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Time-varying linear DAEs transferable into standard canonical form

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Abstract

We introduce a solution theory for time-varying linear differential-algebraic equations (DAEs) $E(t)\dot{x} = A(t)x$ which can be transformed into standard canonical form (SCF), i.e. the DAE is decoupled into an ODE $\dot{z}_1 = J(t)z_1$ and a pure DAE $N(t)\dot{z}_1 = z_1$, where N is pointwise strictly lower triangular. This class is a time-varying generalization of time-invariant DAEs where the corresponding matrix pencil is regular. It will be shown in which sense the SCF is a *canonical* form, that it allows for a transition matrix similar to the one for ODEs, and how this can be exploited to derive a variation of constants formula. Furthermore, we show in which sense the class of systems transferable into SCF is equivalent to DAEs which are analytically solvable, and relate SCF to the derivative array approach, differentiation index and strangeness index. Finally, an algorithm is presented which determines the transformation matrices which put a DAE into SCF.

Keywords: Time-varying linear differential algebraic equations, standard canonical form, analytically solvable, generalized transition matrix

1 Introduction

We study time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x} = A(t)x\,, \tag{1.1}$$

where $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n})$ for $n \in \mathbb{N}$ and – throughout the paper – $\mathcal{I} \subseteq \mathbb{R}$ denotes an open interval. For brevity, the tuple (E, A) is identified with the DAE (1.1). A function $x : \mathcal{J} \to \mathbb{R}^n$ is called *solution* of (E, A) if, and only if, x is a continuously differentiable function on the open interval $\mathcal{J} \subseteq \mathcal{I}$ and solves (1.1) for all $t \in \mathcal{J}$; it is called *global* solution if, and only if, $\mathcal{J} = \mathcal{I}$.

If $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$, then it is well-known that $x : \mathcal{J} \to \mathbb{R}^n$ solves (1.1) if, and only if, $z(\cdot) := T(\cdot)^{-1}x(\cdot)$ solves

$$S(t)E(t)T(t)\dot{z} = \Big[S(t)A(t)T(t) - S(t)E(t)\dot{T}(t)\Big]\,z\,.$$

Therefore, we introduce the following equivalence relation.

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Nomenclature

$\mathbb{N},\ \mathbb{N}_0$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\ker A$		the kernel of the matrix $A \in \mathbb{R}^{m\times n}$
$\operatorname{im} A$		the image of the matrix $A \in \mathbb{R}^{m\times n}$
M^*	:=	$\overline{M}^\top$, the Hermitian conjugate of $M \in \mathbb{C}^{m\times n}$
$\mathbf{Gl}_n(\mathbb{R})$		the general linear group of degree n, i.e. the set of all invertible $n \times n$ matrices over $\mathbb{R}$
$\mathcal{C}^k(\mathcal{I}; \mathcal{S})$		the set of k-times continuously differentiable functions $f : \mathcal{I} \to \mathcal{S}$ from an open set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\operatorname{dom} f$		the domain of the function f
$f\,\|_{\mathcal{M}}$		the restriction of the function f on a set $\mathcal{M} \subseteq \operatorname{dom} f$

Definition 1.1 (Equivalence of DAEs [KM06, Def. 3.3])**.** The DAEs $(E_1, A_1), (E_2, A_2) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n})$ are called *equivalent* if, and only if, there exists $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$ such that

$$E_2 = SE_1T\,, \quad A_2 = SA_1T - SE_1\dot{T}\,; \qquad \text{we write} \quad (E_1,\, A_1) \overset{S,T}{\sim} (E_2,\, A_2)\,. \tag{1.2}$$

◇

That the equivalence of DAEs is in fact an equivalence relation (see e.g. [KM06, Lem. 3.4]) follows easily by exploiting

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(T^{-1}) = -T^{-1}\dot{T}T^{-1}, \tag{1.3}$$

which follows from differentiation of the identity $I = T^{-1}T$.

We now make precise the system class studied in the present paper: DAEs transferable into standard canonical form.

Definition 1.2 (Standard canonical form (SCF) [Cam83, CP83])**.** The DAE $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n})$ is called *transferable into standard canonical form* (SCF) if, and only if, there exist $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$ and $n_1, n_2 \in \mathbb{N}$ such that

$$(E,\, A) \overset{S,T}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right), \tag{1.4}$$

where $N : \mathcal{I} \to \mathbb{R}^{n_2\times n_2}$ is pointwise strictly lower triangular and $J : \mathcal{I} \to \mathbb{R}^{n_1\times n_1}$; a matrix N is called *pointwise strictly lower triangular* if, and only if, all entries of $N(t)$ on the diagonal and above are zero for all $t \in \mathcal{I}$. ◇

The paper is organized as follows: In Section 2 we show in which sense the SCF is a canonical form, and that transferability into SCF is, for time-invariant DAEs, equivalent to regularity of the corresponding matrix pencil. In Section 3, the concept of SCF is used to define a unique generalized transition matrix which has similar semi-group properties as the transition matrix for ODEs. Moreover, the generalized transition matrix is exploited to derive a variation of constants formula for inhomogeneous DAEs. In Section 4, transferability into SCF is shown to be "almost" equivalent to other concepts such as analytic solvability, the derivative array approach and the differentiation index. Finally, in Section 5 we present an algorithm to determine the transformation matrices for the SCF.

2 Standard canonical form (SCF)

We show that the SCF in (1.4) is unique in the sense that the formats of the ODE and the pure DAE are unique, and that the ODE and the pure DAE are unique up to some equivalence as in (1.2).

Theorem 2.1 (Uniqueness of SCF)**.** *Let* $n_1, n_2, \tilde{n}_1, \tilde{n}_2 \in \mathbb{N}$, $J_1 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_1 \times n_1})$, $J_2 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1})$ *and pointwise strictly lower triangular* $N_1 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_2 \times n_2})$, $N_2 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{\tilde{n}_2 \times \tilde{n}_2})$. *If, for some* $S \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$, $T \in \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$,

$$\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \overset{S,T}{\sim} \left(\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & I_{\tilde{n}_2} \end{bmatrix} \right),$$

then

(i) $n_1 = \tilde{n}_1$, $n_2 = \tilde{n}_2$,

(ii) $S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}$, $\quad T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}$, $\quad T_{11} = S_{11}^{-1}$,

(iii) $(I_{n_1}, J_1) \overset{T_{11}^{-1}, T_{11}}{\sim} (I_{n_1}, J_2)$, $\quad (N_1, I_{n_2}) \overset{S_{22}, T_{22}}{\sim} (N_2, I_{n_2})$. ⋄

The proof of Theorem 2.1 requires a lemma on the solution of a *pure* DAE, i.e. $n_1 = 0$ in (1.4).

Lemma 2.2 (Solutions of pure DAEs)**.** *Let* $N \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})$ *be pointwise strictly lower triangular. Then* $x(\cdot) = 0$ *is the unique global solution of the pure DAE*

$$N(t)\dot{x} = x\,, \tag{2.1}$$

and every (local) solution $z : \mathcal{J} \to \mathbb{R}^n$ *of* (2.1) *satisfies* $z(t) = 0$ *for all* $t \in \mathcal{J}$.

Proof: Clearly, $x(\cdot) = 0$ solves (2.1) for all $t \in \mathcal{I}$. We show that any solution $z : \mathcal{J} \to \mathbb{R}^n$ of (2.1) satisfies $z = 0$. Consider (2.1) row-wise. Let $N(t) = (n_{ij}(t))_{i,j=1,\ldots,n}$ for $t \in \mathcal{I}$. Then

$$\forall\, i \in \{1, \ldots, n\}\ \forall\, t \in \mathcal{J} \ :\ z_i(t) = \sum_{j=1}^{i-1} n_{ij}(t)\dot{z}_j(t)\,. \tag{2.2}$$

We prove

$$\forall\, i \in \{1, \ldots, n\}\ \forall\, t \in \mathcal{J} \ :\ z_i(t) = 0$$

by induction over i. The assertion holds true for $i = 1$. Suppose it holds for some $i \in \{1, \ldots, n-1\}$. Then $\dot{z}_j(t) = 0$ for all $t \in \mathcal{J}$ and all $j \in \{1, \ldots, i\}$ and therefore,

$$\forall\, t \in \mathcal{J} \ :\ z_{i+1}(t) \overset{(2.2)}{=} \sum_{j=1}^{i} n_{ij}(t)\dot{z}_j(t) = 0.$$

This shows $z = 0$ and completes the proof of the lemma. □

Proof of Theorem 2.1:
Step 1: Assume, without loss of generality, that $n_1 \geq \tilde{n}_1$. In view of (1.3) we have $T^{-1} \in \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$ and therefore we may write

$$T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad \text{where } T_{11} \in \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{n_1 \times n_1}),\ T_{22} \in \mathcal{C}^1(\mathcal{I}; \mathbb{R}^{n_2 \times n_2}),\ T_{12}, T_{21} \text{ appropriate.}$$

We show that

$$\forall\, t\in\mathcal{I}\ :\ T_{21}(t)=0 \quad \wedge \quad \det T_{11}(t)\neq 0 \quad \wedge \quad \det T_{22}(t)\neq 0.$$

Let $(t^0,x^1)\in\mathcal{I}\times\mathbb{R}^{n_1}$. Then

$$x:\mathcal{I}\to\mathbb{R}^n,\ t\mapsto \begin{bmatrix}\Phi_{J_1}(t,t^0)x^1\\ 0\end{bmatrix},$$

where $\Phi_{J_1}(\cdot,\cdot)$ denotes the transition matrix of $\dot z = J_1(t)z$, solves

$$\begin{bmatrix}I_{n_1} & 0\\ 0 & N_1(t)\end{bmatrix}\dot x = \begin{bmatrix}J_1(t) & 0\\ 0 & I_{n_2}\end{bmatrix}x.$$

Then $y(\cdot):=T(\cdot)^{-1}x(\cdot)$ solves

$$\begin{bmatrix}I_{\tilde n_1} & 0\\ 0 & N_2(t)\end{bmatrix}\dot y = \begin{bmatrix}J_2(t) & 0\\ 0 & I_{\tilde n_2}\end{bmatrix}y,$$

and it follows from Lemma 2.2 that $y(\cdot)=\begin{bmatrix}y_1(\cdot)\\ 0\end{bmatrix}$ for some $y_1\in\mathcal{C}^1(\mathcal{I};\mathbb{R}^{\tilde n_1})$. Hence

$$\begin{bmatrix}T_{11}(t^0)x^1\\ T_{21}(t^0)x^1\end{bmatrix} = T(t^0)^{-1}x(t^0) = y(t^0) = \begin{bmatrix}y_1(t^0)\\ 0\end{bmatrix}. \tag{2.3}$$

Since $n_2\leq \tilde n_2$ it follows that $T_{21}(t^0)x^1=0$ and, since $x^1\in\mathbb{R}^{n_1}$ is arbitrary, we conclude $T_{21}(t^0)=0$. Thus $\det T_{11}(t^0)\cdot\det T_{22}(t^0)=\det T(t^0)^{-1}$, and invertibility of $T(t^0)$ yields invertibility of $T_{11}(t^0)$ and $T_{22}(t^0)$.

Step 2: We prove (i). Assume that $n_1>\tilde n_1$. Let α be the last row of $T_{11}(t^0)$, $\alpha^\top\in\mathbb{R}^{n_1}$. Then (2.3) and $n_1>\tilde n_1$ yield $\alpha x^1=0$, and, since x^1 is arbitrary, it follows that $\alpha=0$, which contradicts $\det T_{11}(t^0)\neq 0$.

Step 3: We prove (ii) and (iii). Write

$$S^{-1}=\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}, \quad \text{where } S_{11}\in\mathcal{C}(\mathcal{I};\mathbb{R}^{n_1\times n_1}),\ S_{22}\in\mathcal{C}(\mathcal{I};\mathbb{R}^{n_2\times n_2}),\ \ S_{12},S_{21} \text{ appropriate.}$$

Then

$$\begin{bmatrix}I_{n_1} & 0\\ 0 & N_1\end{bmatrix} = S^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & N_2\end{bmatrix}T^{-1} = \begin{bmatrix}S_{11}T_{11}+S_{12}N_2T_{21}, & S_{11}T_{12}+S_{12}N_2T_{22}\\ S_{21}T_{11}+S_{22}N_2T_{21}, & S_{21}T_{12}+S_{22}N_2T_{22}\end{bmatrix}, \tag{2.4}$$

and

$$\begin{aligned}\begin{bmatrix}J_1 & 0\\ 0 & I_{n_2}\end{bmatrix} &= S^{-1}\begin{bmatrix}J_2 & 0\\ 0 & I_{n_2}\end{bmatrix}T^{-1} - S^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & N_2\end{bmatrix}\tfrac{\mathrm{d}}{\mathrm{d}t}(T^{-1})\\ &= \begin{bmatrix}S_{11}J_2T_{11}+S_{12}T_{21}-S_{11}\dot T_{11}-S_{12}N_2\dot T_{21}, & S_{11}J_2T_{12}+S_{12}T_{22}-S_{11}\dot T_{12}-S_{12}N_2\dot T_{22}\\ S_{21}J_2T_{11}+S_{22}T_{21}-S_{21}\dot T_{11}-S_{22}N_2\dot T_{21}, & S_{21}J_2T_{12}+S_{22}T_{22}-S_{21}\dot T_{12}-S_{22}N_2\dot T_{22}\end{bmatrix}.\end{aligned} \tag{2.5}$$

Step 1 and the equations in the first n_1 columns in (2.4) yield

$$\forall\, t\in\mathcal{I}:\ S_{11}(t)^{-1}=T_{11}(t)\ \wedge\ S_{21}(t)=0\ \wedge\ \det S_{22}(t)\neq 0,$$

and therefore, by (2.4),

$$N_1 = S_{22}N_2T_{22} \tag{2.6}$$

and, by the lower right block in (2.5),

$$I_{n_2} = S_{22}T_{22} - S_{22}N_2\dot{T}_{22}. \tag{2.7}$$

Now suppose we have shown that $T_{12} = S_{12} = 0$. Then (ii) holds true and (2.6) together with (2.7) shows the second claim in (iii). The upper left block in (2.5) yields $J_1 = S_{11}J_2T_{11} - S_{11}\dot{T}_{11}$, and invoking $S_{11} = T_{11}^{-1}$, we find $J_1 = T_{11}^{-1}J_2T_{11} - T_{11}^{-1}\dot{T}_{11}$ which shows the first claim in (iii).

Step 4: It remains to prove $T_{12} = S_{12} = 0$. It follows from (2.7) that $S_{22}^{-1} = T_{22} - N_2\dot{T}_{22}$. Observe that the upper right block in (2.5) yields $0 = S_{11}(J_2T_{12}\dot{T}_{12}) + S_{12}(T_{22} - N_2\dot{T}_{22})$ and thus

$$S_{12} = -S_{11}(J_2T_{12} - \dot{T}_{12})S_{22}. \tag{2.8}$$

Next, the upper right block in (2.4) gives

$$T_{12} = -S_{11}^{-1}S_{12}N_2T_{22} \overset{(2.8)}{=} (J_2T_{12} - \dot{T}_{12})S_{22}N_2T_{22} \overset{(2.6)}{=} (J_2T_{12} - \dot{T}_{12})N_1. \tag{2.9}$$

Therefore

$$T_{12}e_{n_2} \overset{(2.9)}{=} (J_2T_{12} - \dot{T}_{12})N_1e_{n_2} = (J_2T_{12} - \dot{T}_{12}) \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \tag{2.10}$$

and so

$$T_{12}e_{n_2-1} \overset{(2.9)}{=} (J_2T_{12} - \dot{T}_{12})N_1e_{n_2-1} = (J_2T_{12} - \dot{T}_{12}) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \overset{(2.10)}{=} 0.$$

Proceeding in this way gives $T_{12} = 0$ and, invoking (2.8), we find $S_{12} = 0$. This completes the proof of the theorem. □

In the following proposition we show that transferability into SCF is in fact, for *time-invariant* DAEs $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, a generalization of *regularity* of the matrix pencil $sE - A \in \mathbb{R}^{n\times n}[s]$, i.e. $0 \neq \det(sE - A) \in \mathbb{R}[s]$.

Proposition 2.3 (Time-invariant DAEs: SCF $\triangleq$ regularity)**.** *For $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ we have*

$$(E, A) \text{ is transferable into SCF} \iff (E, A) \text{ is regular.}$$

Proof: "$\Leftarrow$": If $sE-A$ is regular, then by the Weierstraß canonical form (see e.g. [Gan59, Thm. XII.3]) there exist $S, T \in \mathbf{Gl}_n(\mathbb{R})$ such that (1.4) holds for some $J \in \mathbb{R}^{n_1\times n_1}$ and nilpotent $N \in \mathbb{R}^{n_2\times n_2}$ in Jordan form. Therefore, (E, A) is transferable into SCF.
"$\Rightarrow$": If (E, A) is transferable into SCF by time-varying $(S, T) \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$ as in (1.4) and $sE - A$ is not regular, then the latter implies (see e.g. [KM06, Thm. 2.14]) that there exists a nontrivial solution $x(\cdot)$ to the initial value problem (1.1), $x(0) = 0$. Thus, $\binom{z_1(\cdot)}{z_2(\cdot)} := T^{-1}(\cdot)x(\cdot)$ solves $\dot{z}_1 = Jz_1$, $z_1(0) = 0$ and $N\dot{z}_2 = z_2$, $z_2(0) = 0$. It now follows from the theory of ordinary differential equations and Lemma 2.2 that the unique solution is $\binom{z_1(\cdot)}{z_2(\cdot)} = 0$; this contradicts the fact that $x(\cdot)$ is non-trivial. □

3 Transition matrix and variation of constants

In this section we exploit uniqueness of the SCF to introduce a generalized transition matrix for (E,A) as a generalization of time-varying ordinary differential equations. This is used to characterize the set of consistent initial values of (E,A) and to derive a variation of constants formula for inhomogeneous DAEs.

Definition 3.1 (Consistent initial values [KM06, Def. 1.1]). The *set of all pairs of consistent initial values* of $(E,A) \in \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})$ is denoted by

$$\mathcal{V}_{E,A} := \left\{ (t^0,x^0) \in \mathbb{R}\times\mathbb{R}^n \;\middle|\; \exists \text{ (local) sln. } x(\cdot) \text{ of } (1.1):\ t^0 \in \operatorname{dom} x(\cdot),\ x(t^0)=x^0 \right\}$$

and the linear subspace of *initial values which are consistent at time* $t^0 \in \mathcal{I}$ is denoted by

$$\mathcal{V}_{E,A}(t^0) := \left\{ x^0 \in \mathbb{R}^n \;\middle|\; (t^0,x^0) \in \mathcal{V}_{E,A} \right\}. \qquad \diamond$$

Note that if $x:\mathcal{J}\to\mathbb{R}^n$ is a solution of (1.1), then $x(t)\in\mathcal{V}_{E,A}(t)$ for all $t\in\mathcal{J}$.

Now we are in a position to characterize, for DAEs transferable into SCF, the set of consistent initial values and to derive a formula for the solution.

Proposition 3.2 (Solutions of homogeneous DAEs). *Suppose that the DAE* $(E,A) \in \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})$ *is transferable into SCF as in* (1.4)*. Then*

(i)
$$(t^0,x^0)\in\mathcal{V}_{E,A} \iff x^0 \in \operatorname{im} T(t^0)\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}. \tag{3.1}$$

(ii) *Any solution of the initial value problem* (1.1), $x(t^0)=x^0$, *where* $(t^0,x^0)\in\mathcal{V}_{E,A}$, *extends uniquely to a global solution* $x(\cdot)$, *and this solution satisfies*

$$x(t) = U(t,t^0)x^0, \qquad U(t,t^0) := T(t)\begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}, \quad t\in\mathcal{I}, \tag{3.2}$$

where $\Phi_J(\cdot,\cdot)$ *denotes the transition matrix of* $\dot z = J(t)z$.

Proof: Let throughout $x(\cdot)$ be given as in (3.2).
Step 1: We show that $x(\cdot)$ solves (1.1) for all $t\in\mathcal{I}$:

$$\begin{aligned}
E(t)\dot x(t) &= E(t)\left(\dot T(t)\begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} + E(t)T(t)\begin{bmatrix} J(t)\Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix}\right) T(t^0)^{-1}x^0 \\
&\overset{(1.4)}{=} \left(E(t)\dot T(t)\begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} + S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix}\begin{bmatrix} J(t)\Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix}\right) T(t^0)^{-1}x^0 \\
&= \left(E(t)\dot T(t) + S(t)^{-1}\begin{bmatrix} J(t) & 0 \\ 0 & I_{n_2} \end{bmatrix}\right)\begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 \\
&\overset{(1.4)}{=} \left(E(t)\dot T(t) + A(t)T(t) - E(t)\dot T(t)\right)\begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 \\
&= A(t)T(t)\begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 = A(t)x(t).
\end{aligned}$$

Step 2: We show that $x(t^0) = x^0$ if, and only, $x^0 \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$. For

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} := T(t^0)^{-1} x^0, \qquad \text{where } \alpha \in \mathbb{R}^{n_1},\ \beta \in \mathbb{R}^{n_2},$$

we have

$$x(t^0) = T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1} x^0 = T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = x^0 - T(t^0) \begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

and hence $x(t^0) = x^0$ if, and only if, $\beta = 0$ or, equivalently, $x^0 \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$.

Step 3: We show that every solution $z : \mathcal{J} \to \mathbb{R}^n$ of (1.1) such that $z(t^0) = x^0$, $(t^0, x^0) \in \mathcal{V}$, fulfills $z = x\,|_{\mathcal{J}}$.
Clearly, $(z - x) : \mathcal{J} \to \mathbb{R}^n$ solves $E(t)\frac{\mathrm{d}}{\mathrm{d}t}(z - x)(t) = A(t)(z - x)(t)$ for all $t \in \mathcal{J}$. Then

$$[y_1^\top, y_2^\top]^\top = y := T^{-1}(z - x) \quad \text{solves} \quad \dot y_1 = J(t)\, y_1, \qquad N(t)\, \dot y_2 = y_2,$$

and by Lemma 2.2 it follows that $y_2(t) = 0$ for all $t \in \mathcal{J}$. An application of $y(t^0) = T(t^0)^{-1}(x^0 - x(t^0))$ gives

$$\begin{aligned} 0 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} y(t^0) &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t^0)^{-1} \left(x^0 - T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1} x^0 \right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t^0)^{-1} x^0 = T(t^0)^{-1}(x^0 - x(t^0)) = y(t^0). \end{aligned}$$

Hence $y_1(t) = 0$ for all $t \in \mathcal{J}$ and therefore $z = x\,|_{\mathcal{J}}$. This completes the proof. ☐

Next it is shown that the operator $U(\cdot,\cdot)$ defined in (3.2) is unique.

Proposition 3.3 (Uniqueness of U). *Suppose $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I}; \mathbb{R}^{n\times n})$ is transferable into SCF. Then $U(\cdot,\cdot)$ defined in (3.2) is independent of the choice of (S, T) in (1.4).*

Proof: Let (E, A) be transferable into SCF as in (1.4) for some $(S_1, T_1), (S_2, T_2) \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$. Then, in view of Theorem 2.1,

$$\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \overset{S_1^{-1}, T_1^{-1}}{\sim} (E,\ A) \overset{S_2, T_2}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right),$$

where $N_1, N_2 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_2\times n_2})$ are pointwise strictly lower triangular and $J_1, J_2 \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n_1\times n_1})$. This gives

$$\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \overset{S_2 S_1^{-1}, T_1^{-1} T_2}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right),$$

and hence, by Theorem 2.1(ii) and (iii), there exist $T_a, T_b \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_{n_1}(\mathbb{R}))$, $S_b \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_{n_2}(\mathbb{R}))$ such that

$$(S_2 S_1^{-1}, T_1^{-1} T_2) = \left(\begin{bmatrix} T_a^{-1} & 0 \\ 0 & S_b \end{bmatrix}, \begin{bmatrix} T_a & 0 \\ 0 & T_b \end{bmatrix} \right), \qquad J_2 = T_a^{-1} J_1 T_a - T_a^{-1} \dot T_a.$$

The latter is equivalent to, see e.g. [HP05, (3.3.26)],

$$\forall\, t,s \in \mathcal{I} \;:\; \Phi_{J_2}(t,s) = T_a^{-1}(t)\Phi_{J_1}(t,s)T_a(s)\,.$$

This yields

$$\forall\, t,s \in \mathcal{I} \;:\; \begin{bmatrix} \Phi_{J_2}(t,s) & 0 \\ 0 & 0 \end{bmatrix} = \left(T_1(t)^{-1}T_2(t)\right)^{-1} \begin{bmatrix} \Phi_{J_1}(t,s) & 0 \\ 0 & 0 \end{bmatrix} T_1(s)^{-1}T_2(s)$$

and a straightforward calculation gives

$$\forall\, t,s \in \mathcal{I} \;:\; T_1(t) \begin{bmatrix} \Phi_{J_1}(t,s) & 0 \\ 0 & 0 \end{bmatrix} T_1(s)^{-1} \;=\; T_2(t) \begin{bmatrix} \Phi_{J_2}(t,s) & 0 \\ 0 & 0 \end{bmatrix} T_2(s)^{-1}\,.$$

This completes the proof. □

Now Proposition 3.3 ensures that the following is well defined.

Definition 3.4 (Generalized transition matrix)**.** Suppose $(E,A) \in \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})$ is transferable into SCF as in (1.4) for some $(S,T) \in \mathcal{C}(\mathcal{I};\mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1(\mathcal{I};\mathbf{Gl}_n(\mathbb{R}))$. Then the *generalized transition matrix* $U(\cdot\,,\cdot)$ of system (1.1) is defined by

$$U(t,s) := T(t) \begin{bmatrix} \Phi_J(t,s) & 0 \\ 0 & 0 \end{bmatrix} T(s)^{-1}\,, \qquad t,s \in \mathcal{I}\,.$$

◇

Semi-group properties of the generalized transition matrix hold similarly to those of the transition matrix for ODEs:

Proposition 3.5 (Properties of $U(\cdot\,,\cdot)$)**.** *Let $(E,A) \in \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n}) \times \mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})$ be transferable into SCF with generalized transition matrix $U(\cdot\,,\cdot)$. Then we have, for all $t,r,s \in \mathcal{I}$,*

(i) $E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = A(t)U(t,s)$,

(ii) $\operatorname{im} U(t,s) = \mathcal{V}_{E,A}(t)$,

(iii) $U(t,r)U(r,s) = U(t,s)$,

(iv) $U(t,t)^2 = U(t,t)$,

(v) $\forall\, x \in \mathcal{V}_{E,A}(t) :\; U(t,t)x = x$.

Proof: Property (i) is proved similar to Step 1 of the proof of Proposition 3.2. The proofs of Properties (ii) and (iii) follow easily from the definition of $U(\cdot\,,\cdot)$. Property (iv) follows from (iii) and to see (v), let $x \in \mathcal{V}(t)$. Then (ii) gives $x \in \operatorname{im} U(t,t)$ and hence there exists $y \in \mathbb{R}^n$ such that $U(t,t)y = x$. Therefore,

$$U(t,t)x = U(t,t)^2 y \overset{\text{(iv)}}{=} U(t,t)y = x.$$

This completes the proof of the proposition. □

The concept of generalized transition matrix sets us in a position to derive, similar to ODEs, a vector space isomorphism between $\mathcal{V}_{E,A}(t^0)$ (this is $\mathbb{R}^n$ for ODEs) and the set of all global solutions of (1.1).

Theorem 3.6 (Vector space isomorphism). *If* $(E,A)\in\mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})\times\mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})$ *is transferable into SCF and* $t^0\in\mathcal{I}$*, then the linear map*

$$\begin{array}{rcl}\varphi:\mathcal{V}_{E,A}(t^0) & \to & \{\, x:\mathcal{I}\to\mathbb{R}^{n\times n} \mid x(\cdot) \textit{ is a global solution of } (1.1)\,\}\\ x^0 & \mapsto & U(\cdot,t^0)x^0\end{array}$$

is a vector space isomorphism.

Proof: Set

$$\mathcal{B}_{E,A}:=\left\{\, x:\mathcal{I}\to\mathbb{R}^{n\times n} \;\middle|\; x(\cdot) \text{ is a global solution of } (1.1)\,\right\}.$$

Step 1: Since $U(\cdot\,,\cdot)$ is well-defined and Proposition 3.2 gives

$$\forall\, x^0\in\mathcal{V}_{E,A}(t^0):\ \big(\mathcal{I}\ni t\mapsto U(t,t^0)x^0\big)\in\mathcal{B}_{E,A},$$

$\varphi(\cdot)$ is well-defined.
Step 2: We show that $\varphi(\cdot)$ is surjective. Let $x(\cdot)\in\mathcal{B}_{E,A}$. Then $x(t^0)\in\mathcal{V}_{E,A}(t^0)$ and from Proposition 3.2 (ii) it follows that

$$\forall\, t\in\mathcal{I}:\ x(t)=U(t,t^0)x(t^0),$$

and therefore $\varphi(x(t^0))(\cdot)=x(\cdot)$.
Step 3: We show that $\varphi(\cdot)$ is injective. Let $x^1,x^2\in\mathcal{V}_{E,A}(t^0)$ such that $\varphi(x^1)(\cdot)=\varphi(x^2)(\cdot)$. Then

$$x^1\overset{\text{Prop. 3.5 (v)}}{=}U(t^0,t^0)x^1=\varphi(x^1)(t^0)=\varphi(x^2)(t^0)=U(t^0,t^0)x^2\overset{\text{Prop. 3.5 (v)}}{=}x^2.$$

This completes the proof. □

As an immediate consequence of Theorem 3.6 we record:

Corollary 3.7 (Constant dimension of $\mathcal{V}_{E,A}(\cdot)$).
$\dim\mathcal{V}_{E,A}(\cdot)$ *is constant if* $(E,A)\in\mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})\times\mathcal{C}(\mathcal{I};\mathbb{R}^{n\times n})$ *is transferable into SCF.* ◇

Corollary 3.7 does, in general, not hold true for DAEs which are not transferable into SCF; this follows from the following example.

Example 3.8. Consider the initial value problem

$$t\dot{x}=(1-t)x,\quad x(t^0)=x^0,\qquad t\in\mathbb{R}\,, \tag{3.3}$$

for $(t^0,x^0)\in\mathbb{R}^2$. In passing, note that $t\mapsto(E(t),A(t))=(t,t-1)$ is real analytic. For $t^0\neq 0$, $x^0\in\mathbb{R}$, the unique global solution $x(\cdot)$ of (3.3) is

$$x:\mathbb{R}\to\mathbb{R},\ t\mapsto\frac{te^{-t}}{t^0e^{-t^0}}x^0.$$

For $t^0=x^0=0$ the problem (3.3) has infinitely many global solutions and every (local) solution $x:\mathcal{J}\to\mathbb{R}$ can be uniquely extended to a global solution

$$x_c:\mathbb{R}\to\mathbb{R},\ t\mapsto cte^{-t},\qquad \text{where } c=\frac{e^{\tau}}{\tau}\,x(\tau) \text{ for some } \tau\in\mathcal{J}\setminus\{0\}.$$

The solutions $x_c(\cdot)$ are the only global solutions of the initial value problem (3.3), $t^0=x^0=0$. Furthermore, any initial value problem (3.3), $t^0=0$, $x^0\neq 0$ does not have a solution. Therefore, we have

$$\mathcal{V}_{E,A}(t)=\begin{cases}\mathbb{R}, & t\neq 0\\ \{0\}, & t=0\,.\end{cases}$$

◇

We conclude this section with a variation of constants formula for inhomogeneous time-varying linear differential-algebraic initial value problems

$$E(t)\dot{x} = A(t)x + f(t)\,, \qquad x(t^0) = x^0\,, \tag{3.4}$$

where $(t^0, x^0) \in \mathbb{R} \times \mathbb{R}^n$ and $f \in \mathcal{C}(\mathcal{I}; \mathbb{R}^n)$.

Theorem 3.9 (Solutions of inhomogeneous DAEs)**.** *Suppose that the DAE* $(E, A) \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n\times n}) \times \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n\times n})$ *is transferable into SCF by some* $(S, T) \in \mathcal{C}^n(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^n(\mathcal{I}; \mathbf{Gl}_n(\mathbb{R}))$*. Then the following statements hold for* $f \in \mathcal{C}^{n_2}(\mathcal{I}; \mathbb{R}^n)$*:*

(i) *The initial value problem* (3.4) *has a solution if, and only if,*

$$x^0 + T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left(\sum_{k=0}^{n_2-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}] S(\cdot) f(\cdot) \right)\Bigg|_{t=t^0} \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}. \tag{3.5}$$

(ii) *Any solution of* (3.4) *such that* (3.5) *holds can be uniquely extended to a global solution* $x(\cdot)$*, and this solution satisfies, for the generalized transition matrix* $U(\cdot\,,\cdot)$ *of* (E, A)*,*

$$x(t) = U(t, t^0)x^0 + \int_{t^0}^t U(t, s)T(s)S(s)f(s)\,\mathrm{d}s - T(t) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_2-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}] S(t) f(t), \quad t \in \mathcal{I}. \tag{3.6}$$

Proof: First note that since E, A, S, T are n-times continuously differentiable, we have $N \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n_2\times n_2})$.

Step 1: We show that, for any $g \in \mathcal{C}^{n_2}(\mathcal{J}; \mathbb{R}^{n_2})$, $\mathcal{J} \subseteq \mathcal{I}$,

$$\forall\, t \in \mathcal{J}:\ \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2} g(t) = 0. \tag{3.7}$$

Write

$$g_0 := \dot{g}, \qquad g_{k+1} := N\,\dot{g}_k + \dot{N}\,g_k, \qquad \text{for } k = 0, \ldots, n_2 - 2\,.$$

Then $\left(N\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2} g = N\,g_{n_2-1}$, and thus, for some $\alpha_{j_0,j_1,\ldots,j_{n_2}} \in \mathbb{R}$, $(j_0, ..., j_{n_2}) \in \{0, ..., n_2 - 1\}^{n_2+1}$,

$$\forall\, t \in \mathcal{J} \ :\ \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2} g(t) = \sum_{j_0=0}^{n_2-1} \cdots \sum_{j_{n_2}=0}^{n_2-1} \alpha_{j_0,j_1,\ldots,j_{n_2}} N^{(j_0)}(t) \cdots N^{(j_{n_2-1})}(t) g^{(j_{n_2}+1)}(t)\,. \tag{3.8}$$

Since N is pointwise strictly lower triangular, the derivatives of N are also pointwise strictly lower triangular. Obviously, the product of n_2 strictly lower triangular matrices of size $n_2 \times n_2$ must be zero and therefore (3.7) follows from (3.8).

Step 2: We show that $x(\cdot)$ as in (3.6) solves (3.4) for all $t \in \mathcal{I}$.
Set

$$\begin{aligned}
I_1(t) \ &:= \ E(t)\,U(t,t)\,T(t)\,S(t)\,f(t) \overset{(1.4)}{=} S(t)^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} S(t)f(t) \\
I_2(t) \ &:= \ E(t)\,\dot{T}(t) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_2-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]\,S(t)\,f(t) \\
I_3(t) \ &:= \ E(t)\,T(t) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_2-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right) \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]\,S(t)\,f(t)\,.
\end{aligned}$$

Then

$$\begin{aligned}
E(t)\dot{x}(t) \quad &\overset{(3.6)}{=} \quad E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,t^0)x^0 + E(t)\int_{t^0}^{t}\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s)T(s)S(s)f(s)\,\mathrm{d}s + E(t)U(t,t)T(t)S(t)f(t) \\
&\quad -E(t)\dot{T}(t)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_2-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t) \\
&\quad -E(t)T(t)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_2-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t) \\
&\overset{\text{Prop. 3.5 (i)}}{=} \quad A(t)U(t,t^0)x^0 + A(t)\int_{t^0}^{t}U(t,s)T(s)S(s)f(s)\,\mathrm{d}s + I_1(t) - I_2(t) - I_3(t)\,. \qquad (3.9)
\end{aligned}$$

Substituting

$$\begin{aligned}
I_3(t) \quad &= \quad E(t)T(t)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_2-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t) \\
&\overset{(1.4)}{=} \quad S(t)^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & N(t)\end{bmatrix}\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_2-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t) \\
&= \quad S(t)^{-1}\sum_{k=0}^{n_2-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k+1}\end{bmatrix}S(t)f(t) \\
&= \quad S(t)^{-1}\left(\sum_{k=0}^{n_2-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n_2}\end{bmatrix}\right)S(t)f(t) \\
&\overset{(3.7)}{=} \quad S(t)^{-1}\sum_{k=0}^{n_2-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}S(t)f(t) - S(t)^{-1}\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}S(t)f(t)
\end{aligned}$$

in (3.9) yields

$$\begin{aligned}
E(t)\dot{x}(t) \quad &= \quad A(t)U(t,t^0)x^0 + A(t)\int_{t^0}^{t}U(t,s)T(s)S(s)f(s)\,\mathrm{d}s + S(t)^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}S(t)f(t) \\
&\quad -E(t)\dot{T}(t)\sum_{k=0}^{n_2-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}S(t)f(t) \\
&\quad -S(t)^{-1}\sum_{k=0}^{n_2-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}S(t)f(t) + S(t)^{-1}\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}S(t)f(t) \\
&= \quad A(t)U(t,t^0)x^0 + A(t)\int_{t^0}^{t}U(t,s)T(s)S(s)f(s)\,\mathrm{d}s + f(t) \\
&\quad -(E(t)\dot{T}(t) + S(t)^{-1})\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_2-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t) \\
&\overset{(1.4)}{=} \quad A(t)U(t,t^0)x^0 + A(t)\int_{t^0}^{t}U(t,s)T(s)S(s)f(s)\,\mathrm{d}s + f(t) \\
&\quad -\left(A(t)T(t) - S(t)^{-1}\begin{bmatrix}J(t) & 0\\ 0 & I_{n_2}\end{bmatrix} + S(t)^{-1}\right)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\cdot\sum_{k=0}^{n_2-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(1.4)}{=} \quad A(t)U(t,t^0)x^0 + A(t)\int_{t^0}^t U(t,s)T(s)S(s)f(s)\,\mathrm{d}s + f(t)\\
&\qquad -A(t)T(t)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_2-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(t)f(t)\\
&\overset{(3.6)}{=} \quad A(t)x(t) + f(t)\,.
\end{aligned}
$$

Step 3: We show that $x(t^0) = x^0$ for $x(\cdot)$ as in (3.6) if, and only, (3.5) holds. Set

$$
\eta := T(t^0)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\left(\sum_{k=0}^{n_2-1}\left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(\cdot)f(\cdot)\right)\Bigg|_{t=t^0}, \qquad \begin{bmatrix}\alpha\\ \beta\end{bmatrix} := T(t^0)^{-1}\left(x^0 + \eta\right) \tag{3.10}
$$

for $\alpha \in \mathbb{R}^{n_1}$, $\beta \in \mathbb{R}^{n_2}$. Then

$$
\begin{aligned}
x(t^0) &= T(t^0)\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}T(t^0)^{-1}x^0 - \eta\\
&= T(t^0)\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} - T(t^0)\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\left(\sum_{k=0}^{n_2-1}\left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(\cdot)f(\cdot)\right)\Bigg|_{t=t^0} - \eta\\
&= T(t^0)\begin{bmatrix}\alpha\\ 0\end{bmatrix} - \eta\\
&= x^0 - T(t^0)\begin{bmatrix}0\\ \beta\end{bmatrix},
\end{aligned}
$$

and hence $x(t^0) = x^0$ if, and only if, $\beta = 0$ or, equivalently, (3.5) holds.

Step 4: Let $(t^0, x^0) \in \mathcal{I} \times \mathbb{R}^n$ such that (3.4) has a solution. We show that every solution $z : \mathcal{J} \to \mathbb{R}^n$ of (3.4), $z(t^0) = x^0$ fulfills $z = x\,|_{\mathcal{J}}$ for $x(\cdot)$ as in (3.6).
Clearly, $(z - x) : \mathcal{J} \to \mathbb{R}^n$ solves $E(t)\frac{\mathrm{d}}{\mathrm{d}t}(z - x)(t) = A(t)(z - x)(t)$ for all $t \in \mathcal{J}$. Then Proposition 3.2 gives $(z - x)(t^0) \in \operatorname{im} T(t^0)\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}$, and since, by Step 2, $x^0 - x(t^0) = T(t^0)\begin{bmatrix}0\\ \beta\end{bmatrix} \in \operatorname{im} T(t^0)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}$, we conclude

$$
z(t^0) - x(t^0) \in \operatorname{im} T(t^0)\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix} \cap \operatorname{im} T(t^0)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix} = \{0\}.
$$

Therefore, a repeated application of Proposition 3.2 yields $z = x\,|_{\mathcal{J}}$. This concludes the proof. □

A consequence of Theorem 3.9 is the following corollary which treats a characterization of consistent initial values and a variation of constants analogue for pure DAEs.

Corollary 3.10 (Solutions of inhomogeneous pure DAEs)**.** *Let $N \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^{n\times n})$ be pointwise strictly lower triangular, $f \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$ and $(t^0, x^0) \in \mathcal{I} \times \mathbb{R}^n$. Then the initial value problem*

$$
N(t)\dot{x} = x + f(t), \quad x(t^0) = x^0, \tag{3.11}
$$

has a solution if, and only if,

$$
-\sum_{k=0}^{n-1}\left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k f(\cdot)\Bigg|_{t=t^0} = x^0. \tag{3.12}
$$

Any solution of (3.11) can be uniquely extended to a global solution $x(\cdot)$, and this solution satisfies

$$
x(t) = -\sum_{k=0}^{n-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k f(t), \quad t \in \mathcal{I}. \tag{3.13}
$$

Proof: Put $n_1 = 0$, $n_2 = n$, and $T = S = I$ in (3.5) and (3.6). $\square$

Remark 3.11 (Consistent initial values for inhomogeneous DAEs)**.**

(i) Note that a consequence of Corollary 3.10 is that the only initial value consistent at time $t^0 \in \mathcal{I}$ of a pure homogeneous initial value problem (3.11), i.e. $f = 0$, is $x^0 = 0$.

(ii) For η as in (3.10), condition (3.5) reads $x^0 + \eta \in \mathcal{V}_{E,A}(t^0)$. Hence the *set of initial values which are consistent at time* t^0 of (3.4) is the affine subspace

$$-\eta + \mathcal{V}_{E,A}(t^0) = -T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left(\sum_{k=0}^{n_2-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}] S(\cdot) f(\cdot) \right)\Bigg|_{t=t^0} + \mathcal{V}_{E,A}(t^0).$$

$\diamond$

4 Analytic solvability, derivative array approach and differentiation index

In this section we study the relationship of DAEs transferable into SCF to that of other subclasses of time-varying DAEs. Such concepts as analytic solvability, the derivative array approach, differentiation index and strangeness index will be investigated.

Definition 4.1 (Analytic solvability [CP83])**.** Let $(E, A) \in \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n}) \times \mathcal{C}(\mathcal{I}; \mathbb{R}^{n \times n})$. Then the DAE (3.4) is called *analytically solvable* if, and only if, we have, for all $f \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$:

(i) $\exists$ solution to (3.4),

(ii) $\forall$ solutions $y : \mathcal{J} \to \mathbb{R}^n$ of (3.4) : $\exists$ global solution $x(\cdot)$ of (3.4) with $x\,|_{\mathcal{J}} = y$,

(iii) $\forall$ global solutions $x_1(\cdot),\ x_2(\cdot)$ of (3.4) : $\left[\exists\, t^0 \in \mathcal{I} :\ x_1(t^0) \neq x_2(t^0)\right] \ \Rightarrow\ [\forall\, t \in \mathcal{I} :\ x_1(t) \neq x_2(t)]$.

Remark 4.2.

(a) Roughly speaking, system (3.4) is analytically solvable if, and only if, for any inhomogeneity $f \in \mathcal{C}^n(\mathcal{I}; \mathbb{R}^n)$ there exist solutions to (3.4) and solutions, if they exist, can be extended to all of $\mathcal{I}$ and are uniquely determined by their value at any $t^0 \in \mathcal{I}$.

(b) Conditions (i) and (ii) in Definition 4.1 do not imply (iii). This follows from Example 3.8 which shows that an initial value problem (3.4) may have infinitely many global solutions and every local solution can be uniquely extended to one of the global solutions (i.e. there do not exist further solutions with finite escape time or other singular behavior). $\diamond$

The next example – which is due to [CP83, Ex. 2] but is presented with some gaps – shows that a system (E, A) with smooth coefficients may be analytically solvable, but not necessarily transferable into SCF.

Example 4.3. We show that the system

$$E(t)\dot{x} = -x + f(t), \qquad t \in \mathcal{I} = (-\infty, 1)\,, \tag{4.1}$$

where

$$E(t) := t^3 \begin{bmatrix} \sin(t^{-1}) \\ \cos(t^{-1}) \end{bmatrix} [\cos(t^{-1}), -\sin(t^{-1})], \qquad E(0) = 0,$$

is analytically solvable and not transferable into SCF. Note that $E \in \mathcal{C}^1(\mathcal{I};\mathbb{R}^{2\times 2})$. It is easily verified that

$$E^2 \equiv 0, \quad \dot{E}(0) = 0 \quad \text{and} \quad E(t)\dot{E}(t) = -tE(t) \text{ for all } t \in \mathcal{I} \setminus \{0\}.$$

The above is essential to show that, for any $f \in \mathcal{C}^2(\mathcal{I};\mathbb{R}^2)$,

$$x : \mathcal{I} \to \mathbb{R}^2,\ t \mapsto f(t) + \frac{1}{t-1} E(t)\dot{f}(t)$$

is continuously differentiable and the unique global solution of (4.1). Furthermore, any local solution of (4.1) can be uniquely extended to $x(\cdot)$, and therefore the system (4.1) is analytically solvable.
It remains to show that (4.1) is not transferable into SCF. Seeking a contradiction, assume $(S,T) \in \mathcal{C}(\mathcal{I};\mathbb{R}^{2\times 2}) \times \mathcal{C}^1(\mathcal{I};\mathbb{R}^{2\times 2})$ transform (4.1) into SCF. Since $E(0) = 0$, equation (1.4) together with Theorem 2.1 yields that $n_1 = 0$ and therefore,

$$\forall\, t \in \mathcal{I} \ :\ S(t)E(t)T(t) = N(t) = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}$$

and

$$\begin{aligned} \forall\, t \in \mathcal{I} \setminus \{0\} \ :\ t^3 \sin(t^{-1})[\cos(t^{-1}), -\sin(t^{-1})] \begin{pmatrix} T_{12}(t) \\ T_{22}(t) \end{pmatrix} &= (1,0)E(t)T(t) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\ &= (1,0)S(t)^{-1} \left[\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 0\,. \end{aligned} \tag{4.2}$$

This gives

$$t_k^3\,(-1)\,T_{22}(t_k) \overset{(4.2)}{=} 0 \qquad \text{for} \qquad t_k := \frac{1}{2k\pi + \pi/2},\ k \in \mathbb{N},$$

and so, by continuity of $T(\cdot)$, we may conclude $T_{22}(0) = \lim_{k\to\infty} T_{22}(t_k) = 0$. Applying (4.2) again, we have

$$s_k^3\, \sin(\pi/4)\, \cos(\pi/4)\, T_{12}(s_k) \overset{(4.2)}{=} 0 \qquad \text{where} \qquad s_k := \frac{1}{2k\pi + \pi/4},\ k \in \mathbb{N},$$

and again by continuity of $T(\cdot)$, we may conclude that $T_{12}(0) = \lim_{k\to\infty} T_{12}(s_k) = 0$; this contradicts invertibility of $T(0)$. ◇

We now show that transferability into SCF and analytic solvability are equivalent for real analytic (E, A).

Theorem 4.4 ((E,A) real analytic: SCF $\triangleq$ analytic solvability)**.** *Suppose $E, A : \mathcal{I} \to \mathbb{R}^{n\times n}$ are real analytic. Then*

$$(3.4) \textit{ is analytically solvable} \quad \Longleftrightarrow \quad (1.1) \textit{ is transferable into SCF.}$$

For "$\Leftarrow$", it suffices to assume $E, A \in \mathcal{C}^n(\mathcal{I};\mathbb{R}^n)$ and $S, T \in \mathcal{C}^n(\mathcal{I};\mathbb{R}^n)$ so that (1.4) holds.
For "$\Rightarrow$", real analyticity of E and A can, in general, not be dispensed.

Proof: "$\Leftarrow$" follows immediately from Theorem 3.9. Note that it is sufficient that E, A, S, T are n-times continuously differentiable.
"$\Rightarrow$": In [CP83, Thm. 2] it is shown that

$$(E, A) \overset{S,T}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \qquad \text{for some real analytic} \quad S, T : \mathcal{I} \to \mathbf{Gl}_n(\mathbb{R}),$$

where $N : \mathcal{I} \to \mathbb{R}^{n_2 \times n_2}$ is real analytic and pointwise strictly *upper* triangular and $J : \mathcal{I} \to \mathbb{R}^{n_1 \times n_1}$ is real analytic. Therefore,

$$(E, A) \overset{\tilde{S}^{-1}S, T\tilde{T}^{-1}}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & \tilde{N} \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \qquad \text{for} \quad \tilde{S} = \tilde{T} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \end{bmatrix}$$

where $\tilde{N} : \mathcal{I} \to \mathbb{R}^{n_2 \times n_2}$ is real analytic and pointwise strictly lower triangular, and the claim is proved. Example 4.3 shows that "$\Rightarrow$" does not hold in general, if (E, A) are not real analytic. □

In the remainder of this section we compare the concept of SCF with that of the *differentiation index* and the *derivative array*. We now allow for complex-valued $E, A \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times n})$ since this is treated in the literature. To avoid technicalities, we assume that the functions involved are infinitely many times differentiable.

We first state a technical definition on matrices.

Definition 4.5 (1-fullness [KM06, Def. 3.35])**.** Let $k, \ell, n \in \mathbb{N}$ and $M \in \mathcal{C}(\mathcal{I}; \mathbb{C}^{kn \times \ell n})$. Then M is called *smoothly 1-full w.r.t. n* if, and only if,

$$\exists\, R \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_{kn}(\mathbb{C})) : \; RM = \begin{bmatrix} I_n & 0 \\ 0 & * \end{bmatrix}.$$

◇

Definition 4.6 (Derivative array [KM06, (3.28)-(3.30)] and differentiation index [KM06, Def. 3.37])**.** For $E, A \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times n})$, the *derivative array* is defined as the sequence of matrix functions $M_\ell \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{(\ell+1)n \times (\ell+1)n})$, $N_\ell \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{(\ell+1)n \times (\ell+1)n})$ given by

$$\begin{aligned} (M_\ell)_{i,j} &= \tbinom{i}{j} E^{(i-j)} - \tbinom{i}{j+1} A^{(i-j-1)}, \quad i, j = 0, \ldots, \ell, \\ (N_\ell)_{i,j} &= \begin{cases} A^{(i)} & \text{for } i = 0, \ldots, \ell, \; j = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4.3}$$

and the *differentiation index* of (E, A) is the smallest number $\nu \in \mathbb{N}_0$ (if it exists) for which M_ν is smoothly 1-full w.r.t. n and has constant rank. ◇

The notion of 1-fullness and derivative array go back to [Cam85] and [Cam87], resp.

If the differentiation index ν is well-defined for (E, A), then one may construct an underlying ODE of the given DAE (3.4) as follows, cf. [KM06, p. 97-98]: By 1-fullness of M_ν there exists $R \in \mathcal{C}(\mathcal{I}; \mathbf{Gl}_{(\nu+1)n}(\mathbb{C}))$ such that

$$RM_\nu = \begin{bmatrix} I_n & 0 \\ 0 & * \end{bmatrix}.$$

Define $z_j := x^{(j)}$, $g_j := f^{(j)}$ for $j = 0, \ldots, \nu$. Then

$$M_\nu(t)\dot{z} = N_\nu(t) z + g(t), \qquad t \in \mathcal{I},$$

and we obtain the ODE

$$\dot{x} = [I_n, \; 0]\, R(t)\, M_\nu(t)\, \dot{z} = [I_n, \; 0]\, R(t)\, N_\nu(t)\, [I_n, \; 0]^\top x + [I_n, \; 0]\, R(t)\, g(t),$$

which is the so called *underlying ordinary differential equation*. Here x is the same variable as in (3.4) and hence any solution of (3.4) is also a solution of this ODE. Therefore, solving the DAE can be reduced to solving an ODE.

Next we introduce a hypothesis of a certain finite reduction procedure. This hypothesis guarantees that the reduction procedure of the *derivative array approach* presented in [KM06, Sec. 3.2] can be carried out and no consistency condition for the inhomogeneity or free solution components are present.

Hypothesis 4.7 ([KM06, Hypothesis 3.48])**.** *There exist $\mu, a, d \in \mathbb{N}_0$ such that (M_μ, N_μ) defined in Definition 4.6 has the following properties:*

(i) $\forall\, t \in \mathcal{I}:\ \operatorname{rk} M_\mu(t) = (\mu+1)n - a$; *choose $Z_2 \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{(\mu+1)n \times a})$ with pointwise maximal rank and $Z_2^* M_\mu = 0$.*

(ii) $\forall\, t \in \mathcal{I}:\ \operatorname{rk} A_2(t) = a$, *where $A_2 := Z_2^* N_\mu [I_n, 0, \dots, 0]^*$; choose $T_2 \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times d})$, $d = n - a$, with pointwise maximal rank and $A_2 T_2 = 0$.*

(iii) $\forall\, t \in \mathcal{I}:\ \operatorname{rk} E(t) T_2(t) = d$; *choose $Z_1 \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times d})$ with pointwise maximal rank and* $\operatorname{rk} E_1 T_2 = d$, *$E_1 = Z_1^* E$.* ◇

If Hypothesis 4.7 holds true, then [KM06, p. 109] have shown that a solution x of the DAE (3.4) is also a solution of

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x + \hat{f}(t), \qquad \text{where } E_1 = Z_1^* E, \quad A_1 = Z_1^* A, \quad A_2 = Z_2^* N_\mu [I_n, 0, \dots, 0]^* \tag{4.4}$$

and $\hat{f}$ is determined by f and its derivatives. The derived system (4.4) is a so called *strangeness free* DAE (cf. [KM06, Def. 3.15 and p. 93].

The following theorem shows in particular that if (E, A) is real analytic, then transferability into SCF is equivalent to Hypothesis 4.7 and to a well-defined differentiation index.

Theorem 4.8. *Let $E, A \in \mathcal{C}^\infty(\mathcal{I}; \mathbb{C}^{n \times n})$ and consider system* (3.4)*. Then the following conditions are equivalent:*

(i) (3.4) *is analytically solvable.*

(ii) *The differentiation index ν is well-defined for (E, A).*

(iii) (E, A) *satisfies Hypothesis 4.7.*

Proof: The assumptions in [KM06, Thm. 3.39] are equivalent to analytic solvability of (3.4) (Note that [KM06, Thm. 3.39] requires in addition that the solutions depend smoothly on the inhomogeneities and the initial conditions, but this is not needed in the proof, see [Cam87, Thm. 2.1]). Then it follows from [KM06, Thm. 3.45] that (i)⇒(ii) holds true. The conclusion (ii)⇒(iii) is identical to [KM06, Thm. 3.50] and finally (iii)⇒(i) follows from [KM06, p. 111-112]. This completes the proof. ☐

We finalize this section with a remark on the strangeness index as developed in [KM06, Sec. 3.1].

Remark 4.9. The existence of a well-defined strangeness index (see [KM06, Def. 3.15]) guarantees the equivalence of (E, A) to an DAE in a certain canonical form presented in [KM06, Thm. 3.21]. It turns out that there exist systems with a well-defined strangeness index which are not transferable into SCF (see [KM06, Ex. 3.23]); there also exist systems which are transferable into SCF and have no well-defined strangeness index (see [KM06, Ex. 3.54]). ◇

5 Computing SCF

In this section we present an algorithm in "quasi-MATLAB code" for computing the transformation matrices as well as the SCF for real analytic DAEs (E, A); the algorithm also determines whether (E, A) is transferable in SCF or not. This algorithm is indicated by some comments in [CP83]; here we make it precise.

Algorithm 5.1 Function transfSCF

function $[S, T, N, J] =$ **transfSCF**(E, A)
reachedSCF := 0; % initial value for global variable
$[S_1, T_1, N_1, J_1] :=$ getSCF(E, A);
$r :=$ size(J);
$S := \begin{bmatrix} I_r & 0 \\ 0 & \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \end{bmatrix} S_1$; $T := T_1 \begin{bmatrix} I_r & 0 \\ 0 & \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \end{bmatrix}$;
$N := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} N_1 \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$; $J := J_1$;

Algorithm 5.2 Function getSCF

function $[S, T, N, J] =$ **getSCF**(E, A)
$[E_1, E_2, A_1, A_2, G, P, Q] :=$ reduce(E, A);
if reachedSCF$= 0$ **then**
  $[S_1, T_1, J_1, N_1] :=$ getSCF(E_1, A_1);
else if $E \equiv 0$ **then**
  $N_1 := 0$; $J_1 := \emptyset$; $S_1 := T_1 := I$; % we define $M := \emptyset$ if the matrix M should be absent
else
  $N_1 := \emptyset$; $J_1 := E_1^{-1} A_1$; $S_1 := E_1^{-1}$; $T_1 := I$;
end if
$r_1 :=$ size(J_1); $r_2 :=$ size(N_1); % the sizes of empty matrices are 0
$\begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{bmatrix} := S_1 E_2$ s.t. $\tilde{E}_i$ has r_i rows, $i = 1, 2$;
$\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix} := S_1 A_2$ s.t. $\tilde{A}_i$ has r_i rows, $i = 1, 2$;
$S := \begin{bmatrix} I_{r_1} & 0 & \dot{\tilde{E}}_1 + J_1 \tilde{E}_1 - \tilde{A}_1 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{r_2} & -\tilde{A}_2 G^{-1} \\ 0 & 0 & G^{-1} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & I_{\text{size}(E) - r_1 - r_2} \end{bmatrix} P$;
$T := Q \begin{bmatrix} T_1 & 0 \\ 0 & I_{\text{size}(E) - r_1 - r_2} \end{bmatrix} \begin{bmatrix} I_{r_1} & 0 & -\tilde{E}_1 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & I_{\text{size}(E) - r_1 - r_2} \end{bmatrix}$;
$N := \begin{bmatrix} N_1 & \tilde{E}_2 \\ 0 & 0 \end{bmatrix}$; $J := J_1$;

Algorithm 5.3 Function reduce

1: **function** $[E_1, E_2, A_1, A_2, G, P, Q] = \textbf{reduce}(E, A)$
2: **if** $E \equiv 0$ or $(\forall\, t \in \mathcal{I}:\ \det E(t) \neq 0)$ **then**
3: $\quad E_1 := E;\ A_1 := A;\ E_2 := A_2 := G := \emptyset;\ P := Q := I;$
4: $\quad$ reachedSCF := 1;
5: **else if** not$(\forall\, t \in \mathcal{I}:\ \det E(t) = 0)$ **then**
6: $\quad$ **print** "not transferable into SCF!" **STOP**
7: **else**
8: $\quad$ determine (minimal) $r < n := \text{size}(E)$ s.t. $\text{rk}\, E(T) \leq r < n$ for all $t \in \mathcal{I}$ and $P : \mathcal{I} \to \mathbb{R}^{n \times n}$ real analytic s.t. $PE = \begin{bmatrix} \hat{E}_1 & \hat{E}_2 \\ 0 & 0 \end{bmatrix}$, where $\hat{E}_1(t) \in \mathbb{R}^{r \times r}$;
9: $\quad \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} := PA$, where $\hat{A}_{11}(t) \in \mathbb{R}^{r \times r}$;
10: $\quad$ **if** not$(\forall\, t \in \mathcal{I}:\ \text{rk}[\hat{A}_{21}(t), \hat{A}_{22}(t)] = n - r = \max)$ **then**
11: $\quad\quad$ **print** "not transferable into SCF!" **STOP**
12: $\quad$ **else**
13: $\quad\quad$ determine $P : \mathcal{I} \to \mathbb{R}^{n \times n}$ real analytic s.t. $[\hat{A}_{21}, \hat{A}_{22}]Q = [0_{(n-r)\times r}, G]$, where $\det G(t) \neq 0$ for all $t \in \mathcal{I}$;
14: $\quad\quad [E_1, E_2] := [\hat{E}_1, \hat{E}_2]Q;$
15: $\quad\quad [A_1, A_2] := [\hat{A}_{21}, \hat{A}_{22}]Q - [\hat{E}_1, \hat{E}_2]\dot{Q};$
16: $\quad$ **end if**
17: **end if**

Proposition 5.4. *Suppose $E, A : \mathcal{I} \to \mathbb{R}^{n \times n}$ are real analytic. Then Algorithm 5.1 either terminates after finitely many steps with "not transferable into SCF!" or returns real analytic transformation matrices $S, T : \mathcal{I} \to \mathbf{Gl}_n(\mathbb{R})$, $J : \mathcal{I} \to \mathbb{R}^{n_1 \times n_1}$ and $N : \mathcal{I} \to \mathbb{R}^{n_2 \times n_2}$ such that N is pointwise strictly lower triangular and*

$$(E,\ A) \overset{S,T}{\sim} \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right). \tag{5.1}$$

Proof: We consider two cases.

Case 1: Suppose (1.1) is transferable into SCF. Then, in view of Theorem 4.4, the DAE (3.4) is analytically solvable. Therefore, the tests in lines 5 and 10 of Algorithm 5.3 will always fail for (E, A) and every reduced pair (E_1, A_1) (cf. lines 2 and 4 of Algorithm 5.2), see the proof of [CP83, Thm. 2]. Note also that (E_1, A_1) is again analytically solvable. Hence the algorithm does not stop in line 6 or 11 of Algorithm 5.3 with "not transferable into SCF!", and therefore the reduction procedure continues until the test in line 2 of Algorithm 5.3 succeeds at some point. Since the reduction procedure reduces the dimension of (E, A) by at least 1 in each step, we must arrive at this point after at most n reduction steps. Then the SCF for the pair at lowest level (absolutely reduced) is calculated in lines 6 and 8 of Algorithm 5.2; and a simple calculation shows that the SCF of a DAE at a given level is calculated in lines 10–15 of Algorithm 5.2 provided that the SCF for the reduced pair is given; see also the proof of [CP83, Thm. 2]. Feasibility of lines 8 and 13 of Algorithm 5.3 is due to [SB70, Thm. 1] and also shown in the proof of [CP83, Thm. 2]. Invertibility of G in line 13 of Algorithm 5.3 follows from

$$n - r = \text{rk}[\hat{A}_{21}(t),\ \hat{A}_{22}(t)] = \text{rk}[\hat{A}_{21}(t),\ \hat{A}_{22}(t)]Q = \text{rk}\, G.$$

So the algorithm stops and returns S, T and J, N of the SCF such that (5.1) holds. Since N constructed by Algorithms 5.2 and 5.3 is strictly upper triangular, the transformation in lines 4–6 of Algorithm 5.1

finally assures that N is strictly lower triangular.
Case 2: Suppose (1.1) is not transferable into SCF. Assume that the tests in lines 5 and 10 of Algorithm 5.3 will always fail for (E, A) and every reduced pair (E_1, A_1). Then, in view of Case 1, the algorithm stops and returns S, T and the matrices J, N of the SCF, N strictly lower triangular, such that (5.1) holds. Hence (1.1) would be transferable into SCF, a contradiction. Therefore, one of the tests must fail at some point and the algorithm stops with "not transferable into SCF!". □

Remark 5.5.

(i) In practice, it is not easy to implement Algorithm 5.1 for the whole class of real analytic functions. The main problem is to find P, Q such that the conditions in lines 8 and 13 of Algorithm 5.3 are fulfilled. However, if (E, A) has polynomial entries, then there are efficient (actually, polynomial time) algorithms which solve this problem; see [QV95, Sec. 5].

(ii) A numerically verifiable algorithm for testing analytic solvability is given in [Cam87]. Due to Theorem 4.4, this algorithm also tests transferability into SCF for real analytic (E, A). However, this algorithm does not compute the transformation matrices. ⋄

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