{k-s}, x_k, x_{k-s+d}. Hence $x_{k-s+d-1} = 1$ and $k-s+d+r < n-d$. Moreover, all further entries seen by $x_{k-s+d-1}$ satisfy $x_{k-s+d+1} = x_{k-s+d+2} = \dots = x_{k-s+2d-1} = 1$. Furthermore, since x_{k+d} sees three 0 entries, $x_{k-s+2d+1} = \dots = x_{k+2d} = 1$. By the definition of s, $x_{k+2d+1} = \dots = x_{k+2d+r-s-1} = 1$. So, indeed, $x_{k-s+d+r}$ sees only the two 0-entries x_{k-s+d} and x_{k-s+2d} and is irregular. $\square$

We are now prepared to prove Theorem 5.

Proof of Theorem 5:
Let $X^* = X'(\mathbf{u}^*)$ be as in (2). For contradiction, we assume that $X = (x_0, x_1, \dots, x_{n-1})$ is a cyclic binary sequence in $\mathcal{X}$ of smallest order n having minimum density $\mu(X) = g(d, 2d-3)$, and that X is not the concatenation of copies of X^*. Clearly, $\mu(X) \leq \mu(X^*) = \frac{(2d-3)^2-1}{2((2d-3)d-1)}$. Since a 1-entry of X must see at least $2d-3$ other 1-entries, we get for $n \leq 2d$ that $\mu(X) = \frac{n_1}{n} \geq \frac{2d-2}{n} \geq 1 - \frac{1}{d} > \mu(X^*)$, a contradiction. So we may assume that $n \geq 2d+1$.

If X contains no pair of overlapping chain ends, then, by Lemma 9 (i), every interval I of length $2d+1$ of X contains an irregular entry. Since every irregular entry contributes to $2d+1$ such intervals, we get by double-counting

$$n \leq (2d+1)n^+, \tag{3}$$

thus, by Lemma 7, $\mu(X) \geq \frac{2d-3}{2d} + \frac{1}{2d(2d+1)} > \mu(X^*)$ which is a contradiction.

Hence we may assume that X contains a pair of overlapping ends of chains.

First we assume that X contains more than one such pair. By cyclicity, we may assume that $x_{n-1}x_0$ and $x_{k-1}x_k$ are pairs of overlapping chain ends of X. Let

$$X' = x_0x_1\dots x_{k-1}$$

and

$$X'' = x_kx_{k+1}x_{k+2}\dots x_{n-1}.$$

By Lemma 8 (iii), X' and X'', considered as cyclic sequences, are both in $\mathcal{X}$, because each entry sees the same entries as in X. Since X has minimum density $\mu(X)$ and $\mu(X)$ is a weighted average of the densities $\mu(X')$ and $\mu(X'')$, we obtain $\mu(X') = \mu(X'') = \mu(X)$. Since X' and X'' have smaller lengths than X, by our initial assumption, each of X' and X'' are the concatenation of copies of X^*. Hence X is the concatenation of copies of X^* which is a contradiction.

Therefore, X has exactly one pair of overlapping chain ends, say $x_{n-1}x_0$. Let $\mathcal{J}$ be the set of intervals of length $2d+1$ of X. Let $\mathcal{J}_0 \subseteq \mathcal{J}$ denote the set of those intervals containing a regular chain end and let $\mathcal{J}_2 \subseteq \mathcal{J}$ denote the set of those intervals containing an irregular chain end. By Lemma 9, each interval in $\mathcal{J}_2 \setminus \mathcal{J}_0$ contains at least two irregular entries, while only the intervals in $\mathcal{J}_0 \setminus \mathcal{J}_2$ can contain no irregular entry. If X contains more than one chain, then X contains two different irregular chain ends, hence $|\mathcal{J}_2| \geq 2d+2$ while $|\mathcal{J}_0| \leq 2d+2$. Double-counting the incidences interval/irregular entry we obtain

$$n \leq n + |\mathcal{J}_2| - |\mathcal{J}_0| = n + |\mathcal{J}_2 \setminus \mathcal{J}_0| - |\mathcal{J}_0 \setminus \mathcal{J}_2| \leq (2d+1)n^+,$$

as in (3), which again contradicts $\mu(X) \leq \mu(X^*)$.

So X has a single chain both ends of which overlap. By Lemma 10, X contains at least $d-3$ irregular entries. Hence, by Lemma 7, $\mu(X) \geq \frac{2d-3}{2d} + \frac{d-3}{2dn}$. Since $\mu(X) \leq \mu(X^*) = \frac{(2d-3)^2-1}{2(d(2d-3)-1)}$, this implies that

$$n \geq d(2d-3) - 1,$$

i.e. the length of X is at least the length of X^*. By Lemma 8 (iv), each of the $n-(2d+2)$ intervals of length $2d+1$ in $\mathcal{J} \setminus \mathcal{J}_0$ contains at least one irregular entry. Hence $n^+ \geq \frac{n-(2d+2)}{2d+1}$ and, by Lemma 7,

$$\begin{aligned} \mu(X) &\geq \frac{2d-3}{2d} + \frac{n-(2d+2)}{2d(2d+1)n} = \frac{2d-3}{2d} + \frac{1}{2d(2d+1)} - \frac{2d+2}{2d(2d+1)n} \\ &\geq \frac{(2d-3)^2-1}{2(d(2d-3)-1)} = \mu(X^*). \end{aligned}$$

Since $\mu(X) \leq \mu(X^*)$, we obtain $\mu(X) = \mu(X^*)$. Therefore, $n = d(2d-3)-1$, each irregular entry sees exactly $(2d-2)$ 1-entries, and each of the $2d+2$ intervals in $\mathcal{J}_0$ contains no irregular entry while all intervals in $\mathcal{J} \setminus \mathcal{J}_0$ contain exactly one irregular entry. Hence the irregular entries must be exactly $x_{2d+1}, x_{4d+2}, \ldots, x_{(2d+1)(d-3)}$.

So the irregular entries of X and X^*, with the notation of (2), are located at the same positions and, by Lemma 8 (iii), the intervals $x_{n-2d+1} \ldots x_{n-1} x_0 \ldots x_{2d-2}$ of X and $x'_{n-2d+1} \ldots x'_{n-1} x'_0 \ldots x'_{2d-2}$ of X^* are equal.

We assume that for some $i \geq 2d-2$, the intervals $x_{i-2d+1} \ldots x_i$ of X and $x'_{i-2d+1} \ldots x'_i$ of X^* are equal. Now we show that $x_{i+1} = x'_{i+1}$. Indeed, since $x_{i-d+1} = x'_{i-d+1}$ has the same regularity status within X and X^* and sees the same entries in X and X^*, respectively, except possibly at position $i+1$, it follows that $x_{i+1} = x'_{i+1}$. Therefore, $X = X^*$ contradicting the assumption that X is a counterexample. This completes the proof. $\square$

If we define the quantity $\tilde{\delta}(X)$ for a cyclic binary sequence $X = (x_0, x_1, \ldots, x_{n-1})$ as

$$\tilde{\delta}(X) = \min \left\{ \sum_{j=1}^{d} (x_{i+j} + x_{i-j}) \mid 0 \leq i \leq n-1 \right\}$$

and $\tilde{g}(d,k)$ for $d,k \in \mathbb{N}$ with $k \leq 2d$ as the infimum density of a cyclic binary sequence X with $\tilde{\delta}(X) \geq k$, then $g(d,k) \leq \tilde{g}(d,k)$. A simple double-counting implies $\tilde{g}(d,k) \leq \frac{k}{2d}$.

The example described after (1) implies $g(d,k) = \tilde{g}(d,k)$ for $k \geq d+1$ with k even. Furthermore, the comment after Conjecture 4 concerning $k = 2d-1$ and Lemma 6 (i) imply $g(d, 2d-1) = \tilde{g}(d, 2d-1)$ and $g(d, 2d-3) = \tilde{g}(d, 2d-3)$, respectively. Finally, it is easy to check that $\tilde{\delta}(X(\mathbf{u})) \geq k$ for every shifted sequence $X(\mathbf{u})$ for every $\mathbf{u} \in U_k^d$ which does not contain two consecutive 0-entries.

Therefore, Conjecture 4 would - if true - imply that $g(d,k) = \tilde{g}(d,k)$ for all $d+1 \leq k \leq 2d$.

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