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Quasistatic inflation processes within compliant tubes

Part 1: Analytical investigations

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Abstract

Continuing former work [8], [9] the authors consider a mechanical system that models a segment of a live or artificial worm or a balloon for angioplasty that is placed within a cylindrical compliant tube (vein). The statics of the inflation process is based on the Principle of Minimal Potential Energy. This is handled as an optimal control problem with state constraint. Certain peculiarities make the necessary optimality conditions go beyond those from classical textbooks. A careful analysis of the conditions leads to a boundary value problem describing the shape of the inflated system and to the determination of the contact forces between balloon and vein. Simulation results are to be presented in a forthcoming Part 2.

Key words Biomechanics, optimal control, state constraint.
MSC (2000) 49J15, 49M05, 74F10, 74K15.

1 Problem formulation

1.1 Introduction

In this paper we continue investigations published in [8] and [9]. There, the authors considered the statical behavior of compliant mechanical elements called "segments". Such an element has a hull that consists of two rigid circular discs connected by a deformable membrane of circular cylindrical original shape. When this (stress-free) cylinder is filled with some (incompressible) fluid of fixed volume greater than that of the cylinder the segment deforms into some body of revolution. The membrane enters a state of stress, the discs longitudinally displace, and a hydrostatic pressure arises within the fluid. Under some working hypotheses concerning the kind of compliance of the membrane (in particular meridional inextensibility) a boundary value problem - derived either from local equilibrium conditions or from an equivalent variational problem - governs this inflation process. During this process the segment could either be free to expand radially or it could be restricted by a surrounding rigid tube, cylindrical or showing a constriction. In the latter case the membrane of the inflating segment more and more presses against the tube thereby simulating what happens during dilation of a vessel in medical surgery (disregarding the severe falsification of reality by the supposed rigidity of the tube).

In what follows we shall diminish this restrictive assumption by replacing the rigid tube with a thin-walled compliant one. Now the tube is able to deform radially when the inflated segment presses from inside, and this may give a more realistic image of a dilation process. In regard of this interpretation we shall call this tube a *vein*. It is just for the sake of keeping the analytical effort bounded that we assume the vein to have constant original thickness (thereby of course excluding stenoses from investigation) and the same kind of compliance as the segment, a possibly inevitable meridional extensibility could be captured by an *elastic* fastening of the ends of the tube to rigid walls. Following this description we may expect a scenario as sketched in Figure 1.

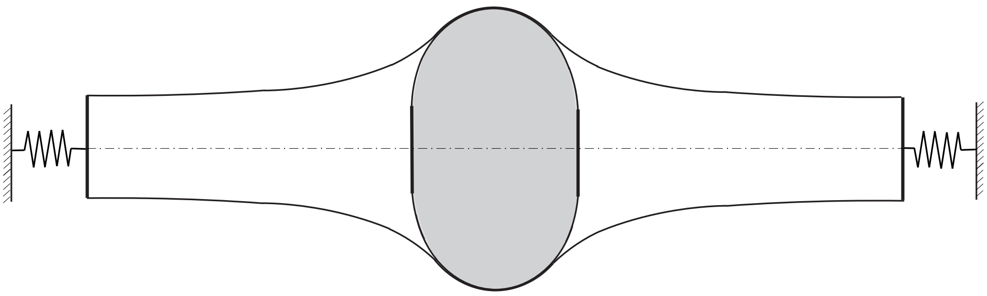

Figure 1: Inflated segment within vein.

As in the foregoing paper [9] mentioned above we encounter a well-known inconvenience immanent to the problem formulation: the region of contact segment-vein (interval of meridian) is of course not known in advance rather it depends on the hydrostatic pressure within the segment and so does the pressure between membrane and vein along this region. Moreover, the shape of the deformed membrane is strongly influenced by the vein (and vice versa) on the region of contact while it is not directly influenced outside of this region. These facts could be a bit unpleasant if the differential equations which govern the problem were to be gained from local equilibrium conditions (synthetical method). Another unfortunate fact is the lack of knowledge about the smoothness of the deformed (thin!) membranes.

We overcome these difficulties in the same analytical way as it has been done in the former paper. The treatment of the problem is based on the Principle of Minimal Potential Energy for the total system. This Principle shows up as a variational problem or, equivalently, as an optimal control problem under state constraint (radius of segment no greater than radius of vein). The crucial point in this formulation is that, first, only minimal smoothness suppositions are needed, and that, second, the unknown pressure of contact, being the *reaction to the state constraint* does not enter the Principle. A careful analysis of the optimality conditions then yields clear statements about the actual smoothness properties, and differential equations which determine the shape of the overall system. Finally, after this has been managed, the geometry of both segment and vein on the contact region is well-known, and the constraint pressure follows from the Lagrange multiplier corresponding to the state constraint (with a comparing glance at the membrane equations of shell theory, [4]).

The central optimal control problem exhibits some features which make it a "non-classical" problem: the cost functional is the sum of two integrals with different integration intervals, and the state constraint is given only on the smaller of these intervals. Therefore the necessary optimality conditions deserve a very careful deduction. These investigations are the contents of the unpublished paper [1], their results are adapted to the problem of the present paper.

It does not bring about essential troubles if we consider an augmented mechanical problem by allowing for additional arbitrary longitudinal forces $\pm F_1$ acting upon the side discs of the segment. Now think of the segment being inflated (and dilating the surrounding vein) under zero longitudinal forces. Then, while keeping the internal pressure fixed, non-zero forces, generated inside the segment or from outside by wire, and pressing or pulling the segment, change the shape of the segment, the dilation of the vein, and the constraint pressure between membrane and vein. Such a scenario could be seen in correspondence with some procedure to run at the tip of an endoscope.

The following considerations start with a sketch of geometry and mechanics of hyperelastic skin-like membrane shells, leading to an expression of the potential energy that can be seen as a mathematical model of the system to be investigated. A normalization makes the model applicable to systems of arbitrary dimension. The analysis then is (we hope) strictly mathematical and avoids any physical arguments. A physical interpretation of the results is given at the end.

1.2 Geometry, rheology, and potential energy

Supposing both segment and vein in deformed state to be of rotational symmetry we describe their surfaces of revolution by means of surface coordinates $\phi \in [0, 2\pi)$ (latitude) and s (arc-length of meridian). So we have the radius vectors (with functions of a sufficient smoothness class)
of the membrane:

$$\mathbf{r}_1(\phi, s_1) = x_1(s_1)\mathbf{e}_x + y_1(s_1)\{\cos\phi\mathbf{e}_z + \sin\phi\mathbf{e}_y\},\ s_1 \in [-s_{10}, s_{11}], \tag{1}$$

of the vein:

$$\mathbf{r}_2(\phi, s_2) = x_2(s_2)\mathbf{e}_x + y_2(s_2)\{\cos\phi\mathbf{e}_z + \sin\phi\mathbf{e}_y\},\ s_2 \in [-s_{20}, s_{21}], \tag{2}$$

In either case the standard meridian ($\phi = 0$) is given by its natural equation

$$\frac{dx}{ds} = \cos u,\ \ \frac{dy}{ds} = \sin u,\ \ \frac{du}{ds} = \kappa, \tag{3}$$

where $u(s)$ is the angle from $\mathbf{e}_x$ to the tangent vector of the meridian, and $\kappa(s)$ is the curvature of the meridian at that point. The moving frame is ($\frac{dx}{ds} =: \dot{x}$, etc.)

$$\begin{aligned}
&\mathbf{g}_1 := \mathbf{r}_{,\phi} = y(s)\{-\sin\phi\mathbf{e}_y + \cos\phi\mathbf{e}_z\},\\
&\mathbf{g}_2 := \mathbf{r}_{,s} = \dot{x}(s)\mathbf{e}_x + \dot{y}(s)\{\cos\phi\mathbf{e}_y + \sin\phi\mathbf{e}_z\},\\
&\mathbf{n} = -\dot{y}(s)\mathbf{e}_x + \dot{x}(s)\{\cos\phi\mathbf{e}_y + \sin\phi\mathbf{e}_z\},
\end{aligned} \tag{4}$$

It entails the metric tensor $g_{\alpha\beta}:\ g_{11} = y^2,\ g_{12} = 0,\ g_{22} = 1$, and the 2nd fundamental tensor $b_{\alpha\beta}:\ b_{11} = -y\dot{x},\ b_{12} = 0,\ b_{22} = \kappa = \dot{x}\ddot{y} - \ddot{x}\dot{y}$.

The compliant segment is in particular characterized by the hypothesis that its deformable latitudinal hull statically behaves like a membrane shell, i.e., like a solid shell with no stress couples (i.e., with no resistance against bending = change of curvature). The *membrane* from above then is nothing else but the middle surface of this shell.

The local equilibrium of the membrane under the action of the *external force per unit area*

$$\mathbf{P} = P^\alpha \mathbf{g}_\alpha + P_n \mathbf{n}$$

is governed by the *membrane equations* [4]

$$\nabla_\beta N^{\alpha\beta} + P^\alpha = 0, \quad N^{\alpha\beta} = N^{\beta\alpha}, \quad b_{\alpha\beta} N^{\alpha\beta} + P_n = 0.$$

Here, $N^{\alpha\beta}$ are the stress resultants per unit of length, they determine the stress vector $d\mathbf{T} = dT^\rho \mathbf{g}_\rho$ acting at the one-dimensional cut element $d\mathbf{f} = df^\alpha \mathbf{g}_\alpha$: $dT^\rho = N^{\rho\sigma} df_\sigma$, $df_\sigma = g_{\sigma\alpha} df^\alpha$. ∇ is covariant derivation, $\nabla_\gamma N^{\alpha\beta} = N^{\alpha\beta},_\gamma + \Gamma^\alpha_{\rho\gamma} N^{\rho\beta} + \Gamma^\beta_{\rho\gamma} N^{\alpha\rho}$, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols.

Remark. Since equilibrium takes place in the actual state of the segment, area and length are those in the *deformed* membrane!

Under the **assumptions**

- surface of revolution,
- only normal forces acting ($P^\alpha = 0$),
- rotationally symmetric state of stress ($N^{\alpha\beta},_\phi = 0$)

the membrane equations appear as

$$\left.\begin{array}{l} \dot{N}^{12} + 2\frac{\dot{y}}{y} N^{12} = 0, \\ \dot{N}^{22} + \frac{\dot{y}}{y} N^{22} - y\dot{y} N^{11} = 0, \\ -y\dot{x} N^{11} + (\dot{x}\ddot{y} - \ddot{x}\dot{y}) N^{22} + P_n = 0. \end{array}\right\} \tag{5}$$

As to the rheological behavior of the segment's hull we adopt the *working hypotheses* from [9]:

- The hull has a *constant* original thickness h;
- the membrane is *skin-like*, i.e., any state of the segment is stable only if the stress resultants are tensile, $N^{11} \geq 0$, $N^{22} \geq 0$, else a breakdown occurs (total flexibility);
- the membrane is *meridionally inextensible*;
- *latitudinally*, the membrane is *homogeneously hyperelastic*.

So the principal strain in s-direction vanishes, $\varepsilon_s = 0$, and N^{22} appears as the reaction to this constraint. The meridians keep their length , the arc-length s is an invariant during deformation.

The principal strain in ϕ-direction is constant along any circle of latitude, it is given by the original (r) and actual (y) radius,

$$\varepsilon_\phi(s) = (y(s) - r)/r. \tag{6}$$

If σ_ϕ denotes the principal stress in latitudinal direction then hyperelasticity means

$$\sigma_\phi = E\chi(\varepsilon_\phi) \tag{7}$$

($\chi(\varepsilon) = \varepsilon$ for Hooke material). Generally, E denotes some constant Young's modulus that is either *given (*in case of Hooke material) *or fictitious* and to be suitably chosen. $\chi(\cdot)$ is a smooth function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $\chi(0) = 0$, monotonically increasing in most cases. It is given by experiments or suitably chosen in theory (see in [8]).
Note that σ_ϕ means force (at a cut $\phi = const$) in ϕ-direction divided by the *original* area of the cut element. Thus (recall ds invariant under deformation) with h as the original thickness of the membrane shell and $\mathbf{g}_1^0 := \mathbf{g}_1 / \parallel \mathbf{g}_1 \parallel$ there holds $\sigma_\phi h ds \mathbf{g}_1^0 = N^{11} df_1 \mathbf{g}_1$, $d\mathbf{f} = df^1 \mathbf{g}_1 = ds\mathbf{g}_1^0$, $df_1 = g_{11} df^1 = y ds$, and it follows

$$y^2 N^{11} = hE\chi(\frac{y}{r} - 1). \tag{8}$$

Just as a quick supplement we note that the second membrane equation yields

$$yN^{22} = \int^y \eta^2 N^{11}(\eta) d\eta$$

and if there are no twisting forces at the left and right boundaries then it is easy to conclude $N^{12} = 0$.

Let us assume that the *above working hypotheses qualitatively also hold for the vein.* That means, segment and vein are distinguishable by thickness h, elasticity modulus E, and the hyperelastic characteristic $\chi(\cdot)$. Let us use labels 1 and 2 for quantities corresponding to the segment and to the vein, respectively. Then all the foregoing equations are valid with adequate indices.

With regard to later calculations it is promising to skip to quantities of physical dimension "1". For this end we could fix any suitable L_0 as unit of length (i.e., put $x = L_0\widetilde{x}$, etc., and drop the tilda after introduction); we choose *the segment's meridional length L_1 as the unit of length.* Moreover (for both segment and vein) let us use the

$$Normalization: \boxed{\begin{array}{c} N^{11} = (\frac{h_1 E_1}{L_1^2}) n^{11}, \quad N^{22} = (h_1 E_1) n^{22}, \\ P_n = (\frac{2 h_1 E_1}{L_1}) p_n, \quad F_1 = (2\pi h_1 E_1 L_1) f_1, \end{array}} \tag{9}$$

(the parenthesized quantities are now the respective units of measurement, and $n^{\alpha\beta}$, p_n, f_1 to be used in calculations take real numbers as their values). The constitutive laws now take the normalized forms

$$\begin{array}{lll} n_1^{11} = \psi_1(y_1)/y_1^2, & \text{where} & \psi_1(y) := \chi_1(\frac{y}{r_1} - 1), \\ n_2^{11} = \psi_2(y_2)/y_2^2, & \text{where} & \psi_2(y) := \beta\chi_2(\frac{y}{r_2} - 1). \end{array} \tag{10}$$

Roughly, the factor $\beta := \frac{h_2 E_2}{h_1 E_1}$ can be given the interpretation

$$\begin{array}{l} \beta > 1 \ : \textit{thick-walled vein} \\ \beta < 1 \ : \textit{thin-walled vein} \end{array} . \tag{11}$$

We conclude this section by giving an expression for the potential energy needed to formulate the Principle that is to serve as the basis of all further analysis.

First we determine the potential energy stored in the deformed membrane. Per original volume unit this energy is (no normalization yet)

$$\int_0^{\varepsilon_\phi} \sigma_\phi(\varepsilon) d\varepsilon = \int_0^{\varepsilon_\phi} E_1 \chi_1(\varepsilon) d\varepsilon = \frac{E_1}{r_1} \int_{r_1}^{y_1} \psi_1(\eta) d\eta,$$

an original volume $h_1\, ds\, r_1 d\phi$ then contains $h_1 E_1\, ds\, d\phi \int_{r_1}^{y_1} \psi(\eta) d\eta$, and the total energy follows by integration about *meridian* $\times (0, 2\pi)$. With normalization there results the total potential energy of the deformed membrane

$$W_1[y_1] = \int \Psi_1(y_1(s)) ds, \quad \text{measured in units } \ 2\pi h_1 E_1 L_1^2, \tag{12}$$

where

$$\Psi_1(y_1) := \int_{r_1}^{y_1} \psi_1(\eta) d\eta. \tag{13}$$

Integration is along the full (normalized) length of a meridian. Formally the same expression W_2 we have for the potential energy stored in the deformed vein. The potential energy of the total system is $W_1 + W_2$ supplemented by the energy of the inflating fluid (= pressure times volume of segment), the energy of the longitudinal forces $\pm f_1$, and the energy of the elastic springs at the ends of the vein.

We confine the investigations to systems of longitudinal symmetry: suppose physically identical springs, a centered relative position of the segment within the vein, and $x_i(0) = 0,\ x_i(\cdot)$ odd, $y_i(\cdot)$ even functions. The latter supposition yields that the inclinations u_i are odd functions which in particular implies

$$u_i(-0) = -u_i(+0).$$

So we obtain

$$W = \begin{cases} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Psi_1(y_1)ds_1 - p\int_{-\frac{1}{2}}^{\frac{1}{2}} y_1^2 \cos u_1 ds_1 + 2f_1\xi_1 \\ \qquad + \int_{-\frac{l}{2}}^{\frac{l}{2}} \Psi_2(y_2)ds_2 + k(\xi_2 - \frac{l}{2})^2, \end{cases} \tag{14}$$

where $\xi_1 = x_1(\frac{1}{2})$, $\xi_2 = x_2(\frac{l}{2})$, $l = L_2/L_1 > 1$ (the normed original length of the vein), and k is the stiffness of the springs (measured in units $2\pi h_1 E_1$). W has to be seen as a functional of $u_1(\cdot),\ y_1(\cdot),\ y_2(\cdot),\ x_1(\cdot),\ x_2(\cdot)$ that depends on the parameters $l,\ k,\ p$, and f_1.

Remark. It is clear what happens if we drop the symmetry supposition: the integral bounds may change and become unequal in magnitude, the force and the spring term either split in two.

1.3 Variational problem

The potential energy W has now to be minimized under certain side conditions formulated below and in particular guaranteeing the condition *"radius of segment no greater than radius of vein"*. In view of this state constraint (where the $y_{1,2}$-values to be compared are at points lying on top of each other - so having the same $x_{1,2}$- values but different s-values in general) the above representation of W does not match with a handy representation of the constraint. Therefore we shall attack the problem using a $x \to y(x)$ representation of the meridians which then admits the constraint in the simple form $y_1(x) \le y_2(x)$. Geometric formulae have to be adapted in corresponding way: $ds^2 = dx^2 + dy^2,\ y' := dy/dx = \tan u,\ etc.$

To proceed in this way the tacit supposition that the meridians are schlicht curves with respect to the x-axis is required. Having the physical system in mind this is certainly not a severe restriction for the vein meridian, but apparently the force f_1 which directly influences the shape of the inflated segment has to be suitably bounded above (else giving the segment the form of a tire), see Supposition 2 and remarks following Figure 3. A slight inconvenience enters because the integral bounds lose their constancy, and isoperimetric side conditions occur.

The potential energy to be minimized now writes

$$W = \int_{-\xi_1}^{\xi_1} \{\Psi_1(y_1)\sqrt{1+y_1'^2} - py_1^2 + f_1\}dx + \int_{-\xi_2}^{\xi_2} \Psi_2(y_2)\sqrt{1+y_2'^2}dx + k(\xi_2 - \frac{l}{2})^2. \tag{15}$$

The task then is the following:[1]

Find $y_1 \in D^1[-\xi_1, \xi_1],\ y_2 \in D^1[-\xi_2, \xi_2]$ with free $\xi_1, \xi_2,\ 0 < \xi_1 < \xi_2$, such that for fixed parameters $p > 0,\ f_1 \in \mathbb{R},\ k > 0,\ l > 1,\ r_2 \geq r_1 > 0$

$$W \to \min$$

under the restrictions

$$\begin{array}{ll} (i) & y_1(-\xi_1) = y_1(\xi_1) = r_1, \\ (ii) & y_2(-\xi_2) = y_2(\xi_2) = r_2, \\ (iii) & y_2(x) \geq y_1(x), \quad x \in [-\xi_1, \xi_1], \\ (iv) & \int_{-\xi_1}^{\xi_1} \sqrt{1+y_1'^2(x)}dx = 1,\ \int_{-\xi_2}^{\xi 2} \sqrt{1+y_2'^2(x)}dx = l. \end{array} \tag{16}$$

This is a Bolza-type variational problem featured by two different integration intervals, partially fixed boundaries, a state constraint on one of the integration intervals, and isoperimetric side conditions. The class D^1 of the $y_{1,2}$ guarantees that the meridians are piecewise smooth arcs which are allowed to show finitely many edges.

Of course, by appropriate continuation of the first integrand the problem could be made a problem with *one* common integration interval $[-\xi_2, \xi_2]$ but possibly discontinuous integrand and showing the peculiarity of a state constraint (iii) on a proper subinterval.

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[1]Smoothness classes: $D^0[a, b]$ = set of piecewise continuous functions $[a, b] \to \mathbb{R}$; with $k \in \mathbb{N}$: $D^k[a, b]$ = set of continuous functions $[a, b] \to \mathbb{R}$ which have continuous derivatives up to order $k - 1$ and a piecewise continuous kth derivative.

The last isoperimetric condition above clearly implies

$$2\xi_2 \leq l. \tag{17}$$

Equivalently, the isoperimetric conditions can be fit into a Lagrange formulation via additional differential equations and boundary conditions.

In the sequel the variational problem is reformulated as an optimal control problem for which the necessary optimality conditions were essentially prepared in [1] and [2].

We shall treat this problem as a self-contained one, we avoid references to the background physics as means of conclusion. Physical meanings of some suppositions and facts are discussed, after the investigations are finished, in section 3.

2 Optimal control problem

We use the inclination angles $x \to u_i(x)$, $i = 1, 2$, of the meridians as controls. Corresponding to the $y \in D^1$ assumption above and $y_i' = \tan u_i$ we start with

- **Supposition 1:** $u_i(\cdot) \in D^0$, $u_i(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $i = 1, 2$.

This matches the supposition of the meridians to be schlicht curves. Then, dropping indices, there holds $\sqrt{1 + y'^2} = 1/\cos u$, and, putting $w(x) := \int_0^x \frac{1}{\cos u(t)} dt$, the first isoperimetric condition can be replaced by $w' = 1/\cos u$, $w(-\xi_1) = -\frac{1}{2}$, $w(\xi_1) = \frac{1}{2}$ (second one treated analogously).

2.1 Problem formulation

The **potential energy**

$$W = \int_{-\xi_1}^{\xi_1} \{\frac{1}{\cos u_1}\Psi_1(y_1) - py_1^2 + f_1\} dx + \int_{-\xi_2}^{\xi_2} \frac{1}{\cos u_2}\Psi_2(y_2) dx + k(\xi_2 - \frac{l}{2})^2 \tag{18}$$

is composed of two integral terms with different integration intervals and a Bolza term

$$g_{02}(\xi_2) := k(\xi_2 - \frac{l}{2})^2. \tag{19}$$

The **task** is now the following:

With free $\xi_1, \xi_2,\ 0 < \xi_1 < \xi_2$, find $u_1 \in D^0[-\xi_1, \xi_1],\ u_2 \in D^0[-\xi_2, \xi_2]$, both functions odd w.r.t. $x = 0$, and $y_1,\ w_1 \in D^1[-\xi_1, \xi_1],\ y_2,\ w_2 \in D^1[-\xi_2, \xi_2]$, such that for fixed parameters $p > 0,\ f_1 \in \mathbb{R},\ k > 0,\ l > 1,\ r_2 > r_1 > 0$

$$W \to \min$$

under the restrictions (on respective intervals)

$$\begin{array}{ll} (i) & y_1' = \tan u_1,\ \ y_2' = \tan u_2, \\ (ii) & w_1' = 1/\cos u_1,\ \ w_2' = 1/\cos u_2, \\ (iii) & y_1(\pm\xi_1) = r_1,\ \ y_2(\pm\xi_2) = r_2, \\ (iv) & w_1(-\xi_1) = -\frac{1}{2},\ \ w_2(-\xi_2) = -\frac{l}{2},\ \ w_1(0) = w_2(0) = 0, \\ (v) & S(y_2(x), y_1(x)) := y_2(x) - y_1(x) \geq 0,\ \ x \in [-\xi_1, \xi_1]. \end{array} \tag{20}$$

Remind that the skin property of the membranes demands $y_1(x) \geq r_1$ (no negative strains possible in any stable configuration). We shall not cope with these inequalities as additional constraints. Instead, if the optimality conditions yielded solutions with $y_1(x) < r_1$ at some x then these solutions would be dropped with regard to their physical insignificance.

2.2 Optimality conditions

The necessary optimality conditions given and utilized in the following are essentially prepared in [2]. The peculiarity of the different integration intervals together with a state constraint on a proper subinterval makes the problem a non-familiar one that is, without any close connection to some physical background treated in the forthcoming paper [1]. In what follows we present and utilize the adaptation to our problem of the general necessary optimality conditions given there.

The *relative degree* of the problem is $h = 1 :\ S = y_2 - y_1$ yields

$$R_0(u) := S' = \tan u_2 - \tan u_1, \tag{21}$$

hence $rank(R_{0,u}) = rank(-\cos^{-2} u_1, \cos^{-2} u_2) = 1$.

First, with a $\mathbb{R}^{1+4+1}-$valued multiplier (l_0, λ, ρ) we define the *Hamiltonian* in the following way:

$$\begin{array}{l} H_1(l_0, \lambda_1, \lambda_3, y_1, u_1) := l_0\{\frac{1}{\cos u_1}\Psi_1(y_1) - py_1^2 + f_1\} + \lambda_1 \tan u_1 + \lambda_3 \frac{1}{\cos u_1}, \\ H_2(l_0, \lambda_2, \lambda_4, y_2, u_2) := l_0 \frac{1}{\cos u_2}\Psi_2(y_2) + \lambda_2 \tan u_2 + \lambda_4 \frac{1}{\cos u_2}, \end{array} \tag{22}$$

and, along any feasible function $x \longmapsto (y_1(x), .., \lambda_4(x))$,

$$H := \begin{cases} H_2, \;\; x \in [-\xi_2, -\xi_1) \cup (\xi_1, \xi_2], \\ H_1 + H_2 + \rho(\tan u_2 - \tan u_1), \; x \in [-\xi_1, \xi_1]. \end{cases} \tag{23}$$

The **optimality conditions** then are the following:

Let $(u_1, u_2, y_1, y_2, w_1, w_2)$ be a solution of the optimal control problem. Then there exists a multiplier $(l_0, \lambda_1(\cdot), \lambda_2(\cdot), \lambda_3(\cdot), \lambda_4(\cdot), \rho(\cdot))$, with $l_0 \in \mathbb{R}^+$, $\lambda_{1,3} \in D^1[-\xi_1, \xi_1]$, $\lambda_2 \in D^1([-\xi_2, \xi_2] \setminus \{\xi_1\})$, $\lambda_4 \in D^1[-\xi_2, \xi_2]$, and $\rho \in C^1([-\xi_1, \xi_1] \setminus \{x : \; S = 0\})$, such that

$$\textit{(o)} \quad \begin{cases} (l_0, \lambda_2(x), \lambda_4(x)) \neq 0, \;\; x \in [-\xi_2, -\xi_1) \cup (\xi_1, \xi_2], \\ (l_0, \lambda_1(x), \lambda_2(x), \lambda_3(x), \lambda_4(x), \rho(x)) \neq 0, \;\; x \in [-\xi_1, \xi_1], \end{cases}$$

$$\textit{(i)} \quad \left.\begin{cases} \lambda_1' = -H_{,y_1} \;\; = -l_0\{\psi_1(y_1)/\cos u_1 - 2py_1\} \\ \lambda_3' = -H_{,w_1} \qquad\qquad\qquad\qquad = 0 \end{cases}\right\} \textit{piecewise on } [-\xi_1, \xi_1]$$

$$\textit{(ii)} \quad \begin{cases} \left.\begin{array}{l} \lambda_2' = -H_{,y_2} \;\; = -l_0\psi_2(y_2)/\cos u_2 \\ \lambda_4' = -H_{,w_2} \;\; = 0 \end{array}\right\} \textit{piecewise on } [-\xi_2, \xi_2], \\ \lambda_2(\xi_1 - 0) + \rho(\xi_1) = \lambda_2(\xi_1 + 0) \end{cases}$$

$$\textit{(iii)} \quad \begin{cases} H(..., \overline{u}_1, \overline{u}_2) \geq H(..., u_1, u_2) \textit{ on } [-\xi_1, \xi_1], \\ H_2(..., \overline{u}_2) \geq H_2(..., u_2) \textit{ on } [-\xi_2, -\xi_1) \cup (\xi_1, \xi_2], \end{cases}$$

$$\textit{(iv)} \quad 0 = H_{,u_i} \;, i = 1, 2 \;:$$

$$\begin{cases} 0 = \{(l_0\Psi_1(y_1) + \lambda_3)\sin u_1 + \lambda_1 - \rho\} \;\; \textit{on } [-\xi_1, \xi_1], \\ 0 = \begin{cases} (l_0\Psi_2(y_2) + \lambda_4)\sin u_2 + \lambda_2 + \rho \;\; \textit{on } [-\xi_1, \xi_1], \\ (l_0\Psi_2(y_2) + \lambda_4)\sin u_2 + \lambda_2 \;\; \textit{on } [-\xi_2, -\xi_1) \cup (\xi_1, \xi_2]. \end{cases} \end{cases}$$

$$\textit{(v)} \qquad H(l_0, \lambda(\cdot), \rho(\cdot), y(\cdot), u(\cdot)) \in D^1[-\xi_2, \xi_2],$$

$$\textit{(vi)} \qquad \tfrac{d}{dx} H = H_{,x} = 0, \textit{ on } [-\xi_2, \xi_2],$$

$$\textit{(vii)} \quad \begin{cases} \textit{transversality at } x = \pm\xi_2\textit{:} \\ l_0 k(\frac{l}{2} - \xi_2) - H_2 \mid_{\pm\xi_2} = 0, \end{cases}$$

$$\textit{(viii)} \begin{cases} (\rho S) \mid_{x=-\xi_1} = 0 \\ \rho' S = 0 \textit{ on } [-\xi_1, \xi_1] \setminus \{x : \; S = 0\} \\ \rho \text{ non-decreasing on } [-\xi_1, \xi_1] \text{ if } l_0 \neq 0. \end{cases}$$

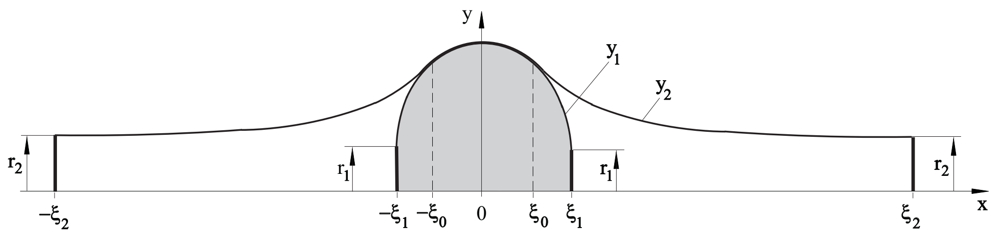

Figure 2: Important points.

Complementary remarks:

re (i), (ii): By means of the continuity of λ_3 and λ_4 it follows

$$\lambda_3 = const \text{ on } [-\xi_1, \xi_1], \quad \lambda_4 = const \text{ on } [-\xi_2, \xi_2].$$

re (ii),(iv): The λ_2-jump-condition at ξ_1 ensures continuity of $H_{,u_2}$ at ξ_1 if u_2 is continuous at ξ_1 (shown below).

re (vi): This means "energy" conservation,

$$H = const := c \text{ on } [-\xi_2, \xi_2].$$

For problems with degree $h > 1$ it need not be valid, [1].

re (vii): The restrictions (ii), (iv) in (20) imply $\xi_2 = \frac{l}{2}$ iff $u_2 \equiv 0$ hence $y_2(x) = r_2$. The 2nd and 3rd term of the potential energy (18) then vanish and this means that the two subsystems segment and vein are without mutual influence. Then one might guess the constraint to be active, $S = 0$, in at most isolated points (touch points). Therefore we shall focus on the case

$$\frac{l}{2} - \xi_2 > 0 \tag{24}$$

in the sequel (active constraint on some open interval).

re (viii): $r_2 > r_1$ implies $S(y_2(-\xi_1), y_1(-\xi_1)) > 0$ and by continuity $S(y_2(x), y_1(x)) > 0$ in a right neighborhood of $-\xi_1$, hence $\rho'(x) = 0$ and $\rho(x) = 0$ for $x \in [-\xi_1, -\xi_0]$, where $-\xi_0$ is the utmost left junction point. Moreover $\rho(x) \geq 0$ for $x \in (-\xi_0, , \xi_1]$ and $\rho(x) = const$ on $(\xi_0, \xi_1]$. (ξ_0 is, by symmetry, the utmost right disjunction point.) If ρ' does not exist at some $x \in (-\xi_0, \xi_0)$ - where $S = 0$ - then the equation $\rho' S = 0$ is to be considered formally.

2.3 Discussion of the optimality conditions

The following analysis of the optimality conditions is based on

- **Supposition 2:**

  (a) *The fixed parameters are confined to*

$$pr_1^2 - f_1 > 0. \tag{25}$$

  (b) *If the state restriction* $S(y_2, y_1) \geq 0$ *shows a non-empty activity domain, then there is exactly one junction point* $-\xi_0 \in (-\xi_1, 0)$ *(accompanied by the disjunction point* ξ_0*).* $\xi_0 = 0$ *describes one touch point, it is not investigated separately but considered as the limit case during inflation from zero pressure or during deflation from big pressure.*

So it holds $y_2(x) - y_1(x) > 0$ for $x \in [-\xi_1, -\xi_0) \cup (\xi_0, \xi_1]$ and $y_2(x) - y_1(x) = 0$ on $[-\xi_0, \xi_0]$, entailing $u_2(x) = u_1(x)$ on the non-empty interval $(-\xi_0, \xi_0)$.

The following investigations focus on the interval $[-\xi_2, 0]$, results concerning $y_{1,2}$ and $u_{1,2}$ then apply to $(0, \xi_2]$ according to symmetry. But note that the supposed geometric symmetry is in general not reflected in a symmetry of the hamiltonian and the multipliers, observe, e.g., the monotonicity of ρ and the jump of λ_2.

1) About normality

We show that $l_0 = 0$ yields a contradiction, so the problem is a normal one.

Suppose $l_0 = 0$ and observe what happens at $x = -\xi_1$. (Denote the respective values by $\overline{\lambda}_1$, $\overline{\lambda}_2$, $\overline{u}_1$, and limits by $u_2^{\pm}$.)

i) $H_{,u_1} |_{-\xi_1+0} = 0 : \quad \lambda_3 \sin \overline{u}_1 + \overline{\lambda}_1 = 0,$

ii) $H_{,u_2} |_{-\xi_1-0} = 0 : \quad \lambda_4 \sin u_2^- + \overline{\lambda}_2 = 0,$

iii) $H |_{-\xi_1-0} = c : \quad \overline{\lambda}_2 \sin u_2^- + \lambda_4 = c,$

iv) $H |_{-\xi_1+0} = c : \quad (\overline{\lambda}_2 \sin u_2^+ + \lambda_4) \frac{1}{\cos u_2^+} + (\overline{\lambda}_1 \sin \overline{u}_1 + \lambda_3) \frac{1}{\cos \overline{u}_1} = c.$

The optimality condition (vii) with $l_0 = 0$ yields $c = 0$. Then ii) and iii) form a system of homogeneous linear equations with the unique trivial solution $\overline{\lambda}_2 = \lambda_4 = 0$, and $\overline{\lambda}_1 = \lambda_3 = 0$ follows from i) and iv) in the same way. So we have $(0, \lambda, \rho) |_{-\xi_1} = 0 :$ contradiction to optimality! Let, therefore, in all what follows

$$l_0 = 1.$$

2) Continuity of the controls:

Preliminary note:

From optimality condition (iv), $H_{,u_i} = 0$, we see that $H_{,u_i}$ is continuous at any $x_0 \in [-\xi_2, \xi_2]$. So we have for the one-sided limits at x_0 (y_{i0}, λ_{i0} values, $u_i^{\pm}$ limits at x_0)

$$\left.\begin{array}{ll} x_0 \in [-\xi_1, -\xi_0): & [\Psi_1(y_{10}) + \lambda_3] \sin u_1^{\pm} + \lambda_{10} = 0, \\ x_0 \in [-\xi_2, -\xi_0): & [\Psi_2(y_{20}) + \lambda_4] \sin u_2^{\pm} + \lambda_{20} = 0, \\ x_0 \in (-\xi_0, 0]: & [\Psi_1(y_0) + \lambda_3] \sin u^{\pm} + \lambda_{10} = \rho^{\pm}, \\ x_0 \in (-\xi_0, 0]: & [\Psi_2(y_0) + \lambda_4] \sin u^{\pm} + \lambda_{20} = -\rho^{\pm}. \end{array}\right\} \quad (26)$$

This shows, first, that the limits $\rho^{\pm}$ exist and are finite. Moreover we get the

Proposition 1 *At every $x_0 \neq -\xi_0$ there hold the implications*

$$x_0 < -\xi_0 : \begin{cases} [\Psi_1(y_{10}) + \lambda_3] \neq 0 \;\Rightarrow\; u_1 \textit{ continuous at } x_0, \\ \\ [\Psi_2(y_{20}) + \lambda_4] \neq 0 \;\Rightarrow\; u_2 \textit{ continuous at } x_0, \end{cases}$$

$$-\xi_0 < x_0 \leq 0 : \begin{cases} \textit{Sum of brackets non-zero } \Rightarrow u \textit{ and } \rho \textit{ continuous at } x_0, \\ \textit{One of brackets zero } \Rightarrow \rho \textit{ continuous at } x_0, \\ u \textit{ continuous } \Rightarrow \rho \textit{ continuous at } x_0. \end{cases}$$

As a consequence we observe:

$$\rho \text{ is continuous on } (-\xi_0, 0].$$

The boundary point $-\xi_0$ deserves extra consideration.

Now let us check the facts at various x_0. For the sake of brevity in writing we introduce the *temporary convention*: In the course of the following proofs we use for the recurring brackets the abbreviations

$$[\Psi_1(y_{10}) + \lambda_3] =: B_1, \;\; [\Psi_2(y_{20}) + \lambda_4] =: B_2.$$

a) $x_0 \in (-\xi_2, -\xi_1)$. $B_2 = 0$ implies $\lambda_{20} = 0$ and $c = H\,|_{x_0} = H_2\,|_{x_0} = 0$: a contradiction for $c > 0$, hence u_2 continuous at x_0, whereas for $c = 0$ we have $u_2(x) \equiv 0$, hence trivial continuity.

b) $x_0 = -\xi_1$. Claim: u_2 continuous at $-\xi_1$.

If we write (26.2) as separate equations then they can be seen as a system of homogeneous linear equations for B_2 and λ_{20} with coefficient matrix

$$M = \begin{pmatrix} \sin u_2^- & 1 \\ \sin u_2^+ & 1 \end{pmatrix}.$$

Continuity of u_2 means $\det M = 0$ whereas discontinuity, $\det M \neq 0$, would imply a trivial solution, $B_2 = 0,\ \lambda_{20} = 0$. But then we get the same conclusion as under **a)**.

Supplement: The continuity of u_2 at x implies continuity of H_2 at x. Then, with $x = -\xi_1$, $H_2 \mid_{-\xi_1 - 0} = (H_1 + H_2) \mid_{-\xi_1+0}$ yields in particular

$$H_1 \mid_{-\xi_1+0} = 0.$$

c) $\underline{x_0 \in (-\xi_1, -\xi_0)}$. (The following reasoning also applies to (ξ_0, ξ_1) after $\lambda_1,\ \lambda_2$ are replaced by $\lambda_1 - \rho$ and $\lambda_2 + \rho$, respectively, with constant ρ.)

$\boldsymbol{\alpha}$) Claim: The brackets B_1 and B_2 cannot vanish simultaneously.

Proof: Assume $B_1 = 0$ and $B_2 = 0$. Then (26) entails $\lambda_{10} = \lambda_{20} = 0$ and $H \mid_{x_0 \pm 0} = -py_{10}^2 + f_1 = c$, a contradiction: $0 \leq c = -py_{10}^2 + f_1 < -pr_1^2 + f_1 < 0$.

Consequently, u_1 and u_2 cannot simultaneously be discontinuous.

$\boldsymbol{\beta}$) W.l.o.g. x_0 can be taken as the utmost left discontinuity of the $u's$, for $u_i \in D^0$ allows at most finitely many jumps of the controls. *Assume u_2 discontinuous at x_0.* Then (26) entails $B_2 = \lambda_{20} = 0$ and $H_2 \mid_{x_0 \pm 0} = 0$. Now we know that H_2 is continuous on $[-\xi_1, x_0)$ and $H_2 \mid_{-\xi_1} = c \geq 0$. Lemma 5 and the final remark in the Appendix yield $\frac{d}{dx} H_2 = 0$ and thus $H_2 = c$ on $[-\xi_1, x_0)$. Hence we get $c = 0$, and this means, following the preliminary remark preceding (25), $u_2(x) = 0$ for every $x \in [-\xi_2, \xi_2]$: u_2 is continuous throughout, contradiction!

A similar reasoning starting with discontinuous u_1 yields the same result.

Summarizing: u_1 and u_2 are continuous on $(-\xi_1, -\xi_0)$.

Now we can conclude, again exploiting the Lemma from Appendix 1 and the final remark therein,

$$H_1 = 0, \quad H_2 = c \ \text{ on } [-\xi_1, -\xi_0). \tag{27}$$

d) $\underline{x = -\xi_0 < 0}$. Remind that $y_1 = y_2 =: y$ and $u_1 = u_2 =: u$ in a right neighborhood of the junction point $-\xi_0$. Limits of H are

$$\begin{aligned} &(H_1 + H_2) \mid_{-\xi_0 - 0} \\ &\quad = \tfrac{1}{\cos u_1^-} \{B_1 + \lambda_{10} \sin u_1^-\} - py_0^2 + f_1 \\ &\qquad + \tfrac{1}{\cos u_2^-} \{B_2 + \lambda_{20} \sin u_2^-\} = c, \end{aligned} \tag{28}$$

$$\begin{aligned} &(H_1 + H_2) \mid_{-\xi_0 + 0} \\ &\quad = \tfrac{1}{\cos u^+} \{B_1 + B_2 + (\lambda_{10} + \lambda_{20}) \sin u^+\} - py_0^2 + f_1 = c. \end{aligned} \tag{29}$$

Limits of $H_{,u_i} = 0$ can be deduced from (26):

$$\begin{array}{ll} 0 = H_{,u_1}|_{-\xi_0-0}: & B_1 \sin u_1^- + \lambda_{10} = 0, \\ 0 = H_{,u_2}|_{-\xi_0-0}: & B_2 \sin u_2^- + \lambda_{20} = 0, \\ 0 = H_{,u_1}|_{-\xi_0+0}: & B_1 \sin u^+ + \lambda_{10} - \rho^+ = 0, \\ 0 = H_{,u_2}|_{-\xi_0+0}: & B_2 \sin u^+ + \lambda_{20} + \rho^+ = 0. \end{array}$$

Eliminating $\lambda_{1,2}$ there follow

$$\left.\begin{array}{r} B_1(\sin u^+ - \sin u_1^-) = \rho^+ \\ B_2(\sin u^+ - \sin u_2^-) = -\rho^+ \\ B_1(\cos u^+ - \cos u_1^-) + B_2(\cos u^+ - \cos u_2^-) = 0. \end{array}\right\} \tag{30}$$

The 3rd equation together with the sum of the first two can be seen as a system of homogeneous linear equations for the two brackets. Its determinant is

$$\Delta = \sin(u^+ - u_2^-) - \sin(u^+ - u_1^-) + \sin(u_2^+ - u_1^-).$$

$\boldsymbol{\alpha}$) Assume $u_1^- \neq u_2^-$. Then $\Delta \neq 0$, whence both brackets are zero and this leads to a contradiction as under $\mathbf{c}\boldsymbol{\alpha}$) above. So let

$\boldsymbol{\beta}$) $u_1^- = u_2^-$. This yields $\Delta = 0$ and there remains

$$\{B_1 + B_2\}(\cos u^+ - \cos u_2^-) = 0.$$

If we had $u^+ \neq u_1^-$ then $B_1 = B_2 = 0$ would follow, leading to the well-known contradiction $-py_0^2 + f_1 = c$.
Therefore we obtain the continuity $u^+ = u_1^- = u_2^-$ and moreover $\rho^+ = 0$.

e) Let $x_0 \in (-\xi_0, \xi_0)$. Let $\xi_0 > 0$. Then we have $y_1 = y_2 =: y$, $u_1 = u_2 =: u$ on this contact interval. We consider the equations $(H_1 + H_2)_{,u_i} = 0$, and $H_1 + H_2 = c$ at x_0. In the limits $x_0 \pm 0$ these are

$$\begin{array}{l} B_1 \sin u^\pm + \lambda_{10} - \rho^\pm = 0, \\ B_2 \sin u^\pm + \lambda_{20} + \rho^\pm = 0, \\ B_1 + B_2 + (\lambda_{10} + \lambda_{20}) \sin u^\pm = (c + py_0^2 - f_1) \cos u^\pm. \end{array}$$

Eliminating the $\lambda's$ in the third equation we get

$$(B_1 + B_2) \cos u^\pm = (c + py_0^2 - f_1).$$

Now either $B_1 + B_2 = 0$, giving the usual contradiction, or it holds $u^+ = u^-$and moreover $\rho^+ = \rho^-$.

For odd $u_i(\cdot)$we already know $u_i(-0) = -u_i(+0)$, and the continuity then yields

$$u(0) = 0. \tag{31}$$

Summarizing we have found the

Proposition 2 *The controls are continuous at every x: $u_1 \in C^0[-\xi_1, \xi_1]$, $u_2 \in C^0[-\xi_2, \xi_2]$. That means geometrically, the meridians are smooth curves. In particular they do not show an edge at the junction (disjunction) point. The describing functions $y_{1,2}$ are of class C^1. Furthermore $\rho \in C^0[-\xi_1, \xi_0]$, (at ξ_0 a jump may happen).*

5) Differentiability of the controls

On the respective intervals we exploit the constancy of $H_1,\ H_2,\ H$ and the vanishing partial derivatives w.r.t. $u_{1,2}$.

For $\underline{x \in [-\xi_1, -\xi_0]}$ we have $H_1 = 0,\ H_{1,u_1} = 0$, i.e.,

$$\begin{aligned} &[\Psi_1(y_0) + \lambda_3]\tfrac{1}{\cos u_1} + \lambda_1 \tan u_1 = py_1^2 - f_1, \\ &[\Psi_1(y_0) + \lambda_3]\tfrac{\sin u_1}{\cos^2 u_1} + \lambda_1 \tfrac{1}{\cos^2 u_1} = 0. \end{aligned}$$

Taking this as a system of linear equations, it solves for the λ,

$$\begin{aligned} \lambda_1 &= -(py_1^2 - f_1)\tan u_1, \\ \lambda_3 &= -\Psi_1(y_1) + (py_1^2 - f_1)/\cos u_1 \end{aligned}\ ,\ x \in [-\xi_1, -\xi_0]. \tag{32}$$

In the same way we obtain from $H_2 = c,\ H_{2,u_2} = 0$

$$\begin{aligned} \lambda_2 &= c\tan u_2, \\ \lambda_4 &= -\Psi_2(y_2) + c/\cos u_2 \end{aligned}\ ,\ x \in [-\xi_2, -\xi_0]. \tag{33}$$

Now, with a focus first on u_1, we inspect

$$[\Psi_1(y_1) + \lambda_3]\cos u_1 = py_1^2 - f_1. \tag{34}$$

Since the right hand side is positive we re-encounter the fact that $\Psi_1(y_1) + \lambda_3 \neq 0$ (even > 0).

We know $y_1(\cdot) \in C^1$, the function ψ_1 has been supposed smooth, say $\psi_1 \in C^n,\ n \geq 1$, thus $\Psi_1 = \int \psi_1 \in C^{n+1}$. So it follows from the last relation

$$u_1 \in C^1([-\xi_1, -\xi_0], (-\frac{\pi}{2}, \frac{\pi}{2}))$$

(at $-\xi_0$ there is of course only a left derivative which is left-continuous). In the same way we obtain

$$u_2 \in C^1([-\xi_2, -\xi_0], (-\frac{\pi}{2}, \frac{\pi}{2})).$$

Finally, for $\underline{x \in [-\xi_0, 0]}$ there hold $H_1 + H_2 = c$, $(H_1 + H_2)_{,u_1} = (H_1 + H_2)_{,u_2} = 0$ (with $u_1 = u_2 = u$, $y_1 = y_2 = y$),

$$\begin{aligned}
&\lambda_3 + \lambda_4 + (\lambda_1 + \lambda_2)\sin u = -(\Psi_1(y) + \Psi_2(y)) + (py^2 - f_1)\cos u,\\
&\lambda_3 \sin u \ + \lambda_1 - \rho = -\Psi_1(y)\sin u,\\
&\lambda_4 \sin u \ + \lambda_2 + \rho = -\Psi_2(y)\sin u.
\end{aligned}$$

These equations yield the ρ-free representations

$$\begin{aligned}
&\lambda_1 + \lambda_2 = -(c + py^2 - f_1)\tan u,\\
&\lambda_3 + \lambda_4 = -(\Psi_1(y) + \Psi_2(y)) + (c + py^2 - f_1)/\cos u
\end{aligned}, \ x \in [-\xi_0, 0],$$

which are analogues to (33). So the same reasoning as above yields

$$u \in C^1([-\xi_0, 0], (-\frac{\pi}{2}, \frac{\pi}{2})).$$

Now it holds iteratively $y_1' = \tan u_1 \in C^1 \Rightarrow y_1 \in C^2 \Rightarrow$(by (33):) $u_1 \in C^2 \Rightarrow y_1 \in C^3 \Rightarrow ... \Rightarrow u_1 \in C^{n+1}$. The same arguments work for $y_2, u_2,\ y, u$ on their domains. Summarizing, we come up with

Proposition 3 *If the functions ψ_1 and ψ_2 which describe the hyperelasticity of segment and vein, respectively, are of class C^n, $n \geq 1$, then the controls u_1 and u_2 are of class C^{n+1} on their domains with exception of the junction points $\pm\xi_0$, where only continuity is ensured. Correspondingly, y_1 and y_2 are C^{n+2} for $x \neq \pm\xi_0$.*

Finally, it is easy matter to find the values of the constants λ_3 and λ_4 from (32) and (33) by looking to the left boundaries of their domains. With

$$\alpha_1 := u_1(-\xi_1), \ \ \alpha_2 := u_2(-\xi_2), \ c = k(\frac{l}{2} - \xi_2)$$

we obtain

$$\lambda_3 = (pr_1^2 - f_1)/\cos\alpha_1, \ \ \lambda_4 = c/\cos\alpha_2. \tag{35}$$

Moreover we get some knowledge about the smoothness of the multipliers $\lambda_{1,2}$ and ρ. It follows from $0 = H_{,u_1} = H_{,u_1}$,

$$0 = \begin{cases} \lambda_1 + [\Psi_1(y_1) + \lambda_3] \sin u_1 \ , & x \in [-\xi_1, -\xi_0), \\ \lambda_1 - \rho + [\Psi_1(y_1) + \lambda_3] \sin u_1 \ , & x \in [-\xi_0, 0], \end{cases}$$
$$0 = \begin{cases} \lambda_2 + [\Psi_2(y_2) + \lambda_4] \sin u_2 \ , & x \in [-\xi_2, -\xi_0), \\ \lambda_2 + \rho + [\Psi_2(y_2) + \lambda_4] \sin u_2 \ , & x \in [-\xi_0, 0], \end{cases}$$

that the smoothness of $u_{1,2}$ passes to the multipliers, in particular λ_1 and λ_2 are C^{n+1} on $[-\xi_1, -\xi_0)$ and $[-\xi_2, -\xi_0)$, respectively, whereas $\lambda_1 + \lambda_2$ is C^{n+1},and ρ is D^1 on $[-\xi_0, 0]$.

By means of the latter fact we obtain by simple calculation a complete description of how the hamiltonian parts behave along an extremal:

$$\begin{aligned} \tfrac{d}{dx} H_1 &= \begin{cases} 0, & x \in [-\xi_1, -\xi_0], \\ -\rho' \tan u_1, & x \in [-\xi_0, 0], \end{cases} \\ \tfrac{d}{dx} H_2 &= \begin{cases} 0, & x \in [-\xi_2, -\xi_0], \\ \rho' \tan u_2, & x \in [-\xi_0, 0]. \end{cases} \end{aligned} \tag{36}$$

2.4 Differential equations of the extremals

Let us take up the optimality condition (4), remove the denominators, and differentiate w.r.t. x (on $[-\xi_0, 0]$ at least piecewise allowed) observing the optimality condition $\lambda'_{1,2} = -H_{,y_{1,2}}$ and the geometric restrictions of the form $y' = \tan u$. The result is

$$\left.\begin{array}{c} 2py_1 - \psi_1(y_1)\cos(u_1) + [\Psi_1(y_1) + \lambda_3]\cos(u_1)u_1' = 0, \\ y_1' = \tan u_1, \end{array}\right\} x \in (-\xi_1, -\xi_0),$$

$$\left.\begin{array}{c} -\psi_2(y_2)\cos(u_2) + [\Psi_2(y_2) + \lambda_4]\cos(u_2)u_2' = 0, \\ y_2' = \tan u_2, \end{array}\right\} x \in (-\xi_2, -\xi_0),$$

$$\left.\begin{array}{c} 2py_1 - \psi_1(y_1)\cos(u_1) + [\Psi_1(y_1) + \lambda_3]\cos(u_1)u_1' = \rho', \\ y_1' = \tan u_1, \\ -\psi_2(y_2)\cos(u_2) + [\Psi_2(y_2) + \lambda_4]\cos(u_2)u_2' = -\rho', \\ y_2' = \tan u_2, \end{array}\right\} x \in (-\xi_0, 0).$$

ρ' disappears by adding the respective equations. On the interval $(-\xi_0, 0)$ there holds $u_1 = u_2 =: u$ and $y_1 = y_2 =: y$, and we let

$$\psi_{12} := \psi_1 + \psi_2, \ \Psi_{12} := \Psi_1 + \Psi_2.$$

Then the outcome is, finally, a set of differential equations

$$\left.\begin{array}{l}\left.\begin{array}{l}2py_1 - \psi_1(y_1)\cos(u_1) + [\Psi_1(y_1) + \lambda_3]\cos(u_1)u_1' = 0,\\ y_1' = \tan u_1,\end{array}\right\} x \in (-\xi_1, -\xi_0),\\ \left.\begin{array}{l}-\psi_2(y_2)\cos(u_2) + [\Psi_2(y_2) + \lambda_4]\cos(u_2)u_2' = 0,\\ y_2' = \tan u_2,\end{array}\right\} x \in (-\xi_2, -\xi_0),\\ \left.\begin{array}{l}2py - \psi_{12}(y)\cos(u) + [\Psi_{12}(y) + \lambda_3 + \lambda_4]\cos(u)u' = 0,\\ y' = \tan u,\end{array}\right\} x \in (-\xi_0, 0).\end{array}\right\} \tag{37}$$

Remind the values $\lambda_3 = (pr_1^2 - f_1)/\cos\alpha_1$, $\lambda_4 = c/\cos\alpha_2 = k(\frac{l}{2} - \xi_2)/\cos\alpha_2$, to be inserted above.

Together with the *boundary conditions*

$$\begin{array}{l} y_1(-\xi_1) = r_1,\ u_1(-\xi_1) = \alpha_1,\\ y_2(-\xi_2) = r_2,\ u_2(-\xi_2) = \alpha_2,\\ u(0) = 0,\end{array} \tag{38}$$

and *junction conditions*

$$\begin{array}{l} y_1(-\xi_0) = y_2(-\xi_0) = y(-\xi_0),\\ u_1(-\xi_0) = u_2(-\xi_0) = u(-\xi_0),\end{array} \tag{39}$$

the differential equations form a somewhat unusual *parameter dependent boundary value problem.* Parameters are $p,\ f_1,\ k,\ l,\ r_1,\ r_2$ (fixed), and $\xi_1,\ \xi_2,\ \xi_0,\ \alpha_1,\ \alpha_2$ (to be matched). Some of the latter (initial values $\xi_2,\ \alpha_1,\ \alpha_2$) enter the differential equations.

If the boundary problem has a solution then it describes *the left half* of the meridians, the right half is obtained by continuation to positive x, even functions y, odd functions u.

Besides the hamiltonian which is constant along the extremal we construct another function appearing as a **conserved quantity** in the following way. Take the hamiltonian parts H_1 and H_2 and eliminate the multipliers λ_1 and λ_2 by means of the optimality condition (iv). Thereby we define the piecewise smooth *state-control functions* (not depending on any non-constant multipliers)

$$\left.\begin{array}{l}\Phi_1(u_1, y_1) := -(py_1^2 - f_1) + [\Psi_1(y_1) + \lambda_3]\cos u_1,\\ \Phi_2(u_2, y_2) := [\Psi_2(y_2) + \lambda_4]\cos u_2,\end{array}\right\} \tag{40}$$

and we put

$$\varphi(x) := \left\{\begin{array}{c}\Phi_2(u_2(x), y_2(x)),\quad x \in [-\xi_2, -\xi_1),\\ \Phi_1(u_1(x), y_1(x)) + \Phi_2(u_2(x), y_2(x)),\quad x \in (-\xi_1, 0].\end{array}\right. \tag{41}$$

It is simple calculation to prove the following

Proposition 4 *φ is a **first integral** on $[-\xi_2, 0]$: $\frac{d}{dx}\varphi(x) = 0$ piecewise along any solution of the foregoing differential equations.*

The separate behavior of Φ_1 and Φ_2 along any solution of the boundary value problem is governed by

$$\left.\begin{array}{l} \Phi_1(u_1(x), y_1(x)) = 0,\ x \in [-\xi_1, -\xi_0], \\ \Phi_2(u_2(x), y_2(x)) = k(\frac{l}{2} - \xi_2),\ x \in [-\xi_2, -\xi_0], \\ \frac{d}{dx}\Phi_1(u, y) = -\rho' \tan u,\ \ \frac{d}{dx}\Phi_2(u, y) = \rho' \tan u,\ \ x \in [-\xi_0, 0]. \end{array}\right\} \tag{42}$$

The rates of change of Φ_1 and Φ_2 are calibrated in such a way that

$$\frac{d}{dx}[\Phi_1(u, y) + \Phi_2(u, y)] = 0.$$

With a glance at the originating hamiltonian parts H_1 and H_2 the foregoing relations do not look very exciting, cf. (36), but interestingly these equations enjoy a nice physical interpretation (see next Section).

First, we deduce another property of the multiplier ρ. In the present symmetry case the $\Phi_i(u_i(\cdot), y_i(\cdot))$ are even functions of x, thus $\frac{d}{dx}\Phi_i(u_i(\cdot), y_i(\cdot))$ are odd. Since, furthermore, u_i and y_i are C^{n+1}, this implies

$$\rho' \geq 0 \ is\ an\ even\ C^n - function\ on\ (-\xi_0, \xi_0).$$

For numerical treatment it may be effective to use the arc lengths of the meridians as the independent variable, i.e., to deal with the (formerly rejected) parameter representation $x(s),\ y(s)$ of the meridians. The main advantage arises from the known and constant domains of all functions. Disadvantage may come from the increased number of differential equations.

Abusing notation the problem takes the following form where $u' \cos u = u'\dot{x} = \dot{u}$ is now the curvature of the meridians. Furthermore we eliminate the brackets containing Ψ by means of the first integral. Let $-t_0$ be the arc length of the junction point, i.e., $-\xi_0 = x_1(-t_0) = x_2(-t_0)$, then we obtain

$$\boxed{\begin{array}{c} \dot{x}_1 = \cos u_1,\ \dot{y}_1 = \sin u_1, \\ \dot{u}_1 = \{-2py_1 + \psi_1(y_1)\cos(u_1)\}\cos u_1/[py_1^2 - f_1], \end{array}}\ s \in [-\frac{1}{2}, -t_0], \tag{43}$$

$$\boxed{\begin{array}{c} \dot{x}_2 = \cos u_2,\ \dot{y}_2 = \sin u_2, \\ \dot{u}_2 = \psi_2(y_2)\cos^2(u_2)/k(\frac{l}{2} - \xi_2), \end{array}}\ s \in [-\frac{l}{2}, -t_0], \tag{44}$$

$$\boxed{\begin{array}{c}\dot{x} = \cos u,\ \dot{y} = \sin u,\\ \dot{u} = \{-2py + \psi_{12}(y)\cos(u)\}\cos u/[py^2 - f_1 + k(\frac{l}{2} - \xi_2)],\end{array}}\ s \in [-t_0, 0]. \tag{45}$$

$$\boxed{\begin{array}{l} y_1(-\frac{1}{2}) = r_1,\\ y_2(-\frac{l}{2}) = r_2,\ \ x_2(-\frac{l}{2}) = -\xi_2,\\ u(0) = 0.\end{array}} \tag{46}$$

$$\boxed{\begin{array}{l} u_1(-t_0) = u_2(-t_0) = u(-t_0),\\ y_1(-t_0) = y_2(-t_0) = y(-t_0),\\ x_1(-t_0) = x_2(-t_0) = x(-t_0).\end{array}} \tag{47}$$

Parameters to be matched are t_0 and ξ_2, only the latter enters the differential equations. After a solution has been found then the interesting data $-\xi_1 = x_1(-\frac{1}{2})$, $\alpha_1 = u_1(-\frac{1}{2})$, $\alpha_2 = u_2(-\frac{l}{2})$ can be determined.

The differential equations (43),(44),(45) are now exactly the natural equations (see (3)) of the meridians. Letting $s \to -t_0 \pm 0$ it is simple to find ($y_0,\ u_0$ common values at $-t_0$)

$$[py_0^2 - f_1](\kappa_1^- - \kappa^+) + k(\frac{l}{2} - \xi_2)(\kappa_2^- - \kappa^+) = 0 \tag{48}$$

as a linear relation (with non-negative coefficients) for the curvature jumps at the junction point.

3 Some physical interpretations

The membrane equations (5) together with the hyperelasticity relations (10) yield

$$y_1 n_1^{22} = \Psi_1(y_1) + c_1^{22},\ \ y_2 n_2^{22} = \Psi_2(y_2) + c_2^{22},$$

connecting the stress resultants n^{22} with the state y . The constants c^{22} are determined by the equilibrium of longitudinal forces at the ends of segment and vein,

$$c_1^{22} = (pr_1^2 - f_1)/\cos\alpha_1,\ \ c_2^{22} = k(\frac{l}{2} - \xi_2)/\cos\alpha_2.$$

(At first it might be amazing that these constants equal the constant multipliers λ_3 and λ_4. It becomes natural if we recall the place of $\lambda_{3,4}$ within

the hamiltonian and the meaning of the n^{22} as reactions to the constraint of *meridional inextensibility.*)
Accordingly, we can write for the state-control functions (40)

$$\begin{aligned}\Phi_1 &= -(py_1^2 - f_1) + y_1 n_1^{22} \cos u_1, \\ \Phi_2 &= -k(\tfrac{l}{2} - \xi_2) + y_2 n_2^{22} \cos u_2,\end{aligned}$$

and the conservation law $\varphi(x) = k(\frac{l}{2} - \xi_2)$ expresses nothing but the equilibrium of longitudinal forces at the left part of the system cut at x.

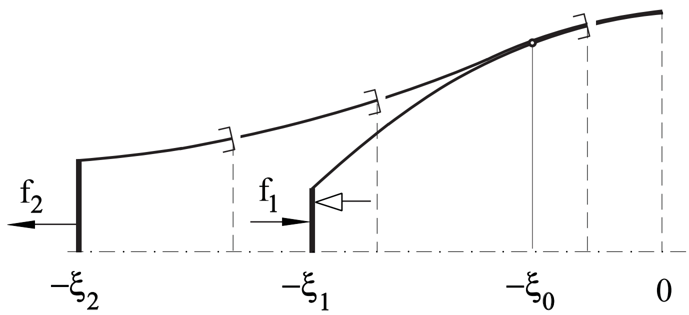

Figure 3. Cuts at various x, $f_2 = k(\frac{l}{2} - \xi_2)$.

The non-constancy of Φ_1 and Φ_2 for $x \in (-\xi_0, 0]$ is due to the influence of the internal constraint (contact) forces $\pm z$ acting upon the membranes. With (42) in Proposition 4 it becomes obvious that z depends significantly on ρ', a fact that is investigated in detail below.

If we make a cut at $x = -\xi_1 + 0$ it becomes clear that the equilibrium of one of the side discs of the segment (acted upon by p, f_1, and the resultant meridional cut force) yields $pr_1^2 - f_1 > 0$ iff the inclination $u_1(-\xi_1)$ of the meridian is smaller than $\pi/2$. So the Supposition 2 gets a reasonable meaning corresponding to the meridians being schlicht curves.

Finally, a cut at $x = -\xi_2 + 0$ exhibits that $c = \frac{l}{2} - \xi_2 > 0$ means that the springs at the vein ends are under tension.

4 The contact force

We start by recalling the last of the general membrane equations (5). After normalization according to (9) it writes

$$-\dot{x}yn^{11} + (\dot{x}\ddot{y} - \ddot{x}\dot{y})n^{22} + 2p_n = 0.$$

Taking this equation for the membranes of segment and vein separately we have

$$\begin{aligned} -\cos u_1 y_1 n_1^{11} + \dot{u}_1 n_1^{22} + 2p - 2z = 0, \\ -\cos u_2 y_2 n_2^{11} + \dot{u}_2 n_2^{22} + 2z = 0. \end{aligned} \tag{49}$$

$\pm z$ are the (normalized) *forces per unit of area* acting in normal direction between the contacting membranes (upon segment inwards, upon vein outwards), $z > 0$ on the contact area, $z = 0$ else. If we introduce the hyperelasticity laws for n^{11} and n^{22} and confine our considerations to the contact area ($x \in [-\xi_0, 0],\ y_1 = y_2 = y,\ u_1 = u_2 = u,\ \dot{u} = u' \cos u$) it follows after a multiplication by y

$$\begin{aligned} -\psi_1(y)\cos u + u' \cos u[\Psi_1(y) + \lambda_3] + 2yp = 2yz, \\ -\psi_2(y)\cos u + u' \cos u[\Psi_2(y) + \lambda_4] = -2yz. \end{aligned}$$

Now it is evident that either of these equations allows to calculate $z(x)$ as soon as the functions u and y are known. Equivalently it holds

$$4yz = 2yp - [\psi_1(y) - \psi_2(y)]\cos u + [\Psi_1(y) - \Psi_2(y) + \lambda_3 - \lambda_4]\dot{u}. \tag{50}$$

If we compare with the first version of the differential equation (37) it turns out that ρ' has a clear physical meaning,

$$\frac{d}{dx}\rho = 2yz. \tag{51}$$

In particular this reflects the facts $z \geq 0$ and ρ non-decreasing. Constancy of ρ is equivalent to zero contact force.

5 Conclusion

In this paper we set up and investigate a mathematical model of a balloon-like compliant mechanical device '*segment*' that is inflated within a (long) cylindrical compliant tube (*'vein'*). Put in concrete terms, compliance means hyperelasticity with a special anisotropy. The background system can be seen as part of a worm crawling in a compliant tube or as a system in medical endoscopy. The investigations continue former work, [8], [9] that concerned freely inflating segments and rigid surrounding tubes, respectively. As before, the mathematical treatment is based on the Principle of Minimal Potential Energy formulated as an optimal control problem with state constraint. The latter shows some features which put it beyond textbook problems, the respective optimality conditions are derived in [1]. In comparison to a formulation by means of the theory of membrane shells this treatment allows to keep the smoothness assumptions (which demand some care when dealing with *skin-like* membranes) on a general level. Any nice smoothness properties then are *deduced* from the optimality conditions. Moreover, contrasting common use, see, e.g., [6], [7], [10], no presuppositions about the shape of the inflated system are introduced.

The analysis of the optimality condition ends up with a 9-dimensional ordinary boundary value problem where several *given* parameters (the internal pressure of the segment in the first place) and two *to-be-matched* parameters enter the differential equations and the boundary conditions as well. All geometrical and physical quantities are appropriately normalized so that the boundary value problem applies to segment-vein systems of arbitrary absolute size and elasticity. A solution of the boundary value problem describes the shape of the deformed system, afterwards the internal force between the contacting segment and vein can be determined utilizing a formula. The paper does not present any numerical exploitation of the final mathematical model yet.

Various improvements of the presented model are at hand. We list some samples.

- Drop the constraint of meridional inextensibility; this might give a more realistic rheology - but at the expense of losing the maximum volume configuration of the inflated ($p = \infty$) segment. (Fortunately, this configuration is given by quadrature and serves as a comfortable start in iteration procedures, [9]).
- Allow for asymmetry of the system: eccentric position of the segment within the vein, or two non-compensating forces instead of $\pm f_1$ equilibrated by tangential forces in the contact area, or different spring stiffnesses. The optimality conditions remain essentially unchanged.
- Replace the isobaric process *'change of shape by variable f_1 at constant*

pressure p_0' by an isochoric one, *'change of shape by variable* f_1 *at constant volume* v_0 *of the segment*'. Then the pressure p varies in the neighborhood of the pre-adjusted pressure p_0 (corresponding to v_0), and it is governed by the additional isoperimetric side condition $\int_{-\xi_1}^{\xi_1} y_1^2 dx = v_0$, [8].

As to the background application problems it is clear that the presented investigations plus forthcoming numerical results are only a first step towards a description of, e.g., stenosis dilatation, [10]. Until now nothing has been done to capture a complicated rheology and rotational non-symmetry (or randomness) of the constricting plaque. And concerning worm-like motion, at least a concatenation of segments within a rigid or compliant surrounding demands an intensified theoretical attention.

6 Appendix

About a lemma on differentiating a composite function under lack of chain-rule.

Lemma 5 *1)* $h \in C^0([t_1, t_2] \times \mathbb{R}^m, \mathbb{R}^1) \;:\; (t, u) \mapsto h(t, u)$.
2) $\exists\, h_{,t}(\cdot, u) \in C^0([t_1, t_2], \mathbb{R}^1)$ *for* $u \in U \subset \mathbb{R}^m$.
3) $u \in D^0([t_1, t_2], U)$, *w.l.o.g. left continuous.*

If u solves a minimum principle

$$h(t, u(t)) := \min_{v \in U} h(t, v), \quad t \in [t_1, t_2], \quad (*)$$

then, on $[t_1, t_2]$,

$$\begin{array}{l} (i) \;\; h(\cdot, u(\cdot)) \in D^1, \\ (ii) \;\; \frac{d}{dt} h(t, u(t)) = h_{,t}(t, u(t)) \;\; \textit{piecewise}, \\ (iii) \; h(t, u(t)) = \int_{t_1}^{t} h_{,s}(s, u(s)) ds + h(t_1, u(t_1)). \end{array}$$

Proof: (see [5], p.77)
a) Claim: $h(\cdot, u(\cdot))$ continuous.
Let $\tau > 0,\ t, t+\tau \in [t_1, t_2]$. Then (*) implies

$$h(t, u(t) \le h(t, u(t+\tau)), \quad h(t+\tau, u(t+\tau)) \le h(t+\tau, u(t)).$$

With $\tau \to +0$ there results

$$h(t, u(t+0)) \leq h(t, u(t)) \leq h(t, u(t+0)),$$

i.e., right-continuity at t. Left-continuity is trivial by **3)**.
b) Claim: u continuous at $t_0 \;\Rightarrow \exists\, \frac{d}{dt}h(t_0, u(t_0))$.
Consider the difference

$$\delta := h(t_0+\tau, u(t_0+\tau)) - h(t_0, u(t_0)).$$

(*) implies

$$h(t_0+\tau, u(t_0+\tau)) - h(t_0, u(t_0+\tau)) \leq \delta \leq h(t_0+\tau, u(t_0)) - h(t_0, u(t_0)).$$

Mean-value theorem: $\exists\, \vartheta_1, \vartheta_2 \in (0,1)$ s.t.

$$\tau \cdot h_{,t}\,(t_0+\vartheta_1\tau, u(t_0+\tau)) \leq \delta \leq \tau \cdot h_{,t}\,(t_0+\vartheta_2\tau, u(t_0)).$$

Let u be continuous at t_0, let $\tau \neq 0$. Then **3)** implies that δ/τ is bounded below and above by $h_{,t}\,(t_0, u(t_0)) + o(1)_{\tau\to 0}$. Thus $\exists \lim_{\tau\to 0} \frac{\delta}{\tau} = \frac{d}{dt}h(t_0, u(t_0))$.
c) We have shown the claimed equation (ii) above at every $t \in [t_1, t_2]$ where u is continuous, so $\frac{d}{dt}h$ is D^0. Then (iii) follows trivially.

Application to present context

Connection with foregoing optimal control problem: Let

$$h(t, v) := H(t, x(t), \lambda(t), v),$$

where $H = f_0(t,x,v) + \lambda f(t,x,v)$ is the Hamiltonian of a control problem, $x \in D^1$ the state, $\lambda \in D^1$ multipliers such that $\dot{x} = H_{,\lambda}$, $\dot{\lambda} = -H_{,x}$. Suppose the premises of the above lemma to be fulfilled. Furthermore let

$$h(t, u(t)) := \min_{v\in U} H(t, x(t), \lambda(t), v).$$

Then the lemma yields $H(\cdot, x(\cdot), \lambda(\cdot).u(\cdot)) \in D^1$ and $\frac{d}{dt}H = H_{,t}$ $\;p.w.$.

Remark: All this applies to the segment-vein problem. Here v stands for (u_1, u_2) and, though H is supplemented by the term ρR_0 this does not disturb the above reasoning: ρ is constant on $(-\xi_1, -\xi_0) \cup (\xi_0, \xi_1)$ whereas R_0 is zero on $(-\xi_0, \xi_0)$. Since either H_i depends only on u_i, the minimum principle applies to H_1 and H_2 separately and, thus, entails the $t-$differentiability of these functions.

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