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ASYMPTOTIC BOUNDARY CONDITIONS FOR THE FINITE ELEMENT MODELLING OF TWO-DIMENSIONAL UNBOUNDED FIELD PROBLEMS WITH UNCOMPENSATED SOURCES

S. Gratkowski, K. Stawicki

West Pomeranian University of Technology, Sikorskiego 37, 70-313 Szczecin, Poland

ABSTRACT

This paper describes theoretical considerations and numerical tests for asymptotic boundary conditions (ABCs) suitable for the finite element modelling of two-dimensional open boundary static and quasi-static electromagnetic field problems with uncompensated sources when the solution at infinity behaves logarithmically. Different shapes of outer boundaries of the finite element region are considered. A simple test example is given. The results using ABC-s are compared with analytical solutions and solutions obtained by boundary truncation techniques.

Index Terms – Finite-element method, open boundary problems, asymptotic boundary conditions

1. INTRODUCTION

Many electromagnetic field problems can be considered as being of the exterior form. That is the problem domain extends to infinity. The finite-element method needs special techniques to represent the exterior region for open boundary problems. A common engineering approach is simple truncation in which the infinite domain is terminated at a finite position and the appropriate infinite boundary condition imposed. Hardly ever such boundary conditions are obvious. Usually, the unknown potential or the normal derivative of the potential is assumed to be equal to zero on the artificial boundary. While this approach is very easy to realize, its accuracy is acceptable only for sufficiently far away outer boundaries. This in turn results in excessive computational costs. Over the last three decades, various methods of analysis for open boundary problems have been investigated (combination of boundary element and finite element method, matching to analytical solutions, mappings, ballooning, generalised multipole technique, iterative procedures, infinite elements, asymptotic boundary conditions). Among the methods, asymptotic boundary conditions (ABCs) provide one of the most attractive alternatives for dealing with static and quasi-static electromagnetic fields in unbounded domains [1]. Most of the published papers have been restricted to applying the ABCs to circular boundaries, which are computationally inefficient in many cases.

In the present paper, we describe different formulations of ABCs for the finite element analysis of 2D open boundary static and quasi-static electromagnetic field problems with uncompensated sources when the solution to Laplace's equation behaves logarithmically at infinity.

2. GENERAL INFORMATION

To solve elliptic boundary value problems in a two-dimensional infinite domain by the finite element method, it is usual to divide the infinite domain by an artificial boundary Γ_e into an interior region Ω_i and a residual exterior region Ω_e. When using the finite element method in the interior region Ω_i, some boundary conditions must be imposed on Γ_e. In Ω_e (and in the outermost part of Ω_i) we seek the solution of the following boundary value problem:

$$\nabla^2 V = 0\,, \quad V \to \ln r \;\text{ as }\; r = \sqrt{x^2 + y^2} \to \infty\,, \quad (1)$$

where: V is the electric potential (electrostatic fields) or z-component of the magnetic vector potential (magnetostatic fields) and r is a distance from the fixed (but arbitrary) origin.

Solution for this problem in the exterior region Ω_e can be written as:

$$V(r,\varphi) = a_0 \ln r + \sum_{n=1}^{\infty} r^{-n}\left(a_n \cos n\varphi + b_n \sin n\varphi\right). \quad (2)$$

This solution can be used to obtain boundary conditions on the artificial boundary Γ_e. In [2] Givoli devised an exact non-local boundary condition on a circular artificial boundary, assuming that V tends to zero at infinity. The same method can be also used when the solution behaves logarithmically at infinity.

3. A NON-LOCAL BOUNDARY CONDITION FOR A CIRCULAR BOUNDARY

We consider Dirichlet problem in the domain Ω_e when Γ_e is a circle of radius d. That is, we consider (1) together with the boundary condition:

$$V = V(d,\varphi) \quad \text{on } \Gamma_e. \quad (3)$$

The function $V(d, \varphi)$ can be expanded in a Fourier series:

$$V(d,\varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left(A_n\cos n\varphi+B_n\sin n\varphi\right), \quad (4)$$

where:

$$\begin{pmatrix}A_n\\ B_n\end{pmatrix}=\frac{1}{\pi}\int_0^{2\pi}V(d,\varphi)\begin{pmatrix}\cos n\varphi\\ \sin n\varphi\end{pmatrix}\mathrm{d}\varphi, \quad n=1,2,3,\ldots,$$

$$A_0=\frac{1}{\pi}\int_0^{2\pi}V(d,\varphi)\,\mathrm{d}\varphi. \quad (5)$$

Equating (2) at $r = d$ to (4) gives:

$$a_n=d^nA_n, \qquad b_n=d^nB_n, \qquad n=1,2,3,\ldots,$$

$$\frac{A_0}{2}=a_0\ln d=\frac{1}{2\pi}\int_0^{2\pi}V(d,\varphi)\,\mathrm{d}\varphi. \quad (6)$$

Upon combining (2), (5) and (6) we obtain the solution in the exterior region Ω_e:

$$V(r,\varphi)=\frac{1}{2\pi}\int_0^{2\pi}V(d,\varphi')\,\mathrm{d}\varphi'\frac{\ln r}{\ln d}+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\pi}\left(\frac{d}{r}\right)^n\int_0^{2\pi}V(d,\varphi')\cos n(\varphi-\varphi')\,\mathrm{d}\varphi'. \quad (7)$$

Differentiating (7) with respect to r and setting $r = d$ yields:

$$\frac{\partial V(d,\varphi)}{\partial r}=\frac{1}{2\pi}\int_0^{2\pi}V(d,\varphi')\,\mathrm{d}\varphi'\frac{1}{d\ln d}+$$

$$-\sum_{n=1}^{\infty}\int_0^{2\pi}k_n(\varphi,\varphi')V(d,\varphi')\,\mathrm{d}\varphi'. \quad (8)$$

This is the exact boundary condition on the artificial boundary $r = d$ and this relation can be used as a boundary condition when solving the field problem in the interior region Ω_i by applying the finite element method. In practice (8) must be approximated by truncating the series at a finite value N.

4. LOCAL BOUNDARY CONDITIONS FOR A CIRCULAR BOUNDARY

Set $\upsilon = V - a_0 \ln r$. In such a case we have:

$$\upsilon(r,\varphi)=\sum_{n=1}^{\infty}r^{-n}F_n(\varphi). \quad (9)$$

The solution (9) can be used to derive local ABCs on the artificial boundary Γ_e. The conditions are exactly correct when imposed at infinity but only approximately correct when imposed at a finite boundary. It can be easily shown that manipulation of the general solution (9) leads to the following boundary conditions on a circular artificial boundary Γ_e of radius d:

$$\frac{\partial\upsilon(d,\varphi)}{\partial r}+\frac{1}{d}\upsilon(d,\varphi)=O\left(1/d^3\right), \quad (10)$$

which is recognised as the first-order ABC, and

$$\frac{\partial^2\upsilon(d,\varphi)}{\partial r^2}+\frac{4}{d}\frac{\partial\upsilon(d,\upsilon)}{\partial r}+\frac{2}{d^2}\upsilon(d,\varphi)=O\left(1/d^5\right), \quad (11)$$

which is recognised as the second-order ABC.

We have also found the general ABC of the order N:

$$\frac{\partial^N\upsilon(d,\varphi)}{\partial r^N}-\sum_{m=1}^{N}\frac{\alpha_m^{(N)}}{d^{N-m+1}}\frac{\partial^{m-1}\upsilon(d,\varphi)}{\partial r^{m-1}}=O\left(d^{-2N-1}\right),$$

$$\alpha_m^{(N)}=-\frac{\overbrace{N!\,N\cdot(N-1)\cdot\ldots}^{m\ \text{factors}}}{\left((m-1)!\right)^2}. \quad (12)$$

This boundary condition is equivalent to the well known Bayliss-Gunzburger-Turkel (BGT) sequence of differential operators [3]:

$$B_1\,\upsilon(r,\varphi)=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\upsilon(r,\varphi),$$

$$B_N=\prod_{m=1}^{N}\left(\frac{\partial}{\partial r}+\frac{2m-1}{r}\right)\equiv\left(\frac{\partial}{\partial r}+\frac{2N-1}{r}\right)B_{N-1}, \quad (13)$$

$$N=2,3,\ldots, \qquad B_N\,\upsilon(d,\varphi)=O\left(d^{-2N-1}\right).$$

The first order local ABC for the potential V (the most important from the practical point of view) can be written as:

$$\frac{\partial V(d,\varphi)}{\partial r}=\frac{1}{d\ln d}V(d,\varphi). \quad (14)$$

The boundary contribution in the finite element formulation enters into a surface integral representation over the outer boundary, where the integrand is the product of a testing function and the normal derivative of V. Hence, the asymptotic boundary condition needs to be imposed on the normal derivative of V. For a circular outer boundary, the normal derivative is simply the radial one.

Figure 1 shows an exemplary grid of finite elements used in the interior region for the artificial circular boundary. The unknown function V within the segment 1-2 (see Fig. 1) can be approximated as:

$$V_{1-2}=N_1V_1+N_2V_2, \quad (15)$$

where: N_1 and N_2 are shape functions:

$$N_{1,2}(d,\varphi)=\frac{1}{2}\left(1\mp\frac{\varphi-\varphi_c}{\alpha_c}\right), \quad \varphi_{1,2}=\varphi_c\mp\alpha_c, \quad (16)$$

where the upper and lower signs correspond to the subscripts 1 and 2, respectively.

The elemental boundary matrix is given by:

$$[K]=-\frac{1}{\ln d}\int_{\varphi_c-\alpha_c}^{\varphi_c+\alpha_c}\begin{bmatrix}N_1^2 & N_1N_2\\ N_1N_2 & N_2^2\end{bmatrix}\mathrm{d}\varphi. \quad (17)$$

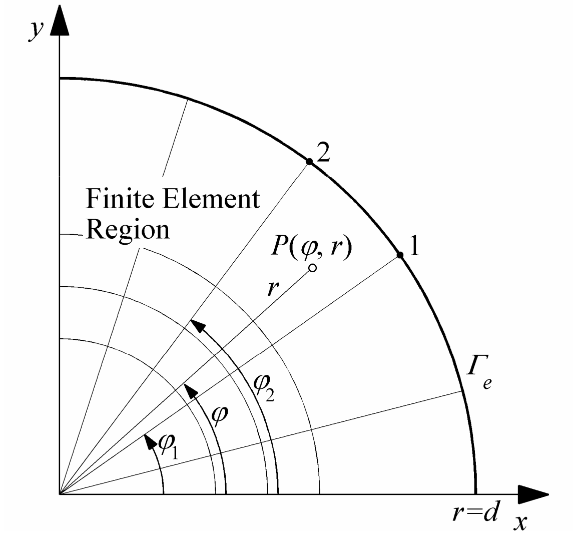

Figure 1 An exemplary grid of finite elements for a circular boundary

The matrix $[K]$ can be evaluated analytically and the result is:

$$[K]=\frac{\alpha_c}{3\ln d}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}. \qquad (18)$$

5. LOCAL BOUNDARY CONDITIONS FOR A NONCIRCULAR BOUNDARY

A circular outer boundary is uneconomical for problems with large aspect ratios. Here we shall consider a more general case. Figure 2 shows the arbitrary shape finite element region.

Let us consider the outermost finite element with nodes 1, 2 and 3. For convenience of analysis, we introduce an auxiliary rectangular system (u, v) with the v-axis parallel to the segment 1-2. The azimuthal angle ψ is measured anticlockwise from the axis u. The length of the segment 1-2 is $2l$. Taking into account that:

$$u=r\cos\psi\,,\ v=r\sin\psi\,, \qquad (19)$$

we can write $\partial V/\partial r$ as:

$$\frac{\partial V}{\partial r}=\frac{\partial V}{\partial u}\frac{\partial u}{\partial r}+\frac{\partial V}{\partial v}\frac{\partial v}{\partial r}=\frac{\partial V}{\partial u}\cos\psi+\frac{\partial V}{\partial v}\sin\psi\,. \qquad (20)$$

Substituting (20) into (14) gives:

$$\frac{\partial V}{\partial u}\cos\psi+\frac{\partial V}{\partial v}\sin\psi=\frac{1}{r\ln r}V\,, \qquad (21)$$

and then:

$$\frac{\partial V}{\partial u}=\frac{1}{u\ln\sqrt{u^2+v^2}}V-\frac{v}{u}\frac{\partial V}{\partial v}\,. \qquad (22)$$

For $u = d$ we obtain the following first-order ABC on the segment 1-2:

$$\frac{\partial V}{\partial u}=\frac{2}{d\ln\left(u^2+v^2\right)}V-\frac{v}{d}\frac{\partial V}{\partial v}\,. \qquad (23)$$

We can proceed to formulate the elemental boundary matrix. The unknown function V within the segment 1-2 can be approximated by (15), however, the shape functions now are:

$$N_1=\frac{1}{2}\left(1+\frac{b}{l}-\frac{v}{l}\right),\ N_2=\frac{1}{2}\left(1-\frac{b}{l}+\frac{v}{l}\right). \qquad (24)$$

The elemental boundary matrix is given by (the internal element sides do not contribute the global matrix and only those residing on the boundary Γ_e have nontrivial contributions):

$$[K]=-\frac{2}{d}\int_{b-l}^{b+l}\frac{1}{\ln\left(d^2+v^2\right)}\begin{bmatrix}N_1^2 & N_1N_2\\ N_1N_2 & N_2^2\end{bmatrix}\mathrm{d}v+$$

$$+\frac{1}{d}\int_{b-l}^{b+l}v\begin{bmatrix}N_1\frac{\partial N_1}{\partial v} & N_1\frac{\partial N_2}{\partial v}\\ N_2\frac{\partial N_1}{\partial v} & N_2\frac{\partial N_2}{\partial v}\end{bmatrix}\mathrm{d}v\,. \qquad (25)$$

The first integral has to be calculated numerically, while the second one can be evaluated analytically and the result is:

$$[K]^{(2)}=\frac{1}{6d}\begin{bmatrix}l-3b & -l+3b\\ -l-3b & l+3b\end{bmatrix}. \qquad (26)$$

Unfortunately, the matrix is symmetric only if $b = 0$. It corresponds to the circular outer boundary of the finite element region.

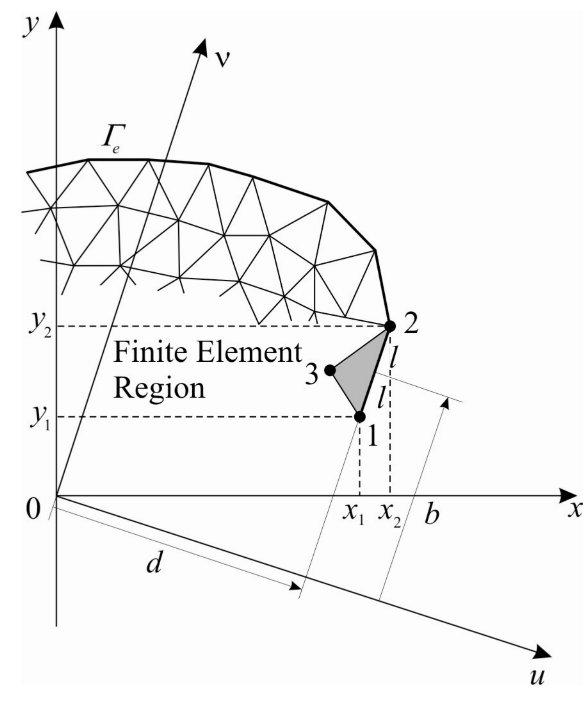

Figure 2 Finite-element region and noncircular artificial boundary

6. A NUMERICAL EXAMPLE

A very simple problem with a known closed-form solution was considered. This was the magnetic field due to currents in two long straight wires (Fig. 3).

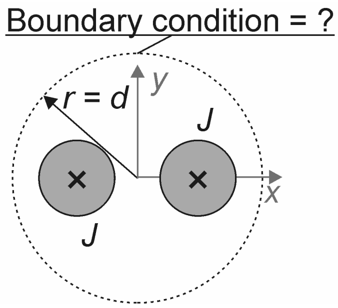

Figure 3 Test problem

The problem was solved for circular and rectangular artificial boundaries Γ_e, using elemental boundary matrices (18) and (25), respectively. The numerical results have been compared with the well-known analytical solution.

In Figs. 4 and 5 the magnetic flux lines are shown for the circular shape of the finite element region and for different conditions on the outer boundary. It can be seen that the ABC gives more realistic magnetic flux density lines than zero Dirichlet boundary condition. Figure 6 shows the exact solution for the x-component of the magnetic flux density along line $x = 0$, the solution obtained by using Dirichlet b.c. and for the ABC. The ABC solution is hardly distinguishable from the exact solution, while the solution using Dirichlet b.c. is off about 20% at $y = \pm$ 30 cm. The relevant results for the rectangular outer boundary are shown in Figs. 7 – 9.

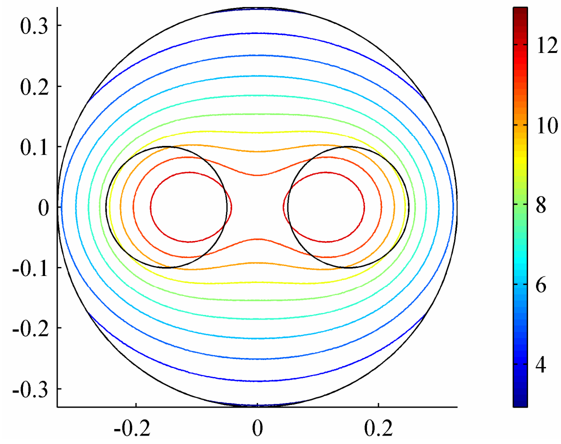

Figure 5 Magnetic flux density lines for the ABC on the outer boundary – matrix (18)

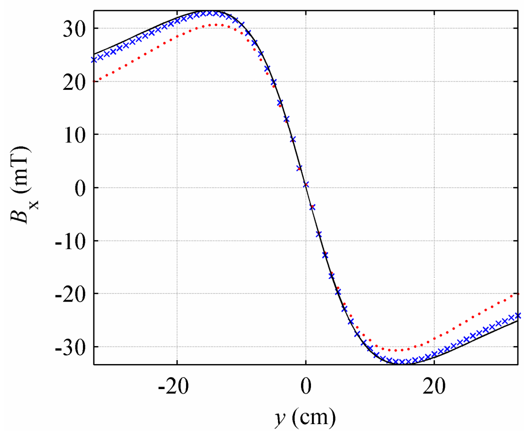

Figure 6 Component B_x of the magnetic flux density along line x = 0; — exact, ··· Dirichlet b.c., ××× ABC – matrix (18)

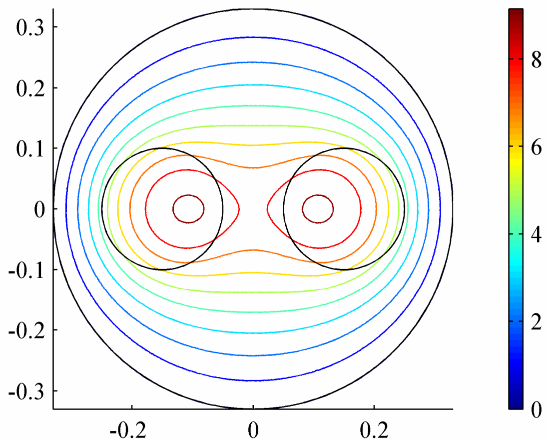

Figure 4 Magnetic flux density lines for the Dirichlet boundary condition on the outer boundary

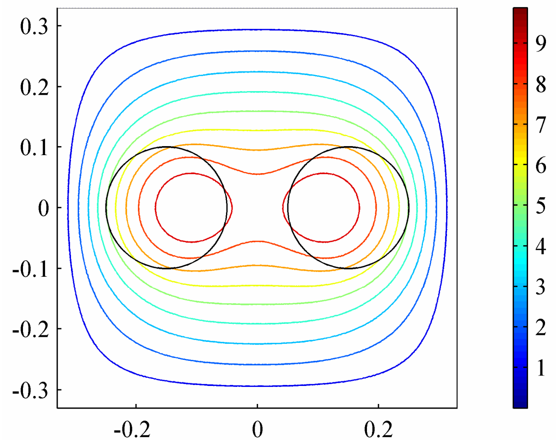

Figure 7 Magnetic flux density lines for the Dirichlet boundary condition on the outer boundary

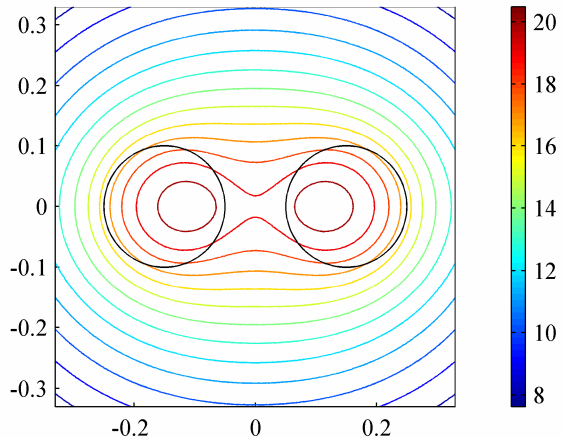

Figure 8 Magnetic flux density lines for the ABC on the outer boundary – matrix (25)

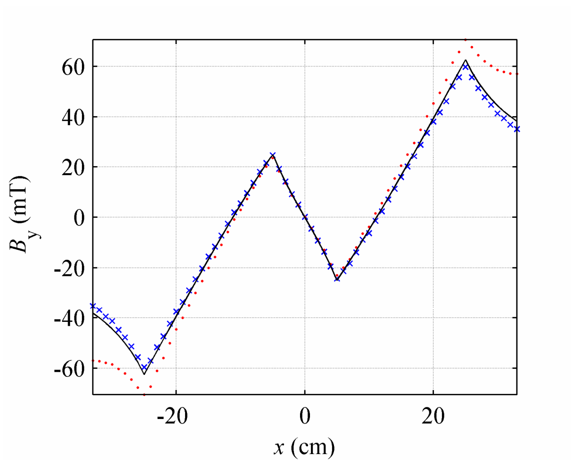

Figure 9 Component B_y of the magnetic flux density along line y = 0; − exact, ··· Dirichlet b.c., ××× ABC – matrix (25)

7. CONCLUSIONS

The asymptotic boundary conditions provide a reasonable unbounded domain representation and due to their simplicity and accuracy are highly recommended. Because of the simple expressions for the coefficients of the boundary matrices the ABCs are computationally only little more time consuming to apply than enforcing Dirichlet zero boundary condition at the outer boundary.

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