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Output Feedback Control and Robustness in the Gap Metric

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Zusammenfassung

Die vorgelegte Arbeit behandelt den Entwurf und die Robustheit von drei verschiedenen Regelstrategien für lineare Differentialgleichungssysteme mit mehrdimensionalen Ein- und Ausgangssignalen. Es werden folgende drei Regler betrachtet: Rückführungen des Ausgangssignals und dessen Ableitungen, λ-tracking und Funnel-Regelung. Für alle drei Regler werden bestimmte strukturelle Voraussetzungen an die linearen Systeme gestellt, auf die der Regler angewendet werden soll. Für die zuerst vorgestellte Ausgangs–Ableitungs–Rückführung wird vorausgesetzt, dass der Relativgrad des Systems bekannt ist, dass das System minimalphasig ist, und dass die sogenannte „high-frequency gain“ Matrix positiv definit ist. Für λ-tracking und Funnel-Regelung fordert man die selben Voraussetzung und zusätzlich, dass der Relativgrad nicht nur bekannt, sondern gleich eins ist.

Einer Einleitung folgend, werden im zweiten Kapitel der Arbeit diese strukturellen Eigenschaften linearer Systeme analysiert. Für Systeme mit mehrdimensionalen Ein- und Ausgängen wird der sogenannte Vektorrelativgrad vorgestellt und für den Fall, dass dieser nicht strikt ist, eine Normalform hergeleitet, die gleiche strukturelle Eigenschaften besitzt, wie die bekannte Byrnes–Isidori–Normalform für Systeme mit eindimensionalen Ein- und Ausgängen oder Systemen mit mehrdimensionalen Ein- und Ausgängen und striktem Relativgrad.

Diese Normalform ist essentiell für die Konstruktion eines Reglers mit Ausgangs–Ableitungs–Rückführung für Systeme mit mehrdimensionalen Ein- und Ausgängen und nicht striktem Relativgrad im dritten Kapitel.

In den Kapitel vier und fünf werden bekannte Resultate für λ-tracking und Funnel-Regelung verallgemeinert und neu bewiesen, um so die Robustheitsanalyse für beide Regler in den zwei abschließenden Kapiteln der Arbeit zu ermöglichen.

Ergebnisse zur robusten Stabilität, die in dieser Arbeit vorgestellt werden, basieren auf der Verwendung der sogenannten Gap-Metrik: salopp gesprochen, bauen diese Robustheitsresultate auf das Messen von Abständen zwischen Systemen oder Reglern auf. Genauer gesagt, wird die Gap-Metrik in der vorliegenden Arbeit benutzt, um Abstände zwischen den Graphen von Operatoren, die ein System beschreiben – das

sind Mengen von Trajektoren aus zuvor gewählten Signalräumen, die „Lösung“ eines Systems oder eines Regler sind – zu messen. Diese Graphen sind Unterräume der Signalräume.

Robuste Stabilität beschreibt im Allgemeinen dann folgendes Prinzip: falls ein geschlossener Kreis aus einem linearen System und einem Regler stabil ist und der Abstand (die Gap-Metrik) zwischen dem im geschlossenen Kreis betrachteten System und einem anderen „neuen“ System hinreichend klein ist (und einige weitere technische Voraussetzungen für den geschlossenen Kreis erfüllt sind), so ist der geschlossene Kreis aus dem „neuen“ System und dem Regler wieder stabil. Die gleiche Aussage stimmt auch für den Fall, dass man den Regler und nicht das System austauscht.

In der vorliegenden Arbeit wird Robustheit für die drei oben genannten Regler in ihrer Anwendung auf (lineare) Systeme untersucht.

Für die Ausgangs–Ableitungs–Rückführung wird gezeigt, dass, falls diese ein System stabilisiert, die auftretenden Ableitungen des Ausgangs durch Euler-Approximationen der Ableitungen ersetzt werden können, insofern diese hinreichend genau sind. In diesem Fall wird also ein „neuer“ Regler auf das selbe System angewandt. Das Ergebnis zur robusten Stabilisierung gilt sogar für nichtlineare Systeme und wird auf den im dritten Kapitel für lineare Systeme mit striktem Relativgrad vorgestellten konkreten Regler angewendet.

Bei den Untersuchungen zu λ-tracking und Funnel-Regelung bleibt der Regler jeweils unverändert. Hier werden die linearen Systeme, auf die der Regler angewendet wird, ersetzt. Es wird gezeigt, dass beide Regler auch für die Stabilisierung linearer Systeme verwendet werden können, die nicht die sonst geforderten Voraussetzungen erfüllen. Hier ist dann allerdings gefordert, dass ein solches System einen geringen Abstand zu einem System hat, dass die notwendigen Voraussetzungen aus Kapitel vier und fünf erfüllt.

Abstract

The present thesis considers the design and robustness analysis of three different control strategies for linear systems of differential equations with multidimensional input and output signals. These three control strategies are the following: high-gain output derivative feedback control, λ-tracking and funnel control. Every control strategy requires certain structural properties of linear systems which it will applied to. For high-gain output derivative feedback control it is required that the system's relative degree is known, that the system is minimum phase and has a positive so-called "high-frequency gain" matrix. The same properties and additionally, that the relative degree is not only known but equal to one, are required for λ-tracking and funnel control.

Following a short introduction, structural properties of linear systems are considered in detail. For systems with multidimensional inputs and outputs a definition and characterization for the so-called (vector) relative degree is presented. For systems with non-strict relative degree a normal form is developed. This normal form has the same structural properties as the well-known Byrnes–Isidori normal form for systems with one-dimensional inputs and outputs or systems with multidimensional input and outputs and strict relative degree.

This normal form for linear systems with non-strict relative degree is crucial for the design of the high-gain output derivative feedback controller. It is important to note that this controller stabilizes any system which (vector) relative degree is known, provided the system is minimum phase and the high-frequency gain matrix is positive definite.

In chapters four and five, respectively, known results for λ-tracking and funnel control are generalized with regard to the analysis of robustness for both control strategies in the concluding chapters of the thesis. It is shown that the λ-tracker and funnel controller may be applied to any system from the class of minimum phase linear systems with strict relative degree one and positive definite high-frequency gain matrix to achieve certain control objectives. Robustness then means that one may apply the controllers to systems which are close (in some sense) to any system from the above class but not in this class of systems.

The result on robustness and robust stability considered in this thesis are based on the application of the so-called gap metric: loosely speak-

ing, all robustness results rely on the measuring of distances of systems or controllers. More precisely, the gap metric is used to determine the distances between the graphs of operators representing a system or a controller, i.e. the set of all trajectories from prespecified signal spaces which are "solutions" for the system or the controller. This graphs are subsets of the considered signal spaces.

In view of this gap metric, a robustness result will be considered as follows: if a closed-loop system represented by the application of a controller to a linear system is stable (in some sense), and the distance (i.e. the gap metric) between the system considered in the closed-loop and a different "new" system is sufficiently small (provided some more technical requirments hold true for the closed-loop system), then the closed-loop system represented by the application of the controller to the "new" system is again stable. This conclusion holds also true when changing the roles of system and controller, i.e. applying a "new" controller, which is "close" (in terms of the gap metric) to the original controller, to the same system.

In the present thesis, robustness results are presented for all three previously introduced control strategies when applied to linear systems.

For high-gain output derivative feedback control it is shown that the designed controller still stabilizes the system when the derivatives of the output are replaced by Euler approximations of the derivatives provided the approximation is sufficiently precise, i.e. if the step size is sufficiently small the gap between the derivative feedback and the feedback with approximations of the derivatives (delay feedback) is sufficiently small. In this case a "new" controller is applied to the same system. The general robustness results holds also true for nonlinear systems and it is applied to the concrete controller for linear systems with strict relative degree, presented in chapter three.

For robustness results for λ-tracking and funnel control the control strategies are not changed but the systems which the controllers are applied to. It is proved that the controller may be applied to systems which are "close" (in terms of a small gap) to any system from the class of systems considered in chapters four and five, i.e. the class of minimum phase systems with relative degree one and positive definite high-frequency gain matrix. The "new" systems may not satisfy any of these classical assumptions.

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1 Introduction

The subject of this thesis is the study of several strategies for the regulation of the output of linear time-invariant MIMO-systems. A linear time-invariant MIMO-system is a system of the form

$$\dot{x} = A\,x + B\,u\,, \qquad y = C\,x\,,$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$ for some integers $n, m \in \mathbb{N}$ with $n \geq m$. Here u is the input of the system and y denotes the system's output; both have m components. For linear systems (A, B, C) of the above form it is rather simple to calculate a solution. Given an initial value $x(0) = x^0 \in \mathbb{R}^n$ and an input signal $u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ from a the set of locally integrable functions, the output y is as follows:

$$t \mapsto y(t) = C\,e^{At}x^0 + C\int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s\,.$$

If the matrices A, B and C are known and have certain properties, for example (A, B) is controllable and (A, C) is observable, see definitions in, for example, [Son98, Ch. 3 and 6], one may choose the input function u such that the output y has some desired properties.

However, in general applications the system's matrices may be unknown. Anyway, it is possible to find inputs u to control or stabilize the system in some ways. One potential ansatz is to use the system's output y (and maybe derivatives of the output) to design such controllers. Such control strategies are presented in, for example, [IP98, ITT04, ITT05, SHWS05, SWS06, IS09], see also the survey article [IR08]. There only structural properties of systems are used. Three control strategies, which are applied to linear systems, are presented in this thesis (Chapters 3–5). Furthermore, this thesis gives robustness results for the considered control strategies (Chapters 7–9) which are proven by utilizing the concept of the gap metric (Chapter 6).

The application of all control strategies to a linear system requires certain structural properties of the system: for example that the *relative degree* is known. In the first part of the present thesis these structural properties are analyzed.

System class.

The definition of a relative degree of linear systems goes back to frequency representations of linear systems [Isi95, p. 139] and earlier. In Chapter 2 definitions and characterizations for the relative degree of linear SISO- and MIMO-systems are introduced. Here it is important to distinguish cases where the system has one-dimensional or, for $m \geq 2$, m-dimensional inputs and outputs. Moreover, in the case of m-dimensional u and y, it is important to know whether the relative degree is or is not *strict*, see the following paragraphs for details.

The system's relative degree can be utilized to construct a coordinate transformation which leads to the so-called Byrnes–Isidori *normal form* for the system, see [Isi95]. This normal form has new system matrices $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ which have nice structural properties in view of decoupling the system's input and output. Therefore the normal form of a system assists the design of suitable control strategies for the system.

In Section 3.3 a new normal form for linear MIMO-systems with non-strict (vector) relative degree is presented. This normal form generalizes the well known results for linear SISO-systems and linear MIMO-systems with strict relative degree. For systems with non-strict relative degree the construction of the normal form and the underlying proofs become much more involved than for SISO-systems, however, all the structural properties as in the case of SISO-systems are preserved.

One benefit of the Byrnes–Isidori normal form is that structural characteristics of the system can be read off very easily from the normal form. For example, the system's *zero dynamics* and *right-invertibility* can be characterized in terms of the normal form. Having a simple characterization of the system's zero dynamics and, therefore, having a straightforward specification of stability of the zero dynamics is crucial for the control strategies which are considered in this thesis. Linear systems with exponentially stable zero dynamics are called *minimum phase*. A linear system is minimum phase if, and only if, the associated transfer function has no zeros in the closed right half complex plane $\overline{\mathbb{C}}_+$, see, for example, [Oga02, p. 509].

The control strategies, considered in this thesis, require, additionally to the minimum phase property, two more structural assumptions to the system: the relative degree must be known (in case of the controller presented in Chapter 3 or must be one (in case of the controllers in Chapters 4 and 5) and the so-called *high-frequency gain* must be positive (positive definite in case of MIMO-systems).

Control strategies: derivative output feedback.

First, high-gain output derivative feedback stabilization is introduced. This means that the input u is designed as

$$u(t) = -\sum_{i=0}^{r-1} k_{i+1} y^{(i)}(t)\,,$$

where $k_1, \ldots, k_r \in \mathbb{R}$ are suitable parameters and $r \in \mathbb{N}$. Suppose the system's matrices are unknown but the system has the following structural properties: its relative degree is r, it is minimum phase and has positive *high-frequency gain*. Then one can show that there exist parameters $k_1, \ldots, k_r$ such that the above controller applied to a linear system stabilizes the system in the sense that the solution x of the controlled system is asymptotically stable. This approach is well known for linear SISO-systems and MIMO-systems with strict relative degree, see, for example, [Isi95, Isi99], however, a new result for linear MIMO-systems with non-strict relative degree is given in Section 3.3. The reader will find that stabilization results for SISO-systems and MIMO-systems with strict relative degree are also slightly improved, when compared to the results in the literature.

For the above output derivative feedback the parameters $k_1, \ldots, k_r$ are fixed and could be very large. One can show existence of such parameters but it is not straight forward to determine the parameters for an example system with unknown system matrices. Therefore, for systems with relative degree one, adaptive control strategies have been developed which will derive suitable "design parameter" k on its own. One possible strategy is to design the input as

$$u(t) = -k(t)\, y(t)\,, \qquad \dot{k}(t) = \|y(t)\|^2\,, \qquad k(0) = k^0\,,$$

where $k^0 \in \mathbb{R}$, see the seminal work of [Mar84, Mor83, WB84] and also the survey [Ilc91]. However, this controller has the shortcomings that, if tracking is the control objective, it needs to be combined with an internal model (thus becoming much more involved) and, more importantly, fails for stabilizing in the presence of additive arbitrarily small input or output L^∞-disturbances.

Control strategies: λ-tracking.

The adaptive λ-tracker, which was first introduced by [IR94], is presented in Chapter 4. One designs the input u as follows:

$$u(t) = -k(t)\,y(t)\,, \qquad \dot{k}(t) = \operatorname{dist}([-\lambda,\lambda], y(t))\|y(t)\|\,, \qquad k(0) = k^0\,,$$

where $\lambda > 0$, $k^0 \in \mathbb{R}$ and $\operatorname{dist}([-\lambda,\lambda], e) = \max\{0, \|e\| - \lambda\}$ for $e \in \mathbb{R}^m$. Note that λ-tracking requires that a system has to have relative degree one, which is more restrictive than in the case of output derivative feedback stabilization. The other requirements, i.e. that the system is minimum phase and has positive high-frequency gain, remain also for λ-tracking.

λ-tracking has two significant shortcomings: (i) the tracking error $\lambda > 0$ will only be achieved asymptotically and (ii) the new dynamic k is monotonically increasing.

To overcome this, funnel control for linear minimum phase systems with relative degree one and positive high-frequency gain was introduced in the control literature, see [IRS02b].

Control strategies: funnel control.

Funnel control is presented in Chapter 5. Applying the funnel controller

$$u(t) = -k(t)\,y(t)\,, \qquad k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y(t)\|}\,,$$

where $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a suitable function, to a linear system which satisfies the above structural assumptions achieves better tracking results as λ-tracking and the "design parameter" k is not necessarily increasing. Intuitively speaking, the funnel controller works as follows: if $\|y(t)\| \to 1/\varphi(t)$, i.e. the output approaches the funnel boundary, then $k(t)$ becomes large such that the intrinsic high-gain property of the system precludes boundary contacts.

All the control strategies introduced above achieve some stability results when applied to linear systems which satisfy certain structural requirements. However: 'Are these strategies *robust*?' which directly leads to the question: 'What does robustness in this context mean?' and finally: 'How can robustness be measured?'

The concept of the gap metric.

An answer to the latter question gives the robustness analysis of feedback systems by T. Georgiou and M. Smith, see [GS93, GS97]. It is possible to measure the *gap metric* between systems or controllers. Chapter 6 provides the required terminology for an application of robustness results which are based on the gap metric.

Actually the gap metric does not measure distances between systems but measures the distance between the graphs of systems; the graph of a system is, loosely speaking, the set of all input and output trajectories from a suitable function space which satisfy the equations of a linear system (A, B, C) or also a controller.

The approach of measuring distances between subspaces as graphs of operators to obtain stability theorems goes back to works of J. C. Gohberg and M. G. Krein, H. O. Cordes and J. P. Labrousse, and T. Kato, see [Kat76, Sec. IV.2.1] and the references therein ([Kat76, p. 197]).

Figure 1.1 may give an idea of this approach. There the directed gap between the subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathbb{R}^3$ is illustrated. For finite vector spaces, the directed gap is defined as

$$\vec{\delta}(\mathcal{V}_1, \mathcal{V}_2) := \sup_{v_1 \in \mathcal{V}_1,\, \|v_1\|=1} \operatorname{dist}^*(v_1, \mathcal{V}_2)\,,$$

where $\operatorname{dist}^*(y, X) := \inf\{\|y - x\| \mid x \in X\}$ for some $y \in \mathbb{R}^n$ and $X \subset \mathbb{R}^n$. In general the directed gap is not symmetric. Therefore, define the gap of two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ as

$$\delta(\mathcal{V}_1, \mathcal{V}_2) := \max\left\{\vec{\delta}(\mathcal{V}_1, \mathcal{V}_2), \vec{\delta}(\mathcal{V}_2, \mathcal{V}_1)\right\}\,.$$

Note that since graphs of systems are subsets of infinite dimensional function spaces the definition for the gap between graphs becomes much

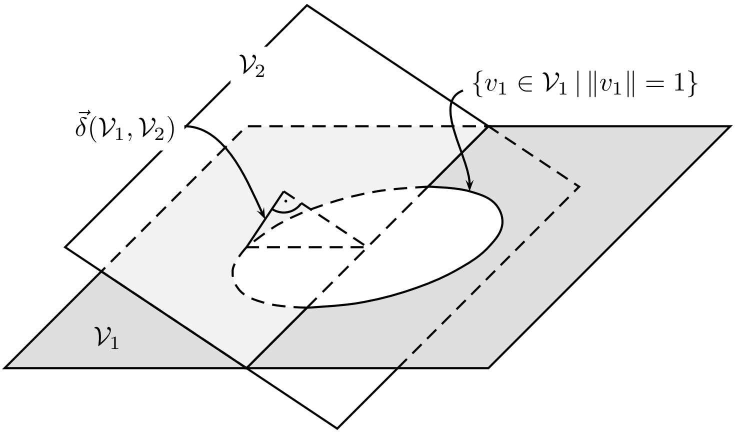


Figure 1.1: Directed gap from [Kat76, Sec. IV.2.1]

more involved, see Section 6.3.

The essential idea of robustness analysis is, loosely speaking, the following: if a system P, for which one knows that a controller C stabilizes the system, is close to another system P_1, in the sense that the gap $\delta(P, P_1)$ is sufficiently small, then the controller C stabilizes also the system P_1.

With robust stability result which are based on measuring the gap between systems one can answer the questions on robust stability of the control strategies considerer in the present paper.

First one should answer the question what robustness means for the considered controllers:

Robustness of output derivative feedback.

For high-gain output derivative feedback stabilization, one natural matter of robustness analysis is the presence of derivatives. The derivatives of the output have to be measured or approximated in some ways. So a robustness analysis for derivative feedback stabilization is to replace the derivatives $y^{(1)}, \ldots, y^{(r-1)}$ by approximations of the derivatives. This is done in Chapter 7: the *delay* feedback

$$u(t) = -\sum_{i=0}^{r-1} k_{i+1}(\Delta_h^i y)(t)\,,$$

where, for $h > 0$, $\Delta_h^0 y = y$ and $\Delta_h^i y$, for $i \geq 1$, denotes the Euler approximation of the ith derivative of y (see the definition in Section 7.2), is considered and robust stability results are presented: for sufficiently small $h > 0$ the delay feedback stabilizes any linear system which can be stabilized by the appropriate derivative feedback. This seems to be a reasonable result, however, in the literature one cannot find any results for systems with relative degree $r \geq 3$. for linear systems with relative degree 2 one can find a stability result in [IS04], however the authors do not utilize results on robust stability which are based on the gap metric.

It is important to note that the main result in Chapter 7, i.e. robust stability of the output derivative feedback controller, is actually much more general than a proof that linear systems can be stabilized by output delay feedback control. It is shown that every linear or even nonlinear system P which can be stabilized (in some sense, namely in terms of gain stability, see Section 6.4) by output derivative feedback also can be stabilized by output delay feedback for sufficiently small delay $h > 0$.

Robustness of λ-tracking and funnel control.

For λ-tracking and funnel control questions on robust stability are different as for output derivative feedback: derivatives of the output are not involved in this control strategies. A natural question is how robust the control strategies are when omitting several structural properties of the linear system (A, B, C). In other words, can the λ-tracker and funnel controller be applied to systems which are not necessarily minimum phase, have higher relative degree and negative high-frequency gain, such that the controlled systems achieve similar stability results as for systems which satisfy the classical assumptions for λ-tracking and funnel control.

Chapters 8 and 9 give positive answers to these problems. Here it is shown that, in presence of sufficiently small initial values and input/output disturbances, the closed-loop system of any stabilizable and detectable linear system (A, B, C) and λ-tracker or funnel controller, respectively, achieves robust stability if the system is sufficiently close – in sense of a small gap metric – to a minimum phase system which has relative degree one and positive high-frequency gain.

2 Linear systems: relative degree and normal form

In this chapter definitions and characterizations for the relative degree of linear systems with one-dimensional or multidimensional input/output are given, see Section 2.1 for single-input single-output (SISO) systems and Section 2.2 for systems with m inputs and m outputs (MIMO). Given the relative degree of a SISO- or MIMO-system a normal form is constructed. The normal forms for SISO- and MIMO-systems are not only structurally simple but allow characterization of the systems' zero dynamics for the design of feedback controllers. These characterizations in terms of the normal forms are given in Section 2.3.

2.1 SISO-systems

The relative degree of (nonlinear) systems with one-dimensional input and output is well studied in literature, see [Isi95]. The concept of the relative degree goes back to linear systems theory in the frequency domain. A linear single-input single-output (SISO) system may be described in the frequency domain by

$$q(s)Y(s) = p(s)U(s)\,, \tag{2.1.1}$$

for polynomials $p, q \in \mathbb{R}[s]$ where Y and U are the Laplace transforms of scalar input/output functions $u, y\colon \mathbb{R} \to \mathbb{R}$, respectively. The difference $r = \deg\, q - \deg\, p$ is called relative degree of the system (2.1.1).

A lot of concepts in systems and control theory are well understood in the frequency domain but often it is worth to have a look at different approaches. All results of this thesis will deal with systems in the time domain. If $r \geq 1$, a realization of $p(s)/q(s)$ in the time domain is given by

$$\left.\begin{aligned} \dot{x} &= A\,x + b\,u \\ y &= c\,x\,, \end{aligned}\right\} \tag{2.1.2}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c^T \in \mathbb{R}^n$.

Hence $q(s)/p(s) = c(sI_n - A)^{-1}b = \sum_{k=0}^{\infty} cA^k b s^{-(k+1)}$. Thus, it is easy to see that

$$\begin{aligned} &r = \deg q - \deg p \quad \text{ if, and only if,} \\ &\qquad cA^k b = 0, \text{ for all } k = 0, \ldots, r-2, \text{ and } cA^{r-1}b \neq 0. \end{aligned} \tag{2.1.3}$$

On the other hand, Isidori writes in [Isi95, Ch. 4.1] that a (nonlinear) SISO-system has relative degree r if "r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t^0$ in order to have the value $u(t^0)$ of the input explicitly appearing."

In other words (see also [LMS02, Def. 2] and [IM07, Def. 2.2]): defining functions, for $k \in \mathbb{N}$,

$$\begin{aligned} H_0 &: \mathbb{R}^n \to \mathbb{R}\,, \qquad\qquad x \mapsto cx\,, \\ H_{k+1} &: \mathbb{R}^n \times \mathbb{R}^{k+1} \to \mathbb{R}\,, \\ &\quad (x, u_0, \ldots, u_k) \mapsto \tfrac{\partial H_k}{\partial x}(Ax + bu_0) + \textstyle\sum_{j=0}^{k-1} \tfrac{\partial H_k}{\partial u_j} u_{j+1}\,. \end{aligned}$$

Then $y(t) = H_0(t)$ and $y^{(k)}(t) = H_k\big(x(t), u(t), \ldots, u^{(k-1)}(t)\big)$ for all $k \in \mathbb{N}$, and (2.1.2) has relative degree $r \in \mathbb{N}$ if, and only if,

$$\left.\begin{aligned} &\text{(i) } \forall\, k = 1, \ldots, r-1\ \forall\, i = 0, \ldots, k-1 \\ &\quad\ \forall\,(x, u_0, \ldots, u_{k-1}) \in \mathbb{R}^n \times \mathbb{R}^k : \tfrac{\partial H_k}{\partial u_i}(x, u_0, \ldots, u_{k-1}) = 0\,, \\ &\text{(ii) } \forall\,(x, u_0, \ldots, u_{r-1}) \in \mathbb{R}^n \times \mathbb{R}^r : \tfrac{\partial H_r}{\partial u_0}(x, u_0, \ldots, u_{r-1}) \neq 0\,. \end{aligned}\right\} \tag{2.1.4}$$

This definition for the relative degree is taken from results about nonlinear time-varying SISO-systems [IM07, Def. 2.2]. For linear time invariant systems (A, b, c) of form (2.1.2) one can simplify the above. Note that, for $r \geq 2$,

$$\begin{aligned} \tfrac{\partial H_1}{\partial u_0}(x, u_0) &= \tfrac{\partial}{\partial u_0}(cAx + cbu_0) = cb = 0\,, \\ \tfrac{\partial H_2}{\partial u_0}(x, u_0, u_1) &= \tfrac{\partial}{\partial u_0}(cA^2x + cAbu_0 + cbu_1) = cAb = 0\,, \end{aligned}$$

and so forth, which equals the relative degree characterization (2.1.3) and also leads to the following definition:

Definition 2.1.1 *A linear system* (A, b, c) *of form* (2.1.2) *has* relative degree $r \in \mathbb{N}$ *if, and only if,*

(i) $\forall\, k \in \{0, \dots, r-2\} \;:\; cA^k b = 0,$

(ii) $cA^{r-1} b \neq 0.$

The relative degree of a system leads to a normal form: the Byrnes–Isidori normal form, which was introduced in [BI84] for nonlinear and linear SISO-systems. For linear SISO-systems of form (2.1.2) the normal form is well known, see [IRT07], and is implicitly contained in [Isi95, Ch. 4.1].

Lemma 2.1.2 *[IRT07, Lem. 3.5] Consider a linear system* (A, b, c) *of form* (2.1.2) *with relative degree* $r \in \mathbb{N}$*. Then there exists an invertible matrix* $U \in \mathbb{R}^{n \times n}$ *such that the coordinate transformation* $\binom{\xi}{\eta} = Ux$ *converts* (A, b, c) *into*

$$\left.\begin{aligned} \tfrac{d}{dt}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \underbrace{\left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ R_1^1 & \dots & & R_r^1 & S^1 \\ \hline P_1 & 0 & \dots & 0 & Q \end{array}\right]}_{=:\widetilde{A}} \begin{pmatrix}\xi\\ \eta\end{pmatrix} + \underbrace{\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ cA^{r-1}b \\ \hline 0 \end{array}\right]}_{=:\widetilde{b}} u \\ y &= \underbrace{[1, 0, \dots, 0]}_{=:\widetilde{c}} \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \end{aligned}\right\} \tag{2.1.5}$$

where $R_1^1, \dots, R_r^1 \in \mathbb{R}$, $S^1 \in \mathbb{R}^{1 \times n-r}$, $P_1 \in \mathbb{R}^{n-r}$ *and* $Q \in \mathbb{R}^{(n-r)\times(n-r)}$ *can be presented explicitly in terms of the system matrices* A, b *and* c.

Proof. Define matrices

$$\mathcal{C} := \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{r-1} \end{bmatrix} \in \mathbb{R}^{r \times n}, \qquad \mathcal{B} := [b, Ab, \dots, A^{r-1}b] \in \mathbb{R}^{n \times r}.$$

Then by Definition 2.1.1

$$\mathcal{C}\mathcal{B} = \begin{bmatrix} cb & \dots & cA^{r-2}b & cA^{r-1}b \\ cAb & \dots & cA^{r-1}b & cA^r b \\ \vdots & \iddots & \vdots & \\ cA^{r-1}b & cA^r b & \dots & cA^{2r-1}b \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & cA^{r-1}b \\ \vdots & \iddots & cA^{r-1}b & cA^r b \\ 0 & \iddots & \iddots & \vdots \\ cA^{r-1}b & cA^r b & \dots & cA^{2r-1}b \end{bmatrix}. \tag{2.1.6}$$

Thus $\operatorname{rk} \mathcal{C}\mathcal{B} = r$ and therefore $\mathcal{C}$ and $\mathcal{B}$ have full rank r. To complete $\mathcal{C}$ to a basis transformation of $\mathbb{R}^n$ one may choose a matrix $\mathcal{V} \in \mathbb{R}^{n \times r}$ of full rank with $\operatorname{im} \mathcal{V} = \ker \mathcal{C}$. Define matrices

$$U := \left[\frac{\mathcal{C}}{\mathcal{N}}\right], \qquad \mathcal{N} := (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T[I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}].$$

It is easy to see that $U^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} | \mathcal{V}]$. Note that

$$\left[\frac{\mathcal{C}}{\mathcal{N}}\right] b = \begin{bmatrix} cb \\ \vdots \\ cA^{r-2}b \\ cA^{r-1}b \\ \hline (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T[b - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}b] \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ cA^{r-1}b \\ \hline (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T[b - b] \end{bmatrix},$$

$$c\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \middle| \mathcal{V}\right] = \left[1, 0, \dots, 0 \middle| c\mathcal{V}\right],$$

which yields $\widetilde{b}$ and $\widetilde{c}$ in (2.1.5). Furthermore,

$$\left[\frac{\mathcal{C}}{\mathcal{N}}\right] A\left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \middle| \mathcal{V}\right] = \left[\begin{array}{c|c} \mathcal{C}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} & \mathcal{C}A\mathcal{V} \\ \hline \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} & \mathcal{N}A\mathcal{V} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ R_1^1 & \dots & & R_r^1 & S^1 \\ \hline P_1 & P_2 & \dots & P_3 & Q \end{array}\right],$$

where $[R_1^1, \dots, R_r^1 | S^1] = cA^r[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} | \mathcal{V}]$, $[P_1, \dots, P_r] = \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}$ and $Q = \mathcal{N}A\mathcal{V}$.

Recall that $\mathcal{N}\mathcal{B} = 0_{r\times r}$. By (2.1.6) follows

$$\begin{aligned}[P_1, \dots, P_r] &= \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \\ &= [0, \dots, 0, \mathcal{N}A^r b] \begin{bmatrix} * & & (cA^{-1}b)^{-1} \\ & \cdot^{\cdot^{\cdot}} & \\ (cA^{-1}b)^{-1} & & 0 \end{bmatrix},\end{aligned}$$

hence $P_2 = \cdots = P_r = 0_{(n-r)\times 1}$, which yields $\widetilde{A}$ in (2.1.5) and completes the proof. □

The Byrnes–Isidori normal form for nonlinear/linear SISO-systems is widely used in control theory for the design of local and global feedback stabilization of nonlinear systems [BI85, BI88, BI89], for the design of adaptive observers [NT89], for the design of adaptive controllers [IT93, IR94], to name but a few applications. Thus constructing of a normal form for linear MIMO-systems will assist the design of controllers and observers for this systems. For this a characterization of the (vector) relative degree of linear MIMO-systems in sense of Definition 2.1.1 is required which is given in the following section. Moreover, a normal form for linear MIMO-systems is presented.

2.2 MIMO-systems

In this section linear systems with m inputs and m outputs of the form

$$\left.\begin{aligned} \dot{x} &= Ax + \underbrace{\left[b_1^{(n)}, \dots, b_m^{(n)}\right]}_{=B} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}}_{=u} \\ \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{=y} &= \underbrace{\begin{bmatrix} c_{(n)}^1 \\ \vdots \\ c_{(n)}^m \end{bmatrix}}_{=C} x \end{aligned}\right\} \tag{2.2.1}$$

are considered, where $n, m \in \mathbb{N}$ with $m \le n$ and $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$.

2.2.1 Relative degree

As observed in the previous section a linear single-input single-output system (2.1.2), i.e. system (2.2.1) with $m = 1$, has relative degree $r \in \mathbb{N}$ if, and only if, r is the least number of times one has to differentiate the output to have the input appear explicitly, see the characterization (2.1.4).

In case of MIMO-system (2.2.1) one can consider the SISO-system relating input u_j to output y_i, for all $(i, j) \in \{1, \dots, m\} \times \{1, \dots, m\}$, given by

$$\left.\begin{array}{rcl} \dot{x} & = & Ax + b_j^{(n)} u_j \\ y_i & = & c_{(n)}^i x\,, \qquad\qquad i, j \in \{1, \dots, m\}\,. \end{array}\right\} \tag{2.2.2}$$

Let $r_{i,j} \in \mathbb{N}$ be the relative degree of (2.2.2). Then, for $i \in \{1, \dots, m\}$, $r_i := \min_{j \in \{1,\dots,m\}} r_{i,j}$ is the least number one has to differentiate the i-th output to have at least one of the m inputs appear explicitly in the sense of (2.1.4). The vector $(r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$ is called the vector relative degree of the MIMO-system (2.2.1) if, for all $j \in \{1, \dots, m\}$, the rows $c_{(n)}^j A^{r_j - 1} B$ are linearly independent, see Definition 2.2.1(a).

Isidori [Isi95] presents a local definition of the vector relative degree for nonlinear MIMO-systems.

Liberzon et al. [LMS02] give a generalization of the definition of the relative degree for time-invariant nonlinear systems which is extended in [IM07] for time-varying linear and nonlinear systems. However in these papers only SISO-systems and MIMO-systems with strict relative degree (see also Definition 2.2.1(c)) are considered. To the author's best knowledge, Isidori [Isi95] is the only reference in the literature where a definition for the vector relative degree for (nonlinear) systems can be found.

For linear MIMO-systems the (vector) relative degree is defined as follows:

Definition 2.2.1 *A linear system* (A, B, C) *of form* (2.2.1) *has*

(a) (vector) relative degree $r = (r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$ *if, and only if,*

(i) $\forall\, j \in \{1, \dots, m\}\ \forall\, k \in \{0, \dots, r_j - 2\}\ :\ c_{(n)}^j A^k B = 0_{1\times m}$*,*

$$(ii)\ rk \begin{bmatrix} c^1_{(n)} A^{r_1-1} B \\ c^2_{(n)} A^{r_2-1} B \\ \vdots \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix} = m.$$

(b) ordered (vector) relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ *if, and only if,* (2.2.1) *has (vector) relative degree* $r = (r_1, \ldots, r_m)$ *with* $r_1 \geq r_2 \geq \ldots \geq r_m$.

(c) strict relative degree $\varrho \in \mathbb{N}$ *if, and only if,* (2.2.1) *has (vector) relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ *with* $\varrho = r_1 = r_2 = \ldots = r_m$.

Remark 2.2.2 *(i) Note that Definition 2.2.1(a) coincides with the definition of the vector relative degree for nonlinear systems with multidimensional inputs and outputs, see [Isi95, Ch. 5.1].*

(ii) The linear independence of the rows $c^j_{(n)} A^{r_j-1} B$*, although a quite restrictive requirement, is significant for the construction of a coordinate transformation and with it the normal form. For systems that do not satisfy both conditions in Definition 2.2.1(a) the vector relative degree does not exist and thus one cannot construct the normal form* (2.2.3)–(2.2.4).

(iii) Note that in the literature sometimes the relative degree is called uniform instead of strict.

The following lemma shows that it is not restrictive to consider systems with ordered vector relative degree. It is shown that any linear MIMO-systems (A, B, C) of form (2.2.1) with arbitrarily vector relative degree $r \in \mathbb{N}^{1\times m}$ can easily be transformed in a system with ordered relative degree $\widetilde{r} = (\widetilde{r}_1, \ldots, \widetilde{r}_m) \in \mathbb{N}^{1\times m}$ by a permutation of the output signal.

Lemma 2.2.3 *Let* (A, B, C) *be a linear system of form* (2.2.1) *with vector relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$*. Then there exists a permutation matrix* $P \in \mathbb{R}^{m\times m}$ *such that the system* (A, B, PC) *has ordered vector relative degree* $rP = (\widetilde{r}_1, \ldots, \widetilde{r}_m)$.

Proof. Let $\sigma\colon \{1,\dots,m\} \to \{1,\dots,m\}$ be a permutation such that $r_{\sigma(1)} \geq r_{\sigma(2)} \geq \cdots \geq r_{\sigma(m)}$. Furthermore set

$$P := \begin{bmatrix} e_{(m)}^{\sigma(1)} \\ \vdots \\ e_{(m)}^{\sigma(m)} \end{bmatrix}.$$

Then

$$PC = \begin{bmatrix} e_{(m)}^{\sigma(1)} \\ \vdots \\ e_{(m)}^{\sigma(m)} \end{bmatrix} \begin{bmatrix} c_{(m)}^{1} \\ \vdots \\ c_{(m)}^{m} \end{bmatrix} = \begin{bmatrix} c_{(m)}^{\sigma(1)} \\ \vdots \\ c_{(m)}^{\sigma(1)} \end{bmatrix},$$

and by the assumption on the relative degree it follows that

$$\forall\, j \in \{1,\dots,m\}\ \forall\, k \in \{0,\dots,r_{\sigma(j)}-2\}\ : \quad (PC)_{(n)}^{j} A^k B = c_{(n)}^{\sigma(j)} A^k B = 0_{1\times m}\,,$$

and

$$\operatorname{rk} \begin{bmatrix} (PC)_{(n)}^{1} A^{r_1-1} B \\ \vdots \\ (PC)_{(n)}^{m} A^{r_m-1} B \end{bmatrix} = m\,.$$

This shows that the linear system (A, B, PC) has relative degree $Pr = (r_{\sigma(1)},\dots,r_{\sigma(m)})$ with $r_{\sigma(1)} \geq \cdots \geq r_{\sigma(m)}$ and the proof is complete. □

2.2.2 Normal form

Isidori [Isi95, Ch. 5] presents a local normal form for nonlinear MIMO-systems. In [Isi99, Ch. 11] a proof is given to specify the diffeomorphism to produce the normal form in terms of the system data of the nonlinear system. Moreover, for nonlinear systems that satisfy certain assumptions, namely commutativity of certain vector fields which is automatically satisfied for linear systems, Isidori [Isi99, Prop. 11.5.2] gives a normal form which coincides with the normal form for linear systems given in the present work. However, the corresponding results for linear systems cannot be found in literature. One could translate the nonlinear results for linear systems, but the machinery for nonlinear systems, e.g. Lie-derivatives of smooth functions, is not necessary to prove the linear results. The proof given in the present work only uses standard linear algebra. The transformation matrix is given in terms of the linear system matrices A, B and C and leads to “many zeros and ones” in the normal form and allows to read off the zero dynamics very easily; the reader will find that the normal form (2.2.3) for linear MIMO-systems is, roughly speaking, structured as a “diagonal form of m copies of SISO normal forms (2.1.5)”. Furthermore the matrices of the normal form and transformation will be characterized explicitly by the system matrices.

The following theorem presents a normal form for linear MIMO-systems (A, B, C) of form (2.2.1) with ordered vector relative degree. The normal form has similar structural properties as the normal form for linear SISO-systems and linear MIMO-systems with strict relative degree, respectively, see (2.1.5).

Theorem 2.2.4 *(i) Consider a linear system* (A, B, C) *of form* (2.2.1) *with ordered vector relative degree* $r = (r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$. *Set* $r^s := \sum_{j=1}^m r_j$. *Then there exists an invertible matrix* $U \in \mathbb{R}^{n\times n}$ *such that the coordinate transformation* $\left(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\right) := Ux$ *with*

$$\xi(t) = \Big(y_1(t), \dots, y_1^{(r_1-1)}(t)\Big| \dots \Big| y_m(t), \dots, y_m^{(r_m-1)}(t)\Big)^T \in \mathbb{R}^{r^s},$$
$$\eta(t) \in \mathbb{R}^{n-r^s},$$

converts (A, B, C) *into*

$$\left.\begin{aligned} \tfrac{d}{dt}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \widetilde{A}\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \widetilde{B}u \\ y &= \widetilde{C}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \end{aligned}\right\} \tag{2.2.3}$$

where

$$
\left[\begin{array}{c|c} \widetilde{A} & \widetilde{B} \\ \hline \widetilde{C} & 0 \end{array}\right] =
$$

$$
\left[\begin{array}{c|c}
\left[\begin{array}{ccc|ccc|c|ccc|c}
0 & 1 & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1\times(n-r^s)} \\
\vdots & \ddots & \ddots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots\ 0 & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1\times(n-r^s)} \\
R^1_{1,1} & \cdots & R^1_{1,r_1} & R^1_{2,1} & \cdots & R^1_{2,r_2} & & R^1_{m,1} & \cdots & R^1_{m,r_m} & S^1 \\ \hline
0 & \cdots & 0 & 0 & 1 & 0 & & 0 & \cdots & 0 & 0_{1\times(n-r^s)} \\
\vdots & & \vdots & \vdots & \ddots & \ddots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots\ 0 & 1 & & 0 & \cdots & 0 & 0_{1\times(n-r^s)} \\
R^2_{1,1} & \cdots & R^2_{1,r_1} & R^2_{2,1} & \cdots & R^2_{2,r_2} & & R^2_{m,1} & \cdots & R^2_{m,r_m} & S^2 \\ \hline
 & \vdots & & & & & \ddots & & & & \vdots \\ \hline
0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & 1 & 0 & 0_{1\times(n-r^s)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots\ 0 & 1 & 0_{1\times(n-r^s)} \\
R^m_{1,1} & \cdots & R^m_{1,r_1} & R^m_{2,1} & \cdots & R^m_{2,r_2} & & R^m_{m,1} & \cdots & R^m_{m,r_m} & S^m \\ \hline
P_1 & 0\ \cdots & 0 & P_2 & 0\ \cdots & 0 & \cdots & P_m & 0\ \cdots & 0 & Q
\end{array}\right]
&
\left[\begin{array}{c}
0_{1\times m} \\ \vdots \\ 0_{1\times m} \\ c^1_{(n)}A^{r_1-1}B \\ \hline
0_{1\times m} \\ \vdots \\ 0_{1\times m} \\ c^2_{(n)}A^{r_2-1}B \\ \hline
\vdots \\ \hline
0_{1\times m} \\ \vdots \\ 0_{1\times m} \\ c^m_{(n)}A^{r_m-1}B \\ \hline
0_{(n-r^s)\times m}
\end{array}\right]
\\ \hline
\left[\begin{array}{ccc|ccc|c|ccc|c}
1 & 0\ \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & \\
0 & \cdots & 0 & 1 & 0\ \cdots & 0 & & \vdots & & \vdots & \\
\vdots & & \vdots & 0 & \cdots & 0 & \cdots & & & & 0_{m\times(n-r^s)} \\
 & & & \vdots & & \vdots & & 0 & \cdots & 0 & \\
0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & 0\ \cdots & 0 &
\end{array}\right]
& 0
\end{array}\right]
\tag{2.2.4}
$$

and $R^j_{i,k} \in \mathbb{R}$, *for* $i, j \in \{1, \ldots, m\}$ *and* $k \in \{1, \ldots, r_i\}$, $S^1, \ldots, S^m \in \mathbb{R}^{1\times(n-r^s)}$, $P_1, \ldots, P_m \in \mathbb{R}^{n-r^s}$ *and* $Q \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$.

(ii) In the following, the entries of the matrices in (2.2.4) *and the entries of the transformation matrix* U *are expressed explicitly in terms of the system matrices* A, B *and* C*: Set*

$$
m_i := \#\{r_j \mid r_j \geq i\,,\ j \in \{1, \ldots, m\}\}\,, \quad i \in \{1, \ldots, r_1\}\,, \tag{2.2.5}
$$

the number of r_j*'s,* $j \in \{1, \ldots, m\}$*, such that* $r_j \geq i$*, and define*

$$
\Gamma := \begin{bmatrix} c^1_{(n)}A^{r_1-1}B \\ \vdots \\ c^m_{(n)}A^{r_m-1}B \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{2.2.6}
$$

$$\mathcal{C} := \left[\begin{array}{c} c^1_{(n)} \\ \vdots \\ c^1_{(n)}A^{r_1-1} \\ \hline c^2_{(n)} \\ \vdots \\ c^2_{(n)}A^{r_2-1} \\ \hline \vdots \\ \hline c^m_{(n)} \\ \vdots \\ c^m_{(n)}A^{r_m-1} \end{array}\right] \in \mathbb{R}^{r^s \times n} \tag{2.2.7}$$

$$\mathcal{B} := \Big[B\Gamma^{-1}\left[e^{(m)}_1, \ldots, e^{(m)}_{m_1}\right], AB\Gamma^{-1}\left[e^{(m)}_1, \ldots, e^{(m)}_{m_2}\right], \ldots, A^{r_1-1}B\Gamma^{-1}\left[e^{(m)}_1, \ldots, e^{(m)}_{m_{r_1}}\right]\Big] \in \mathbb{R}^{n \times r^s} \tag{2.2.8}$$

$$\mathcal{V} \in \mathbb{R}^{n \times (n-r^s)} \;:\; im\mathcal{V} = ker\mathcal{C}, \quad and \quad rk\mathcal{V}^T\mathcal{V} = n - r^s \tag{2.2.9}$$

$$\widehat{U} := \left[\begin{array}{c} \mathcal{C} \\ \hline \mathcal{N} \end{array}\right] \in \mathbb{R}^{n \times n} \quad and \quad \mathcal{N} := (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right] \tag{2.2.10}$$

$$T_i := \left[\begin{array}{c|c|c|c} 0_{(r_i+n-r^s)\times(\sum_{j=1}^{i-1} r_j)} & \begin{array}{c} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{array} & 0_{(r_i+n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & \begin{array}{c} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{array} \end{array}\right] \in \mathbb{R}^{(r_i+n-r^s)\times n} \tag{2.2.11}$$

$$\widehat{\mathcal{C}}_i := \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \in \mathbb{R}^{r_i \times (r_i+n-r^s)} \tag{2.2.12}$$

$$\widehat{\mathcal{B}}_i := \Big[e^{(r_i+n-r^s)}_{r_i}, \left(T_i\widehat{U}A\widehat{U}^{-1}T_i^T\right)e^{(r_i+n-r^s)}_{r_i}, \tag{2.2.13}$$

$$\ldots, \left(T_i\widehat{U}A\widehat{U}^{-1}T_i^T\right)^{r_i-1}e^{(r_i+n-r^s)}_{r_i}\Big] \in \mathbb{R}^{(r_i+n-r^s)\times r_i} \tag{2.2.14}$$

$$\widehat{\mathcal{N}}_i := \left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \left[I_{r_i+n-r^s} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i\right] \qquad (2.2.15)$$
$$\in \mathbb{R}^{(n-r^s)\times(r_i+n-r^s)}$$

$$\widehat{U}_i := \left[\begin{array}{c|c} I_{r^s} & 0_{r^s\times(n-r^s)} \\ \hline 0_{(n-r^s)\times(\sum_{j=1}^{i-1} r_j)}, \widehat{\mathcal{N}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix}, 0_{(n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & I_{n-r^s} \end{array}\right]$$
$$\in \mathbb{R}^{n\times n}, \qquad (2.2.16)$$

for $i \in \{1,\ldots,m\}$, *and finally*

$$U := \widehat{U}_m \cdot \widehat{U}_{m-1} \cdot \ldots \cdot \widehat{U}_1 \cdot \widehat{U}\,. \qquad (2.2.17)$$

Then, for $i, j \in \{1,\ldots m\}$, *the entries in* (2.2.3) *are given by*

$$\left[R^i_{j,1},\ldots,R^i_{j,r_j}\right] = \left[c^i_{(n)}A^{r_i}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\begin{bmatrix} 0_{(\sum_{\mu=1}^{j-1} r_\mu)\times r_j} \\ I_{r_j} \\ 0_{(\sum_{\mu=j+1}^{m} r_\mu)\times r_j}\end{bmatrix} + c^i_{(n)}A^{r_i}\mathcal{V}\left[0_{(n-r^s)\times r_j}, I_{n-r^s}\right]\widehat{\mathcal{B}}_j(\widehat{\mathcal{C}}_j\widehat{\mathcal{B}}_j)^{-1}\right] \qquad (2.2.18)$$

$$S^i = c^i_{(n)}A^{r_i}\mathcal{V} \qquad (2.2.19)$$

$$[P_i, 0, \ldots, 0] = \widehat{\mathcal{N}}_i\left(T_i\widehat{U}A\widehat{U}^{-1}T_i^T\right)\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1} \qquad (2.2.20)$$

$$Q = \mathcal{N}A\mathcal{V} \overset{(2.2.10)}{=} (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right]A\mathcal{V}\,. \qquad (2.2.21)$$

It is easy to see that the normal form (2.1.5) for linear single-input single-output systems (A, b, c) of form (2.1.2) is contained in the normal form (2.2.3)–(2.2.4) explicitly: if $m = 1$ the matrix $\widetilde{A}$ reduces to the two left and right topmost and two left and right lowermost blocks of the original matrix. Moreover, $\widetilde{B}$ reduces to the first row of the topmost and lowermost blocks and $\widetilde{C}$ reduces to the topmost column of its left and right block.

For linear MIMO-systems with strict relative degree $r_1 = \cdots = r_m = r \in \mathbb{N}$ all $m_1, \ldots, m_{r_1}$ from (2.2.5) are equal to the dimension m of the input and output. Thus all $m \times m$ left upper blocks of $\widetilde{A}$ have dimension $r \times r$. Moreover, the lower row blocks $[P_i, 0, \ldots, 0] \in \mathbb{R}^{(n-rm)\times r}$ have the same dimensions for all $i \in \{1, \ldots, m\}$, as well as the right column blocks $\begin{bmatrix} 0_{1\times(n-r^s)} \\ \vdots \\ 0_{1\times(n-r^s)} \\ S^i \end{bmatrix} \in \mathbb{R}^{r\times(n-rm)}$ and $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$. By this observation, one may obtain the following well know normal form for linear MIMO-systems with strict relative degree. Note that the normal form matrices $(\overline{A}, \overline{B}, \overline{C})$ from Corollary 2.2.5 differ slightly from the normal form (2.2.3)–(2.2.4): they are structured as the normal form (2.1.5) for SISO-systems, which is more convenient for applications, see, for example, Theorem 3.2.1.

Corollary 2.2.5 *[IRT07, Lem. 3.5] Consider a linear system (A, B, C) of form* (2.2.1) *with strict relative degree $r \in \mathbb{N}$. Then there exists an invertible matrix $\overline{U} \in \mathbb{R}^{n\times n}$ such that the coordinate transformation $\binom{\xi}{\eta} = \overline{U}x$ converts (A, B, C) into*

$$\left.\begin{aligned} \tfrac{d}{dt}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \underbrace{\left[\begin{array}{cccc|c} 0 & I_m & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & I_m & 0 \\ R_1 & \ldots & & R_r & S \\ \hline P & 0 & \ldots & 0 & Q \end{array}\right]}_{=:\overline{A}} \begin{pmatrix}\xi\\ \eta\end{pmatrix} + \underbrace{\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ \hline 0 \end{array}\right]}_{=:\overline{B}} u \\ y &= \underbrace{[I_m, 0, \ldots, 0]}_{=:\overline{C}} \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \end{aligned}\right\} \tag{2.2.22}$$

where $R_1, \ldots, R_r \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times(n-rm)}$, $P \in \mathbb{R}^{(n-rm)\times m}$ and $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$ and the transformation matrix $\overline{U}$ can be presented *explicitly in terms of the system matrices A, B and C as follows:*

$$[R_1,\dots,R_r\,|\,S] = CA^r\left[\overline{\mathcal{B}}(\overline{\mathcal{C}}\overline{\mathcal{B}})^{-1}\middle|\overline{\mathcal{V}}\right], \tag{2.2.23}$$

$$[P,0,\dots,0] = \overline{\mathcal{N}}A\overline{\mathcal{B}}(\overline{\mathcal{C}}\overline{\mathcal{B}})^{-1}, \tag{2.2.24}$$

$$Q = \overline{\mathcal{N}}A\overline{\mathcal{V}}, \tag{2.2.25}$$

where

$$\overline{\mathcal{C}} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1}\end{bmatrix} \in \mathbb{R}^{rm\times n},$$

$$\overline{\mathcal{B}} := [B, AB, \dots, A^{r-1}B] \in \mathbb{R}^{n\times rm},$$

$$\overline{\mathcal{V}} \in \mathbb{R}^{n\times rm} \quad \textit{with} \quad \mathit{im}\,\overline{\mathcal{V}} = \mathit{ker}\,\overline{\mathcal{C}},$$

and

$$\overline{U} := \left[\frac{\overline{\mathcal{C}}}{\overline{\mathcal{N}}}\right], \qquad \overline{\mathcal{N}} := (\overline{\mathcal{V}}^T\overline{\mathcal{V}})^{-1}\overline{\mathcal{V}}^T[I_n - \overline{\mathcal{B}}(\overline{\mathcal{C}}\overline{\mathcal{B}})^{-1}\overline{\mathcal{C}}].$$

Proof. As observed above: for strict relative degree $r\in\mathbb{N}$, $r_1=\dots=r_m=r$ and by (2.2.5) follows $m_1=\dots=m_{r_1}=m$. Moreover, $r^s=rm$ and by (2.2.6)–(2.2.8) and (2.2.16)

$$\Gamma := CA^{r-1}B$$

$$\mathcal{C} = \left[\begin{bmatrix} c^1_{(n)} \\ \vdots \\ c^1_{(n)}A^{r-1}\end{bmatrix}\Big/\begin{bmatrix} c^2_{(n)} \\ \vdots \\ c^2_{(n)}A^{r-1}\end{bmatrix}\Big/\cdots\Big/\begin{bmatrix} c^m_{(n)} \\ \vdots \\ c^m_{(n)}A^{r-1}\end{bmatrix}\right] \in \mathbb{R}^{rm\times n}$$

$$\begin{aligned}\mathcal{B} &= \left[B\Gamma^{-1}I_m, AB\Gamma^{-1}I_m, \dots, A^{r_1-1}B\Gamma^{-1}I_m\right] \in \mathbb{R}^{n\times rm} \\ &= \left[B, AB, \dots, A^{r_1-1}B\right]\Gamma^{-1} = \overline{\mathcal{B}}\Gamma^{-1}\end{aligned}$$

$$\widehat{U}_i = \left[\begin{array}{c|c} I_{rm} & 0_{rm\times(n-rm)} \\ \hline 0_{(n-rm)\times(i-1)m},\ \widehat{\mathcal{N}}_i\begin{bmatrix} I_r \\ 0_{(n-rm)\times r}\end{bmatrix},\ 0_{(n-rm)\times(m-i)r} & I_{n-rm}\end{array}\right],$$

for $i \in \{1, \dots, m\}$. Choose a permutation matrix

$$\Pi := \left[\begin{array}{c} e^{1}_{(n)} \\ e^{r+1}_{(n)} \\ \vdots \\ e^{(m-1)r+1}_{(n)} \\ \hline e^{2}_{(n)} \\ e^{r+2}_{(n)} \\ \vdots \\ e^{(m-1)r+2}_{(n)} \\ \hline \vdots \\ \hline e^{r}_{(n)} \\ e^{2r}_{(n)} \\ \vdots \\ e^{mr}_{(n)} \\ \hline e^{mr+1}_{(n)} \\ e^{mr+2}_{(n)} \\ \vdots \\ e^{n}_{(n)} \end{array}\right] = \left[\begin{array}{c|c} \begin{array}{cccccccccccc} 1 & 0 & \dots & & & & & & & & & 0 \\ 0 & & \dots & 0 & 1 & 0 & \dots & & & & & 0 \\ \vdots & & & & & \ddots & & & & & & \vdots \\ 0 & & & & & & \dots & 0 & 1 & 0 & \dots & 0 \\ \hline 0 & 1 & 0 & \dots & & & & & & & & 0 \\ 0 & & & \dots 0 & 1 & 0 & \dots & & & & & 0 \\ \vdots & & & & & & \ddots & & & & & \vdots \\ 0 & & & & & & & \dots & 0 & 1 & 0 \dots & 0 \\ \hline & & & & & & \vdots & & & & & \\ \hline 0 & \dots & 0 & 1 & 0 \dots & & & & & & & 0 \\ 0 & & & & \dots & 0 & 1 & 0 & \dots & & & 0 \\ \vdots & & & & & & & \ddots & & & & \\ 0 & & & & & & & & & \dots & 0 & 1 \end{array} & 0_{rm\times(n-rm)} \\ \hline 0_{(n-rm)\times rm} & I_{n-rm} \end{array}\right],$$

with inverse

$$\begin{aligned} \Pi^{-1} &= \Pi^{T} \\ &= \left[e^{(n)}_{1}, e^{(n)}_{r+1}, \dots, e^{(n)}_{(m-1)r+1} \middle| \dots \middle| e^{(n)}_{r}, e^{(n)}_{2r}, \dots, e^{(n)}_{mr} \middle| e^{(n)}_{mr+1}, e^{(n)}_{mr+2}, \dots, e^{(n)}_{n} \right] \end{aligned}$$

and let

$$\Pi_1 := \left[I_{rm}, 0_{rm\times(n-rm)} \right] \Pi \begin{bmatrix} I_{rm} \\ 0_{(n-rm)\times rm} \end{bmatrix}.$$

For all $i \in \{1, \dots, m\}$ follows $\Pi \widehat{U}_i = \Pi$ and $\Pi_1 \mathcal{C} = \overline{\mathcal{C}}$. Thus, by (2.2.17) $\Pi U = \Pi \widehat{U}$, and (2.2.10) yields

$$\begin{aligned} \mathcal{N} &= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}\left[I_n - \overline{\mathcal{B}}\Gamma^{-1}\left(\Pi_1^{-1}\overline{\mathcal{C}\mathcal{B}}\Gamma^{-1}\right)^{-1}\Pi_1^{-1}\overline{\mathcal{C}} \right] \\ &= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}\left[I_n - \overline{\mathcal{B}}\Gamma^{-1}\Gamma\left(\overline{\mathcal{C}\mathcal{B}}\right)^{-1}\Pi_1\Pi_1^{-1}\overline{\mathcal{C}} \right] \end{aligned}$$

and, setting $\overline{\mathcal{V}} := \mathcal{V}$, for $\mathcal{V} \in \mathbb{R}^{n\times(n-rm)}$ from (2.2.9), gives $\overline{\mathcal{N}} = \mathcal{N}$ and thus

$$\overline{U} = \Pi\widehat{U} = \Pi U = \left[\begin{array}{c}\overline{\mathcal{C}}\\ \hline \overline{\mathcal{N}}\end{array}\right] \tag{2.2.26}$$

and it is easy to see that the coordinate transformation $\binom{\xi}{\eta} = \overline{U}x$ converts (A, B, C) into $(\overline{A}, \overline{B}, \overline{C}) = (\Pi\widetilde{A}\Pi^{-1}, \Pi\widetilde{B}, \widetilde{C}\Pi^{-1})$ in (2.2.22), where $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is given by (2.2.4).

It remains to show (2.2.23)–(2.2.25).

Note that for systems with strict relative degree $r \in \mathbb{N}$ the definition of $\widehat{U}_i$, $i \in \{1, \dots, m\}$, yields

$$\widehat{U}_m \cdot \widehat{U}_{m-1} \cdot \ldots \cdot \widehat{U}_1 = \left[\begin{array}{c|c} I_{rm} & 0_{rm\times(n-rm)} \\ \hline \widehat{\mathcal{N}}_1 \begin{bmatrix} I_r \\ 0_{(n-rm)\times r}\end{bmatrix}, \widehat{\mathcal{N}}_2 \begin{bmatrix} I_r \\ 0_{(n-rm)\times r}\end{bmatrix}, \dots, \widehat{\mathcal{N}}_m \begin{bmatrix} I_r \\ 0_{(n-rm)\times r}\end{bmatrix} & I_{n-rm}\end{array}\right],$$

By (2.2.26) and the invertibility of Π follows that $\widehat{U} = U = \widehat{U}_m \cdot \widehat{U}_{m-1} \cdot \ldots \cdot \widehat{U}_1 \cdot \widehat{U}$ and thus invertibility of $\widehat{U}$ yields $\widehat{U}_m \cdot \widehat{U}_{m-1} \cdot \ldots \cdot \widehat{U}_1 = I_n$. Hence, for all $i \in \{1, \dots, m\}$, $\widehat{\mathcal{N}}_i \begin{bmatrix} I_r \\ 0_{(n-rm)\times r}\end{bmatrix} = 0_{(n-rm)\times r}$ and by (2.2.15)

$$\begin{aligned} &0_{(n-rm)\times r} \\ &\quad = \left[0_{(n-rm)\times r}, I_{n-rm}\right] \left[I_{r+n-rm} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i\right] \begin{bmatrix} I_r \\ 0_{(n-rm)\times r}\end{bmatrix} \\ &\overset{(2.2.12)}{=} - \left[0_{(n-rm)\times r}, I_{n-rm}\right] \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1} \end{aligned} \tag{2.2.27}$$

which, in view of (2.2.18) and (2.2.19), yields (2.2.23). Note that $\overline{U}^{-1} = [\overline{\mathcal{B}}(\overline{\mathcal{C}}\overline{\mathcal{B}})^{-1} \mid \overline{\mathcal{V}}]$. By (2.2.20), for $i \in \{1, \dots, m\}$, follows

$$[P_i, 0, \dots, 0] = \widehat{\mathcal{N}}_i \left(T_i\Pi^{-1}\overline{U}A\overline{U}^{-1}\Pi T_i^T\right) \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}$$

and

$$
\begin{aligned}
&[P_i, 0, \dots, 0] \\
&\overset{(2.2.11)}{=} \Bigg[0_{(n-rm)\times rm} \;\Bigg|\; [0_{(n-rm)\times r}, I_{n-rm}] \\
&\qquad\qquad \cdot \left[I_{r+n-rm} - \widehat{\mathcal{B}}_i (\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i)^{-1} \widehat{\mathcal{C}}_i \right] \begin{bmatrix} 0_{r\times(n-rm)} \\ I_{n-rm} \end{bmatrix} \Bigg] \\
&\qquad \cdot \Pi^{-1} \overline{U} A \overline{U}^{-1} \Pi \begin{bmatrix} 0_{(i-1)r\times(r+n-rm)} \\ \hline [I_r, 0_{r\times(n-rm)}] \\ \hline 0_{(m-i)r\times(r+n-rm)} \\ \hline [0_{(n-rm)\times r}, I_{n-rm}] \end{bmatrix} \widehat{\mathcal{B}}_i (\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i)^{-1} \\
&\overset{(2.2.12)}{\underset{(2.2.27)}{=}} \left[0_{(n-rm)\times rm} \,\middle|\, I_{n-rm} \right] \Pi^{-1} \begin{bmatrix} \overline{\mathcal{C}} \\ \hline \overline{\mathcal{N}} \end{bmatrix} A [\overline{\mathcal{B}} (\overline{\mathcal{C}}\overline{\mathcal{B}})^{-1} \mid \overline{\mathcal{V}}] \Pi \begin{bmatrix} 0_{(i-1)r\times r} \\ \hline I_r \\ \hline 0_{(m-i)r\times r} \\ \hline 0_{(n-rm)\times r} \end{bmatrix},
\end{aligned}
$$

which, in view of the definition of permutation matrix Π, yields (2.2.24). Finally (2.2.21) leads directly to (2.2.25) and the proof is complete. □

The remainder of this section contains the proof for Theorem 2.2.4. It is structured as follows: First linearly independence of the matrices $\mathcal{C}$ and $\mathcal{B}$ defined by (2.2.7) and (2.2.8), respectively, is shown. Then the proof for the normal form including the construction of the coordinate transformation is given.

2.2.3 Linearly independence of $\mathcal{C}$ and $\mathcal{B}$

Recall the matrices $\mathcal{C} \in \mathbb{R}^{r^s \times n}$ defined by (2.2.7) and $\mathcal{B} \in \mathbb{R}^{n \times r^s}$ defined by (2.2.8). Note that, for m_i, $i \in \{1, \dots, r_1\}$, defined by (2.2.5) it holds true that $m = m_1 \geq m_2 \geq \dots \geq m_{r_1} \geq 1$ and

$$
\begin{aligned}
r^s &= \sum_{j=1}^{m} r_j = \sum_{j=1}^{m} \sum_{i=1}^{r_1} \frac{\max\{r_j - i + 1, 0\}}{\max\{r_j - i + 1, 1\}} = \sum_{i=1}^{r_1} \underbrace{\sum_{j=1}^{m} \frac{\max\{r_j - i + 1, 0\}}{\max\{r_j - i + 1, 1\}}}_{\#\{r_j \mid r_j \geq i,\ j \in \{1,\dots,m\}\}} \\
&= \sum_{i=1}^{r_1} m_i \,.
\end{aligned}
$$

The following lemma shows that $\mathcal{C}$ and $\mathcal{B}$ have full rank.

Lemma 2.2.6 *If a linear system* (A, B, C) *of form* (2.2.1) *has ordered vector relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$*, then* $\mathcal{C}$ *and* $\mathcal{B}$ *defined by* (2.2.7) *and* (2.2.8)*, respectively, have full rank.*

Proof. Note that $\sum_{j=1}^m r_j \le n$. It suffices to show that $\mathcal{CB} \in \mathbb{R}^{r^s \times r^s}$ is invertible.

First consider the first $m_1 = m$ rows of $\mathcal{CB}$. Since (A, B, C) has relative degree $r = (r_1, \ldots, r_m) \in N^{1\times m}$ it follows that

$$\mathcal{C}B = \left[\begin{array}{c} c^1_{(n)} \\ \vdots \\ c^1_{(n)} A^{r_1-1} \\ \hline \vdots \\ \hline c^m_{(n)} \\ \vdots \\ c^m_{(n)} A^{r_m-1} \end{array}\right] B = \left[\begin{array}{c} c^1_{(n)} A^0 B \\ \vdots \\ c^1_{(n)} A^{r_1-1} B \\ \hline \vdots \\ \hline c^m_{(n)} A^0 B \\ \vdots \\ c^m_{(n)} A^{r_m-1} B \end{array}\right] = \left[\begin{array}{c} 0_{(r_1-1)\times m_1} \\ \Gamma^1_{(m_1)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m_1} \\ \Gamma^m_{(m_1)} \end{array}\right] \tag{2.2.28}$$

where $\Gamma^i_{(m_1)}$, $i \in \{1, \ldots, m\}$, is the i-th row of Γ. Thus

$$\mathcal{C}B\Gamma^{-1}\left[e^{(m)}_1, \ldots, e^{(m)}_{m_1}\right] = \left[\begin{array}{c} 0_{(r_1-1)\times m_1} \\ \Gamma^1_{(m_1)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m_1} \\ \Gamma^m_{(m_1)} \end{array}\right] \Gamma^{-1} I_{m_1} = \left[\begin{array}{c} 0_{(r_1-1)\times m_1} \\ e^1_{(m_1)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m_1} \\ e^{m_1}_{(m_1)} \end{array}\right],$$

which shows that $\mathcal{C}B\Gamma^{-1}\left[e^{(m)}_1, \ldots, e^{(m)}_{m_1}\right]$ has rank $m_1 = m$.

Next consider $\mathcal{C}A^{i-1}B\Gamma^{-1}\left[e^{(m)}_1, \ldots, e^{(m)}_{m_i}\right]$, for $i \in \{2, \ldots, r_1\}$. Since (A, B, C) has relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ it follows with the conventions

(i) $0_{(r_j-i)\times m_i}$ is of dimension zero, if $i \ge r_j$, $j \in \{1, \ldots, m\}$, and

(ii) $\Gamma^{j}_{(m)}$ and $e^{j}_{(m_i)}$ do not exist in the following matrices if $j > m_i$, $j \in \{1, \ldots, m\}$, and

(iii) $\mathcal{X}_{\mu\times\nu} \in \mathbb{R}^{\mu\times\nu}$ is an arbitrarily matrix of dimension $\mu \times \nu$,

that

$$\begin{aligned}
&\mathcal{C}A^{i-1}B\Gamma^{-1}\left[e_1^{(m)},\ldots,e_{m_i}^{(m)}\right]\\
&= \begin{bmatrix} c^1_{(n)} \\ \vdots \\ c^m_{(n)}A^{r_m-1} \end{bmatrix} A^{i-1}\left[b_1^{(n)},\ldots,b_m^{(n)}\right]\Gamma^{-1}\left[e_1^{(m)},\ldots,e_{m_i}^{(m)}\right]\\
&= \left[\begin{bmatrix} c^1_{(n)}A^{i-1}B \\ \vdots \\ c^1_{(n)}A^{r_1-1}B \\ \vdots \\ c^1_{(n)}A^{r_1+i-2}B \end{bmatrix} \Big/ \cdots \Big/ \begin{bmatrix} c^m_{(n)}A^{i-1}B \\ \vdots \\ c^m_{(n)}A^{r_m-1}B \\ \vdots \\ c^m_{(n)}A^{r_m+i-2}B \end{bmatrix}\right]\Gamma^{-1}\left[e_1^{(m)},\ldots,e_{m_i}^{(m)}\right]\\
&= \left[\begin{array}{c} 0_{(r_1-i)\times m} \\ \Gamma^1_{(m)} \\ \mathcal{X}_{(\min\{r_1,i-1\})\times m} \\ \hline \vdots \\ \hline 0_{(r_m-i)\times m} \\ \Gamma^m_{(m)} \\ \mathcal{X}_{(\min\{r_m,i-1\})\times m} \end{array}\right]\Gamma^{-1}\left[e_1^{(m)},\ldots,e_{m_i}^{(m)}\right]\\
&= \left[\begin{array}{c} 0_{(r_1-i)\times m_i} \\ e^1_{(m_i)} \\ \mathcal{X}_{(\min\{r_1,i-1\})\times m_i} \\ \hline \vdots \\ \hline 0_{(r_m-i)\times m_i} \\ e^m_{(m_i)} \\ \mathcal{X}_{(\min\{r_m,i-1\})\times m_i} \end{array}\right], \qquad i \in \{2,\ldots,r_1\}\,.
\end{aligned}$$

Thus, for all $i \in \{1, \ldots, r_1\}$, the m_i rows of $\mathcal{C}A^{i-1}B\Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_i}^{(m)}\right]$ are linearly independent, and since

$$\mathcal{C}\mathcal{B} =$$

$$\left[\begin{array}{c|c|c|c|c}
\begin{matrix} 0_{(r_1-1)\times m_1} \\ e^1_{(m_1)} \end{matrix} & \begin{matrix} 0_{(r_1-2)\times m_2} \\ e^1_{(m_2)} \\ \mathcal{X}_{1\times m_2} \end{matrix} & \begin{matrix} \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} \end{matrix} & \begin{matrix} 0_{1\times m_{r_1-1}} \\ e^1_{(m_{r_1-1})} \\ \mathcal{X}_{(r_1-2)\times m_{r_1-1}} \end{matrix} & \begin{matrix} e^1_{(m_{r_1})} \\ \mathcal{X}_{(r_1-1)\times m_{r_1}} \end{matrix} \\ \hline
\begin{matrix} 0_{(r_2-1)\times m_1} \\ e^2_{(m_1)} \end{matrix} & \begin{matrix} 0_{(r_2-2)\times m_2} \\ e^2_{(m_2)} \\ \mathcal{X}_{1\times m_2} \end{matrix} & \begin{matrix} \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} \end{matrix} & \begin{matrix} 0_{(r_2-r_1+1)\times m_{r_1-1}} \\ e^2_{(m_{r_1-1})} \\ \mathcal{X}_{(\min\{r_1-2,r_2\})\times m_{r_1-1}} \end{matrix} & \begin{matrix} e^2_{(m_{r_1})} \\ \mathcal{X}_{(\min\{r_1-1,r_2\})\times m_{r_1}} \end{matrix} \\ \hline
\vdots & \vdots & & \vdots & \vdots \\ \hline
\begin{matrix} 0_{(r_m-1)\times m_1} \\ e^m_{(m_1)} \end{matrix} & \begin{matrix} 0_{(r_m-2)\times m_2} \\ e^m_{(m_2)} \\ \mathcal{X}_{1\times m_2} \end{matrix} & \begin{matrix} \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} \end{matrix} & \begin{matrix} 0_{(r_m-r_1+1)\times m_{r_1-1}} \\ e^m_{(m_{r_1-1})} \\ \mathcal{X}_{(\min\{r_1-2,r_m\})\times m_{r_1-1}} \end{matrix} & \begin{matrix} e^m_{(m_{r_1})} \\ \mathcal{X}_{(\min\{r_1-1,r_m\})\times m_{r_1}} \end{matrix}
\end{array}\right]
\begin{matrix} \}\, r_1 \\ \}\, r_2 \\ \vdots \\ \}\, r_m \end{matrix}$$

$$\underbrace{\qquad}_{m_1}\ \underbrace{\qquad}_{m_2}\cdots\underbrace{\qquad\qquad}_{m_{r_1-1}}\ \underbrace{\qquad\qquad}_{m_{r_1}} \tag{2.2.29}$$

it follows that $\mathcal{C}\mathcal{B}$ is invertible. □

As an immediate consequence of Lemma 2.2.6 it follows that for linear systems (A, B, C) of form (2.2.1) with vector relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$, the matrices $C \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ have full rank m.

2.2.4 Proof of the normal form

Lemma 2.2.6 shows that the rows of $\mathcal{C}$ qualify as basis, which, if $r^s = \sum_{j=1}^m r_j < n$, has to be completed, for a coordinate transformation in $\mathbb{R}^n$. Consider a matrix $\mathcal{V} \in \mathbb{R}^{n\times(n-r^s)}$ given by (2.2.9). For $\widehat{U}$ and $\mathcal{N}$

given by (2.2.10) it follows from

$$\begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}\right] = I_n$$

that $\widehat{U}$ has the inverse

$$\widehat{U}^{-1} = \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}\right] . \tag{2.2.30}$$

Although $\widehat{U}$ already qualifies as coordinate transformation in $\mathbb{R}^n$ we do not obtain a normal form which has the same structure properties as the normal form (2.1.5) for linear SISO-systems (2.1.2), i.e. the transformation matrix $\widehat{U}$ will not lead in general to a matrix $\widetilde{A}$ as in (2.2.4). Therefore, it is necessary to consider the transformation matrix U given by (2.2.17) and $T_i, \widehat{\mathcal{C}}_i, \widehat{\mathcal{B}}_i, \widehat{\mathcal{N}}_i, \widehat{U}_i$, for $i \in \{1, \ldots, m\}$, defined in (2.2.11)–(2.2.16), respectively.

Proof of Theorem 2.2.4. *Step 1*: First it is shown that the coordinate transformation $\left(\begin{smallmatrix} \chi \\ \zeta \end{smallmatrix}\right) := \widehat{U}x$ with

$$\begin{aligned} &\chi(t) = \left(y_1(t), \ldots, y_1^{(r_1-1)}(t) \,\Big|\, \ldots \,\Big|\, y_m(t), \ldots, y_m^{(r_m-1)}(t)\right)^T \in \mathbb{R}^{r^s} , \\ &\zeta(t) \in \mathbb{R}^{n-r^s} \end{aligned}$$

given by (2.2.9) and (2.2.10) converts (2.2.1) into

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} &= \widehat{A} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} + \widetilde{B}u \\ y &= \widetilde{C} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} , \end{aligned}\right\} \tag{2.2.31}$$

where

$$\widehat{A} = \left[\begin{array}{cccc|c} \widehat{A}_{1,1} & \widehat{A}_{1,2} & \ldots & \widehat{A}_{1,m} & \widehat{S}_1 \\ \widehat{A}_{2,1} & \widehat{A}_{2,2} & \ldots & \widehat{A}_{2,m} & \widehat{S}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \widehat{A}_{m,1} & \widehat{A}_{m,2} & \ldots & \widehat{A}_{m,m} & \widehat{S}_m \\ \hline \widehat{P}_1 & \widehat{P}_2 & \ldots & \widehat{P}_m & \widehat{Q} \end{array}\right], \tag{2.2.32}$$

and, for $i, j \in \{1, \ldots, m\}$,

$$\widehat{A}_{i,i} := \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1 \\ \widehat{R}^i_{i,1} & \ldots & & \widehat{R}^i_{i,r_i} \end{bmatrix} \in \mathbb{R}^{r_i \times r_i},$$

$$\widehat{A}_{i,j} := \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \widehat{R}^i_{j,1} & \ldots & \widehat{R}^i_{j,r_j} \end{bmatrix} \in \mathbb{R}^{r_i \times r_j}, \quad j \neq i,$$

where $\widehat{R}^j_{i,k} \in \mathbb{R}$, for $k \in \{1, \ldots, r_i\}$ and $i, j \in \{1, \ldots, m\}$, and, for $i \in \{1, \ldots, m\}$,

$$\left.\begin{aligned} &\widehat{S}_i := \begin{bmatrix} 0_{(r_i-1)\times(n-r^s)} \\ S^i \end{bmatrix} \in \mathbb{R}^{r_i \times (n-r^s)}, \\ &\widehat{P}_i \in \mathbb{R}^{(n-r^s)\times r_i}, \\ &\widehat{Q} \in \mathbb{R}^{(n-r^s)\times(n-r^s)}. \end{aligned}\right\} \tag{2.2.33}$$

Step 1a): First the structure of $\widehat{A}$ is proven. By the definition of $\widehat{U}$, see (2.2.10), it follows that

$$\widehat{A} = \widehat{U} A \widehat{U}^{-1} = \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} A \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}\right] = \left[\begin{array}{c|c} \mathcal{C}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} & \mathcal{C}A\mathcal{V} \\ \hline \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} & \mathcal{N}A\mathcal{V} \end{array}\right].$$

Thus

$$\left[\widehat{P}_1, \ldots, \widehat{P}_m\right] = \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \in \mathbb{R}^{(n-r^s)\times r^s} \tag{2.2.34}$$

$$\widehat{Q} = \mathcal{N}A\mathcal{V} \in \mathbb{R}^{(n-r^s)\times(n-r^s)}. \tag{2.2.35}$$

Therefore, the definitions of $\mathcal{C}$ and $\mathcal{B}$, see (2.2.7) and (2.2.8), respectively, yield

$$
\mathcal{C}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = \left[\begin{array}{c} c^1_{(n)}A \\ \vdots \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline c^m_{(n)}A \\ \vdots \\ c^m_{(n)}A^{r_m} \end{array}\right] \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}
$$

$$
= \left[\begin{array}{c} \mathcal{C}^2_{(n)} \\ \vdots \\ \mathcal{C}^{r_1}_{(n)} \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline \mathcal{C}^{\sum_{j=1}^{m-1} r_j+2}_{(n)} \\ \vdots \\ \mathcal{C}^{r^s}_{(n)} \\ c^m_{(n)}A^{r_m} \end{array}\right] \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = \left[\begin{array}{c} (\mathcal{C}\mathcal{B})^2_{(r^s)} \\ \vdots \\ (\mathcal{C}\mathcal{B})^{r_1}_{(r^s)} \\ c^1_{(n)}A^{r_1}\mathcal{B} \\ \hline \vdots \\ \hline (\mathcal{C}\mathcal{B})^{\sum_{j=1}^{m-1} r_j+2}_{(r^s)} \\ \vdots \\ (\mathcal{C}\mathcal{B})^{r^s}_{(r^s)} \\ c^m_{(n)}A^{r_m}\mathcal{B} \end{array}\right] (\mathcal{C}\mathcal{B})^{-1}
$$

$$
= \left[\begin{array}{c} e^2_{(r^s)} \\ \vdots \\ e^{r_1}_{(r^s)} \\ c^1_{(n)}A^{r_1}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \\ \hline \vdots \\ \hline e^{\sum_{j=1}^{m-1} r_j+2}_{(r^s)} \\ \vdots \\ e^{r^s}_{(r^s)} \\ c^m_{(n)}A^{r_m}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \end{array}\right]. \tag{2.2.36}
$$

Furthermore, invoking $\operatorname{im}\mathcal{V} = \ker\mathcal{C}$, it follows that

$$\begin{aligned}
\mathcal{C}A\mathcal{V} \quad &= \left[\begin{array}{c} c^1_{(n)}A \\ \vdots \\ c^1_{(n)}A^{r_1-1} \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline c^m_{(n)}A \\ \vdots \\ c^m_{(n)}A^{r_m-1} \\ c^m_{(n)}A^{r_m} \end{array}\right]\mathcal{V} = \left[\begin{array}{c} \mathcal{C}^2_{(n)} \\ \vdots \\ \mathcal{C}^{r_1}_{(n)} \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline (\mathcal{C})^{\sum_{j=1}^{m-1} r_j+2}_{(n)} \\ \vdots \\ (\mathcal{C})^{r^s}_{(n)} \\ c^m_{(n)}A^{r_m} \end{array}\right]\mathcal{V} = \left[\begin{array}{c} 0_{1\times(n-r^s)} \\ \vdots \\ 0_{1\times(n-r^s)} \\ c^1_{(n)}A^{r_1}\mathcal{V} \\ \hline \vdots \\ \hline 0_{1\times(n-r^s)} \\ \vdots \\ 0_{1\times(n-r^s)} \\ c^m_{(n)}A^{r_m}\mathcal{V} \end{array}\right] \\
&\overset{(2.2.19)}{=} \left[\begin{array}{c} 0_{(r_1-1)\times(n-r^s)} \\ S^1 \\ \hline \vdots \\ \hline 0_{(r_m-1)\times(n-r^s)} \\ S^m \end{array}\right],
\end{aligned} \tag{2.2.37}$$

Hence (2.2.36) and (2.2.37) and setting, for $i \in \{1, \dots, m\}$,

$$\left[\widehat{R}^i_{1,1}, \dots, \widehat{R}^i_{1,r_1} \,\middle|\, \dots \,\middle|\, \widehat{R}^i_{m,1}, \dots, \widehat{R}^i_{m,r_m}\right] := c^i_{(n)}A^{r_i}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \tag{2.2.38}$$

yield the structure of $\widehat{A}$ as given in (2.2.32)–(2.2.33).

Step 1b): Next the structure of $\widetilde{B}$ is proven. By the definition of $\widehat{U}$, see (2.2.10), it follows that

$$\widetilde{B} = \widehat{U}B = \begin{bmatrix} \mathcal{C}B \\ (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[B - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}B\right] \end{bmatrix}.$$

Recall (2.2.28), i.e.

$$\mathcal{C}B = \left[\begin{array}{c} 0_{(r_1-1)\times m} \\ \Gamma^1_{(m)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m} \\ \Gamma^m_{(m)} \end{array}\right] = \left[\begin{array}{c} 0_{(r_1-1)\times m} \\ c^1_{(n)}A^{r_1-1}B \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m} \\ c^m_{(n)}A^{r_m-1}B \end{array}\right].$$

Furthermore

$$\begin{aligned}
&(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[B-\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}B\right]\\
&=(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\Big[B\Gamma^{-1}\Gamma-\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\underbrace{B\Gamma^{-1}\left[e_1^{(m)},\dots,e_m^{(m)}\right]}_{=\left[\mathcal{B}_1^{(n)},\dots,\mathcal{B}_m^{(n)}\right]}\Gamma\Big]\\
&=(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\Big[B\Gamma^{-1}\Gamma-\mathcal{B}\underbrace{(\mathcal{C}\mathcal{B})^{-1}\left[(\mathcal{C}\mathcal{B})_1^{(r^s)},\dots,(\mathcal{C}\mathcal{B})_m^{(r^s)}\right]}_{=\left[e_1^{(r^s)},\dots,e_m^{(r^s)}\right]}\Gamma\Big]\\
&=(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\Big[\left[\mathcal{B}_1^{(n)},\dots,\mathcal{B}_m^{(n)}\right]-\left[\mathcal{B}_1^{(n)},\dots,\mathcal{B}_m^{(n)}\right]\Big]\right)\Gamma\\
&=0_{(n-r^s)\times m}\,,
\end{aligned}$$

which shows the structure of $\widetilde{B}$ as in (2.2.4).

Step 1c): Now the structure of $\widetilde{C}$ is shown. Since the rows of C are also rows of $\mathcal{C}$, i.e.

$$C=\begin{bmatrix} c^1_{(n)}\\ c^2_{(n)}\\ \vdots\\ c^m_{(n)}\end{bmatrix}=\begin{bmatrix}\mathcal{C}^1_{(n)}\\ \mathcal{C}^{r_1+1}_{(n)}\\ \vdots\\ \mathcal{C}^{r^s-r_m+1}_{(n)}\end{bmatrix}$$

and since $\operatorname{im}\mathcal{V}=\ker\mathcal{C}$ it follows that $C\mathcal{V}=0_{m\times(n-r^s)}$. Furthermore

$$\begin{aligned}
C\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}&=\begin{bmatrix}(\mathcal{C}\mathcal{B})^1_{(r^s)}\\ (\mathcal{C}\mathcal{B})^{r_1+1}_{(r^s)}\\ \vdots\\ (\mathcal{C}\mathcal{B})^{r^s-r_m+1}_{(r^s)}\end{bmatrix}(\mathcal{C}\mathcal{B})^{-1}\\
&=\begin{bmatrix}
1 & 0_{1\times(r_1-1)} & 0 & 0_{1\times(r_2-1)} & 0 & \dots & 0 & 0_{1\times(r_m-1)}\\
0 & 0_{1\times(r_1-1)} & 1 & 0_{1\times(r_2-1)} & 0 & \dots & 0 & 0_{1\times(r_m-1)}\\
0 & 0_{1\times(r_1-1)} & 0 & 0_{1\times(r_2-1)} & 1 & & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & & \ddots & 0 & 0_{1\times(r_m-1)}\\
0 & 0_{1\times(r_1-1)} & 0 & 0_{1\times(r_2-1)} & 0 & & 1 & 0_{1\times(r_m-1)}
\end{bmatrix}.
\end{aligned}$$

Hence

$$\widetilde{C} = C\widehat{U}^{-1} = \left[C\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, C\mathcal{V}\right]$$

yields the structure of $\widetilde{C}$ as in (2.2.4).

Step 2: Finally it is shown that the coordinate transformation $\left(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\right) := Ux$ given by (2.2.9)–(2.2.17) with

$$\begin{aligned}
&\xi(t) = \left(y_1(t), \dots, y_1^{(r_1-1)}(t) \middle| \dots \middle| y_m(t), \dots, y_m^{(r_m-1)}(t)\right)^T \in \mathbb{R}^{r^s},\\
&\eta(t) \in \mathbb{R}^{n-r^s},
\end{aligned}$$

converts the linear system (A, B, C) of form (2.2.1) into (2.2.3) with $\widetilde{A}, \widetilde{B}, \widetilde{C}$ as in (2.2.4) with matrix components of $\widetilde{A}$ as in (2.2.18)–(2.2.21).

Recall the structure of $\widehat{A}$ given by (2.2.32)–(2.2.33). Consider, for $i \in \{1, \dots, m\}$, the matrices

$$\begin{aligned}
\widehat{A}_i &:= \left[\begin{array}{cccc|c}
0 & 1 & \dots & 0 & \\
\vdots & & \ddots & & \\
0 & \dots & 0 & 1 & \widehat{S}_i \\
\widehat{R}^i_{i,1} & \dots & & \widehat{R}^i_{i,r_i} & \\
\hline
& & \widehat{P}_i & & \widehat{Q}
\end{array}\right] = \left[\begin{array}{c|c} \widehat{A}_{i,i} & \widehat{S}_i \\ \hline \widehat{P}_i & \widehat{Q} \end{array}\right] = T_i \widehat{A} T_i^T \\
&= T_i \widehat{U} A \widehat{U}^{-1} T_i^T \in \mathbb{R}^{(r_i+n-r^s)\times(r_i+n-r^s)},\\
\widehat{C}_i &:= \left[1, 0_{1\times(r_i+n-r^s-1)}\right] = e^1_{(r_i+n-r^s)} = e^i_{(m)} \widetilde{C} T_i^T \in \mathbb{R}^{1\times(r_i+n-r^s)},\\
\widehat{B}_i &:= \left[\begin{array}{c} 0_{(r_i-1)\times 1} \\ 1 \\ \hline 0_{(n-r^s)\times 1} \end{array}\right] = e_{r_i}^{(r_i+n-r^s)} = T_i \widetilde{B} \Gamma^{-1} e_i^{(m)} \in \mathbb{R}^{r_i+n-r^s}.
\end{aligned}$$

Then

$$\begin{bmatrix} \widehat{C}_i \\ \widehat{C}_i \widehat{A}_i \\ \vdots \\ \widehat{C}_i \widehat{A}_i^{r_i-1} \end{bmatrix} = \begin{bmatrix} e^1_{(r_i+n-r^s)} \\ e^2_{(r_i+n-r^s)} \\ \vdots \\ e^{r_i}_{(r_i+n-r^s)} \end{bmatrix} = \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \overset{(2.2.12)}{=} \widehat{\mathcal{C}}_i \in \mathbb{R}^{r_i\times(r_i+n-r^s)}$$

and

$$\left[\widehat{B}_i, \widehat{A}_i \widehat{B}_i, \ldots, \widehat{A}_i^{r_i-1} \widehat{B}_i\right] = \left[e_{r_i}^{(r_i+n-r^s)}, \ldots, \widehat{A}_i^{r_i-1} e_{r_i}^{(r_i+n-r^s)}\right] \overset{(2.2.14)}{=} \widehat{\mathcal{B}}_i \in \mathbb{R}^{(r_i+n-r^s)\times r_i}.$$

More precisely $\widehat{\mathcal{B}}_i$ is structured as follows:

$$\widehat{\mathcal{B}}_i = \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 0 & \ldots & 0 & 1 \\ \vdots & \iddots & 1 & * \\ 0 & \iddots & \iddots & \vdots \\ 1 & * & \ldots & * \\ * & \ldots & & * \end{matrix} \\ \hline 0_{(n-r^s)\times 1} & \mathcal{X}_{(n-r^s)\times(r_i-1)} \end{array}\right] \in \mathbb{R}^{(r_i+n-r^s)\times r_i}. \tag{2.2.39}$$

Since $\widehat{C}_i \widehat{A}_i^j \widehat{B}_i = 0$, for all $j \in \{0, \ldots, r_i - 2\}$, and $\widehat{C}_i \widehat{A}_i^{r_i-1} \widehat{B}_i = 1$, it follows that the linear SISO-system

$$\left.\begin{aligned} \dot{z} &= \widehat{A}_i z + \widehat{B}_i v \\ w &= \widehat{C}_i z \end{aligned}\right\}$$

has relative degree r_i. Furthermore, it follows that

$$\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ \vdots & \iddots & \iddots & * \\ 0 & 1 & \iddots & \vdots \\ 1 & * & \ldots & * \end{bmatrix} \quad \text{and} \quad \left(\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i\right)^{-1} = \begin{bmatrix} * & \ldots & * & 1 \\ \vdots & \iddots & \iddots & 0 \\ * & 1 & \iddots & \vdots \\ 1 & 0 & \ldots & 0 \end{bmatrix} \tag{2.2.40}$$

and thus

$$\widehat{\mathcal{B}}_i \left(\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i\right)^{-1} = \left[\begin{array}{c|c} \begin{matrix} 1 & 0 & \ldots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \ldots & * & 1 \\ * & \ldots & & * \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{matrix} \\ \hline \mathcal{X}_{(n-r^s)\times(r_i-1)} & 0_{(n-r^s)\times 1} \end{array}\right] \in \mathbb{R}^{(r_i+n-r^s)\times r_i} \tag{2.2.41}$$

and

$$\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \widehat{\mathcal{B}}_i \left(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i\right)^{-1} = \left[\mathcal{X}_{(n-r^s)\times(r_i-1)}, 0_{(n-r^s)\times 1}\right]. \quad (2.2.42)$$

Set $\widehat{\mathcal{V}}_i := \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix}$. Then $\ker \widehat{\mathcal{C}}_i = \operatorname{im} \widehat{\mathcal{V}}_i$ and thus

$$\begin{bmatrix} \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \\ \underbrace{\left(\widehat{\mathcal{V}}_i\widehat{\mathcal{V}}_i^T\right)^{-1}\widehat{\mathcal{V}}_i^T}_{=\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right]} \left[I_{r_i+n-r^s} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i\right] \end{bmatrix} \overset{(2.2.12),\,(2.2.15)}{=} \begin{bmatrix} \widehat{\mathcal{C}}_i \\ \widehat{\mathcal{N}}_i \end{bmatrix} \quad (2.2.43)$$

is invertible with inverse

$$\begin{bmatrix} \widehat{\mathcal{C}}_i \\ \widehat{\mathcal{N}}_i \end{bmatrix}^{-1} = \left[\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}, \widehat{\mathcal{V}}_i\right]. \quad (2.2.44)$$

Furthermore, it follows that

$$\widehat{\mathcal{C}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} = \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} = 0_{r_i\times(n-r^s)}$$

$$\widehat{\mathcal{C}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix} = \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix} = I_{r_i}$$

and, in view of (2.2.43),

$$\begin{aligned} \widehat{\mathcal{N}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} &= \left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \\ &\quad \cdot \left[\begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix}\right] \\ &= I_{n-r^s}. \end{aligned}$$

This leads, for $i \in \{1, \ldots, m\}$, to the structural representation of the transformation matrix $\widehat{U}_i$, namely

$$\left[\begin{array}{c|c|c|c} I_{\sum_{j=1}^{i-1} r_j} & 0_{(\sum_{j=1}^{i-1} r_j)\times r_i} & 0_{(\sum_{j=1}^{i-1} r_j)\times(\sum_{j=i+1}^{m} r_j)} & 0_{(\sum_{j=1}^{i-1} r_j)\times(n-r^s)} \\ \hline 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\widehat{\mathcal{C}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i}\end{bmatrix}}_{=I_{r_i}} & 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\widehat{\mathcal{C}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s}\end{bmatrix}}_{=0_{r_i\times(n-r^s)}} \\ \hline 0_{(\sum_{j=i+1}^{m} r_j)\times(\sum_{j=1}^{i-1} r_j)} & 0_{(\sum_{j=i+1}^{m} r_j)\times r_i} & I_{\sum_{j=i+1}^{m} r_j} & 0_{(\sum_{j=i+1}^{m} r_j)\times(n-r^s)} \\ \hline 0_{(n-r^s)\times(\sum_{j=1}^{i-1} r_j)} & \widehat{\mathcal{N}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i}\end{bmatrix} & 0_{(n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & \underbrace{\widehat{\mathcal{N}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s}\end{bmatrix}}_{=I_{n-r^s}} \end{array}\right]$$

$$\overset{(2.2.16)}{=} \widehat{U}_i \,.$$

Since

$$\left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1} = \widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1} = I_{r_i} \,,$$

it follows from (2.2.43) and (2.2.44) that the inverse of the transformation matrix $\widehat{U}_i$ is given by

$$\widehat{U}_i^{-1} = \left[\begin{array}{c|c|c|c} I_{\sum_{j=1}^{i-1} r_j} & 0_{(\sum_{j=1}^{i-1} r_j)\times r_i} & 0_{(\sum_{j=1}^{i-1} r_j)\times(\sum_{j=i+1}^{m} r_j)} & 0_{(\sum_{j=1}^{i-1} r_j)\times(n-r^s)} \\ \hline 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\begin{matrix}\left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \\ \cdot\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\end{matrix}}_{=I_{r_i}} & 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \widehat{\mathcal{V}}_i}_{=0_{r_i\times(n-r^s)}} \\ \hline 0_{(\sum_{j=i+1}^{m} r_j)\times(\sum_{j=1}^{i-1} r_j)} & 0_{(\sum_{j=i+1}^{m} r_j)\times r_i} & I_{\sum_{j=i+1}^{m} r_j} & 0_{(\sum_{j=i+1}^{m} r_j)\times(n-r^s)} \\ \hline 0_{(n-r^s)\times(\sum_{j=1}^{i-1} r_j)} & \begin{matrix}\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \\ \cdot\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\end{matrix} & 0_{(n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & \underbrace{\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \widehat{\mathcal{V}}_i}_{=I_{n-r^s}} \end{array}\right] . \tag{2.2.45}$$

Recall $\widehat{A} = \widehat{U}A\widehat{U}^{-1}$ given by (2.2.32)–(2.2.33). First apply the transformation $\widehat{U}_1$. Then, omitting the dimensions of the zeros and identity matrices in $\widehat{U}_1$, it follows that

$$
\begin{aligned}
&\widehat{U}_1\widehat{A}\widehat{U}_1^{-1} \\
&= \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline \widehat{\mathcal{N}}_1 \begin{bmatrix} I\\0\end{bmatrix} & 0 & \underbrace{\widehat{\mathcal{N}}_1 \begin{bmatrix}0\\I\end{bmatrix}}_{=I} \end{array}\right]
\left[\begin{array}{c|c|c} \widehat{A}_{1,1} & \dots\ \widehat{A}_{1,m} & \widehat{S}_1 \\ \hline \vdots & \ddots\ \ \vdots & \vdots \\ \widehat{A}_{m,1} & \dots\ \widehat{A}_{m,m} & \widehat{S}_m \\ \hline \widehat{P}_1 & \dots\ \ \widehat{P}_m & \widehat{Q}\end{array}\right]
\left[\begin{array}{c|c|c} \underbrace{[I,0]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}}_{=I} & 0 & 0 \\ \hline 0 & I & 0 \\ \hline [0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & 0 & I\end{array}\right] \\
&= \left[\begin{array}{c|c|c} \widehat{A}_{1,1} & \widehat{A}_{1,2} \dots \widehat{A}_{1,m} & \widehat{S}_1 \\ \hline \widehat{A}_{2,1} & \widehat{A}_{2,2}\ \dots\ \widehat{A}_{2,m} & \widehat{S}_2 \\ \vdots & \vdots \qquad \vdots & \vdots \\ \widehat{A}_{m,1} & \widehat{A}_{m,2} \dots \widehat{A}_{m,m} & \widehat{S}_m \\ \hline \widehat{\mathcal{N}}_1 \begin{bmatrix} I\\0\end{bmatrix}\widehat{A}_{1,1} + \widehat{\mathcal{N}}_1 \begin{bmatrix} 0\\I\end{bmatrix}\widehat{P}_1 & \begin{array}{c}\widehat{\mathcal{N}}_1 \begin{bmatrix} I\\0\end{bmatrix}\left[\widehat{A}_{1,2},\dots,\widehat{A}_{1,m}\right] \\ +I\left[\widehat{P}_2,\dots,\widehat{P}_m\right]\end{array} & \widehat{\mathcal{N}}_1 \begin{bmatrix} I\\0\end{bmatrix}\widehat{S}_1 + \widehat{\mathcal{N}}_1 \begin{bmatrix} 0\\I\end{bmatrix}\widehat{Q} \end{array}\right] \\
&\quad \cdot \left[\begin{array}{c|c|c} \underbrace{[I,0]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}}_{=I} & 0 & 0 \\ \hline 0 & I & 0 \\ \hline [0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & 0 & I\end{array}\right] \\
&= \left[\begin{array}{c|c|c} \widehat{A}_{1,1}I + \widehat{S}_1\,[0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & \widehat{A}_{1,2} \dots \widehat{A}_{1,m} & \widehat{S}_1 \\ \hline \begin{bmatrix}\widehat{A}_{2,1}\\ \vdots \\ \widehat{A}_{m,1}\end{bmatrix} + \begin{bmatrix}\widehat{S}_2\\ \vdots \\ \widehat{S}_m\end{bmatrix}\,[0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & \begin{array}{c}\widehat{A}_{2,2}\ \dots\ \widehat{A}_{2,m} \\ \vdots \qquad \vdots \\ \widehat{A}_{m,2} \dots \widehat{A}_{m,m}\end{array} & \begin{array}{c}\widehat{S}_2 \\ \vdots \\ \widehat{S}_m\end{array} \\ \hline \begin{array}{l}\widehat{\mathcal{N}}_1\left[\begin{bmatrix} I\\0\end{bmatrix}\widehat{A}_{1,1} + \begin{bmatrix} 0\\I\end{bmatrix}\widehat{P}_1\right][I,0]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} \\ +\widehat{\mathcal{N}}_1\left[\begin{bmatrix} I\\0\end{bmatrix}\widehat{S}_1 + \begin{bmatrix} 0\\I\end{bmatrix}\widehat{Q}\right][0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}\end{array} & \begin{array}{c}\widehat{\mathcal{N}}_1 \begin{bmatrix} I\\0\end{bmatrix}\left[\widehat{A}_{1,2},\dots,\widehat{A}_{1,m}\right] \\ +\left[\widehat{P}_2,\dots,\widehat{P}_m\right]\end{array} & \widehat{\mathcal{N}}_1 \begin{bmatrix} I\\0\end{bmatrix}\widehat{S}_1 + \widehat{Q} \end{array}\right].
\end{aligned}
\tag{2.2.46}
$$

Following, the entries of $\widehat{U}_1\widehat{A}\widehat{U}_1^{-1}$ are considered in detail. For the two most upper blocks of the left column block of $\widehat{U}_1\widehat{A}\widehat{U}_1^{-1}$ it follows that, for $j \in \{2, \ldots, m\}$, the entries are given by

$$\begin{aligned}
&\widehat{A}_{1,1} I_{r_1} + \widehat{S}_1 \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad = \quad \widehat{A}_{1,1} + \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ S^1 \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(2.2.37)}{=} \begin{bmatrix} 0_{(r_1-1)\times 1}, I_{r_1-1} \\ \left[\widehat{R}^i_{1,1}, \ldots, \widehat{R}^i_{1,r_1}\right] \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ c^1_{(n)} A^{r_1} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(2.2.38)}{=} \begin{bmatrix} 0_{(r_1-1)\times 1}, I_{r_1-1} \\ c^1_{(n)} A^{r_1} \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} I_{r_1} \\ 0_{(r_s-r_1)\times r_1} \end{bmatrix} \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ c^1_{(n)} A^{r_1} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \qquad (2.2.47)
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{A}_{j,1} + \widehat{S}_j \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(2.2.37)}{=} \begin{bmatrix} 0_{(r_j-1)\times r_1} \\ \left[\widehat{R}^j_{1,1}, \ldots, \widehat{R}^j_{1,r_1}\right] \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0_{(r_j-1)\times(n-r^s)} \\ c^j_{(n)} A^{r_j} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(2.2.38)}{=} \begin{bmatrix} 0_{(r_j-1)\times r_1} \\ c^j_{(n)} A^{r_j} \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} I_{r_1} \\ 0_{(r_s-r_1)\times r_1} \end{bmatrix} \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0_{(r_j-1)\times(n-r^s)} \\ c^j_{(n)} A^{r_j} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1}. \qquad (2.2.48)
\end{aligned}$$

The entries of the two most right blocks of the bottom row block of

$\widehat{U}_1\widehat{A}\widehat{U}_1^{-1}$ are, for $j \in \{2, \dots, m\}$, given by

$$\begin{aligned}
&\widehat{\mathcal{N}}_1 \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \widehat{A}_{1,j} \\
&\overset{(2.2.43)}{=} \begin{bmatrix} 0_{(n-r^s)\times r_1}, I_{n-r^s} \end{bmatrix} \begin{bmatrix} I_{r_1+n-r^s} - \widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}\widehat{\mathcal{C}}_1 \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \begin{bmatrix} \widehat{R}^1_{j,1}, \dots, \widehat{R}^1_{j,r_j} \end{bmatrix} \end{bmatrix} \\
&\overset{(2.2.12)}{=} \begin{bmatrix} 0_{(n-r^s)\times r_1}, I_{n-r^s} \end{bmatrix} \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \begin{bmatrix} \widehat{R}^1_{j,1}, \dots, \widehat{R}^1_{j,r_j} \end{bmatrix} \end{bmatrix} \\
&\qquad - \begin{bmatrix} 0_{(n-r^s)\times r_1}, I_{n-r^s} \end{bmatrix} \widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} \begin{bmatrix} I_{r_1}, 0_{r_1\times(n-r^s)} \end{bmatrix} \\
&\qquad\quad \cdot \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \begin{bmatrix} \widehat{R}^1_{j,1}, \dots, \widehat{R}^1_{j,r_j} \end{bmatrix} \end{bmatrix} \\
&\overset{(2.2.42)}{=} 0_{r_1\times r_j} - \begin{bmatrix} \mathcal{X}_{(n-r^s)\times(r_1-1)}, 0_{(n-r^s)\times 1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \begin{bmatrix} \widehat{R}^1_{j,1}, \dots, \widehat{R}^1_{j,r_j} \end{bmatrix} \end{bmatrix} \\
&= \quad 0_{r_1\times r_j}
\end{aligned} \tag{2.2.49}$$

and

$$\begin{aligned}
&\widehat{\mathcal{N}}_1 \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \widehat{S}_j \\
&\overset{(2.2.37)}{=} \begin{bmatrix} 0_{(n-r^s)\times r_1}, I_{n-r^s} \end{bmatrix} \begin{bmatrix} I_{r_1+n-r^s} - \widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}\widehat{\mathcal{C}}_1 \end{bmatrix} \\
&\qquad \cdot \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ S^1 \end{bmatrix} \\
&\overset{(2.2.49)}{=} 0_{r_1\times r_j} \,.
\end{aligned} \tag{2.2.50}$$

Finally, the left bottom block of the first column block of $\widehat{U}_1\widehat{A}\widehat{U}_1^{-1}$ is given by

$$
\begin{aligned}
&\widehat{\mathcal{N}}_1 \left[\begin{bmatrix} I \\ 0 \end{bmatrix} \widehat{A}_{1,1} + \begin{bmatrix} 0 \\ I \end{bmatrix} \widehat{P}_1\right] [I,0]\, \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&+ \widehat{\mathcal{N}}_1 \left[\begin{bmatrix} I \\ 0 \end{bmatrix} \widehat{S}_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} \widehat{Q}\right] [0,I]\, \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&= \widehat{\mathcal{N}}_1 \left(\begin{bmatrix} \widehat{A}_{1,1} \\ \widehat{P}_1 \end{bmatrix} [I,0] + \begin{bmatrix} \widehat{S}_1 \\ \widehat{Q} \end{bmatrix} [0,I] \right) \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&= \widehat{\mathcal{N}}_1 \widehat{A}_1 \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&= \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T\right)^{-1} \widehat{\mathcal{V}}_1^T \left[I_{r_1+n-r^s} - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \widehat{\mathcal{C}}_1 \right] \widehat{A}_1 \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&= \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T\right)^{-1} \widehat{\mathcal{V}}_1^T \left[\widehat{A}_1 \widehat{\mathcal{B}}_1 - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \widehat{\mathcal{C}}_1 \widehat{A}_1 \widehat{\mathcal{B}}_1 \right] (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&= \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T\right)^{-1} \widehat{\mathcal{V}}_1^T \Big[\left[(\widehat{\mathcal{B}}_1)_2^{(r_1+n-r^s)}, \dots, (\widehat{\mathcal{B}}_1)_{r_1}^{(r_1+n-r^s)}, * \right] \\
&\qquad - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \left[(\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)_2^{(r_1)}, \dots, (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)_2^{(r_1)}, * \right] \Big] (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&= \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T\right)^{-1} \widehat{\mathcal{V}}_1^T \Big[\left[(\widehat{\mathcal{B}}_1)_2^{(r_1+n-r^s)}, \dots, (\widehat{\mathcal{B}}_1)_{r_1}^{(r_1+n-r^s)}, * \right] \\
&\qquad - \widehat{\mathcal{B}}_1 \begin{bmatrix} 0 & * \\ I_{r_1-1} & * \end{bmatrix} \Big] (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\overset{(2.2.40)}{=} \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T\right)^{-1} \widehat{\mathcal{V}}_1^T [0, \dots, 0, *] \begin{bmatrix} * & \dots & * & 1 \\ \vdots & \iddots & \iddots & 0 \\ * & 1 & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix} \\
&= \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T\right)^{-1} \widehat{\mathcal{V}}_1^T [*, 0, \dots, 0]\,. \qquad (2.2.51)
\end{aligned}
$$

Hence the representation of $\widehat{U}_1 \widehat{A} \widehat{U}_1^{-1}$, see (2.2.46), and the equations (2.2.47)–(2.2.51) show that only the first r_1 columns of $\widehat{A}$ change when applying the transformation $\widehat{U}_1$. Furthermore the first r_1 columns of $\widehat{U}_1 \widehat{A} \widehat{U}_1^{-1}$ are equal to the first r_1 columns of $\widetilde{A}$ and by (2.2.47)–(2.2.51) equations (2.2.18) and (2.2.20) hold for $i = 1$.

Moreover, an application of the transformation $\widehat{U}_i$, $i \in \{2, \dots m\}$, has the similar effect as in (2.2.46)–(2.2.51) on the r_i columns from column number $\sum_{j=1}^{i-1} r_j + 1$ to column number $\sum_{j=1}^{i} r_j$ of the (already by $\widehat{U}_1, \dots, \widehat{U}_{i-1}$ transformed) matrix $\left[\widehat{U}_{i-1} \dots \widehat{U}_1 \widehat{A} \widehat{U}_1^{-1} \dots \widehat{U}_{i-1}^{-1}\right]$, which, when finally all m transformation matrices $\widehat{U}_i$ are applied, yields (2.2.4) and (2.2.18)–(2.2.21). This completes the proof. □

Although the proof of Theorem 2.2.4 is very technical only basic concepts from linear algebra are used. It might be also possible to prove the result using the ideas for nonlinear MIMO-systems from [Isi99, Sec. 11]. However, the proof given here is independent of any knowledge about nonlinear systems, so the theorem is an autonomous result. Moreover, some of the following results of the present thesis are conclusions for linear MIMO-systems which are shown using the normal form. So, it is reasonable to have some better understanding of the structure of such systems which is successfully accomplished by the above.

2.3 Zero dynamics and right-invertibility

This section comprises first applications of the normal form (2.2.3)–(2.2.4) for linear MIMO-systems (A, B, C) of form (2.2.1): characterizations of the system's zero dynamics and right-invertibility in terms of the normal form.

The notion of *zero dynamics* for (nonlinear) systems goes back to works of C. I. Byrnes and A. Isidori, see [BI84]. In [Isi95] the author considers the "*Problem of Zeroing the Output*: Find [...] pairs consisting of an initial state x^0 and an input function $u^0(\cdot)$ [...], such that the corresponding output $y(t)$ is identically zero for all t [...]".

The definition for the zero dynamics of a linear system given in the present work is very similar to Isidori [Isi95, pp. 163-164]. While Isidori's definition is based on the normal form already, here the zero dynamics are considered as a subset of the system's *behaviour*, compare, for example, [IM07, Def. 4.1].

Furthermore, exponential stability of linear systems and exponential stability of the zero dynamics is defined as follows:

Definition 2.3.1 *Introduce the* behaviour *of a linear system* (A, B, C) *of form* (2.2.1)*:*

$$\begin{aligned}\mathfrak{B}(A,B,C) \\ := \left\{ \begin{array}{r} (x,u,y) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \\ \times\, \mathcal{C}_{pw}([0,\infty) \to \mathbb{R}^m) \\ \times\, \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \end{array} \;\middle|\; (x,u,y) \text{ solves } (2.2.1) \right\}.\end{aligned}$$

(i) The zero dynamics *of a linear system* (A, B, C) *of form* (2.2.1) *are defined as the real vector space of trajectories*

$$\mathcal{ZD}(A,B,C) := \left\{ (x,u,y) \in \mathfrak{B}(A,B,C) \;\middle|\; y \equiv 0 \text{ on } [0,\infty) \right\}.$$

(ii) A linear system $\dot{x} = Ax$*, for* $A \in \mathbb{R}^{n\times n}$*, is called* exponentially stable *on* $[0,\infty)$ *if, and only if,*

$$\exists\, M, \lambda > 0 \;\forall\, t \geq 0 \;:\; \|x(t)\| \leq M e^{-\lambda t} \|x(0)\|,$$

for all solutions x *of* $\dot{x} = Ax$.

A *is called* Hurwitz *if, and only if, all its eigenvalues have a negative real part, see [Son98, Def. C.5.2], which is equivalent to* $\dot{x} = Ax$ *being exponentially stable, see [Son98, Prop. 5.5.5].*

(iii) The zero dynamics of a linear system (A, B, C) *of form* (2.2.1) *are called* exponentially stable *if, and only if,*

$$\begin{aligned}\exists\, M, \lambda > 0 \;\forall\, (x,u,y) \in \mathcal{ZD}(A,B,C) \;\forall\, t \geq 0 \;: \\ \|x(t)\| \leq M e^{-\lambda t} \|x(0)\|.\end{aligned}$$

(iv) A linear system (A, B, C) *is called* minimum phase *if, and only if, the system's zero dynamics are exponentially stable.*

For linear systems (A, B, C) of form (2.2.1) with ordered vector relative degree $r \in \mathbb{N}^{1\times m}$ the zero dynamics of (A, B, C) can be read off from normal form (2.2.3) given by Theorem 2.2.4. Corollary 2.3.2 provides a characterization of the system's zero dynamics in terms of the normal form. Furthermore exponential stability of the zero dynamics of (A, B, C) will be characterized.

Corollary 2.3.2 *For any linear system* (A, B, C) *of form* (2.2.1) *with ordered relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ *and normal form given by* (2.2.3)–(2.2.4) *the following holds:*

(i) For $S := \left[{S^1}^T, \ldots, {S^m}^T \right]^T$*, with* $S^1, \ldots, S^m$ *defined in* (2.2.19), Γ *defined by* (2.2.6), $\mathcal{V}$ *defined by* (2.2.9) *and* Q *defined in* (2.2.21), *the zero dynamics of* (A, B, C) *are given by*

$$\mathcal{ZD}(A,B,C) = \left\{ \begin{array}{r} \left(\mathcal{V}\eta, -\Gamma^{-1}S\eta, 0\right) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \\ \times\, \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \\ \times\, \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \end{array} \,\middle|\, \dot\eta = Q\eta \right\},$$

(ii) System (A, B, C) *is minimum phase if, and only if,* Q *is Hurwitz.*

(iii) System (A, B, C) *is minimum phase if, and only if,*

$$\forall\, s \in \overline{\mathbb{C}}_+ \;:\; \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0\,.$$

Note that in the control literature the latter characterization (iii) of the minimum phase property of linear systems is often considered as definition, see [Oga02, p. 509]. In view of (iii) it is easy to verify that a system is minimum phase if the associated transfer function has only zeros in the open left half complex plane $\mathbb{C}_-$.

Proof. (i) Set

$$\mathcal{Z} = \left\{ \begin{array}{r} (\mathcal{V}\eta, -\Gamma^{-1}S\eta, 0) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \\ \times\, \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \\ \times\, \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \end{array} \,\middle|\, \dot\eta = Q\eta \right\}.$$

"$\subseteq$": If $(x, u, y) \in \mathcal{ZD}(A, B, C)$ then $y \equiv 0$ on $[0, \infty)$ and so

$$\xi = \left(y_1, y_1^{(1)}, \ldots, y_1^{(r_1-1)} \,\middle|\, y_2, \ldots, y_2^{(r_2-1)} \,\middle|\, \ldots \,\middle|\, y_m, \ldots, y_m^{(r_m-1)} \right)^T \equiv 0\,,$$

which, in view of (2.2.3)–(2.2.4), yields

$$0_{m\times 1} = \underbrace{\begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix}}_{=:S} \eta + \begin{bmatrix} c^1_{(n)} A^{r_1-1} B \\ \vdots \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix} u\,, \qquad \dot\eta = Q\eta\,,$$

thus (2.2.6) yields $u = -\Gamma^{-1} S\eta$. Since $x = U^{-1}\left(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\right)$ it follows from the representiation of the inverse of $\widehat{U}$ and $\widehat{U}_1, \dots, \widehat{U}_m$, see (2.2.30) and (2.2.45), that $(x,u,y) = (\mathcal{V}\eta, -\Gamma^{-1}S\eta, 0)$ for η being a solution of $\dot\eta = Q\eta$ and therefore $(x,u,y) \in \mathcal{Z}$.

"$\supseteq$": Let $(\tilde x, \tilde u, \tilde y) = (\mathcal{V}\eta, -\Gamma^{-1}S\eta, 0) \in \mathcal{Z}$. By (2.2.9), $0 \equiv \tilde y = C\tilde x = C\mathcal{V}\eta$ thus

$$\tilde\xi = \left(\tilde y_1, \dots, \tilde y_1^{(r_1-1)} \,\middle|\, \tilde y_2, \dots, \tilde y_2^{(r_2-1)} \,\middle|\, \dots \,\middle|\, \tilde y_m, \dots, \tilde y_m^{(r_m-1)}\right)^T \equiv 0\,,$$

and therefore $\left(\left(\begin{smallmatrix}0\\ \eta\end{smallmatrix}\right), \tilde u, 0\right)$ solves (2.2.3), hence

$$(\tilde x, \tilde u, \tilde y) = \left(U^{-1}\begin{pmatrix}0\\ \eta\end{pmatrix}, \tilde u, 0\right) = (\mathcal{V}\eta, \tilde u, 0) \in \mathcal{ZD}(A,B,C)\,.$$

(ii) From (i) it follows that

$$x = \mathcal{V}\eta \qquad \text{and} \qquad \eta = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T x\,,$$

where η is a solution of $\dot\eta = Q\eta$. Thus the zero dynamics of (A,B,C) are exponentially stable if, and only if, $\dot\eta = Q\eta$ is an exponentially stable system. By Definition 2.3.1(iv) then follows that (A,B,C) is minimum phase.

(iii) For $U \in \mathbb{R}^{n\times n}$ defined by (2.2.17) it follows that

$$\begin{aligned}\det\begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} &= \det\begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} U^{-1} & 0 \\ 0 & I_m \end{bmatrix} \\ &= \det\begin{bmatrix} sI_n - \widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0 \end{bmatrix}.\end{aligned}$$

Deriving the determinant of $\left[\begin{smallmatrix} sI_n - \widetilde{A} & \widetilde{B} \\ \widetilde{C} & 0 \end{smallmatrix}\right]$ by lines of the system's normal form (2.2.3)–(2.2.4) and starting with the last m lines leads to

$$
\begin{aligned}
&\det \left[\begin{array}{c|c} sI_n - \widetilde{A} & \widetilde{B} \\ \hline \widetilde{C} & 0 \end{array}\right] \\
&= \det \left[\left[\begin{array}{ccc|c|ccc|c}
-1 & & 0 & & 0 & \dots & 0 & 0 \\
s & \ddots & & \dots & \vdots & & \vdots & \vdots \\
0 & \ddots & -1 & & 0 & \dots & 0 & 0 \\
-R^1_{1,2} & \dots s & -R^1_{1,r_1} & & -R^1_{m,2} & \dots & -R^1_{m,r_m} & -S^1 \\ \hline
& \vdots & & \ddots & & \vdots & & \vdots \\ \hline
0 & \dots & 0 & & -1 & & 0 & 0 \\
\vdots & & \vdots & \dots & s & \ddots & & \vdots \\
0 & \dots & 0 & & 0 & \ddots & -1 & 0 \\
-R^m_{1,2} & \dots & -R^m_{1,r_1} & & -R^m_{m,2} & \dots s & -R^m_{m,r_m} & -S^m \\ \hline
0 & \dots & 0 & \dots & 0 & \dots & 0 & sI - Q
\end{array}\right] \left| \left[\begin{array}{c}
0 \\ \vdots \\ 0 \\ c^1_{(n)} A^{r_1 - 1} B \\ \hline \vdots \\ \hline 0 \\ \vdots \\ 0 \\ c^m_{(n)} A^{r_m - 1} B \\ \hline 0
\end{array}\right] \right. \right] \\
&= (-1)^{\sum_{j=1}^m (r_j - 1)} \cdot \det(sI - Q) \cdot \det \Gamma \,,
\end{aligned}
$$

where Γ is defined by (2.2.6). Thus, in view of (ii), the system is minimum phase if, and only if, Q has only eigenvalues in $\mathbb{C}_-$. This completes the proof. □

For systems (A, B, C) with ordered relative degree $r = (r_1, \dots, r_m) \in \mathbb{N}^{1 \times m}$ it follows that, in view of Corollary 2.3.2(i), the input is given by $u(t) = -\Gamma^{-1} S \eta(t) = -\Gamma^{-1} S (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T x(t)$ where S, Γ and $\mathcal{V}$ are given by the system's normal form, namely $S = \left[{S^1}^T, \dots, {S^m}^T \right]^T$ is given by (2.2.19), Γ is defined by (2.2.6) and $\mathcal{V}$ is defined by (2.2.9). Then Corollary 2.3.2(ii) yields for systems with exponentially stable zero dynamics that

$$
\exists\, M, \lambda > 0 \ \forall\, (x, u, y) \in \mathcal{ZD}(A, B, C) \ \forall\, t \geq 0 \ : \ \|u(t)\| \leq M e^{-\lambda t} \|x(0)\| \,.
$$

The "Problem of Zeroing the Output" is related to the question whether a preassigned reference signal can be tracked instantly by an appropriate input function, i.e. whether it is possible to determine an input $u = u_R$ such that the output y of (2.2.1) matches a given sufficiently smooth reference signal $y_R \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. One can do so by using the normal form (2.2.3)–(2.2.4). A system (A, B, C) is called *right-invertible* if this tracking problem can be solved [SJK97]. The following corollary presents a solution to this problem for linear MIMO-systems with ordered relative degree.

Corollary 2.3.3 *Consider a linear system* (A, B, C) *of form* (2.2.1) *with ordered relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ *and normal form given by* (2.2.3)–(2.2.4). *Let* $y_R = (y_{R1}, \ldots, y_{Rm})^T \in \mathcal{C}^{r_j}([0,\infty) \to \mathbb{R}^m)$. *Let* y *be the output of* (2.2.1). *Then the following are equivalent*

(i) $y = y_R$,

(ii) the input u *of* (2.2.1) *is given by*

$$u = u_R = \Gamma^{-1}\left(\begin{bmatrix} y_{R1}^{(r_1)} \\ \vdots \\ y_{Rm}^{(r_m)} \end{bmatrix} - \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi - \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta \right) \tag{2.3.1}$$

where, for arbitrary $\eta^0 \in \mathbb{R}^{n-r^s}$, η *is a solution of the initial value problem*

$$\dot\eta = Q\eta + [P_1, \ldots, P_m]\, y_R\,, \qquad \eta(0) = \eta^0\,, \tag{2.3.2}$$

$\xi = \left(y_{R1}, \ldots, y_{R1}^{(r_1-1)} \middle| y_{R2}, \ldots, y_{R2}^{(r_2-1)} \middle| \ldots \middle| y_{Rm}, \ldots, y_{Rm}^{(r_m-1)}\right)^T$, Q *is defined in* (2.2.21), Γ *is defined in* (2.2.6) *and* P_j *is defined in* (2.2.20), S^j *is defined in* (2.2.19) *and, for* $R^j_{i,k}$ *from* (2.2.18), $R^j := \left[R^j_{1,1}, \ldots, R^j_{1,r_1} \middle| \ldots \middle| R^j_{m,1}, \ldots, R^j_{m,r_m}\right]$, $j \in \{1, \ldots, m\}$.

Proof. "$\Rightarrow$": If $y = y_R$ then by (2.2.3) the new coordinates ξ are given by $\xi = \left(y_{R1}, \ldots, y_{R1}^{(r_1-1)} \middle| \ldots \middle| y_{Rm}, \ldots, y_{Rm}^{(r_m-1)}\right)^T$ and furthermore

$$y_{Rj}^{(r_j)} = y_j^{(r_j)} = \dot\xi_{\sum_{i=1}^j r_i} = R^j\xi + S^j\eta + c^j_{(m)} A^{r_j-1}Bu\,, \quad j \in \{1, \ldots, m\}\,,$$

thus

$$\begin{bmatrix} y_{R1}^{(r_1)} \\ \vdots \\ y_{Rm}^{(r_m)} \end{bmatrix} = \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi + \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta + \underbrace{\begin{bmatrix} c_{(m)}^1 A^{r_1-1} B \\ \vdots \\ c_{(m)}^m A^{r_m-1} B \end{bmatrix}}_{=\Gamma} u$$

hence (2.3.1) and (2.3.2).

"$\Leftarrow$": Assume that (ii) holds. By (2.2.3) it follows that, for $j \in \{1, \ldots, m\}$,

$$\begin{aligned} &\dot{\xi}_{\sum_{i=1}^{j} r_i} \\ &= R^j \xi + S^j \eta + \underbrace{c_{(m)}^j A^{r_j-1} B \Gamma^{-1}}_{=e_{(m)}^j} \left(\begin{bmatrix} y_{R1}^{(r_1)} \\ \vdots \\ y_{Rm}^{(r_m)} \end{bmatrix} - \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi - \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta \right) \\ &= y_{Rj}^{(r_j)} \end{aligned}$$

where, for any $\eta^0 \in \mathbb{R}^{n-r^s}$, η is a solution of the initial value problem

$$\dot{\eta} = [P_1, \ldots, P_m]\, y_R + Q\eta\,, \qquad \eta(0) = \eta^0\,.$$

Thus (2.2.3) yields $\xi = \left(y_{R1}, \ldots, y_{R1}^{(r_1-1)} \middle| \ldots \middle| y_{Rm}, \ldots, y_{Rm}^{(r_m-1)} \right)^T$ and so $y = \left(\xi_1, \xi_{r_1+1}, \ldots, \xi_{\sum_{i=1}^{m-1} r_i + 1} \right)^T = y_R$ which completes the proof. $\square$

Zero dynamics and right-invertibility of systems are very useful concepts in control theory. Note that system (2.3.1)–(2.3.2) is called the *inverse system* of system (2.2.1) [Isi95]. Corollary 2.3.3 shows that it is possible to track any sufficiently smooth reference signal instantly by the system's output provided that the system's relative degree and all entries of normal form matrices are known explicitly. Thus, in view of (2.2.6) and (2.2.18)–(2.2.21), it is sufficient to know the original system's matrices explicitly to evaluate the normal form matrices and so the solution η of (2.3.2) and the required input u in (2.3.1).

In general applications the system's data is often not explicitly available. So, one cannot construct an input as in Corollary 2.3.3 to track a given reference signal. However, in the following sections of the present thesis control strategies for stabilization and/or tracking for systems, are presented which accomplish the shortcoming of unavailable system matrices. For this strategies only some structural properties of the system, such as the system's relative degree, a "sign", e.g. positive definiteness, of the matrix Γ and exponential stability of the system's zero dynamics, are required.

The system's normal forms and characterizations of the system's zero dynamics from this section will be utilized to prove that this control strategies work.

2.4 Notes and references

Although, Byrnes–Isidori normal forms for linear SISO- and MIMO-systems with strict relative are well-known in the control literature, see [IRT07] and also [Isi95, Ch. 4.1], for linear MIMO-systems with non-strict (vector) relative degree the main result of the present chapter is a new result. It is also available in [Mue09a].

The Byrnes–Isidori normal form for nonlinear SISO-systems is presented in [Isi95, Ch. 5]. For nonlinear MIMO-systems one can find a result in [Isi99, Ch. 11] which comprises also the case of MIMO-system with non-strict relative degree. However, it is not straightforward to deduce the main result of the present chapter from the nonlinear case.

3 Stabilization by high-gain output derivative feedback

Derivative feedback controllers for stabilizing linear and/or nonlinear systems are well known in control theory; at least for single-input single-output (SISO) systems and multi-input multi-output (MIMO) systems with strict relative degree. To design these feedback laws one can use the invertibility of the system for which the system's data must be known explicitly, see Corollary 2.3.3 and also [Mor73]. For linear MIMO systems without relative degree one can find feedback strategies in [SS87] where the authors use the "special coordinate basis" which is based on the explicit knowledge of the system matrices, too.

In this chapter control strategies for stabilizing linear systems via feedback of the output and its derivatives are introduced. The feedback laws considered here require only structural properties of the system such as known vector relative degree, stable zero dynamics and "positive" high-frequency gain. It is important that the explicit knowledge of the system's data is not required. One can find some related ideas for nonlinear SISO- and MIMO-systems in [Isi95, Ch. 9] and [Isi99, Ch. 12].

Note that the present chapter is divided into three sections, on SISO-systems, MIMO-systems with strict relative degree and MIMO-systems with non-strict (vector) relative degree, however the feedback controllers for all three cases are rather similar – for SISO-systems and MIMO-systems with strict relative degree the controllers are in fact identical. Due to the complexity of the normal form for MIMO-systems with non-strict relative degree, it is actually a bit surprising that for this systems the controller is still very simple and has almost the same structural properties as for the case of SISO-systems.

In all cases similar ideas and methods, i.e. the systmes' normal forms, pole placement theorems, Lyapunov stability arguments and linear algebra methods are used to prove that the controllers operate as desired.

3.1 SISO-systems

In this section the simple high-gain derivative feedback controller for linear single-input single-output systems of the form

$$u(t) = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1}\, y^{(i)}(t) \tag{3.1.1}$$

is introduced, where $k_1, \ldots, k_m \in \mathbb{R}$ are suitable design parameters which are independent of the system's data and $\kappa > 0$ is a sufficiently large number. This controller will be applied to a linear system (A, b, c) with the following structural properties: known relative degree, positive high-frequency gain and exponentially stable zero dynamics.

Recall the linear SISO-system (A, b, c) of form (2.1.2), i.e.

$$\left.\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx\,, \end{aligned}\right\} \tag{3.1.2}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c^T \in \mathbb{R}^n$. Suppose that the relative degree $r \in \mathbb{N}$ of (A, b, c) is known. Then, see Definition 2.1.1, the value $cA^{r-1}b \neq 0$ is either positive or negative. For an output derivative feedback (3.1.1) which should stabilize the system, the sign of $cA^{r-1}b$ must be known and, moreover, it must be positive. However, in view of the normal form (2.1.5) of (A, b, c) it is easy to see that a restriction to the class of systems with positive $cA^{r-1}b$ is no loss of generality: recall the normal form

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \underbrace{\left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ R_1^1 & \ldots & & R_r^1 & S^1 \\ \hline P_1 & 0 & \ldots & 0 & Q \end{array}\right]}_{=:\widetilde{A}} \begin{pmatrix}\xi\\ \eta\end{pmatrix} + \underbrace{\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ cA^{r-1}b \\ \hline 0 \end{array}\right]}_{=:\widetilde{b}} u \\ y &= \underbrace{[1, 0, \ldots, 0]}_{=:\widetilde{c}} \begin{pmatrix}\xi\\ \eta\end{pmatrix}, \end{aligned}\right\} \tag{3.1.3}$$

where $R_1^1, \dots, R_r^1 \in \mathbb{R}$, $S^1 \in \mathbb{R}^{1\times(n-r)}$, $P_1 \in \mathbb{R}^{n-r}$ and also $Q \in \mathbb{R}^{(n-r)\times(n-r)}$ may be presented explicitly in terms of the system matrices A, b and c, see Lemma 2.1.2. Here the input u is multiplied by $cA^{r-1}b$ so a change of the sign of $cA^{r-1}b$ can easily be nullified by multiplying u by -1. Thus, if the controller (3.1.1) stabilizes any linear minimum phase system (i.e. the system must have exponentially stable zero dynamics) with positive high-frequency gain and known relative degree then the controller (3.1.1) multiplied by -1 stabilizes any linear minimum phase system with *negative* high-frequency gain and known relative degree.

For linear MIMO-systems with strict or non-strict relative degree "positivity" of the high-frequency gain matrix ($CA^{r-1}B$ in case of strict relative degree) may be interpreted as positive definiteness. In the present thesis a matrix $M \in \mathbb{C}^{n\times n}$ is called *positive definite* if, and only if, its Hermitian part (symmetric part in case of real matrices) $1/2(M + M^*)$ is positive definite, i.e. $x^*(M + M^*)x > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$, see also the list of symbols. Thus it is not necessarily assumed that M is Hermitian/symmetric.

The analogue results as in the SISO-case holds: if controller (3.1.1) for systems with strict relative degree or controller (3.3.1) for systems with non-strict relative degree stabilizes the respective system with positive definite high-frequency gain matrix, known relative degree and exponentially stable zero dynamics then the controller (3.1.1) or (3.3.1) multiplied by -1 stabilizes the respective system with negative definite high-frequency gain matrix, known relative degree and exponentially stable zero dynamics. Since the normal forms for SISO-systems and MIMO-systems with strict relative degree have the same structure this is straightforward. Considering matrix $\widetilde{B}$ from the normal form (2.2.3)–(2.2.4) for general MIMO-systems the above is also easy to see.

For this reason all results in this chapter are restricted to systems with "positive" high-frequency gain.

To get an idea that controller (3.1.1) works, i.e. stabilizes any linear system with the mentioned structural properties, consider the easier feedback law

$$u(t) = \sum_{i=0}^{r-1} k_{i+1} y^{(i)}(t)\,. \tag{3.1.4}$$

An application of (3.1.4) to (3.1.3) yields the closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0\\ \vdots & \ddots & \ddots & & \vdots\\ 0 & \dots & 0 & 1 & 0\\ R_1 + cA^{r-1}bk_1 & \dots & & R_r + cA^{r-1}bk_r & S\\ \hline P & 0 & \dots & 0 & Q\end{array}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

and if (3.1.3) has exponentially stable zero dynamics, that is Q is a Hurwitz matrix, a Lyapunov function argument shows that, for suitable $k_1, \dots, k_r$, the above system is exponentially stable. Here the high-frequency gain $cA^{r-1}b \in \mathbb{R}$ must be known explicitly; only knowing the sign of $cA^{r-1}b$ is not sufficient.

Isidori shows that there exist design parameters $\kappa, k_1, \dots, k_r$ such that the feedback $u(t) = -\sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1} y^{(i)}(t)$ stabilizes SISO-systems with known lower bound for the high-frequency gain [Isi95, Thm. 9.3.1.], see also [Isi99, Thm. 12.1.1.] for a proof for MIMO-systems with strict relative degree. In the present thesis it is assumed that the high-frequency gain $cA^{r-1}b$ is unknown but only the sign of $cA^{r-1}b$ is known. Therefore proving that the feedback law (3.1.1) with sufficiently large $\kappa > 0$ is stabilizing linear systems becomes much more involved, see Lemma 3.2.2, Lemma 3.2.3 and the proof of Theorem 3.2.1.

Now high-gain derivative feedback stabilization is formulated for linear SISO-systems as a corollary of Theorem 3.2.1, i.e. the equivalent result for linear MIMO-systems with strict relative degree.

Corollary 3.1.1 *Suppose that system* (A, b, c) *of form* (3.1.2) *has relative degree* $r \in \mathbb{N}$, *positive* $cA^{r-1}b$ *and is minimum phase, i.e. has exponentially stable zero dynamics. Then for any monic Hurwitz polynomial* $\left(s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i\right) \in \mathbb{R}[s]$, *there exists* $\kappa^* \geq 1$ *such that, for all* $\kappa > \kappa^*$, *the feedback*

$$u(t) = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1}\, y^{(i)}(t)$$

applied to (3.1.2) *yields an exponentially stable closed-loop system.*

Proof. Corollary 3.1.1 is a direct consequence of Theorem 3.2.1. □

A detailed proof is omitted here: due to scalar input and output it would be a bit less involved than the proof of Theorem 3.2.1. However, since Theorem 3.2.1 is true for any $m \in \mathbb{N}$ it is in particular true for $m = 1$.

In the following section the above result is generalized for linear MIMO-systems with strict relative degree.

3.2 MIMO-systems with strict relative degree

Due to the identical structure of the normal forms for linear SISO-systems and MIMO-systems with strict relative degree, compare (2.1.5) and (2.2.22), it is easy to generalize the idea of derivative feedback stabilization of SISO-systems to stabilization of MIMO-systems with strict relative degree. Because of redundancy a detailed proof for SISO-systems was omitted in the previous section. In the following a proof is given for linear minimum phase MIMO-systems (A, B, C) of form

$$\left.\begin{aligned} \dot{x} &= Ax + \underbrace{\left[b_1^{(n)}, \dots, b_m^{(n)}\right]}_{=B} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}}_{=u} \\ \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{=y} &= \underbrace{\begin{bmatrix} c^1_{(n)} \\ \vdots \\ c^m_{(n)} \end{bmatrix}}_{=C} x\,, \end{aligned}\right\} \tag{3.2.1}$$

where $n, m \in \mathbb{N}$ with $m \leq n$ and $A \in \mathbb{R}^{n \times n}$, $B, C^T \in \mathbb{R}^{n \times m}$, which have strict relative degree $r \in \mathbb{N}$ and positive definite high-frequency gain matrix $CA^{r-1}B$ with the additional assumption that there exist m linearly independent eigenvectors for $CA^{r-1}B$. The latter assumption is due to technical reasons with respect to the proof of Lemma 3.2.3.

It is easy to see that the set of matrices in $\mathbb{R}^{m \times m}$ with m pairwise distinct eigenvalues is dense in the set of all matrices: for any matrix $M \in \mathbb{R}^{m \times m}$ with characteristic polynomial $\chi_M(\cdot) = \prod_{i=1}^m (\cdot - \lambda_i)$ (for

$i \neq j$ it may be that $\lambda_i = \lambda_j$) and any neighbourhood $\mathcal{U}(M)$ there exists a matrix $\widetilde{M} \in \mathcal{U}(M)$ whose characteristic polynomial $\chi_{\widetilde{M}} = \prod_{i=1}^{m}(\cdot - (\lambda_i + \delta_i))$ has m pairwise distinct zeros for some $\delta_i \in \mathbb{C}$ with $|\delta_i|$, $i \in \{1, \ldots, m\}$, sufficiently small. Thus, for matrices in $\mathbb{R}^{m \times m}$, the property of having m linearly independent eigenvectors is generic.

3.2.1 Main result: feedback stabilization for MIMO-systems with strict relative degree

Now the main result of this section is presented. Note that only the above structural requirements and no explicit knowledge of the system's data are necessary to show that an application of the high-gain derivative feedback (3.1.1) to a linear MIMO-system (3.2.1) leads to an exponentially stable closed-loop system.

Theorem 3.2.1 *Suppose that the system* (A, B, C) *of form* (3.2.1) *has strict relative degree* $r \in \mathbb{N}$, *positive definite* $CA^{r-1}B \in \mathbb{R}^{m \times m}$ *and is minimum phase, i.e. has exponentially stable zero dynamics. Moreover, suppose that* $CA^{r-1}B$ *has* m *linearly independent eigenvectors. Then for any monic Hurwitz polynomial* $\left(s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i\right) \in \mathbb{R}[s]$, *there exists* $\kappa^* \geq 1$ *such that, for all* $\kappa > \kappa^*$, *the feedback*

$$u(t) = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1}\, y^{(i)}(t) \tag{3.2.2}$$

applied to (3.2.1) *yields an exponentially stable closed-loop system.*

Theorem 3.2.1 is not a corollary of the main result of this chapter, i.e. Theorem 3.3.1 for linear MIMO-systems with non-strict relative degree. In Theorem 3.3.1 an additional design parameter $\nu > 0$ appears in the feedback law (3.3.1). Stabilizing by the feedback (3.2.2) requires only one design parameter $\kappa > 0$ which has to be sufficiently large.

Due to the slightly different structure of the normal forms for MIMO-systems with strict and non-strict relative degree, compare (2.2.22) and (2.2.3)–(2.2.4), the proof of Theorem 3.2.1 for the strict relative degree case is not as technically involved as in the non-strict relative degree case but, in view of Lemma 3.2.3, requires additional considerations to show that one design parameter $\kappa > 0$ is sufficient.

The proof for Theorem 3.2.1 uses the systems' normal form. Moreover, some technical lemmata are required which are presented in the following subsections.

3.2.2 Simple root-locus lemma

First a useful property of Hurwitz polynomials is shown. A polynomial p is called *Hurwitz* if, and only if, all zeros of p lie in the open left complex half-plane $\mathbb{C}_-$. To prove the stabilization result for systems with strict relative degree, it is sufficient to show that, given a Hurwitz polynomial p with degree $r-1$, $s \mapsto s^r + \kappa\, p(s)$ is again Hurwitz for sufficiently large $\kappa > 0$. The following lemma is actually a corollary of the more advanced root-locus result and will be used in the proof of Theorem 3.2.1. Lemma 3.3.2 considers the general case of a sum of arbitrarily many polynomials which is required to proof derivative feedback stabilization of MIMO-systems with non-strict relative degree. Recall that $\mathcal{Z}(p) = \{s \in \mathbb{C} \,|\, p(s) = 0\}$ is the set of zeros of $p \in \mathbb{C}[s]$.

Lemma 3.2.2 *Let* $\left(s \mapsto p(s) := \sum_{i=0}^{r-1} p_i s^i\right) \in \mathbb{C}[s]$, *with* $\mathrm{Re}(p_{r-1}) > 0$, *be Hurwitz. Then*

$$\exists\, \delta, \kappa_0 > 0\ \forall\, \kappa > \kappa_0\ :\ \max\{\mathrm{Re}\ s \,|\, s \in \mathbb{C}\ :\ s^r + \kappa p(s) = 0\} < -\delta/2\,.$$

Moreover, setting $\mathcal{Z}(s \mapsto s^r + \kappa p(s)) = \{\lambda_1(\kappa), \ldots, \lambda_r(\kappa)\}$,

$$\exists\, c_0, \kappa_0 > 0\ \forall\, \kappa > \kappa_0\ :\quad \{\lambda_1(\kappa), \ldots, \lambda_{r-1}(\kappa)\} \subset \mathcal{B}_{c_0}(0)\ \textit{and}\ \lambda_r(\kappa) \notin \mathcal{B}_{c_0}(0)\,.$$

The first statement of Lemma 3.2.2 is a consequence of Lemma 3.3.2. The second statement, which is required for the proof of Theorem 3.2.1, can be deduced easily from the proof of Lemma 3.3.2. However, it is worth to present an independent short proof. One may consider the lemma as a consequence of [HP05, Prop. 4.1.3].

Proof of Lemma 3.2.2. Write $p(s) = p_{r-1} \prod_{j=1}^{\ell} (s - \lambda_j)^{m_j}$ – the canonical factorization of p with $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C}$ pairwise distinct and

$m_1, \ldots, m_\ell \in \mathbb{N}$. Then there exists $\varepsilon^* > 0$ such that

$$\forall\, i, j \in \{1, \ldots, \ell\},\, i \neq j \;:\; \mathcal{B}_{\varepsilon^*}(\lambda_i) \cap \mathcal{B}_{\varepsilon^*}(\lambda_j) = \emptyset\,.$$

Since p is Hurwitz there exists $\delta > 0$ such that $\max\{\mathrm{Re}\, s \,|\, s \in \mathbb{C} \;:\; p(s) = 0\} < -\delta$. Set $c_0 := \max_{i\in\{1,\ldots,\ell\}}\{|\lambda_i|\} + \delta$. Write, for $\gamma > 0$, $q[\gamma](s) = \gamma s^r + p(s) = \prod_{j=1}^r (s - s_j[\gamma])$. Then, [HP05, Prop. 4.1.3] and suitable numbering of the zeros $s_j[\gamma]$ of $q[\gamma]$ implies

$$\begin{aligned}
&\forall\, \varepsilon \in (0, \min\{\varepsilon^*, \delta/2, c_0^{-1}\,\mathrm{Re}(p_{r-1})\})\; \exists\, \gamma^* > 0\; \forall\, \gamma \in (0, \gamma^*) \;: \\
&\qquad \{s_1[\gamma], \ldots, s_{r-1}[\gamma]\} \subset \bigcup_{j\in\{1,\ldots,\ell\}} \mathcal{B}_\varepsilon(\lambda_j)\,, \\
&\qquad\qquad s_r[\gamma] \in \{s \in \mathbb{C} \,|\, \mathrm{Re}\; s < -\varepsilon^{-1}\,\mathrm{Re}(p_{r-1})\}\,.
\end{aligned}$$

Setting $\kappa = \gamma^{-1}$ yields $s^r + \kappa p(s) = \kappa(\kappa^{-1} s^r + p(s)) = \kappa q[\kappa^{-1}](s) = \kappa q[\gamma](s)$ for all $s \in \mathbb{C}$. Hence, for all $\kappa, \gamma > 0$ and $s \in \mathbb{C}$, $s^r + \kappa p(s) = 0$ if, and only if, $q[\gamma](s) = 0$, whence, setting $\kappa_0 = \gamma_0^{-1}$ completes the proof. $\square$

The above lemma could also be proved with the Routh–Hurwitz criterion (see details on the criterion in [Gan86, Ch. 16]). However, a proof applying the Routh–Hurwitz criterion is much more technical and several lines longer.

3.2.3 Boundedness of the solution of a parameterized Lyapunov equation

The following lemma gives a uniform bound for the solution $P(\kappa)$ of the Lyapunov equation $P(\kappa)A(\kappa) + A(\kappa)^T P(\kappa) = -I_{rm}$, where $A(\kappa) = \left[\begin{smallmatrix} 0 & I \\ A_1(\kappa) & A_2(\kappa) \end{smallmatrix}\right]$ is a stable matrix in multi-companion form: given a number $\kappa_0 > 0$ then, for all $\kappa > \kappa_0$, the spectrum of $A(\kappa)$ is in complex half-plane $\mathbb{C}_{<-\delta}$, for some $\delta > 0$. The bound is equal for all $\kappa > \kappa_0$ and depends only on δ. Recall that $\mu(A) := \max\{\mathrm{Re}\; s \,|\, s \in \mathrm{spec}(A)\}$ denotes the largest real part of the eigenvalues of $A \in \mathbb{C}^{n\times n}$. Moreover, recall that $\mathcal{Z}(p) = \{s \in \mathbb{C} \mid p(s) = 0\}$ denotes the set of zeros of a $p \in \mathbb{C}[s]$.

Lemma 3.2.3 *Let* $r, m \in \mathbb{N}$ *and* $\Gamma \in \mathbb{R}^{m\times m}$ *be positive definite, i.e.* $\Gamma + \Gamma^T > 0$*, with* $\operatorname{spec}(\Gamma) = \{\gamma_1, \ldots, \gamma_m\}$ *and* m *linearly independent eigenvectors. Let, for* $k_1, \ldots, k_r \in \mathbb{R}$*,*

$$A(\kappa) := \left[\begin{array}{cccc} 0_{m\times m} & I_m & & 0_{m\times m} \\ \vdots & \ddots & \ddots & \\ 0_{m\times m} & \ldots & 0_{m\times m} & I_m \\ \hline -\Gamma\kappa k_1, & -\Gamma\kappa k_2, & \ldots, & -\Gamma\kappa k_r \end{array}\right], \qquad \kappa \in \mathbb{R},$$

and suppose that

$$\exists\, \delta, \kappa_0 > 0\ \forall\, \kappa > \kappa_0\ :\ \mu(A(\kappa)) < -\delta\,.$$

Let $\operatorname{spec}(A(\kappa)) = \{\lambda_1(\kappa), \ldots, \lambda_{rm}(\kappa)\}$ *and suppose that, for suitable numbering, for all* $j \in \{1, \ldots, m\}$ *holds*

$$\mathcal{Z}\left(s \mapsto s^r + \gamma_j \kappa \sum_{i=0}^{r-1} k_{i+1} s^i\right) \\ = \left\{\lambda_{(j-1)(r-1)+1}(\kappa), \ldots, \lambda_{(j-1)(r-1)+r-1}(\kappa), \lambda_{(m-1)r+j}(\kappa)\right\},$$

and

$$\begin{aligned} \exists\, c_0 > 0\ \forall\, \kappa > \kappa_0\ \forall\, i \in \{1, \ldots, (r-1)m\}\ &:\ |\lambda_i(\kappa)| < c_0 \\ \forall\, i \in \{(r-1)m+1, \ldots, rm\}\ &:\ |\lambda_i(\kappa)| \geq c_0\,. \end{aligned}$$

Then the unique, symmetric, positive definite solution $P(\kappa) \in \mathbb{R}^{rm\times rm}$ *of*

$$P(\kappa)A(\kappa) + A(\kappa)^T P(\kappa) = -I_{rm}$$

satisfies

$$\exists\, c_1 > 0\ \forall\, \kappa > \kappa_0\ :\ P(\kappa) \leq \frac{c_1}{\delta} I_{rm}\,. \tag{3.2.3}$$

Note that the proof requires Γ with m linearly independent eigenvectors. This is due to technical convenience. A proof for Γ with chains of generalized eigenvectors might be also possible but would be a lot more complicated. Recall that having m pairwise distinct eigenvalues and therefore m eigenvectors is a generic property of positive definite matrices $\Gamma \in \mathbb{R}^{m\times m}$.

Proof. *Step 1*: Let $\operatorname{spec}(\Gamma) = \{\gamma_1, \dots, \gamma_m\}$ and $x_1, \dots, x_m \in \mathbb{C}^m$. be the corresponding eigenvectors. Consider the characteristic polynomial of $A(\kappa)$ given by

$$\begin{aligned}\chi_{A(\kappa)}(s) &= \det\left(sI_n - \begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ -\kappa k_1\Gamma & \dots & -\kappa k_{r-1}\Gamma & -\kappa k_r\Gamma \end{bmatrix}\right) \\ &= \det\left(s^r I_m + \left(\sum_{i=0}^{r-1} \kappa k_{i+1} s^i\right)\Gamma\right).\end{aligned}$$

Since Γ has m linearly independent eigenvectors, one may choose a Jordan decomposition $\Gamma = XJX^{-1}$, where $J = \operatorname{diag}(\gamma_1, \dots, \gamma_m) \in \mathbb{C}^{m\times m}$ and $X = [x^1, \dots, x^m] \in \mathbb{C}^{m\times m}$ is invertible, see [GvL96, Thm. 7.1.9]. Note that $\gamma_i = \gamma_j$ is allowed for some $i \neq j$. Then

$$\begin{aligned}\chi_{A(\kappa)}(s) &= \det\left(s^r I_m + \left(\sum_{i=0}^{r-1} \kappa k_{i+1} s^i\right) XJX^{-1}\right) \\ &= \det X \det\left(s^r X^{-1}X + \left(\sum_{i=0}^{r-1} \kappa k_{i+1} s^i\right) J\right) \det X^{-1} \\ &= \det\left(s^r I_m + \left(\sum_{i=0}^{r-1} \kappa k_{i+1} s^i\right) J\right) \\ &= \prod_{j=1}^{m}\left(s^r + \gamma_j\kappa \sum_{i=0}^{r-1} k_{i+1} s^i\right).\end{aligned}$$

Step 2: Consider a Jordan decomposition of $A(\kappa)$, namely

$$A(\kappa) = V(\kappa)\Lambda(\kappa)V(\kappa)^{-1},$$

where for some $\xi_1, \dots, \xi_{rm-1} \in \{0, 1\}$,

$$\Lambda(\kappa) = \operatorname{diag}\big(\lambda_1(\kappa), \dots, \lambda_{rm}(\kappa)\big) + \begin{bmatrix} 0 & \xi_1 & & \\ & \ddots & \ddots & \\ & & & \xi_{rm-1} \\ & & & 0 \end{bmatrix}$$

$$V(\kappa) = \big[v^1(\kappa), \dots, v^{rm}(\kappa)\big].$$

Next it is shown that for any eigenvalue $\lambda_i(\kappa)$, $i \in \{1, \dots, rm\}$, of $A(\kappa)$ the associated eigenvector may be considered as

$$v^i(\kappa) = \begin{pmatrix} x^\nu \\ \lambda_i(\kappa) x^\nu \\ \vdots \\ \lambda_i(\kappa)^{r-1} x^\nu \end{pmatrix}, \tag{3.2.4}$$

where x^ν, $\nu \in \{1, \dots, m\}$, is an eigenvector of Γ.

Note that, for fixed $\nu \in \{1, \dots, m\}$, vectors $v^{i_1}(\kappa), \dots, v^{i_r}(\kappa)$ are columns of a Vandermonde matrix, and thus are linearly independent for r pairwise distinct $\lambda_{i_1}(\kappa), \dots, \lambda_{i_r}(\kappa)$, see, for example, [HJ90, Sec. 0.9].

Since $\lambda_i(\kappa)$ is a zero of $\chi_{A(\kappa)}(\cdot)$ there exists some eigenvalue γ_ν, $\nu \in \{1, \dots, m\}$, of Γ such that $\lambda_i(\kappa)^r + \kappa\gamma_\nu \sum_{j=0}^{r-1} k_{j+1}\lambda_i(\kappa)^j = 0$. One may choose x^ν such that $\Gamma x^\nu = \gamma_\nu x^\nu$. Then

$$\begin{aligned}
&\big(A(\kappa) - \lambda_i(\kappa) I_{rm}\big) v^i(\kappa) \\
&= \begin{pmatrix} \lambda_i(\kappa) x^\nu \\ \vdots \\ \lambda_i(\kappa)^{r-1} x^\nu \\ -\kappa\Gamma\left[\sum_{j=0}^{r-1} k_{j+1}\lambda_i(\kappa)^j\right] x^\nu \end{pmatrix} - \begin{pmatrix} \lambda_i(\kappa) x^\nu \\ \vdots \\ \lambda_i(\kappa)^{r-1} x^\nu \\ \lambda_i(\kappa)^r x^\nu \end{pmatrix} \\
&= \begin{pmatrix} 0_{m\times 1} \\ \vdots \\ 0_{m\times 1} \\ -\left(\lambda_i(\kappa)^r + \kappa\gamma_\nu \sum_{j=0}^{r-1} k_{j+1}\lambda_i(\kappa)^j\right) x^\nu \end{pmatrix} = 0_{rm},
\end{aligned}$$

which shows (3.2.4).

Step 3: Let $v^i(\kappa)$ be an eigenvector of $A(\kappa)$ to the eigenvalue $\lambda_i(\kappa)$ and suppose that, for some $\ell_i \in \{1, \ldots, r-1\}$, there exists a chain of generalized eigenvectors $v^i(\kappa), v^{i+1}(\kappa), \ldots, v^{i+\ell_i}(\kappa)$ to $\lambda_i(\kappa)$, which satisfy

$$\forall\, \mu \in \{1, \ldots, \ell_i\} \;:\; \big(A(\kappa) - \lambda_i(\kappa) I_{rm}\big) v^{i+\mu}(\kappa) = v^{i+\mu-1}(\kappa)\,.$$

Then

$$\forall\, \mu \in \{0, \ldots, \ell_i\} \;:\; v^{i+\mu}(\kappa) = \frac{1}{\mu!} \frac{\mathrm{d}^\mu}{\mathrm{d}\lambda_i(\kappa)^\mu} \begin{pmatrix} x^\nu \\ \lambda_i(\kappa) x^\nu \\ \vdots \\ \lambda_i(\kappa)^{r-1} x^\nu \end{pmatrix}. \tag{3.2.5}$$

This claim will be proved by induction: for $\mu = 0$, (3.2.5) follows obviously from (3.2.4). Suppose (3.2.5) holds for $\mu \in \{0, \ldots, \ell_i - 1\}$ and show (3.2.5) for $\mu + 1$. Adopt the convention that, for $n, m \in \mathbb{N}$ with $m > n$, the binomial coefficient $\binom{n}{m} = 0$. Then, in view of $\lambda_i(\kappa)$ being a zero of $s \mapsto \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}s^{\mu+1}} \Big(s^r + \kappa\gamma_\nu \sum_{j=0}^{r-1} k_{j+1} s^j\Big)$ and omitting the κ in $\lambda_i(\kappa)$ to improve readability,

$$\begin{aligned}
&\big(A(\kappa) - \lambda_i I_{rm}\big) v^{i+\mu+1}(\kappa) \\
&= \begin{pmatrix} \frac{1}{(\mu+1)!} \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}\lambda_i^{\mu+1}} \big(\lambda_i\big) x^\nu \\ \vdots \\ \frac{1}{(\mu+1)!} \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}\lambda_i^{\mu+1}} \big(\lambda_i^{r-1}\big) x^\nu \\ -\kappa\Gamma \left[\sum_{j=0}^{r-1} k_{j+1} \frac{1}{(\mu+1)!} \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}\lambda_i^{\mu+1}} \big(\lambda_i^j\big)\right] x^\nu \end{pmatrix} - \begin{pmatrix} \frac{\lambda_i}{(\mu+1)!} \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}\lambda_i^{\mu+1}} \big(\lambda_i^0\big) x^\nu \\ \vdots \\ \frac{\lambda_i}{(\mu+1)!} \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}\lambda_i^{\mu+1}} \big(\lambda_i^{r-2}\big) x^\nu \\ \frac{\lambda_i}{(\mu+1)!} \frac{\mathrm{d}^{\mu+1}}{\mathrm{d}\lambda_i^{\mu+1}} \big(\lambda_i^{r-1}\big) x^\nu \end{pmatrix} \\
&= \begin{pmatrix} \frac{(\mu+1)!}{(\mu+1)!} \binom{1}{\mu+1} \lambda_i^{1-(\mu+1)} x^\nu \\ \vdots \\ \frac{(\mu+1)!}{(\mu+1)!} \binom{r-1}{\mu+1} \lambda_i^{r-1-(\mu+1)} x^\nu \\ -\kappa\gamma_\nu \left[\sum_{j=0}^{r-1} k_{j+1} \frac{(\mu+1)!}{(\mu+1)!} \binom{j}{\mu+1} \lambda_i^{j-(\mu+1)}\right] x^\nu \end{pmatrix} - \begin{pmatrix} \lambda_i \binom{0}{\mu+1} \lambda_i^{0-(\mu+1)} x^\nu \\ \vdots \\ \lambda_i \binom{r-2}{\mu+1} \lambda_i^{r-2-(\mu+1)} x^\nu \\ \lambda_i \binom{r-1}{\mu+1} \lambda_i^{r-1-(\mu+1)} x^\nu \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \binom{0}{\mu}\lambda_i^{0-\mu}x^\nu \\ \vdots \\ \binom{r-2}{\mu}\lambda_i^{r-2-\mu}x^\nu \\ \Big[-\Big(\binom{r}{\mu+1}-\binom{r}{\mu+1}+\binom{r-1}{\mu+1}\Big)\lambda_i^{r-(\mu+1)} \\ -\kappa\gamma_\nu\sum_{j=0}^{r-1}k_{j+1}\binom{j}{\mu+1}\lambda_i^{j-(\mu+1)}\Big]x^\nu \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\mu!}\frac{\mathrm{d}^\mu}{\mathrm{d}\lambda_i^\mu}\left(\lambda_i^0\right)x^\nu \\ \vdots \\ \frac{1}{\mu!}\frac{\mathrm{d}^\mu}{\mathrm{d}\lambda_i^\mu}\left(\lambda_i^{r-2}\right)x^\nu \\ \binom{r-1}{\mu}\lambda_i^{r-1-\mu}x^\nu \end{pmatrix} = \frac{1}{\mu!}\frac{\mathrm{d}^\mu}{\mathrm{d}\lambda_i^\mu}\begin{pmatrix} \lambda_i^0 x^\nu \\ \vdots \\ \lambda_i^{r-2}x^\nu \\ \lambda_i^{r-1}x^\nu \end{pmatrix} = v^{i+\mu}(\kappa)\,,$$

which shows (3.2.5).

By assumption $\{x^1,\ldots,x^m\}$ are linearly independent. Thus, there are at most m chains of generalized eigenvectors $v^i(\kappa),\ldots,v^{i+\ell_i}(\kappa)$ associated with some x^ν, $\nu\in\{1,\ldots,m\}$, with length $\ell_i+1\le r$. Since, for all $j\in\{1,\ldots,m\}$, the polynomial $s\mapsto\left(s^r+\gamma_j\kappa\sum_{i=0}^{r-1}k_{i+1}s^i\right)$ has exactly one zero $\lambda_{(m-1)r+j}(\kappa)$ with $|\lambda_{(m-1)r+j}(\kappa)|\ge c_0$ and exactly $r-1$ zeros $\{\lambda_{(j-1)(r-1)+1}(\kappa),\ldots,\lambda_{(j-1)(r-1)+r-1}(\kappa)\}\subset\mathcal{B}_{c_0}(0)$, any chain of generalized eigenvectors $v^i(\kappa),\ldots,v^{i+\ell_i}(\kappa)$ has length $\ell_i+1\le r-1$, hence $\ell_i\le r-2$.

Step 4: Derive the inverse of the Jordan transformation matrix $V(\kappa)$.

Recall $V(\kappa)=\left[v^1(\kappa),\ldots,v^{rm}(\kappa)\right]$, where the columns are the (generalized) eigenvectors given by (3.2.4) and (3.2.5), respectively. Note that the last m columns $v^{(r-1)m+1}(\kappa),\ldots,v^{rm}(\kappa)$ of $V(\kappa)$ are the eigenvectors associated with the eigenvalues $\lambda_{(r-1)m+1}(\kappa),\ldots,\lambda_{rm}(\kappa)$, where, in view of (3.2.4), one may choose

$$\forall\, j\in\{1,\ldots,m\}\;:\; v^{(r-1)m+j}(\kappa)=\begin{pmatrix} x^j \\ \lambda_{(r-1)m+j}(\kappa)x^j \\ \vdots \\ \lambda_{(r-1)m+j}(\kappa)^{r-1}x^j \end{pmatrix}. \qquad (3.2.6)$$

Divide the matrices $V(\kappa)$ and $\Lambda(\kappa)$ into parts related to the first

$(r-1)m$ and last m eigenvalues of $A(\kappa)$, respectively:

$$\Lambda(\kappa) = \left[\begin{array}{c|c} \widehat{\Lambda}(\kappa) & 0 \\ \hline 0 & \operatorname{diag}(\lambda_{(r-1)m+1}(\kappa), \ldots, \lambda_{rm}(\kappa) + N \end{array}\right]$$
$$V(\kappa) = \left[\widehat{V}(\kappa) \,\middle|\, v^{(r-1)m+1}(\kappa), \ldots, v^{rm}(\kappa)\right] = \left[\widehat{V}(\kappa) \,\middle|\, \widetilde{V}(\kappa)\right],$$

where $N = 0_{m\times m}$ since the last m colums of $\widetilde{V}(\kappa)$ consists of the m eigenvectors $v^{(r-1)m+1}(\kappa), \ldots, v^{rm}(\kappa)$ of $A(\kappa)$.

Assumption $\lambda_1(\kappa), \ldots, \lambda_{(r-1)m}(\kappa) \in \mathcal{B}_{c_0}(0)$ yields, in view of (3.2.4) and (3.2.5), that the associated eigenvectors and generalized eigenvectors $v^1(\kappa), \ldots, v^{(r-1)m}(\kappa)$ are also bounded in $\mathbb{C}^{rm}$. So

$$\exists\, c_2 > 0\ \forall\, \kappa > \kappa_0\ :\ \|\widehat{V}(\kappa)\| < c_2\,. \tag{3.2.7}$$

Recall that all chains of generalized eigenvectors $v^i(\kappa), \ldots, v^{i+\ell_i}$ have length $\ell_i + 1 \le r-1$. Thus, all columns of $\left[I_{(r-1)m}, 0_{rm\times m}\right]\widehat{V}(\kappa) \in \mathbb{C}^{(r-1)m\times(r-1)m}$ are non-zero and, in view of (3.2.4) and (3.2.5), linearly independent. Hence $\det\left(\left[I_{(r-1)m}, 0_{(r-1)m\times m}\right]\widehat{V}(\kappa)\right) \neq 0$. Then

$$\begin{aligned}
&\det V(\kappa)\\
&= \det\left(\left[I_{(r-1)m}, 0_{(r-1)m\times m}\right]\widehat{V}(\kappa)\right)\\
&\quad \cdot \det\Bigg(\left[\lambda_{(r-1)m+1}(\kappa)^{r-1}x^1, \ldots, \lambda_{rm}(\kappa)^{r-1}x^m\right]\\
&\qquad\qquad - \left[0_{m\times(r-1)m}, I_m\right]\widehat{V}(\kappa)\left(\left[I_{(r-1)m}, 0_{(r-1)m\times m}\right]\widehat{V}(\kappa)\right)^{-1}\\
&\qquad\qquad\quad \cdot \left[I_{(r-1)m}, 0_{(r-1)m\times m}\right]\widetilde{V}(\kappa)\Bigg)
\end{aligned} \tag{3.2.8}$$

may be considered as multivariate polynomial in $\mathbb{C}[s_1, \ldots, s_m]$ in the m pairwise distinct variables $\lambda_{(r-1)m+1}(\kappa), \ldots, \lambda_{rm}(\kappa)$ which satisfies

$$\begin{aligned}
&\forall\, j \in \{1, \ldots, m\}\ :\\
&\qquad \deg\left(\det V(\kappa)(s_1, \ldots, s_{j-1}, \cdot, s_{j+1}, \ldots, s_m)\right) = r-1\,,
\end{aligned} \tag{3.2.9}$$

for fixed $(s_1, \ldots, s_m) \in \mathbb{C}^{1\times m}$. Note that $\lambda_{(r-1)m+i}(\kappa) = \lambda_{(r-1)m+j}(\kappa)$ is allowed for some $i \neq j$, however, the determinant of $V(\kappa)$ is always considered as multivariate polynomial in m variables.

The inverse of $V(\kappa)$ is given as follows:

$$V(\kappa)^{-1} = \frac{1}{\det V(\kappa)} \operatorname{adj} V(\kappa)\,,$$

where, omitting the κ in $\lambda_{(r-1)m+1}(\kappa), \ldots, \lambda_{rm}(\kappa)$,

$$\begin{aligned} &\operatorname{adj} V(\kappa) \\ &= \begin{bmatrix} \psi_{1,1}(\lambda_{(r-1)m+1}, \ldots, \lambda_{rm}) & \ldots & \psi_{1,rm}(\lambda_{(r-1)m+1}, \ldots, \lambda_{rm}) \\ \vdots & \ddots & \vdots \\ \psi_{rm,1}(\lambda_{(r-1)m+1}, \ldots, \lambda_{rm}) & \ldots & \psi_{rm,rm}(\lambda_{(r-1)m+1}, \ldots, \lambda_{rm}) \end{bmatrix}, \end{aligned}$$

for some multivariate polynomials $\psi_{i,j} \in \mathbb{C}[s_1, \ldots, s_m]$, for i and $j \in \{1, \ldots, rm\}$, with

$$\begin{aligned} &\forall\, i,j \in \{1, \ldots, rm\}\ \forall\, \nu \in \{1, \ldots, m\}\ : \\ &\qquad\qquad \deg(\psi_{i,j}(s_1, \ldots, s_{\nu-1}, \cdot, s_{\nu+1}, \ldots, s_m)) \le r-1\,, \\ &\forall\, i \in \{(r-1)m+1, \ldots, rm\}\ \forall\, j \in \{1, \ldots, rm\}\ : \\ &\qquad\qquad \deg(\psi_{i,j}(s_1, \ldots, s_{i-(r-1)m-1}, \cdot, s_{i-(r-1)m+1}, \ldots, s_m)) = 0\,. \end{aligned}$$

Step 5: Finally, boundedness of $\|P(\cdot)\|$ is shown.

Partition $V(\kappa)^{-1}$ as follows

$$V(\kappa)^{-1} = \frac{1}{\det V(\kappa)} \left[\begin{array}{c} \widehat{W}(\kappa) \\ \hline \widetilde{W}(\kappa) \end{array}\right]$$

where $\big((s_1, \ldots, s_m) \mapsto \widehat{W}(\kappa)(s_1, \ldots, s_m)\big) \in \mathbb{C}[s_1, \ldots, s_m]^{(r-1)m \times rm}$ satisfies $\deg \widehat{W}(\kappa)(s_1, \ldots, s_{\nu-1}, \cdot, s_{\nu+1}, \ldots, s_m) \le r-1$ for every $\nu \in \{1, \ldots, m\}$, and

$$\widetilde{W}(\kappa) = \begin{bmatrix} w^{(r-1)m+1}(\kappa) \\ \vdots \\ w^{rm}(\kappa) \end{bmatrix} \in \mathbb{C}[s_1, \ldots, s_m]^{m \times rm},$$

has the property $\deg\big(w^{(r-1)m+\nu}(\kappa)(s_1,\dots,s_{\nu-1},\cdot,s_{\nu+1},\dots,s_m)\big) = 0$ for all $\nu \in \{1,\dots,m\}$. Then, in view of (3.2.9)

$$\exists\, c_3 > 0\ \forall\, \kappa > \kappa_0\ :\ \left\| \frac{1}{\det V(\kappa)} \widehat{W}(\kappa) \right\| < c_3\,. \tag{3.2.10}$$

Moreover, in view of (3.2.6),

$$\begin{aligned}&\forall\, j,\nu \in \{1,\dots,m\}\ :\\ &\quad \deg\Big(\big(v^{(r-1)m+j}(\kappa)w^{(r-1)m+j}(\kappa)\big)(s_1,\dots,s_{\nu-1},\cdot,s_{\nu+1},\dots,s_m)\Big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le r-1\,,\end{aligned}$$

hence, for $t \ge 0$ and omitting some κ for ease of notation,

$$\begin{aligned}&\frac{1}{\det V(\kappa)} \widetilde{V}(\kappa)\,\mathrm{diag}\left(e^{\lambda_{(r-1)m+1}t},\dots,e^{\lambda_{rm}t}\right)\widetilde{W}(\kappa)\\ &= \frac{1}{\det V(\kappa)} \left[e^{\lambda_{(r-1)m+1}t}v^{(r-1)m+1},\dots,e^{\lambda_{rm}t}v^{rm}\right] \begin{bmatrix} w^{(r-1)m+1}\\ \vdots \\ w^{rm}\end{bmatrix}\\ &= \frac{1}{\det V(\kappa)} \sum_{j=1}^{m} e^{\lambda_{(r-1)m+j}t} v^{(r-1)m+j}(\kappa) w^{(r-1)m+j}(\kappa)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\exists\, c_4 > 0\ \forall\, \kappa > \kappa_0\ :\\ &\quad\left\| \frac{1}{\det V(\kappa)} \widetilde{V}(\kappa)\,\mathrm{diag}\left(e^{\lambda_{(r-1)m+1}t},\dots,e^{\lambda_{rm}t}\right)\widetilde{W}(\kappa)\right\| < c_4\,.\end{aligned} \tag{3.2.11}$$

Thus

$$\begin{aligned}P(\kappa) &= \int_0^\infty e^{A(\kappa)^T t} e^{A(\kappa)t}\,\mathrm{d}t\\ &\le \int_0^\infty \left\| V(\kappa)e^{\Lambda(\kappa)t}V(\kappa)^{-1}\right\|^2\,\mathrm{d}t\, I_{rm}\,,\end{aligned}$$

and (3.2.7), (3.2.10) and (3.2.11) give that, for $c_1 := (c_2 c_3 + c_4)^2$,

$$\begin{aligned}
P(\kappa) &\leq \int_0^\infty \left\| \left[\widehat{V}(\kappa) \,\middle|\, \widetilde{V}(\kappa) \right] \begin{bmatrix} e^{\widehat{\Lambda}(\kappa)t} & 0 \\ 0 & e^{\operatorname{diag}(\lambda_{(r-1)m+1}(\kappa),\dots,\lambda_{rm}(\kappa))t} \end{bmatrix} \right. \\
&\qquad\qquad \left. \cdot \frac{1}{\det V(\kappa)} \begin{bmatrix} \widehat{W}(\kappa) \\ \widetilde{W}(\kappa) \end{bmatrix} \right\|^2 \mathrm{d}t\, I_{rm} \\
&= \int_0^\infty \left\| \left[\widehat{V}(\kappa) e^{\widehat{\Lambda}(\kappa)t} \,\middle|\, \widetilde{V}(\kappa) \operatorname{diag}\left(e^{\lambda_{(r-1)m+1}(\kappa)t}, \dots, e^{\lambda_{rm}(\kappa)t} \right) \right] \right. \\
&\qquad\qquad \left. \cdot \frac{1}{\det V(\kappa)} \begin{bmatrix} \widehat{W}(\kappa) \\ \widetilde{W}(\kappa) \end{bmatrix} \right\|^2 \mathrm{d}t\, I_{rm} \\
&= \int_0^\infty \left\| \left[\widehat{V}(\kappa) e^{\widehat{\Lambda}(\kappa)t} \frac{1}{\det V(\kappa)} \widehat{W}(\kappa) \right. \right. \\
&\qquad\qquad \left. \left. + \frac{1}{\det V(\kappa)} \widetilde{V}(\kappa) \begin{bmatrix} e^{\lambda_{(r-1)m+1}(\kappa)t} & & \\ & \ddots & \\ & & e^{\lambda_{rm}(\kappa)t} \end{bmatrix} \widetilde{W}(\kappa) \right] \right\|^2 \mathrm{d}t\, I_{rm} \\
&\leq \int_0^\infty c_1 e^{-\delta\, t} \,\mathrm{d}t\, I_{rm} \\
&= c_1/\delta\, I_{rm}\,,
\end{aligned}$$

which shows (3.2.3) and completes the proof. □

The above result is essential for the proofs of the derivative feedback stabilization in the case of linear MIMO-systems with strict relative degree. The uniform boundedness of the Lyapunov equation solution enables the choice of arbitrarily large $\kappa > 0$ in the control strategy (3.1.1) to control all terms in the closed-loop systems, see Step 3 of the proof of Theorem 3.2.1 for detail.

Note that the result has one shortcoming: it is required that Γ has m linearly independent eigenvectors. A proof for Γ having (chains of) generalized eigenvectors might be also possible, however it would become much more technical than the above proof.

Note also that a similar result might be true for any multi-companion matrices which appear when proving that the control strategy (3.3.1) applied to linear MIMO-systems with non-strict relative degree yields an exponentially stable closed-loop system, see the proof of Theorem 3.3.1. However, a proof would be a lot more technical and is not included in the present thesis. Therefore, the control strategy (3.3.1) has the shortcoming that an additional design parameter $\nu > 0$ is required.

3.2.4 Proof of the main result

In this subsection a proof for the output derivative feedback stabilization result for linear MIMO-systems with strict relative degree is presented. Since there is an additional design parameter $\nu > 0$ in (3.3.1) Theorem 3.2.1 is not simply a corollary of Theorem 3.3.1. However, both proofs use similar ideas. Owing to the complexity of the result for systems with non-strict relative degree, the results about robustness of high-gain feedback stabilization in Chapter 7 are restricted to MIMO-systems with strict relative degree.

First, consider another standard property of positive definite matrices which is required throughout the proof. Since mostly the definition of positive definiteness of matrices requires that the matrix has to be Hermitian [HJ90] (symmetric for real matrices; in [GvL96, Sec. 4.2.2] one can find a very short section on the unsymmetrical case), a proof was not found in standard literature. There might be a proof of this lemma in some books on linear algebra, however, due to completeness of the present thesis, a proof is given here.

Lemma 3.2.4 *Any positive definite* $A \in \mathbb{R}^{n\times n}$ *satisfies* $\operatorname{spec}(A) \subset \mathbb{C}_+$.

Proof. Let $\lambda \in \operatorname{spec}(A) = \operatorname{spec}(A^T) \subset \mathbb{C}$ and let $v \in \mathbb{C}^n \setminus \{0\}$ be such that $Av = \lambda v$. Then $A\bar{v} = \bar{\lambda}\bar{v}$ and the definition of positive definiteness, see the list of symbols, yields

$$\begin{aligned} 0 < v^* A v + v^* A^T v = v^* \lambda v + (A\bar{v})^T v &= \lambda \|v\|^2 + (\bar{\lambda}\bar{v})^T v \\ &= \lambda \|v\|^2 + \bar{\lambda}\|v\|^2 = 2\operatorname{Re}(\lambda)\|v\|^2 , \end{aligned}$$

whence $\operatorname{Re}(\lambda) > 0$. □

The proof of Theorem 3.2.1 is structured as follows: first the system's normal form is utilized, then another coordinate transformation will be applied to design, in view of the root-locus Lemma 3.2.2, solutions of two Lyapunov equations which are finally used to create a Lyapunov function to prove exponential stability of the closed-loop system.

In the following let $\mu(p(\cdot)) := \max\{\mathrm{Re}\, s \mid s \in \mathbb{C}, p(s) = 0\}$ denote the largest real part of the zeros of $p \in \mathbb{C}[s]$ and let $\mathbb{R}^H[s] := \{p \in \mathbb{R}[s] \mid \mu(p) < 0\}$ be the set of all Hurwitz polynomials. Moreover, recall that $\mathcal{Z}(p) = \{s \in \mathbb{C} \mid p(s) = 0\}$ is the set of zeros of $p \in \mathbb{C}[s]$.

Proof of Theorem 3.2.1. Let $x(\cdot)$ be a solution of (3.2.1).

Step 1: Representation of (3.2.1) in normal form.

By Corollary 2.2.5 there exists an invertible $\overline{U} \in \mathbb{R}^{n\times n}$ such that the coordinate transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := \overline{U} x$$

converts (3.2.1) into $(\overline{A}, \overline{B}, \overline{C})$ with

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \underbrace{\left[\begin{array}{cccc|c} 0 & I_m & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & I_m & 0 \\ R_1 & \dots & & R_r & S \\ \hline P & 0 & \dots & 0 & Q \end{array}\right]}_{=:\overline{A}=\overline{U}A\overline{U}^{-1}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \underbrace{\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ \hline 0 \end{array}\right]}_{=:\overline{B}=\overline{U}B} u \\ y &= \underbrace{[I_m, 0, \dots, 0]}_{=:\overline{C}=C\overline{U}^{-1}} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \end{aligned}\right\} \tag{3.2.12}$$

where $P \in \mathbb{R}^{(n-rm)\times m}$, $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$, $S \in \mathbb{R}^{m\times(n-rm)}$ and $R_1, \dots, R_r \in \mathbb{R}^{m\times m}$ are given by (2.2.23)–(2.2.25). In the following let $\Gamma := CA^{r-1}B$.

Step 2: Scaling the state vector.

Section $\xi = (\xi_1/\xi_2/\dots/\xi_r)$ and $\zeta = (\zeta_1/\zeta_2/\dots/\zeta_r)$ with $\xi_i(t), \zeta_i(t) \in \mathbb{R}^m$ for $i \in \{1, \dots, r\}$. Setting $\zeta_i = \kappa^{-i+1}\xi_i$ for $i \in \{1, \dots, r\}$, yields

$$\dot{\zeta}_i \;=\; \kappa^{-i+1}\dot{\xi}_i = \kappa^{-i+1}\xi_{i+1} = \kappa\zeta_{i+1}\,, \qquad \text{for } i \in \{1, \dots, r-1\}\,,$$

and

$$\begin{aligned}\dot{\zeta}_r &= \kappa^{-r+1}\dot{\xi}_r\\ &= \kappa^{-r+1}\big[(R_1 - \kappa k_1 \kappa^r \Gamma)\xi_1 + \cdots + (R_{r-1} - \kappa k_{r-1}\kappa^2\Gamma)\xi_{r-1}\\ &\qquad + (R_r - \kappa k_r \kappa \Gamma)\xi_r\big] + \kappa^{-r+1} S\eta\\ &= \kappa\left[\left(\frac{1}{\kappa^r}R_1 - \kappa k_1 \Gamma\right)\zeta_1 + \cdots + \left(\frac{1}{\kappa^2}R_{r-1} - \kappa k_{r-1}\Gamma\right)\zeta_{r-1}\right.\\ &\qquad \left. + \left(\frac{1}{\kappa}R_r - \kappa k_r \Gamma\right)\zeta_r\right] + \kappa^{-r+1} S\eta\,.\end{aligned}$$

Thus, for $\kappa \geq 1$, additional scaling

$$\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \underbrace{\operatorname{diag}\left(I_m, \kappa^{-1}I_m, \ldots, \kappa^{-r+1}I_m, I_{n-rm}\right)}_{=:\, U_\kappa}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

leads to

$$\left.\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \eta\end{pmatrix} &= \underbrace{\left[\begin{array}{c|c}\kappa\begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m\\ \frac{R_1}{\kappa^r} - \kappa k_1\Gamma & \ldots & \frac{R_{r-1}}{\kappa^2} - \kappa k_{r-1}\Gamma & \frac{R_r}{\kappa} - \kappa k_r \Gamma\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\\ \kappa^{-r+1}S\end{matrix}\\ \hline \begin{matrix} P & 0 & \ldots & 0\end{matrix} & Q\end{array}\right]}_{=:\, \overline{A}_{\Gamma,k,\kappa}}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}\\ y &= (I_m, 0, \ldots, 0)\begin{pmatrix}\zeta\\ \eta\end{pmatrix}.\end{aligned}\right\} \tag{3.2.13}$$

Step 3: Design of positive definite solutions of two Lyapunov equations.

Since Γ is positive definite and $\operatorname{spec}(\Gamma) = \{\gamma_1, \ldots, \gamma_m\} \subset \mathbb{C}_+$, see Lemma 3.2.4, one may choose a Jordan decomposition of Γ such that, in view of [GvL96, Thm. 7.1.9], $\Gamma = XJX^{-1}$ for an invertible matrix $X \in \mathbb{C}^{m\times m}$ and the Jordan normal form $J \in \mathbb{C}^{m\times m}$ of Γ.

In view of Step 1 of the proof of Lemma 3.2.3 it follows that

$$\det\left(sI_n - \begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ -\kappa k_1\Gamma & \dots & -\kappa k_{r-1}\Gamma & -\kappa k_r\Gamma \end{bmatrix} \right) = \prod_{j=1}^{m} \left(s^r + \gamma_j \sum_{i=0}^{r-1} \kappa k_{i+1} s^i \right).$$

Recall the assumption $\left(s \mapsto p(s) := \sum_{i=0}^{r-1} k_{i+1}\, s^i\right) \in \mathbb{R}^H[s]$ which, setting

$$\delta := -\mu(p(\cdot)) > 0\,,$$

together with $\gamma_j > 0$, for all $j \in \{1, \dots, m\}$, and Lemma 3.2.2 then yields

$$\begin{aligned} &\exists\, \kappa^* > 0\ \forall\, \kappa > \kappa^*\ : \\ &\quad \mu\left(s \mapsto \det\left(s^r I_m + \left(\sum_{i=0}^{r-1} \kappa k_{i+1} s^i \right) \Gamma \right) \right) \\ &\qquad = \mu\left(s \mapsto \prod_{j=1}^{m} \left(s^r + \gamma_j \kappa \sum_{i=0}^{r-1} k_{i+1} s^i \right) \right) < -\delta/2\,. \end{aligned}$$

Thus, and since $\operatorname{spec}(Q) \subset \mathbb{C}_-$, one may choose, for all $\kappa > \kappa^*$, positive definite matrices $N_\zeta(\kappa) = N_\zeta(\kappa)^T \in \mathbb{R}^{mr\times mr}$ and $N_\eta = N_\eta^T \in \mathbb{R}^{(n-mr)\times(n-mr)}$ such that

$$\begin{aligned} N_\zeta(\kappa) \begin{bmatrix} 0_{(r-1)m\times m} & I_{(r-1)m} \\ -\kappa k_1\Gamma & \dots - \kappa k_r\Gamma \end{bmatrix} \\ + \begin{bmatrix} 0_{(r-1)m\times m} & I_{(r-1)m} \\ -\kappa k_1\Gamma & \dots - \kappa k_r\Gamma \end{bmatrix}^T N_\zeta(\kappa) &= -I_{mr}\,, && (3.2.14a) \\ N_\eta Q + Q^T N_\eta &= -I_{n-mr}\,. && (3.2.14b) \end{aligned}$$

Moreover, setting $\{\lambda_{j,1}(\kappa), \dots, \lambda_{j,r}(\kappa)\} = \mathcal{Z}(s \mapsto s^r + \gamma_j\kappa \sum_{i=0}^{r-1} k_{i+1} s^i)$

for $j \in \{1,\dots,m\}$, Lemma 3.2.2 yields the existence of a constant $c_0 > 0$ such that, for all $j \in \{1,\dots,m\}$,

$$\forall\, \kappa > \kappa^* \;:\; \{\lambda_{j,1}(\kappa),\dots,\lambda_{j,r-1}(\kappa)\} \subset \mathcal{B}_{c_0}(0) \text{ and } |\lambda_r(\kappa)| \geq c_0\,.$$

Since Γ has m linearly independent eigenvectors, Lemma 3.2.3 yields that there exists a constant $c_1 > 0$ such that, for all $\kappa > \kappa^*$,

$$\|N_\zeta(\kappa)\| \leq 2c_1/\delta\,.$$

Step 4: Design of a Lyapunov function to show exponential stability. The derivative of

$$t \mapsto V(t) \;:=\; \tfrac{1}{2}\,\zeta(t)^T N_\zeta(\kappa)\zeta(t) + \tfrac{1}{2}\,\eta(t)^T N_\eta \eta(t)$$

along the solution of

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t) = \overline{A}_{\Gamma,k,\kappa}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t)$$

yields, for all $t \geq 0$, and omitting the argument t for brevity,

$$\begin{aligned}
\dot V(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\,\zeta^T N_\zeta(\kappa)\zeta + \tfrac{1}{2}\,\eta^T N_\eta \eta\right)\\
&= \zeta^T N_\zeta(\kappa)\left(\kappa\begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ \frac{R_1}{\kappa^r} - \kappa k_1\Gamma, & \dots, & \frac{R_{r-1}}{\kappa^2} - \kappa k_{r-1}\Gamma, & \frac{R_r}{\kappa} - \kappa k_r\Gamma \end{bmatrix}\zeta + \begin{bmatrix} 0\\ \vdots \\ 0 \\ \frac{1}{\kappa^{r-1}} S\eta\end{bmatrix}\right)\\
&\quad + \eta^T N_\eta\left(Q\eta + P\zeta_1\right)\\
&\overset{(3.2.14)}{\leq} -\tfrac{\kappa}{2}\|\zeta\|^2 + \kappa\zeta N_\zeta(\kappa)\begin{bmatrix} 0_{m(r-1)\times mr} \\ \frac{R_1}{\kappa^r} \;\cdots\; \frac{R_r}{\kappa}\end{bmatrix}\zeta + \frac{1}{\kappa^{r-1}}\|N_\zeta(\kappa)\|\,\|S\|\,\|\zeta\|\,\|\eta\|\\
&\quad -\tfrac{1}{2}\|\eta\|^2 + \|N_\eta P\|\,\|\eta\|\,\|\zeta_1\|\\
&\overset{\kappa\geq 1}{\leq} -\tfrac{\kappa}{2}\|\zeta\|^2 + \|N_\zeta(\kappa)\|\,\|(R_1,\dots,R_r)\|\,\|\zeta\|^2 + \tfrac{1}{\kappa^{r-1}}\|N_\zeta(\kappa)\|\,\|S\|\,\|\zeta\|^2\\
&\quad + \tfrac{1}{\kappa^{r-1}}\|N_\zeta(\kappa)\|\,\|S\|\,\|\eta\|^2 - \tfrac{1}{2}\|\eta\|^2 + \tfrac{1}{4}\|\eta\|^2 + 4\|N_\eta P\|\,\|\zeta_1\|^2\\
&\leq -\Big(\tfrac{\kappa}{2} - \|N_\zeta(\kappa)\|\,\|(R_1,\dots,R_r)\| - \|N_\zeta(\kappa)\|\,\|S\| - 4\|N_\eta P\|\Big)\|\zeta\|^2\\
&\quad - \left(\tfrac{1}{4} - \tfrac{\|N_\zeta(\kappa)\|\,\|S\|}{\kappa^{r-1}}\right)\|\eta\|^2\,.
\end{aligned}$$

Setting

$$\kappa^{**} := \max\left\{\tfrac{1}{4} + 2(2c_1 \,\|(R_1,\ldots,R_r)\|/\delta - 2c_1 \,\|S\|/\delta - 4\|N_\eta P\|),\right.$$
$$\left.(16\, c_1 \,\|S\|)^{-r+1}/\delta, \kappa^*\right\},$$
$$\alpha := \min\left\{\frac{\delta}{16\, c_1}, \frac{1}{8\|N_\eta\|}\right\},$$

yields, for all $t \geq 0$ and $\kappa > \kappa^{**}$,

$$\begin{aligned}\dot V(t) &\leq -\tfrac{1}{8}\|\zeta(t)\|^2 - \tfrac{1}{8}\|\eta(t)\|^2\\ &\leq -\tfrac{1}{8\|N_\zeta(\kappa)\|}\zeta(t)^T N_\zeta(\kappa)\zeta(t) - \tfrac{1}{8\|N_\eta\|}\eta(t)^T N_\eta \eta(t) \leq -\alpha V(t),\end{aligned}$$

whence, since the initial value $x(t_0) = x^0$ for (3.2.1) leads to the initial value $\left(\begin{smallmatrix}\zeta\\ \eta\end{smallmatrix}\right)(t_0) = U_\kappa \overline{U} x^0$ for (3.2.13),

$$\forall\, t \geq t_0 \;\forall\, t_0 \geq 0 \;:$$
$$\left\|\begin{pmatrix}\zeta(t)\\ \eta(t)\end{pmatrix}\right\| \leq \exp\left(-\alpha(t-t_0)\right)\sqrt{\frac{\max\operatorname{spec}\left[\begin{smallmatrix}N_\zeta(\kappa) & 0\\ 0 & N_\eta\end{smallmatrix}\right]}{\min\operatorname{spec}\left[\begin{smallmatrix}N_\zeta(\kappa) & 0\\ 0 & N_\eta\end{smallmatrix}\right]}}\;\left\|\begin{pmatrix}\zeta(t_0)\\ \eta(t_0)\end{pmatrix}\right\|,$$

which completes the proof of the theorem. □

The above proof of high-gain derivative feedback stabilization for linear MIMO-systems with strict relative degree will be generalized to MIMO-systems with non-strict relative degree in the following section. However, the stabilizing feedback comes with one additional design parameter $\nu > 0$. This is due to the fact that a generalization of Lemma 3.2.3 for more involved multi-companion matrices appearing in the proof of Theorem 3.3.1 could not be proved yet.

3.3 MIMO-systems with non-strict relative degree

Stabilization by output derivative feedback of linear MIMO-systems (A, B, C) with non-strict relative degree is shown. Recall the linear system (A, B, C) with m inputs and m outputs of form (3.2.1), i.e.

$$\begin{aligned} \dot{x} &= Ax + \underbrace{\left[b_1^{(n)}, \ldots, b_m^{(n)}\right]}_{=B} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}}_{=u} \\ \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{=y} &= \underbrace{\begin{bmatrix} c_{(n)}^1 \\ \vdots \\ c_{(n)}^m \end{bmatrix}}_{=C} x \,, \end{aligned}$$

where $n, m \in \mathbb{N}$ with $m \leq n$ and $A \in \mathbb{R}^{n \times n}$, $B, C^T \in \mathbb{R}^{n \times m}$. For the remainder of this chapter the linear system (A, B, C) will satisfy the following three assumptions: it has (i) (non-strict vector) relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1 \times m}$, see Definition 2.2.1, (ii) a positive definite high-frequency gain matrix $\left[c_{(n)}^1 A^{r_1 - 1} B \big/ c_{(n)}^2 A^{r_2 - 1} B \big/ \ldots \big/ c_{(n)}^m A^{r_m - 1} B\right]$ and (iii) is minimum phase, i.e. has exponentially stable zero dynamics, see Definition 2.3.1.

As in the case of SISO-systems and MIMO-systems with strict relative degree no explicit knowledge of the system's data and only the structural properties (i)–(iii) are required.

In Section 3.2 stabilization by high-gain output derivative feedback of linear MIMO-systems with strict relative degree was shown. Due to the higher complexity of the normal form (2.2.3)–(2.2.4) for MIMO-systems with non-strict relative degree, the control strategy (3.2.2) applied in Theorem 3.2.1 is not easily applicable to systems with non-strict relative degree. It could be possible to prove such stabilization results for systems with known upper bound for the (vector) relative degree, see [Hop07]. However, this is not part of the present thesis.

It is not at all hard to get an idea why controller (3.2.2) does not fit for MIMO-systems with non-strict relative degree: consider a system (A, B, C) with $m = 2$, i.e.

$$\begin{aligned} \dot{x} &= Ax + \left[b_1^{(n)}, b_2^{(n)}\right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{bmatrix} c_{(n)}^1 \\ c_{(n)}^2 \end{bmatrix} x \,, \end{aligned}$$

which has relative degree $r = (r_1, r_2) \in \mathbb{N}^{1\times 2}$ with $r_1 > r_2$. Recall that the normal form (2.2.3)–(2.2.4) for this system is given by

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \left[\begin{array}{cccc|cccc|c}
0 & 1 & & 0 & 0 & \dots & & 0 & 0_{1\times(n-r_1-r_2)}\\
\vdots & \ddots & \ddots & & \vdots & & & \vdots & \vdots\\
0 & \dots & 0 & 1 & 0 & \dots & & 0 & 0_{1\times(n-r_1-r_2)}\\
R^1_{1,1} & \dots & & R^1_{1,r_1} & R^1_{2,1} & \dots & & R^1_{2,r_2} & S^1\\
\hline
0 & \dots & & 0 & 0 & 1 & & 0 & 0_{1\times(n-r_1-r_2)}\\
\vdots & & & \vdots & \vdots & \ddots & \ddots & & \vdots\\
0 & \dots & & 0 & 0 & \dots & 0 & 1 & 0_{1\times(n-r_1-r_2)}\\
R^2_{1,1} & \dots & & R^2_{1,r_1} & R^2_{2,1} & \dots & & R^2_{2,r_2} & S^2\\
\hline
P_1 & 0 & \dots & 0 & P_2 & 0 & \dots & 0 & Q
\end{array}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \left[\begin{array}{c}
0_{1\times 2}\\ \vdots\\ 0_{1\times 2}\\ c^1_{(n)}A^{r_1-1}B\\ \hline 0_{1\times 2}\\ \vdots\\ 0_{1\times 2}\\ c^2_{(n)}A^{r_2-1}B\\ \hline 0_{(n-r_1-r_2)\times 2}
\end{array}\right] u\\
y &= \left[\begin{array}{cccc|cccc|c}
1 & 0 & \dots & 0 & 0 & \dots & & 0 & \multirow{2}{*}{$0_{2\times(n-r_1-r_2)}$}\\
0 & \dots & & 0 & 1 & 0 & \dots & 0 &
\end{array}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix},
\end{aligned}
$$

Recall the controller (3.2.2):

$$
u(t) = -\kappa \sum_{i=0}^{\varrho-1} \kappa^{\varrho-i} k_{i+1}\, y^{(i)}(t)\,,
$$

with suitable design parameters $k_1, \dots, k_m \in \mathbb{R}$ and sufficiently large $\kappa > 0$. Note that, in view of Theorem 3.2.1, $\varrho \in \mathbb{N}$ should be equal to either r_1 or r_2. In view of the proof of Theorem 3.2.1, it is not possible to choose $\varrho = r_2$ since $r_2 < r_1$, and so stabilization of the "upper subsystem" (system concerning the first component of y) cannot be achieved. Hence one has to choose $\varrho = r_1$. But then showing stabilization of the "lower subsystem" might be complicated. However, in view of the normal form for MIMO-systems with non-strict relative degree, Theorem 3.3.1 presents a natural generalization for the stabilizing feedback law.

3.3.1 Main result: feedback stabilization for MIMO-systems with non-strict relative degree

The main result of this section, namely stabilization of linear MIMO-systems with non-strict vector relative degree by feedback law (3.3.1), is on the one hand a generalization of the stabilization result for systems with strict relative degree: Theorem 3.2.1. On the other hand (3.3.1) has one strong shortcoming: there is another design parameter $\nu > 0$. This is due to the fact that a generalization of Lemma 3.2.3 is not available yet: it might be possible to prove a similar statement for more involved multi-companion matrices, however, a proof would be extremely technical.

Theorem 3.3.1 *Suppose that system* (A, B, C) *of form* (3.2.1) *has vector relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$, *positive definite high-frequency gain matrix* $\left[c^1_{(n)} A^{r_1-1}B / c^2_{(n)} A^{r_2-1}B / \ldots / c^m_{(n)} A^{r_m-1}B\right]$ *and is minimum phase, i.e. has exponentially stable zero dynamics. Then, for any* m *Hurwitz polynomials*

$$\left(s \mapsto \sum_{i=0}^{r_j-1} k_{j,i+1}\, s^i\right) \in \mathbb{R}[s]\,, \quad j = 1, \ldots, m\,,$$

there exists $\nu^* \geq 1$ *such that, for all* $\nu > \nu^*$, *there exists* $\kappa^* \geq 1$ *such that, for all* $\kappa > \kappa^*$, *the feedback*

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix} = -\nu \begin{pmatrix} \sum_{i=0}^{r_1-1} \kappa^{r_1-i} k_{1,i+1}\, y_1^{(i)}(t) \\ \vdots \\ \sum_{i=0}^{r_m-1} \kappa^{r_m-i} k_{m,i+1}\, y_m^{(i)}(t) \end{pmatrix} \tag{3.3.1}$$

applied to (3.2.1) *yields an exponentially stable closed-loop system.*

Because of the higher complexity of the normal form for linear MIMO-systems with non-strict relative degree the proof of the theorem is still similar to the proof of Theorem 3.2.1 but much more involved. A generalization of Lemma 3.2.2 and a very technical property of the determinant of a parameterized matrix are required. These follow next.

3.3.2 Advanced root-locus lemma

Recall the simple root-locus result Lemma 3.2.2: it was shown that, for $p \in \mathbb{R}[s]$ Hurwitz with $\deg p = r - 1$, $s \mapsto s^r + \kappa p(s)$ is again Hurwitz for sufficiently large $\kappa > 0$. Next this lemma will be generalized. For a polynomial $p \in \mathbb{R}[s]$ write

$$p(s) = p_n \prod_{j=1}^{\ell} (s - s_j)^{m_j}\,, \quad \text{can. fact.}$$

for the canonical factorization of p with $s_1, \ldots, s_\ell \in \mathbb{C}$ pairwise distinct and $m_1, \ldots, m_\ell \in \mathbb{N}$. It will be shown that a sum of d products of polynomials p_i of decreasing degree $n-i$ and an increasing power of a sufficiently large parameter $\nu > 0$ is a Hurwitz polynomial if the polynomial p_d with the smallest degree and a polynomial build from the leading coefficients of all polynomials are Hurwitz. This result is a generalization of [HP05, Thm. 4.1.2]. Recall that $\mu(p(\cdot)) := \max\{\mathrm{Re}\, s \mid s \in \mathbb{C}, p(s) = 0\}$ denotes the largest real part of the zeros of $p \in \mathbb{C}[s]$.

Lemma 3.3.2 *For $\delta > 0$, $d \in \mathbb{N}$ and $i \in \{0, \ldots, d\}$, let*

$$\big(s \mapsto p_i(s) = p_{i,n-i} s^{n-i} + p_{i,n-i-1} s^{n-i-1} + \cdots + p_{i,1} s + p_{i,0}\big) \in \mathbb{R}[s]\,,$$

such that

$$p_{d,n-d} > 0\,, \qquad p_{0,n} > 0\,, \qquad \mu(p_d(\cdot)) < -\delta$$

and

$$\big(\nu \mapsto \widehat{p}(\nu) = p_{0,n}\nu^d + p_{1,n-1}\nu^{d-1} + \cdots + p_{d-1,n-d+1}\nu + p_{d,n-d}\big) \in \mathbb{R}[\nu]$$

with $\mu(\widehat{p}(\cdot)) < -\delta$. Then

$$\exists\, \nu_0 > 0\ \forall\, \nu > \nu_0\ :\ \mu\left(\sum_{k=0}^{d} \nu^k p_k(\cdot)\right) < -\delta/2\,. \tag{3.3.2}$$

Proof. Write

$$p_d(s) = p_{d,n-d} \prod_{j=1}^{\ell_1} (s - \zeta_j)^{m_j}\,, \quad \text{can. fact.}$$

and

$$\widehat{p}(\nu) = p_{0,n} \prod_{j=1}^{\ell_2} (\nu - \xi_j)^{n_j}\,, \quad \text{can. fact.}$$

and, for $\gamma > 0$,

$$q[\gamma](s) := \sum_{k=0}^{d} \gamma^{d-k} p_k(s) = \gamma^d p_{0,n} \prod_{j=1}^{n} (s - s_j[\gamma]) \,, \qquad \text{with } s_1[\gamma], \ldots, s_n[\gamma] \in \mathbb{C} \,.$$

Suppose it is shown that, for $\alpha := \max_{j\in\{1,\ldots,\ell_1\}}\{|\zeta_j|\} + \delta$ and suitable numbering of the zeros $s_j[\gamma]$,

$$\begin{aligned} &\exists\, \gamma_0 > 0\ \forall\, \gamma \in (0, \gamma_0)\ : \\ &\qquad \{s_1[\gamma], \ldots, s_{n-d}[\gamma]\} \subset \bigcup_{j\in\{1,\ldots,\ell_1\}} \mathcal{B}_{\delta/2}(\zeta_j) \subset \mathbb{C}_- \,, && (3.3.3a) \\ &\qquad \{s_{n-d+1}[\gamma], \ldots, s_n[\gamma]\} \subset \{s \in \mathbb{C} \mid \operatorname{Re} s < -\alpha\} \subset \mathbb{C}_- \,. && (3.3.3b) \end{aligned}$$

Then there exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$ all zeros of $q[\gamma]$ are in $\mathbb{C}_{-\delta/2}$. Setting $\nu = \gamma^{-1}$ yields $p[\nu] = \nu^d q[\nu^{-1}] = \nu^d q[\gamma]$. Hence, for all $\nu, \gamma > 0$ and $s \in \mathbb{C}$, $p[\nu](s) = 0$ if, and only if, $q[\gamma](s) = 0$. Thus setting $\nu_0 = \gamma_0^{-1}$ yields (3.3.2).

In the remainder of the proof (3.3.3) is shown. One may choose $\varepsilon^* > 0$ such that

$$\begin{aligned} &\forall\, i, j \in \{1, \ldots, \ell_1\},\ i \neq j\ :\ \mathcal{B}_{\varepsilon^*}(\zeta_i) \cap \mathcal{B}_{\varepsilon^*}(\zeta_j) = \emptyset \,, \\ &\forall\, i, j \in \{1, \ldots, \ell_2\},\ i \neq j\ :\ \mathcal{B}_{\varepsilon^*}(\xi_i) \cap \mathcal{B}_{\varepsilon^*}(\xi_j) = \emptyset \,. \end{aligned}$$

Then, an application of [HP05, Thm. 4.1.2] to the polynomial $q[\gamma](\cdot) = \sum_{k=0}^{d-1} \gamma^{d-k} p_k(\cdot) + p_d(\cdot)$ and suitable numbering of the zeros $s_j[\gamma]$ of $q[\gamma]$ implies

$$\begin{aligned} &\forall\, \varepsilon \in (0, \min\{\varepsilon^*, \delta/2\})\ \exists\, \gamma^* = \gamma^*(\varepsilon) > 0\ \forall\, \gamma \in (0, \gamma^*)\ : \\ &\qquad \{s_1[\gamma], \ldots, s_{n-d}[\gamma]\} \subset \textstyle\bigcup_{j\in\{1,\ldots,\ell_1\}} \mathcal{B}_\varepsilon(\zeta_j) \,, \\ &\qquad \{s_{n-d+1}[\gamma], \ldots, s_n[\gamma]\} \subset \mathbb{C} \setminus \mathcal{B}_{1/\varepsilon}(0) \,. \end{aligned}$$

Now $\mu(p_d(\cdot)) < -\delta$ yields (3.3.3a).

For $\gamma > 0$ setting $x = \gamma s$ yields

$$
\begin{aligned}
&q[\gamma](s) \\
&= q[\gamma](\gamma^{-1}x) \\
&= \gamma^d \left(p_{0,n}\gamma^{-n}x^n + p_{0,n-1}\gamma^{-n+1}x^{n-1} + \cdots + p_{0,1}\gamma^{-1}x + p_{0,0}\right) \\
&\quad + \gamma^{d-1} \left(p_{1,n-1}\gamma^{-n+1}x^{n-1} + \cdots + p_{1,1}\gamma^{-1}x + p_{1,0}\right) \\
&\quad \vdots \\
&\quad + \gamma \left(p_{d-1,n-d+1}\gamma^{-n+d-1}x^{n-d+1} + \cdots + p_{d-1,1}\gamma^{-1}x + p_{d-1,0}\right) \\
&\quad + \left(p_{d,n-d}\gamma^{-n+d}x^{n-d} + \cdots + p_{d,1}\gamma^{-1}x + p_{d,0}\right) \\
&= \gamma^{-n+d} \Big(x^{n-d} \left(p_{0,n}x^d + p_{1,n-1}x^{d-1} + \cdots + p_{d-1,n-d+1}x + p_{d,n-d}\right) \\
&\qquad\qquad + \gamma\Big[\left(p_{0,n-1}x^{n-1} + \cdots + \gamma^{n-1}p_{0,0}\right) \\
&\qquad\qquad\qquad + \left(p_{1,n-2}x^{n-2} + \cdots + \gamma^{n-2}p_{1,0}\right) \\
&\qquad\qquad\qquad + \cdots + \left(p_{d,n-d-1}x^{n-d-1} + \cdots + \gamma^{n-d-1}p_{d,0}\right) \Big]\Big) .
\end{aligned}
$$

Write

$$
\begin{aligned}
\widehat{q}[\gamma](x) &:= \gamma^{n-d}q[\gamma](\gamma^{-1}x) \\
&= x^{n-d}\widehat{p}(x) + \gamma \sum_{k=0}^{d} \sum_{i=0}^{n-k-1} \left(\gamma^{n-k-1-i}p_{k,i}\, x^i\right) \\
&= p_{0,n} \prod_{j=1}^{n} (x - x_j[\gamma]) .
\end{aligned}
$$

Thus, and by [HP05, Thm. 4.1.2] and suitable numbering of the zeros $x_j[\gamma]$ of the polynomial $\widehat{q}[\gamma]$,

$$
\begin{aligned}
\forall\, \varepsilon \in (0, \min\{\varepsilon^*, \delta/2\})\ \exists\, \gamma_0 \in \left(0, \min\left\{\gamma^*, \varepsilon\alpha^{-1}, 1\right\}\right)\ \forall\, \gamma \in (0, \gamma_0)\ :& \\
\{x_1[\gamma], \ldots, x_{n-d}[\gamma]\} \subset \mathcal{B}_\varepsilon(0)\,,& \\
\{x_{n-d+1}[\gamma], \ldots, x_n[\gamma]\} \subset \textstyle\bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_\varepsilon(\xi_j)\,.&
\end{aligned}
$$

Hence $\mu(\widehat{p}(\cdot)) < -\delta$ yields, for suitable numbering of the zeros $x_j[\gamma]$,

$$\forall\,\gamma \in (0,\gamma_0)\ :\quad \{x_{n-d+1}[\gamma],\ldots,x_n[\gamma]\} \in \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_{\delta/2}(\xi_j) \subset \mathbb{C}_-\,. \qquad (3.3.4)$$

Furthermore, $\mu(\widehat{p}(\cdot)) < -\delta$ and $\gamma_0 < 1$ yields

$$\forall\,\gamma \in (0,\gamma_0)\ \forall\,x_0 \in \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_{\delta/2}(\xi_j)\ :\quad |\gamma^{-1}x_0| > |\gamma_0^{-1}x_0| > \varepsilon^{-1}\alpha\delta/2 > \alpha\,,$$

thus

$$\forall\,\gamma \in (0,\gamma_0)\ \forall\,x_0 \in \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_{\delta/2}(\xi_j)\ :\ \gamma^{-1}x_0 \notin \bigcup_{j\in\{1,\ldots,\ell_1\}} \mathcal{B}_{\delta/2}(\zeta_j)$$

and by (3.3.4), for suitable numbering of the zeros $x_j[\gamma]$,

$$\forall\,\gamma \in (0,\gamma_0)\ :\ \{\gamma^{-1}x_{n-d+1}[\gamma],\ldots,\gamma^{-1}x_n[\gamma]\} \in \{s \in \mathbb{C} \mid \operatorname{Re}(s) < -\alpha\}\,,$$

whence, noting that for every $\gamma > 0$ and $x_0 \in \mathbb{C}$, $\widehat{q}[\gamma](x_0) = 0$ if, and only if, $q[\gamma](\gamma^{-1}x_0) = 0$, (3.3.3b), which completes the proof. □

One can find various results about Hurwitz polynomials in the literature which are related to Lemma 3.3.2, for example Hurwitz properties under coefficient perturbation [Bar84, BG85, GB83, Soh89, WY87]. The results published in these papers use the Routh–Hurwitz criterion, see, for example [Gan86, Ch. 16]. However, all these results do not provide the necessary statements to prove the main results Theorem 3.2.1 and Theorem 3.3.1.

3.3.3 Determinant of a parameterized matrix

The following property of the determinant of the sum of a general matrix and a diagonal matrix multiplied by a parameter t, which is used in to prove the main result Theorem 3.3.1 of this chapter, is maybe a standard result in linear algebra. However, a proof for the following was not found in the literature. A reason for this might be that the proof given here is rather technical and it might be hard to simplify the used ideas and methods. A similar result for the characteristic polynomial of a matrix can be found in [Pen87].

First note that the principal minors of a matrix $A = [a_{i,j}]_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$ are defined as follows: for a set of indices $\{i_1,\dots,i_k\}$ with $1 \le i_1 < \dots < i_k \le n$, $k \in \{0,\dots,n\}$, let

$$\operatorname{minor}(A\,;\,\{i_1,\dots,i_k\}) := \det \begin{bmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \dots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{i_{k-1},i_k} \\ a_{i_k,i_1} & \dots & a_{i_k,i_{k-1}} & a_{i_k,i_k} \end{bmatrix},$$

$$\operatorname{minor}(A\,;\,\emptyset) := 1\,.$$

With the minors of a matrix A one can write the determinant of $A + t\ \operatorname{diag}(b_1,\dots,b_n)$ as polynomial in t as follows:

Lemma 3.3.3 *Let* $A = [a_{i,j}]_{i,j=1,\dots,m} \in \mathbb{R}^{m\times m}$ *and* $b_1,\dots,b_m \in \mathbb{R}$. *Then*

$$\det(A + t\ \operatorname{diag}(b_1,\dots,b_m)) = \sum_{k=0}^{m}\left(\sum_{1\le i_1<\dots<i_k\le m} \operatorname{minor}\left(A\,;\,\{i_1,\dots,i_k\}\right) \prod_{\substack{i\in\{1,\dots,m\}\\ i\notin\{i_1,\dots,i_k\}}} b_i\right) t^{m-k}\,. \quad (3.3.5)$$

Proof. Write, for $t \in \mathbb{C}$,

$$\det(A + t\ \operatorname{diag}(b_1,\dots,b_m)) = p_m t^m + p_{m-1}t^{m-1} + \dots + p_1 t + p_0$$

and define the function $f\colon \mathbb{R}^m \to \mathbb{R}$ by

$$(t_1,\dots,t_m) \mapsto f(t_1,\dots,t_m) = \det(A + \operatorname{diag}(b_1t_1,\dots,b_mt_m))\,.$$

Then

$$
\begin{aligned}
f(t_1,\dots,t_m) &= \det \begin{bmatrix} a_{1,1}+b_1t_1 & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2}+b_2t_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{m-1,m} \\ a_{m,1} & \cdots & a_{m,m-1} & a_{m,m}+b_mt_m \end{bmatrix} \\
&= \sum_{\sigma\in\Sigma_m} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} \left(a_{i,\sigma(i)} + b_{i,\sigma(i)}t_i\right),
\end{aligned}
$$

where

$$
b_{i,\sigma(i)} := \begin{cases} b_i\,, & \text{if } i=\sigma(i) \\ 0\,, & \text{if } i \neq \sigma(i) \end{cases}
$$

and Σ_m is the set of all permutations of $\{1,\dots,m\}$. Then

$$
\begin{aligned}
f(t,\dots,t) &= \det(A + t\ \operatorname{diag}(b_1,\dots,b_m))\,, \\
p_0 = f(0,\dots,0) &= \det(A) = \operatorname{minor}(A\,;\,\{1,\dots,m\})\,.
\end{aligned}
$$

The coefficients p_k are given via the partial derivatives of f, that is, for $k \in \{1,\dots,m\}$,

$$
p_k = \sum_{1\le i_1<\dots<i_k\le m} \frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} f(t_1,\dots,t_m)\Big|_{t_1=\dots=t_m=0}\,.
$$

Moreover, it follows that

$$
\begin{aligned}
&\frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} f(t_1,\dots,t_m)\Big|_{t_1=\dots=t_m=0} \\
&= \frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} \left(\sum_{\sigma\in\Sigma_m} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} \left(a_{i,\sigma(i)} + b_{i,\sigma(i)}t_i\right) \right)\Bigg|_{t_1=\dots=t_m=0} \\
&= \sum_{\sigma\in\Sigma_m} \operatorname{sgn}(\sigma) \frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} \prod_{i=1}^{m} \left(a_{i,\sigma(i)} + b_{i,\sigma(i)}t_i\right)\Big|_{t_1=\dots=t_m=0}
\end{aligned}
$$

$$
= \sum_{\sigma \in \Sigma_m} \operatorname{sgn}(\sigma) \frac{\partial^{k-1}}{\partial t_{i_1} \cdots \partial t_{i_{k-1}}} \left. \begin{cases} 0\,, & \text{if } \sigma(i_k) \neq i_k \\ b_{i_k} \prod\limits_{\substack{i=1 \\ i \neq i_k}}^m \big(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\big) \\ + (a_{i_k,i_k} + b_{i_k} t_{i_k}) \\ \cdot \underbrace{\frac{\partial}{\partial t_{i_k}} \prod\limits_{\substack{i=1 \\ i \neq i_k}}^m \big(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\big)}_{=0}, & \text{if } \sigma(i_k) = i_k \end{cases} \right|_{\substack{t_1 = \ldots \\ = t_m \\ = 0}}
$$

$$
= \sum_{\sigma \in \Sigma_m} \operatorname{sgn}(\sigma) \left. \begin{cases} 0\,, & \text{if } \exists\, j \in \{1, \ldots, k\} : \\ & \quad \sigma(i_j) \neq i_j \\ \prod\limits_{j=1}^k b_{i_j} \prod\limits_{\substack{i=1 \\ i \neq i_1, \ldots, i_k}}^m \big(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\big), & \text{if } \forall\, j \in \{1, \ldots, k\} : \\ & \quad \sigma(i_j) = i_j \end{cases} \right|_{\substack{t_1 = \ldots \\ = t_m \\ = 0}}
$$

$$
= \sum_{\substack{\sigma \in \Sigma_m \\ \{i_1, \ldots, i_k\} \subset \sigma}} \operatorname{sgn}(\sigma) \prod_{j=1}^k b_{i_j} \prod_{\substack{i=1 \\ i \neq i_1, \ldots, i_k}}^m a_{i,\sigma(i)}
$$

$$
= \prod_{j=1}^k b_{i_j} \ \operatorname{minor}(A\,;\, \{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\})\,.
$$

Hence, for $k = 1, \ldots, m$,

$$
p_k = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \prod_{j=1}^k b_{i_j} \ \operatorname{minor}(A\,;\, \{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\})\,,
$$

and thus

$$
p_{m-k} = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \prod_{\substack{i \in \{1, \ldots, m\} \\ i \notin \{i_1, \ldots, i_k\}}} b_i \ \operatorname{minor}(A\,;\, \{i_1, \ldots, i_k\})\,,
$$

which shows (3.3.5) and completes the proof. □

With the above lemma and Lemma 3.3.2 all required lemmata to prove Theorem 3.3.1 are given.

3.3.4 Proof of the main result

The proof of Theorem 3.3.1 is structured similar to the proof of derivative feedback stabilization for MIMO-systems with strict relative degree, i.e. Theorem 3.2.1: first the system's normal form and another coordinate transformation are utilized to design, in view of the general root-locus Lemma 3.3.2, solutions of two Lyapunov equations which are finally used to create a Lyapunov function to prove exponential stability of the closed-loop system.

Recall that $\mu(p(\cdot)) := \max\{\mathrm{Re}\ s \mid s \in \mathbb{C}, p(s) = 0\}$ denotes the largest real part of the zeros of $p \in \mathbb{C}[s]$. Let $\mu(A) := \max\{\mathrm{Re}\ s \mid s \in \mathrm{spec}(A)\}$ be the largest real part of the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ and recall that $\mathbb{R}^H[s] := \{p \in \mathbb{R}[s] \mid \mu(p) < 0\}$ denotes the set of all Hurwitz polynomials.

Proof of Theorem 3.3.1. Without loss of generality suppose that the linear system (3.2.1) has ordered relative degree $r = (r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$, see Definition 2.1.1(b), otherwise note that for a linear system (A, B, C) of form (3.2.1) with vector relative degree $r = (r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$ there exists a permutation matrix $P \in \mathbb{R}^{m\times m}$ such that the system (A, B, PC) has ordered vector relative degree $rP = (\widetilde{r}_1, \dots, \widetilde{r}_m)$, see also Lemma 2.2.3. Thus it is sufficient to prove the statement of Theorem 3.3.1 for systems with ordered vector relative degree.

Step 1: Next it is shown that, for

$$\begin{bmatrix} \Gamma_1^1 & \Gamma_2^1 & \dots & \Gamma_m^1 \\ \Gamma_1^2 & \Gamma_2^2 & \dots & \Gamma_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_1^m & \Gamma_2^m & \dots & \Gamma_m^m \end{bmatrix} := \Gamma = \begin{bmatrix} c_{(n)}^1 A^{r_1-1} B \\ c_{(n)}^2 A^{r_2-1} B \\ \vdots \\ c_{(n)}^m A^{r_m-1} B \end{bmatrix},$$

$R_{i,k}^j \in \mathbb{R}$, for $i, j \in \{1, \dots, m\}$ and $k \in \{1, \dots, r_i\}$, $S^1, \dots, S^m \in \mathbb{R}^{1\times(n-r^s)}$, $P_1, \dots, P_m \in \mathbb{R}^{n-r^s}$ and $Q \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$, where $r^s :=$

$\sum_{j=1}^{m} r_j$, as in normal form (2.2.3)–(2.2.4) with (2.2.18)–(2.2.21) and

$$
A_{\Gamma,k,\kappa,\nu} :=
\left[\begin{array}{c|c|c|c}
\kappa\begin{bmatrix} 0 & 1 \ldots & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 \\ \frac{R^1_{1,1}}{\kappa^{r_1}} - \Gamma^1_1 \nu\, k_{1,1} & & \ldots & \frac{R^1_{1,r_1}}{\kappa} - \Gamma^1_1 \nu\, k_{1,r_1} \end{bmatrix} & \ldots & \kappa\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \frac{R^1_{m,1}}{\kappa^{r_m}} - \Gamma^1_m \nu\, k_{m,1} & \ldots & \frac{R^1_{m,r_m}}{\kappa} - \Gamma^1_m \nu\, k_{m,r_m} \end{bmatrix} & \kappa\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{\kappa^{r_m}} S^1 \end{bmatrix} \\
\hline
\vdots & & \vdots & \vdots \\
\hline
\kappa\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \frac{R^m_{1,1}}{\kappa^{r_1}} - \Gamma^m_1 \nu\, k_{1,1} & \ldots & \frac{R^m_{1,r_1}}{\kappa} - \Gamma^m_1 \nu\, k_{1,r_1} \end{bmatrix} & \ldots & \kappa\begin{bmatrix} 0 & 1 \ldots & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 \\ \frac{R^m_{m,1}}{\kappa^{r_m}} - \Gamma^m_m \nu\, k_{m,1} & & \ldots & \frac{R^m_{m,r_m}}{\kappa} - \Gamma^m_m \nu\, k_{m,r_m} \end{bmatrix} & \kappa\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{\kappa^{r_m}} S^m \end{bmatrix} \\
\hline
\left[\kappa^{-r_1+r_m} P_1 \; 0 \ldots \; 0\right] & \ldots & \left[\kappa^{-r_m+r_m} P_m \; 0 \ldots \; 0\right] & Q
\end{array}\right],
$$

the closed-loop system (3.2.1), (3.3.1) is equivalent to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \zeta \\ \vartheta \end{pmatrix} = A_{\Gamma,k,\kappa,\nu}\begin{pmatrix} \zeta \\ \vartheta \end{pmatrix} \tag{3.3.6}
$$

in the sense that there exists a matrix $W \in \mathbb{R}^{n\times n}$ such that $\left(\begin{smallmatrix} \zeta \\ \vartheta \end{smallmatrix}\right) = Wx$.

By Theorem 2.2.4 there exists an invertible $U \in \mathbb{R}^{n\times n}$ such that the coordinate transformation $\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right) = Ux$ converts (3.2.1) into normal form (2.2.3)–(2.2.4). Thus the closed-loop system (3.2.1), (3.3.1) is equivalent to the system (2.2.3), (2.2.4), (3.3.1).

Split ξ as follows:

$$
\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^m \end{pmatrix} \qquad \text{and} \qquad \xi^j = \begin{pmatrix} \xi^j_1 \\ \vdots \\ \xi^j_{r_j} \end{pmatrix} \in \mathbb{R}^{r_j}\,, \; j \in \{1,\ldots,m\}\,.
$$

For ease of notation define vectors

$$
R^j_\mu := \left[R^j_{\mu,1},\ldots,R^j_{\mu,r_\mu}\right] \in \mathbb{R}^{1\times r_\mu}\,, \quad j,\mu \in \{1,\ldots,m\}\,,
$$

where, for $i \in \{1, \ldots, r_\mu\}$, $R^j_{\mu,i}$, is defined by (2.2.18).

Setting $\zeta_i^j = \kappa^{r_j - r_1 - i + 1} \xi_i^j$, for $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, r_j\}$, yields

$$
\begin{aligned}
\xi_i^j &= \kappa^{-r_j + r_1 + i - 1} \zeta_i^j\,, \qquad \text{for } j \in \{1, \ldots, m\} \text{ and } i \in \{1, \ldots, r_j\}\,,\\
\xi^j &= \begin{pmatrix} \xi_1^j \\ \vdots \\ \xi_{r_j}^j \end{pmatrix}\\
&= \begin{pmatrix} \kappa^{-r_j + r_1 + 1 - 1} \zeta_1^j \\ \vdots \\ \kappa^{-r_j + r_1 + r_j - 1} \zeta_{r_j}^j \end{pmatrix}\\
&= \begin{bmatrix} \kappa^{r_1 - r_j} & 0 & \cdots & 0 \\ 0 & \kappa^{r_1 - r_j + 1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \kappa^{r_1 - 1} \end{bmatrix} \underbrace{\begin{pmatrix} \zeta_1^j \\ \vdots \\ \zeta_{r_j}^j \end{pmatrix}}_{=:\zeta^j}, \quad \text{for } j \in \{1, \ldots, m\}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\dot\zeta_i^j &= \kappa^{r_j - r_1 - i + 1} \dot\xi_i^j\\
&= \kappa^{r_j - r_1 - i + 1} \xi_{i+1}^j\\
&= \kappa^{r_j - r_1 - i + 1} \kappa^{r_1 - r_j + i + 1 - 1} \zeta_{i+1}^j = \kappa \zeta_{i+1}^j\,, \qquad \text{for } j \in \{1, \ldots, m\}\\
&\qquad\qquad\qquad\qquad \text{and } i \in \{1, \ldots, r_j - 1\}\,,\\
\dot\zeta_{r_j}^j &= \kappa^{-r_1 + 1} \dot\xi_{r_j}^j\\
&= \kappa^{-r_1 + 1} \left[\sum_{\mu=1}^m R_\mu^j \xi^\mu + S^j \eta + c_{(n)}^j A^{r_j - 1} B u \right], \quad \text{for } j \in \{1, \ldots, m\}\,.
\end{aligned}
$$

Thus, by (3.3.1) and $y_j^{(i-1)} = \xi_i^j$, for all $j \in \{1, \ldots, m\}$ and $i \in$

$\{1,\dots,r_j\}$, it follows that, for $j\in\{1,\dots,m\}$,

$$
\begin{aligned}
\dot\zeta^j_{r_j} &= \kappa\left[\sum_{i=1}^m \kappa^{-r_1} R^j_i \begin{bmatrix} \kappa^{r_1-r_i} & 0 & \dots & 0\\ 0 & \kappa^{r_1-r_i+1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & \kappa^{r_1-1}\end{bmatrix}\zeta^i + \kappa^{-r_1}S^j\eta \right.\\
&\qquad\left. -\nu\,\kappa^{-r_1} c^j_{(n)} A^{r_j-1}B \begin{pmatrix} \sum_{i=0}^{r_1-1}\kappa^{r_1-i}k_{1,i+1}\,\xi^1_{i+1}\\ \vdots\\ \sum_{i=0}^{r_m-1}\kappa^{r_m-i}k_{m,i+1}\,\xi^m_{i+1}\end{pmatrix}\right]\\
&= \kappa\left[\sum_{i=1}^m\left[\kappa^{-r_i}R^j_{i,1},\kappa^{-r_i+1}R^j_{i,2},\dots,\kappa^{-1}R^j_{i,r_i}\right]\zeta^i + \kappa^{-r_1}S^j\eta\right.\\
&\qquad\left. -\nu\,\kappa^{-r_1}\left[\Gamma^j_1,\Gamma^j_2,\dots,\Gamma^j_m\right]\begin{pmatrix}\sum_{i=0}^{r_1-1}\kappa^{r_1-i}\kappa^{r_1-r_1+i}k_{1,i+1}\,\zeta^1_{i+1}\\ \vdots\\ \sum_{i=0}^{r_m-1}\kappa^{r_m-i}\kappa^{r_1-r_m+i}k_{m,i+1}\,\zeta^m_{i+1}\end{pmatrix}\right]\\
&= \kappa\left[\sum_{i=1}^m\left(\left[\frac{R^j_{i,1}}{\kappa^{r_i}},\frac{R^j_{i,2}}{\kappa^{r_i-1}},\dots,\frac{R^j_{i,r_i}}{\kappa}\right]-\Gamma^j_i\nu\left[k_{i,1},\dots,k_{i,r_i}\right]\right)\zeta^i+\frac{1}{\kappa^{r_1}}S^j\eta\right]
\end{aligned}
$$

Setting $\vartheta=\kappa^{r_m-r_1}\eta$ yields

$$
\begin{aligned}
\dot\vartheta &= \kappa^{r_m-r_1}\dot\eta = \kappa^{r_m-r_1}\left(\sum_{i=1}^m P_i\xi^i_1 + Q\eta\right)\\
&= \kappa^{r_m-r_1}\left(\sum_{i=1}^m \kappa^{r_1-r_i}P_i\zeta^1_i + \kappa^{r_1-r_m}Q\vartheta\right) = \sum_{i=1}^m \kappa^{r_m-r_i}P_i\zeta^1_i + Q\vartheta\,.
\end{aligned}
$$

Thus, in view of the coordinate transformation,

$$
\begin{aligned}
\zeta^j_i &= \kappa^{r_j-r_1-i+1}\xi^j_i\,, \qquad \text{for } j\in\{1,\dots,m\}\,,\ i\in\{1,\dots,r_j\}\,,\\
\vartheta &= \kappa^{r_m-r_1}\eta\,,
\end{aligned}
$$

and setting $\zeta=\left[\zeta^1/\zeta^2/\dots/\zeta^m\right]$ (the first r^s entries of the new coordinates) the closed-loop system (2.2.3), (2.2.4), (3.3.1) is equivalent to (3.3.6).

Step 2: For $r^s = \sum_{i=1}^{m} r_i$ and

$$
\mathfrak{A}_{\Gamma,k,\nu} :=
\left[\begin{array}{c|c|c|c}
\begin{bmatrix} 0 & 1 \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots 0 & 1 \\ -\nu\,\Gamma_1^1 \big[k_{1,1} \dots & & k_{1,r_1}\big] \end{bmatrix} &
\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\nu\,\Gamma_2^1 \big[k_{2,1} \dots k_{2,r_2}\big] \end{bmatrix} & \dots &
\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\nu\,\Gamma_m^1 \big[k_{m,1} \dots k_{m,r_m}\big] \end{bmatrix} \\ \hline
\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\nu\,\Gamma_1^2 \big[k_{1,1} \dots k_{1,r_1}\big] \end{bmatrix} &
\begin{bmatrix} 0 & 1 \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots 0 & 1 \\ -\nu\,\Gamma_2^2 \big[k_{2,1} \dots & & k_{2,r_2}\big] \end{bmatrix} & \dots &
\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\nu\,\Gamma_m^2 \big[k_{m,1} \dots k_{m,r_m}\big] \end{bmatrix} \\ \hline
\vdots & \vdots & & \vdots \\ \hline
\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\nu\,\Gamma_1^m \big[k_{1,1} \dots k_{1,r_1}\big] \end{bmatrix} &
\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ -\nu\,\Gamma_2^m \big[k_{2,1} \dots k_{2,r_2}\big] \end{bmatrix} & \dots &
\begin{bmatrix} 0 & 1 \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots 0 & 1 \\ -\nu\,\Gamma_m^m \big[k_{m,1} \dots & & k_{m,r_m}\big] \end{bmatrix}
\end{array}\right].
$$

it is shown that

$$
\exists\, \delta > 0 \;\exists\, \nu^* \geq 1 \;\forall\, \nu > \nu^* \;:\; \mu\left(\mathfrak{A}_{\Gamma,k,\nu}\right) < -\delta/2\,. \tag{3.3.7}
$$

Set, for $i = 1, \dots, r_1$, $m_i := \#\left\{ r_j \,\middle|\, r_j \geq i, j \in \{1, \dots, m\} \right\}$, the number of r_j's, $j \in \{1, \dots, m\}$, such that $r_j \geq i$. Define permutation matrices

$$
\begin{aligned}
\Pi_R := \Bigg[& e_1^{(r^s)}, e_{r_1+1}^{(r^s)}, \dots, e^{(r^s)}_{\sum_{i=1}^{m_1-1} r_i+1} \,\Bigg|\, e_2^{(r^s)}, e_{r_1+2}^{(r^s)}, \dots, e^{(r^s)}_{\sum_{i=1}^{m_2-1} r_i+2} \,\Bigg| \\
& \qquad \dots \,\Bigg|\, e_{r_1}^{(r^s)}, e_{r_1+r_2}^{(r^s)}, \dots, e^{(r^s)}_{\sum_{i=1}^{m_{r_1}-1} r_i+r_1} \Bigg],
\end{aligned}
$$

$$
\Pi_L := \left[\begin{bmatrix} e^1_{(r^s)} \\ e^{r_1+1}_{(r^s)} \\ \vdots \\ e_{(r^s)}^{\sum_{i=1}^{m_2-1} r_i+1} \end{bmatrix} \Bigg/ \begin{bmatrix} e^2_{(r^s)} \\ e^{r_1+2}_{(r^s)} \\ \vdots \\ e_{(r^s)}^{\sum_{i=1}^{m_3-1} r_i+2} \end{bmatrix} \Bigg/ \dots \Bigg/ \begin{bmatrix} e^{r_1-1}_{(r^s)} \\ e^{r_1+r_2-1}_{(r^s)} \\ \vdots \\ e_{(r^s)}^{\sum_{i=1}^{m_{r_1}-1} r_i+(r_1-1)} \end{bmatrix} \Bigg/ \begin{bmatrix} e^{r_1}_{(r^s)} \\ e^{r_1+r_2}_{(r^s)} \\ \vdots \\ e_{(r^s)}^{\sum_{i=1}^{m_1} r_i} \end{bmatrix} \right],
$$

and

$$
\begin{aligned}
\Pi_D := \Bigg[& e_1^{(r^s)}, e_2^{(r^s)}, \dots, e_{m_2}^{(r^s)} \,\Bigg|\, e_{m_1}^{(r^s)}, \dots, e_{m_1+m_3}^{(r^s)} \,\Bigg|\, \dots \,\Bigg|\, e^{(r^s)}_{\sum_{i=1}^{r_1-2} m_i+1}, \dots, e^{(r^s)}_{\sum_{i=1}^{r_1-2} m_i + m_{r_1}} \,\Bigg| \\
& e^{(r^s)}_{r^s - m_{r_1}+1}, \dots, e^{(r^s)}_{r^s-m_{r_1}+m_{r_1}} \,\Bigg| \\
& e^{(r^s)}_{\sum_{i=1}^{r_1-2} m_i + m_{r_1}+1}, \dots, e^{(r^s)}_{\sum_{i=1}^{r_1-1} m_i} \,\Bigg|\, \dots \,\Bigg|\, e^{(r^s)}_{\sum_{i=1}^{r_1-r_1} m_i + m_{r_1-(r_1-2)}+1}, \dots, e^{(r^s)}_{\sum_{i=1}^{r_1-(r_1-1)} m_i} \Bigg]
\end{aligned}
$$

Then, for $s \in \mathbb{C}$,

$$
\Pi_D \Pi_L \left(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\nu} \right) \Pi_R =
$$

$$
\left[
\begin{array}{c|c|c|c|c||}
sI_{m_2} & \begin{bmatrix} -I_{m_3} \\ 0_{(m_2-m_3)\times m_3} \end{bmatrix} & 0_{m_2\times m_4} & \cdots & 0_{m_2\times m_{r_1}} \\ \hline
0_{m_3\times m_2} & sI_{m_3} & \begin{bmatrix} -I_{m_4} \\ 0_{(m_3-m_4)\times m_4} \end{bmatrix} & 0_{m_3\times m_5} & \cdots \\ \hline
\vdots & 0_{m_4\times m_3} & sI_{m_4} & \ddots & \ddots \\ \hline
 & \vdots & \ddots & \ddots & \begin{bmatrix} -I_{m_{r_1}} \\ 0_{(m_{r_1-1}-m_{r_1})\times m_{r_1}} \end{bmatrix} \\ \hline
0_{m_{r_1}\times m_2} & 0_{m_{r_1}\times m_3} & \cdots & 0_{m_{r_1}\times m_{r_1-1}} & sI_{m_{r_1}} \\ \hline\hline
\Gamma \begin{bmatrix} \begin{matrix} \nu k_{1,1} & & 0 \\ & \ddots & \\ 0 & & \nu k_{m_2,1} \end{matrix} \\ \hline 0_{(m_1-m_2)\times m_2} \end{bmatrix} &
\Gamma \begin{bmatrix} \begin{matrix} \nu k_{1,2} & & 0 \\ & \ddots & \\ 0 & & \nu k_{m_3,2} \end{matrix} \\ \hline 0_{(m_1-m_3)\times m_3} \end{bmatrix} &
\Gamma \begin{bmatrix} \begin{matrix} \nu k_{1,3} & & 0 \\ & \ddots & \\ 0 & & \nu k_{m_4,3} \end{matrix} \\ \hline 0_{(m_1-m_4)\times m_4} \end{bmatrix} &
\cdots &
\Gamma \begin{bmatrix} \begin{matrix} \nu k_{1,r_1-1} & & 0 \\ & \ddots & \\ 0 & & \nu k_{m_{r_1},r_1-1} \end{matrix} \\ \hline 0_{(m_1-m_{r_1})\times m_{r_1}} \end{bmatrix}
\end{array}
\right.
$$

$$
\left.
\begin{array}{c|c|c|c|c}
0_{m_2\times m_{r_1}} & \cdots & & \begin{bmatrix} 0_{m_3\times(m_2-m_3)} \\ -I_{m_2-m_3} \end{bmatrix} & 0_{m_2\times(m_1-m_2)} \\ \hline
\vdots & & \iddots & & \vdots \\ \hline
0_{m_{r_1-1}\times m_{r_1}} & \begin{bmatrix} 0_{m_{r_1}\times(m_{r_1-1}-m_{r_1})} \\ -I_{m_{r_1-1}-m_{r_1}} \end{bmatrix} & & \vdots & \\ \hline
\begin{bmatrix} -I_{m_{r_1}} \end{bmatrix} & 0_{m_{r_1}\times(m_{r_1-1}-m_{r_1})} & \cdots & 0_{m_{r_1}\times(m_2-m_3)} & 0_{m_{r_1}\times(m_1-m_2)} \\ \hline\hline
\multicolumn{5}{c}{sI_{m_1} + \Gamma \underbrace{\operatorname{diag}\left(\begin{array}{r} \nu k_{1,r_1}, \ \dots, \ \nu k_{m_{r_1},r_1}, \nu k_{m_{r_1}+1,r_1-1}, \ \dots, \ \nu k_{m_{r_1-1},r_1-1}, \\ \dots, \nu k_{m_2+1,1}, \ \dots, \ \nu k_{m_1,1} \end{array} \right)}_{=: \mathfrak{K}_{k,\nu}}}
\end{array}
\right], \tag{3.3.8}
$$

and

$$
\begin{aligned}
&\det(\Pi_D \Pi_L \left(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\nu}\right) \Pi_R) \\
&= \det(sI_{m_2}) \\
&\quad \det \left[\begin{array}{c|c|c||c}
sI_{m_3} & \begin{bmatrix} -I_{m_4} \\ 0_{(m_3-m_4)\times m_4} \end{bmatrix} & & \left[\begin{array}{c|c|c} 0_{m_3 \times m_4} & \begin{matrix} 0_{m_3\times (m_3-m_4)} \\ I_{m_3-m_4} \end{matrix} & 0_{m_3\times(m_1-m_3)} \end{array}\right] \\
\hline
0_{\left(\sum_{i=4}^{r_1} m_i\right)\times m_3} & & \cdots & \cdots \\
\hline\hline
\Gamma \left[\begin{array}{c} \begin{matrix} \nu\, k_{1,2} + \frac{1}{s}\nu\, k_{1,1} & & 0 \\ & \ddots & \\ 0 & & \nu\, k_{m_3,2} + \frac{1}{s}\nu\, k_{m_3,1} \end{matrix} \\ \hline 0_{(m_1-m_3)\times m_3} \end{array}\right] & \Gamma \left[\begin{array}{c} \begin{matrix} \nu\, k_{1,3} & & 0 \\ & \ddots & \\ 0 & & \nu\, k_{m_4,3} \end{matrix} \\ \hline 0_{(m_1-m_4)\times m_4} \end{array}\right] & & \begin{array}{l} sI_{m_1} + \Gamma \mathfrak{K}_{k,\nu} \\ + \dfrac{\Gamma}{s} \left[\begin{array}{c|c} \begin{array}{c} 0_{m_3\times(m_2-m_3)} \\ \hline \begin{matrix} \nu\, k_{m_3+1,1} & & 0 \\ & \ddots & \\ 0 & & \nu\, k_{m_2,1} \end{matrix} \\ \hline 0_{(m_1-m_2)\times(m_2-m_3)} \end{array} & 0 \end{array}\right] \end{array}
\end{array} \right] \\
&= \ldots \\
&= s^{\sum_{i=2}^{r_1-1} m_i} \det(sI_{m_{r_1}}) \\
&\quad \cdot \det \Bigg[sI_{m_1} + \Gamma \operatorname{diag}\Bigg(\sum_{i=1}^{r_1} \nu\, k_{1,i} s^{-(r_1-i)}, \ldots, \sum_{i=1}^{r_1} \nu\, k_{m_{r_1},i} s^{-(r_1-i)}, \\
&\qquad \sum_{i=1}^{r_1-1} \nu\, k_{m_{r_1}+1,i} s^{-(r_1-1-i)}, \ldots, \sum_{i=1}^{r_1-1} \nu\, k_{m_{r_1-1},i} s^{-(r_1-1-i)}, \\
&\qquad \ldots, \sum_{i=1}^{1} \nu\, k_{m_2+1,i} s^{-(1-i)}, \ldots, \sum_{i=1}^{1} \nu\, k_{m_1,i} s^{-(1-i)} \Bigg) \Bigg] \\
&= \det \Bigg[\operatorname{diag}\left(s^{r_1}, \ldots, s^{r_m}\right) \\
&\qquad + \Gamma \operatorname{diag}\left(\sum_{i=1}^{r_1} \nu\, k_{1,i} s^{i-1}, \sum_{i=1}^{r_2} \nu\, k_{2,i} s^{i-1}, \ldots, \sum_{i=1}^{r_{m_1}} \nu\, k_{m_1,i} s^{i-1} \right) \Bigg].
\end{aligned}
$$

Recall that $m = m_1$. Setting, for $j \in \{1, \ldots, m\}$, $k^j(s) := \sum_{i=1}^{r_j} k_{j,i} s^{i-1}$ an application of Lemma 3.3.3 leads to

$$
\begin{aligned}
&\det(\Gamma) \det(\Pi_D \Pi_L \left(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\nu}\right) \Pi_R) \\
&= \det \begin{bmatrix}
s^{r_1}\left(\Gamma^{-1}\right)_{1,1} + \nu k^1(s), & s^{r_2}\left(\Gamma^{-1}\right)_{1,2}, & \ldots, & s^{r_m}\left(\Gamma^{-1}\right)_{1,m} \\
s^{r_1}\left(\Gamma^{-1}\right)_{2,1}, & s^{r_2}\left(\Gamma^{-1}\right)_{2,2} + \nu k^2(s) & \ddots & \vdots \\
\vdots & \ddots & \ddots & s^{r_m}\left(\Gamma^{-1}\right)_{m-1,m} \\
s^{r_1}\left(\Gamma^{-1}\right)_{m,1}, & & \ldots, s^{r_m-1}\left(\Gamma^{-1}\right)_{m,m-1}, & s^{r_m}\left(\Gamma^{-1}\right)_{m,m} + \nu k^m(s)
\end{bmatrix}
\end{aligned}
$$

$$= \sum_{j=0}^{m} \left(\sum_{1 \le i_1 < \cdots < i_j \le m} s^{\left(\sum_{l=1}^{j} r_{i_l}\right)} \operatorname{minor}\left(\Gamma^{-1}\,;\,\{i_1,\ldots,i_j\}\right) \prod_{\substack{i \in \{1,\ldots,m\} \\ i \notin \{i_1,\ldots,i_j\}}} k^i(s) \right) \nu^{m-j}. \tag{3.3.9}$$

Observe that, for fixed $j = 0, \ldots, m$, every summand in (3.3.9) is a polynomial in $\mathbb{R}[s]$ with degree $r^s - m + j$. Moreover, for $j = 0$, the summand in (3.3.9) equals $p(s) := \prod_{i=1}^{m} k^i(s)$ which is the product of m Hurwitz polynomials and thus $p \in \mathbb{R}^H[s]$, hence

$$\exists\, \delta^* > 0 \;:\; \mu(p(\cdot)) < -\delta^*\,.$$

Let $\{\gamma_1, \ldots, \gamma_m\} = \operatorname{spec}(\Gamma) \subset \mathbb{C}$ and $J \in \mathbb{C}^{m \times m}$ be a Jordan canonical form of Γ^{-1}, see [GvL96, Thm. 7.1.9]. Since $\det(\Gamma^{-1} + \nu I_m) = \det(J + \nu I_m) = \prod_{j=1}^{m}(\nu + \gamma_j^{-1})$ and Γ and thus Γ^{-1} are positive definite, whence, in view of Lemma 3.2.4, $\operatorname{spec}(\Gamma^{-1}) \subset \mathbb{C}_+$, Lemma 3.3.3 yields

$$\exists\, \delta^{**} > 0 \;:$$
$$\mu\left(\nu \mapsto \sum_{j=0}^{m} \left(\sum_{1 \le i_1 < \cdots < i_j \le m} \operatorname{minor}\left(\Gamma^{-1}\,;\,\{i_1,\ldots,i_j\}\right) \right) \nu^{m-j} \right) < -\delta^{**}$$

Setting $\delta = \min\{\delta^*, \delta^{**}\}$ and since

$$\begin{aligned} &\det(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\nu}) \\ &= \underbrace{\det(\Pi_D)\det(\Pi_L)\det(\Pi_R)}_{\in \{-1,+1\}} \det(\Pi_D \Pi_L \left(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\nu}\right) \Pi_R)\,, \quad s \in \mathbb{C}\,, \end{aligned}$$

Lemma 3.3.2 and (3.3.9) yield (3.3.7).

Step 3: It is shown that, for any initial value $x(t_0) = x^0$ for (3.2.1) or, equivalently, $\left(\begin{smallmatrix}\zeta\\ \vartheta\end{smallmatrix}\right)(t_0) = Wx^0$ for (3.3.6),

$$\begin{aligned} &\exists\, \nu^* > 0\; \forall\, \nu > \nu^*\; \exists\, \alpha > 0\; \exists\, M > 0 \\ &\qquad \exists\, \kappa^* \ge 1\; \forall\, \kappa > \kappa^*\; \forall\, t_0 \ge 0\; \forall\, t \ge t_0 \;: \\ &\qquad \left\| \begin{pmatrix} \zeta(t) \\ \vartheta(t) \end{pmatrix} \right\| \le \exp\left(-\alpha(t - t_0)\right) M \left\| \begin{pmatrix} \zeta(t_0) \\ \vartheta(t_0) \end{pmatrix} \right\|. \end{aligned} \tag{3.3.10}$$

By Step 2 and since system (3.2.1) has stable zero dynamics which, in view of Corollary 2.3.2, is equivalent to $\mathrm{spec}(Q) \subset \mathbb{C}_-$, one may choose symmetric positive definite matrices $N_\zeta(\nu) = N_\zeta(\nu)^T \in \mathbb{R}^{r^s \times r^s}$, for all $\nu > \nu^*$, and $N_\vartheta = N_\vartheta^T \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$ such that

$$N_\zeta(\nu)\,\mathfrak{A}_{\Gamma,k,\nu} + \mathfrak{A}_{\Gamma,k,\nu}^T\,N_\zeta(\nu) = -I_{r^s}\,, \quad N_\vartheta\,Q + Q^T\,N_\vartheta = -I_{n-r^s}\,, \tag{3.3.11}$$

where $\zeta = \left[\zeta^1/\zeta^2/\ldots/\zeta^m\right]$. Moreover

$$\forall\ \nu > \nu^*\ :\ \|N_\zeta(\nu)\| < \infty\,.$$

Let $\left(\begin{smallmatrix}\zeta\\ \vartheta\end{smallmatrix}\right)$ be an arbitrary solution of the closed system (3.3.6). Set

$$\mathfrak{R}_\kappa := \kappa \left[\begin{array}{c|c|c} \begin{bmatrix} 0 & 1 & \ldots & 0\\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1\\ \frac{R^1_{1,1}}{\kappa^{r_1}} & \ldots & & \frac{R^1_{1,r_1}}{\kappa} \end{bmatrix} & \cdots & \begin{bmatrix} 0 & \ldots & 0\\ \vdots & & \vdots\\ 0 & \ldots & 0\\ \frac{R^1_{m,1}}{\kappa^{r_m}} & \ldots & \frac{R^1_{m,r_m}}{\kappa} \end{bmatrix}\\ \hline \vdots & & \vdots\\ \hline \begin{bmatrix} 0 & \ldots & 0\\ \vdots & & \vdots\\ 0 & \ldots & 0\\ \frac{R^m_{1,1}}{\kappa^{r_1}} & \ldots & \frac{R^m_{1,r_1}}{\kappa} \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 1 & \ldots & 0\\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1\\ \frac{R^m_{m,1}}{\kappa^{r_m}} & \ldots & & \frac{R^m_{m,r_m}}{\kappa} \end{bmatrix} \end{array}\right], \tag{3.3.12}$$

$$\mathfrak{S}_\kappa := \kappa \left[\begin{bmatrix} 0_{1\times(n-r^s)}\\ \vdots\\ 0_{1\times(n-r^s)}\\ \frac{1}{\kappa^{r_m}} S^1 \end{bmatrix} \Bigg/ \cdots \Bigg/ \begin{bmatrix} 0_{1\times(n-r^s)}\\ \vdots\\ 0_{1\times(n-r^s)}\\ \frac{1}{\kappa^{r_m}} S^m \end{bmatrix}\right]. \tag{3.3.13}$$

Then differentiation of

$$t \mapsto V(t) := \frac{1}{2}\zeta(t)^T N_\zeta(\nu)\zeta(t) + \frac{1}{2}\vartheta(t)^T N_\vartheta \vartheta(t)$$

along $(\zeta^T, \vartheta^T)^T$ yields, for all $t \geq 0$, and omitting the argument t for brevity,

$$
\begin{aligned}
\dot V(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\zeta^T N_\zeta(\nu)\zeta + \frac{1}{2}\vartheta^T N_\vartheta \vartheta\right) \\
&= \frac{1}{2}\left(\dot\zeta^T N_\zeta(\nu)\zeta + \zeta^T N_\zeta(\nu)\dot\zeta\right) + \frac{1}{2}\left(\dot\vartheta^T N_\vartheta\vartheta + \vartheta^T N_\vartheta\dot\vartheta\right) \\
&= \frac{1}{2}\Big(\left[\kappa\,\mathfrak{A}_{\Gamma,k,\nu}\zeta + \mathfrak{R}_\kappa\zeta + \mathfrak{S}_\kappa\vartheta\right]^T N_\zeta(\nu)\zeta \\
&\qquad + \zeta^T N_\zeta(\nu)\left[\kappa\,\mathfrak{A}_{\Gamma,k,\nu}\zeta + \mathfrak{R}_\kappa\zeta + \mathfrak{S}_\kappa\vartheta\right]\Big) \\
&\quad + \frac{1}{2}\left(\left(Q\vartheta + \sum_{j=1}^m \frac{P_j}{\kappa^{r_1 - r_m}}\zeta_1^j\right)^T N_\vartheta\vartheta \right.\\
&\qquad \left. + \vartheta^T N_\vartheta\left(Q\vartheta + \sum_{j=1}^m \frac{P_j}{\kappa^{r_1 - r_m}}\zeta_1^j\right)\right) \\
&= \frac{1}{2}\Big(\kappa\,\zeta^T\left(\mathfrak{A}_{\Gamma,k,\nu}^T N_\zeta(\nu) + N_\zeta(\nu)\mathfrak{A}_{\Gamma,k,\nu}\right)\zeta + \zeta^T \mathfrak{R}_\kappa^T N_\zeta(\nu)\zeta \\
&\qquad + \zeta^T N_\zeta(\nu)\mathfrak{R}_\kappa\zeta + \vartheta^T\mathfrak{S}_\kappa^T N_\zeta(\nu)\zeta + \zeta^T N_\zeta(\nu)\mathfrak{S}_\kappa\vartheta\Big) \\
&\quad + \frac{1}{2}\left(\vartheta^T\left(Q^T N_\vartheta + N_\vartheta Q\right)\vartheta \right.\\
&\qquad \left. + \sum_{j=1}^m\left[(\zeta_1^j)^T\frac{P_j^T}{\kappa^{r_1-r_m}}N_\vartheta\vartheta + \vartheta^T N_\vartheta\frac{P_j}{\kappa^{r_1-r_m}}\zeta_1^j\right]\right)
\end{aligned}
$$

whence, in view of (3.3.11),

$$
\begin{aligned}
\dot V(t) = &-\frac{1}{2}\kappa\,|\zeta|^2 + \zeta^T N_\zeta(\nu)\mathfrak{R}_\kappa\zeta + \zeta^T N_\zeta(\nu)\mathfrak{S}_\kappa\vartheta \\
&- \frac{1}{2}|\vartheta|^2 + \sum_{j=1}^m \vartheta^T N_\vartheta\frac{P_j}{\kappa^{r_1-r_m}}\zeta_1^j\,,
\end{aligned}
$$

and since $r_1 \geq \cdots \geq r_m \geq 1$ and $\kappa \geq 1$ it follows that

$$\begin{aligned}
\dot V(t) \leq & -\frac{1}{2}\kappa \,\|\zeta\|^2 + \|N_\zeta(\nu)\|\,\|\mathfrak{R}_1\|\,\|\zeta\|^2 + \|N_\zeta(\nu)\|\,\|\mathfrak{S}_1\|\,\|\zeta\|\,\|\vartheta\| \\
& -\frac{1}{2}\|\vartheta\|^2 + \sum_{j=1}^m \frac{\|N_\vartheta\|\,\|P_j\|}{\kappa^{r_1-r_m}}\|\zeta_1^j\|\,\|\vartheta\| \\
\leq & -\frac{1}{2}\kappa \,\|\zeta\|^2 + \|N_\zeta(\nu)\|\,\|\mathfrak{R}_1\|\,\|\zeta\|^2 + 2\|N_\zeta(\nu)\|^2\|\mathfrak{S}_1\|^2\|\zeta\|^2 \\
& + \frac{1}{8}\|\vartheta\|^2 - \frac{1}{2}\|\vartheta\|^2 + \sum_{j=1}^m \left(2m\|N_\vartheta\|^2\|P_j\|^2\|\zeta_1^j\|^2 + \frac{1}{8m}\|\vartheta\|^2\right) \\
\leq & -\frac{1}{2}\kappa \,\|\zeta\|^2 + \|N_\zeta(\nu)\|\,\|\mathfrak{R}_1\|\,\|\zeta\|^2 + 2\|N_\zeta(\nu)\|^2\|\mathfrak{S}_1\|^2\|\zeta\|^2 \\
& + 2m\|N_\vartheta\|^2\sum_{j=1}^m \|P_j\|^2\|\zeta\|^2 - \frac{1}{2}\|\vartheta\|^2 + \frac{1}{8}\|\vartheta\|^2 + m\frac{1}{8m}\|\vartheta\|^2 \\
\leq & -\Bigg(\frac{1}{2}\kappa - \|N_\zeta(\nu)\|\,\|\mathfrak{R}_1\| - 2\|N_\zeta(\nu)\|^2\|\mathfrak{S}_1\|^2 \\
& \qquad -2m\|N_\vartheta\|^2\sum_{j=1}^m \|P_j\|^2\Bigg)\,\|\zeta\|^2 - \frac{1}{4}\|\vartheta\|^2 ,
\end{aligned}$$

where $\mathfrak{R}_1$, $\mathfrak{S}_1$ are defined setting $\kappa = 1$ in (3.3.12) and (3.3.13). Setting

$$\begin{aligned}
\kappa^* &:= \max\Bigg\{\frac{1}{4} + 2\Bigg(\|N_\zeta(\nu)\|\|\mathfrak{R}_1\| - 2\,\|N_\zeta(\nu)\|^2\|\mathfrak{S}_1\|^2 \\
&\qquad\qquad - 2m\|N_\vartheta\|^2 \textstyle\sum_{j=1}^m \|P_j\|^2\Bigg), \nu^*\Bigg\}, \\
\alpha &:= \min\left\{\frac{1}{8\,\|N_\zeta(\nu)\|}, \frac{1}{8\|N_\vartheta\|}\right\}, \\
M &:= \sqrt{\frac{\max\operatorname{spec}\left[\begin{smallmatrix} N_\zeta(\nu) & 0 \\ 0 & N_\vartheta\end{smallmatrix}\right]}{\min\operatorname{spec}\left[\begin{smallmatrix} N_\zeta(\nu) & 0 \\ 0 & N_\vartheta\end{smallmatrix}\right]}}
\end{aligned}$$

yields, for all $t \geq 0$ and all $\kappa > \kappa^*$,

$$\begin{aligned}\dot{V}(t) &\leq -\frac{1}{8}\|\zeta(t)\|^2 - \frac{1}{8}\|\vartheta(t)\|^2 \\ &\leq -\frac{1}{8\|N_\zeta(\nu)\|}\zeta(t)^T N_\zeta(\nu)\zeta(t) - \frac{1}{8\|N_\vartheta\|}\vartheta(t)^T N_\vartheta \vartheta(t) \leq -\alpha V(t)\,,\end{aligned}$$

hence (3.3.10) which shows the exponential stability of the closed-loop system and the proof is complete. □

The above proof gives evidence that high-gain derivative feedback stabilization works for linear MIMO-systems with non-strict relative degree. Any linear minimum phase system (A, B, C) the relative degree of which is know and having a positive definite high-frequency gain matrix can be stabilized by the derivative feedback (3.3.1) with sufficiently large design parameters $\nu > 0$ and $\kappa = \kappa(\nu) > 0$. Any other parameter $k_{j,i} \in \mathbb{R}$, $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, r_j\}$, can be chosen almost arbitrarily: $s \mapsto \sum_{i=0}^{r_j-1} k_{j,i+1}s^i$ must be a Hurwitz polynomial. Although the normal form for MIMO-systems with non-strict relative degree is involved and the proof for the high-gain stabilization result is very technical, the control strategy (3.3.1) is surprisingly simple. One shortcoming is that two parameters $\nu, \kappa > 0$ required. It might be possible to obtain a stabilization result with only one parameter $\kappa > 0$ as for systems with strict relative degree, however, to show boundedness of the solution $N_\zeta(\kappa)$ of the first Lyapunov equation in (3.3.11), one has to prove a generalization of Lemma 3.2.3 which might be extremely technical. Another shortcoming of the high-gain feedback is that the actual $\nu^*, \kappa^* > 0$ for which stabilization can be guaranteed are possibly very large. Moreover, it would be nice if a controller can "find" the sufficient ν and κ adaptively.

The following chapters will pursue this question. Two more control strategies are presented: an adaptive controller namely the λ-tracker, see Chapter 4, and the so-called funnel controller, see Chapter 5. On the one hand some more restrictive assumptions to the class of systems as for high-gain derivative feedback stabilization are made to apply these strategies: a system shall also have a positive high-frequency gain matrix and stable zero dynamics, but moreover the system has to have (strict) relative degree one. This restriction is essential for the proofs given in the present thesis. Future research might give more general results for λ-tracking and funnel control, however this is not part of the thesis. On the other hand the benefit of λ-tracking and funnel control is that, roughly speaking, these controllers do not required that a special parameter κ is provided; these controllers "will find their κ automatically".

Another question in control theory is the robustness analysis of control strategies. In Chapter 7 one robustness problem will be analyzed: may high-gain derivative output feedback be replaced by "high-gain delay output feedback"? In the feedback law (3.2.2) the derivatives of the output signal will be replaced by approximations of the derivatives and it will be shown that the new delay system is again stable for sufficiently small approximation step size. This will be proved using the terminology of the gap metric, see Chapter 6.

3.4 Notes and references

Stabilization of linear MIMO-systems by high-gain output derivative feedback seems to be well-known in control theory. However, to the author's best knowledge the results of the present chapter for MIMO-systems with strict and particularly non-strict relative degree cannot be found in the literature. There are similar results for nonlinear SISO- and MIMO-systems in [Isi95, Ch. 9] and [Isi99, Ch. 12]. The results and proofs from the present chapter are revised versions of the results within the submitted paper [Mue09b].

4 Adaptive λ-tracking

For m-input, m-output, finite-dimensional, linear systems satisfying the classical assumptions of adaptive control, that is that the system (i) is minimum phase (i.e. has stable zero dynamics), (ii) has (strict) relative degree one and (iii) has a "positive" high-frequency gain, it is well known (see [IR04]) that the adaptive λ-tracker '$u = -k\,e$, $\dot{k} = \max\{0, \|e\| - \lambda\}\|e\|$' achieves λ-tracking of the tracking error e if applied to such a system: all states of the closed-loop system are bounded and $\|e\|$ is ultimately bounded by λ, where $\lambda > 0$ is prespecified and may be arbitrarily small.

In the present chapter λ-tracking is introduced in detail. Due to later robustness analysis of λ-tracking in Chapter 8, a special λ-tracking result and a new proof are presented. This is required for the result of Chapter 8: there it is shown that the λ-tracker is robust in terms of the gap metric, see Chapter 6.

The present chapter is structured as follows. First some well known adaptive feedback strategies are introduced. Then follows a new λ-tracking result which is essential for the later robustness analysis in Chapter 8.

4.1 Adaptive feedback control

Consider linear n-dimensional, m-input, m-output (MIMO) systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t)\,, \qquad x(0) = x^0, \\ y(t) &= C\,x(t)\,, \end{aligned}\right\} \tag{4.1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$ and $x^0 \in \mathbb{R}^n$ is an arbitrary initial value.

There may be additive input/output disturbances u_0, y_0, respectively, from signal spaces specified in due course. To improve readability of the following, input u and output y of system (A, B, C) will be denoted as u_1, y_1, respectively. The interconnection equations

$$u_0 = u_1 + u_2, \qquad y_0 = y_1 + y_2\,, \tag{4.1.2}$$

lead to a closed-loop system as depicted in Figure 4.1: the linear system (A, B, C) maps the interior input signal u_1 to the interior output signal y_1 and the *controller* C maps the interior output signal y_2 (i.e. the difference of system's output y_1 and the output disturbance y_0) to the interior input signal u_2. Controller C, specified in due course, always denotes a feedback law.

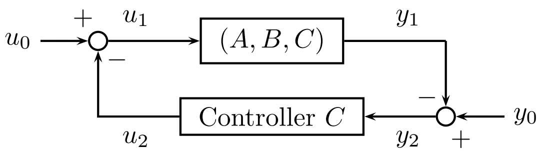


Figure 4.1: The closed-loop system: (A, B, C) with controller C.

By Theorem 3.2.1 one knows that in case of zero disturbances $u_0 \equiv y_0 \equiv 0$, any linear system (A, B, C) of form (4.1.1) can be stabilized by proportional high-gain ($k \gg 0$) output feedback

$$u_2(t) = -k\, y_2(t)\,, \tag{4.1.3}$$

provided that (4.1.1) is *minimum phase*, i.e. has stable zero dynamics or equivalently

$$\forall\, s \in \overline{\mathbb{C}}_+ \;:\; \det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0\,,$$

see Definition 2.3.1 and Corollary 2.3.2, and has *strict relative degree* one with "positive" high-frequency gain, i.e. CB is positive definite (not necessarily symmetric). For notational convenience write $CB + (CB)^T > 0$ if CB is positive definite.

The class of systems which satisfy the above structural properties is denoted, for $n, m \in \mathbb{N}$ with $n \geq m$, as

$$\widetilde{\mathcal{M}}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \;\middle|\; \begin{array}{l} CB + (CB)^T > 0\,, \\ \forall\, s \in \overline{\mathbb{C}}_+ \,:\, \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \right\}.$$

Note that the state space dimension $n \in \mathbb{N}$ needs not to be known but only the input/output dimension $m \in \mathbb{N}$. Most importantly, only structural assumptions are required but the entries of the system's matrices may be completely unknown, see Chapter 3. A sufficiently large high-gain k in (4.1.3) can be found adaptively. More precisely, any system $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$ can be stabilized adaptively, in the presence of L^2 input/output disturbances, by the controller (ubiquitous in the adaptive control literature)

$$\left. \begin{array}{rl} \dot{k}(t) = \|y_2(t)\|^2\,, & \quad k(0) = k^0 \in \mathbb{R}\,, \\ u_2(t) = -k(t)\, y_2(t)\,, & \end{array} \right\} \tag{4.1.4}$$

in the sense that all states of the closed-loop (4.1.1), (4.1.2), (4.1.4) are bounded and $\lim_{t\to\infty} y_1(t) = 0$. This approach has been introduced by the seminal work of [Mar84, Mor83, WB84], see also the survey [Ilc91].

The surprising property of the controller (4.1.4) is not only its simplicity but also its robustness: it is also applicable in the presence of additive L^2 input/output disturbances and it may stabilize systems (4.1.1) not belonging to $\widetilde{\mathcal{M}}_{n,m}$ but sufficiently "close" – in terms of the *gap metric* defined in Chapter 6 – to some (A, B, C) belonging to $\widetilde{\mathcal{M}}_{n,m}$. This has been proved in [FIR06].

However, the controller (4.1.4) has the shortcomings that, if tracking is the control objective, it needs to be combined with an internal model (thus becoming more involved) and, more importantly, fails for stabilizing non-linear systems or in the presence of additive arbitrarily small input or output L^∞-disturbances. To overcome these shortcomings, the so called λ-tracker

$$\left. \begin{array}{rl} \dot{k}(t) = \operatorname{dist}\left(y_2(t), [-\lambda, \lambda]\right) \cdot \|y_2(t)\|\,, & \quad k(0) = k^0\,, \\ u_2(t) = -k(t) y_2(t)\,, & \end{array} \right\} \tag{4.1.5}$$

for $\lambda > 0$, $k^0 \in \mathbb{R}$ and $\mathrm{dist}\colon \mathbb{R}^m \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ defined by $(y,\lambda) \mapsto \mathrm{dist}(y, [-\lambda, \lambda]) := \max\{0, \|y\| - \lambda\}$, has been introduced by [IR94].

The application of the λ-tracker (4.1.5) to any system (4.1.1) belonging to $\widetilde{\mathcal{M}}_{n,m}$, via (4.1.2), satisfies, in the presence of arbitrary input/output disturbances u_0, y_0 which are essentially bounded and have essentially bounded derivatives, arbitrary initial conditions $x^0 \in \mathbb{R}^n$, $k^0 \in \mathbb{R}$ and any arbitrarily small design parameter $\lambda > 0$, the control objectives of λ-*tracking*:

- all signals of the closed-loop system (4.1.1), (4.1.2), (4.1.5) and their derivatives are bounded;
- $\limsup_{t\to\infty} \mathrm{dist}(y_2(t), [-\lambda, \lambda]) = 0.$

This result has been generalized to nonlinear and infinite dimensional systems [IRS02a] and applied, to name but a few, to regulate biogas tower reactors [IP98], chemical reactors [ITT04], insulin delivery for diabetic patients [BFS+00] by preserving the simplicity of the control strategy. Note also that it is a tracking result without invoking an internal model: set $y_0(\cdot) \equiv y_{\mathrm{ref}}(\cdot)$ as the prespecified reference signal.

Due to better applicability, the result in the present chapter is restricted to systems in Byrnes–Isidori normal form, see Corollary 2.2.5 for the normal form for linear systems with strict relative degree, instead of systems $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$. That is, for each $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ the matrix

$$\overline{U} = \begin{bmatrix} C \\ (\overline{\mathcal{V}}^T \overline{\mathcal{V}})^{-1} \overline{\mathcal{V}}^T [I_n - B(CB)^{-1}C] \end{bmatrix},$$

where $\overline{\mathcal{V}} \in \mathbb{R}^{n\times(n-m)}$ with $\mathrm{rk}\,\overline{\mathcal{V}} = n-m$ and $\mathrm{im}\,\overline{\mathcal{V}} = \ker C$, converts (4.1.1) via the coordinate transformation $\binom{y_1}{\eta} = \overline{U}x$ into

$$\begin{aligned} \dot y_1 &= A_1 y_1 + A_2 \eta + CB\, u_1\,, & y_1(0) &= y_1^0 \in \mathbb{R}^m, \\ \dot\eta &= A_3 y_1 + A_4 \eta\,, & \eta(0) &= \eta^0 \in \mathbb{R}^{n-m}, \end{aligned} \quad \begin{pmatrix} y_1^0 \\ \eta^0 \end{pmatrix} = \overline{U}x^0, \tag{4.1.6}$$

where

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} := \overline{U} A \overline{U}^{-1}, \quad \begin{bmatrix} B_1 \\ 0_{(n-m)\times m} \end{bmatrix} := \begin{bmatrix} CB \\ 0 \end{bmatrix} = \overline{U}B\,.$$

and

$$\begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix} = C\overline{U}^{-1}.$$

By the minimum-phase property Corollary 2.3.1 yields that A_4 is Hurwitz, i.e. has spectrum in the open left half complex plane $\mathbb{C}_-$. Now introduce, for $n, m \in \mathbb{N}$ with $n \geq m$, the system class

$$\mathcal{M}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \middle| \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C = \begin{bmatrix} I_m & 0 \end{bmatrix}, \; B_1, A_1 \in \mathbb{R}^{m\times m}, \\ \operatorname{spec}(A_4) \subset \mathbb{C}_-, \; B_1 + B_1^T > 0 \end{array} \right\}.$$

In the following section the properties of the closed-loop system generated by the application of the λ-tracker (4.1.5) to systems (A, B, C) of class $\mathcal{M}_{n,m}$ and in the presence of disturbances (u_0, y_0) from signal spaces $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, satisfying the interconnection equations (4.1.2), are studied. The closed-loop system (4.1.6), (4.1.5), (4.1.2) is depicted in Figure 4.2.

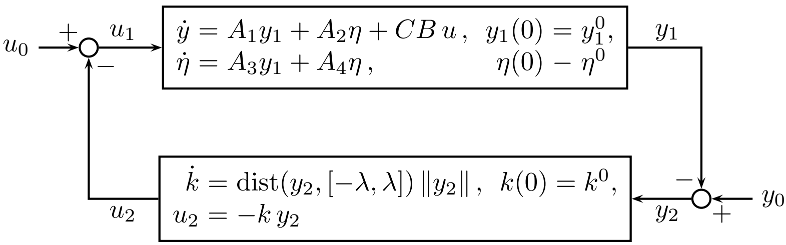


Figure 4.2: The adaptive closed-loop system.

4.2 λ-tracking

In this section it is shown that the control strategy (4.1.5) applied to any linear system (A, B, C) of class $\mathcal{M}_{n,m}$ achieves λ-tracking in the presence of $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ input/output disturbances, see Figure 4.2 for the closed-loop system. To be more precise it is shown that all signals and their derivatives of the closed-loop system (4.1.6), (4.1.5), (4.1.2) are essentially bounded and that $\|y_2\|$ asymptotically approaches a λ-strip at zero, i.e. $\limsup_{t\to\infty} \operatorname{dist}(y_2(t), [-\lambda, \lambda]) = 0$.

The latter is the actual tracking result: consider $y_0 = y_{\text{ref}}$ as reference signal and the error $e = y_2 = y_{\text{ref}} - y_1$ between reference signal y_{ref} and solution y_1 of the controlled system. Recall that $\lambda > 0$ is prespecified and so may be arbitrarily small. Thus the norm of the error $\|e(t)\|$ becomes, in view of $\limsup_{t\to\infty} \operatorname{dist}(e(t), [-\lambda, \lambda]) = 0$, also arbitrarily small as t tends to infinity. Hence $y_1(t)$ is close to the reference signal $y_{\text{ref}}(t)$ for sufficiently large $t > 0$.

Furthermore, Theorem 4.2.1 shows not only that all signals of the closed-loop systems (4.1.6), (4.1.5), (4.1.2) and their derivatives are essentially bounded but that they are uniformly essentially bounded in terms of the linear system's matrices, initial values and disturbance signals. This feature is very important for the robustness analysis of λ-tracking in terms of the gap metric: alone the uniform boundedness leads to the *gain-function stability* of the closed-loop systems, see Chapter 8 and Section 6.4, which is essential for the robustness analysis.

Set, for $n, m \in \mathbb{N}$ with $n \geq m$,

$$\mathcal{D}_{n,m} := \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$$

the set of all tuples of systems, initial values y_1^0, η^0 of the linear system and k^0 of the controller and input/output disturbances (u_0, y_0).

Theorem 4.2.1 *Let* $m, n \in \mathbb{N}$ *with* $n \geq m$ *and* $\lambda > 0$. *Then there exists a continuous map* $\nu \colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ *such that, for all tuples* $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix} \right], B, C, (y_1^0, \eta^0, k^0), u_0, y_0 \right) \in \mathcal{D}_{n,m}$, *the associated closed-loop initial value problem* (4.1.6), (4.1.2), (4.1.5) *satisfies*

$$\|(u_2, y_2, \eta, k)\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n+1})} \leq \nu(d) \tag{4.2.1}$$

and

$$\limsup_{t\to\infty} \|y_2(t)\| \leq \lambda \,. \tag{4.2.2}$$

That λ-tracking achieves the second control objective from Section 4.1 for systems from the class $\mathcal{M}_{n,m}$ goes back to [IR94] and tracking with input disturbances are considered in [Ilc98]. However, to prove robustness of the λ-tracker in Chapter 8, the existence of a continuous function $\nu(\cdot)$ satisfying (4.2.1) is crucial. Therefore, a new proof showing (4.2.1) and (4.2.2) follows.

The proof is structured as follows: first existence and uniqueness of solutions of the closed-loop initial value problem (4.1.6), (4.1.2), (4.1.5) is proved. Then estimates for the y_2- and η-dynamics are given which are used to show boundedness of k, y_2, u_2 and η. With this follows that the solution exists on whole $\mathbb{R}_{\geq 0}$. Then it is easy to show essential boundedness of the derivatives of k, y_2, u_2 and η. The last step proves (4.2.2).

Proof of Theorem 4.2.1. Consider any element from the set $\mathcal{D}_{n,m}$, i.e. let $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0, k^0), u_0, y_0\right) \in \mathcal{D}_{n,m}$ and set, for notational convenience,

$$\begin{array}{rcl} h(\cdot) & := & \dot{y}_0(\cdot) - A_1\, y_0(\cdot) - CB\, u_0(\cdot)\,, \\ e(\cdot) & := & y_2(\cdot) \end{array}$$

and define $d_\lambda \colon \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $e \mapsto d_\lambda(e) := \max\{0, \|e\| - \lambda\}$. The closed-loop initial value problem (4.1.6), (4.1.2), (4.1.5) is then given by

$$\left.\begin{array}{ll} \dot{e} = A_1\, e - A_2\, \eta - k\, CB\, e + h\,, & e(0) = e^0 := y_0(0) - y_1^0\,, \\ \dot{\eta} = -A_3\, e + A_4\, \eta + A_3\, y_0\,, & \eta(0) = \eta^0\,, \\ \dot{k} = d_\lambda(e)\, \|e\|\,, & k(0) = k^0\,. \end{array}\right\} \quad (4.2.3)$$

The proof is divided into ten steps.

Step 1: Existence and uniqueness of a solution of (4.2.3) is shown.

The right hand side of (4.2.3) is continuous and locally Lipschitz, i.e.

$$f \colon \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R} \to \mathbb{R}^{n+1}\,,$$
$$(t, e, \eta, k) \mapsto \begin{pmatrix} A_1\, e - A_2\, \eta - kCBe + h \\ -A_3\, e + A_4\, \eta + A_3\, y_0 \\ d_\lambda(e)\, \|e\| \end{pmatrix},$$

satisfies a local Lipschitz condition on the relatively open set $\mathbb{R}_{\geq 0} \times \mathbb{R}^m \times$

$\mathbb{R}^{n-m} \times \mathbb{R}$ in the sense that, for all $(\tau, \xi, \zeta, \kappa) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$, there exists an open neighbourhood O of $(\tau, \xi, \zeta, \kappa)$ and a constant $L > 0$ such that

$$\forall\, (t, e, \eta, k) \in O \;:\; \|f(t, e, \eta, k) - f(t, \xi, \zeta, \kappa)\| \leq L(\|e - \xi\| + \|\eta - \zeta\| + \|k - \kappa\|)\,.$$

Therefore, standard theory of ordinary differential equations, see, for example, [Wal98, Thm. III.11.III], yields that (4.2.3) has an absolutely continuous solution

$$(e, \eta, k)\colon [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$$

for some $\omega \in (0, \infty]$. Moreover, the solution is unique and the solution can be extended up to the boundary of $\mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$. In other words: for every compact $\mathcal{K} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$ exists $t \in [0, \omega)$ such that $(t, e(t), \eta(t), k(t)) \notin \mathcal{K}$. This means that either $\omega = \infty$ or if $\omega < \infty$ then for every $\varepsilon > 0$ there exists $t \in (0, \omega)$ such that $\|(e(t), \eta(t), k(t)\| > 1/\varepsilon$.

Step 2: Next some constants are defined. These are used in the following steps of the proof.

Since $\operatorname{spec}(A_4) \subset \mathbb{C}_-$ it follows that

$$\exists\, M_1, \mu > 0 \quad \forall\, t \geq 0 \;:\; \|\exp(A_4\, t)\| \leq M_1 \exp(-\mu t)\,. \tag{4.2.4}$$

Set

$$\begin{aligned}
\sigma_1 &:= \min \operatorname{spec}\left(CB + (CB)^T\right)/2\\
M_2 &:= M_1 + M_1\|A_3\|\big(\|y_0\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \lambda + \mu\big)/\mu\\
M_3 &:= M_2\left(1 + \|\eta^0\|\right)/\lambda + M_2\left(1 + 1/\mu\right)\\
M_4 &:= \|A_1\| + \|A_2\| + \|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}/\lambda\\
M_5 &:= |k^0| + 2\left(M_4 + M_3 M_4 + 1\right)/\sigma_1\\
M_6 &:= M_5 + |k^0| + \|e^0\|^2/2\\
M_7 &:= \big(d_\lambda(e^0)^2 + 2(M_6 + |k^0|)\left[\sigma_1\big(M_6 + |k^0|\big)/2 + M_4 + M_3 M_4\right]\big)^{\frac{1}{2}} + \lambda\\
M_8 &:= M_2\left(1 + \|\eta^0\| + M_7/\mu\right).
\end{aligned}$$

Step 3: It is shown that the η-dynamics can be estimated as

$$\forall\, t \in [0,\omega) \;:\; \int_0^t d_\lambda(e(\tau))\,\|\eta(\tau)\|\,\mathrm{d}\tau \le\; M_3\,\left[k(t) - k^0\right]. \tag{4.2.5}$$

Applying Variation of Constants to the second equation in (4.2.3) and invoking (4.2.4) gives, for all $t \in [0,\omega)$,

$$\begin{aligned}
&\|\eta(t)\| \\
&\le M_1\, e^{-\mu t}\|\eta^0\| + \int_0^t M_1\, e^{-\mu(t-\tau)}\|A_3\|\,(\|e(\tau)\| + \|y_0(\tau)\|)\,\mathrm{d}\tau \\
&\le M_1\, e^{-\mu t}\|\eta^0\| \\
&\quad + M_1\|A_3\| \int_0^t e^{-\mu(t-\tau)}\,\big(d_\lambda(e(\tau)) + \|y_0\|_{L^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R}^m)} + \lambda\big)\,\mathrm{d}\tau \\
&\le M_1\, e^{-\mu t}\|\eta^0\| + M_1\|A_3\| \int_0^t e^{-\mu(t-\tau)}\|y_0\|_{L^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R}^m)}\,\mathrm{d}\tau \\
&\quad + M_1\|A_3\| \int_0^t e^{-\mu(t-\tau)}\lambda\,\mathrm{d}\tau + M_1\|A_3\| \int_0^t e^{-\mu(t-\tau)} d_\lambda(e(\tau))\,\mathrm{d}\tau \\
&\le M_1\,\|\eta^0\| + \frac{M_1\|A_3\|}{\mu}\big(\|y_0\|_{L^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R}^m)} + \lambda\big) \\
&\quad + M_1\|A_3\| \int_0^t e^{-\mu(t-\tau)} d_\lambda(e(\tau))\,\mathrm{d}\tau \\
&\le M_2\,\left[1 + \|\eta^0\| + \int_0^t e^{-\mu(t-\tau)} d_\lambda(e(\tau))\,\mathrm{d}\tau\right].
\end{aligned} \tag{4.2.6}$$

Let

$$\forall\, t \in [0,\infty)\ \forall\, \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}) \;:\; (L * \varphi)(t) := \int_0^t e^{-\mu(t-\tau)}\varphi(\tau)\,\mathrm{d}\tau\,.$$

Invoking the well known inequality, see for example [Vid93, p. 298],

$$\begin{aligned}
\forall\, t \ge 0 \;:\; \|L * \varphi\|_{L^2([0,t)\to\mathbb{R})} &\le \|e^{-\mu\,\cdot}\|_{L^1(\mathbb{R}_{\ge 0}\to\mathbb{R})}\|\varphi\|_{L^2([0,t)\to\mathbb{R})} \\
&= \tfrac{1}{\mu}\|\varphi\|_{L^2([0,t)\to\mathbb{R})}
\end{aligned}$$

and the fact that

$$\forall\, e \in \mathbb{R}^m \;:\; d_\lambda(e)^2 \le d_\lambda(e)\, \|e\|$$

yields, in view of (4.2.3), (4.2.6) and the Cauchy–Schwarz inequality, for all $t \in [0,\omega)$,

$$\begin{aligned}
&\int_0^t d_\lambda(e(\tau))\, \|\eta(\tau)\|\, \mathrm{d}\tau \\
&\le M_2 \int_0^t d_\lambda(e(\tau)) \left[1 + \|\eta^0\| + (L * d_\lambda(e))(\tau)\right] \mathrm{d}\tau \\
&\le M_2 \left[1 + \|\eta^0\|\right] \frac{1}{\lambda} \int_0^t d_\lambda(e(\tau))\, \|e(\tau)\|\, \mathrm{d}\tau \\
&\quad + M_2 \left[\|d_\lambda(e)\|^2_{L^2([0,t)\to\mathbb{R})} + \|L * d_\lambda(e)\|^2_{L^2([0,t)\to\mathbb{R})}\right] \\
&\le M_2 \left[1 + \|\eta^0\|\right] \frac{1}{\lambda} \int_0^t d_\lambda(e(\tau))\, \|e(\tau)\|\, \mathrm{d}\tau \\
&\quad + M_2 \left(1 + \frac{1}{\mu}\right) \int_0^t d_\lambda(e(\tau))^2\, \mathrm{d}\tau\,.
\end{aligned}$$

This proves (4.2.5).

Step 4: It is shown that the e-dynamics can be estimated as

$$\begin{aligned}
&\forall\, t \in [0,\omega) \;: \\
&\frac{1}{2} d_\lambda(e(t))^2 \le \frac{1}{2} d_\lambda(e^0)^2 \\
&\qquad - \left(k(t) - k^0\right) \left[\frac{\sigma_1}{2}\left(k(t) + k^0\right) - M_4 - M_3 M_4\right]. \qquad (4.2.7)
\end{aligned}$$

By (4.2.3) and Step 2 follows, omitting the argument t,

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} d_\lambda(e(t))^2\right) \\
&= d_\lambda(e)\, \|e\|^{-1}\, e^T\, \dot{e} \\
&= d_\lambda(e)\, \|e\|^{-1} e^T \left[A_1\, e - A_2\, \eta - k\, CB\, e + h\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq d_\lambda(e)\,\|e\|\,\|A_1\| + d_\lambda(e)\,\|\eta\|\,\|A_2\| + d_\lambda(e)\,\|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\quad - k\,d_\lambda(e)\,\|e\|^{-1}\,e^T\left(\tfrac{1}{2}(CB+(CB)^T)\right)e \\
&\leq -(k\,\sigma_1 - \|A_1\|)d_\lambda(e)\,\|e\| + \|A_2\|\,d_\lambda(e)\,\|\eta\| + d_\lambda(e)\,\|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq -(k\,\sigma_1 - \|A_1\|)d_\lambda(e)\,\|e\| + \|A_2\|\,d_\lambda(e)\,\|\eta\| \\
&\quad + d_\lambda(e)\frac{\|e\|}{\lambda}\,\|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq -(k\,\sigma_1 - M_4)d_\lambda(e)\,\|e\| + M_4\,d_\lambda(e)\,\|\eta\|\,,
\end{aligned}
$$

and hence, by integration and invoking (4.2.5), one arrives at

$$
\begin{aligned}
\forall\, t\in[0,\omega)\ :\ \frac{1}{2}d_\lambda(e(t))^2 &\leq \frac{1}{2}d_\lambda(e^0)^2 \\
&\quad - \int_0^t (k(\tau)\,\sigma_1 - M_4)\dot{k}(\tau)\,\mathrm{d}\tau + M_3M_4\left[k(t)-k^0\right]
\end{aligned}
$$

which yields (4.2.7).

Step 5: Boundedness of k is shown:

$$
\forall\ t\in[0,\omega)\ :\ k(t)\leq M_6 \tag{4.2.8}
$$

Suppose there exists $T\in[0,\omega)$ such that $k(T)=M_5$, otherwise inequality (4.2.8) is obvious. Then, by monotonicity of k, it follows from (4.2.7) that, for all $t\in[T,\omega)$,

$$
\begin{aligned}
0 &\leq \frac{1}{2}d_\lambda(e(t))^2 \\
&\leq \frac{1}{2}d_\lambda(e^0)^2 - \frac{\sigma_1}{2}\left(k(t)-k^0\right)\left[k(t)+k^0-\frac{2}{\sigma_1}(M_4+M_3M_4)\right] \\
&\leq \frac{1}{2}d_\lambda(e^0)^2 - \frac{\sigma_1}{2}\left(k(t)-k^0\right)\left[M_5+k^0-\frac{2}{\sigma_1}(M_4+M_3M_4)\right] \\
&= \frac{1}{2}d_\lambda(e^0)^2 - \frac{\sigma_1}{2}\left(k(t)-k^0\right)\left[|k^0|+k^0+\frac{2}{\sigma_1}\right] \\
&\leq \frac{1}{2}d_\lambda(e^0)^2 - \left(k(t)-k^0\right)\,,
\end{aligned}
$$

and thus

$$\forall\, t \in [T,\omega) \;:\; k(t) - k^0 \le \frac{1}{2} d_\lambda(e^0)^2 \le \frac{1}{2}\|e^0\|^2$$

and

$$\forall\, t \in [0,T) \;:\; k(t) - k^0 \le M_5 - k^0\,,$$

whence (4.2.8).

Step 6: Boundedness of e is shown:

$$\forall\, t \in [0,\omega) \;:\; \|e(t)\| \le M_7\,. \tag{4.2.9}$$

An application of (4.2.8) to (4.2.7) gives, for all $t \in [0,\omega)$,

$$\begin{aligned}
&\|e(t)\| \le d_\lambda(e(t)) + \lambda \\
&\le \left(d_\lambda(e^0)^2 - 2\left(k(t)-k^0\right)\left[\frac{\sigma_1}{2}\left(k(t)+k^0\right) - M_4 - M_3M_4\right]\right)^{\frac{1}{2}} + \lambda \\
&\le \left(d_\lambda(e^0)^2 + 2(M_6+|k^0|)\left[\frac{\sigma_1}{2}\left(M_6+|k^0|\right) + M_4 + M_3M_4\right]\right)^{\frac{1}{2}} + \lambda\,.
\end{aligned}$$

Note that the argument of the root in the second line is nonnegative, see Step 5. Now (4.2.9) follows from Step 2.

Step 7: Boundedness of η in the form

$$\forall\, t \in [0,\omega) \;:\; \|\eta(t)\| \le M_2\left[1 + \|\eta^0\| + \int_0^t e^{-\mu(t-\tau)} M_7 \,\mathrm{d}\tau\right] \le M_8 \tag{4.2.10}$$

follows from applying (4.2.9) to (4.2.6).

Step 8: It is shown that $\omega = \infty$.

Seeking a contradiction suppose that $\omega < \infty$. Then, in view of inequalities (4.2.8)–(4.2.10),

$$\mathcal{K} := \left\{ \begin{array}{l} (t,e,\eta,k) \\ \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R} \end{array} \;\middle|\; \begin{array}{l} t \in [0,\omega]\,, \\ \|e,\eta,k\| \le \sqrt{M_6^2 + M_7^2 + M_8^2} \end{array} \right\}$$

is a compact subset of $\mathbb{R}_{\ge 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}$ with $(t, e(t), \eta(t), k(t)) \in \mathcal{K}$ for all $t \in [0,\omega)$, which contradicts the fact that the closure of graph $\left((e,\eta,k)|_{[0,\omega)}\right)$ is not a compact set, see Step 1. Therefore, $\omega = \infty$

as required.

Step 9: Inequality (4.2.1) is shown.

Recall that $y_2 = e$. It follows from Step 5–8 that (u_2, y_2, η, k) is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0, k^0), u_0, y_0\right)$. Moreover, applying Step 5–8 again and invoking (4.2.3) yields uniform boundedness of $\left(\dot{u}_2, \dot{y}_2, \dot{\eta}, \dot{k}\right)$ in terms of d. Now the existence of a continuous function $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that (4.2.1) holds is straightforward by invoking the constants from Step 2.

Step 10: Finally, (4.2.2) is shown.

Since, in view of (4.2.1), $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ it follows from the equality $\|d_\lambda(y_2)\,\|y_2\|\|_{L^1([0,t)\to\mathbb{R})} = k(t) - k^0$ that $d_\lambda(y_2)\,\|y_2\| \in L^1(\mathbb{R}_{\geq 0} \to \mathbb{R})$.

Since $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ there exists a constant $M > 0$ such that $\operatorname{ess\,sup}_{t\geq 0} |(\dot{y}_2)_i(t)| < M$, for all $i \in \{1, \ldots, m\}$, which gives

$$\forall\, s \geq 0\ \forall\, i \in \{1. \ldots, m\}\ \forall\, t \in [0, s)\ \exists\, \tau_i \in (t, s)\ :\quad (\dot{y}_2)_i(\tau_i) = \frac{(y_2)_i(s) - (y_2)_i(t)}{s - t} < M$$

and so

$$\forall\, i \in \{1. \ldots, m\}\ \forall\, t \in [0, s)\ :\ |(y_2)_i(s) - (y_2)_i(t)| < M(s - t)\,.$$

In view of setting $\delta = \frac{\varepsilon}{M}$, one arrives at

$$\forall\, i \in \{1. \ldots, m\}\ \forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, t, s \in \mathbb{R}_{\geq 0} \text{ with } |t - s| < \delta\ :\quad |(y_2)_i(t) - (y_2)_i(s)| < \varepsilon\,,$$

i.e. y_2 is uniformly continuous. Boundedness and uniform continuity of y_2 and the continuity of the map $e \mapsto d_\lambda(e)\,\|e\|$ gives uniform continuity of $t \mapsto d_\lambda(y_2(t))\,\|y_2(t)\|$. So Barbălat's Lemma, see [Bar59], gives

$$\lim_{t\to\infty} d_\lambda(y_2(t))\,\|y_2(t)\| = 0\,,$$

which yields (4.2.2) and completes the proof. □

The statement of Theorem 4.2.1 includes the classical λ-tracking result. Moreover, uniform boundedness of the system's signals and their derivatives is shown. This is essential for the robustness analysis in terms of the gap metric, see Chapter 8.

Although an important result, λ-tracking has two significant shortcomings: (i) the tracking error $\lambda > 0$ will only be achieved asymptotically, i.e. the distance between reference signal and output is only asymptotically small, and (ii), though bounded, the system's dynamic k is increasing. The latter is easy to see since $\dot{k} = d_\lambda(y_2)\|y_2\|$ is always nonnegative. The λ-tracker cannot overcome this shortcomings. Therefore, funnel control is introduced in the following chapter.

The funnel controller overcomes both shortcomings: (i) one can prescribe almost arbitrary positive, bounded, locally Lipschitz continuous functions which constrain the error between reference signal and the system's output, for example functions which approach a prespecified bound $\lambda > 0$ in prespecified time $T > 0$, and (ii) the funnel controller has no additional dynamic $\dot{k}(t)$ which has to be positive for all $t \geq 0$, but applies a map k in $u_2 = -k\,y_2$ which will be become large if necessary and may decrease afterwards.

4.3 Notes and references

λ-tracking has been introduced by [IR94] and has been generalized to nonlinear and infinite dimensional systems [IRS02a]. The results of the present chapter are from [IM08], where the attention lies on the robustness analysis of λ-tracking. Therefore, the main result of the present chapter is designed to provide all requirements to show robust stability of λ-tracking in Chapter 8. In particular it is shown that the system's signals are uniformly bounded in terms of the system's data.

5 Funnel control

In the present chapter funnel control is introduced. Loosely speaking, funnel control exploits the high-gain property of a system from class $\widetilde{\mathcal{M}}_{n,m}$ by designing a proportional feedback $u(t) = -k(t)\, e(t)$ in such a way that $k(t)$ becomes large if $\|e(t)\|$ approaches a prespecified performance funnel boundary ψ from a large class of boundary functions, thereby precluding contact with the funnel boundary.

As for λ-tracking in the previous chapter m-input, m-output (MIMO), finite-dimensional, linear systems satisfying the classical assumptions of adaptive control, i.e. having (i) stable zero dynamics (i.e. being minimum phase), (ii) (strict) relative degree one and (iii) "positive" high-frequency gain are considered. The well known funnel controller '$u = -k\, e,\ k = 1/(\psi - \|e\|)$' achieves tracking of a tracking error e within a prescribed performance funnel with boundaries ψ and $-\psi$ where ψ is from a large class of positive, bounded, locally Lipschitz continuous functions, see also Figure 5.2. Moreover, if applied to the class of systems which satisfy properties (i)–(iii), the funnel controller also leads to that all signals and states of the closed-loop system are essentially bounded.

Funnel control has been introduced by [IRS02b] not only for linear systems but a rather general system class including nonlinear systems, nonlinear delay systems, systems with hysteresis and infinite-dimensional regular linear systems. In [IRT06] also linear systems with higher relative degree are considered: to apply the funnel controller to linear systems with (strict) relative degree $r \geq 2$ the authors have to apply an additional filter.

In Section 5.2 a well known but slightly modified funnel control result is presented: additionally it is shown that all states and signals of a funnel controlled system are uniformly bounded in terms of the matrices of the linear system, the initial values, input/output disturbances and the funnel boundary function. This is required for the robustness analysis of funnel control in Chapter 9 by invoking the conceptual framework of nonlinear gap metric.

5.1 Preliminaries for funnel control

As in the previous chapter on λ-tracking, consider linear n-dimensional, m-input, m-output (MIMO) systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u_1(t)\,, \qquad x(0) = x^0, \\ y_1(t) &= C\,x(t)\,, \end{aligned}\right\} \tag{5.1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$ and $x^0 \in \mathbb{R}^n$. As depicted in Figure 5.1 consider additive input/output disturbances u_0, y_0, respectively, which in view of the interconnection equations

$$u_0 = u_1 + u_2, \qquad y_0 = y_1 + y_2\,, \tag{5.1.2}$$

lead to a closed-loop system of linear system and funnel controller, where the funnel controller is specified in due course.

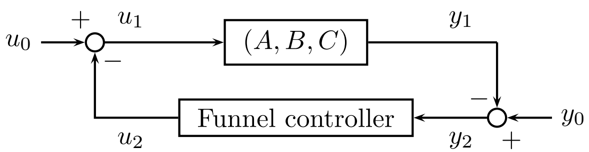


Figure 5.1: The closed-loop system (A, B, C) with funnel controller.

As for λ-tracking consider, for $n, m \in \mathbb{N}$ with $n \geq m$, linear systems from the class of systems

$$\widetilde{\mathcal{M}}_{n,m} = \left\{ \begin{array}{l} (A, B, C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \;\middle|\; \begin{array}{l} CB + (CB)^T > 0\,, \\ \forall\, s \in \overline{\mathbb{C}}_+ \,:\, \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \right\},$$

i.e. the class of all linear minimum phase systems with relative degree one and positive definite high-frequency gain matrix CB.

As highlighted in the conclusion of Chapter 4 one shortcoming of the λ-tracker (4.1.5) is the increasing k. It would be more efficient having a control strategy which uses k that is large if necessary and small else. Therefore the control objectives, defined below, will be captured in terms of the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\},$$

determined by a function $\varphi(\cdot)$ belonging to

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}) \;\middle|\; \begin{array}{l} \varphi(0) = 0\,,\ \forall\, t > 0\,:\, \varphi(t) > 0\,, \\ \forall\, \varepsilon > 0 : \varphi|_{[\varepsilon,\infty)}(\cdot)^{-1} \text{ is globally} \\ \text{Lipschitz},\ \liminf_{t\to\infty} \varphi(t) > 0 \end{array} \right\}.$$

Note that the funnel boundary is given by $\varphi(t)^{-1}$, $t > 0$; see Figure 5.2. The concept of performance funnel had been introduced by [IRS02b]. There it is not assumed that $\varphi(\cdot)$ has the Lipschitz condition as given in Φ; this mild assumption is incorporated for technical reasons. The assumption $\varphi(0) = 0$ allows to start with arbitrarily large initial conditions x_0 and output disturbances y_0. If for special applications the initial value and y_0 are known, then $\varphi(0) = 0$ may be relaxed by $\varphi(0)\|y_0(0) - Cx^0\| < 1$, see also the examples in Subsection 9.2.1.

The funnel controller, for prespecified $\varphi(\cdot) \in \Phi$, is given by

$$\left.\begin{aligned} k(t) &= \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|}\,, \quad \varphi \in \Phi\,, \\ u_2(t) &= -k(t)y_2(t)\,, \end{aligned}\right\} \tag{5.1.3}$$

and will be applied to linear systems (5.1.1) or (5.1.4).

If the funnel controller (5.1.3), for prespecified $\varphi \in \Phi$ determining the funnel boundary, is applied to any system (5.1.1), belonging to the class $\widetilde{\mathcal{M}}_{n,m}$, in the presence of input/output disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (5.1.2), then the closed-loop system (5.1.1), (5.1.2), (5.1.3), as depicted in Figure 5.3, is supposed to meet the following control objectives:

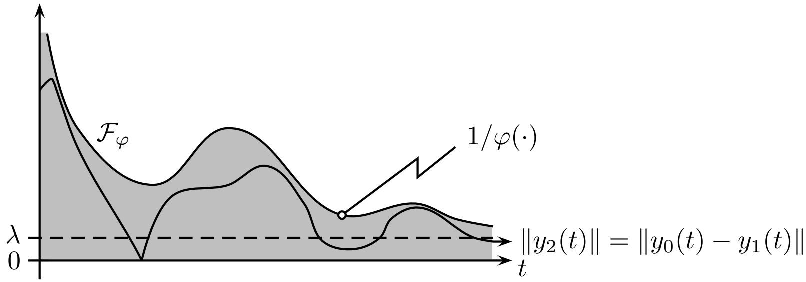


Figure 5.2: Funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ and $\inf_{t>0} \varphi(t)^{-1} = \lambda$

- all signals of the closed-loop system are essentially bounded;
- $\forall\, t \geq 0 : (t, y_2(t)) \in \mathcal{F}_\varphi = \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}$.

A proof for these will be given in Section 5.2. Note that, for $t = 0$, in view of the set Φ the value $y_2(0)$ may be arbitrarily large which implies that one may choose every initial value $y_1^0 \in \mathbb{R}^m$ and every bounded output disturbance y_0 such that $y_2(0) = y_0(0) - y_1^0 \in \mathbb{R}^m$.

The reciprocal funnel boundary $\varphi \in \Phi$ may be chosen arbitrarily. So one can set φ such that $\lim_{t\to\infty} \varphi^{-1}(t) = \lambda$, for arbitrarily small $\lambda > 0$. Then, as in the case of λ-tracking, one would also arrive at

$$\limsup_{t\to\infty} \operatorname{dist}(y_2(t), [-\lambda, \lambda]) = 0\,.$$

Furthermore, one can choose φ such that $\varphi^{-1}(t) = \lambda$ for all t larger then some specified time $T > 0$. Then

$$\forall\, t > T \;:\; \operatorname{dist}(y_2(t), [-\lambda, \lambda]) = 0\,,$$

which is much better than the usual tracking result.

Note that the funnel controller (5.1.3) is not an adaptive control strategy. There is no new dynamic $\dot{k}$ in the closed-loop system as for λ-tracking (4.1.5). The new variable k just measures the distance between $y_2 = y_0 - y_1$ and the funnel boundary φ^{-1}. If this becomes large then “some stabilizing effect” on the system will be executed, if k is small then y_2 is well in the funnel and has some distance to the funnel boundary.

As in the previous chapter and due to more convenient applicability the main result of the present chapter is restricted to systems in Byrnes–Isidori normal form, see Corollary 2.2.5 for the normal form for linear systems with strict relative degree, instead of systems $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$. Recall the system class

$$\mathcal{M}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \left| \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C = \begin{bmatrix} I_m & 0 \end{bmatrix},\ B_1, A_1 \in \mathbb{R}^{m\times m}, \\ \operatorname{spec}(A_4) \subset \mathbb{C}_-\,,\ B_1 + B_1^T > 0 \end{array} \right. \right\}.$$

from Section 4.1, i.e. the class of all linear minimum phase systems with relative degree one and positive definite high-frequency gain matrix CB which are already in Byrnes–Isidori normal form

$$\begin{aligned} \dot y_1 &= A_1 y_1 + A_2 \eta + CB\, u_1\,, & y_1(0) &= y_1^0 \in \mathbb{R}^m, \\ \dot\eta &= A_3 y_1 + A_4 \eta\,, & \eta(0) &= \eta^0 \in \mathbb{R}^{n-m}, \end{aligned} \quad \begin{pmatrix} y_1^0 \\ \eta^0 \end{pmatrix} = \overline{U} x^0, \quad (5.1.4)$$

where

$$\overline{U} = \begin{bmatrix} C \\ (\overline{\mathcal{V}}^T \overline{\mathcal{V}})^{-1} \overline{\mathcal{V}}^T [I_n - B(CB)^{-1} C] \end{bmatrix},$$

for $\overline{\mathcal{V}} \in \mathbb{R}^{n\times(n-m)}$ with $\operatorname{rk} \overline{\mathcal{V}} = n - m$ and $\operatorname{im} \overline{\mathcal{V}} = \ker C$ and

$$\begin{aligned} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} &:= \overline{U} A \overline{U}^{-1}, \\ \begin{bmatrix} B_1 \\ 0_{(n-m)\times m} \end{bmatrix} &:= \begin{bmatrix} CB \\ 0 \end{bmatrix} = \overline{U} B\,, \\ \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix} &= C \overline{U}^{-1}. \end{aligned}$$

In the following section properties of the closed-loop system generated by the application of the funnel controller (5.1.3) to systems (5.1.1) of class $\mathcal{M}_{n,m}$ in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (5.1.2) are studied. The closed-loop system (5.1.4), (5.1.2), (5.1.3) is depicted in Figure 5.3.

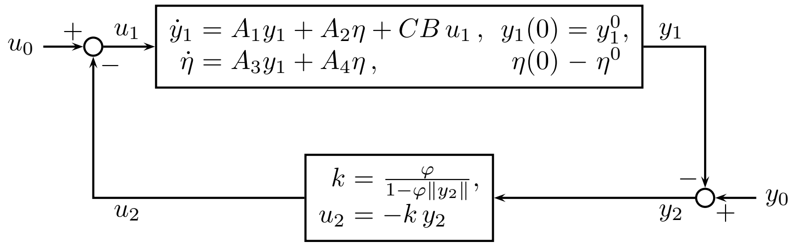


Figure 5.3: The funnel control closed-loop system.

5.2 Funnel control result

In this section it is explicitly shown that the funnel controller (5.1.3) applied to any linear system (A, B, C) of class $\mathcal{M}_{n,m}$ achieves in presence of $L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ input/output disturbances u_0, y_0, respectively, the control objectives of funnel control: y_2 is forced to proceed within a funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ and all signals and states of the closed-loop (5.1.4), (5.1.2), (5.1.3), as depicted in Figure 5.3, remain essentially bounded. Moreover, it is shown that the derivatives of the output signals y_1, y_2 and the state $\left(\begin{smallmatrix} y_1 \\ \eta \end{smallmatrix}\right)$ are essentially bounded, too.

Furthermore, it is proved that all signals and states of the closed-loop system are uniformly essentially bounded in terms of the system matrices A, B, C, the initial values y_1^0, η^0, the function φ and the input/output disturbances u_0, y_0, respectively. This is required for the robustness analysis of funnel control, see Chapter 9.

Set, for $n, m \in \mathbb{N}$ with $n \geq m$,

$$\mathcal{D}_{n,m}$$
$$:= \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m}) \times \Phi \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m),$$

the set of all tuples of systems, initial values y_1^0, η^0 of the linear system, reciprocal funnel boundary function φ and input/output disturbances (u_0, y_0).

Theorem 5.2.1 *Let $n, m \in \mathbb{N}$ with $n \geq m$ and $\varphi \in \Phi$. Then there exists a continuous map $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that, for all tuples $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}_{n,m}$, the associated closed-loop initial value problem* (5.1.4), (5.1.2), (5.1.3) *satisfies*

$$\|(k, u_2, y_2, \eta)\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^{1+m}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n-m})} \leq \nu(d)\,, \tag{5.2.1}$$

and

$$\forall\, t \geq 0 \;: \quad (t, y_2(t)) \in \mathcal{F}_\varphi := \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\,. \tag{5.2.2}$$

That funnel control works for the class of systems $\mathcal{M}_{n,m}$ goes back to [IRS02b] where also input disturbances are considered. However, to prove robustness of funnel control in terms of the gap metric, see Chapter 9, the existence of a continuous function $\nu(\cdot)$ satisfying (5.2.1) is crucial. Therefore, a new proof showing (5.2.1) is presented. This proof uses some ideas from [HIR09]. In this paper the authors consider funnel control with input saturation and give a proof which is much more elementary than the proofs one can find in [IRS02b, IRT05, IR06, IRT06, IRT07, RST08].

Proof of Theorem 5.2.1. Consider any element from the set $\mathcal{D}_{n,m}$, i.e. let $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}_{n,m}$. The closed-loop initial value problem (5.1.4), (5.1.2), (5.1.3) is then given by

$$\left.\begin{aligned} \dot{y}_2 &= A_1\, y_2 - A_2\, \eta - k\, CB\, y_2 + \dot{y}_0 - A_1\, y_0 - CB\, u_0\,, & y_2(0) &= y_2^0\,,\\ \dot{\eta} &= -A_3\, y_2 + A_4\, \eta + A_3\, y_0\,, & \eta(0) &= \eta^0\,,\\ k &= \frac{\varphi}{1 - \varphi\|y_2\|}\,, & &\\ u_2 &= -k y_2\,, & & \end{aligned}\right\} \tag{5.2.3}$$

where $y_2^0 := y_0(0) - y_1^0$. The proof is divided into 5 steps.

Step 1: Existence and uniqueness of a solution of (5.2.3) is shown.

In view of the definition of $\mathcal{F}_\varphi$, see (5.2.2), it is easy to see that

$$\forall\, y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\; \forall\, y_1^0 \in \mathbb{R}^m \;:\; (0, y_2(0)) = (0, y_0(0) - y_1^0) \in \mathcal{F}_\varphi\,.$$

In view of the equations in (5.2.3) the closed-loop initial value problem (5.1.4), (5.1.2), (5.1.3) may be written as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_2 \\ \eta \end{pmatrix} = f(t, y_2, \eta)\,, \qquad \begin{pmatrix} 0 \\ y_2(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ y_0(0) - y_1^0 \\ \eta^0 \end{pmatrix} \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}\,, \tag{5.2.4}$$

where, in view of $\varphi|_{[\varepsilon,\infty)}(\cdot)^{-1}$ being globally Lipschitz for every $\varepsilon > 0$ and $\varphi(0) = 0$,

$$\begin{aligned} &f\colon \mathcal{F}_\varphi \times \mathbb{R}^{n-m} \to \mathbb{R}^n\,, \\ &\quad (t, y_2, \eta) \\ &\quad \mapsto \begin{pmatrix} A_1 y_2 - A_2\eta - CB\frac{\varphi(t)}{1-\varphi(t)\|y_2\|} y_2 + \dot y_0(t) - A_1 y_0(t) - CBu_0(t) \\ -A_3 y_2 + A_4\eta + A_3 y_0(t) \end{pmatrix}, \end{aligned}$$

satisfies a local Lipschitz condition on the relatively open set $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ in the sense that, for all $(\tau, \xi, \zeta) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$, there exists an open neighbourhood O of (τ, ξ, ζ) and a constant $L > 0$ such that

$$\forall\, (t, y, \eta) \in O \;:\; \|f(t, y, \eta) - f(t, \xi, \zeta)\| \leq L(\|y - \xi\| + \|\eta - \zeta\|)\,.$$

Therefore, standard theory of ordinary differential equations, see, for example, [Wal98, Thm. III.11.III], yields that (5.2.3) has an absolutely continuous solution $(y_2, \eta)\colon [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ for some $\omega \in (0, \infty]$, which satisfies $(t, y_2(t), \eta(t)) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ for all $t \in [0, \omega)$. Moreover, the solution is unique and can be extended up to the boundary of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$. In other words: the closure of graph $\left((y_2, \eta)|_{[0,\omega)}\right)$ is not a compact subset of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$, i.e. for every compact $\mathcal{K} \subset \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ exists $t \in [0, \omega)$ such that $(t, y_2(t), \eta(t)) \notin \mathcal{K}$. This means that either $\omega = \infty$ or if $\omega < \infty$ then for every $\varepsilon > 0$ there exists $t \in (0, \omega)$ such that $1 - \varphi(t)\|y_2(t)\| < \varepsilon$ or $\|\eta(t)\| > 1/\varepsilon$.

Step 2: Some technicalities, crucial for the following steps, are presented.

From the existence of a solution and the properties of φ it follows that

$$\exists\, \delta = \delta(d) > 0\ \forall\, t \in [0, \delta]\ :\ \begin{cases} \|y_2(t)\| \le \|y_2(0)\| + 1 & \text{and} \\ 1 - \varphi(t)\|y_2(t)\| \ge \max\{1/2, \varphi(t)\}\,, \end{cases} \tag{5.2.5}$$

By definition of Φ there exists a global Lipschitz constant $L_\delta > 0$ of the reciprocal $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$ and $\lambda := \inf\big\{\varphi(t)^{-1}\,|\, t > 0\big\}$. Moreover, $(t, y_2(t)) \in \mathcal{F}_\varphi$ for all $t \in [0, \omega)$ yields

$$\forall\, t \in [0, \omega)\ :\ \|y_2(t)\| \le \max\Big\{\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\|_{L^\infty}, \|y_0(0) - y_1^0\| + 1\Big\}. \tag{5.2.6}$$

By the minimum phase property of (5.1.4), i.e. $\operatorname{spec} A_4 \subset \mathbb{C}_-$,

$$\exists\, \alpha, \beta > 0\ \forall\, t \ge 0\ :\ \|e^{A_4}t\| \le \beta e^{-\alpha t}\,. \tag{5.2.7}$$

In view of positive definiteness of CB, define $\gamma_{CB} > 0$ by

$$\forall\, v \in \mathbb{R}^m \setminus \{0\}\ :\ \langle v, CBv\rangle \ge \gamma_{CB}\|v\|^2\,.$$

Step 3: It is shown:

$$\forall\, t \in [\delta, \omega)\ :\ \varphi(t)^{-1} - \|y_2(t)\| \ge \varepsilon\,, \tag{5.2.8}$$

where $\delta > 0$ is defined by (5.2.5) and, for $\gamma_{CB}, \lambda, L_\delta, \alpha$ and β defined in Step 2,

$$\begin{aligned} \varepsilon := \min\Bigg\{\frac{1}{2}, \frac{\lambda}{2}, \frac{\gamma_{CB}\,\lambda}{2}, \Bigg[L_\delta + \bigg(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\bigg) \\ \cdot \Big(\|y_0\|_{L^\infty} + \|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\|_{L^\infty}\Big) \\ + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg]^{-1}\Bigg\}. \end{aligned} \tag{5.2.9}$$

Seeking a contradiction, suppose that

$$\exists\, t_1 \in [\delta, \omega) \;:\; \varphi(t_1)^{-1} - \|y_2(t_1)\| < \varepsilon\,. \tag{5.2.10}$$

Since $t \mapsto \varphi(t)\|y_2(t)\|$ is continuous on $[0,\omega)$ and in view of (5.2.5) it follows that

$$\exists\, t_0 \geq \delta \;:\; t_0 = \max\left\{ t \in [\delta, t_1) \mid \varphi(t) - \|y_2(t)\| = \varepsilon \right\}.$$

Thus, by definition of Φ,

$$\forall\, t \in [t_0, t_1] \;:\; \begin{cases} \varphi(t)^{-1} - \|y_2(t)\| \leq \varepsilon & \text{and} \\ \|y_2(t)\| \geq \varphi(t)^{-1} - \varepsilon \geq \lambda - \lambda/2 \end{cases} \tag{5.2.11}$$

and hence

$$\forall\, t \in [t_0, t_1] \;:\; \frac{\|y_2(t)\|}{\varphi(t)^{-1} - \|y_2(t)\|} \geq \frac{\lambda}{2}\,\varepsilon^{-1}\,. \tag{5.2.12}$$

By Variation of Constants the second line of (5.2.3) yields

$$\forall\, t \geq 0 \;:\; \eta(t) = e^{A_4 t}\eta^0 + \int_0^t e^{A_4(t-s)} A_3\left(y_0(s) - y_2(s)\right) \mathrm{d}s\,, \tag{5.2.13}$$

thus the first line of (5.2.3) writes

$$\begin{aligned} &\text{for a.a. } t \geq 0 \;: \\ &\quad \dot y_2(t) = -A_1(y_0(t) - y_2(t)) \\ &\qquad\qquad + A_2 \int_0^t e^{A_4(t-s)} A_3\left(y_0(s) - y_2(s)\right) \mathrm{d}s - A_2 e^{A_4 t}\eta^0 \\ &\qquad\qquad + \dot y_0(t) - CBu_0(t) + CB\left(\frac{-\varphi(t)}{1 - \varphi(t)\|y_2(t)\|}\, y_2(t)\right). \end{aligned}$$

Hence, by (5.2.6), (5.2.7), (5.2.12) and (5.2.9), conclude, for almost all

$t \in [t_0, t_1]$,

$$
\begin{aligned}
&\langle y_2(t), \dot{y}_2(t) \rangle \\
&\leq \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\| \, \|A_3\| \frac{\beta}{\alpha} \right) \left[\|y_0\|_{L^\infty} + \left\| \varphi|_{[\delta,\infty)}(\cdot)^{-1} \right\|_{L^\infty} \right] \\
&\qquad\qquad + \|A_2\| \beta \|\eta^0\| + \|\dot{y}_0\|_{L^\infty} + \|CB\| \, \|u_0\|_{L^\infty} \Bigg] \\
&\quad - \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \langle y_2(t), CB y_2(t) \rangle \\
&\leq \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\| \, \|A_3\| \frac{\beta}{\alpha} \right) \left[\|y_0\|_{L^\infty} + \left\| \varphi|_{[\delta,\infty)}(\cdot)^{-1} \right\|_{L^\infty} \right] \\
&\qquad\qquad + \|A_2\| \beta \|\eta^0\| + \|\dot{y}_0\|_{L^\infty} + \|CB\| \, \|u_0\|_{L^\infty} \Bigg] \\
&\quad - \frac{\varphi(t) \gamma_{CB} \|y_2(t)\|}{\varphi(t) \left(\varphi(t)^{-1} - \|y_2(t)\| \right)} \|y_2(t)\| \\
&\leq \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\| \, \|A_3\| \frac{\beta}{\alpha} \right) \left[\|y_0\|_{L^\infty} + \left\| \varphi|_{[\delta,\infty)}(\cdot)^{-1} \right\|_{L^\infty} \right] \\
&\qquad\qquad + \|A_2\| \beta \|\eta^0\| + \|\dot{y}_0\|_{L^\infty} + \|CB\| \, \|u_0\|_{L^\infty} \Bigg] \\
&\quad - \frac{\gamma_{CB} \lambda}{2} \varepsilon^{-1} \|y_2(t)\| \\
&\leq -L_\delta \|y_2(t)\| \,.
\end{aligned}
\tag{5.2.14}
$$

Thus

$$
\begin{aligned}
\|y_2(t_1)\| - \|y_2(t_0)\| &= \int_{t_0}^{t_1} \frac{1}{\|y_2(\tau)\|} \langle y_2(\tau), \dot{y}_2(\tau) \rangle \, \mathrm{d}\tau \\
&\leq -L_\delta (t_1 - t_0) \leq -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \\
&\leq \varphi(t_1)^{-1} - \varphi(t_0)^{-1} \,,
\end{aligned}
$$

whence the contradiction

$$
\varepsilon = \varphi(t_0)^{-1} - \|y_2(t_0)\| \leq \varphi(t_1)^{-1} - \|y_2(t_1)\| < \varepsilon \,.
$$

This proves (5.2.8).

Step 4: It is shown that $\omega = \infty$.

Let $\sigma := \min\left\{1, \inf_{t\in[\delta,\omega)} \varphi(t)\right\} > 0$. By (5.2.8) follows that

$$\forall\, t \in [\delta, \omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon\varphi(t) \geq \varepsilon\sigma\,,$$

where $\varepsilon > 0$ is defined by (5.2.9) and so, in view of (5.2.5),

$$\forall\, t \in [0, \omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon\sigma\,.$$

Seeking a contradiction, suppose $\omega < \infty$. By (5.2.6) and (5.2.13) follows that $\eta \in L^\infty([0,\omega) \to \mathbb{R}^{n-m})$ with $\left\|\eta|_{[0,\omega)}\right\|_{L^\infty} \leq c$ for some $c > 0$. Then

$$\mathcal{K} := \left\{(t, y, z) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m} \;\middle|\; t \in [0,\omega]\,,\; 1 - \varphi(t)\|y\| \geq \varepsilon\sigma\,,\; \|z\| \leq c\right\}$$

is a compact subset of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ with $(t, y_2(t), \eta(t)) \in \mathcal{K}$ for all $t \in [0,\omega)$, which contradicts the fact that the closure of graph $\Big((y_2,\eta)|_{[0,\omega)}\Big)$ is not a compact set, see Step 1. Therefore, $\omega = \infty$.

Step 5: Inequality (5.2.1) is shown.

Step 4 yields $\omega = \infty$. Then Step 3 and (5.2.5) guarantee that $(t, y_2(t)) \in \mathcal{F}_\varphi$ for all $t \geq 0$. Moreover, for some $\delta = \delta(d) > 0$ defined by (5.2.5), $\|y_2(t)\| \leq \varphi^{-1}(t) - \varepsilon$ for all $t \geq \delta$, and, in view of (5.2.5), is $\|y_2(t)\| \leq \|y_2(0)\| + 1 \leq \|y_0(0)\| + \|y_1^0\| + 1$ for all $t \in [0,\delta]$. Thus $y_2 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right)$. Moreover, (5.2.8) and (5.2.5) yield

$$\forall\, t \geq 0 \;:\; 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon\varphi(t)$$

and so

$$\forall\, t \geq 0 \;:\; k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \leq \varepsilon^{-1}$$

which gives $k \in L^\infty(R_{\geq 0} \to \mathbb{R})$ and, in view of (5.2.5), $\|k\|_{L^\infty} \leq \frac{1}{\varepsilon}$, thus k is uniformly bounded in terms of d. Hence, $u_2 = -k\, y_2 \in L^\infty(\mathbb{R}_{\geq 0} \to$

$\mathbb{R}^m$) is also uniformly bounded in terms of d. By (5.2.13) follows

$$
\begin{aligned}
&\forall\, t \geq 0 \;: \\
&\qquad \|\eta(t)\| = \left\| e^{A_4 t}\eta^0 + \int_0^t e^{A_4(t-s)} A_3 \left(y_0(s) - y_2(s)\right) \mathrm{d}s \right\| \\
&\qquad\qquad \leq \beta e^{-\alpha t}\|\eta^0\| + \int_0^t \beta \|A_3\| e^{-\alpha(t-s)} \left(\|y_0\|_{L^\infty} - \|y_2\|_{L^\infty}\right) \mathrm{d}s \\
&\qquad\qquad \leq \beta\|\eta^0\| e^{-\alpha t} + \frac{\beta}{\alpha}\|A_3\| \left(\|y_0\|_{L^\infty} - \|y_2\|_{L^\infty}\right)\left(1 + e^{-\alpha t}\right) ,
\end{aligned}
$$

hence $\eta \in L^\infty\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^{n-m}\right)$ and moreover, η is uniformly bounded in terms of the system matrices and the L^∞-norms of y_0 and y_2 which yields that η is uniformly bounded in terms of $d \in \mathcal{D}_{n,m}$.

Finally, in view of (5.2.3), it follows that the derivatives of y_2 and η are also uniformly bounded in terms of d which yields that $(y_2, \eta) \in W^{1,\infty}\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^m \times \mathbb{R}^{n-m}\right)$. Moreover, this proves the existence of a continuous function $\nu \colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that (5.2.1) holds true.

Step 6: Finally, (5.2.2) is shown.

By Step 5 one has $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Thus, and since y_2 is continuous, it follows that, for all $t \geq 0$, $1 - \varphi(t)\|y_2(t)\| > 0$, which shows (5.2.2) and completes the proof. □

Funnel control is actually not a new result. However, a proof for Theorem 5.2.1 becomes a lot easier with the ideas from [HIR09]; here the authors consider funnel control with input saturation. Most proofs for funnel control in the literature, see [IRS02b, IRT05, IR06, IRT06, IRT07, RST08] are much more technical and cannot give uniform boundedness of the system's signals in terms of the system's data, initial values, funnel boundary function and disturbance signals.

Note that the second statement of Theorem 5.2.1, i.e. (5.2.2), can be formulated even more restrictive:

$$
\forall\, t \geq 0 \;:\; (t, y_2(t)) \in \left\{ (t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| \leq 1 - \varepsilon \right\} ,
$$

where $\varepsilon > 0$ is defined in terms of $d \in \mathcal{D}_{n,m}$ by (5.2.9). Loosely speaking: one can show that there exists a constant $\varepsilon = \varepsilon(d) > 0$ (only depending on the closed-loop system's data) such that the distance between funnel boundary $\varphi(t)^{-1}$ and norm of the error/output $y_2(t)$ is larger than ε, for all $t > 0$, and infinity for $t = 0$.

Statement (5.2.1) is very important for the robustness analysis of funnel control in the terminology of the gap metric, see Chapter 9. The requirements for the robustness analysis not only of funnel control, but also λ-tracking and high-gain output derivative feedback follow in the next chapter, where the concept of the gap metric is introduced and all important details are presented.

5.3 Notes and references

The results of the present section are from [IM09] and the "new" proof for funnel control relies on an idea from [HIR09], where the authors show funnel control of MIMO-systems with input saturation. Classical proofs for funnel control can be found in [IRS02b, IRT05, IR06, IRT06, IRT07, RST08]. However, the results from these works do not provide a proof which may lead to uniform boundedness of the systems signals in terms of the system's data which is required for the robustness analysis in Chapter 9.

6 The concept of the gap metric

In the previous chapters several control strategies have been introduced: high-gain derivative feedback, λ-tracking and funnel control. These strategies work for more or less different classes of systems. The main idea is feeding back the output signal, and maybe derivatives of the output, in an appropriate way via the input of the system. The so resulting closed-loop systems are "stable" in different senses. An important question might be: 'What can be said about the robustness of a "stable" closed-loop system?' That means does it remain stable under sufficiently small disturbances of, for example, the system's data. Therefore, another question arises: 'How can anything be said about the robustness of a closed-loop system?'

Consider the classical feedback configuration shown in Figure 6.1, where the *plant* P denotes a system which is controlled by the *controller* C.

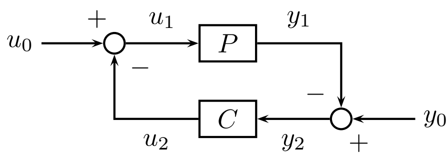


Figure 6.1: The closed-loop system $[P, C]$

One can find answers to questions concerning robustness of a closed-loop system $[P, C]$ by, loosely speaking, measuring distances between systems. To be more precise: given a "stable" closed-loop system $[P, C]$ with known P and C, one can consider a new plant P_1 and measure the distance between P and P_1 in some sense. If this distance is sufficiently small then one can guarantee that the new closed-loop system $[P_1, C]$ is again "stable". The big advantage of this approach is that almost no knowledge of the closed-loop system $[P_1, C]$ is required. This means in particular that it is not necessary to study how controller C effects P_1. Moreover, this makes it possible to consider systems P_1 which do not satisfy the classical assumptions required for an application of the control strategy C. This idea is applied to λ-tracking and funnel control and some robustness results for these control strategies are presented, see Chapter 8 and Chapter 9.

Note that the closed-loop system $[P, C]$ in Figure 6.1 is symmetric. Therefore, it is possible to switch the roles of P and C: instead of considering different plants P and P_1 one may consider different controllers C and C_1. So, the idea of measuring distances can also applied to different control strategies which is done for high-gain derivative feedback control, see Chapter 7.

One possibility of measuring the distance between two systems is the so-called *gap metric*, firstly introduced by [ZES80]. Further important papers on gap metric are the works by T. Georgiou and M. Smith [GS90, GS93, GS97]. Other robustness concepts which are related to the gap metric are, for example, w-stability introduced by T. Georgiou and M. Smith in [GS89], or the ν-gap distance introduced by G. Vinnecombe, see, for example [Vin93, Vin99, Vin01]. Some more thoughts on these are added in the conclusion of the present chapter.

The terminology and results of the present chapter are based on the gap metric basics from [GS97, Sec. II], [Fre08, Sec. 2], [FIR06, Sec. 2] and [FIM09, Sec. 2]. Moreover, the terminology in the present chapter is generalized for signal spaces of continuous functions. Therefore, results on robust stability from [GS97, Sec. III] and [Fre08, Sec. 5] which are required for the robustness analysis for λ-tracking and funnel control are revisited.

6.1 Generalized signal spaces

In this section the so-called *extended* space $\mathcal{V}_e$ and the *ambient* space $\mathcal{V}_a$ of a signal space $\mathcal{V}$ are introduced. Loosely speaking, these spaces are required for the terminology of the gap metric to deal with systems which produce unbounded outputs in some sense. For example, consider the system $\dot{y}_1 = y_1 + u_1$ with initial value $y_1(0) = 0$, which can be regarded as a plant P which maps a signal u_1 to $y_1 = Pu_1 = \left(t \mapsto \int_0^t e^{t-s} u_0(s)\,\mathrm{d}s\right)$. Then, for $u_0 = (t \mapsto 1) \in \mathcal{V} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, $y_1 = (t \mapsto e^t - 1)$ which is not in $L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$. However, the extended space of $\mathcal{V} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is $\mathcal{V}_e = L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and so $y_1 = (t \mapsto e^t - 1) \in \mathcal{V}_e$. Here one can see that it becomes necessary to consider generalized signal spaces. Note that, for $\mathcal{V} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, one would find that the ambient space $\mathcal{V}_a = \cup_{0<\omega\leq\infty} L^\infty_{\text{loc}}([0,\omega) \to \mathbb{R}^m)$ which becomes important for plants P that produce even outputs with finite escape time.

To be more precise, let $\mathcal{X}$ be a nonempty set. For $0 < \omega \leq \infty$ let $\mathcal{S}_\omega$ denote the set of all locally integrable maps in $\text{map}([0,\omega) \to \mathcal{X})$. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \leq \infty$ define a *truncation operator* T_τ as follows:

$$T_\tau : \mathcal{S}_\omega \to \mathcal{S}\,, \quad v \mapsto T_\tau v := \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau) \\ 0, & t \in [\tau,\infty) \end{cases}\right).$$

Then, as in [Fre08, FIR06, GS97], one may associate with $\mathcal{V} \subset \mathcal{S}$ spaces as follows:

$$\begin{aligned}
\mathcal{V}_e^{\text{old}} &= \left\{ v \in \mathcal{S} \,\middle|\, \forall\,\tau > 0 \;:\; T_\tau v \in \mathcal{V} \right\}, \quad \textit{the “old” extended space}\,; \\
\mathcal{V}_\omega^{\text{old}} &= \left\{ v \in \mathcal{S}_\omega \,\middle|\, \forall\,\tau \in (0,\omega) \;:\; T_\tau v \in \mathcal{V} \right\},\ 0 < \omega \leq \infty\,; \\
\mathcal{V}_a^{\text{old}} &= \bigcup\nolimits_{\omega\in(0,\infty]} \mathcal{V}_\omega, \quad \textit{the “old” ambient space.}
\end{aligned}$$

However note that these definitions are not applicable for subspaces of continuous functions as considered in the present thesis. This is due to the fact that for continuous $v \in W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ (see the list of symbols for a definition of this Sobolev space), with $r \in \mathbb{N}$ and $p \in [1,\infty]$, the truncation $T_\tau v$, $\tau > 0$, does not necessarily belong to $W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$.

Therefore, define, in addition to the truncation operator, for $0 < \tau < \omega \leq \infty$, the *restriction* of maps as follows:

$$(\cdot)\big|_{[0,\tau)} : \mathcal{S}_\omega \to \mathcal{S}_\tau\,, \quad v \mapsto v\big|_{[0,\tau)} := (t \mapsto v(t), \quad t \in [0,\tau))\ .$$

Consider next a vector space $\mathcal{V} \subset \mathcal{S}$ of maps defined on $[0,\infty)$ with norm $\|\cdot\|_{\mathcal{V}} \colon \mathcal{V} \to \mathbb{R}_{\geq 0}$. One may introduce the norm $\|\cdot\|_{\mathcal{V}|_{[0,\tau)}} : \{v|_{[0,\tau)} \mid v \in \mathcal{V}\} \to \mathbb{R}_{\geq 0}$ where $\|v|_{[0,\tau)}\|_{\mathcal{V}|_{[0,\tau)}}$ denotes the norm on the restriction $[0,\tau) \subset \mathbb{R}_{\geq 0}$, and write, for ease of notation, $\|T_\tau v\|_{\mathcal{V}} = \|v|_{[0,\tau)}\|_{\mathcal{V}|_{[0,\tau)}}$ for $v \in \mathcal{V}$.

Now associate with $\mathcal{V} \subset \mathcal{S}$ spaces as follows:

$$\mathcal{V}[0,\tau) = \left\{ v \in \mathcal{S}_\tau \,\middle|\, \exists\, w \in \mathcal{V} \text{ with } \|T_\tau w\|_{\mathcal{V}} < \infty \;:\; v = w|_{[0,\tau)} \right\}, \quad \text{for } \tau > 0\,;$$

$$\mathcal{V}_e = \left\{ v \in \mathcal{S} \,\middle|\, \forall\, \tau > 0 \;:\; v|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\}, \ \textit{the extended space}\,;$$

$$\mathcal{V}_\omega = \left\{ v \in \mathcal{S}_\omega \,\middle|\, \forall\, \tau \in (0,\omega) \;:\; v|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\}, \ \text{for } 0 < \omega \leq \infty\,;$$

$$\mathcal{V}_a = \bigcup\nolimits_{\omega \in (0,\infty]} \mathcal{V}_\omega\,, \ \textit{the ambient space}\,.$$

In case of $\mathcal{V} = W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ this means in view of the $W^{r,p}$-norm:

$$\|T_\tau w\|_{W^{r,p}} = \sum_{i=0}^{r} \|T_\tau w^{(i)}\|_{L^p}\,.$$

For L^p spaces these definitions coincide with the above "old" definitions of ambient and extended spaces $\mathcal{V}_a^{\text{old}}$ and $\mathcal{V}_e^{\text{old}}$, respectively.

If $v, w \in \mathcal{V}_a$ with $v|_I = w|_I$ on $I = \text{dom}(v) \cap \text{dom}(w)$, then write $v = w$. For $(u,y) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of u and y may be different; adopt the convention

$$\text{dom}(u,y) \;:=\; \text{dom}(u) \cap \text{dom}(y)\,.$$

The set $\mathcal{V} \subset \mathcal{S}$ is a said to be a *signal space* if, and only if, it is a) a normed vector space and b) $\sup_{\tau \geq 0} \|T_\tau v\|_{\mathcal{V}} < \infty$ implies $v \in \mathcal{V}$.

Following an example: in some later applications, $\mathcal{V}$ may be the space $L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ for $p \in [1,\infty]$, which obviously satisfies the aforementioned assumptions a) and b): L^p is a normed space and if $\sup_{\tau \geq 0} \|T_\tau v\|_{L^p} < \infty$ then $v \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Note that this also holds for any Sobolev space $W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $r \in \mathbb{N}$, $p \in [1,\infty]$, which are also used in later applications.

If $\mathcal{V} = L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ then it follows that $\mathcal{V}_e = L^p_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{V}_\omega = L^p_{\text{loc}}([0,\omega) \to \mathbb{R}^m)$ for $\omega \in (0,\infty]$, and $\mathcal{V}_a = \cup_{0<\omega\leq\infty} L^p_{\text{loc}}([0,\omega) \to \mathbb{R}^m)$. It is important to note that $\mathcal{V}_\omega \supsetneq L^p([0,\omega) \to \mathbb{R}^m)$.

For a normed signal space $\mathcal{U}$ and the Euclidean space $\mathbb{R}^l$, $l \in \mathbb{N}$, also subsets of $\mathcal{V} = \mathbb{R}^l \times \mathcal{U}$ will be considered, which, on identifying each $\theta \in \mathbb{R}^l$ with the constant signal $t \mapsto \theta$, can be thought of as a normed signal space with norm given by $\|(\theta, x)\|_{\mathcal{V}} = \sqrt{|\theta|^2 + \|x\|^2_{\mathcal{U}}}$.

The results in Chapters 7–9 handle different types of signal spaces are considered. These are not specified here but in the respective chapter.

6.2 Well posedness

The notion of well posedness goes back to a definition by J. Hadamard, see [Par94], for models of physical phenomena which should satisfy the assumptions that there exists a unique solution which depends continuously on the systems data, in some reasonable topology. In the present section this idea will be applied to closed-loop systems $[P, C]$, as depicted in Figure 6.1, and softened in some sense: existence and uniqueness of solutions is still required, but one may consider local, global or regular well posedness, see below.

For this some more terminology is presented. A mapping $Q \colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causal* if, and only if,

$$\forall\ x, y \in \mathcal{U}_a\ \forall\ \tau \in \operatorname{dom}(x,y) \cap \operatorname{dom}(Qx, Qy)\ :\\ \Big[\ x|_{[0,\tau)} = y|_{[0,\tau)}\ \Rightarrow\ (Qx)|_{[0,\tau)} = (Qy)|_{[0,\tau)}\ \Big]\ .$$

This means that, loosely speaking, the "present" may only depend on the "past". For better comprehension note that there exist operators which do not satisfy this causality assumption. For example, let $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and consider the translation operator $T \colon x(\cdot) \mapsto x(\cdot + 1)$. Then, for $x = (t \mapsto 1)$, $y = \left(t \mapsto \begin{cases} 1\,, & t\in[0,1] \\ 0\,, & t>1 \end{cases}\right)$ and $\tau = 1$, it follows that $x|_{[0,\tau)} = y|_{[0,\tau)}$ but $(Tx)|_{[0,\tau)} \equiv 0|_{[0,\tau)} \neq 1|_{[0,\tau)} \equiv (Ty)|_{[0,\tau)}$, see, for example, [Kur05].

Consider $P \colon \mathcal{U}_a \to \mathcal{Y}_a$, $u_1 \mapsto y_1$, and $C \colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$ being causal mappings representing the plant and the controller, respectively, and satisfying the closed-loop equations:

$$[P,C] \ : \quad y_1 = Pu_1, \quad u_2 = Cy_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2 \,, \quad (6.2.1)$$

corresponding to the closed-loop shown in Figure 6.1.

For $w_0 = (u_0, y_0) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$ a pair $(w_1, w_2) = ((u_1, y_1), (u_2, y_2)) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a *solution* if, and only if, (6.2.1) holds on $\mathrm{dom}(w_1, w_2)$. The (possibly empty) set of solutions is denoted by

$$\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{ solves } (6.2.1)\}$$

The closed-loop system $[P,C]$, given by (6.2.1), is said to have:

- the *existence property* if, and only if, $\mathcal{X}_{w_0} \neq \emptyset$;
- the *uniqueness property* if, and only if,

$$\forall \ w_0 \in \mathcal{W} \ : \ \Big[(\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \Longrightarrow$$
$$(\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}\,(\tilde{w}_1, \tilde{w}_2) \Big] \,.$$

Assume that $[P,C]$ has the existence and uniqueness property. For each $w_0 \in \mathcal{W}$, define $\omega_{w_0} \in (0, \infty]$, by the property

$$[0, \omega_{w_0}) := \cup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \mathrm{dom}(\hat{w}_1, \hat{w}_2)$$

and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\mathrm{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This construction induces the closed-loop operator

$$H_{P,C} \colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad w_0 \mapsto (w_1, w_2) \,.$$

For $\Omega \subset \mathcal{W}$ the closed-loop system $[P,C]$, given by (6.2.1), is said to be:

- *locally well posed on* Ω if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C}\big|_{\Omega} \colon \Omega \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;

- *locally well posed* if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;

- *globally well posed on* Ω if, and only if, it is locally well posed on Ω and $H_{P,C}(\Omega) \subset \mathcal{W}_e \times \mathcal{W}_e$;

- *globally well posed* if, and only if, it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e$;

- $\mathcal{W}$*-stable* if, and only if, it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$;

- *regularly well posed* if, and only if, it is locally well posed and

$$\forall\, w_0 \in \mathcal{W} \ : \quad \Big[\ \omega_{w_0} < \infty \ \Rightarrow\ \big\| (H_{P,C} w_0)|_{[0,\tau)} \big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0}\ \Big]. \tag{6.2.2}$$

Note that it is differentiated between well posedness of $[P, C]$ on a subset Ω of $\mathcal{W}$ and sheer well posedness of $[P, C]$. This is required due to technical reasons when applying the terminology to the robustness analysis in the following chapters.

If $[P, C]$ is globally well posed, then for each $w_0 \in \mathcal{W}$ the solution $H_{P,C}(w_0)$ exists on the half line $\mathbb{R}_{\geq 0}$. Regular well posedness means that if the closed-loop system has a finite escape time $\omega_{w_0} > 0$ for some disturbance $w_0 \in \mathcal{W}$, then at least one of the components u_1, u_2 or y_1, y_2 is not a restriction to $[0, \omega_{w_0})$ of a function in $\mathcal{U}$ or $\mathcal{Y}$, respectively. If $[P, C]$ is regularly well posed and satisfies

$$\forall\, w_0 \in \mathcal{W} \ : \quad \Big[\ \omega_{w_0} < \infty \ \Rightarrow\ H_{P,C}(w_0)|_{[0,\omega_{w_0})} \in \mathcal{W}[0, \omega_{w_0}) \times \mathcal{W}[0, \omega_{w_0})\Big],$$

there does not exist a solution of $[P, C]$ with a finite escape time, and therefore $[P, C]$ is globally well posed. However, global well posedness does not guarantee that each solution belongs to $\mathcal{W} \times \mathcal{W}$; the latter is ensured by $\mathcal{W}$-stability of $[P, C]$. Note also that neither regular nor global well posedness implies the other.

6.3 Graphs and the nonlinear gap metric

To measure the distance between two plants P and P_1 it is necessary to find sets associated with the plant operators within some space where one may define a map which identifies the gap. These set are the *graphs* of the operators: for the plant operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and the controller operator $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ define the *graph* $\mathcal{G}_P$ of the plant and the *graph* $\mathcal{G}_C$ of the controller, respectively, as follows:

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \;\middle|\; u \in \mathcal{U},\, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}\,,$$
$$\mathcal{G}_C := \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \;\middle|\; Cy \in \mathcal{U},\, y \in \mathcal{Y} \right\} \subset \mathcal{W}\,.$$

Note that $\mathcal{G}_P$ and $\mathcal{G}_C$ are, strictly speaking, not subsets of $\mathcal{W}$; however, abusing the notation one may identify $\mathcal{G}_P \ni \left(\begin{smallmatrix} u \\ Pu \end{smallmatrix}\right) = (u, Pu) \in \mathcal{W}$ and $\mathcal{G}_C \ni \left(\begin{smallmatrix} Cy \\ y \end{smallmatrix}\right) = (Cy, y) \in \mathcal{W}$.

An operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causally extendible* [GS93] (or stabilizable in [Fre08]) if, and only if,

$$\forall\, \tau > 0 \;\forall\, w_1 = (u_1, y_1) \in \mathcal{W}_a \text{ with } T_\tau y_1 = T_\tau P u_1 \;\exists\, w_1^* \in \mathcal{G}_P \;: \qquad T_\tau w_1 = T_\tau w_1^*\,.$$

The essence of Chapters 7–9 is a study of robust stability in a specific control context. Robust stability is the property that the stability properties of a globally well posed closed-loop system $[P, C]$ persists under "sufficiently small" perturbations of the plant. In other words, robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$, when the plant P is replaced by any plant P_1 sufficiently "close" to P. In the context of this thesis, plants P and P_1 are deemed to be close if, and only if, their respective graphs are *close* in the gap sense of [GS97]. The nonlinear gap is defined as follows:

Definition 6.3.1 *Let, for signal spaces* $\mathcal{U}$ *and* $\mathcal{Y}$*,*

$$\Gamma(\mathcal{U},\mathcal{Y}) := \big\{ P : \mathcal{U}_a \to \mathcal{Y}_a \;\big|\; P \text{ is causal} \big\}$$

and, for $P_1, P_2 \in \Gamma$*, define the (possibly empty) set*

$$\mathcal{O}_{P_1,P_2} := \big\{ \Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \;\big|\; \Phi \text{ is causal, surjective, and } \Phi(0) = 0 \big\}.$$

The directed nonlinear gap $\vec{\delta} \colon \Gamma(\mathcal{U},\mathcal{Y}) \times \Gamma(\mathcal{U},\mathcal{Y}) \to [0,\infty]$ *is given by*

$$(P_1,P_2) \mapsto$$
$$\vec{\delta}(P_1,P_2) := \inf_{\Phi \in \mathcal{O}_{P_1,P_2}} \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\},\, \tau > 0} \left(\frac{\| T_\tau (\Phi - I)|_{\mathcal{G}_{P_1}}(x) \|_{\mathcal{U}\times\mathcal{Y}}}{\| T_\tau x \|_{\mathcal{U}\times\mathcal{Y}}} \right),$$

with the convention that $\vec{\delta}(P_1,P_2) := \infty$ *if* $\mathcal{O}_{P_1,P_2} = \emptyset$*, and the* nonlinear gap δ *is*

$$\delta \colon \Gamma(\mathcal{U},\mathcal{Y}) \times \Gamma(\mathcal{U},\mathcal{Y}) \to [0,\infty],$$
$$(P_1,P_2) \mapsto \delta(P_1,P_2) := \max\{ \vec{\delta}(P_1,P_2), \vec{\delta}(P_2,P_1) \}.$$

In the following subsection the above definitions are illustrated. The graphs of two different systems are considered and an upper bound for the gap is derived.

6.3.1 Example: the gap of two linear systems

In this subsection the previously introduced concepts of graphs and the nonlinear gap are illustrated by two simple example plants. Consider linear systems given by the transfer functions $\frac{1}{s-\alpha}$ and $\frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}$, respectively. Let associated operators

$$P_{\alpha;x^0} : \mathcal{U}_a \to \mathcal{Y}_a, \; u_1 \mapsto y_1 \quad \text{and} \quad P_{N,M,\alpha;\widetilde{x}^0} : \mathcal{U}_a \to \mathcal{Y}_a, \; \widetilde{u}_1 \mapsto \widetilde{y}_1$$

be induced by state space systems

$$P_{\alpha;x^0} \quad : \quad \left.\begin{array}{l} \dot{x} = \alpha x + u_1\,, \qquad x(0) = x^0 \\ y_1 = x \end{array}\right\} \tag{6.3.1}$$

$$P_{N,M,\alpha;\widetilde{x}^0} : \quad \left.\begin{array}{l} \dot{x} = \widetilde{A}\,x + \widetilde{b}\,\widetilde{u}_1\,, \qquad x(0) = \widetilde{x}^0 \\ \widetilde{y}_1 = \widetilde{c}\,x \end{array}\right\} \tag{6.3.2}$$

for $x^0 \in \mathbb{R}$, $\widetilde{x}^0 \in \mathbb{R}^3$, $\alpha, N, M > 0$ and $\big(\widetilde{A}, \widetilde{b}, \widetilde{c}\big)$ given by

$$\widetilde{A} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha NM, & -NM + \alpha N + \alpha M, & \alpha - N - M \end{bmatrix}, \quad \widetilde{b} := \begin{bmatrix} 0 \\ 0 \\ N \end{bmatrix},$$
$$\widetilde{c} := [M, -1, 0]\,.$$

Throughout this subsection assume that $x^0 = 0$ and $\widetilde{x}^0 = 0_{3\times 1}$. The purpose of this example is to show that $P_{\alpha;0}$ is close to $P_{N,M,\alpha;0}$ in the sense

$$\limsup_{M\to\infty} \vec{\delta}(P_{\alpha;0}, P_{2M,M,\alpha;0}) = 0\,. \tag{6.3.3}$$

The graphs of $P_{\alpha;0}$ and $P_{N,M,\alpha;0}$ are given, respectively, by

$$\mathcal{G}_{P_{\alpha;0}} = \left\{ \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \,\middle|\, (u_1, y_1) \in \mathcal{U} \times \mathcal{Y} \,:\, y_1 \text{ solves (6.3.1) for } x^0 = 0 \right\},$$
$$\mathcal{G}_{P_{N,M,\alpha;0}} = \left\{ \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{y}_1 \end{pmatrix} \,\middle|\, (\widetilde{u}_1, \widetilde{y}_1) \in \mathcal{U} \times \mathcal{Y} \,:\, \widetilde{y}_1 \text{ solves (6.3.2) for } \widetilde{x}^0 = 0 \right\},$$

for some signal spaces $\mathcal{U}$ and $\mathcal{Y}$ specified in due course. To determine an upper bound for the gap between $P_{\alpha;0}$ and $P_{N,M,\alpha;0}$, consider the bijective mapping Φ from graph $\mathcal{G}_{P_{\alpha;0}}$ to graph $\mathcal{G}_{P_{N,M,\alpha;0}}$ given by

$$\Phi \colon \mathcal{G}_{P_{\alpha;0}} \to \mathcal{G}_{P_{N,M,\alpha;0}}\,,$$
$$\begin{pmatrix} u \\ \int_0^{\cdot} e^{\alpha(\cdot - s)}\, u(s)\,\mathrm{d}s \end{pmatrix} \mapsto \begin{pmatrix} u \\ \widetilde{c}\int_0^{\cdot} e^{\widetilde{A}(\cdot - s)}\widetilde{b}\, u(s)\,\mathrm{d}s \end{pmatrix}.$$

By the definition of the nonlinear gap, see Section 6.3, one arrives at

$$\vec{\delta}(P_{\alpha;0}, P_{N,M,\alpha;0}) \le \sup_{w \in \mathcal{G}_{P_{\alpha;0}} \setminus \{0\},\, \tau > 0} \frac{\|T_\tau (\Phi - I)(w)\|_{\mathcal{W}}}{\|T_\tau w\|_{\mathcal{W}}}\,, \tag{6.3.4}$$

where $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and, for $w = (u, y) \in \mathcal{W}$, the norm of $\mathcal{W}$ is defined by

$$\|(u, y)\|_{\mathcal{W}} := \|u\|_{\mathcal{U}} + \|y\|_{\mathcal{Y}}\,.$$

In the following let $\mathcal{U}$ and $\mathcal{Y}$ be any of the spaces $L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$ or $W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$. To estimate

$$\|(\Phi - I)(w)(t)\| \quad \text{for} \quad w := \begin{pmatrix} u \\ \int_0^\cdot e^{\alpha(\cdot - s)}\, u(s)\, \mathrm{d}s \end{pmatrix} \in \mathcal{G}_{P_{\alpha;0}}$$

calculate that the output $\widetilde{y}_1$ of (6.3.2) is given, for all $t \ge 0$, by

$$\begin{aligned}
\widetilde{y}_1(t) &= \widetilde{c} \int_0^t e^{\widetilde{A}(t-s)} \widetilde{b}\, \widetilde{u}_1(s)\, \mathrm{d}s \\
&= \int_0^t \frac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} e^{\alpha(t-s)}\, \widetilde{u}_1(s)\, \mathrm{d}s \\
&\quad + \int_0^t \frac{N(N+M)}{(N-M)(\alpha+N)} e^{-N(t-s)}\, \widetilde{u}_1(s)\, \mathrm{d}s \\
&\quad + \int_0^t \frac{-2NM}{(N-M)(\alpha+M)} e^{-M(t-s)}\, \widetilde{u}_1(s)\, \mathrm{d}s
\end{aligned}$$

and thus, for all $t \ge 0$,

$$\begin{aligned}
\|(\Phi - I)(w)(t)\| \le &\left\| \left(\tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right) \int_0^t e^{\alpha(t-s)}\, u(s)\, \mathrm{d}s \right\| \\
&+ \left\| \tfrac{N(N+M)}{(N-M)(\alpha+N)} \int_0^t e^{-N(t-s)}\, u(s)\, \mathrm{d}s \right\| \\
&+ \left\| \tfrac{-2NM}{(N-M)(\alpha+M)} \int_0^t e^{-M(t-s)}\, u(s)\, \mathrm{d}s \right\|
\end{aligned}$$

and moreover, for all $t \geq 0$,

$$\begin{aligned}
&\|(\Phi - I)(w)(t)\| \\
&\leq \left\| \tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right\| \left\| \int_0^t e^{\alpha(t-s)}\, u(s)\, \mathrm{d}s \right\| \\
&\quad + \left(\left\| \tfrac{N(N+M)}{(N-M)(\alpha+N)} \int_0^t e^{-N(t-s)}\, \mathrm{d}s \right\| \right. \\
&\qquad \left. + \left\| \tfrac{-2NM}{(N-M)(\alpha+M)} \int_0^t e^{-M(t-s)}\, \mathrm{d}s \right\| \right) \|u\|_{\mathcal{U}} \\
&\leq \left\| \tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right\| \left\| \int_0^\cdot e^{\alpha(\cdot-s)}\, u(s)\, \mathrm{d}s \right\|_{\mathcal{Y}} \\
&\quad + \left(\left\| \tfrac{N+M}{(N-M)(\alpha+N)} \right\| + \left\| \tfrac{2N}{(N-M)(\alpha+M)} \right\| \right) \|u\|_{\mathcal{U}} \\
&\leq \left[\left\| \tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right\| + \left\| \tfrac{N+M}{(N-M)(\alpha+N)} \right\| + \left\| \tfrac{2N}{(N-M)(\alpha+M)} \right\| \right] \\
&\quad \cdot \left(\|u\|_{\mathcal{U}} + \left\| \int_0^\cdot e^{\alpha(\cdot-s)}\, u(s)\, \mathrm{d}s \right\|_{\mathcal{Y}} \right).
\end{aligned}$$

Hence, in view of (6.3.4),

$$\begin{aligned}
&\vec{\delta}(P_{\alpha;0}, P_{N,M,\alpha;0}) \\
&\qquad \leq \left| \tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right| + \left| \tfrac{N+M}{(N-M)(\alpha+N)} \right| + \left| \tfrac{2N}{(N-M)(\alpha+M)} \right| \qquad (6.3.5)
\end{aligned}$$

which yields (6.3.3).

In Chapters 8 and 9 the above example plants are visited again to illustrate the robustness analysis of λ-tracking and funnel control.

6.4 Gain stability and gain-function stability

Two concepts of stability in the terminology of closed-loop systems $[P, C]$ given by (6.2.1) are introduced in detail: *gain stability* and *gain-function stability*. Given normed signal spaces $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and causal operators $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, recall from Section 6.2 that the closed-loop system $[P, C]$ with the associated operator

$H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$ is $\mathcal{W}$-stable if, and only if, it is globally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$. Then $\mathcal{W}$-gain stability of $[P,C]$ is defined as follows:

Definition 6.4.1 *(i) Given normed signal spaces $\mathcal{X}$ and $\mathcal{V}$ and $\Omega \subset \mathcal{X}$, a causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is said to be* gain stable on Ω *if, and only if, $Q(\Omega) \subset \mathcal{V}$, $Q(0) = 0$ and*

$$\left\| Q|_{\Omega} \right\|_{\mathcal{X},\mathcal{V}} := \sup \left\{ \frac{\|(Qx)|_{[0,\tau)}\|_{\mathcal{V}_\tau}}{\|x|_{[0,\tau)}\|_{\mathcal{X}_\tau}} \;\middle|\; x \in \Omega\,,\ \tau > 0\,,\ x|_{[0,\tau)} \neq 0 \right\} < \infty\,.$$

*(ii) The closed-loop system $[P,C]$ given by (6.2.1) with the associated operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$ is said to be $\mathcal{W}$-*gain stable *if, and only if, it is $\mathcal{W}$-stable and $H_{P,C}$ is gain stable on $\mathcal{W}$.*

Next, associate with the closed-loop system $[P,C]$ given by (6.2.1) the following two parallel projection operators:

$$\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a\,,\ w_0 \mapsto w_1 \quad \text{and} \quad \Pi_{C//P}\colon \mathcal{W} \to \mathcal{W}_a\,,\ w_0 \mapsto w_2\,.$$

Clearly, $H_{P,C} = \big(\Pi_{P//C}\,, \Pi_{C//P}\big)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Note that gain stability of either $\Pi_{P//C}$ and $\Pi_{C//P}$ implies $\mathcal{W}$-gain stability of the closed-loop system $[P,C]$ and that $\left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}}, \left\|\Pi_{C//P}\right\|_{\mathcal{W},\mathcal{W}} \geq 1$ since $\Pi_{P//C} = \Pi_{P//C}^2$, $\Pi_{C//P} = \Pi_{C//P}^2$.

The following definition of gain-function stability goes back to [GS97]. Note that the concept of gain-function stability is required to attain robustness results for λ-tracking and funnel control.

Definition 6.4.2 *(i) A causal operator $F\colon \mathcal{X} \to \mathcal{V}_a$, where $\mathcal{X}, \mathcal{V}$ are subsets of normed signal spaces, is said to be* gain-function stable *if, and only if, $F(\mathcal{X}) \subset \mathcal{V}$ and the following nonlinear so-called* gain-function *is well defined:*

$$\begin{aligned} g[F] : (r_0, \infty) &\to \mathbb{R}_{\geq 0}, \\ r \mapsto g[F](r) &= \sup \left\{ \|T_\tau F x\|_{\mathcal{V}} \;\middle|\; x \in \mathcal{X}\,,\ \|T_\tau x\|_{\mathcal{X}} \in (r_0, r],\ \tau > 0 \right\}, \end{aligned} \tag{6.4.1}$$

where $r_0 := \inf_{x \in \mathcal{X}} \|x\|_{\mathcal{X}} < \infty$.

(ii) A closed-loop system $[P, C]$ *is said to be* gain-function stable *if, and only if, it is globally well posed and* $H_{P,C} \colon \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e$ *is gain-function stable.*

Observe that $\|T_\tau F x\|_{\mathcal{V}} \leq g[F](\|T_\tau x\|_{\mathcal{X}})$ and note the following facts:

(i) global well posedness of $[P, C]$ implies that $\operatorname{im} H_{P,C} \subset \mathcal{W}_e \times \mathcal{W}_e$;

(ii) stability of $[P, C]$ implies $\mathcal{W}$-stability of $[P, C]$;

(iii) if $[P, C]$ is $\mathcal{W}$-stable, then $H_{P,C} \colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is a bijective operator with inverse $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

To see (iii), note that $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$ implies that $H_{P,C}(\mathcal{W}) \subset \mathcal{G}_P \times \mathcal{G}_C$, and since, for any $w_1 \in \mathcal{G}_P \subset \mathcal{W}$, $w_2 \in \mathcal{G}_C \subset \mathcal{W}$ one has $w_1 + w_2 \in \mathcal{W}$, it follows that $H_{P,C}(\mathcal{W}) \supset \mathcal{G}_P \times \mathcal{G}_C$. Therefore, think of a gain-function stable $H_{P,C}$ as a surjective operator $H_{P,C} \colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$. The inverse of $H_{P,C} \colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is obviously $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

Finally, recall $\Pi_{P//C} \colon w_0 \mapsto w_1$ and $\Pi_{C//P} \colon w_0 \mapsto w_2$ and that $H_{P,C} = \big(\Pi_{P//C}, \Pi_{C//P}\big)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Therefore, gain-function stability of one of the operators $\Pi_{P//C}$ and $\Pi_{C//P}$ implies the gain-function stability of the other, and so gain-function stability of either operator implies gain-function stability of the closed-loop system $[P, C]$.

In the following robustness results from [GS97] and [Fre08] are rewritten into the present terminology. These results are crucial for the robustness analysis in the following chapters.

6.5 Robust stability

This section reviews some results from [GS97] and [Fre08]. These theorem about robust stability are rewritten in the terminology of the present chapter.

6.5.1 [GS97, Thm. 1] revisited

Now a robust stability theorem is proved on which the main results of Chapter 7 are based. This result is based on [GS97, Thm. 1], but extends the scope of that result in several directions. First, the result is established in the language of ambient signal spaces to handle finite escape times (cf. [GS97, Thm. 8]). More importantly, the implicit requirement in [GS97] of well posedness of $[P_1, C]$ is extended to include the often weaker requirement of regular well posedness. This eases the application of the result in general, as global well posedness is non-trivial to verify a priori, and regular well posedness is often easier to establish (for L^∞ as signal space regular well posedness follows from standard results on the finite escape time properties of differential equations).

Note that this theorem is presented in a form where stability of $[P_1, C]$ is inferred from $[P, C]$, however, in the sequel the theorem will be apply in the setting whereby stability of $[P, C_1]$ is to be inferred from $[P, C]$. Such applications of the theorem follow from a trivial interchange of P and C and $\mathcal{U}$, $\mathcal{Y}$; to follow the convention of the literature and since, in contrast to the results in Chapter 7, most applications of such robust stability results concern uncertainty in the plant P, the theorem is presented in the context of P, P_1.

Theorem 6.5.1 *Let $\mathcal{U}$, $\mathcal{Y}$ be signal spaces and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $P_1\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ with $P(0) = 0$, $C(0) = 0$. Suppose $[P, C]$ is gain stable on $\mathcal{W}$, P_1 is causally extendible and $[P_1, C]$ is either a) globally or b) regularly well posed. If*

$$\vec{\delta}(P, P_1) < \left\|\Pi_{P//C}\right\|^{-1}_{\mathcal{W},\mathcal{W}} \tag{6.5.1}$$

then the closed-loop system $[P_1, C]$ is gain stable on $\mathcal{W}$ with

$$\left\|\Pi_{P_1//C}\right\|_{\mathcal{W},\mathcal{W}} \;\leq\; \left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \frac{1 + \vec{\delta}(P, P_1)}{1 - \left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \vec{\delta}(P, P_1)} \,. \tag{6.5.2}$$

Proof. Since $\left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \geq 1$, it follows that $\vec{\delta}(P, P_1) < \infty$ and hence there exists a causal surjective mapping $\Phi\colon \mathcal{G}_P \to \mathcal{G}_{P_1}$ such that

$$\gamma := \|(\Phi - I)\Pi_{P//C}\|_{\mathcal{W},\mathcal{W}} \leq \|(\Phi - I)\|_{\mathcal{W},\mathcal{W}} \cdot \|\Pi_{P//C}\|_{\mathcal{W},\mathcal{W}} < 1. \tag{6.5.3}$$

Let $w \in \mathcal{W}$ and let $[0, \omega_w)$ be the maximal interval of existence for

$H_{P_1,C}(w)$. Let $0 < \tau < \omega_w$. Consider the equation

$$w|_{[0,\tau)} = ((I + (\Phi - I)\Pi_{P//C})(x))|_{[0,\tau)} \tag{6.5.4}$$
$$= ((\Pi_{C//P} + \Phi\Pi_{P//C})(x))|_{[0,\tau)}. \tag{6.5.5}$$

By either well posedness assumption a) or b), one knows that $[P_1, C]$ is locally well posed, and hence satisfies the existence and uniqueness properties on $[0, \tau)$. Hence there exists $w_1 = (u_1, y_1), w_2 = (u_2, y_2) \in \mathcal{W}_{\omega_w}$ such that $y_1 = P_1 u_1$, $u_2 = Cy_2$ and $w|_{[0,\tau)} = w_1|_{[0,\tau)} + w_2|_{[0,\tau)}$. Since P_1 is causally extendible, there exists $w_1'' \in \mathcal{G}_{P_1}$, such that $w_1''|_{[0,\tau)} = w_1|_{[0,\tau)}$. The definition of $\mathcal{W}_{\omega_w}$ yields that $w_2|_{[0,\tau)} \in \mathcal{W}[0, \tau)$ and hence there exists $w_2' \in \mathcal{W}$ such that $w_2'|_{[0,\tau)} = w_2|_{[0,\tau)}$. Since Φ is surjective it follows that there exists $w_1' \in \mathcal{G}_P$ such that $\Phi(w_1') = w_1''$ and hence $(\Phi(w_1'))|_{[0,\tau)} = w_1''|_{[0,\tau)} = w_1|_{[0,\tau)}$. It can now be seen that $x = w_1' + w_2' \in \mathcal{W}$ satisfies $x|_{[0,\tau)} = (w_1' + w_2)|_{[0,\tau)}$ and x is a solution of (6.5.5).

Since Φ, $\Pi_{P_1//C}$, $\Pi_{P//C}$, $\Pi_{C//P}$ are causal, it follows from (6.5.5) that

$$(\Pi_{P_1//C}(w))|_{[0,\tau)} = \left(\Pi_{P_1//C}\left(\Pi_{C//P}x + \Phi\Pi_{P//C}(x)\right)\right)|_{[0,\tau)}$$
$$= \left(\Phi\Pi_{P//C}(x)\right)|_{[0,\tau)}. \tag{6.5.6}$$

It follows from (6.5.4) that $\left\|x|_{[0,\tau)}\right\|_{\mathcal{W}_\tau} \leq \frac{1}{1-\gamma}\left\|w|_{[0,\tau)}\right\|_{\mathcal{W}_\tau}$, hence, in view of inequalities (6.5.1), (6.5.3) and equation (6.5.6),

$$\begin{aligned}
&\left\|\Pi_{P_1//C}(w)|_{[0,\tau)}\right\|_{\mathcal{W}_\tau}\\
&\quad= \left\|\Phi\Pi_{P//C}(x)|_{[0,\tau)}\right\|_{\mathcal{W}_\tau}\\
&\quad\leq \left\|\Pi_{P//C}(x)|_{[0,\tau)}\right\|_{\mathcal{W}_\tau} + \left\|(\Phi - I)\Pi_{P//C}(x)|_{[0,\tau)}\right\|_{\mathcal{W}_\tau}\\
&\quad\leq \left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \frac{1 + \vec{\delta}(P, P_1)}{1 - \left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}}\vec{\delta}(P, P_1)} \left\|w|_{[0,\tau)}\right\|_{\mathcal{W}_\tau}.
\end{aligned} \tag{6.5.7}$$

If $[P_1, C]$ is globally well posed, $\omega_w = \infty$, so inequality (6.5.7) holds for all $\tau > 0$, and the proof is complete.

Suppose $[P_1, C]$ is regularly well posed. Since, in view of (6.5.7), it is shown that $(\Pi_{P_1//C}(w))\big|_{[0,\tau)} \in \mathcal{W}[0,\tau)$ is uniformly bounded for all $\tau \in (0, \omega_w)$ and since $[P_1, C]$ is regularly well posed, it follows that $\omega_w = \infty$ so inequality (6.5.7) holds for all $\tau > 0$. This completes the proof. □

6.5.2 [Fre08, Thm. 5.1–5.3] revisited

The results in this subsection form the theoretical basis for deducing robustness results for closed-loop systems $[P, C]$ which are not $\mathcal{W}$-gain stable but gain-function stable, see Theorems 6.5.2 and 6.5.3, and provide a robustness result for systems with non-zero initial conditions. The following theorems are extracted from [Fre08, Sec. V] and rewritten in the terminology of the present chapter. This becomes necessary due to the more general definitions of extended and ambient spaces in view of the truncation and restriction operators, see Section 6.1.

Some more definitions are required: let

$$\begin{aligned}\mathcal{K}_\infty &:= \left\{ k \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}) \,\middle|\, k(0) = 0,\ \lim_{t\to\infty} k(t) = \infty \right\},\\ \mathcal{B}_r &:= \{ w \in \mathcal{W} \mid \|w\|_{\mathcal{W}} \leq r \}.\end{aligned}$$

The following theorem rewrites [Fre08, Thm. 5.1] in the present terminology of restriction and truncation operators, see Section 6.1. Note that only one additional assumption is made: for signal spaces $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ it is required that $\mathcal{W}[0,\tau)$ is complete for all $\tau \in (0,\infty)$. Theorem 6.5.2 is required to proof Theorem 6.5.3.

Theorem 6.5.2 *([Fre08, Thm. 5.1] revisited) Let $\mathcal{U}$, $\mathcal{Y}$ be signal spaces and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ such that $\mathcal{W}[0,\tau)$ is complete for all $\tau \in (0,\infty)$. Suppose that the closed-loop $[P, C]$ given by (6.2.1) is gain-function stable and the closed-loop $[P_1, C]$ is regularly well posed. Let $\mathcal{D} \subset \mathcal{G}_P$, $\mathcal{D}_1 \subset \mathcal{G}_{P_1}$, $\mathcal{X} \subset \mathcal{W}$ be convex and $r > 0$. Suppose that $\Pi_{P//C}\mathcal{X} \subset \mathcal{D}$ and there exists a causal, gain-function stable map $\Psi\colon \mathcal{D} \to \mathcal{D}_1$ such that, for all $\tau > 0$, the following hold:*

(1) $(I - \Psi)\Pi_{P//C}(\cdot)\big|_{[0,\tau)} \colon \mathcal{X} \to \mathcal{W}[0,\tau)$ is causal and compact;

(2) $\forall\, x, w \in \mathcal{X}[0,\tau)\,:\, w|_{[0,\tau)} + \big((I-\Psi)\Pi_{P//C}(x)\big)\,|_{[0,\tau)} \in \mathcal{X}[0,\tau)$*;*

(3) $\exists\, \kappa \in \mathcal{K}_\infty\ \forall\, x \in \mathcal{X} \cap \mathcal{B}_r\,:\, \|T_\tau(I-\Psi)\Pi_{P//C}(x)\|_{\mathcal{W}} \le (1+\kappa)^{-1}(r).$ (6.5.8)

Then $H_{P_1,C}\big|_{\mathcal{X}\cap\mathcal{B}_r} : \mathcal{X}\cap\mathcal{B}_r \to \mathcal{W}\times\mathcal{W}$ *is gain-function stable and*

$$\forall\, \tau > 0\ \forall\, w \in \mathcal{X}\cap\mathcal{B}_r\,:\, \|T_\tau \Pi_{P_1//C} w\|_{\mathcal{W}} \le g[\Psi]\circ g[\Pi_{P//C}]\circ(1+\kappa^{-1})(r)\,. \tag{6.5.9}$$

Proof. The proof follows the outline of the proof of [Fre08, Thm. 5.1].
Let $w \in \mathcal{X}$ with $\|w\|_{\mathcal{W}} \le r$ and let $[0,\omega_w)$ be the maximal of existence for $H_{P_1,C}w$. Let $\tau \in (0,\omega_w)$. Consider the equation

$$\begin{aligned} w|_{[0,\tau)} &= ((I+(\Psi-I)\Pi_{P//C})(x))|_{[0,\tau)} \\ &= ((\Pi_{C//P}+\Psi\Pi_{P//C})(x))|_{[0,\tau)}. \end{aligned} \tag{6.5.10}$$

It is claimed that this equation has a solution $x \in V$, where

$$V := \left\{ x \in \mathcal{X}[0,\tau) \,\middle|\, \|x\|_{\mathcal{W}} \le (1+\kappa^{-1})(r) \right\}\,.$$

Note that $(t \mapsto 0)\big|_{[0,\tau)} \in V \ne \emptyset$. Moreover, it is easy to see that V is closed and bounded and, in view of convexity of $\mathcal{X}$, V is convex, too. Consider the operator

$$Q_w\colon V \to \mathcal{X}[0,\tau)\,, \quad x \mapsto w|_{[0,\tau)} + ((I-\Psi)\Pi_{P//C}(x))|_{[0,\tau)}$$

where observe, by assumption (2), it follows that $Q_w(V) \subset \mathcal{X}[0,\tau)$ as required. In view of assumption (3) there exists $\kappa \in \mathcal{K}_\infty$ such that, for all $x \in V$,

$$\begin{aligned} \|Q_w x\|_{\mathcal{W}_\tau} &= \|w|_{[0,\tau)} + ((I-\Psi)\Pi_{P//C}(x))|_{[0,\tau)}\|_{\mathcal{W}_\tau} \\ &\le \|T_\tau w\|_{\mathcal{W}} + \|T_\tau((I-\Psi)\Pi_{P//C}(x))\|_{\mathcal{W}} \\ &\le \|T_\tau w\|_{\mathcal{W}} + (1+\kappa)^{-1}\left(\|T_\tau x\|_{\mathcal{W}}\right) \\ &\le r + (1+\kappa)^{-1}\circ(1+\kappa^{-1})(r) \\ &= (1+\kappa^{-1})(r)\,. \end{aligned}$$

Therefore, $Q_w(V) \subset V$. Since, by assumption (1), $(I-\Psi)\Pi_{P//C}(\cdot)\big|_{[0,\tau)}$ is compact it then follows that Q_w is compact. Then, in view of completeness of $\mathcal{W}[0,\tau)$, $\mathcal{W}[0,\tau)$ is a Banach space. Hence, and since V is nonempty, closed, bounded and convex, Schauder's fixed point theorem [Zei86, Thm. 2.A] yields that Q_w has a fixed point in V. Hence (6.5.10) has a solution $x \in V \subset \mathcal{X}[0,\tau)$ as claimed.

Since $\Psi\Pi_{P//C}(x) \in \mathcal{G}_{P_1}$, $\Pi_{C//P}(x) \in \mathcal{G}_C$ and $\Psi, \Pi_{P_1//C}, \Pi_{P//C}$ and $\Pi_{C//P}$ are causal it follows from (6.5.10) that

$$\begin{aligned}(\Pi_{P_1//C}(w))\big|_{[0,\tau)} &= \big(\Pi_{P_1//C}\big(\Pi_{C//P}(x) + \Psi\Pi_{P//C}(x)\big)\big)\big|_{[0,\tau)} \\ &= \big(\Psi\Pi_{P//C}(x)\big)\big|_{[0,\tau)}\,.\end{aligned}$$

Hence, since $x \in V$,

$$\begin{aligned}\|(\Pi_{P_1//C}(w))\big|_{[0,\tau)}\|_{\mathcal{W}_\tau} &= \|T_\tau\Pi_{P_1//C}(w)\|_{\mathcal{W}} \\ &= \|T_\tau\Psi\Pi_{P//C}(x)\|_{\mathcal{W}} \\ &\le g[\Psi]\circ g[\Pi_{P//C}]\big(\|T_\tau x\|_{\mathcal{W}}\big) \\ &\le g[\Psi]\circ g[\Pi_{P//C}]\circ(1+\kappa^{-1})(x)\,.\end{aligned}$$

As $\mathcal{W}$ is a signal space, thus has the property that $\sup_{\tau\ge 0}\|T_\tau w\|_{\mathcal{W}} < \infty$, and since $\tau \in (0,\omega_w)$ was arbitrary it follows that $(\Pi_{P_1//C}(w))\big|_{[0,\omega_w)} \in \mathcal{W}[0,\omega_w)$ and so $(H_{P_1,C}(w))\big|_{[0,\omega_w)} \in \mathcal{W}[0,\omega_w)\times\mathcal{W}[0,\omega_w)$. Since $[P_1,C]$ is regularly well posed it follows that $\omega_w = \infty$, see Section 6.2, and thus $\Pi_{P_1//C}(w) \in \mathcal{W}$. Since $w \in \mathcal{X}\cap\mathcal{B}_r$ was arbitrary it follows that (6.5.9) holds and hence, $H_{P_1,C}\colon \mathcal{X}\cap\mathcal{B}_r \to \mathcal{W}\times\mathcal{W}$ is gain-function stable. Therefore, the proof is complete. □

The following theorem applies the above gain-function stability result to linear plants with zero initial conditions. Robustness is shown with respect to a sufficiently small gap. Theorem 6.5.3 together with Theorem 6.5.3 provide the theoretical basis for the robustness analysis of λ-tracking and funnel control in Chapters 8 and 9. However, both theorems achieve general robustness results for all closed-loop systems which arise from the application of any causal controller C to any stabilizable and detectable linear plant P.

Recall that a linear time-invariant system $(A,B,C) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times$

$\mathbb{R}^{m\times n}$ is *stabilizable* if, and only if, there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $A+BF$ is Hurwitz, see, for example, [Son98, Ch. 5.5]. Moreover, recall that $(A,B,C) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times n}$ is *detectable* if, and only if, (A^T, C^T, B^T) is stabilizable, see [Son98, Ch. 7.1].

Introduce the class of stabilizable and detectable m-input, m-output linear systems: for $n,m \in \mathbb{N}$ with $n \geq m$, define

$$\mathcal{P}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times n} \end{array} \middle| \begin{array}{l} (A,B,C) \text{ is stabilizable} \\ \text{and detectable} \end{array} \right\},$$

where (A,B,C) has, as in the previous chapters, the (standard) form

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u_1(t)\,, \qquad x(0) = x^0, \\ y_1(t) &= C\,x(t)\,, \end{aligned}\right\} \tag{6.5.11}$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$ and $x^0 \in \mathbb{R}^n$ is an arbitrary initial value.

For any normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$, associate with a system $\theta = (A,B,C) \in \mathcal{P}_{n,m}$ and initial value $x^0 \in \mathbb{R}^n$ associate a plant operator

$$P(\theta,x^0)\colon \mathcal{U}_a \to \mathcal{Y}_a\,, \qquad u_1 \mapsto P(\theta,x^0)(u_1) := y_1\,, \tag{6.5.12}$$

where $u_1 \in \mathcal{U}_a$ and $y_1 \in \mathcal{Y}_a$ satisfy (6.5.11). Note that P is a map from $\bigcup_{n\geq m}(\mathcal{P}_{n,m}\times\mathbb{R}^n)$ to the space of maps $\mathcal{U}_a \to \mathcal{Y}_a$.

To establish gap margin results, it is required to construct the *augmented* plant and controller operators as in [Fre08, FIR06].

For $m,n \in \mathbb{N}$ with $n \geq m$, consider $\mathcal{P}_{n,m}$ as a subspace of the Euclidean space $\mathbb{R}^{n^2+2mn}$ by identifying a plant $\theta = (A,B,C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. Define, for any signal spaces $\mathcal{U}$ and $\mathcal{Y}$, the space $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2nm}\times\mathcal{U}$ and let $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}}\times\mathcal{Y}$, which can be considered as signal spaces by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$ and endowing $\widetilde{\mathcal{U}}$ with the norm $\|(\theta,u)\|_{\widetilde{\mathcal{U}}} := \sqrt{\|\theta\|^2 + \|u\|^2_{\mathcal{U}}}$. For given $P(\theta,0)$ as in (6.5.12), define the (augmented) plant operator as

$$\widetilde{P}\colon \widetilde{\mathcal{U}}_a \to \mathcal{Y}_a\,, \quad (\theta,u_1) = \widetilde{u}_1 \mapsto y_1 = \widetilde{P}(\widetilde{u}_1) := P(\theta,0)(u_1)\,. \tag{6.5.13}$$

Define, for any causal controller operator $C\colon \mathcal{Y} \to \mathcal{U}$, $y_2 \mapsto u_2$, the (augmented) controller operator as

$$\widetilde{C}\colon \mathcal{Y}_a \to \widetilde{\mathcal{U}}_a\,, \quad y_2 \mapsto \widetilde{u}_2 = \widetilde{C}(y_2) := (0, C(y_2)) = (0, u_2)\,. \tag{6.5.14}$$

Now, one is in the position to state the revision of [Fre08, Thm. 5.2] for a larger class of signal spaces than in [Fre08], namely Sobolev spaces $W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $r \in \mathbb{N}$ and $1 \leq 0 \leq \infty$.

Theorem 6.5.3 *([Fre08, Thm. 5.2] revisited) Let $n, q, m, r \in \mathbb{N}$, $p \in [1, \infty]$ and let $\mathcal{U}$, $\mathcal{Y}$ be any of the signal spaces $L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ or $W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Set $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and let $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$ be any causal controller operator. Consider the plant operator $P(\theta, x^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ defined by* (6.5.12) *for $(\theta, 0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ or $(\theta_1, 0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$. Suppose that the closed-loop system $[P(\theta_1, 0), C]$ is regularly well posed for all $\theta_1 \in \mathcal{P}_{q,m}$. Define the (augmented) plant and controller operators as in* (6.5.13) *and* (6.5.14)*:*

$$\begin{aligned} \widetilde{P} &: (\mathbb{R}^{n^2+2nm} \times \mathcal{U})_a \to \mathcal{Y}_a\,, \quad (\vartheta, u_1) \mapsto y_1 = \widetilde{P}(\vartheta, u_1) := P(\vartheta, 0)(u_1)\,, \\ \widetilde{C} &: \mathcal{Y}_a \to (\mathbb{R}^{n^2+2nm} \times \mathcal{U})_a\,, \qquad y_2 \mapsto \widetilde{C}(y_2) := (0, C(y_2)) = (0, u_2)\,. \end{aligned}$$

Let $\Omega \subset \mathbb{R}^{n^2+2nm}$ be closed. Suppose that $H_{\widetilde{P},\widetilde{C}}\big|_{\Omega \times \mathcal{W}}$ is gain-function stable and $\Pi_{\widetilde{P}//\widetilde{C}}(\cdot)\big|_{[0,\tau)}$ is continuous for all $\tau > 0$.

Then there exists a continuous function $\mu\colon \mathbb{R}_{\geq 0} \times \Omega \to (0, \infty)$ such that

$$\forall\, (\theta_1, w_0, \varrho) \in \mathcal{P}_{n,m} \times \mathcal{W} \times (0, \infty)\ :\quad \left.\begin{array}{l} \|w_0\|_{\mathcal{W}} \leq \varrho \\ \vec{\delta}\big(P(\theta, 0), P(\theta_1, 0)\big) \leq \mu(\varrho, \theta) \end{array}\right\} \;\Rightarrow\; H_{P(\theta_1,0),C} w_0 \in \mathcal{W} \times \mathcal{W}\,. \tag{6.5.15}$$

A proof of Theorem 6.5.3 is omitted here since it is equivalent to the proof of [Fre08, Thm. 5.2] with the one exception: apply Theorem 6.5.2 instead of [Fre08, Thm. 5.1]. Note that the additional assumption of Theorem 6.5.2, namely that $\mathcal{W}[0, \tau)$ is complete for all $\tau \in (0, \infty)$, obviously holds true for $\mathcal{U}, \mathcal{Y}$ being any of the signal spaces $L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ or $W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, see also [AF03, Ch. 3].

The third result in this section shows that robust stability of closed-loop systems with zero initial conditions also guarantees stability margins in presence of sufficiently small initial values $x^0 \in \mathbb{R}^n$ of the linear plant (A, B, C).

Theorem 6.5.4 *([Fre08, Thm. 5.3] revisited) Let $n, m, r \in \mathbb{N}$, $p \in [1, \infty]$ and let $\mathcal{U}$, $\mathcal{Y}$ be any of the signal spaces $L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ or $W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Set $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and let $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$ be any causal controller operator. Consider, for $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, the plant operator $P(\theta, x^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ defined by* (6.5.12). *Suppose that there exists $\varrho > 0$ such that $H_{P(\theta,0),C} w_0 \in \mathcal{W} \times \mathcal{W}$ for all $w_0 \in \mathcal{W}$ with $\|w_0\|_{\mathcal{W}} \leq \varrho$. Then*

$$\begin{aligned}\exists\, \lambda > 0\ \forall\, (\theta, w_0, x^0) \in \mathcal{P}_{n,m} \times \mathcal{W} \times \mathbb{R}^n \ :& \\ \lambda \|x^0\| + \|w_0\|_{\mathcal{W}} \leq \varrho \ \Rightarrow\ H_{P(\theta,x^0),C} w_0 \in \mathcal{W} \times \mathcal{W}\,.& \qquad (6.5.16)\end{aligned}$$

Proof. The proof follows the steps of the proof of [Fre08, Thm. 5.3].

Since system $(A, B, C) = \theta \in \mathcal{P}_{n,m}$ is stabilizable there exists $F \in \mathbb{R}^{m \times n}$ such that $\widehat{A} = A + BF$ is Hurwitz. Then one may define maps $\widetilde{N}\colon \mathcal{U} \to \mathcal{U}$, $u_0 \mapsto u_1$, and $M\colon \mathcal{U} \to \mathcal{Y}$, $u_0 \mapsto y_1$ such that the tuples $(u_0, u_1) = (u_0, \widetilde{N} u_0)$ and $(u_0, y_1) = (u_0, M u_0)$ satisfy

$$\left.\begin{aligned} \dot{x} &= (A + BF)x + B\,u_0\,, \qquad x(0) = 0 \\ u_1 &= F\,x + u_0\,, \\ y_1 &= C\,x\,. \end{aligned}\right\} \qquad (6.5.17)$$

Step 1: It is shown that $\widetilde{N}(\mathcal{U}) = \mathcal{V} := \{u \in \mathcal{U} \mid P(\theta, 0)u \in \mathcal{Y}\}$:

Suppose $u \in \mathcal{V}$, i.e. $u \in \mathcal{U}$ with $P(\theta, 0)u \in \mathcal{Y}$. Then $\mathcal{Y} \ni P(\theta, 0)u = C\,x =: y$ for x being a solution of $\dot{x} = A\,x + B\,u$, $x(0) = 0$. Since (A, B, C) is detectable, there exists $L \in \mathbb{R}^{n \times m}$ such that $A + LC$ is Hurwitz. Since $y \in \mathcal{Y}$ and $u \in \mathcal{U}$ writing

$$\dot{x} = (A + LC)x - LC\,x + B\,u = (A + LC)x - L\,y + B\,u$$

yields that $x \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)$ for every $p \in [1, \infty]$. Thus $u_0 := u - Fx \in \mathcal{U}$ and (6.5.17) then yields that $u = u_1 = \widetilde{N}(u_0) \in \widetilde{N}(\mathcal{U})$, which gives $\mathcal{V} \subset \widetilde{N}(\mathcal{U})$.

Conversely, suppose $u \in \widetilde{N}(\mathcal{U})$. Then there exists $u_0 \in \mathcal{U}$ such that $u_0 = u - Fx \in \mathcal{U}$. Since $\operatorname{spec}(A + BF) \subset \mathbb{C}_-$ it follows by (6.5.17) that $P(\theta, 0)u = y = Cx \in \mathcal{Y}$. Hence $\widetilde{N}(\mathcal{U}) \subset \mathcal{V}$.

Now $N \colon \mathcal{U} \to \mathcal{V}$, $u_0 \mapsto (1\ \ 0)\, \Pi_{P(\theta,0)//C} \left(\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right)$, is well defined and writing

$$\begin{aligned} \dot{x} &= A\,x + B\,u_1\,, \qquad x(0) = 0 \\ u_0 &= F\,x - u_1\,, \end{aligned}$$

directly gives that N is invertible and $P(\theta, 0) = MN^{-1}$.

Step 2: Characterization of the graph $\mathcal{G}_{P(\theta,x^0)}$: it is shown that

$$\begin{aligned} \mathcal{G}_{P(\theta,x^0)} &= Q \\ &:= \left\{ \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\widehat{A}\cdot)x^0 \\ C\exp(\widehat{A}\cdot)x^0 \end{pmatrix} \in \mathcal{W} \;\middle|\; \begin{array}{l} v \in \mathcal{U}\,,\ N, M, F \\ \text{and } \widehat{A} \text{ as in Step 1} \end{array} \right\}. \end{aligned}$$

Suppose, for any $v \in \mathcal{U}$, $q_v = \left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right) v + \left(\begin{smallmatrix} F\exp(\widehat{A}\cdot)x^0 \\ C\exp(\widehat{A}\cdot)x^0 \end{smallmatrix}\right) \in Q$. Let $u = Nv + F\exp(\widehat{A}\cdot)x^0$. Since $Nv \in \mathcal{U}$ and, for any $r \in \mathbb{N}$ and $p \in [1, \infty]$, $F\exp(\widehat{A}\cdot) \in W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)^n \subset \mathcal{U}^n$ it follows that $u \in \mathcal{U}$. Observe

$$\dot{x} = A\,x + B(F\exp(\widehat{A}\cdot)x^0)\,, \qquad x(0) = x^0 \in \mathbb{R}^n$$

has the solution $x(\cdot) = \exp(\widehat{A}\cdot)x^0$. Thus it follows that

$$P(\theta, x^0)\big(F\exp(\widehat{A}\cdot)x^0\big) = C\exp(\widehat{A}\cdot)x^0\,. \tag{6.5.18}$$

Hence, by $C\exp(\widehat{A}\cdot) \in W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)^n \subset \mathcal{Y}^n$, it follows that

$$\begin{aligned} P(\theta, x^0)(u) &= P(\theta, x^0)(Nv) + P(\theta, x^0)\big(F\exp(\widehat{A}\cdot)x^0\big) - P(\theta, x^0)(0) \\ &= P(\theta, 0)(Nv) + P(\theta, x^0)\big(F\exp(\widehat{A}\cdot)x^0\big) \\ &= (MN^{-1})(Nv) + C\exp(\widehat{A}\cdot)x^0 \\ &= Mv + C\exp(\widehat{A}\cdot)x^0 \in \mathcal{Y}\,. \end{aligned} \tag{6.5.19}$$

Thus $q_v = \big(u, P(\theta, x^0)u\big) \in \mathcal{U} \times \mathcal{Y}$ and so $q_v \in \mathcal{G}_{P(\theta,x^0)}$ and $Q \subset \mathcal{G}_{P(\theta,x^0)}$.

Conversely, suppose $\binom{u}{P(\theta,x^0)u} \in \mathcal{G}_{P(\theta;x^0)}$. Then, in view of (6.5.18),

$$P(\theta,0)\big(u - F\exp(\widehat{A}\cdot)x^0\big) = P(\theta,x^0)(u) - P(\theta,x^0)\big(F\exp(\widehat{A}\cdot)x^0\big) \in \mathcal{Y}\,.$$

Therefore, Step 1 yields that $u - F\exp(\widehat{A}\cdot)x^0 \in \mathcal{V} = \mathrm{im}(N)$ and so there exists $v \in \mathcal{U}$ such that $Nv = u - F\exp(\widehat{A}\cdot)x^0$. Therefore, equation (6.5.19) holds, hence

$$\begin{pmatrix} u \\ P(\theta,x^0)u \end{pmatrix} = \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\widehat{A}\cdot)x^0 \\ C\exp(\widehat{A}\cdot)x^0 \end{pmatrix} \in Q$$

and so $\mathcal{G}_{P(\theta,x^0)} \subset Q$. Therefore, $\mathcal{G}_{P(\theta,x^0)} = Q$ as claimed.

Step 3: Finally, (6.5.16) is shown.

Let $\lambda := \big\|(F\exp(\widehat{A}\cdot), C\exp(\widehat{A}\cdot))\|_{\mathcal{U}^n\times\mathcal{Y}^n}$. Suppose that $w_0 \in \mathcal{W}$ and $x^0 \in \mathbb{R}^n$ satisfy $\lambda\|x^0\| + \|w_0\|_{\mathcal{W}} \le \varrho$. Then, by letting

$$w_0^{'} = w_0 - w_0^{''}\,, \quad w_0^{''} = \big(F\exp(\widehat{A}\cdot)x^0, C\exp(\widehat{A}\cdot)x^0\big)^T,$$

it follows that $\|w_0^{'}\|_{\mathcal{W}} \le \lambda\|x^0\| + \|w_0\|_{\mathcal{W}} \le \varrho$. Hence, by assumption,

$$H_{P(\theta,0),C}(w_0^{'}) = (w_1, w_2) \in \mathcal{G}_{P(\theta,0)} \times \mathcal{G}_C\,.$$

In particular, $w_0^{'} = w_1 + w_2$ and, by rearranging, it follows that $w_0 = (w_1 + w_0^{''}) + w_2$. Since $w_1 \in \mathcal{G}_{P(\theta,0)}$ there exists $v \in \mathcal{U}$ such that, in view of N, M defined in Step 1,

$$w_1 = \begin{pmatrix} N \\ M \end{pmatrix} v\,,$$

hence $w_1 + w_0^{''} \in Q = \mathcal{G}_{P(\theta,x^0)}$. Thus, and since $w_2 \in \mathcal{G}_C$, it follows $H_{P(\theta,x^0),C}w_0 = (w_1 + w_0^{''}, w_2) \in \mathcal{W}\times\mathcal{W}$ and the proof is complete. $\square$

In the following chapters the results of the present subsection are applied to the robustness analysis of high-gain derivative feedback stabilization, λ-tracking and funnel control.

6.6 Notes and references

The idea of measuring distances between graphs of operators to determine stability results in some sense goes back to works of J. C. Gohberg and M. G. Krein, H. O. Cordes and J. P. Labrousse, and T. Kato, see the references within [Kat76, p. 197].

The gap metric used in the present thesis, firstly introduced by the work of G. Zames and A. El-Sakkary [ZES80], is based on the works by T. Georgiou and M. Smith [GS90, GS93, GS97]. However, due to the application in the various signal space settings of Chapters 7–9, the terminology and results of the present chapter are based on the terminology of [FIM09, Sec. 2]. In particular, the use of continuously differential signals which do not allow for the terminology of [GS97, Sec. II], [Fre08, Sec. 2] or [FIR06, Sec. 2].

Finally, some additional comments on w-Stability and ν-gap distance follow:

T. Georgiou and M. Smith introduce in their paper [GS89] the idea of w-Stability of closed-loop systems $[P, C]$ as "a generalization of a concept of well posedness discussed by J. Willems [Wil71]". w-stability of systems means that the systems should remain stable for a certain class of perturbations which may not necessarily be small when measured by the gap metric. However, w-stability of a system is defined in terms of the so-called *approximate identities*, see [GS89, Sec. 2] for the definitions. Due to a strict frequency domain terminology in [GS89], the concept of w-stability is limited to linear plants and linear controllers. It might be worth to study whether w-stability can be adopted to find robustness results for high-gain derivative feedback stabilization, however, this is not part of the present thesis.

G. Vinnicombe introduces the ν-gap distance in his thesis [Vin92] for linear time-invariant systems and gives a generalization for nonlinear systems in [Vin99]. The ν-gap metric $\delta_\nu(P, P_1)$ of two systems P and P_1 is the smallest metric for which certain robustness results hold. Unlike the gap metric the ν-gap is introduced in a strict frequency domain terminology which makes it much more complex when applying to the nonlinear feedback systems considered in the present thesis.

7 Robustness of output feedback stabilization

The main result in this chapter establishes that if the high-gain output derivative feedback controller

$$C\colon y \mapsto u = \sum_{i=0}^{r-1} k_{i+1} y^{(i)} ,$$

introduced in Chapter 3, globally stabilizes (in some sense) a (nonlinear) plant P, then global stabilization of P can also be achieved by an output feedback controller $C[h]$, given by

$$C[h]\colon y \mapsto u = \sum_{i=0}^{r-1} k_{i+1} \Delta_h^i y ,$$

that means where the output derivatives in C are replaced by an Euler approximation with sufficiently small delay $h > 0$.

This is proved within the conceptual framework of the nonlinear gap metric approach to robust stability. It is shown that, for sufficiently small $h > 0$, the closed-loop system $[P, C[h]]$ of plant P and controller $C[h]$ is gain stable (recall the definition of gain stability from Section 6.4). The main result is then applied to linear minimum phase systems (A, B, C) with unknown coefficients but relative degree r and "positive" high-frequency gain to prove exponential stability of the delay differential system consisting of the linear system controlled by the delay output feedback.

7.1 Derivative and delay feedback

The problem of robust stability of high-gain output derivative feedback is studied in the setup of the feedback configuration shown in Figure 7.1, see also Chapter 6. Here P may be any (nonlinear) plant and the controllers C_k and $C_k^{\text{Euler}}[h]$ are specified in due course. One is concerned

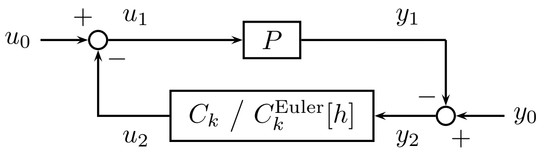


Figure 7.1: The closed-loop system $[P, C]$.

with the concept of gain stability from Section 6.4 of the closed-loop systems $[P, C_k]$ and $[P, C_k^{\text{Euler}}[h]]$, respectively. Recall, for some appropriate choices of signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r}$, the high-gain output derivative feedback controller

$$C_k \colon \mathcal{Y} \to \mathcal{U}\,, \qquad y_2 \mapsto u_2 = -\sum_{i=0}^{r-1} k_{i+1}\, y_2^{(i)}\,. \tag{7.1.1}$$

In the following section it is shown that stabilization of P can also be achieved by replacing C_k by the delay (Euler) feedback controller $C_k^{\text{Euler}}[h]$ for sufficiently small $h > 0$, given by

$$C_k^{\text{Euler}}[h] \colon \mathcal{Y} \to \mathcal{U}\,, \qquad y_2 \mapsto u_2 = -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i y_2\,, \tag{7.1.2}$$

where $\Delta_h^0 y_2 = y_2$ and $\Delta_h^i y_2$ for $i \geq 1$, denotes the Euler approximation of the ith derivative of y_2 defined by

$$\Delta_h^i y_2 = \underbrace{\Delta_h \circ \cdots \circ \Delta_h}_{i \text{ times}} y_2\,, \qquad \text{where} \quad (\Delta_h y_2)(t) = \frac{y_2(t) - y_2(t-h)}{h}\,.$$

The main result presents conditions under which a feedback controller based on the measured output and its derivatives can be replaced by a feedback controller based on the measured output and *numerical* derivatives.

The signal spaces $\mathcal{U}$ and $\mathcal{Y}$ for which these result holds depend on structural properties of the controller C_k. For concreteness: since C_k maps the output y_2 onto a sum of derivatives of y_2 some regularity of $\mathcal{Y}$ is required. On the other side, in view of the application of the main result Theorem 7.2.1 to m-input, m-output linear plants, which are minimum phase and have strict relative degree $r \geq 1$, see Section 7.3, choices of signal spaces $\mathcal{U} = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $\mathcal{Y} = CW^{\varrho,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, where $\varrho = r$ if $p = \infty$ and $\varrho = r$ or either $\varrho = \infty$ if $p < \infty$, are valid.

For the application of Theorem 7.2.1, the stabilizing high-gain output derivative feedback (3.2.2) for linear MIMO-systems with strict relative degree r is considered and replaced by an according high-gain output delay feedback and an explicit upper bound on the permitted delay is given. The results on robust stability are also extended to incorporate systems with non-zero initial conditions. One can find a similar result on stabilizing by output delay feedback for systems with relative degree 2 in [IS04]. Some more comments about related literature on stabilizing by delays follow in the conclusion of this section.

The results are established by computing the gap distance between C_k and $C_k^{\text{Euler}}[h]$ and using the framework of the nonlinear robust stability theory from Chapter 6 to deduce the stability of the closed-loop system containing the Euler controller from the stability of the derivative feedback controlled closed-loop system.

7.2 Robust stabilization by delay feedback

First, the result on robust stabilization by output delay feedback is presented: the main result of the present chapter establishes conditions under which a derivative feedback controller (7.1.1) may be replaced by the Euler controller (7.1.2). First formally define, for $m \in \mathbb{N}$ and $h > 0$, the Euler approximation

$$\Delta_h \colon \operatorname{map}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \to \operatorname{map}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\,,$$
$$(t \mapsto y(t)) \mapsto \left(t \mapsto \frac{y(t) - y(t-h)}{h}\right), \quad \text{where } y(s) = 0 \text{ if } s < 0,$$

of the derivative of y and, for higher derivatives $y^{(i)}$, $i \in \mathbb{N}$,

$$\Delta_h^i : \operatorname{map}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \to \operatorname{map}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\,,$$
$$y \mapsto \Delta_h^i(y) := \begin{cases} \Delta_h^{i-1}(\Delta_h(y)) & \text{if } i \geq 2 \\ \Delta_h(y) & \text{if } i = 1 \\ y & \text{if } i = 0\,. \end{cases} \tag{7.2.1}$$

Now the underlying signal spaces for the results of this section are presented, see also the list of symbols. Let, for $p \in [1, \infty]$ and $r \in \mathbb{N} \cup \{\infty\}$,

$$CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) := \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \cap L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$$

$$CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) := W^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \cap \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$$

$$CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) := \left\{ y \in CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \;\middle|\; \begin{array}{l} y^{(i)}(0) = 0 \text{ for} \\ \text{all } i \in \{0, \dots, r-1\} \end{array} \right\}$$

$$CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) := \left\{ y \in CW^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \;\middle|\; \forall\, i \in \mathbb{N}_0 \,:\, y^{(i)}(0) = 0 \right\}$$

The results in the present section will hold in the following three signal space settings (A)–(C):

$$
\left.\begin{aligned}
&\textbf{(A)}\ \ \mathcal{W} = \mathcal{U}\times\mathcal{Y} \ \ = CL^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m) \ \ \times CW^{r,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\,,\\
&\qquad \mathcal{W}_0 = \mathcal{U}_0\times\mathcal{Y}_0 = CL^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m) \ \ \times CW_0^{r,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r\in\mathbb{N}\,,\ p=\infty\,,\\
&\textbf{(B)}\ \ \mathcal{W} = \mathcal{U}\times\mathcal{Y} \ \ = CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m) \ \ \times CW^{r,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\,,\\
&\qquad \mathcal{W}_0 = \mathcal{U}_0\times\mathcal{Y}_0 = CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m) \ \ \times CW_0^{r,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r\in\mathbb{N}\,,\ p\in[1,\infty)\,,\\
&\textbf{(C)}\ \ \mathcal{W} = \mathcal{U}\times\mathcal{Y} \ \ = CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\,,\\
&\qquad \mathcal{W}_0 = \mathcal{U}_0\times\mathcal{Y}_0 = CW_0^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\times CW_0^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p\in[1,\infty)\,.
\end{aligned}\right\}
\tag{7.2.2}
$$

The spaces $\mathcal{W}_0$ will be utilized for results whereby the systems' initial conditions are zero, whilst the spaces $\mathcal{W}$ are utilized in the general setting with non-zero initial conditions. The spaces of type (A) and (B) are standard, the need for spaces with constrained derivatives arises from the setting whereby derivative based controllers are being considered. The application of the robust stability result Theorem 7.2.1 to linear minimum phase systems in Section 7.3 a legitimates the spaces of type (C), which allows for more general controllers (one will not require $k_r = 0$ for controller C_k given by (7.1.1) as for signal spaces of type (B), see below) at the price of greater regularity constraints on the disturbances.

Next, the main result of this section is stated: if C_k gain stabilizes a plant P it follows that $C_k^{\text{Euler}}[h]$ is also a gain stabilizing controller for P for sufficiently small $h > 0$. The idea behind the proof is to show that the gap $\vec{\delta}(C_k, C_k^{\text{Euler}}[h])$ is small if $h > 0$ is small and hence deduce the result from Theorem 6.5.1. Recall the definition of the closed-loop operators $\Pi_{C_k//P}$ for $[P, C_k]$ and $\Pi_{C_k^{\text{Euler}}[h]//P}$ for $[P, C_k^{\text{Euler}}[h]]$ from Section 6.4.

Theorem 7.2.1 *Let $p \in [1,\infty]$, $m, r \in \mathbb{N}$ and consider signal spaces $\mathcal{U}_0$, $\mathcal{Y}_0$ and $\mathcal{W}_0$ of type (A), (B) or (C) in* (7.2.2)*. Suppose that there exists $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r} \setminus \{0\}$, with $k_r = 0$ for case (B), such that controller $C_k : \mathcal{Y}_{0a} \to \mathcal{U}_{0a}$ given by* (7.1.1) *applied to a causal plant*

$P\colon \mathcal{U}_{0a} \to \mathcal{Y}_{0a}$, $u_1 \mapsto y_2$ with $P(0) = 0$ yields, in view of the closed-loop equations (6.2.1)*, a closed-loop system $[P, C_k]$ which is gain stable on $\mathcal{W}_0 := \mathcal{U}_0 \times \mathcal{Y}_0$ with*

$$1 \ \le \gamma := \left\|\Pi_{C_k//P}\right\|_{\mathcal{W}_0,\mathcal{W}_0} \ < \ \infty\,.$$

Suppose $h^* > 0$ satisfies

$$\begin{aligned} h^* &\le \left(\gamma \sum_{i=1}^{r-1} |k_{i+1}| \cdot i\eta_{p,m}(h^*, i)\right)^{-1} \\ &\text{where, for } h > 0,\ \eta_{p,m}(h,i) := \begin{cases} m\,, & \text{in case (A)},\\ 2m(1+ihp)^{1/p}\,, & \text{in case (B)},\\ 2m(1+ihp)\,, & \text{in case (C)}. \end{cases} \end{aligned} \tag{7.2.3}$$

Let $h \in (0, h^*)$ and suppose that the closed-loop system $[P, C_k^{Euler}[h]]$ is either globally or regularly well posed, where controller $C_k^{Euler}[h]\colon \mathcal{Y}_{0a} \to \mathcal{U}_{0a}$ is given by (7.1.2)*. Then the closed-loop system $[P, C_k^{Euler}[h]]$ is gain stable on $\mathcal{W}_0$ with*

$$\begin{aligned} &\left\|\Pi_{C_k^{Euler}[h]//P}\right\|_{\mathcal{W}_0,\mathcal{W}_0} \\ &\le \left\|\Pi_{C_k//P}\right\|_{\mathcal{W}_0,\mathcal{W}_0} \frac{1 + h\sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_{p,m}(h,i)}{1 - \ \left\|\Pi_{C_k//P}\right\|_{\mathcal{W}_0,\mathcal{W}_0} h \sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_{p,m}(h,i)}\,. \end{aligned} \tag{7.2.4}$$

In all three signal space settings (A), (B) and (C) the condition (7.2.3) on h^* can always be met for sufficiently small $h^* > 0$, for example by taking

$$h^* = \left(\gamma\, m \sum_{i=1}^{r-1} |k_{i+1}| \cdot i\right)^{-1}$$

in case (A) and by taking

$$h^* = \min\left\{\frac{1}{rp}, \left(4m\gamma \sum_{i=1}^{r-1} |k_{i+1}| \cdot i\right)^{-1}\right\}$$

in cases (B) and (C).

The condition that the nominal closed-loop gain is bounded enforces key attenuation properties, which varies with the different choices of the signal space $\mathcal{W}_0$. Within the context of linear systems a key purpose of Section 7.3 is to explore how these properties can be enforced by structural requirements on the relative degree and the order of the controller. For example, Theorem 7.3.2 shows that in the case of the signal space setting (A) a stabilizing controller whose order is less than or equal to the relative degree of the plant can be replaced by suitable a Euler controller. In the signal space setting (B) Theorem 7.3.2 shows that a stabilizing controller is required to have a order strictly less than the relative degree of the plant. Theorem 7.3.2 also shows that the third signal space setting (C) overcomes the structural limitation of the choice (B) by again allowing stabilizing controllers whose order is less than or equal to the relative degree of the plant, but with considerable extra signal regularity requirements.

The extra requirement that $k_r = 0$ in the signal space setting (B) arises from the application of the Mean Value Theorem in the proof of Theorem 7.3.2. In the case of $p = \infty$, i.e. signal space setting (A), it follows from the Mean Value Theorem that $\|y(\cdot) - y(\cdot - h)\|_{L^\infty} \leq h\|\dot{y}\|_{L^\infty} \leq h\|y\|_{W^{1,\infty}}$, whereas in the case of $p < \infty$, i.e. signal space setting (B), again by the Mean Value Theorem, $\|y(\cdot) - y(\cdot - h)\|_{L^p} \leq h\|M_h[\dot{y}](\cdot)\|_{L^p}$ which in view of Proposition 7.2.2 below yields a bound of the form: $\|y(\cdot) - y(\cdot - h)\|_{L^p} \leq 2h\|M_h[y](\cdot)\|_{W^{2,p}}$. The requirement to bound an extra derivative then leads to the additional requirement that $k_r = 0$ (signal space setting (A)), or alternatively, that derivatives of all orders are bounded (signal space setting (C)).

Before giving the proof of Theorem 7.2.1 one has to establish the key bound which will be required in the proof of Theorem 7.2.1 for the signal space choices (B) and (C) as discussed above.

Proposition 7.2.2 *For $y \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $\varrho > 0$, define the function*

$$M_\varrho[y] \colon \mathbb{R}_{\geq 0} \to \mathbb{R}\,, \quad t \mapsto \max_{\tau \in [t-\varrho, t]} |y(\tau)|\;, \quad \text{where } y(s) = 0 \text{ if } s < 0\,. \tag{7.2.5}$$

Then, for every $y \in CW_0^{1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ *and* $p \in [1, \infty)$,

$$\forall\, T > 0 \ : \|M_\varrho[y]\|^p_{L^p([0,T)\to\mathbb{R})} \\ \leq 2\|y\|^{p-1}_{L^p([0,T)\to\mathbb{R})} \left(\|y\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\|\dot{y}\|_{L^p([0,T)\to\mathbb{R})}\right). \tag{7.2.6}$$

Proof. Let $y \in CW_0^{1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, $p \in [1, \infty)$, $T > 0$ and $\varepsilon > 0$. By the density of $C_0^2([0,T] \to \mathbb{R})$ in $C_0([0,T] \to \mathbb{R})$ it follows from [Riv69, Thm. 4.12] applied on the interval $[0,T]$ that there exists a (piecewise cubic) function $G_0 \colon [0,T] \to \mathbb{R}$ such that G_0 is nowhere locally constant, and

$$|G_0(t) - y(t)| \leq \varepsilon\,, \quad |\dot{G}_0(t) - \dot{y}(t)| \leq \varepsilon\,, \quad t \in [0,T]\,.$$

Define $G \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R})$ by $G = T_T G_0$, where T_T is the truncation operator defined in Section 6.1 for time $T > 0$. Suppose, for the time being,

$$\|M_\varrho[G]\|^p_{L^p([0,T)\to\mathbb{R})} \\ \leq 2\,\|G\|^{p-1}_{L^p([0,T)\to\mathbb{R})} \Big(\|G\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\|\dot{G}\|_{L^p([0,T)\to\mathbb{R})}\Big). \tag{7.2.7}$$

Then in view of

$$\begin{aligned} M_\varrho[y](t) &= M_\varrho[y + G - G](t) \\ &\leq M_\varrho[G](t) + M_\varrho[y - G](t) \\ &\leq M_\varrho[G](t) + M_\varrho[\varepsilon](t)\,, \end{aligned}$$

and since $M_\varrho[\varepsilon](t) = \varepsilon$ for $t \geq 0$, it follows that

$$\|M_\varrho[y]\|^p_{L^p([0,T)\to\mathbb{R})} \\ \leq 2\|G\|^{p-1}_{L^p([0,T)\to\mathbb{R})} \Big(\|G\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\|\dot{G}\|_{L^p([0,T)\to\mathbb{R})}\Big) + T\varepsilon^p\,.$$

Since

$$\begin{aligned} \|G\|_{L^p([0,T)\to\mathbb{R})} &\leq \|G - y\|_{L^p([0,T)\to\mathbb{R})} + \|y\|_{L^p([0,T)\to\mathbb{R})} \\ &\leq T^{1/p}\,\varepsilon + \|y\|_{L^p([0,T)\to\mathbb{R})}\,, \end{aligned}$$

$$\begin{aligned}\|\dot{G}\|_{L^p([0,T)\to\mathbb{R})} &\leq \|\dot{G}-\dot{y}\|_{L^p([0,T)\to\mathbb{R})} + \|\dot{y}\|_{L^p([0,T)\to\mathbb{R})} \\ &\leq T^{1/p}\,\varepsilon + \|\dot{y}\|_{L^p([0,T)\to\mathbb{R})}\,,\end{aligned}$$

it follows that

$$\begin{aligned}&\|M_\varrho[y]\|^p_{L^p([0,T)\to\mathbb{R})} \\ &\quad\leq 2\big(T^{1/p}\,\varepsilon + \|y\|_{L^p([0,T)\to\mathbb{R})}\big)^{p-1} \\ &\qquad\cdot\Big(T^{1/p}\,\varepsilon + \|y\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\big(T^{1/p}\,\varepsilon + \|\dot{y}\|_{L^p([0,T)\to\mathbb{R})}\big)\Big) + T\varepsilon^p\end{aligned}$$

As this holds for all $\varepsilon > 0$, inequality (7.2.6) follows as required.

It remains to show (7.2.7). Let

$$\mathcal{R}(G) := \big\{t \in [0,T] \;\big|\; |G(t)| \text{ is a local maximum of } |G|\big\}\,.$$

Since $|G|$ is piecewise polynomial, $G \not\equiv 0$, $\mathcal{R}(G)$ is non-empty and has a finite or countable number of elements. To every point $t \in \mathcal{R}(G)$ define

$$\begin{aligned}t^M &:= \inf\big(\{T\} \cup \big\{\tau \in [t,T] \;\big|\; |G(\tau)| \text{ is a local minimum of } |G|\big\}\big)\,,\\ t^R &:= \min\big\{t+\varrho\,, T\,,\ \inf\big\{\tau \in \mathcal{R}(G) \,\big|\, \tau > t\big\}\big\}\,.\end{aligned}$$

Next, the L^p-norm of $M_\varrho[G]$ is estimated by the L^p-norm of G and the sum of parts of the areas of the hatched boxes, see Figure 7.2. By the definition of $M_\varrho[G]$ it follows that

$$\begin{aligned}&\|M_\varrho[G]\|^p_{L^p([0,T)\to\mathbb{R})} = \int_0^T \Big(\max_{\tau\in[t-\varrho,t]} |G(\tau)|\Big)^p \,\mathrm{d}t \\ &\leq \int_0^T |G(t)|^p \,\mathrm{d}t + \sum_{t\in\mathcal{R}(G)} \Big(\big[t^R - t\big]\,|G(t)|^p + \min\{t+\varrho-t^R, T-t^R\} \\ &\qquad\qquad\qquad\qquad\qquad\cdot \max\big\{0, |G(t)|^p - |G(t^R)|^p\big\}\Big),\end{aligned}$$

where $\big(\big[t^R - t\big]\,|G(t)|^p\big)$ is the area of the hatched box of height $|G(t)|^p$ between the local maximum t and either the following local maximum t^R on the right or the minimum of the points T or $t+\varrho$. Furthermore, $\min\{t+\varrho-t^R, T-t^R\}\cdot\max\big\{0, |G(t)|^p - |G(t^R)|^p\big\}$ is the area of the box which remains by subtracting a box with the height $|G(t^R)|^p$ of the

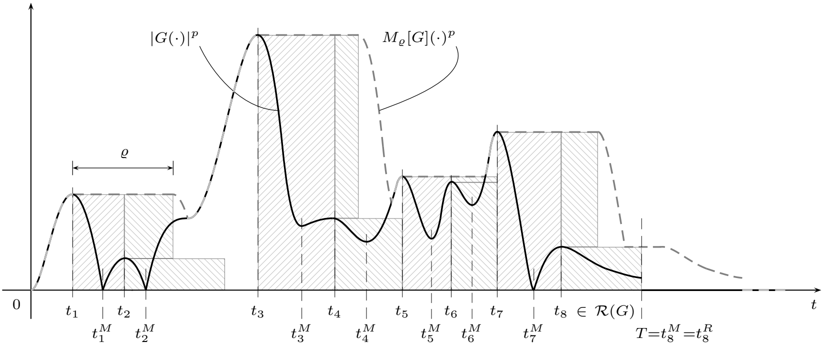


Figure 7.2: Example function $|G|^p$ and $M_\varrho[G]^p$, here: $t_1^R = t_2$, $t_2^R = t_2 + \varrho$, $t_3^R = t_4$, $t_4^R = t_5$, $t_5^R = t_6$, $t_6^R = t_7$, $t_7^R = t_8$ and $t_8^R = T = t_8^M$.

following maximum value t^R from a box with height $|G(t)|^p$ and length $\min\{t+\varrho-t^R, T-t^R\}$. Since $|G(t^R)| \geq |G(t^M)|$, $(t,t^R)\cap(s,s^R)=\emptyset$ for all $t,s\in\mathcal{R}(G)$ and $t^R \leq T$, follows

$$\sum_{t\in\mathcal{R}(G)} \big[t^R - t\big]|G(t^M)|^p \leq \sum_{t\in\mathcal{R}(G)} \int_t^{t^R} |G(t)|^p \,\mathrm{d}t \leq \int_0^T |G(t)|^p \,\mathrm{d}t\,,$$

and hence

$$\begin{aligned}
&\|M_\varrho[G]\|^p_{L^p([0,T)\to\mathbb{R})}\\
&\leq \int_0^T |G(t)|^p \,\mathrm{d}t + \sum_{t\in\mathcal{R}(G)} \Big(\big[t^R - t\big]\big(|G(t)|^p - |G(t^M)|^p\big) + \big[t^R - t\big]|G(t^M)|^p \\
&\qquad + \min\{t+\varrho-t^R, T-t^R\}\cdot \max\big\{0, |G(t)|^p - |G(t^R)|^p\big\}\Big)\\
&\leq \int_0^T |G(t)|^p \,\mathrm{d}t + \sum_{t\in\mathcal{R}(G)} \Big(\big[t^R - t\big]\big(|G(t)|^p - |G(t^M)|^p\big) + \big[t^R - t\big]|G(t^M)|^p \\
&\qquad + \big[t+\varrho-t^R\big]\big(|G(t)|^p - |G(t^M)|^p\big)\Big)\\
&\leq 2\int_0^T |G(t)|^p \,\mathrm{d}t + \varrho \sum_{t\in\mathcal{R}(G)} \big(|G(t)|^p - |G(t^M)|^p\big). \qquad (7.2.8)
\end{aligned}$$

Since $G\big|_{(t,t^M)}$ is either strictly positive or negative, $|G|$ is continuously differentiable on (t,t^M), partial integration yields

$$\begin{aligned}\sum_{t\in\mathcal{R}(G)} &\big(|G(t)|^p - |G(t^M)|^p\big) \\ &\le \sum_{t\in\mathcal{R}(G)} \int_t^{t^M} p\,|G(t)|^{p-1}\,|\dot G(t)|\,\mathrm{d}t \\ &\le p\|G^{p-1}\dot G\|_{L^1([0,T)\to\mathbb{R})}\,, \quad (7.2.9)\end{aligned}$$

where the second inequality above follows from $(t,t^M)\cap(s,s^M)=\emptyset$ for all $t,s\in\mathcal{R}(G)$ and since $t^M\le T$. Let $1<q<\infty$ satisfy $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality

$$\begin{aligned}\|G^{p-1}\dot G\|_{L^1([0,T)\to\mathbb{R})} &\\ &\le \|G^{p-1}\|_{L^q([0,T)\to\mathbb{R})}\|\dot G\|_{L^p([0,T)\to\mathbb{R})} \\ &= \|G\|^{p-1}_{L^p([0,T)\to\mathbb{R})}\|\dot G\|_{L^p([0,T)\to\mathbb{R})}. \quad (7.2.10)\end{aligned}$$

Finally, inequalities (7.2.8), (7.2.9) and (7.2.10) give the claimed inequality (7.2.7) and the proof is complete. □

The following inequalities for the norms of vector valued L^p-functions are required to prove Theorem 7.2.1.

Lemma 7.2.3 *Let, for $m\in\mathbb{N}$ and $p\in[1,\infty]$, $x\in L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$. Then*

$$\|x\|_{L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^m)} \le \begin{cases} 2^{\frac{p-1}{p}}\sum_{\nu=1}^m \|x_\nu\|_{L^p(\mathbb{R}_{\ge0}\to\mathbb{R})}\,, & \text{if } p<\infty,\\ \sum_{\nu=1}^m \|x_\nu\|_{L^\infty(\mathbb{R}_{\ge0}\to\mathbb{R})}\,, & \text{if } p=\infty,\end{cases} \quad (7.2.11)$$

Furthermore, let, for $r\in\mathbb{N}\cup\{\infty\}$, $x\in W^{r,p}(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$. Then

$$\sum_{\nu=1}^m \|x_\nu\|_{W^{r,p}(\mathbb{R}_{\ge0}\to\mathbb{R})} \le m\|x\|_{W^{r,p}(\mathbb{R}_{\ge0}\to\mathbb{R}^m)}\,. \quad (7.2.12)$$

Proof. Let $p=\infty$. Recall that $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$. Then, in view of the well-known inequality $\|\xi\|\le\sum_{\nu=1}^m|\xi_\nu|$ for $\xi\in\mathbb{R}^m$, it follows that

$$\begin{aligned}\|x\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} &= \operatorname*{ess\,sup}_{t\geq 0} \|x(t)\| \\ &\leq \operatorname*{ess\,sup}_{t\geq 0} \sum_{\nu=1}^m |x_\nu(t)| = \sum_{\nu=1}^m \|x_\nu\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})}.\end{aligned}$$

Let $p \in [1,\infty)$. By the definition of the L^p-norm of vector valued functions, see the nomenclature, and, for example, [AF03, Lem. 2.2] it follows that

$$\begin{aligned}\|x\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} &= \left(\int_0^\infty \|x(t)\|^p \,\mathrm{d}t\right)^{1/p} \\ &= \left(\int_0^\infty \left(\sum_{\nu=1}^m |x_\nu(t)|^2\right)^{p/2} \mathrm{d}t\right)^{1/p} \\ &\leq \left(\int_0^\infty \left(\sum_{\nu=1}^m |x_\nu(t)|\right)^{p} \mathrm{d}t\right)^{1/p} \\ &\leq \left(\int_0^\infty 2^{p-1}\left(\sum_{\nu=1}^m |x_\nu(t)|^p\right) \mathrm{d}t\right)^{1/p} \\ &\leq 2^{\frac{p-1}{p}}\left(\sum_{\nu=1}^m \|x_\nu\|^p_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\right)^{1/p} \\ &\leq 2^{\frac{p-1}{p}} \sum_{\nu=1}^m \|x_\nu\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})},\end{aligned}$$

which yields (7.2.11). Inequality (7.2.12) from the definition of the $W^{r,p}$-norm: write for brevity $\|\cdot\|_{W^{r,p}} = \|\cdot\|_{W^{r,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})}$ and $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}$. Then $\sum_{\nu=1}^m \|x_\nu\|_{W^{r,p}} = \sum_{\nu=1}^m \sum_{i=0}^r \|x_\nu\|_{L^p} \leq \sum_{i=0}^r m\,\|x\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} = m\,\|x\|_{W^{r,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}$, and the proof is complete. □

With the crucial property of continuous differential shown by Proposition 7.2.2 one is now in a position to prove Theorem 7.2.1.

Proof of Theorem 7.2.1. Let $p \in [1,\infty]$, $r \in \mathbb{N}$, signal spaces $\mathcal{U}_0$, $\mathcal{Y}_0$

and $\mathcal{W}_0$ of type (A), (B) or (C) given by (7.2.2), and $k = (k_1, \dots, k_r) \in \mathbb{R}^{1\times r}$, $k_r = 0$ in case (B). First, it is claimed that if $h \in (0, h^*)$, then

$$\vec{\delta}\left(C_k, C_k^{\text{Euler}}[h]\right) \leq h \sum_{i=1}^{r-1} |k_{i+1}| \cdot i\eta_{p,m}(h,i)\,, \tag{7.2.13}$$

and hence,

$$\begin{aligned}\vec{\delta}(C_k, C_k^{\text{Euler}}[h]) &\overset{(7.2.13)}{\leq} h \sum_{i=0}^{r-1} |k_{i+1}| \cdot i\eta_{p,m}(h,i) \\ &\overset{(7.2.3)}{<} \gamma^{-1} = \left\|\Pi_{C_k//P}\right\|_{\mathcal{W}_0,\mathcal{W}_0}^{-1}.\end{aligned}$$

By assumption P is causal. It is easy to see that C_k and $C_k^{\text{Euler}}[h]$ are causal, see, for example, [Kur05]. Moreover, by assumption $P(0) = C_k(0) = C_k^{\text{Euler}}[h](0) = 0$, $[P, C_k]$ is gain stable on $\mathcal{W}_0$ and $[P, C_k^{\text{Euler}}[h]]$ is either globally or regularly well posed. Finally, since $C_k^{\text{Euler}}[h](\mathcal{Y}_0) \subset \mathcal{U}_0$ it follows that $C_k^{\text{Euler}}[h]$ is causally extendible, see Section 6.3. Applying Theorem 6.5.1 with the roles of P and C interchanged one sees that (7.2.4) is a consequence of inequality (6.5.2) and inequality (7.2.13).

It remains to show (7.2.13).

Step 1: The graphs of C_k and $C_k^{\text{Euler}}[h]$ are given by

$$\mathcal{G}_{C_k} = \left\{ \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)} \\ y \end{pmatrix} \;\middle|\; \begin{array}{l} -\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)} \in \mathcal{U} \\ \text{and } y \in \mathcal{Y} \end{array} \right\} \subset \mathcal{U} \times \mathcal{Y}\,,$$

$$\mathcal{G}_{C_k^{\text{Euler}}[h]} = \left\{ \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y) \\ y \end{pmatrix} \;\middle|\; \begin{array}{l} -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y) \in \mathcal{U} \\ \text{and } y \in \mathcal{Y} \end{array} \right\} \subset \mathcal{U} \times \mathcal{Y}\,.$$

Recall that $\mathcal{U}$ and $\mathcal{Y}$ are signal spaces of functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^m$. For $u \in \mathcal{U}$ the norm of u is given by $\|u\|_{\mathcal{U}} = \left(\sum_{\nu=1}^m \|u_\nu\|_{\mathcal{U}'}^2\right)^{1/2}$, where $u_\nu \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the νth component of u and $\mathcal{U}'$ is corresponding to $\mathcal{U}$ the space of scalar valued functions.

Consider the surjective map

$$\Phi_h : \mathcal{G}_{C_k} \to \mathcal{G}_{C_k^{\text{Euler}}[h]}\,, \quad \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)} \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y) \\ y \end{pmatrix}. \tag{7.2.14}$$

Since $\left\| \left(-\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)}, y \right) \right\|_{\mathcal{U}_0 \times \mathcal{Y}_0} \geq \|y\|_{\mathcal{Y}_0}$ and

$$\begin{aligned} &\left\| \left(\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y), y \right) - \left(\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)}, y \right) \right\|_{\mathcal{W}_0} \\ &\quad = \left\| \sum_{i=1}^{r-1} k_{i+1} \left(\Delta_h^i(y) - y^{(i)} \right) \right\|_{\mathcal{U}_0} \leq \sum_{i=1}^{r-1} |k_{i+1}| \left\| \Delta_h^i(y) - y^{(i)} \right\|_{\mathcal{U}_0}, \end{aligned}$$

it follows that

$$\begin{aligned} \vec{\delta}(C_k, C_k^{\text{Euler}}[h]) &\leq \|\Phi_h - I\|_{\mathcal{W}_0, \mathcal{W}_0} \\ &\leq \sup_{y \in \mathcal{Y}_0 \setminus \{0\}} \frac{\sum_{i=1}^{r-1} |k_{i+1}| \left\| \Delta_h^i(y) - y^{(i)} \right\|_{\mathcal{U}_0}}{\|y\|_{\mathcal{Y}_0}}\,. \end{aligned} \tag{7.2.15}$$

Note that (7.2.15) holds for all signal spaces $\mathcal{U}_0$ and $\mathcal{Y}_0$ considered in (A), (B) and (C).

Step 2: Recall that, for $y \in \mathcal{Y}_0$, the definition of $\mathcal{Y}_0$ gives $y^{(i)}(0) = 0$, for all $i \in \{0, \ldots, r-1\}$, in case (A) and (B), and that $y^{(i)}(0) = 0$, for all $i \in \mathbb{N}_0$ in case (C). Also recall that by definition of Δ_h^i one has $\Delta_h^i(y)(t) = 0$ for $t < ih$. To simplify notation, without loss of generality, define $y(t) = 0$ for $t < 0$.

Let $y \in \mathcal{Y}_0$ and fix $i \in \{1, \ldots, r-1\}$. By $i+1$ applications of the Mean Value Theorem there exist, for $j \in \{1, \ldots, i\}$, functions $\xi_j^i : [0, \infty) \to \mathbb{R}^m$ with $\xi_j^i(t) \in (0, jh]^m$ and $\xi_{i+1}^{i,0} : [0, \infty) \to \mathbb{R}^m$ with $\xi_{i+1}^{i,0}(t) \in (0, ih]^m$ such that, for all components of y, i.e. for all $\nu \in \{1,\dots,m\}$ and all $t \geq 0$,

$$\begin{aligned}
&\left|\Delta_h^i(y_\nu)(t) - y_\nu^{(i)}(t)\right| \\
&= \left|\Delta_h^{i-1}\left(\frac{1}{h}\big(y_\nu(\cdot) - y_\nu(\cdot - h)\big)\right)(t) - y_\nu^{(i)}(t)\right| \\
&= \left|\Delta_h^{i-1} y_\nu^{(1)}(t - (\xi_1^i)_\nu(t)) - y_\nu^{(i)}(t)\right| \\
&\vdots \\
&= \left|\frac{1}{h}\left(y_\nu^{(i-1)}(t - (\xi_{i-1}^i)_\nu(t)) - \big(y_\nu^{(i-1)}\big)(t - (\xi_{i-1}^i)_\nu(t) - h)\right) - y_\nu^{(i)}(t)\right| \\
&= \left|y_\nu^{(i)}\big(t - (\xi_i^i)_\nu(t)\big) - y_\nu^{(i)}(t)\right| \\
&\leq ih\left|y_\nu^{(i+1)}(t - (\xi_{i+1}^{i,0})_\nu(t))\right| .
\end{aligned}$$

Furthermore, for signal spaces from case (C), there exist, for all $\mu \in \mathbb{N}$, functions $\xi_{i+1}^{i,\mu}\colon [0,\infty) \to \mathbb{R}^m$ with $\xi_{i+1}^{i,\mu}(t) \in (0, ih]^m$ such that, for all $\nu \in \{1,\dots,m\}$ and all $t \geq 0$,

$$\left|\Delta_h^i(y_\nu^{(\mu)})(t) - y_\nu^{(\mu+i)}(t)\right| \leq ih\left|y_\nu^{(\mu+i+1)}(t - (\xi_{i+1}^{i,\mu})_\nu(t))\right| .$$

Hence, in case (A) for $p = \infty$, $\mu = 0$; in case (B) for $p \in [1,\infty)$, $\mu = 0$; and in case (C) for $p \in [1,\infty)$, $\mu \in \mathbb{N}_0$; the following inequality holds for all $\nu \in \{1,\dots,m\}$,

$$\begin{aligned}
\left\|\Delta_h^i(y_\nu^{(\mu)}) - y_\nu^{(\mu+i)}\right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} &\leq ih\left\|y_\nu^{(\mu+i+1)}(\cdot - (\xi_{i+1}^{i,\mu})_\nu(\cdot))\right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} \\
&\leq ih\left\|M_{ih}[y_\nu^{(\mu+i+1)}](\cdot)\right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})},
\end{aligned}$$

whence, in view of inequality (7.2.11), see Lemma 7.2.3,

$$\begin{aligned}
&\left\|\Delta_h^i(y^{(\mu)}) - y^{(\mu+i)}\right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq ih \begin{cases} 2^{\frac{p-1}{p}} \sum_{\nu=1}^m \left\|M_{ih}[y_\nu^{(\mu+i+1)}](\cdot)\right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}, & \text{if } p < \infty, \\ \sum_{\nu=1}^m \left\|M_{ih}[y_\nu^{(\mu+i+1)}](\cdot)\right\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})}, & \text{if } p = \infty. \end{cases}
\end{aligned} \tag{7.2.16}$$

Step 3: Inequality (7.2.13) in case (A), i.e. for $\mathcal{U}_0 = CL^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $\mathcal{Y}_0 = CW_0^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, is shown.

Let $y \in \mathcal{Y}_0$. Observe that, for all $\nu \in \{1, \dots, m\}$ and all $i \in \{1, \dots, r-1\}$, $\left\|M_{ih}[y_\nu^{(i+1)}](\cdot)\right\|_{\mathcal{U}_0'} = \|y_\nu^{(i+1)}\|_{\mathcal{U}_0'}$ and $\|y^{(i+1)}\|_{\mathcal{U}_0} \leq \|y\|_{\mathcal{Y}_0}$, where $\mathcal{U}_0' = CL^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Thus it follows from inequalities (7.2.15), (7.2.16) and Lemma 7.2.3 that

$$\vec{\delta}(C_k, C_k^{\text{Euler}}[h]) \leq \sup_{y \in \mathcal{Y}_0 \setminus \{0\}} \frac{\sum_{i=1}^{r-1} |k_{i+1}|\, ih\, m \left\|y^{(i+1)}\right\|_{\mathcal{U}_0}}{\|y\|_{\mathcal{Y}_0}} \leq h\, m \sum_{i=1}^{r-1} |k_{i+1}|\, i\,.$$

This completes the proof for case of signal spaces of type (A).

Step 4: Show (7.2.13) in case (B) with $k_r = 0$, that is, for $p \in [1, \infty)$, let $\mathcal{U}_0 = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $\mathcal{Y}_0 = CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$.

Let $y \in \mathcal{Y}_0$, $i \in \{1, \dots, r-2\}$. Since $k = (k_1, \dots, k_{r-1}, 0) \in \mathbb{R}^{1\times r}$ and $y^{(i+1)} \in CW_0^{1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, it follows from (7.2.16), Proposition 7.2.2 and Lemma 7.2.3 that

$$\begin{aligned}
\left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0} &\leq ih\, 2^{\frac{p-1}{p}} \sum_{\nu=1}^m \left\|M_{ih}[y_\nu^{(i+1)}](\cdot)\right\|_{\mathcal{U}_0'} \\
&\leq ih\, 2^{\frac{p-1}{p}} \sum_{\nu=1}^m \left(2\, \|y_\nu^{(i+1)}\|_{\mathcal{U}_0'}^{p-1} \left(\|y_\nu^{(i+1)}\|_{\mathcal{U}_0'} + ihp\|y_\nu^{(i+2)}\|_{\mathcal{U}_0'}\right)\right)^{1/p} \\
&\leq ih\, 2^{\frac{p-1}{p}} \sum_{\nu=1}^m 2^{1/p}(1 + ihp)^{1/p} \|y_\nu\|_{\mathcal{Y}_0'} \\
&\leq 2\, m\, ih(1 + ihp)^{1/p} \|y\|_{\mathcal{Y}_0}\,,
\end{aligned} \tag{7.2.17}$$

where $\mathcal{U}_0' = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $\mathcal{Y}_0' = CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Then (7.2.15) and (7.2.17) yield

$$\vec{\delta}(C_k, C_k^{\text{Euler}}[h]) \leq 2m\, h \sum_{i=0}^{r-2} |k_{i+1}| \cdot i(1 + ihp)^{1/p} \overset{k_r=0}{=} 2m\, h \sum_{i=0}^{r-1} |k_{i+1}| \cdot i(1 + ihp)^{1/p}\,.$$

This completes the proof for signal spaces of type (B) with $k_r = 0$.

Step 5: Show (7.2.13) in case (C), i.e. for $p \in [1,\infty)$ let $\mathcal{U}_0 = \mathcal{Y}_0 = CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$.

Let $y \in \mathcal{Y}_0$. Note that $(\Delta_h^i(y))^{(\mu)} = \Delta_h^i(y^{(\mu)})$ for all $i \in \{1,\ldots,r-1\}$ and all $\mu \in \mathbb{N}_0$. For brevity write $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}$. Then it follows from Proposition 7.2.2, inequality (7.2.16) and Lemma 7.2.3 that, setting $\mathcal{Y}_0' = CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, for all $i \in \{1,\ldots,r-1\}$,

$$
\begin{aligned}
&\left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0} \\
&= \sum_{\mu=0}^{\infty} \left\|\Delta_h^i(y^{(\mu)}) - y^{(i+\mu)}\right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq \sum_{\mu=0}^{\infty} \left(ih\, 2^{\frac{p-1}{p}} \sum_{\nu=1}^{m} \left\|M_{ih}\left[y_\nu^{(i+1+\mu)}\right](\cdot)\right\|_{L^p} \right) \\
&\leq ih\, 2^{\frac{p-1}{p}} \sum_{\mu=0}^{\infty} \left(\sum_{\nu=1}^{m} \Big[2\, \|y_\nu^{(\mu+i+1)}\|_{L^p}^{p-1} \right. \\
&\qquad\qquad \left. \Big(\|y_\nu^{(\mu+i+1)}\|_{L^p} + ihp\|y_\nu^{(\mu+i+2)}\|_{L^p}\Big) \Big]^{1/p} \right) \\
&\leq 2\, ih \sum_{\mu=0}^{\infty} \sum_{\nu=1}^{m} \left(\|y_\nu^{(\mu+i+1)}\|_{L^p} + ihp \|y_\nu^{(\mu+i+1)}\|_{L^p}^{\frac{p-1}{p}} \|y_\nu^{(\mu+i+2)}\|_{L^p}^{1/p} \right) \\
&= 2\, ih \sum_{\nu=1}^{m} \left(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu+i+1)}\|_{L^p} + ihp \sum_{\mu=0}^{\infty} \|y_\nu^{(\mu+i+1)}\|_{L^p}^{\frac{p-1}{p}} \|y_\nu^{(\mu+i+2)}\|_{L^p}^{1/p} \right) \\
&\leq 2\, ih \sum_{\nu=1}^{m} \left(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu+i+1)}\|_{L^p} \right. \\
&\qquad\qquad \left. + ihp \Big(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu+i+1)}\|_{L^p}^{\frac{p-1}{p}} \Big) \Big(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu+i+2)}\|_{L^p}^{1/p} \Big) \right) \\
&\leq 2\, ih \sum_{\nu=1}^{m} \left(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu)}\|_{L^p} + ihp \Big(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu)}\|_{L^p}^{\frac{p-1}{p}} \Big) \Big(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu)}\|_{L^p}^{1/p} \Big) \right) \\
&\leq 2\, ih \Big[\sum_{\nu=1}^{m} (1+ihp) \Big(\sum_{\mu=0}^{\infty} \|y_\nu^{(\mu)}\|_{L^p} \Big),
\end{aligned}
$$

thus, for all $y \in \mathcal{Y}_0$ and $i \in \{1, \ldots, r-1\}$,

$$\left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0} = 2\, ih(1+ihp) \sum_{\nu=1}^{m} \|y_\nu\|_{\mathcal{Y}_0'} = 2\, m\, ih(1+ihp)\|y\|_{\mathcal{Y}_0},$$

and so (7.2.15) yields

$$\vec{\delta}(C_k, C_k^{\mathrm{Euler}}[h]) \leq 2m\, h \sum_{i=0}^{r-1} |k_{i+1}| \cdot i(1+ihp)\,.$$

which completes the proof for signal spaces of type (C) and concludes the proof of the theorem. □

Note that Theorem 7.2.1 requires the following assumptions to the plant P: 1) P must be causal, 2) P must be gain stabilizable by the controller C_k, in other words the closed-loop system $[P, C_k]$ must be gain stable on $\mathcal{W}_0$ for some signal spaces $\mathcal{W}_0$, and 3) the closed-loop system $[P, C_k^{\mathrm{Euler}}[h]]$ must be either globally or regularly well posed. For example, linear minimum phase MIMO-systems (A, B, C) with strict relative degree r and positive definite high-frequency matrix of form (2.2.1) satisfy these assumptions. This is partially shown in Section 3.2: the plant operator $P(A, B, C; x^0)$ corresponding to an initial value problem (7.3.1) with $x^0 \in \mathbb{R}^n$ is causal, and Theorem 3.2.1 shows that, for suitable $k \in \mathbb{R}^{1\times r}$, the controller C_k stabilizes any (A, B, C) (in sense of exponential stability of the closed-loop initial value problem with zero disturbances $u_0 = y_0 \equiv 0$) that is minimum phase, strict relative degree r and positive definite high-frequency matrix. It remains to show that the closed-loop $[P(A, B, C; x^0), C_k]$ is gain stable on $\mathcal{W}_0$, see Subsection 7.3.2, and that the closed-loop $[P(A, B, C; x^0), C_k^{\mathrm{Euler}}[h]]$ of linear system and output delay feedback is globally or regularly well posed, see Subsection 7.3.3.

7.3 Applications to linear minimum phase systems

The main result from the previous chapter, Theorem 7.2.1, is stated for various signal spaces (7.2.2). Now, consider linear systems in detail to illustrate how the choice of signal space is determined by relative degree assumptions on the linear system and the stabilizability requirements in the various signal spaces. In particular consider, for $n, m, r \in \mathbb{N}$, $n \geq rm$, the class $\mathcal{M}_{n,m,r}$ of all triples $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ corresponding to n-dimensional, minimum phase, m-input, m-output systems with known relative degree $r \in \mathbb{N}$ and positive definite high-frequency gain matrix $CA^{r-1}B \in \mathbb{R}^{m\times m}$ which has m linearly independent eigenvectors (note that $\mathcal{M}_{n,m,r}$ is more general than the class of relative degree one systems $\widetilde{\mathcal{M}}_{n,m}$ considered for λ-tracking and funnel control, see Sections 4.1 and 5.1). Let $(A, B, C) \in \mathcal{M}_{n,m,r}$, $x^0 \in \mathbb{R}^n$ and $P(A, B, C; x^0)\colon \mathcal{U}_e \to \mathcal{Y}_e$ be the associated plant operator $u_1 \mapsto y_1$ given by

$$\left.\begin{aligned} \dot{x} &= A\,x + B\,u_1\,, \qquad x(0) = x^0\,, \\ y_1 &= C\,x\,, \end{aligned}\right\} \tag{7.3.1}$$

where $\mathcal{U}$ and $\mathcal{Y}$ are any of the input/output signal spaces pairs given in (7.2.2).

Stability properties for both the nominal closed-loop initial value problem $[P(A, B, C; x^0), C_k]$ and the closed-loop initial value problem with the delay based controller $[P(A, B, C; x^0), C_k^{\mathrm{Euler}}[h]]$ are established. First, recall the exponential stability result for closed-loop initial value problems $[P(A, B, C; x^0), C_k]$ with zero disturbances $u_0 = y_0 \equiv 0$ from Chapter 3. There it is shown that the high-gain output derivative feedback controller

$$C_{k,\kappa}\colon \mathcal{Y}_e \to \mathcal{U}_e\,, \quad y_2 \mapsto u_2 = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1} y_2^{(i)}\,, \tag{7.3.2}$$

where $k = (k_1, \dots k_r) \in R^{1\times r}$ is such that $k_r = 1$ and the polynomial $s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i$ is Hurwitz, i.e. has all roots in $\mathbb{C}_-$, stabilizes any

$(A, B, C) \in \mathcal{M}_{n,m,r}$ provided $\kappa \geq 1$ is suitably large, see Theorem 3.2.1.

Note for better readability that it is always referred to the closed-loop equations (6.2.1) when any closed-loop $[P, C]$ is considered in the present section.

7.3.1 Exponential stability of the 'derivative closed-loop' $[P(A, B, C; x^0), C_{k,\kappa}]$ with $u_0 = y_0 \equiv 0$

The following corollary is a direct consequence of Theorem 3.2.1: the statement of Theorem 3.2.1 is rewritten such that it can be applied to the following results of the present section.

Corollary 7.3.1 *Let, for $n, m, r \in \mathbb{N}$ with $n \geq rm$, $(A, B, C) \in \mathcal{M}_{n,m,r}$ and $x^0 \in \mathbb{R}^n$. Suppose $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r}$ with $k_r = 1$ and $s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i$ Hurwitz. Then, there exists $\kappa^* \geq 1$ such that for all $\kappa > \kappa^*$ the closed-loop initial value problem $[P(A, b, c; x^0), C_{k,\kappa}]$ given by* (7.3.1), (7.3.2), (6.2.1) *with $u_0 \equiv y_0 = 0$ is exponentially stable, in the sense*

$$\exists\, \kappa^* \geq 1\ \exists\, M > 0\ \exists\, \alpha > 0\ \forall\, \kappa \geq \kappa^*\ \forall\, t \geq 0\ \forall\, x^0 \in \mathbb{R}^n\ : \quad \big\|x(t; x^0)\big\| \leq M e^{-\alpha t} \|x^0\|, \tag{7.3.3}$$

where $x(\cdot\,; x^0)$ denotes, for initial condition $x(0) = x^0$, the solution of the closed-loop initial value problem (7.3.1), (7.3.2), (6.2.1) *with $u_0 \equiv y_0 = 0$. Therefore, for every system $(A, B, C) \in \mathcal{M}_{n,m,r}$ of form* (7.3.1), *one may choose $\widetilde{K} \in \mathbb{R}^{m\times rm}$ such that*

$$\operatorname{spec}\Big(A + B\widetilde{K}[C/\ldots/CA^{r-1}]\Big) \subset \mathbb{C}_- \,. \tag{7.3.4}$$

Proof. In view of the closed-loop equations (6.2.1), in particular $y_2 = y_0 - y_1$ and $u_2 = u_0 - u_1$, and the $u_0 = y_0 \equiv 0$, the first statement (7.3.4) follows directly form Theorem 3.2.1.

With the transformation matrix $\overline{U} \in \mathbb{R}^{n\times n}$ given by Corollary 2.2.5 and $\widetilde{K} = \big[\widetilde{k}_1 I_m, \ldots, \widetilde{k}_r I_m\big]$ with $\widetilde{k}_i$ defined by $\widetilde{k}_i := \kappa^{r+2-i} k_i$, $i \in$

$\{1,\ldots,r\}$, (7.3.4) follows from

$$\operatorname{spec}\left(A+B\widetilde{K}[C/\ldots/CA^{r-1}]\right) = \operatorname{spec}\left(\overline{U}A\overline{U}^{-1}+\overline{U}B\big[\widetilde{K}\mid 0_{m\times(n-rm)}\big]\right)\subset\mathbb{C}_-\,,$$

and the proof is complete. □

In the remainder of this section most results are considered for signal spaces of type (A), (B) and/or (C) from (7.2.2). When presenting results in case of signal spaces of type (B) it is assumed, see Theorem 7.2.1, that $\widetilde{k}_r=0$. In this case one cannot refer to Corollary 7.3.1 and so for all results in case of signal spaces of type (B) the matrix $\widetilde{K}\in\mathbb{R}^{m\times rm}$ is chosen such that $\operatorname{spec}\left(A+B\widetilde{K}[C/\ldots/CA^{r-1}]\right)\subset\mathbb{C}_-$.

7.3.2 Stability properties of the 'derivative closed-loop' $[P(A,B,C;x^0),C_k]$

Now, for $(A,B,C)\in\mathcal{M}_{n,m,r}$, $n,m,r\in\mathbb{N}$ with $n\geq rm$, and $k\in\mathbb{R}^{1\times r}$ with the assumption that, for $K=[k_1I_m,\ldots,k_rI_m]\in\mathbb{R}^{m\times rm}$, it holds that $\operatorname{spec}\left(A+BK[C/\ldots/CA^{r-1}]\right)\subset\mathbb{C}_-$, and for appropriate input/output signal spaces of types (A), (B) or (C) in (7.2.2), it is shown that, if $x^0=0$, then the closed-loop system $[P(A,B,C;x^0),C_k]$ is gain stable on $\mathcal{W}_0$. For the input/output signal spaces of type (A) or (B) only, it is also shown that the closed-loop initial value problem $[P(A,B,C;x^0),C_k]$ is $\mathcal{W}$-stable for any initial conditions $x^0\in\mathbb{R}^n$.

Theorem 7.3.2 *Let, for $n,m,r\in\mathbb{N}$ with $n\geq rm$, $(A,B,C)\in\mathcal{M}_{n,m,r}$ given by* (7.3.1) *and choose $k=(k_1,\ldots,k_r)\in\mathbb{R}^{1\times r}$ with the matrix $K=[k_1I_m,\ldots,k_rI_m]\in\mathbb{R}^{m\times rm}$ such that* $\operatorname{spec}\left(A+BK[C/\ldots/CA^{r-1}]\right)\subset\mathbb{C}_-$*. Let signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$ and $\mathcal{W}_0$ be of type (A), (B) or (C) in* (7.2.2)*; in case (B) suppose $k_r=0$. Consider the controller operator $C_k\colon\mathcal{Y}_e\to\mathcal{U}_e$, as defined by* (7.1.1) *and the associated plant operator $P(A,B,C;x^0)\colon\mathcal{U}_e\to\mathcal{Y}_e$ with initial value $x^0\in\mathbb{R}^n$ as defined by* (7.3.1)*. Then the closed-loop system $[P(A,B,C;0),C_k]$ is $\mathcal{W}_0$-gain stable. In the case of signal spaces given by (A) or (B), the closed-loop system $[P(A,B,C;x^0),C_k]$ is also $\mathcal{W}$-stable.*

Proof. *Step 1*: Consider $\mathcal{W}$ of type (A), (B) or (C) given by (7.2.2) and let $(u_0, y_0) \in \mathcal{W}$.

The closed-loop initial value problem $[P(A,B,C;x^0), C_k]$ given by equations (7.3.1), (7.1.1), (6.2.1) is, in view of the coordinate transformation $\binom{\xi}{\eta} = \overline{U}x$, where $\overline{U}$ is given by Corollary 2.2.5, equivalent to (2.2.22), (7.1.1), (6.2.1). Invoking Corollary 2.2.5 and applying Variation of Constants yields

$$\forall\, t \geq 0 \;:\; \begin{pmatrix} \xi \\ \eta \end{pmatrix}(t) = e^{\overline{U}\left(A+BK\begin{bmatrix} C \\ \vdots \\ CA^{r-1}\end{bmatrix}\right)\overline{U}^{-1}t} \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix} + \int_0^t e^{\overline{U}\left(A+BK\begin{bmatrix} C \\ \vdots \\ CA^{r-1}\end{bmatrix}\right)\overline{U}^{-1}(t-s)} \varphi(s)\mathrm{d}s\,, \quad (7.3.5)$$

where $\binom{\xi^0}{\eta^0} = \overline{U}x^0$ and, in view of $u_0 \in \mathcal{U}$ and $y_0 \in \mathcal{Y}$,

$$\varphi(\cdot) := \begin{pmatrix} 0_{(r-1)m\times m} \\ CA^{r-1}B \\ 0_{(n-rm)\times m} \end{pmatrix} \big[u_0(\cdot) + \big(C_k y_0\big)(\cdot)\big] \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)\,. \quad (7.3.6)$$

Step 2: Consider case (A) or (B), i.e. $\mathcal{U} \times \mathcal{Y} = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $p \in [1,\infty]$, $n \geq rm$. Taking norms in (7.3.5) and invoking the well-known inequality $\big\| \int_0^{\cdot} f(\cdot - s)\, g(s)\, \mathrm{d}s \big\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$, for $f \in L^1$ and $g \in L^p$, one obtains, for some $\beta_1, \beta_2 > 0$,

$$\begin{aligned} &\left\| \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^n)} \\ &\leq \beta_1 \left[\left| \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix} \right| + \|\varphi\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^n)} \right] \\ &\leq \beta_1 \left| \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix} \right| + \beta_1\beta_2 \left[\|u_0\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \sum_{i=0}^{r-1} |k_{i+1}| \big\|y_0^{(i)}\big\|_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \right] \end{aligned}$$

and thus,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)\,.$$

Now, by (2.2.22),

$$y_1^{(i)} = \xi_{i+1} \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\,, \qquad \text{for } i = 0, \ldots, r-1$$
$$y_1^{(r)} = \dot{\xi}_r = \left(\sum_{i=1}^{r} \left(R_i - CA^{r-1}Bk_i\right) \xi_i \right) + S\eta + \varphi \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$$

and with (7.3.6) it follows that $y_1 \in CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) = \mathcal{Y}$.

Finally,

$$u_1 = u_0 - C_k(y_2) = u_0 - C_k(y_0) + C_k(y_1) \in CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) = \mathcal{U}\,,$$

which shows that the closed-loop system $[P(A,B,C;x^0), C_k]$ is $\mathcal{W}$-stable for signal spaces of types (A) and (B).

Step 3: Let $x^0 = 0$ and let $\mathcal{W}_0$ be as in (A) or (B), i.e. $\mathcal{W}_0 = \mathcal{U}_0 \times \mathcal{Y}_0 = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $p \in [1,\infty]$, $n \geq rm$. It is straightforward to see that $y^{(i)}(0) = 0$ for $i = 0, \ldots, r$, and hence one can show similarly as in Step 2 that, for some $\beta_1, \ldots, \beta_5 \geq 1$,

$$\begin{aligned}
&\|y_1\|_{CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \\
&\quad \leq \beta_1\beta_2\beta_3 \left[\|u_0\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} + \sum_{i=0}^{r-1} |k_{i+1}| \|y_0^{(i)}\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \right] \\
&\quad \leq \beta_4 \left[\|u_0\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} + \|y_0\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \right]
\end{aligned}$$

and

$$\begin{aligned}
&\|u_1\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \\
&\quad \leq \|u_0\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} + \|C_k y_2\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \\
&\quad \leq \|u_0\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} + \beta_5 \|y_2\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \\
&\quad \leq \|u_0\|_{CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \\
&\qquad + \beta_5 \left[\|y_1\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} + \|y_0\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)} \right].
\end{aligned}$$

Therefore, $\mathcal{W}_0$-gain stability for signal spaces of types (A) and (B) follows.

Step 4: Let $x^0 = 0$ and let $\mathcal{W}_0$ be as in (C), i.e. $\mathcal{W}_0 = \mathcal{U}_0 \times \mathcal{Y}_0 = CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $p \in [1, \infty)$. First note that $\varphi \in CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. By [Yos80, Prop. VI.3.1] one has, for all $i \in \mathbb{N}$ and all $t \geq 0$,

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i} \int_0^t e^{\overline{U}\left(A+BK\begin{bmatrix} C \\ \vdots \\ CA^{r-1}\end{bmatrix}\right)\overline{U}^{-1}(t-s)} \varphi(s)\mathrm{d}s = \int_0^t e^{\overline{U}\left(A+BK\begin{bmatrix} C \\ \vdots \\ CA^{r-1}\end{bmatrix}\right)\overline{U}^{-1}(t-s)} \varphi^{(i)}(s)\mathrm{d}s.$$

Hence it follows from (7.3.5) that $\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\begin{smallmatrix}\xi \\ \eta\end{smallmatrix}\right)\Big|_{t=0} = 0$, and so $y^{(i)}(0) = 0$ for all $i \in \mathbb{N}$. It follows also that $u_1^{(i)}(0) = u_0^{(i)}(0) - C_k(y_0^{(i)}(0)) + C_k(y_1^{(i)}(0)) = 0$ for all $i \in \mathbb{N}$. One can then show similarly as in Step 3 that, for some $\beta_1, \beta_2 \geq 1$,

$$\begin{aligned}
\|y_1\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} &= \sum_{j\geq 0} \|y_1\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq \beta_1 \sum_{j\geq 0}\left[\left\|u_0^{(j)}\right\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \left\|\left(C_k y_0\right)^{(j)}\right\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}\right] \\
&\leq \beta_1 \|u_0\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \beta_2 \sum_{j\geq 0}\sum_{i=0}^{r-1} \left\|y_0^{(i+j)}\right\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq \beta_1 \|u_0\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + r\beta_2 \|y_0\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} .
\end{aligned}$$

An analogous inequality for $\|u_1\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}$ gives $\mathcal{W}_0$-gain stability as required. This completes the proof of the theorem. □

Theorem 7.3.2 shows in combination with Corollary 7.3.1, for signal spaces of type (A) or (C) in (7.2.2), that if $n \geq rm$ and $\kappa^* \geq 1$ is sufficiently large then $[P(A, B, C; x^0), C_{k,\kappa}]$ is $\mathcal{W}_0$-gain stable for all $\kappa > \kappa^*$, with a bound for the gain given by

$$\left\|\Pi_{C_{k,\kappa}//P(A,B,C;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} \leq \beta(k,\kappa)\,, \tag{7.3.7}$$

for some $\beta(k, \kappa) > 0$ determined by the proof of Theorem 7.3.2 and Corollary 7.3.1.

In the signal space setting of type (B) in (7.2.2), i.e. $p, r < \infty$, these stability results are only proved for $k_r = 0$, thus precluding the application of Corollary 7.3.1. Since Corollary 7.3.1 gives stabilizability of plants (A, B, C) in $\mathcal{M}_{n,m,r}$, $n \geq rm$, and since the signal space setting (A) is only applicable when $p = \infty$, the setting (C) has been introduced to allow stability results in the context of $p < \infty$, without the assumption that $k_r = 0$ as in (B). However, the setting (C) does introduce extra regularity requirements on the external disturbances u_0, y_0.

7.3.3 Gain stability of the 'delay closed-loop' with zero initial conditions: $[P(A,B,C;0), C_k^{\mathsf{Euler}}[h]]$

To apply Theorem 7.2.1 one has to show that, for initial condition $x^0 = 0$, the closed-loop system $[P(A, B, C; 0), C_k^{\mathrm{Euler}}[h]]$ is globally or regularly well posed. Note that, for $(A, B, C) \in \mathcal{M}_{n,m,r}$ and $x^0 \in \mathbb{R}^n$, the closed-loop $[P(A, B, C; x^0), C_k^{\mathrm{Euler}}[h]]$ may be written as delay differential equation as follows:

$$\left.\begin{aligned}
\dot{x}(t) &= \left(A + \widetilde{A}^0\right) x(t) + \sum_{j=1}^{r-1} \widetilde{A}^j x(t - jh) \\
&\qquad + B u_0(t) + \sum_{j=0}^{r-1} \widetilde{B}^j y_0(t - jh)\,, \quad x \equiv \varphi \text{ on } [(1-r)h, 0]\,, \\
y_1(t) &= C\, x(t)\,, \\
u_1(t) &= u_0(t) + \sum_{j=0}^{r-1} (-1)^j \sum_{i=j}^{r-1} \tfrac{k_{i+1}}{h^i} \tbinom{i}{j} (y_0 - y_1)(t - jh)
\end{aligned}\right\} \tag{7.3.8}$$

where $\varphi(0) = x^0$ and, in view of the normal form (2.2.22) of (A, B, C), for $j = 0, \ldots, r-1$,

$$\widetilde{A}^j := (-1)^{j+1} C A^{r-1} B \sum_{i=j}^{r-1} \frac{k_{i+1}}{h^i} \binom{i}{j} \overline{U}^{-1} \left[\begin{array}{c|c} \begin{matrix} 0_{(r-1)m \times m} \\ I_m \\ 0_{(n-rm) \times m} \end{matrix} & 0_{n \times (n-m)} \end{array}\right] \overline{U}\,,$$

$$\widetilde{B}^j := (-1)^j \sum_{i=j}^{r-1} \frac{k_{i+1}}{h^i} \binom{i}{j} B\,,$$

where the transformation matrix $\overline{U} \in \mathbb{R}^{n \times n}$ is given by Corollary 7.3.1.

Lemma 7.3.3 *Let, for $n, m, r \in \mathbb{N}$ with $n \geq rm$, $(A, B, C) \in \mathcal{M}_{n,m,r}$ and $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r}$. Let the signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ be of type (A), (B) or (C) in* (7.2.2)*. Then the closed-loop system $[P(A, B, C; 0), C_k^{Euler}[h]]$ which is given by the application of the delay feedback controller operator $C_k^{Euler}[h] \colon \mathcal{Y}_{0e} \to \mathcal{U}_{0e}$ defined in* (7.1.2)*, onto the plant operator $P(A, B, C; 0) \colon \mathcal{U}_{0e} \to \mathcal{Y}_{0e}$ given by* (7.3.1)*, is globally well posed on $\mathcal{W}$.*

Proof. In view of (7.3.8) with $\varphi \equiv 0$ on $[(1-r)h, 0]$, $u_0, y_0 \in \mathcal{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and the additional assumption that u_0, y_0 can be continuously extended to functions in $\mathcal{C}([(1-r)h, \infty) \to \mathbb{R}^m)$, the lemma follows directly from [HVL93, Thm. 1.2.1]. □

Now one is in the position to show that linear minimum phase MIMO-systems $(A, B, C) \in \mathcal{M}_{n,m,r}$, $n, m, r \in \mathbb{N}$, $n \geq rm$, are gain stabilizable on $\mathcal{U} \times \mathcal{Y}$ by the delay feedback $C_k^{\mathrm{Euler}}[h]$ defined in (7.1.2), for suitable $k \in \mathbb{R}^{1\times r}$, for sufficiently small $h > 0$ and for signal spaces of type (A), (B) or (C) in (7.2.2).

Theorem 7.3.4 *Let, for $n, m, r \in \mathbb{N}$ with $n \geq rm$, $(A, B, C) \in \mathcal{M}_{n,m,r}$ and choose $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r}$ with $K = [k_1 I_m, \ldots, k_r I_m] \in \mathbb{R}^{m\times rm}$ such that $\operatorname{spec}\big(A + BK[C/\ldots/CA^{r-1}]\big) \subset \mathbb{C}_-$. Let the signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ be of type (A), (B) or (C) in* (7.2.2)*; in case (B) suppose $k_r = 0$. Then $\gamma := \big\|\Pi_{C_k//P(A,B,C;0)}\big\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty$. Suppose $h \in (0, h^*)$, where $h^* > 0$ satisfies* (7.2.3)*. Then the delay feedback controller $C_k^{Euler}[h] \colon \mathcal{Y}_{0e} \to \mathcal{U}_{0e}$, defined in* (7.1.2)*, applied to the plant $P(A, B, C; 0) \colon \mathcal{U}_{0e} \to \mathcal{Y}_{0e}$ given by* (7.3.1) *yields*

$$\big\|\Pi_{C_k^{Euler}[h]//P(A,B,C;0)}\big\|_{\mathcal{W}_0,\mathcal{W}_0} \leq \gamma \frac{1 + h \sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_{p,m}(h,i)}{1 - h\gamma \sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_{p,m}(h,i)} . \tag{7.3.9}$$

Proof. By Theorem 7.3.2 follows $\gamma := \big\|\Pi_{C_k//P(A,B,C;0)}\big\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty$. The result now follows from Theorem 7.2.1 since, by Lemma 7.3.3, the closed-loop $[P(A, B, C; 0), C_k^{\mathrm{Euler}}[h]]$ is globally well posed and, moreover, $P(0) = 0$. □

In the following let $C_{k,\kappa}^{\text{Euler}}[h]\colon y_2 \mapsto u_2 = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1} \Delta_h^i y_2$ be the delay feedback controller corresponding to controller $C_{k,\kappa}$ given in (7.3.2).

Together with Corollary 7.3.1 and Theorem 7.3.2, Theorem 7.3.4 shows for signal spaces of type (A) or (C) in (7.2.2), that for sufficiently large $\kappa^* \geq 1$ (determined by Corollary 7.3.1) and any $\kappa > \kappa^*$, $\beta(k,\kappa)$ given in (7.3.7) and sufficiently small $h > 0$ (determined by Theorem 7.3.4), the closed-loop system $[P(A,b,c;0), C_{k,\kappa}^{\text{Euler}}[h]]$ is $\mathcal{W}_0$-gain stable. Moreover,

$$\exists\, \kappa^* \geq 1\ \forall\, \kappa \geq \kappa^*\ \exists\, h^* > 0\ \forall\, h \in (0,h^*)\ :$$
$$\left\| \Pi_{C_{k,\kappa}^{\text{Euler}}[h]//P(A,B,C;0)} \right\|_{\mathcal{W}_0,\mathcal{W}_0} \leq \beta(k,\kappa) \frac{1 + h\kappa \sum\limits_{i=1}^{r-1} \kappa^{r-i} |k_{i+1}|\, i\eta_{p,m}(h,i)}{1 - h\beta(k,\kappa)\kappa \sum\limits_{i=1}^{r-1} \kappa^{r-i} |k_{i+1}|\, i\eta_{p,m}(h,i)}\,.$$

7.3.4 Gain stability of the ‘delay closed-loop’ with non-zero initial conditions: $[P(A,B,C;x^0), C_k^{\text{Euler}}[h]]$

To generalize Theorem 7.3.4 by allowing for non-zero initial conditions, the following theorem, which incorporates non-zero initial conditions and will be applied to signal spaces of type (A) or (B) in (7.2.2), is presented. The proof of Theorem 7.3.5 is based on an extension of Theorem 6.5.4.

Theorem 7.3.5 *Let $n,m,r \in \mathbb{N}$ with $n \geq rm$ and consider signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ of type (A) or (B) in* (7.2.2)*. Let $k = (k_1,\ldots,k_r) \in \mathbb{R}^{1\times r}$ and additionally in case (B) suppose $k_r = 0$. Let $(A,B,C) \in \mathcal{M}_{n,m,r}$, $x^0 \in \mathbb{R}^n$ and consider the operator $P(A,B,C;x^0)\colon \mathcal{U}_e \to \mathcal{Y}_e$ as defined in* (7.3.1)*. Suppose that for $h > 0$, applying the feedback controllers*

$$C_k\colon \mathcal{Y}_e \to \mathcal{U}_e \qquad \text{and} \qquad C_k^{Euler}[h]\colon \mathcal{Y}_{0e} \to \mathcal{U}_e$$

as defined in (7.1.1) *and* (7.1.2)*, respectively, to* $P(A,B,C;0)$ *yields*

$$\left\|\Pi_{C_k//P(A,B,C;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty \qquad \textit{and}$$
$$\left\|\Pi_{C_k^{\textit{Euler}}[h]//P(A,B,C;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} =: \gamma < \infty\,.$$

Then

$$\exists\,\lambda > 0\ \forall\,x^0 \in \mathbb{R}^n\ \forall\,w_0 \in \mathcal{W}_0\ : \quad \left\|\Pi_{C_k^{\textit{Euler}}[h]//P(A,B,C;x^0)} w_0\right\|_{\mathcal{W}} \le \lambda|x^0| + \gamma\,\|w_0\|_{\mathcal{W}_0}\,. \tag{7.3.10}$$

Proof. Note that one may consider $P(A,B,C;0)$ as an operator from $\mathcal{U}_e$ to $\mathcal{Y}_e$ or from $\mathcal{U}_e$ to $\mathcal{Y}_{0e}$. Furthermore, note that C_k and $C_k^{\text{Euler}}[h]$ may be considered as operators from $\mathcal{Y}_e$ to $\mathcal{U}$ or from $\mathcal{Y}_{0e}$ to $\mathcal{U}$. Thus one may consider the graphs of $P(A,B,C;0)$, C_k and $C_k^{\text{Euler}}[h]$ in $\mathcal{W}_0$ or in $\mathcal{W}$. To identify in which signal space a graph is considered a superscript $\mathcal{W}_0$ or $\mathcal{W}$ is added such as in $\mathcal{G}^{\mathcal{W}_0}_{P(A,B,C;0)} \subset \mathcal{W}_0$ or $\mathcal{G}^{\mathcal{W}}_{P(A,B,C;0)} \subset \mathcal{W}$. For $x^0 \neq 0$ one has to consider $P(A,B,C;x^0)$ as an operator from $\mathcal{U}_e$ to $\mathcal{Y}_e$ with $\mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)} \subset \mathcal{W}$.

Step 1: For $k = (k_1,\ldots,k_r) \in \mathbb{R}^{1\times r}$ let $K = \left[k_1 I_m,\ldots,k_r I_m\right]$. Let $y_0 \equiv 0$ and consider the map defined by $u_0 \overset{(2.2.22),\ (7.1.1),\ (6.2.1)}{\longmapsto} y_1$ with transfer matrix

$$\begin{aligned} s \mapsto G(s) &= [I_m \,|\, 0_{m\times(n-m)}] \\ &\quad\cdot (sI_n - (\overline{U}A\overline{U}^{-1} + \overline{U}B(K \mid 0_{m\times(n-rm)})))^{-1}\overline{U}B \\ &= [I_m \,|\, 0_{m\times(n-m)}] \\ &\quad\cdot \left(\overline{U}(sI_n - (A + BK[C/\ldots/CA^{r-1}]))\overline{U}^{-1}\right)^{-1}\overline{U}B\,, \end{aligned}$$

where the matrix $\overline{U}$ is given by Corollary 2.2.5.

By boundedness of $\left\|H_{P(A,B,C;0),C_k}\right\|_{\mathcal{W}_0,\mathcal{W}_0\times\mathcal{W}_0}$ and in view of [Fra87, Thm. 2.4.2] it follows that $G(\cdot)$ is stable. Since (A,B,C) is minimum phase, setting $F := K[C/\ldots/CA^{r-1}] \in \mathbb{R}^{m\times n}$, [Cop74, Thm. 10] yields that $\operatorname{spec}(A+BF) = \operatorname{spec}(A + BK[C/\ldots/CA^{r-1}]) \subset \mathbb{C}_-$.

Since $\left\|H_{P(A,B,C;0),C_k}\right\|_{\mathcal{W}_0,\mathcal{W}_0\times\mathcal{W}_0} < \infty$, one may define maps $\widetilde{N}\colon \mathcal{U} \to$

$\mathcal{U}$, $u_0 \mapsto u_1$, and $M\colon \mathcal{U} \to \mathcal{Y}_0$, $u_0 \mapsto y_1$ by

$$\widetilde{N} u_0 = (I_m \,|\, 0_{m\times m})\, \Pi_{P(A,B,C;0)//C_k} \begin{pmatrix} u_0 \\ 0 \end{pmatrix},$$
$$M u_0 = (0_{m\times m} \,|\, I_m)\, \Pi_{P(A,B,C;0)//C_k} \begin{pmatrix} u_0 \\ 0 \end{pmatrix}.$$

Then, Proposition 7.3.1 yields that the tuples $(u_0, u_1) = (u_0, \widetilde{N} u_0)$ and $(u_0, y_1) = (u_0, M u_0)$ satisfy

$$\left.\begin{aligned} \dot{x} &= (A+BF)x + Bu_0\,, \qquad x(0) = 0 \\ u_1 &= Fx + u_0\,, \\ y_1 &= Cx\,. \end{aligned}\right\} \tag{7.3.11}$$

Step 2: It is shown that $\widetilde{N}(\mathcal{U}) = \mathcal{V} := \{u \in \mathcal{U} \,|\, P(A,B,C;0)u \in \mathcal{Y}\}$.

Suppose $u \in \mathcal{V}$, i.e. $u \in \mathcal{U}$ with $P(A,B,C;0)u \in \mathcal{Y}$. Then $\mathcal{Y} \ni P(A,B,C;0)u = C\,x =: y$ for x being a solution of $\dot{x} = A\,x + B\,u$, $x(0) = 0$. Since (A,B,C) is minimum phase, thus (A,C) is detectable, there exists $L \in \mathbb{R}^n$ such that $\operatorname{spec}(A+LC) \subset \mathbb{C}_-$. Since $y \in \mathcal{Y}$ and $u \in \mathcal{U}$ writing

$$\dot{x} = (A+LC)x - LC\,x + B\,u = (A+LC)x - L\,y + B\,u$$

yields that $x \in CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)$. Thus $u_0 := u - Fx \in \mathcal{U}$ and (7.3.11) then yields that $u = u_1 = \widetilde{N}(u_0) \in \widetilde{N}(\mathcal{U})$, which gives $\widetilde{N}(\mathcal{U}) \subset \mathcal{V}$.

Conversely, suppose $u \in N(\mathcal{U})$. Then there exists $u_0 \in \mathcal{U}$ such that $u_0 = u - Fx \in \mathcal{U}$. Since $\operatorname{spec}(A+BF) \subset \mathbb{C}_-$ it follows by (7.3.11) that $P(A,B,C;0)u = y = Cx \in \mathcal{Y}$. Hence $\widetilde{N}(\mathcal{U}) \subset \mathcal{V}$.

Now $N\colon \mathcal{U} \to \mathcal{V}$, $u_0 \mapsto (I_m \,|\, 0_{m\times m})\, \Pi_{P(A,B,C;0)//C_k} \left(\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right)$, is well defined and writing

$$\begin{aligned} \dot{x} &= A\,x + B\,u_1\,, \qquad x(0) = 0 \\ u_0 &= F\,x - u_1\,, \end{aligned}$$

directly gives that N is invertible and $P(A,B,C;0) = MN^{-1}$.

Step 3: Set $\overline{A} := A + BF = (A + BK[C/\ldots/CA^{r-1}])$. Next is shown that

$$\mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)} = Q := \left\{ \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\overline{A}\cdot)x^0 \\ C\exp(\overline{A}\cdot)x^0 \end{pmatrix} \in \mathcal{W} \;\middle|\; \begin{array}{l} v \in \mathcal{U},\ N, M, F \\ \text{and } \overline{A} \text{ as in Step 1} \end{array} \right\}.$$

First, show $Q \subset \mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)}$.

Consider, for any $v \in \mathcal{U}$, $q_v = \left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right) v + \begin{pmatrix} F\exp(\overline{A}\cdot)x^0 \\ C\exp(\overline{A}\cdot)x^0 \end{pmatrix} \in Q$. Let $u = Nv + F\exp(\overline{A}\cdot)x^0$. Since $Nv \in \mathcal{U}$ and $\exp(\overline{A}\cdot) \in CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})^{n \times n}$, it follows that $u \in \mathcal{U}$. Observe

$$\dot{x} = A\,x + B(F\exp(\overline{A}\cdot)x^0)\,, \qquad x(0) = x^0 \in \mathbb{R}^n$$

has the solution $x(\cdot) = \exp(\overline{A}\cdot)x^0$. Thus it follows that

$$P(A,B,C;x^0)F\exp(\overline{A}\cdot)x^0 = C\exp(\overline{A}\cdot)x^0\,.$$

Hence

$$\begin{aligned} P(A,B,C;x^0)u &= P(A,B,C;x^0)Nv + P(A,B,C;x^0)\left(F\exp(\overline{A}\cdot)x^0\right) \\ &\quad - P(A,B,C;x^0)(0) \\ &= P(A,B,C;0)Nv + P(A,B,C;x^0)\left(F\exp(\overline{A}\cdot)x^0\right) \\ &= MN^{-1}Nv + C\exp(\overline{A}\cdot)x^0 \\ &= Mv + C\exp(\overline{A}\cdot)x^0 \in \mathcal{Y}\,. \end{aligned} \tag{7.3.12}$$

Thus $q_v = \left(\begin{smallmatrix} u \\ P(A,B,C;x^0)u \end{smallmatrix}\right) \in \mathcal{U} \times \mathcal{Y}$, so $q_v \in \mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)}$ and $Q \subset \mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)}$.

Next, show $\mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)} \subset Q$.

Consider $\left(\begin{smallmatrix} u \\ P(A,B,C;x^0)u \end{smallmatrix}\right) \in \mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)}$. Then

$$\begin{aligned} &P(A,B,C;0)\left(u - F\exp(\overline{A}\cdot)x^0\right) \\ &\qquad = P(A,B,C;x^0)u - P(A,B,C;x^0)\left(F\exp(\overline{A}\cdot)x^0\right) \in \mathcal{Y}\,. \end{aligned}$$

Therefore $u - F\exp(\overline{A}\cdot)x^0 \in \mathcal{V} = \operatorname{Im}(N)$, and so there exists $v \in \mathcal{U}$ such that $Nv = u - F\exp(\overline{A}\cdot)x^0$. Therefore, equation (7.3.12) holds, hence

$$\begin{pmatrix} u \\ P(A,B,C;x^0)u \end{pmatrix} = \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\overline{A}\cdot)x^0 \\ c\exp(\overline{A}\cdot)x^0 \end{pmatrix} \in Q$$

and so $\mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)} \subset Q$. Therefore, it is shown that $\mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)} = Q$ as claimed.

Step 4: Finally, inequality (7.3.10) is shown.

For $w_0 \in \mathcal{W}_0$ and $x^0 \in \mathbb{R}^n$ let $w_0' := w_0 - v_1 - v_2$ where

$$v_1 := \begin{pmatrix} F\exp(\overline{A}\cdot)x^0 \\ C\exp(\overline{A}\cdot)x^0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} -C_k^{\text{Euler}}[h](C\exp(\overline{A}\cdot)x^0) \\ -C\exp(\overline{A}\cdot)x^0 \end{pmatrix}.$$

Since $C_k^{\text{Euler}}[h](\mathcal{Y}) \subset \mathcal{U}$, it follows that $w_0' \in \mathcal{W}_0$, hence,

$$H_{P(A,B,C;0),C_k^{\text{Euler}}[h]}(w_0') = (w_1, w_2) \in \mathcal{G}^{\mathcal{W}_0}_{P(A,B,C;0)} \times \mathcal{G}^{\mathcal{W}_0}_{C_k^{\text{Euler}}[h]}.$$

In particular, $w_0' = w_1 + w_2$, and by rearranging one has $w_0 = (w_1 + v_1) + (w_2 + v_2)$. Since $w_1 \in \mathcal{G}^{\mathcal{W}_0}_{P(A,B,C;0)} \subset \mathcal{G}^{\mathcal{W}}_{P(A,B,C;0)}$, there exists $v \in \mathcal{U}$ such that $w_1 = \left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right) v$, hence $w_1 + v_1 \in Q = \mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)}$. Since $w_2 \in \mathcal{G}^{\mathcal{W}_0}_{C_k^{\text{Euler}}[h]} \subset \mathcal{G}^{\mathcal{W}}_{C_k^{\text{Euler}}[h]}$ and $v_2 \in \mathcal{G}^{\mathcal{W}}_{C_k^{\text{Euler}}[h]}$, it follows by linearity of $C_k^{\text{Euler}}[h]$ that $w_2 + v_2 \in \mathcal{G}^{\mathcal{W}}_{C_k^{\text{Euler}}[h]}$. Therefore, since the closed-loop $[P(A,B,C;x^0), C_k^{\text{Euler}}[h]]$ has the uniqueness property, see Lemma 7.3.3, $H_{P(A,B,C;x^0),C_k^{\text{Euler}}[h]} \colon \mathcal{W} \to \mathcal{W} \times \mathcal{W}$ is defined and

$$\begin{aligned} &H_{P(A,B,C;x^0),C_k^{\text{Euler}}[h]} w_0 \\ &\qquad = (w_1 + v_1, w_2 + v_2) \in \mathcal{G}^{\mathcal{W}}_{P(A,B,C;x^0)} \times \mathcal{G}^{\mathcal{W}}_{C_k^{\text{Euler}}[h]} \subset \mathcal{W} \times \mathcal{W}. \end{aligned}$$

For

$$\lambda := \sup_{x_0 \in \mathbb{R}^n \setminus \{0\}} \frac{\|v_2\|_{\mathcal{W}}}{\|x^0\|} = \left\| \begin{pmatrix} -C_k^{\text{Euler}}[h](C\exp(\overline{A}\cdot)) \\ -C\exp(\overline{A}\cdot) \end{pmatrix} \right\|_{L^p \times W^{r,p}}, \tag{7.3.13}$$

where $p = \infty$ in case (A) and $p \in [1, \infty)$ in case (B), it follows that

$$\begin{aligned} \left\| \Pi_{C_k^{\text{Euler}}[h]//P(A,B,C;x^0)} w_0 \right\|_{\mathcal{W} \times \mathcal{W}} &\leq \|v_2\|_{\mathcal{W}} + \|w_2\|_{\mathcal{W}} \\ &\leq \lambda \|x^0\| + \gamma \|w_0\|_{\mathcal{W}_0}, \end{aligned}$$

thus concluding the proof. □

Now one can state the robust stability result for the delay feedback controller in the presence of both input/ouput disturbances and initial conditions.

Theorem 7.3.6 *Let, for $n, m, r \in \mathbb{N}$ with $n \geq rm$, $(A, B, C) \in \mathcal{P}_{n,m,r}$ and choose $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r}$ with $K = [k_1 I_m, \ldots, k_r I_m] \in \mathbb{R}^{m\times rm}$ such that $\operatorname{spec}\left(A + BK[C/\ldots/CA^{r-1}]\right) \subset \mathbb{C}_-$. Let the signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$ and $\mathcal{W}_0$ be of type (A) or (B) in (7.2.2); in case (B) suppose $k_r = 0$. Then $\gamma := \left\|\Pi_{C_k//P(A,B,C;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty$. Suppose $h \in (0, h^*)$, where $h^* > 0$ satisfies (7.2.3). Consider for (A, B, C) the plant operator $P(A, B, C; x^0)\colon \mathcal{U}_e \to \mathcal{Y}_e$ given by (7.3.1) and the delay feedback controller $C_k^{Euler}[h]\colon \mathcal{Y}_{0e} \to \mathcal{U}_e$, defined in (7.1.2). Then there exists $\lambda > 0$ such that, for all $w_0 \in \mathcal{W}_0$,*

$$\left\|\Pi_{C_k^{Euler}[h]//P(A,B,C;x^0)} w_0\right\|_{\mathcal{W},\mathcal{W}} \leq \lambda\|x^0\| + \gamma \frac{1 + h\sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_{p,m}(h,i)}{1 - h\gamma\sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_{p,m}(h,i)} \|w_0\|_{\mathcal{W}_0}.$$

Proof. The result follows directly from Theorems 7.3.4 and 7.3.5. □

Together with Corollary 7.3.1 and Theorem 7.3.2, Theorem 7.3.6 shows for signal spaces of type (A) in (7.2.2), that for sufficiently large $\kappa^* \geq 1$ (determined by Corollary 7.3.1) and any $\kappa > \kappa^*$, $\beta(k, \kappa)$ given in (7.3.7) and for sufficiently small $h > 0$ (determined by Theorem 7.3.4) there exists $\lambda > 0$ (determined by equation (7.3.13)) such that, for all $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathcal{W}_0$:

$$\left\|\Pi_{C_{k,\kappa}^{\mathrm{Euler}}[h]//P(A,B,C;x^0)} w_0\right\|_{\mathcal{W},\mathcal{W}} \leq \lambda\|x^0\| + \left(\beta(k,\kappa)\frac{1 + h\kappa\sum_{i=1}^{r-1} \kappa^{r-i}|k_{i+1}|\, i\eta_{p,m}(h,i)}{1 - h\beta(k,\kappa)\kappa\sum_{i=1}^{r-1} \kappa^{r-i}|k_{i+1}|\, i\eta_{p,m}(h,i)}\right) \|w_0\|_{\mathcal{W}_0}.$$

7.3.5 Exponential stability of the 'delay closed-loop' $[P(A,B,C;x^0), C_k^{\text{Euler}}[h]]$ with $u_0 = y_0 \equiv 0$

In Theorem 3.2.1, see also Corollary 7.3.1, it has been shown that the high-gain output derivative feedback controller $C_{k,\kappa}\colon y_2 \mapsto u_2$ leads to an internally stable system, i.e. for $u_0 = y_0 \equiv 0$, any $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times m}$ with $k_r = 1$ and $s \mapsto \sum_{i=0}^{r-1} k_{i+1} s^i$ Hurwitz, Corollary 7.3.1 gives

$$\exists\, \kappa^* \geq 1\ \forall\, \kappa \geq \kappa^*\ :\quad \dot{x} = (A + B\widetilde{K}[C/\ldots/CA^{r-1}])\, x \quad \text{is exponentially stable,}$$

where $\widetilde{K} = \left[\widetilde{k}_1 I_m, \ldots, \widetilde{k}_r I_m\right]$ with $\widetilde{k}_i := \kappa^{r+2-i} k_i$, $i \in \{1, \ldots, r\}$.

Now (as in [IS04] where a more limited class of systems was considered, namely systems with relative degree 2) it is shown that an analogous result holds true if the stabilizing derivative feedback controller $C_k\colon y_2 \mapsto u_2$ is replaced by the delay feedback $C_k^{\text{Euler}}[h]\colon y_2 \mapsto u_2$ for $h > 0$ sufficiently small. Exponential stability for a delay differential equation is defined as follows, see, for example [CZ95, Def. 5.1.1].

Definition 7.3.7 *Let $h > 0$ and, for $n, r \in \mathbb{N}$, $n \geq r$, $A_0, \ldots, A_{r-1} \in \mathbb{R}^{n\times n}$. Then the delay differential initial value*

$$\dot{x} = \sum_{j=0}^{r-1} A_j\, x(t - jh)\,, \qquad x \equiv \varphi \text{ on } [(1-r)h, 0]\,, \tag{7.3.14}$$

is said to be exponentially stable *if, and only if,*

$$\exists\, M, \lambda > 0\ \forall\, t \geq 0\ \forall\, \varphi \in \mathcal{C}_{pw}([(1-r)h, 0] \to \mathbb{R}^n)\ :\quad \|x(t)\| \leq M e^{-\lambda t} \max_{s\in[(1-r)h,0]} \|\varphi(s)\|\,. \tag{7.3.15}$$

The following proposition now shows exponential stability of the closed-loop initial value problem $[P(A,B,C;x^0), C_k^{\text{Euler}}[h]]$ with zero disturbances $u_0 = y_0 \equiv 0$.

Proposition 7.3.8 *Let, for $n, m, r \in \mathbb{N}$ with $n \geq rm$, $(A,B,C) \in \mathcal{P}_{n,m,r}$, $x^0 \in \mathbb{R}^n$. Consider the signal spaces $\mathcal{U} = \mathcal{Y} = CW_0^{\infty,2}(\mathbb{R}_{\geq 0} \to$*

$\mathbb{R}$) and $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and choose $k \in \mathbb{R}^{1\times r}$ and $h > 0$ such that

$$\left\|\Pi_{P(A,B,C;0)//C_k^{Euler}[h]}\right\|_{\mathcal{W},\mathcal{W}} < \infty\,,$$

Then the delay differential system associated with the closed-loop initial value problem $[P(A,B,C;x^0), C_k^{Euler}[h]]$ given by (7.3.1)*,* (7.1.2) *and* (6.2.1) *with $u_0 = y_0 \equiv 0$ is exponentially stable.*

Proof. For $(A,B,C) \in \mathcal{P}_{n,m,r}$, $n,m,r \in \mathbb{N}$ with $n \geq m$ and $h > 0$, the closed-loop system $[P(A,B,C;x^0), C_k^{\mathrm{Euler}}[h]]$ given by (7.3.1), (7.1.2) and (6.2.1) is described by the delay differential equation (7.3.8), see Subsection 7.3.3. Let $G_{P(A,B,C;0)//C_k^{\mathrm{Euler}}[h]} \in \mathbb{R}(s)^{2m\times 2m}$ denote the transfer function of $\Pi_{P(A,B,C;0)//C_k^{\mathrm{Euler}}[h]}$. Then, [Tre04, Thm. 30] or, for example, [Fra87, Thm. 2, Sect. 2.4] yields

$$\sup_{\omega\in\mathbb{R}} \left\|G_{P(A,B,C;0)//C_k^{\mathrm{Euler}}[h]}(i\omega)\right\|_2 = \left\|\Pi_{P(A,B,C;0)//C_k^{\mathrm{Euler}}[h]}\right\|_{\mathcal{W},\mathcal{W}} < \infty\,.$$

Since the denominator of the function $\left(s \mapsto G_{P(A,B,C;0)//C_k^{\mathrm{Euler}}[h]}(s)\right)$ is equal to

$$\det\left(sI - \left(A + \widetilde{A}^0 + e^{-sh}\widetilde{A}^1 + \cdots + e^{-s(r-1)h}\widetilde{A}^{r-1}\right)\right),$$

where $\widetilde{A}^0, \ldots, \widetilde{A}^{r-1}$ as in Subsection 7.3.3, it follows that

$$\forall\, s \in \overline{\mathbb{C}}_+ \;:\; \det\left(sI - \left(A + \widetilde{A}^0 + e^{-sh}\widetilde{A}^1 + \cdots + e^{-s(r-1)h}\widetilde{A}^{r-1}\right)\right) \neq 0\,.$$

Now, [CZ95, Thm. 5.1.5] yields exponential stability of (7.3.8) with $u_0 = y_0 \equiv 0$, and the proof is complete. □

This section is concluded by noting that for sufficiently large $\kappa^* \geq 1$ (determined by Corollary 7.3.1) and any $\kappa > \kappa^*$, $\beta(k,\kappa)$ given in (7.3.7) and for sufficiently small $h > 0$ (determined by Theorem 7.3.4) Proposition 7.3.8 yields that the closed-loop system $[P(A,B,C;x^0), C_{k,\kappa}^{\mathrm{Euler}}[h]]$ with $u_0 = y_0 \equiv 0$ is exponentially stable. Thus, robust stability when replacing derivatives by delays of the high-gain output derivative feedback controller 3.2.2 is proven.

7.4 Example

In this section an example is presented which illustrates Theorem 7.3.4, and its application within a nonlinear context. Consider the linear SISO-system $P(A,b,c;x^0)\colon u_1 \mapsto y_1$ given by (7.3.1) with the system matrices $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $c = [1,0,0]$ and initial value $x^0 \in \mathbb{R}^3$. Note that the system has relative degree $r = 2$, positive high-frequency gain $cA^{r-1}b = 1$, stable zero dynamics and is already in Byrnes-Isidori normal form. An application of $C_{k,\kappa}$ defined by (7.3.2) with $k = (1,1)$ yields the closed-loop system

$$\left.\begin{aligned} \begin{pmatrix} \dot y_1 \\ \ddot y_1 \\ \dot\eta \end{pmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1-\kappa^3 & 1-\kappa^2 & 1 \\ 1 & 0 & -1 \end{bmatrix}}_{=:A_\kappa} \begin{pmatrix} y_1 \\ \dot y_1 \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left[u_0 + \kappa^3 y_0 + \kappa^2 \dot y_0\right], \\ u_1 &= u_0 + \kappa^3 y_0 + \kappa^2 \dot y_0 - (\kappa^3 y_1 + \kappa^2 \dot y_1), \qquad \begin{pmatrix} y_1 \\ \dot y_1 \\ \eta \end{pmatrix}(0) = x^0 . \end{aligned}\right\} \tag{7.4.1}$$

If $\kappa > \sqrt[3]{2}$ then $\operatorname{spec}(A_\kappa) \subset \mathbb{C}_-$. Replacing $C_{k,\kappa}$ by the appropriate delay feedback controller $C^{\text{Euler}}_{k,\kappa}[h]$ yields the closed-loop system

$$\left.\begin{aligned} \begin{pmatrix} \dot y_1 \\ \ddot y_1 \\ \dot\eta \end{pmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 1-\kappa^3-\frac{\kappa^2}{h} & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} y_1 \\ \dot y_1 \\ \eta \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{\kappa^2}{h} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ \dot y_1 \\ \eta \end{pmatrix} (\cdot - h) \\ &\quad + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left[u_0 + \kappa^3 y_0 + \kappa^2 \tfrac{y_0 - y_0(\cdot - h)}{h}\right], \\ u_1 &= u_0 + \kappa^3 y_0 + \kappa^2 \tfrac{y_0 - y_0(\cdot-h)}{h} - \left(\kappa^3 y_1 + \kappa^2 \tfrac{y_1 - y_1(\cdot - h)}{h}\right), \\ &\qquad \begin{pmatrix} y_1 \\ \dot y_1 \\ \eta \end{pmatrix} \equiv \varphi \text{ on } [-h, 0], \end{aligned}\right\} \tag{7.4.2}$$

where $\varphi(0) = x^0$. Theorem 7.3.4 now guarantees that, for sufficiently small $h > 0$, the delay system (7.4.2) is $\mathcal{W}_0$-gain stable for signal spaces

$\mathcal{W}_0$ of type (A) or (C) defined by (7.2.2). Moreover, Proposition 7.3.8 shows that, for $(u_0, y_0) = (0, 0)$, the solution of (7.4.2) is exponentially stable for sufficiently small $h > 0$.

Next the effect of an additional causal and invertible nonlinearity $\Psi\colon \mathcal{U}_0 \to \mathcal{U}_0$ is considered. Connect this in series with the input of the plant, to give the plant operator $P(A, b, c; x^0) \circ \Psi$, see Figure 7.3.

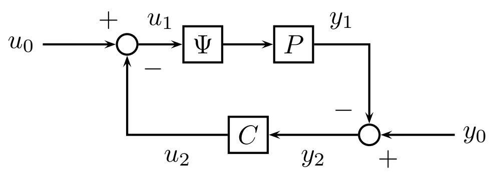


Figure 7.3: The closed-loop system $[P \circ \Psi, C]$

Consider the map $\Phi\colon \mathcal{G}_{P(A,b,c;0)} \to \mathcal{W}_0$ defined by

$$\Phi \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} \Psi^{-1}(u) \\ y \end{pmatrix}, \qquad \text{for all } \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{G}_{P(A,b,c;0)} \, .$$

Since $\Psi^{-1}(u) \in \mathcal{U}_0$ for all $u \in \mathcal{U}_0$, $\Phi \in \mathcal{O}_{P(A,b,c;0),P(A,b,c;0)\circ\Psi}$, hence

$$\vec{\delta}(P(A, b, c; 0), P(A, b, c; 0) \circ \Psi) \le \|I - \Phi\|_{\mathcal{W}_0,\mathcal{W}_0} \le \|I - \Psi^{-1}\|_{\mathcal{U}_0,\mathcal{U}_0}.$$

An application of Theorem 6.5.1 then shows that if the nonlinearity Ψ is sufficiently mild, i.e. if $\|I - \Psi^{-1}\|_{\mathcal{U}_0,\mathcal{U}_0} < \|\Pi_{C_{k,\kappa}//P(A,b,c;0)}\|^{-1}_{\mathcal{W}_0,\mathcal{W}_0}$, then $[P(A, b, c; 0) \circ \Psi, C_{k,\kappa}]$ is also gain stable on $\mathcal{W}_0$. Consequently, Theorem 7.2.1 establishes, for sufficiently small $h > 0$, that $[P(A, b, c; 0) \circ \Psi, C^{\text{Euler}}_{k,\kappa}[h]]$ is $\mathcal{W}_0$-gain stable for signal spaces $\mathcal{W}_0$ of type (A) or (C) defined by (7.2.2).

For example, in the setting of signal space (A), consider $\Psi\colon \mathcal{U}_0 \to \mathcal{U}_0$ defined by $\Psi(u)(t) = \psi(u(t))$ for the memoryless function $\psi\colon \mathbb{R} \to \mathbb{R}$, $\psi(s) = s + \varepsilon \sin(s)$, $0 < \varepsilon \le 1$. Then ψ is bijective and there exist functions $\alpha, \beta\colon (0, 1] \to (0, \infty)$ with the property $\lim_{\varepsilon\to 0} \alpha(\varepsilon) = 1$, $\lim_{\varepsilon\to 0} \beta(\varepsilon) = 1$ such that $\alpha(\varepsilon)s \le \psi(s) \le \beta(\varepsilon)s$. It follows that the inverse ψ^{-1} holds $\frac{1}{\beta(\varepsilon)} s \le \psi^{-1}(s) \le \frac{1}{\alpha(\varepsilon)} s$. Hence the bounded inverse

$\Psi^{-1}\colon \mathcal{U}_0 \to \mathcal{U}_0$ is given by $\Psi^{-1}(u)(t) = \psi^{-1}(u(t))$ and

$$\|I - \Psi^{-1}\| \leq \sup_{s\in\mathbb{R}} \frac{|s-\psi^{-1}(s)|}{|s|} \leq 1 - \max\left\{\frac{1}{\alpha(\varepsilon)}, \frac{1}{\beta(\varepsilon)}\right\} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence for sufficiently small $\varepsilon > 0$ and $h > 0$, $[P(A,b,c;x^0)\circ\Psi, C_{k,\kappa}^{\text{Euler}}[h]]$ is $\mathcal{W}_0$-gain stable.

Robust stability is illustrated in the following simulation study. Recall that $k = (1,1)$ and choose $\kappa = 2$ and $\varepsilon = 1$. The simulations are computed with MATLAB's dde23 solver for delay differential equations, see also [SGT03]. All state variables $y_1, \dot{y}_1, \eta$ of the undisturbed ($u_0 = y_0 = 0$) closed-loop systems are illustrated for $t \in [0, 20]$.

Figure 7.4(a) shows an exponentially stable solution of (7.4.1) with initial conditions $(y_1(0), \dot{y}_1(0), \eta(0)) = {x^0}^T = (1,1,1)$ and $(u_0, y_0) = (0,0)$.

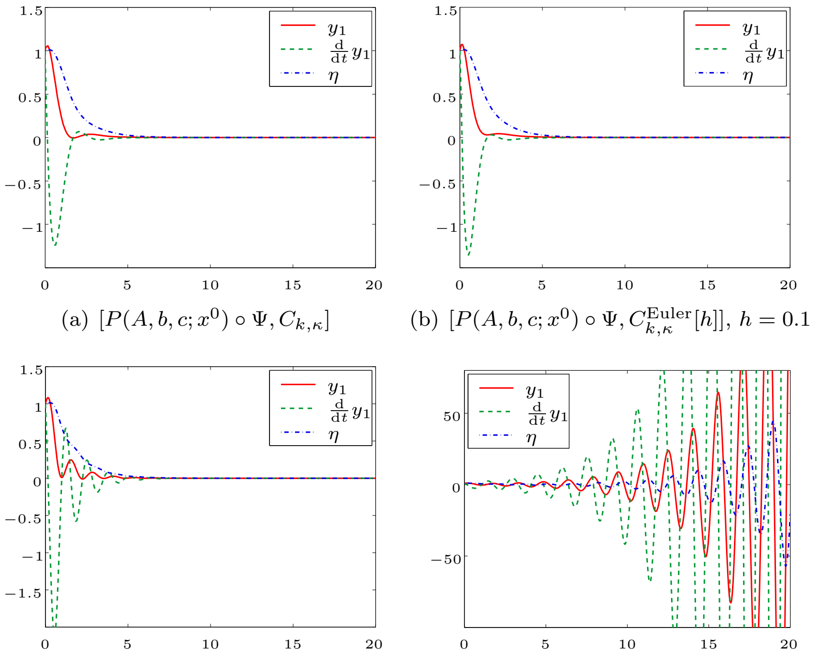
Figures 7.4(b), 7.4(c), 7.4(d) illustrate $[P(A,b,c;x^0)\circ\Psi, C_{k,\kappa}^{\text{Euler}}[h]]$ with $(u_0, y_0) = (0,0)$ and initial function

$$(y_1|_{[-h,0]}, \dot{y}_1|_{[-h,0]}, \eta|_{[-h,0]}) \equiv \varphi^T \equiv (1,1,1)$$

for Euler step sizes $h = 0.1, 0.4, 0.7$, respectively. Note that for $h = 0.7$ the delay system becomes unstable.

7.5 Notes and references

The results from the present chapter are shown in [FIM09]. Perhaps surprisingly, there are relatively few theoretical results available on closed-loop stability for such delay based controllers. For linear time-invariant systems with relative degree 2 controlled by the delay feedback (7.1.2), exponential stability of the resulting closed-loop delay differential system was established in [IS04]. In the very recent work, [Kar08] tries to generalize the results in [IS04] for systems with arbitrary relative degree, however the proof is very technical and seems to be inaccurate. Other analogous results for systems with higher relative degree have not been previously established. Stabilization of (nonlinear) systems via delays has been considered by some authors: in [NM04] the authors give a control strategy with multiple delays that stabilizes a simple system



(a) $[P(A, b, c; x^0) \circ \Psi, C_{k,\kappa}]$
(b) $[P(A, b, c; x^0) \circ \Psi, C_{k,\kappa}^{\text{Euler}}[h]]$, $h = 0.1$
(c) $[P(A, b, c; x^0) \circ \Psi, C_{k,\kappa}^{\text{Euler}}[h]]$, $h = 0.4$
(d) $[P(A, b, c; x^0) \circ \Psi, C_{k,\kappa}^{\text{Euler}}[h]]$, $h = 0.7$

Figure 7.4: Simulations; Robust stability of derivative feedback

of the form $y^{(n)} = u$. In [KNMM05] necessary conditions for multiple delay controllers that stabilize linear systems are shown, but no explicit control strategy is given. In [Kar06] the author considers nonlinear systems with several constraints and gives a control strategy that achieves a bounded output.

Analogous results for sampled versions of $C_k^{\text{Euler}}[h]$ can also be shown utilizing the techniques of this chapter. A variety of sampled versions of these results are presented in [FM09], which also extends the results for fully nonlinear controllers and to the important case of semi-global stabilization.

8 Robustness of λ-tracking

In this chapter it is verified that λ-tracking is robust in the sense that the control objectives (bounded signals and tracking) are still met if the λ-tracker (4.1.5) is applied to any system "close" (in terms of a small gap metric) to a system satisfying the classical assumption for adaptive control, i.e. systems which are (i) minimum phase, have (ii) relative degree one and (iii) "positive" high-frequency gain, as long as the initial conditions and the disturbances are "small". This will be achieved by exploiting the concept of the nonlinear gap metric from Chapter 6.

8.1 Well posedness of the closed-loop system

Recall λ-tracking from Chapter 4: Theorem 4.2.1 shows that linear n-dimensional, m-input, m-output (MIMO) systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u_1(t)\,, \qquad x(0) = x^0, \\ y_1(t) &= C\,x(t)\,, \end{aligned}\right\} \tag{8.1.1}$$

where $(A, B, C) \in \mathcal{M}_{n,m}$, see Section 4.1, and $x^0 \in \mathbb{R}^n$ is an arbitrary initial value, can be "stabilized" in presence of additive input/output disturbances $u_0, y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations

$$u_0 = u_1 + u_2, \qquad y_0 = y_1 + y_2\,, \tag{8.1.2}$$

by application of the adaptive control strategy

$$\left.\begin{aligned} \dot{k}(t) &= \operatorname{dist}\left(y_2(t), [-\lambda, \lambda]\right) \cdot \|y_2(t)\|\,, \quad k(0) = k^0\,, \\ u_2(t) &= -k(t) y_2(t)\,, \end{aligned}\right\} \tag{8.1.3}$$

where $\lambda > 0$, $k^0 \in \mathbb{R}$ and $\operatorname{dist}\colon \mathbb{R}^m \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ defined by $(y, \lambda) \mapsto \operatorname{dist}(y, [-\lambda, \lambda]) := \max\{0, \|y\| - \lambda\}$.

A linear system (A, B, C) from system class $\mathcal{M}_{n,m}$ satisfies the classical assumptions for adaptive control: (A, B, C) given by (8.1.1) is minimum phase, see Definition 2.3.1 and Corollary 2.3.2, has strict relative degree one with positive definite high-frequency gain matrix CB, i.e. $CB + (CB)^T > 0$.

The purpose of the present chapter is to show robustness properties of the λ-tracker in terms of the gap metric. This means that the λ-tracker may also be applied to linear systems which are sufficiently close (in terms of a small gap) to any system from class $\mathcal{M}_{n,m}$ but may not satisfy any of the above assumptions.

For the purpose of illustration recall the example plant from Subsection 6.3.1. A minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$, see Subsection 8.2.1, of the system given by the transfer function

$$s \mapsto \frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}, \qquad \alpha, N, M > 0,$$

(which obviously does not satisfy any of the classical assumptions since it is not minimum phase, has relative degree 2 and negative high-frequency gain) is closer (in sense of the nonlinear gap) to a system in $\mathcal{M}_{1,1}$ the larger N and M are, see (6.3.3). Furthermore, $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ is stabilizable and detectable. Recall, for $n, m \in \mathbb{N}$ with $n \geq m$, the system class

$$\mathcal{P}_{n,m} := \left\{ \begin{array}{l} (A, B, C) \\ \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times n} \end{array} \middle| \begin{array}{l} (A, B, C) \text{ is stabilizable} \\ \text{and detectable} \end{array} \right\},$$

from Subsection 6.5.2 and note that $\mathcal{M}_{n,m} \subsetneq \mathcal{P}_{n,m}$.

For $m, n \in \mathbb{N}$ with $n \geq m$, consider $\mathcal{P}_{n,m}$ as a subspace of the Euclidean space $\mathbb{R}^{n^2+2mn}$ by identifying a plant $\theta = (A, B, C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. With normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, where $x^0 \in \mathbb{R}^n$ is the initial value of a linear system (8.1.1), one may associate the causal plant operator

$$P(\theta, x^0) \colon \mathcal{U}_a \to \mathcal{Y}_a\,, \qquad u_1 \mapsto P(\theta, x^0)(u_1) := y_1\,, \tag{8.1.4}$$

where, for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0, \omega)$, holds $y_1 = Cx$, x being the unique solution of (8.1.1) on $[0, \omega)$. Note that P is a map from

$\bigcup_{n\geq m}(\mathcal{P}_{n,m} \times \mathbb{R}^n)$ to the space of maps $\mathcal{U}_a \to \mathcal{Y}_a$.

Consider, for $\lambda > 0$, the adaptive strategy (8.1.3) and associate the causal controller operator, parameterized by λ and the initial value $k^0 \in \mathbb{R}$, i.e.

$$C(\lambda, k^0)\colon \mathcal{Y}_a \to \mathcal{U}_a\,, \qquad y_2 \mapsto C(\lambda, k^0)(y_2) := u_2\,. \tag{8.1.5}$$

Note that C is a map from $(0,\infty) \times \mathbb{R}$ to the space of causal maps $\mathcal{Y}_a \to \mathcal{U}_a$.

In this section it is shown that, for $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the closed-loop system $[P(\theta, x^0), C(\lambda, k^0)]$ (depicted in Figure 8.1) of any plant of the form (8.1.1) (with associated operator $P(\theta, x^0)$) and adaptive controller (8.1.3) (with associated operator $C(\lambda, k^0)$), where $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $(\lambda, k^0) \in (0,\infty) \times \mathbb{R}$, satisfying the interconnection equations (8.1.2), is regularly well posed.

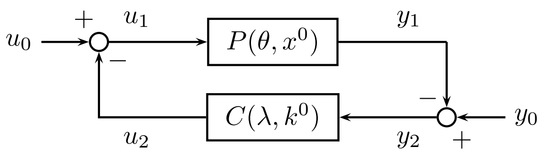


Figure 8.1: The closed-loop system $[P, C]$.

First, it is verified that, for any $\theta \in \mathcal{M}_{n,m}$, see Section 4.1, the closed-loop system $[P(\theta, x^0), C(\lambda, k^0)]$ is globally well posed and $(\mathcal{U}\times\mathcal{Y})$-stable, see Section 6.2.

Proposition 8.1.1 *Let* $m, n \in \mathbb{N}$ *with* $n \geq m$, $\lambda > 0$, $(\theta, x^0, k^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n \times \mathbb{R}$ *and* $u_0, y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. *Then, for plant operator* $P(\theta, x^0)$ *and control operator* $C(\lambda, k^0)$, *given by* (8.1.4) *and* (8.1.5), *respectively, the closed-loop initial value problem* $[P(\theta, x^0), C(\lambda, k^0)]$, *given by* (4.1.6), (8.1.2), (8.1.3), *is globally well posed and additionally* $\big(W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$*-stable.*

Proof. The proposition is a consequence of Proposition 4.2.1. □

Note that, for $(A,B,C) \in \mathcal{P}_{n,m}$, $x^0 \in \mathbb{R}^n$, $\lambda > 0$ and $k^0 \in \mathbb{R}$, the closed-loop initial value problem (8.1.1), (8.1.2), (8.1.3) may be written as

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B[u_0(t) - u_2(t)]\,, & x(0) &= x^0 \in \mathbb{R}^n\,,\\ \dot{k}(t) &= d_\lambda(y_2(t))\,\|y_2(t)\|\,, & k(0) &= k^0 \in \mathbb{R}\,,\\ y_2(t) &= y_0(t) - C\,x(t)\,, & &\\ u_2(t) &= -k(t)y_2(t)\,, & & \end{aligned}\right\} \tag{8.1.6}$$

where $d_\lambda\colon \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is defined by $e \mapsto d_\lambda(e) := \max\{0, \|e\| - \lambda\}$, see also Section 4.2.

Proposition 8.1.2 *Let $m, n \in \mathbb{N}$ with $n \geq m$, $\lambda > 0$, $(\theta, x^0, k^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n \times \mathbb{R}$ and $u_0, y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $P(\theta, x^0)$ and control operator $C(\lambda, k^0)$, given by* (8.1.4) *and* (8.1.5)*, respectively, the closed-loop initial value problem $[P(\theta, x^0), C(\lambda, k^0)]$, given by* (8.1.6)*, has the following properties:*

(i) there exists a unique solution $(x,k)\colon [0,\omega) \to \mathbb{R}^n \times \mathbb{R}$, for some $\omega \in (0,\infty]$, and the solution is maximal in the sense that for every compact $\mathcal{K} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$ exists $t \in [0,\omega)$ such that $(t, x(t), k(t)) \notin \mathcal{K}$;

(ii) if $k \in W^{1,\infty}([0,\omega) \to \mathbb{R})$, then $\omega = \infty$;

(iii) if $y_2 \in W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$, then $\omega = \infty$;

(iv) $[P(\theta, x^0), C(\lambda, k^0)]$ is regularly well posed.

Proof. (i): The right hand side of (8.1.6) is continuous and locally Lipschitz, i.e.

$$f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}\,,$$
$$(t,x,k) \mapsto \begin{pmatrix} A\,x + B[u_0(t) + k(y_0(t) - C\,x)] \\ d_\lambda(y_0(t) - C\,x)\,\|y_0(t) - C\,x\| \end{pmatrix},$$

satisfies a local Lipschitz condition on the relatively open set $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$ in the sense that, for all $(\tau, \xi, \kappa) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$, there exists an open neighbourhood O of (τ, ξ, κ) and a constant $L > 0$ such that

$$\forall\,(t,x,k) \in O \;:\; \|f(t,x,k) - f(t,\xi,\kappa)\| \leq L(\|x-\xi\| + \|k-\kappa\|)\,.$$

Therefore, standard theory of ordinary differential equations, see, for example, [Wal98, Thm. III.11.III], yields that (8.1.6) has an absolutely continuous solution

$$(x,k)\colon [0,\omega) \to \mathbb{R}^n \times \mathbb{R}$$

for some $\omega \in (0,\infty]$, which satisfies $(t,x(t),k(t)) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$. Moreover, the solution is unique and the solution can be extended up to the boundary of $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$. In other words: for every compact $\mathcal{K} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$ exists $t \in [0,\omega)$ such that $(t,x(t),k(t)) \notin \mathcal{K}$, as required.

(ii): Suppose $k \in W^{1,\infty}([0,\omega) \to \mathbb{R})$ and, for contradiction, $\omega < \infty$. Since $d_\lambda(y_2)^2 \leq d_\lambda(y_2)\,\|y_2\| = \dot{k} \in L^\infty([0,\omega) \to \mathbb{R}_{\geq 0})$, it follows that $d_\lambda(y_2) \in L^\infty([0,\omega) \to \mathbb{R}_{\geq 0})$ and $d_\lambda(y_2) + \lambda \in L^\infty([0,\omega) \to \mathbb{R}_{\geq 0})$. Thus, $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$.

Since $k \in L^\infty([0,\omega) \to \mathbb{R})$, Variation of Constants applied to (8.1.6) yields the existence of constants $c_0, c_1 > 0$ such that

$$\begin{aligned}&\forall\, t \in [0,\omega) \;: \\ &\quad \|x(t)\| \leq c_0 \left(e^{c_1\omega} + \int_0^\omega e^{c_1(\omega - s)}\left(\|u_0(s)\| + \|y_2(s)\|\right)\,\mathrm{d}s \right). \end{aligned} \tag{8.1.7}$$

Since $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$ and $u_0 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, it follows from the convolution in (8.1.7) that the right hand side of (8.1.7) is bounded by $c_3 = c_0\big(e^{c_1\omega} + (e^{c_1\omega}+1)(\|u_0\|_{L^\infty([0,\omega)\to\mathbb{R}^m)} + \|y_2\|_{L^\infty([0,\omega)\to\mathbb{R}^m)})/c_1\big) > 0$ on $[0,\omega)$ which gives that

$$\mathcal{K} := \left\{ (t,x,\kappa) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R} \,\middle|\, t \in [0,\omega]\,,\; \|x,\kappa\| \leq \sqrt{c_3^2 + \|k\|_{L^\infty}} \right\}$$

is a compact subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}$ with $(t,x(t),k(t)) \in \mathcal{K}$ for all $t \in [0,\omega)$, which contradicts the fact that the closure of graph$\Big((x,k)|_{[0,\omega)}\Big)$ is not a compact set, see (i). Therefore, $\omega = \infty$.

(iii): Suppose $y_2 \in W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$ and, for contradiction, $\omega <$

∞. Then $\dot{k} = d_\lambda(y_2)\,\|y_2\| \in L^\infty([0,\omega) \to \mathbb{R})$ and, combined with

$$\begin{aligned}\forall\, t \in [0,\omega)\ :\\ k(t) - k^0 &= \int_0^t d_\lambda(y_2(s))\,\|y_2(s)\|\,\mathrm{d}s\\ &\le \int_0^t \|y_2\|^2_{L^\infty([0,\omega)\to\mathbb{R}^m)}\,\mathrm{d}s = \omega\,\|y_2\|^2_{L^\infty([0,\omega)\to\mathbb{R}^m)}\,,\end{aligned}$$

one arrives at $k \in W^{1,\infty}([0,\omega) \to \mathbb{R})$. Now (ii) yields that $\omega = \infty$. This is a contradiction and so $\omega = \infty$.

(iv): For ease of notation, let $\mathcal{W} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$. By (i), the closed-loop $[P(\theta,x^0), C(\lambda,k^0)]$ is locally well posed. To prove that $[P(\theta,x^0), C(\lambda,k^0)]$ is regularly well posed, it suffices to show that (6.2.2) holds, see Section 6.2. For arbitrary $w_0 = (u_0,y_0) \in \mathcal{W}$ consider $(w_1,w_2) = H_{P(\theta,x^0),C(\lambda,k^0)}(w_0)$ where $\mathrm{dom}(w_1,w_2) = [0,\omega)$ is maximal. Suppose, contrary to the right hand side of (6.2.2), that $\big\|(w_1,w_2)\big|_{[0,\omega)}\big\|_{\mathcal{W}_\omega \times \mathcal{W}_\omega} < \infty$. Then $y_2 \in W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$, which, in view of (iii), yields $\omega = \infty$, i.e. the contrary of the left hand side of (6.2.2). Hence the closed-loop system is regularly well posed. □

With the knowledge that the closed-loop system $[P(\theta,x^0), C(\lambda,k^0)]$ is regularly well posed for all $\theta \in \mathcal{P}_{n,m}$, one is in the position to present the main results on robust stability of λ-tracking which follows in the next section

8.2 Robust stability

Theorem 4.2.1 and Proposition 8.1.1 establish that, for $(\theta,x^0,k^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n \times \mathbb{R}$ and $m,n \in \mathbb{N}$ with $n \ge m$, $\lambda > 0$ and $u_0,y_0 \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, the closed-loop system $[P(\theta,x^0), C(\lambda,k^0)]$, given by (4.1.6), (8.1.2), (8.1.3) is globally well posed and has certain stability properties. Furthermore, Theorem 4.2.1 shows λ-tracking for linear systems belonging to class $\mathcal{M}_{n,m}$.

The purpose of this section is to determine conditions under which these properties are maintained when the plant $P(\theta,x^0)$ is perturbed to a plant $P(\widetilde{\theta},\widetilde{x}^0)$ where $(\widetilde{\theta},\widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ for some $q \in \mathbb{N}$, in particular when $\widetilde{\theta} \notin \mathcal{M}_{q,m}$. The main result Theorem 8.2.3 shows that stability properties and λ-tracking persist if (a) the plants $P(\widetilde{\theta},0)$ and $P(\theta,0)$ are sufficiently close (in the gap sense) and (b) the initial data $\widetilde{x}^0$ and disturbance $w_0 = (u_0, y_0) \in \mathcal{W} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ are sufficiently small.

To establish gap margin results, the augmented plant and controller operators as in Subsection 6.5.2 are required. Define, for the concrete signal spaces $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2nm} \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and let $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}} \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, which can be considered as signal spaces by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$ and endowing $\widetilde{\mathcal{U}}$ with the norm $\|(\theta,u)\|_{\widetilde{\mathcal{U}}} := \sqrt{\|\theta\|^2 + \|u\|^2_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)}}$. For $P(\theta,0)$ given as in (8.1.4), define the (augmented) plant operator, as in (6.5.13), by

$$
\begin{aligned}
&\widetilde{P}\colon \widetilde{\mathcal{U}}_a \to W^{1,\infty}_a(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\,, \\
&\qquad (\theta, u_1) = \widetilde{u}_1 \mapsto y_1 = \widetilde{P}(\widetilde{u}_1) := P(\theta,0)(u_1)\,. \qquad (8.2.1)
\end{aligned}
$$

Fix $\lambda > 0$, $k^0 \in \mathbb{R}$ and define, for $C(\lambda, k^0)$ as in (8.1.5), the (augmented) controller operator, as in (6.5.14) but for the concrete λ-tracker, by

$$
\begin{aligned}
&\widetilde{C}\colon W^{1,\infty}_a(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \to \widetilde{\mathcal{U}}_a\,, \\
&\qquad y_2 \mapsto \widetilde{u}_2 = \widetilde{C}(y_2) := \big(0, C(\lambda,k^0)(y_2)\big) = (0, u_2)\;. \qquad (8.2.2)
\end{aligned}
$$

Note that $0 \notin \mathcal{M}_{n,m}$. For each non-empty $\Omega \subset \mathcal{M}_{n,m}$, define

$$
\mathcal{W}^\Omega := \big(\Omega \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \quad \text{and} \quad H^\Omega_{\widetilde{P},\widetilde{C}} := H_{\widetilde{P},\widetilde{C}}|_{\mathcal{W}^\Omega}\,. \qquad (8.2.3)
$$

It follows from Proposition 8.1.1 that $H^\Omega_{\widetilde{P},\widetilde{C}}\colon \mathcal{W}^\Omega \to \widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}}$ is a causal operator for any non-empty $\Omega \subset \mathcal{M}_{n,m}$. Next, Proposition 8.2.1 shows gain-function stability of $H^\Omega_{\widetilde{P},\widetilde{C}}$. This is a supposition of Theorem 6.5.3 and latter is used to show Proposition 8.2.2 and thus the main result Theorem 8.2.3.

Proposition 8.2.1 *Let $m, n \in \mathbb{N}$ with $n \geq m$, $k^0 \in \mathbb{R}$, $\lambda > 0$ and assume $\Omega \subset \mathcal{M}_{n,m}$ is closed. Then, for the closed-loop system $[\widetilde{P}, \widetilde{C}]$ given by* (6.2.1), (8.2.1) *and* (8.2.2), *the operator $H^{\Omega}_{\widetilde{P},\widetilde{C}}$, given by* (8.2.3) *is gain-function stable.*

Proof. Note that $((\theta, u_1), y_1) = ((\theta, u_0), y_0) - ((0, u_2), y_2)$. For the continuous map $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ as in Theorem 4.2.1 and $\mathcal{W}^{\Omega}$ given by (8.2.3), it follows that

$$\begin{aligned}
&\forall\, ((\theta, u_0), y_0) \in \mathcal{W}^{\Omega} \ : \\
&\left\| H^{\Omega}_{\widetilde{P},\widetilde{C}}((\theta, u_0), y_0) \right\|_{\widetilde{\mathcal{W}}\times\widetilde{\mathcal{W}}} = \left\| \big(((\theta, u_1), y_1), ((0, u_2), y_2) \big) \right\|_{\widetilde{\mathcal{W}}\times\widetilde{\mathcal{W}}} \\
&\qquad \leq \|((\theta, u_0), y_0)\|_{\widetilde{\mathcal{W}}} + 2\|((0, u_2), y_2)\|_{\widetilde{\mathcal{W}}} \\
&\qquad \leq \|(u_0, y_0)\|_{\mathcal{W}} + \|\theta\| + 2\,\nu(\theta, (0, k^0), u_0, y_0)\,,
\end{aligned}$$

and so, for $r_0 := \inf_{w \in \mathcal{W}^{\Omega}} \|w\|_{\widetilde{\mathcal{W}}}$ and $r \in (r_0, \infty)$, closedness of Ω yields

$$g\Big[H^{\Omega}_{\widetilde{P},\widetilde{C}}\Big](r) := \sup \left\{ \begin{array}{l} \|(u_0, y_0)\|_{\mathcal{W}} + \|\theta\| \\ +2\,\nu(\theta, (0, k^0), u_0, y_0) \end{array} \,\middle|\, \begin{array}{l} (\theta, u_0, y_0) \in \mathcal{W}^{\Omega}\,, \\ \|(\theta, u_0, y_0)\|_{\widetilde{\mathcal{W}}} \leq r \end{array} \right\} < \infty\,.$$

Thus a gain-function for $H^{\Omega}_{\widetilde{P},\widetilde{C}}$ exists, and the proof is complete. □

The following establishes $\big(W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stability of the closed-loop system $[P(\widetilde{\theta}, \widetilde{x}^0), C(\lambda, k^0)]$ for a system $\widetilde{\theta}$ belonging to the system class $\mathcal{P}_{q,m}$, $q \geq m$, if, for a system θ belonging to $\mathcal{M}_{n,m}$, $n \geq m$, the gap between $P(\widetilde{\theta}, 0)$ and $P(\theta, 0)$, the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ and the input/output disturbances $w_0 = (u_0, y_0) \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ are sufficiently small. The proof is based on results from Subsection 6.5.2, namely Theorems 6.5.3 and 6.5.4.

Proposition 8.2.2 *Let $m, n, q \in \mathbb{N}$ with $n, q \geq m$, signal spaces $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and $\theta \in \mathcal{M}_{n,m}$. For any $(\widetilde{\theta}, \widetilde{x}^0, k^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathbb{R}$ and $\lambda > 0$, consider $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$, and $C(\lambda, k^0)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (8.1.4) *and* (8.1.5), *respectively. Then there exist a continuous function $\eta\colon (0, \infty) \to (0, \infty)$ and a function*

$\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ *such that the following holds:*

$$\forall \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \; :$$
$$\left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r) \end{array}\right\} \Rightarrow \; H_{P(\widetilde{\theta},\widetilde{x}^0), C(\lambda,k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}\,. \tag{8.2.4}$$

It is required to show how the gain-function stability of the augmented closed loop $[\widetilde{P}, \widetilde{C}]$, given by (6.2.1), (8.2.1), (8.2.2), yields the robustness property (8.2.4) for the unaugmented closed-loop $[P(\widetilde{\theta}, \widetilde{x}^0), C(\lambda, k^0)]$.

Proof of Proposition 8.2.2. By Proposition 8.1.2 the closed-loop system $[P(\widetilde{\theta}, \widetilde{x}^0), C(\lambda, k^0)]$ is regularly well-posed for all $\widetilde{\theta} \in \mathcal{P}_{q,m}$. Consider the augmented operators defined by (8.2.1) and (8.2.2), i.e.

$$\begin{array}{lll} \widetilde{P}\colon \; \mathcal{P}_{n,m} \times \mathcal{U}_a \to \mathcal{Y}_a\,, & (\widetilde{\theta}, u_1) \mapsto \widetilde{P}(\widetilde{\theta}, u_1) = P(\widetilde{\theta}, 0)(u_1)\,, \\ \widetilde{C}\colon \; \mathcal{Y}_a \to \mathcal{P}_{n,m} \times \mathcal{U}_a\,, & y_2 \mapsto \widetilde{C}(y_2) = (0, C(\lambda, k^0)(y_2))\,. \end{array}$$

For $\theta \in \mathcal{M}_{n,m}$ set $\Omega = \{\theta\}$. By Proposition 8.2.1, $H^{\Omega}_{\widetilde{P},\widetilde{C}} = H_{\widetilde{P},\widetilde{C}}|_{\mathcal{W}^\Omega}$, given by (8.2.3), is gain-function stable. By, for example, the proof of [Zei86, Thm. 4.D], $\Pi_{P(\theta,0)//C(\lambda,k^0)}(\cdot)\big|_{[0,\tau)}$ is continuous for all $\tau > 0$, which yields that $\big(\Pi_{\widetilde{P}//\widetilde{C}}|_{\mathcal{W}^\Omega}\big)(\cdot)\big|_{[0,\tau)}$ is continuous for all $\tau > 0$.

Then Theorem 6.5.3 for $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ gives the existence of a continuous function $\mu\colon (0,\infty) \times \Omega \to (0,\infty)$ such that

$$\forall \big(\theta, \widetilde{\theta}, w_0, r\big) \in \Omega \times \mathcal{P}_{q,m} \times \mathcal{W} \times (0,\infty) \; :$$
$$\left.\begin{array}{l} \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \mu(r,\theta) \end{array}\right\} \Rightarrow \; H_{P(\widetilde{\theta},0), C(\lambda,k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}\,.$$

Now, Theorem 6.5.4 for $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ yields the existence of a continuous function $\mu\colon (0,\infty) \times \Omega \to (0,\infty)$ and a function

$\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ such that

$$\forall \big(\widetilde{\theta}, \theta, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathcal{M}_{n,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \;:$$

$$\left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \mu\,(r,\theta) \end{array}\right\} \;\Rightarrow\; H_{P(\widetilde{\theta},\widetilde{x}^0),C(\lambda,k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}\,. \tag{8.2.5}$$

Finally, statement (8.2.4) follows on setting $\eta(\cdot) = \mu(\cdot,\theta)$. □

Note that Theorem 6.5.4 requires stabilizability of system $\widetilde{\theta} \in \mathcal{P}_{q,m}$.

Finally, one is in the position to state and prove the main result of the present chapter. Loosely speaking, it is show that the λ-tracker also works for systems $\big(\widetilde{A},\widetilde{B},\widetilde{C}\big) \in \mathcal{P}_{q,m}$ which are not necessarily minimum phase, may have higher relative degree and negative high-frequency gain. However $\big(\widetilde{A},\widetilde{B},\widetilde{C}\big)$ has to be sufficiently close – in the terms of the gap metric – to a system $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ and the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ for $\big(\widetilde{A},\widetilde{B},\widetilde{C}\big)$ and the input/output disturbances (u_0,y_0) have to be sufficiently small.

Theorem 8.2.3 *Let $m,n,q \in \mathbb{N}$ with $n,q \ge m$, $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, $k^0 \in \mathbb{R}$, $\lambda > 0$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta},\widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ consider the associated operators $P(\widetilde{\theta},\widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C(\lambda,k^0)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (8.1.4) *and* (8.1.5)*, respectively, and the closed-loop initial value problem* (8.1.1), (8.1.2), (8.1.3)*. Then there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{q.m} \to (0,\infty)$ such that the following holds:*

$$\forall \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \;:$$

$$\left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r) \end{array}\right\} \;\Rightarrow\; \left\{\begin{array}{l} \limsup_{t\to\infty} \|y_2(t)\| \le \lambda\,, \\ k \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})\,, \\ x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)\,, \end{array}\right. \tag{8.2.6}$$

where (x,k) and y_2 satisfy (8.1.6).

Proof. *Step 1*: It is shown that

$$((u_1, y_1), (u_2, y_2)) = H_{P(\widetilde{\theta}, \widetilde{x}^0), C(\lambda, k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}. \tag{8.2.7}$$

Choose functions $\eta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ from Proposition 8.2.2. Let

$$\big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \;:\; \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \;\wedge\; \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r).$$

Then Proposition 8.2.2 gives (8.2.7).

Step 2: By Proposition 8.1.2 it follows that (8.1.6) has a unique solution

$$(x,k)\colon [0,\omega) \to \mathbb{R}^q \times \mathbb{R}$$

on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$, in the sense that for every compact $\mathcal{K} \subset \mathbb{R}_{\ge 0} \times \mathbb{R}^n \times \mathbb{R}$ exists $t \in [0,\omega)$ such that $(t, x(t), k(t)) \notin \mathcal{K}$. Now, Proposition 8.1.2(iii) together with (8.2.7), in particular $y_2 \in W^{1,\infty}([0,\omega) \to \mathbb{R}^q)$, yields $\omega = \infty$.

Step 3: It is shown that $\dot{k} \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$.

Suppose, for contradiction, that $\dot{k} \notin L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, i.e. there exists a sequence $(t_i) \in (\mathbb{R}_{\ge 0})^{\mathbb{N}}$ with $t_i > t_{i+1}$ and $\lim_{i\to\infty} \dot{k}(t_i) = \infty$. Then

$$\lim_{i\to\infty} d_\lambda(y_2(t_i))\, \|y_2(t_i)\| = \infty$$

and thus

$$\lim_{i\to\infty} \|y_2(t_i)\| = \infty,$$

a contradiction to (8.2.7).

Step 4: It is shown that $k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$.

Suppose, for contradiction, that $k \notin L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, that means that $\lim_{t\to\infty} k(t) = \infty$. Since $u_2 \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, the forth equation in (8.1.6) yields $\lim_{t\to\infty} y_2(t) = 0$, and thus

$$\exists\, T > 0\ \forall\, t \ge T \;:\; \dot{k}(t) = d_\lambda(y_2(t))\, \|y_2(t)\| = 0$$

which contradicts the assumption on k.

Step 5: By Step 3 and 4 one may obtain $k \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$.

Step 6: Proposition 8.2.2 yields in particular that $y_2, \dot{y}_2 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Analogous as in Step 10, of the proof of Theorem 4.2.1, one may establish that y_2 is uniformly continuous.

Step 7: By Step 6 and continuity of the map $e \mapsto d_\lambda(e)\|e\|$ obtain that $t \mapsto d_\lambda(y_2(t))\,\|y_2(t)\|$ is uniformly continuous. Hence, in view of $\dot{k} = d_\lambda(y_2)\|y_2\| \in L^1(\mathbb{R}_{\geq 0} \to \mathbb{R})$, which is equivalent to $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, and Barbălat's Lemma, see [Bar59], $\lim_{t\to\infty} d_\lambda(y_2(t))\,\|y_2(t)\| = 0$ holds. This gives $\limsup_{t\to\infty} \|y_2(t)\| \leq \lambda$.

Step 8: It remains to show that $x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$.

Let $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ associated with (8.1.1). Detectability of system $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ yields the existence of $F \in \mathbb{R}^{q\times m}$ such that $\widetilde{A}+F\widetilde{C}$ is Hurwitz. Setting $g := -\big[F + k\,\widetilde{B}\big]\,(y_0 - y_2) + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0$ gives

$$\dot{x} = \big[\widetilde{A} - k\,\widetilde{B}\widetilde{C}\big]x + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0 = \big[\widetilde{A} + F\widetilde{C}\big]x + g\,. \tag{8.2.8}$$

By Proposition 8.2.2 and Step 5 follows $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $k \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and since $w_0 = (u_0, y_0) \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ it follows that $g \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (8.2.8) obtain $x \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. The first equation in (8.1.6) then gives $\dot{x} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ which shows $x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ and completes the proof. □

8.2.1 Example: robust stability of λ-tracking

Finally, revisit the example plants (8.2.9) and (8.2.10) given by minimal realizations of the transfer functions $s \mapsto \frac{1}{s-\alpha}$ and $s \mapsto \frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}$, $\alpha, N, M > 0$, with the associated operators

$$P_{\alpha;x^0} : \mathcal{U}_a \to \mathcal{Y}_a\,,\ u_1 \mapsto y_1 \quad \text{and} \quad P_{N,M,\alpha;\widetilde{x}^0} : \mathcal{U}_a \to \mathcal{Y}_a\,,\ \widetilde{u}_1 \mapsto \widetilde{y}_1$$

which are induced by state space systems

$$P_{\alpha;x^0} \quad : \quad \left.\begin{aligned} \dot{x} &= \alpha x + u_1\,, & x(0) &= x^0 \\ y_1 &= x \end{aligned}\right\} \tag{8.2.9}$$

$$P_{N,M,\alpha;\widetilde{x}^0} : \quad \left.\begin{aligned} \dot{x} &= \widetilde{A}\,x + \widetilde{b}\,\widetilde{u}_1\,, & x(0) &= \widetilde{x}^0 \\ \widetilde{y}_1 &= \widetilde{c}\,x \end{aligned}\right\} \tag{8.2.10}$$

with $x^0 \in \mathbb{R}$, $\widetilde{x}^0 \in \mathbb{R}^3$ and where

$$\widetilde{A} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha NM, & -NM + \alpha N + \alpha M, & \alpha - N - M \end{bmatrix}, \quad \widetilde{b} := \begin{bmatrix} 0 \\ 0 \\ N \end{bmatrix},$$
$$\widetilde{c} := [M, -1, 0] .$$

The signal spaces in the present chapter, see also Chapter 4, are given as $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. In Subsection 6.3.1 it was shown that the gap metric between $P_{\alpha;0}$ and $P_{2M,M,\alpha;0}$ is small for large $M > 0$, i.e.

$$\limsup_{M \to \infty} \vec{\delta}(P_{\alpha;0}, P_{2M,M,\alpha;0}) = 0 .$$

Note that system (8.2.9) is minimum phase, has relative degree one and positive high-frequency gain, in fact $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$. Furthermore, $(\widetilde{A}, \widetilde{b}, \widetilde{c}) \notin \widetilde{\mathcal{M}}_{3,1}$. Equivalently, for the Byrnes–Isidori normal form of system (8.2.10), given by Lemma 2.1.2, i.e.

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta \end{pmatrix} = \underbrace{U \widetilde{A} U^{-1}}_{=: \widetilde{A}'} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta \end{pmatrix} + \underbrace{U \widetilde{b}}_{=: \widetilde{b}'} u_1 , \qquad y_1 = \underbrace{\widetilde{c} U^{-1}}_{=: \widetilde{c}'} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta \end{pmatrix} , \quad (8.2.11)$$

where, in view of the coordinate transformation $\left(\begin{smallmatrix} \xi_1 \\ \xi_2 \\ \eta \end{smallmatrix}\right) = \underbrace{\left[\begin{smallmatrix} M & -1 & 0 \\ 0 & M & -1 \\ 1 & 0 & 0 \end{smallmatrix}\right]}_{=: U} x$,

$$\widetilde{A}' = \begin{bmatrix} 0 & 1 & 0 \\ \alpha N + 2M(\alpha - M - N) , & \alpha - M - N , & 2M(NN + M^2 - \alpha M - \alpha N) \\ -1 & 0 & M \end{bmatrix},$$
$$\widetilde{b}' = \begin{pmatrix} 0 \\ -N \\ 0 \end{pmatrix}, \qquad \widetilde{c}' = [1, 0, 0] , \quad (8.2.12)$$

holds $(\widetilde{A}', \widetilde{b}', \widetilde{c}') \notin \mathcal{M}_{3,1}$. In particular, (8.2.10) does not satisfy any of the classical structural assumptions for adaptive control: it is not minimum phase, it has relative degree two and its high-frequency gain $-N < 0$ has the “wrong” sign.

However, note that the matrices

$$[\widetilde{b}, \widetilde{A}\widetilde{b}, \widetilde{A}^2\widetilde{b}] = \begin{bmatrix} 0 & 0 & N \\ 0 & N & (\alpha - N - M)N \\ N, & (\alpha - N - M)N, & (-NM + \alpha M + \alpha N + (\alpha - N - M)^2)N \end{bmatrix}$$

$$\begin{bmatrix} \widetilde{c} \\ \widetilde{c}\widetilde{A} \\ \widetilde{c}\widetilde{A}^2 \end{bmatrix} = \begin{bmatrix} M & -1 & 0 \\ 0 & M & -1 \\ -\alpha NM, & NM - \alpha M - \alpha N, & 2M - \alpha + M \end{bmatrix}$$

are invertible, thus system (8.2.10) is controllable and observable and therefore, $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ is stabilizable and detectable. Thus $(\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \mathcal{P}_{3,1}$.

By Theorem 8.2.3 there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{3,1} \to (0,\infty)$ such that

$$\forall\, (\widetilde{x}^0, w_0, r) \in \mathbb{R}^3 \times \mathcal{W} \times (0,\infty) \;\; :$$

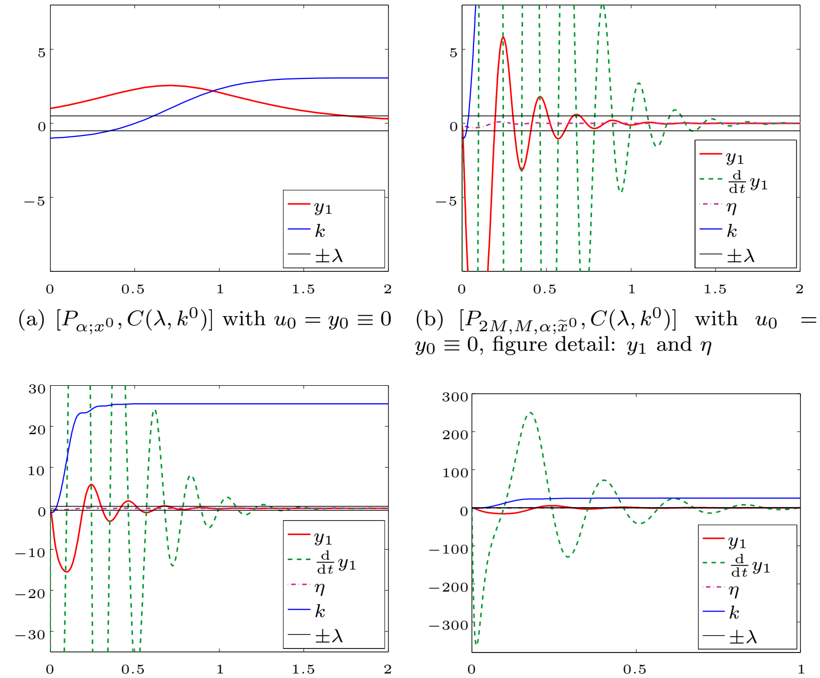
$$\left.\begin{array}{l} \psi\big((\widetilde{A}, \widetilde{b}, \widetilde{c})\big)\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P_{\alpha;0}, P_{N,M,\alpha;\widetilde{x}^0}\big) \le \eta(r) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} k \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})\,, \\ \limsup_{t\to\infty} \|y_0(t) - y_1(t)\| \le \lambda\,, \\ x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^3)\,, \end{array}\right.$$

where $\mathcal{W} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$.

This means in particular that λ-tracking is achieved by the adaptive control strategy (8.1.3) applied to system (8.2.10) despite the fact that it has unstable zero dynamics, has relative degree two and negative high-frequency gain. The only restrictions are that the zero is “far” in the right half complex plane, the initial condition $\widetilde{x}^0$ is “small” and the $W^{1,\infty}$ input/output disturbances u_0 and y_0 are “small”, too.

The result is illustrated by following simulations: recall the Byrnes–Isidori normal form (8.2.11), (8.2.12) of example plant $P_{N,M,\alpha;\widetilde{x}^0}$. Let $\alpha = 1$ and $N = 2M = 100$. Then inequality (6.3.5), see Subsection 6.3.1, yields $\vec{\delta}(P_{\alpha;0}, P_{N,M,\alpha;0}) \le 8/51$. Moreover, let $\lambda = 1/2$.

Then, for initial values $k^0 = -1$ for the controller $C(\lambda, k^0)$, $x^0 = 1$ for system (8.2.9) and $U\widetilde{x}^0 = (-0.1, 0.1, -0.1)^T$ for system (6.3.2) and input/output disturbances $u_0 = y_0 \equiv 0$, obtain Figure 8.2: Figure 8.2(a) shows the solution $t \mapsto y_1(t)$ and $t \mapsto k(t)$ of the closed-loop system (8.2.9), (8.1.3) and (8.1.2) with $u_0 = y_0 \equiv 0$. Figures 8.2(b)–8.2(d) show all components of the solution $t \mapsto \big(\xi(t)^T, \eta(t), k(t)\big)^T =$



(a) $[P_{\alpha;x^0}, C(\lambda, k^0)]$ with $u_0 = y_0 \equiv 0$

(b) $[P_{2M,M,\alpha;\widetilde{x}^0}, C(\lambda, k^0)]$ with $u_0 = y_0 \equiv 0$, figure detail: y_1 and η

(c) $[P_{2M,M,\alpha;\widetilde{x}^0}, C(\lambda, k^0)]$ with $u_0 = y_0 \equiv 0$, figure detail: k

(d) $[P_{2M,M,\alpha;\widetilde{x}^0}, C(\lambda, k^0)]$ with $u_0 = y_0 \equiv 0$, figure detail: $\frac{\mathrm{d}}{\mathrm{d}t} y_1$

Figure 8.2: Simulations; Robustness of λ-tracking

$(y_1(t), \dot{y}_1(t), \eta(t), k(t))$ of the closed-loop system (8.2.11), (8.1.3), (8.1.2) with $u_0 = y_0 \equiv 0$ in different picture details.

Note that Theorem 8.2.3 shows existence of two functions $\psi\colon \mathcal{P}_{n,m} \to (0, \infty)$ and $\eta\colon (0,\infty) \to (0,\infty)$ in (8.2.6) however, it could be hard to find these functions for a given system. Moreover, it is also possible that these functions counteract in some ways. For example: given small $r > 0$ and $\widetilde{\theta} \in \mathcal{P}_{q,m}$ such that $\vec{\delta}\big(P(\theta, 0), P(\widetilde{\theta}, 0)\big) \leq \eta(r)$ it could be possible that $\psi(\widetilde{\theta})$ is very large which requires then a very small initial value $\widetilde{x}_0 \in \mathbb{R}^q$ so that the left hand side of (8.2.6) holds. However, in view of (8.2.6) given that the second inequality holds for r and $\widetilde{\theta}$ it is always possible to choose a sufficiently small initial value.

8.3 Notes and references

The results from this chapter on robust stability for λ-tracking are shown in [IM08]. There it is shown that the λ-tracker (8.1.3) may be applied to a class of linear systems close in the gap metric to linear minimum phase systems with strict relative degree one and "positive" high-frequency gain. Moreover, the λ-tracker copes with certain bounded input/output disturbances. Some inaccuracy is present in [IM08]: although robust stability of λ-tracking is shown in [IM08] by using the terminology from [GS97], [Fre08] and [FIR06] and theorems about robust stability from [Fre08], the underlying signal spaces, namely $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ (as in the present chapter), actually do not fit in the terminology of [FIR06] and [IM08]; since all signals in $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ are absolutely continuous the definitions for the generalized signal spaces in [IM08] are not applicable to this function space. Therefore, more general definitions for extended and ambient spaces, which are also applicable to spaces of continuous functions, are presented in Section 6.1. Moreover, for this generalized signal spaces a revision of the required results from [Fre08, Sec. 5] is given in Subsection 6.5.2. However, the main result of [IM08], namely that λ-tracking is robust, holds true since the inaccuracy can be resolved, as can be seen by the analysis from Chapter 6 and the results from the present chapter.

A robustness analysis of the adaptive controller $\dot{k}(t) = \|y_2(t)\|^2$, $u_2(t) = -k(t)\, y_2(t)$, see also (4.1.4), is presented in [FIR06]. The authors utilize similar techniques as in the present chapter, for example applying the concept of gain-function stability and robustness results as in Subsection 6.5.2 or from [Fre08, Sec. V], respectively.

9 Robustness of funnel control

In this chapter it is verified that funnel control is robust. As in the previous chapter for λ-tracking, the framework of the nonlinear gap metric from Chapter 6 is applied to show that the funnel controller (5.1.3) may be applied to any stabilizable and detectable system as long as the initial conditions and the input/output disturbances are "small" and the system is "close" (in terms of a small gap metric) to a system satisfying the assumptions for funnel control, i.e. linear minimum phase systems with relative degree one and positive definite high-frequency gain matrix.

9.1 Well posedness of the closed-loop system

Recall funnel control from Chapter 5: the closed-loop system of linear system

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u_1(t)\,, \qquad x(0) = x^0, \\ y_1(t) &= C\,x(t)\,, \end{aligned}\right\} \tag{9.1.1}$$

where $(A,B,C) \in \mathcal{M}_{n,m}$, see Section 5.1, and $x^0 \in \mathbb{R}^n$ is an arbitrary initial value, and funnel controller (5.1.3), i.e.

$$\left.\begin{aligned} k(t) &= \frac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|}\,, \\ u_2(t) &= -k(t)y_2(t)\,, \end{aligned}\right\} \tag{9.1.2}$$

where $\varphi \in \Phi$, see Section 5.1, interconnected via the equations

$$u_0 = u_1 + u_2, \qquad y_0 = y_1 + y_2\,, \tag{9.1.3}$$

satisfies, for $u_0 \in \mathcal{U} := L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $y_0 \in \mathcal{Y} := W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the control objectives of funnel control, see Theorem 5.2.1.

Recall also that $(A,B,C) \in \mathcal{M}_{n,m}$ is minimum phase, has strict relative degree one with positive definite high-frequency gain matrix CB, i.e. $CB + (CB)^T > 0$.

The purpose of the present chapter is to show robust stability properties in terms of the gap metric of the funnel controller when applied to linear systems from

$$\mathcal{P}_{q,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{q\times q} \times \mathbb{R}^{q\times m} \times \mathbb{R}^{m\times q} \end{array} \;\middle|\; \begin{array}{l} (A,B,C) \text{ is stabilizable} \\ \text{and detectable} \end{array} \right\},$$

$q, m \in \mathbb{N}$, $q \geq m$, see also Chapter 8, which are sufficiently close to a system from class $\mathcal{M}_{n,m}$ but not necessarily in $\mathcal{M}_{n,m}$, $n \in \mathbb{N}$, $n \geq m$. This is equivalent to the intentions of Chapter 8.

As in Section 8.1 one may associate, for $(\theta, x^0) = ((A,B,C), x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$, the causal plant operator

$$P(\theta, x^0) \colon \mathcal{U}_a \to \mathcal{Y}_a\,, \qquad u_1 \mapsto P(\theta, x^0)(u_1) := y_1\,, \tag{9.1.4}$$

with the initial value problem (9.1.1), where $y_1 = Cx$, x being the unique solution of (9.1.1) on $[0,\omega)$ for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0,\omega)$.

Consider, for $\varphi \in \Phi$, the funnel controller (9.1.2) and associate the causal controller operator

$$C(\varphi) \colon \mathcal{Y}_a \to \mathcal{U}_a\,, \qquad y_2 \mapsto C(\varphi)(y_2) := u_2\,. \tag{9.1.5}$$

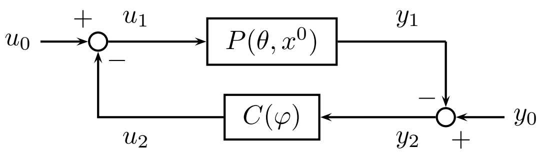


Figure 9.1: The closed-loop system $[P, C]$.

First, it is verified that, for any $\theta \in \mathcal{M}_{n,m}$, see Section 4.1, the closed-loop system $[P(\theta, x^0), C(\lambda, k^0)]$ is globally well posed and $(\mathcal{U}\times\mathcal{Y})$-stable, see Section 6.2.

In this section it is shown that, for $\mathcal{W} = \mathcal{U} \times \mathcal{Y} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, any reciprocal funnel boundary function $\varphi \in \Phi$

and every $(\theta, x^0) \in \mathcal{M}_{n,m} \times x^0$, the closed-loop system $[P(\theta, x^0), C(\varphi)]$ as depicted in Figure 9.1 is globally well posed and $\mathcal{W}$-stable. More important, it is shown that, for all $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $\varphi \in \Phi$, the closed-loop $[P(\theta, x^0), C(\varphi)]$ is regularly well posed.

Proposition 9.1.1 *Let $n, m \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, $(\theta, x^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n$ and $(u_0, y_0) \in \mathcal{W} := L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $P(\theta, x^0)$ and control operator $C(\varphi)$, given by* (9.1.4) *and* (9.1.5)*, respectively, the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$, given by* (5.1.4), (9.1.3), (9.1.2)*, is globally well posed and moreover, the closed-loop $[P(\theta, x^0), C(\varphi)]$ is $\mathcal{W}$-stable.*

Proof. The proposition is a direct consequence of Theorem 5.2.1. □

Note that, for $(A, B, C) \in \mathcal{P}_{n,m}$, $x^0 \in \mathbb{R}^n$ and $\varphi \in \Phi$, the closed-loop initial value problem (9.1.1), (9.1.3), (9.1.2) may be written as

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B[u_0(t) - u_2(t)]\,, \qquad x(0) = x^0 \in \mathbb{R}^n\,, \\ k(t) &= \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|}\,, \\ y_2(t) &= y_0(t) - Cx(t)\,, \\ u_2(t) &= -k(t)y_2(t)\,. \end{aligned}\right\} \tag{9.1.6}$$

Proposition 9.1.2 *Let $n \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $(u_0, y_0) \in \mathcal{W} := L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for the plant operator $P(\theta, x^0)$ and the control operator $C(\varphi)$, given by* (9.1.4) *and* (9.1.5)*, respectively, the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$, given by* (9.1.6)*, has the following properties:*

(i) there exists a unique solution $x\colon [0, \omega) \to \mathbb{R}^n$, for some $\omega \in (0, \infty]$, and the solution is maximal in the sense that for every compact $\mathcal{K} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ exists $t \in [0, \omega)$ such that $(t, x(t)) \notin \mathcal{K}$;

(ii) if $(u_2, y_2) \in L^\infty([0, \omega) \to \mathbb{R}^m) \times W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$ then $\omega = \infty$ and $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and y_2 is uniformly bounded away from the funnel boundary $\varphi(\cdot)^{-1}$;

(iii) $[P(\theta, x^0), C(\varphi)]$ is regularly well posed.

Proof. (i): Set, for $\varphi \in \Phi$ and $y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$,

$$\mathcal{H}_{\varphi,y_0} := \left\{ (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \,\middle|\, \varphi(t)\|y_0(t) - C\,x\| < 1 \right\} .$$

Note that, for all $x^0 \in \mathbb{R}^n$ and $y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the tuple $(0, y_0(0) - C\,x_0) \in \mathcal{H}_{\varphi,y_0}$. Then (9.1.6) may be written as

$$\dot{x}(t) = g(t, x(t)) , \qquad x(0) = x^0 , \tag{9.1.7}$$

where

$$\begin{aligned} &g \colon \mathcal{H}_{\varphi,y_0} \to \mathbb{R}^n , \\ &\qquad (t,x) \mapsto Ax + Bu_0(t) + \frac{\varphi(t)}{1 - \varphi(t)\|y_0(t) - C\,x\|} B(y_0(t) - C\,x) , \end{aligned}$$

satisfies, in view of $\varphi^{-1}|_{[\varepsilon,\infty)}$ being globally Lipschitz for every $\varepsilon > 0$ and $\varphi(0) = 0$, see the definition of Φ in Section 5.1, a local Lipschitz condition on the relatively open set $\mathcal{H}_{\varphi,y_0}$ in the sense that, for all $(\tau, \xi) \in \mathcal{H}_{\varphi,y_0}$, there exists an open neighbourhood O of (τ, ξ) and a constant $L > 0$ such that

$$\forall\, (t,x) \in O \;:\; \|g(t,x) - g(t,\xi)\| \leq L\|x - \xi\| .$$

Therefore, standard theory of ordinary differential equations, see, for example, [Wal98, Thm. III.11.III], yields that (9.1.6) has an absolutely continuous solution $x \colon [0, \omega) \to \mathbb{R}^n$ for some $\omega \in (0, \infty]$, which satisfies $(t, x) \in \mathcal{H}_{\varphi,y_0}$. Moreover, the solution is unique and the solution can be extended up to the boundary of $\mathcal{H}_{\varphi,y_0}$. In other words: for every compact $\mathcal{K} \subset \mathcal{H}_{\varphi,y_0}$ exists $t \in [0, \omega)$ such that $(t, x(t)) \notin \mathcal{K}$, as required.

(ii): Suppose $(u_2, y_2) \in L^\infty([0, \omega) \to \mathbb{R}^m) \times W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$ and, for contradiction, $\omega < \infty$. By boundedness of φ, see the definition of Φ, it follows that there exists $\lambda > 0$ such that $\varphi(t) \leq 1/\lambda$ for all $t \in [0, \omega)$. Thus

$$\begin{aligned} \forall\, t \in [0, \omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \leq 1/2 \;&\Rightarrow\; 1/2 \leq \varphi(t)\|y_2(t)\| \leq \|y_2(t)\|/\lambda \\ &\Rightarrow\; \|y_2(t)\| \geq \lambda/2 \end{aligned}$$

which yields, in view of $y_2 \in L^\infty([0, \omega) \to \mathbb{R}^m)$ and $\frac{-\varphi}{1-\varphi\|y_2\|} y_2 = u_2 \in$

$L^\infty([0,\omega) \to \mathbb{R})$, that

$$\forall\, t \in [0,\omega) \;:\; 1-\varphi(t)\|y_2(t)\| \le 1/2 \quad \Rightarrow$$
$$\|u_2\|_{L^\infty} \ge \frac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|}\|y_2(t)\| \ge \frac{\lambda\varphi(t)}{2(1-\varphi(t)\|y_2(t)\|)}\,,$$

thus $\frac{\varphi}{1-\varphi\|y_2\|}$ is bounded on $\left\{t \in [0,\omega) \,\middle|\, 1-\varphi(t)\|y_2(t)\| \le 1/2\right\}$. Moreover, for all $t \in [0,\omega)$ such that $1-\varphi(t)\|y_2(t)\| > 1/2$, $\left(\frac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|}\right) \le 2/\lambda$. Thus $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty([0,\omega) \to \mathbb{R})$. Hence, by continuity of the solution

$$\exists\, \varepsilon > 0\ \forall\, t \in [0,\omega) \;:\; 1-\varphi(t)\|y_2(t)\| \ge \varepsilon\,.$$

Then, Variation of Constants applied to (9.1.6) yields the existence of constants $c_0 = c_0(B,\lambda,\varepsilon), c_1 = c_1(A) > 0$ such that

$$\forall\, t \in [0,\omega) \;:$$
$$\|x(t)\| \le c_0\left(e^{c_1\omega} + \int_0^\omega e^{c_1(\omega-s)}\left(\|u_0(s)\| + \|y_2(s)\|\right)\,\mathrm{d}s\right). \qquad (9.1.8)$$

Since $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$ and $u_0 \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, it follows from the convolution in (9.1.8) that the right hand side of (9.1.8) is bounded by $c_3 = c_0\big(e^{c_1\omega} + (e^{c_1\omega}+1)(\|u_0\|_{L^\infty([0,\omega)\to\mathbb{R}^m)} + \|y_2\|_{L^\infty([0,\omega)\to\mathbb{R}^m)})/c_1\big) > 0$ on $[0,\omega)$ which gives that

$$\mathcal{K} := \left\{(t,x) \in \mathcal{H}_{\varphi,y_0} \,\middle|\, t \in [0,\omega]\,,\ \|x\| \le c_3\right\}$$

is a compact subset of $\mathcal{H}_{\varphi,y_0}$ with $(t,x(t)) \in \mathcal{K}$ for all $t \in [0,\omega)$, which contradicts the fact that the closure of graph $\left(x|_{[0,\omega)}\right)$ is not a compact set, see (i). Therefore, $\omega = \infty$.

(iii): By (i), the closed-loop initial value problem $[P(\theta,x^0), C(\varphi)]$ is locally well posed. To prove that $[P(\theta,x^0), C(\varphi)]$ is regularly well posed, it suffices to show that (6.2.2) holds. For arbitrary $w_0 = (u_0,y_0) \in \mathcal{W}$ consider $(w_1,w_2) = H_{P(\theta,x^0),C(\varphi)}(w_0)$ where $\mathrm{dom}(w_1,w_2) = [0,\omega)$ is maximal. Suppose, contrary to the right hand side of (6.2.2), that $\left\|(w_1,w_2)|_{[0,\omega)}\right\|_{\mathcal{W}_\omega\times\mathcal{W}_\omega} < \infty$. Then $(u_2,y_2) \in L^\infty([0,\omega) \to \mathbb{R}^m) \times W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$, which, in view of (ii), yields $\omega = \infty$, i.e. the contrary of the left hand side of (6.2.2). Hence the closed-loop system is regularly well posed and the proof is complete. □

Regularly well posedness of the closed-loop system $[P(\theta, x^0), C(\varphi)]$ for all $\theta \in \mathcal{P}_{n,m}$ is crucial to show robust stability for funnel control because of the application of the results from Subsection 6.5.2. The results in the following section are analogous to the robust stability results for λ-tracking in Section 8.2.

9.2 Robust stability

In Section 8.2 it was shown that λ-tracking works also for linear systems $P(\widetilde{\theta}, \widetilde{x}^0)$ with $\widetilde{\theta} \in \mathcal{P}_{q,m}$, $q, m \in \mathbb{N}$, $q \geq m$, and initial value $\widetilde{x}^0 \in \mathbb{R}^q$, if three requirements are satisfied:

1) the system $\widetilde{\theta}$ is close to a system $\theta \in \mathcal{M}_{n,m}$, $n \in \mathbb{N}$, $n \geq m$, in other words, the gap metric $\vec{\delta}(P(\widetilde{\theta}, 0), P(\theta, 0))$ is sufficiently small;

2) the initial value $\widetilde{x}^0$ is sufficiently small;

3) input/output disturbances $(u_0, y_0) \in \mathcal{W}$ are sufficiently small.

An equivalent stability result is shown for funnel control in the present section. Proofs and lemmata are adapted from Section 8.2, too.

As for λ-tracking the augmented plant and controller operators as in Section 8.2 and Subsection 6.5.2 are required. Note that for funnel control and λ-tracking slightly different signals spaces are considered. In the following, let $\mathcal{W} = \mathcal{U} \times \mathcal{Y} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and define $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2mn} \times \mathcal{U}$ and $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}} \times \mathcal{Y}$. Recall that, by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$, the norm of $\widetilde{\mathcal{U}}$ is given by $\|(\theta, u)\|_{\widetilde{\mathcal{U}}} := \sqrt{\|\theta\|^2 + \|u\|^2_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)}}$. For given $P(\theta, 0)$ as in (9.1.4), define the (augmented) plant operator as in (6.5.13) and (8.2.1) by

$$\begin{aligned} \widetilde{P} \colon \widetilde{\mathcal{U}}_a &\to W_a^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ (\theta, u_1) = \widetilde{u}_1 &\mapsto y_1 = \widetilde{P}(\widetilde{u}_1) := P(\theta, 0)(u_1), \end{aligned} \tag{9.2.1}$$

and, for $\varphi \in \Phi$ and for $C(\varphi)$ as in (9.1.5), the (augmented) controller operator, as in (6.5.14) and (8.2.2) but for the funnel controller, by

$$\begin{aligned} \widetilde{C}\colon W_a^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) &\to \widetilde{\mathcal{U}}_a\,, \\ y_2 \mapsto \widetilde{u}_2 = \widetilde{C}(y_2) &:= (0, C(\varphi)(y_2)) = (0, u_2)\,. \end{aligned} \tag{9.2.2}$$

Moreover, define for each non-empty $\Omega \subset \mathcal{M}_{n,m}$, equivalently to (8.2.3),

$$\mathcal{W}^\Omega := (\Omega \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \quad \text{and} \quad H^\Omega_{\widetilde{P},\widetilde{C}} := H_{\widetilde{P},\widetilde{C}}|_{\mathcal{W}^\Omega}\,. \tag{9.2.3}$$

Also equivalently as for λ-tracking in the previous chapter one can show gain-function stability of $H^\Omega_{\widetilde{P},\widetilde{C}}$.

Proposition 9.2.1 *Let $n, m \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$ and assume $\Omega \subset \mathcal{M}_{n,m}$ is closed. Then, for the closed-loop system $[\widetilde{P}, \widetilde{C}]$ given by* (6.2.1)*, and the augmented operators* (9.2.1) *and* (9.2.2)*, the operator $H^\Omega_{\widetilde{P},\widetilde{C}}$ given by* (9.2.3) *is gain-function stable.*

Note that since the proof of Proposition 9.2.1 is identical – with one exception: apply the funnel control result Theorem 5.2.1 instead of the λ-tracking result Theorem 4.2.1 – to the proof of Proposition 8.2.1, it is omitted here.

Next, as for λ-tracking in Section 8.2, it is shown that the closed-loop system $[P(\widetilde{\theta}, \widetilde{x}^0), C(\varphi)]$ for a system $\widetilde{\theta}$ belonging to the system class $\mathcal{P}_{q,m}$ is $\big(L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stable if, for any system θ belonging to $\mathcal{M}_{n,m}$, the gap between $P(\widetilde{\theta}, 0)$ and $P(\theta, 0)$, the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ and the input/output disturbances $w_0 = (u_0, y_0)$ are sufficiently small.

Proposition 9.2.2 *Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0, \varphi) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \Phi$, consider $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$, and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (9.1.4) *and* (9.1.5)*, respectively. Then there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$*

such that the following holds:

$$\forall \left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ :$$
$$\left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \leq r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \leq \eta(r) \end{array}\right\} \implies H_{P(\widetilde{\theta},\widetilde{x}^0),C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W} \,. \tag{9.2.4}$$

Again the proof of Proposition 9.2.2 is identical to the proof of Proposition 8.2.2 if the gain-function stability result Proposition 9.2.1 for funnel control is applied instead of the corresponding result for λ-tracking (Proposition 8.2.1) and one chooses the signal spaces for funnel control (namely $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ instead of $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$) when applying Theorems 6.5.3 and 6.5.4. Therefore, the proof of Proposition 9.2.2 is omitted here.

With the above proposition one can state and prove the main result of this chapter, namely robust stability of funnel control. As for λ-tracking, see Theorem 8.2.3, it is shown that, loosely speaking, funnel control also works for any systems $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ which is sufficiently close to a system $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ and the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ for $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ and the input/output disturbances (u_0, y_0) are sufficiently small.

The system $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ need not necessarily be minimum phase, may have higher relative degree and negative high-frequency gain . Note that the following theorem is the equivalent to Theorem 8.2.3, however, the proof is much shorter.

Theorem 9.2.3 *Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, $\varphi \in \Phi$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ consider the associated operators $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (9.1.4) *and* (9.1.5)*, respectively, and the closed-loop initial value problem* (9.1.1), (9.1.3), (5.1.3)*. Then there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function*

$\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ *such that the following holds:*

$$\forall \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \;:$$
$$\left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r) \end{array}\right\} \;\Rightarrow\; \left\{\begin{array}{l} \forall\, t \ge 0 \;:\; (t, y_2(t)) \in \mathcal{F}_{\varphi}\,, \\ k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})\,, \\ x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)\,, \end{array}\right. \tag{9.2.5}$$

where (x,k) *and* y_2 *satisfy* (9.1.6) *and*

$$\mathcal{F}_{\varphi} = \{(t,y) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\;.$$

Proof. *Step 1*: First, it is shown that

$$\big((u_1,y_1),(u_2,y_2)\big) = H_{P(\widetilde{\theta},\widetilde{x}^0),C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W}\,. \tag{9.2.6}$$

Choose functions $\eta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ from Proposition 9.2.2. Let

$$\big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \;:$$
$$\psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \;\wedge\; \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \;\eta(r)\,.$$

Then Proposition 9.2.2 gives (9.2.6).

Step 2: By Proposition 9.1.2 it follows that (9.1.6) has a unique solution

$$x\colon [0,\omega) \to \mathbb{R}^q$$

on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$. Proposition 9.1.2(iii) yields $\omega = \infty$ and $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, the second assertion of (9.2.5).

Step 3: By Step 2 follows $k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$ which, in view of continuity of $1-\varphi\|y_2\|$ on $(0,\infty)$, yields $1-\varphi(t)\|y_2(t)\| \ge \|\varphi\|_{L^\infty}\|k\|_{L^\infty}^{-1} > 0$. Thus, for all $t \ge 0$, $\varphi(t)\|y_2(t)\| < 1$, which yields the first assertion of (9.2.5).

Step 4: It remains to show that $x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$.

Let $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ associated with (9.1.1). Detectability of system

$(\widetilde{A}, \widetilde{B}, \widetilde{C})$ yields the existence of $F \in \mathbb{R}^q$ such that $\widetilde{A} + F\widetilde{C}$ is Hurwitz. Setting $g := -\left[F + k\,\widetilde{B}\right](y_0 - y_2) + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0$ gives

$$\dot{x} = \left[\widetilde{A} - k\,\widetilde{B}\widetilde{C}\right]x + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0 = \left[\widetilde{A} + F\widetilde{C}\right]x + g\,. \tag{9.2.7}$$

By Proposition 9.2.2 and Step 3 it follows that $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and since $w_0 = (u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ it follows that $g \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (9.2.7) and Variation of Constants one arrives at $x \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. The first equation in (9.1.6) then gives $\dot{x} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ which shows the third assertion in (9.2.5) and the proof is complete. □

Following this result is illustrated by application of funnel control to the same example systems from the previous chapter.

9.2.1 Example: robust stability of funnel control

Revisit the example plants (8.2.9) and (8.2.10) from Subsection 8.2.1 (see also Subsection 6.3.1).

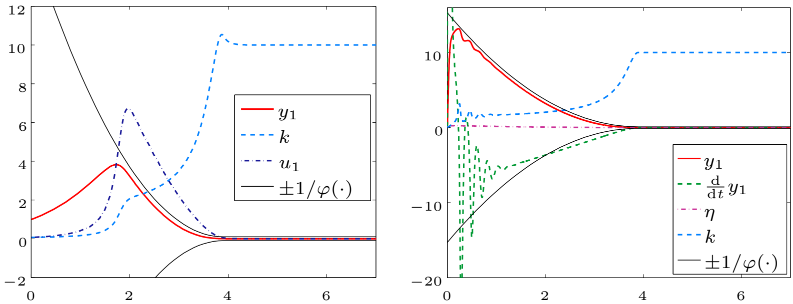
Recall that, for zero initial conditions, the gap between the system $P_{\alpha;0}$ and $P_{N,M,\alpha;0}$ given by (8.2.9) and (8.2.10), respectively, tends to zero as $N = 2M$ and M tends to infinity, see (6.3.3). Moreover, recall that the systems satisfy $(\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \mathcal{P}_{3,1} \setminus \mathcal{M}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$.

By Theorem 9.2.3 there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{3,1} \to (0,\infty)$ such that

$$\forall\, (\widetilde{x}^0, w_0, r) \in \mathbb{R}^3 \times \mathcal{W} \times (0,\infty)\ :$$
$$\left.\begin{array}{l} \psi\big((\widetilde{A}, \widetilde{b}, \widetilde{c})\big)\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \leq r \\ \vec{\delta}\big(P_{\alpha;0}, P_{N,M,\alpha;0}\big) \leq \eta(r) \end{array}\right\} \Rightarrow \begin{cases} \forall\, t \geq 0\ : \\ \qquad (t, y_0(t) - y_1(t)) \in \mathcal{F}_\varphi\,, \\ k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \\ x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^3)\,, \end{cases}$$

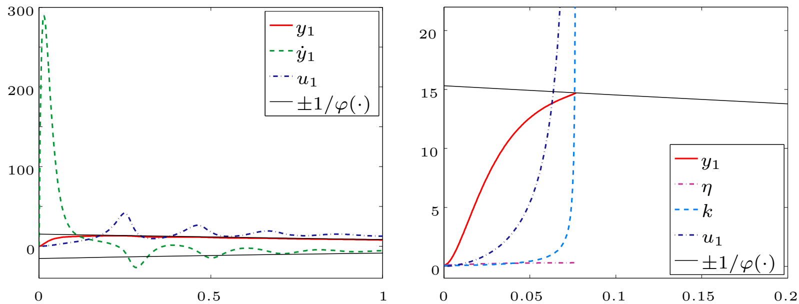
where $\mathcal{W} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Note that Theorem 9.2.3 shows only existence of two continuous functions $\psi\colon \mathcal{P}_{n,m} \to (0,\infty)$ and $\eta\colon (0,\infty) \to (0,\infty)$ in (9.2.5) however, it could be hard to find these functions for a given system.

This result is visualized by MATLAB simulations.



(a) $[P_{\alpha;x^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$, figure detail: y_1, k and u_1

(b) $[P_{N,M,\alpha;\tilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$, figure detail: y_1, η and k



(c) $[P_{N,M,\alpha;\tilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$, figure detail: $\frac{\mathrm{d}}{\mathrm{d}t}$ and u_1

(d) $[P_{N,M,\alpha;\tilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$ and inital value $(0.1, 0.1, 0.1)$

Figure 9.2: Simulations; Robustness of funnel control

Recall normal form (8.2.11)–(8.2.12) of the example plant $P_{N,M,\alpha;\tilde{x}^0}$. Let $\alpha = 1$ and $N = 2M = 100$ then (6.3.5) yields $\vec{\delta}(P_{\alpha;0}, P_{N,M,\alpha;0}) \leq 8/51$. Moreover, let, for $\lambda = 0.1$ and

$$\varphi^{-1} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}\,, \qquad t \mapsto \begin{cases} 15.31 - 7.8\,t + t^2\,, & \text{if } t \in [0, 3.9) \\ \lambda\,, & \text{if } t \geq 3.9. \end{cases}$$

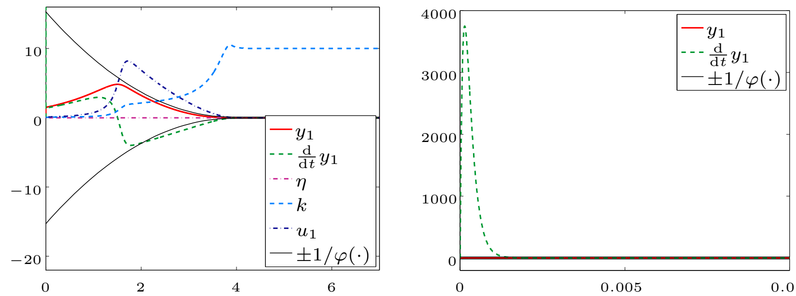
For initial values $x^0 = 1$ for system (8.2.9) and $U\widetilde{x}^0 = (0.1, 0.1, 0.08)^T$ for the second system (8.2.10) and input/output disturbances $u_0 = y_0 \equiv 0$, Figures 9.2(a) shows the solution $t \mapsto y_1(t)$, k and the input u_1 of the closed-loop system (6.3.1), (5.1.3), (9.1.3). Moreover, Figures 9.2(b) and 9.2(c) show the components of the solution $t \mapsto \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ \dot{y}_1(t) \\ \eta(t) \end{pmatrix}$, k and u_1 of the closed-loop system (8.2.11), (5.1.3), (9.1.3), where Figures 9.2(c) indicates that all states (in particular $\xi_2 = \dot{y}_1$) are bounded.

Figure 9.2 illustrates that the funnel controller (5.1.3) may be applied to linear systems, which do not satisfy the classical assumptions for funnel control, but are close in terms of the gap metric to minimum phase systems with relative degree one and positive high-frequency gain, as long as the initial values are small.

Figure 9.2(d) indicates that, for too large initial values, here $U\widetilde{x}^0 = (0.1, 0.1, 0.1)^T$, the output $t \mapsto y_1(t)$ approaches the funnel boundary $1/\varphi(\cdot)$ after some time $t' > 0$ and therefore $\lim_{t \to t'} k(t) = \infty$ and $\lim_{t \to t'} u_1(t) = \infty$. Thus, the solution has (y_1, u_1) of the closed-loop system has a finite escape time.

A shortcoming of the main result is that it shows only existence of functions ψ and η in (9.2.5), compare also with the result for λ-tracking. For a given systems $\widetilde{\theta}$ it is maybe hard to calculate the value $\psi(\widetilde{\theta})$. It could be also possible that this functions counteract in some ways. For example: given small $r > 0$ and $\widetilde{\theta} \in \mathcal{P}_{q,m}$ such that $\vec{\delta}\big(P(\theta, 0), P(\widetilde{\theta}, 0)\big) \leq \eta(r)$ it could be possible that $\psi(\widetilde{\theta})$ is very large which requires then a very small initial value $\widetilde{x}_0 \in \mathbb{R}^q$ so that the left hand side of (9.2.5) holds. However, in view of (9.2.5) given that the second inequality holds for r and $\widetilde{\theta}$ it is always possible to choose a sufficiently small initial value. This is shown with the simulation in Figure 9.3: choose $P_{N,M,\alpha;\widetilde{x}^0}$ with $\alpha = 1$, $N = 2M = 10000$ and the initial value $\widetilde{x}^0 = (0.0001, 0.0001, 0.0001)$.

Figure 9.3(a) shows that the output y_2 is within the funnel and k is bounded. Figure 9.3(b) shows that all states are bounded, although though $\dot{y}_1$ is very large.



(a) $[P_{N,M,\alpha;\tilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$, figure detail: y_1, η, k and u_1

(b) $[P_{N,M,\alpha;\tilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$, figure detail: $\frac{d}{dt} y_1$

Figure 9.3: Simulations; Robustness of funnel control for $P_{N,M,\alpha;\tilde{x}^0}$ with "huge" $N = 2M = 10000$

9.3 Notes and references

Robust stability of the funnel controller is a new result which will be submitted for publication in due course [IM09]. The results are very similar to those for λ-tracking, see [IM08]. Since [IM08] has some inaccuracy which result from the application of the terminology from [GS97], [Fre08] and [FIR06] and theorems on robust stability from [Fre08] on signal spaces $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, see also the comments in Section 8.3, the terminology in [IM09] will be adapted from the present thesis.

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List of Symbols

$\mathbb{N}, \mathbb{N}_0$; the set of natural numbers without and with zero, respectively

$\mathbb{R}$, $\mathbb{C}$; the sets of real/complex numbers

$\mathbb{R}^n$, $\mathbb{C}^n$;
the sets of real/complex vectors

$\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$;
the sets of real/complex matrices

$\mathbb{R}_{\geq\tau} = [\tau, \infty)$;
the set of real numbers larger than $\tau \in \mathbb{R}$

$\mathbb{C}_+ = \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$;
the open right half complex plane

$\mathbb{C}_- = \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$;
the open left half complex plane

$\left[l_1^{(n)}, \ldots, l_m^{(n)}\right] = L \in \mathbb{R}^{n\times m}$;
where $l_i^{(n)} \in \mathbb{R}^n$ denotes the i-th column of L and the superscript (n) remarks the dimension of the vector

$\left[l_{(m)}^1 / \ldots / l_{(m)}^n\right] = L \in \mathbb{R}^{n\times m}$;
where $l_{(m)}^j \in \mathbb{R}^{1\times m}$ denotes the j-th row of L and the subscript (m) remarks the dimension of the row-vector

$e_k^{(n)} = \left[0_{1\times(k-1)}, 1, 0_{1\times(n-k)}\right]^T$;
the k-th row unit vector in $\mathbb{R}^n$

$e_{(m)}^k = \left[0_{1\times(k-1)}, 1, 0_{1\times(m-k)}\right]$;
the k-th row unit vector in $\mathbb{R}^{1\times m}$

$0_{n\times m} \in \mathbb{R}^{n\times m}$;
the 0-matrix of dimension $n \times m$

$\mathcal{X}_{n\times m} \in \mathbb{R}^{n\times m}$;
an arbitrary matrix of dimension $n \times m$; note that the use of this symbol implicates that the specific entries of the matrix are not important but only the dimension

$I_n \in \mathbb{R}^{n\times n}$;
the identity matrix of dimension $n \times n$

$\operatorname{diag}(A_1, \dots, A_m) \in \mathbb{C}^{n\times n}$;
a matrix with $A_i \in \mathbb{C}^{j_i \times j_i}$, $i = 1, \dots, m$, on the diagonal and zeros otherwise

$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} \mid \det(\lambda I_n - A) = 0\}$;
the spectrum of a matrix $A \in \mathbb{C}^{n\times n}$

$\operatorname{im} A$; the image (or range) of a matrix A

$\ker A$; the kernel of a matrix A

$\operatorname{rk} A$; the rank of a matrix A

$A^T \mathbb{C}^{m\times n}, x^T \mathbb{C}^{1\times n}$;
the transpose of a matrix $A \in \mathbb{C}^{n\times m}$ or a vector $x \in \mathbb{C}^n$

$A \in \mathbb{C}^{n\times n}$ is called positive definite;
if, and only if, $x^*(A + A^*)x > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$; write also $A + A^T > 0$

$A \in \mathbb{R}^{n\times n}$ is called Hurwitz;
if, and only if, $\operatorname{spec}(A) \subset \mathbb{C}_-$

$\operatorname{Re} z$, $\operatorname{Im} z$;
the real/imaginary part of a complex number or vector z

$\bar{x}$; the complex conjugate of scalar of vector x

$\operatorname{adj} A$; the adjoint of a matrix $A \in \mathbb{C}^{n\times n}$

$\|x\|$; the euclidian norm of $x \in \mathbb{C}^n$

$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$;
the matrix norm of $A \in \mathbb{C}^{n \times m}$ induced by the euclidian norm

$\|v\|_{\mathcal{V}}$; the norm of $v \in \mathcal{V}$, for any normed vector space $\mathcal{V}$

$\langle \cdot, \cdot \rangle$; the scalar product in $\mathbb{R}^n$, where $n \in \mathbb{N}$

$\mathrm{map}(E \to F)$;
the set of all maps from the set E to the set F

$\mathbb{R}[s] = \left\{ p(s) = \sum_{i=0}^{m} a_i s^i \;\middle|\; m \in \mathbb{N},\ a_0, \ldots, a_m \in \mathbb{R} \right\}$;
the ring of polynomials with real coefficients

$\mathbb{C}[s] = \left\{ p(s) = \sum_{i=0}^{m} a_i s^i \;\middle|\; m \in \mathbb{N},\ a_0, \ldots, a_m \in \mathbb{C} \right\}$;
the ring of polynomials with complex coefficients

$\mathbb{R}^H[s] = \{ p \in \mathbb{R}[s] \mid \mu(p) < 0 \}$;
the set of all Hurwitz polynomials

$\mathbb{R}(s) = \frac{\mathbb{R}[s]}{\mathbb{R}[s] \setminus \{0\}}$;
the quotient field of real rational functions

$\mathcal{C}^r(I \to \mathbb{R}^\ell)$;
the set of r-times continuously differentiable maps from I to $\mathbb{R}^\ell$, where $r \in \mathbb{N}_0 \cup \{\infty\}$ and $I \subset \mathbb{R}$ is an interval

$\mathcal{C}_{\mathrm{pw}}(I \to \mathbb{R}^\ell)$;
the set of piecewise continuous maps from I to $\mathbb{R}^\ell$, where $I \subset \mathbb{R}$ is an interval

$L^p(I \to \mathbb{R}^\ell)$;
the space of p-integrable functions $y \colon I \to \mathbb{R}^\ell$, where $p \in [1, \infty)$, $\ell \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an interval, with norm $\|y\|_{L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)} = \left(\int_0^\infty \|y(t)\|^p \,\mathrm{d}t \right)^{\frac{1}{p}}$

$L^p_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$;
the space of locally p-integrable functions $y \colon I \to \mathbb{R}^\ell$, with $\int_K \|y(t)\|^p \,\mathrm{d}t < \infty$ for all compact $K \subset I$, where $p \in [1, \infty)$, $\ell \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an interval

$L^\infty(I \to \mathbb{R}^\ell)$;
the space of essentially bounded functions $y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$, where $\ell \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an interval, with norm $\|y\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)} = \operatorname{ess\,sup}_{t \geq 0} \|y(t)\|$

$L^\infty_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$;
the space of locally bounded functions $y\colon I \to \mathbb{R}^\ell$ with $\operatorname{ess\,sup}_{t \in K} \|y(t)\| < \infty$ for all compact $K \subset I$, where $\ell \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an interval

$CL^p(I \to \mathbb{R}^\ell) = \mathcal{C}(I \to \mathbb{R}^\ell) \cap L^p(I \to \mathbb{R}^\ell)$;
the set of continuous L^p-functions from I to $\mathbb{R}^\ell$, where $p \in [1, \infty]$, $\ell \in \mathbb{N}$ and $I \subset \mathbb{R}$ an interval, with norm $\|y\|_{CL^p(I \to \mathbb{R}^\ell)} = \|y\|_{L^p(I \to \mathbb{R}^\ell)}$

$W^{r,p}(I \to \mathbb{R}^\ell)$;
the Sobolev space of r-times almost everywhere differentiable maps $y\colon I \to \mathbb{R}^\ell$, where $r \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$, $p \in [1, \infty]$ and $I \subset \mathbb{R}$ an interval, with norm $\|y\|_{W^{r,p}(I \to \mathbb{R}^\ell)} = \sum_{i=0}^{r} \|D^{(i)} y\|_{L^p(I \to \mathbb{R}^\ell)}$, where $D^{(i)} y$ is the i-th weak derivative of y, see, for example [AF03, Para. 1.62]

$CW^{r,p}(I \to \mathbb{R}^\ell) = W^{r,p}(I \to \mathbb{R}^\ell) \cap \mathcal{C}^r(I \to \mathbb{R}^\ell)$;
the set of r-times continuously differentiable $W^{r,p}$-functions, where $r \in \mathbb{N} \cup \{\infty\}$, $\ell \in \mathbb{N}$, $p \in [1, \infty]$ and $I \subset \mathbb{R}$ an interval, with norm $\|y\|_{CW^{r,p}(I \to \mathbb{R}^\ell)} = \sum_{i=0}^{r} \|y^{(i)}\|_{L^p(I \to \mathbb{R}^\ell)}$

$CW_0^{r,p}(I \to \mathbb{R}^\ell) = \left\{ y \in CW^{r,p}(I \to \mathbb{R}^\ell) \;\middle|\; \begin{array}{l} \text{if } 0 \in I \text{ then } y^{(i)}(0) = 0 \\ \text{for all } i \in \{0, \ldots, r-1\} \end{array} \right\}$;
the set of r-times continuously differentiable functions in $W^{r,p}$ which first $r-1$ derivatives are zero at 0, where $r, \ell \in \mathbb{N}$, $p \in [1, \infty]$ and $I \subset \mathbb{R}$ is an interval, with norm $\|y\|_{CW^{r,p}(I \to \mathbb{R}^\ell)}$

$CW_0^{\infty,p}(I \to \mathbb{R}^\ell) = \left\{ y \in CW^{\infty,p}(I \to \mathbb{R}^\ell) \;\middle|\; \begin{array}{l} \text{if } 0 \in I \text{ then} \\ \forall\, i \in \mathbb{N}_0 \,:\, y^{(i)}(0) = 0 \end{array} \right\}$;
the set of infinitely often continuously differentiable functions

in $W^{r,p}$ which are zero at 0 and all derivatives are zero at 0, where $p \in [1, \infty]$, $\ell \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an interval, with norm $\|y\|_{CW^{\infty,p}(I \to \mathbb{R}^\ell)}$

deg p; the degree of a polynomial p in $\mathbb{R}[s]$ or $\mathbb{C}[s]$

$\mu(A) = \max\{\text{Re } s \mid s \in \text{spec}(A)\}$;
the largest real part of the eigenvalues of $A \in \mathbb{C}^{n \times n}$

$\mathcal{Z}(p) = \{s \in \mathbb{C} \mid p(s) = 0\}$;
the set of zeros of $p \in \mathbb{C}[s]$

$\mu(p(\cdot)) = \max\{\text{Re } s \mid s \in \mathcal{Z}(p)\}$;
the largest real part of the zeros of $p \in \mathbb{C}[s]$

$\mathcal{B}_\delta(s_0) = \{s \in \mathbb{C} \mid |s - s_0| < \delta\}$;
the ball in $\mathbb{C}$ of radius $\delta > 0$ around $s_0 \in \mathbb{C}$

Σ_m; the set of permutations of $\{1, \ldots, m\}$, where $m \in \mathbb{N}$

sgn(σ); the sign of a permutation $\sigma \in \mathcal{S}_m$

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