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In the Complement of a Dominating Set

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Preface

Many authors have written on dominating sets (for references we refer to [29, 30]). The distribution of legions in the Roman empire or the placement of queens on chessboards are usually cited as the origins of domination theory. But only a few authors have written on the complement of a dominating set, e.g. [18, 21, 41, 48]. They studied the minimum size of a dominating set whose complement contains a minimum dominating set. In this thesis we study diverse sets that are contained in the complement of a suitable dominating set. It seems that in real life a set in the complement of a dominating set does not influence the dominating set. However, in this thesis the dominating set must usually cooperate with the set in its complement, in order to reach a common goal, e.g. existence or small common size.

The set in the complement that is studied in Chapter 2 is a dominating set, too. For graphs without isolated vertices, Ore observed in 1962 (Theorem 2.1) that the complement of a minimal dominating set is also a dominating set. Hence, every graph without isolated vertices has a dominating set whose complement contains another dominating set. However, by Zelinka [77], no minimum degree condition is sufficient to guarantee a dominating set whose complement contains two or more disjoint dominating sets. In Chapter 2 the dominating set and the dominating set in its complement must cooperate, such that their common cardinality is small. We prove upper bounds for the minimum size of two disjoint dominating sets.

In Chapter 3 we study dominating sets together with total dominating sets in their complement. Henning and Southey [38] characterized all graphs that have a dominating set whose complement contains a total dominating set. We characterize graphs that have a dominating set whose complement contains a total dominating set T and a non-empty vertex set that is disjoint from T.

In Section 4.1 we consider trees and diverse sets in the complement of a suitable dominating set. In Theorem 4.2 we characterize trees with the smallest possible size of two disjoint dominating sets, i.e. again the set in the complement is a dominating set. However the set in the complement that is studied in Observation 4.3 is a minimum dominating set and, additionally, we require that both dominating sets are minimum. We exhibit a tree that does not have two disjoint minimum dominating sets even though no single vertex is in all minimum dominating sets. Both results answer questions of [32].

So far, the dominating sets must cooperate with the set in its complement. But this is different in Theorem 4.5. Here, the set in the complement is an independent dominating set, because we prove that, if T is a tree of order at least 2 and D is a minimum dominating set of T containing at most one leaf of T, then the complement of D contains an independent dominating set. This proves a conjecture of Johnson, Prier, and Walsh [41].

In Subsection 4.1.1 we characterize trees that have a minimum dominating set whose complement contains a maximum independent set.

In Section 4.2 we prove lower bounds of the maximum size of two disjoint independent sets for connected graphs with small average degree. These results imply lower bounds for the independence number for connected graphs with small average degree. In order to motivate the results of this section, the title of this thesis would be better

"In the Complement of an Independent Set"

This title also applies to Theorem 4.5 and Subsection 4.1.1. However, it applies to fewer results of the thesis than the correct title.

Neither the correct title nor the title just mentioned do not apply to the topic of Section 4.3. Probably, no similar title applies to the topic of Section 4.3. Therefore, the question arises, why Section 4.3 is in this thesis. In my opinion, article [59] is the best article that has been written with my assistance during my time as a Ph.D. student. We prove that for connected graphs of order n, the spanning tree congestion is bounded by $n^{\frac{3}{2}}$. The idea of the proof is easy. If a graph has a few edges, then any spanning tree satisfies the bound. Otherwise, if a graph has many edges, then a spanning tree that is similar to a star satisfies the bound. In order to combine both methods we merely need a criterion to distinguish both cases ...

The last chapter of this thesis is devoted to the complexity of decision problems that are related to the topics of the other chapters. Some of these results motivate us to pay attention to bounds for the graph parameter that are studied in this thesis.

Acknowledgment

There are many people I'd like to thank for diverse support during my time as a Ph.D. student, i.e. friends, colleagues or relatives (these sets are not disjoint in my case). But there are too many to list them by name. My special gratitude goes to my parents, Hildegard and Günter Löwenstein, not only for financial support during my education, and to my supervisor, Prof. Dr. Dieter Rautenbach. He has induced me into mathematical research.

Ilmenau, April 2010 Christian Löwenstein

Contents

Chapter 1

Introduction

Before we proceed to our results, we introduce some definitions and notation. While we summarize some basic terminology in Section 1.1, we introduce non-standard terminology used in this thesis in Section 1.2. Furthermore, Section 1.2 contains an overview of the thesis.

1.1 General Terminology

A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a finite set whose elements are called *vertices* of G, $E(G)$ is a subset of $\{\{u,v\} \mid u,v \in V(G), u \neq v\}$ and the elements $uv = \{u,v\}$ of $E(G)$ are called *edges* of G. We always denote an edge by uv instead of $\{u,v\}$, i.e. $\{u,v\}$ is not an edge, but a set of two vertices. The *order* $n(G)$ of G is the cardinality of $V(G)$ and the *size* $m(G)$ of G is the cardinality of $E(G)$. For an edge $e = uv \in E(G)$, we say that e is *incident* to u and v. In this case u is *adjacent* to v and u is *neighbor* of v. For a vertex $v \in V(G)$, the set of neighbors of v is the *neighborhood* $N_G(v)$ of v in G and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v in G. For a vertex set $U \subseteq V(G)$, the neighborhood of U is $N_G(U) = \bigcup_{v \in U} N_G(v)$ and the closed neighborhood of U is $N_G[U] = N_G(U) \cup U$.

The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the order of $N_G(v)$. A vertex $v \in V(G)$ is called *isolated* if $d_G(v) = 0$. The *minimum degree* $\delta(G)$ (*maximum degree* $\Delta(G)$) of a graph is the minimum (maximum) degree of a vertex of the graph. A graph with maximum degree at most 3 is called *subcubic* and a subcubic graph with minimum degree 3 is called *cubic*. The *average degree* of G is $\bar{d}(G) = \frac{2m(G)}{n(G)}$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called *induced* $H = G[V(H)]$ if $E(H) = \{uv \in E(G) \mid u,v \in V(H)\}$. A subgraph H of G is called *spanning* if $V(H) = V(G)$. For $\{u_1, \ldots, u_k\} \subseteq V(H)$, we define $H - u_1 = G[V(H) \setminus \{u_1\}]$ and $H - \{u_1, \ldots, u_k\} = G[V(H) \setminus \{u_1, \ldots, u_k\}]$. For $\{e_1, \ldots, e_k\} \subseteq E(H)$, we define $H - e_1 = (V(H), E(H) \setminus \{e_1\})$ and $H - \{e_1, \ldots, e_k\} = (V(H), E(H) \setminus \{e_1, \ldots, e_k\})$. We define $H + e = (V(H), E(H) \cup \{e\})$ for an suitable edge $e \notin E(H)$. For two graphs G and H, we define their *union* $G \cup H$ as $(V(G) \cup V(H), E(G) \cup E(H))$.

A *path* of length $l \geq 0$ in a graph G is a sequence $P = u_0u_1u_2 \dots u_l$ of $l+1$ distinct vertices of G such that $u_{i-1}u_i \in E(G)$ for $1 \leq i \leq l$. P is also called an u_0-u_l-path. The *endvertices* of P are u_0 and u_l and the *internal vertices* of P are $u_1, \dots, u_{l-1}$. A path $P = u_0u_1u_2 \dots u_l$ in G naturally corresponds to a subgraph P of G with vertex set $\{u_0, u_1, \dots, u_l\}$ and edge set $\{u_0u_1, u_1u_2, \dots, u_{l-1}u_l\}$. A *cycle* of length $l \geq 3$ in a graph G is a sequence $C = u_1u_2 \dots u_{l-1}u_lu_1$ of vertices of G such that $u_i \neq u_j$ for $i \neq j$, $u_{i-1}u_i \in E(G)$ for $2 \leq i \leq l$ and $u_1u_l \in E(G)$. An *n-cycle* (*odd cycle, even cycle*) is a cycle of length n (odd length, even length) and a *triangle* is a cycle of length 3. A cycle $C = u_1u_2 \dots u_{l-1}u_lu_1$ in G naturally corresponds to a subgraph C of G with vertex set $\{u_1, u_2, \dots, u_l\}$ and edge set $\{u_1u_2, u_2u_3, \dots, u_{l-1}u_l, u_lu_1\}$. The *girth* $g(G)$ (*odd girth* $g_{odd}(G)$) of a graph G is the minimum length of a cycle (of a odd cycle) in G. A cycle C (path P) is called *hamiltonian* in a graph G if $n(C) = n(G)$ ($n(P) = n(G)$) and a graph is called *hamiltonian* if it contains a hamiltonian cycle.

A graph G is *connected* if there is an u-v-path for all $u, v \in V(G)$. A *component* of G is a maximal connected subgraph of G. We call an edge $e \in E(G)$ a *bridge* of G if $G - e$ has more components than G. A *tree* T is a connected graph of size $n(T) - 1$. A vertex of degree 1 in a tree is called *leaf*.

A graph is *complete* if each pair of vertices is adjacent. We denote a complete graph on n vertices by K_n. A graph G is *bipartite* with the *partite* sets A, B if $A \cup B = V(G)$, $A \cap B = \emptyset$, and $m(G[A]) = m(G[B]) = 0$. A bipartite graph G with partite sets A, B is *complete bipartite* if $n(G) = |A| \cdot |B|$. We denote such a graph by $K_{n,m}$ where $n = |A|$ and $m = |B|$. A *star* is a complete bipartite graph such that one partite set is of order 1. A connected graph on n vertices with minimum degree and maximum degree exactly 2 is denoted by C_n and for an edge $e \in E(C_n)$, the graph $C_n - e$ is denoted by P_n.

A set of edges $M \subseteq E(G)$ of a graph G is called a *matching* (*perfect matching*) if every vertex of $V(G)$ is incident to at most (exactly) one edge of M.

Let G be a graph. For $D, U \subseteq V(G)$, we say that D *dominates* U if $U \subseteq N_G[D]$. D is a *dominating set* of G if D dominates $V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. For $T, U \subseteq V(G)$, we say that T *totally dominates* U if $U \subseteq N_G(T)$. The set T is a *total dominating set* of G if T totally dominates $V(G)$. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G. A vertex set $I \subseteq V(G)$ is called an *independent set* if $m(G[I]) = 0$. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set of G.

1.2 Special Terminology and Overview of the Thesis

Let G be a graph without isolated vertices. A pair (D_1, D_2) of disjoint sets of vertices $D_1, D_2 \subseteq V(G)$ is said to dominate a vertex set $X \subseteq V(G)$, if both of D_1 and D_2 dominate X. (D_1, D_2) is a *dominating pair* of G if (D_1, D_2) dominates $V(G)$. Hedetniemi, Hedetniemi, Laskar, Markus, and Slater [32] defined the *disjoint domination number* $\gamma\gamma(G)$ of a graph G as follows.

$$\gamma\gamma(G) = \min\{|D_1| + |D_2|\colon (D_1, D_2) \text{ is a dominating pair of } G\}.$$

In Chapter 2 we study upper bounds for the disjoint domination number of graphs, which are

- of minimum degree at least 2,
- of large minimum degree,
- or cubic.

Additionally, in Section 4.1 we answer two problems which were posed in [32] related to disjoint dominating sets.

A *DT-pair* of G is a pair (D, T) of disjoint sets of vertices $D, T \subseteq V(G)$ such that D is a dominating set and T is a total dominating set of G. Using the notation of [32], for a graph G that has a DT-pair, we define $\gamma\gamma_t(G)$ as follows.

$$\gamma\gamma_t(G) = \min\{|D| + |T|\colon (D, T) \text{ is a DT-pair of } G\}.$$

In Chapter 3 we characterize graphs G with $\gamma\gamma_t(G) = n(G)$ which are

- of minimum degree at least 2 and C_5-free
- or of minimum degree at least 3.

Additionally, in Section 5.4 we show that it is NP-complete to decide for a given graph G and a given integer k, whether $\gamma\gamma_t(G) \leq k$.

An (α, γ)-*pair* of G is a pair (I, D) of disjoint sets of vertices $I, D \subseteq V(G)$ such that I is a maximum independent set and D is a minimum dominating set of G. In Section 4.1 we give a constructive characterization of trees with an (α, γ)-pair. Furthermore, we prove that if T is a tree of order at least 2 and D is a minimum dominating set of T containing at most one leaf of T, then there is an independent dominating set I of T which is disjoint from D, which was conjectured in. In Section 5.4 we show that the decision problem whether a given graph has an (α, γ)-pair is NP-hard.

In analogy to $\gamma\gamma(G)$ we define

$$\begin{aligned} \gamma i(G) &= \min\{|D| + |I|\colon (D, I) \text{ is a dominating pair of } G \text{ and } I \text{ is independent}\}, \\ ii(G) &= \min\{|I_1| + |I_2|\colon I_1 \text{ and } I_2 \text{ are disjoint independent dominating sets of } G\}, \\ \alpha\alpha(G) &= \max\{|I_1| + |I_2|\colon I_1 \text{ and } I_2 \text{ are disjoint independent sets of } G\}. \end{aligned}$$

In Sections 5.1, 5.2, and 5.5 we consider decision problems related to $\gamma\gamma(G), \gamma i(G)$, $ii(G)$, and $\alpha\alpha(G)$. In Section 4.2 we prove lower bounds on $\alpha(G)$ and $\alpha\alpha(G)$ in connected graphs with specified odd girth and small average degree.

Let G be a connected graph and let T be a spanning tree of G. For an edge $e \in E(T)$, we consider the *congestion* $c(e,(G,T))$ *of* e *with respect to* (G,T) as the number of edges $uv \in E(G)$ for which e lies on the path in T between u and v. The maximum over $e \in E(T)$ of the congestion of e with respect to (G,T) is denoted by $c(G,T)$. The *tree congestion* of G is defined by

$$t(G) = \min\{c(G,T) \mid T \text{ is a tree}\},$$

and the *spanning tree congestion* of G is defined by

$$s(G) = \min\{c(G,T) \mid T \text{ is a spanning tree of } G\}.$$

In Section 4.3 we show an upper bound for $s(G)$ in terms of $n(G)$ and we show that $\frac{s(G)}{t(G)}$ is linearly bounded in terms of $n(G)$. Furthermore, in Section 5.6 we show that it is NP-complete to decide for a given graph G and a given integer k, whether $s(G) \leq k$.

Chapter 2

Upper Bounds on $\gamma\gamma(G)$

Domination in graphs is a fundamental and well-studied topic. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [29, 30]. In view of its computational hardness (see e.g. [22]), upper bounds on the domination number have been studied and natural arguments for such bounds are the graph's order and minimum degree [2, 3, 9, 62, 64, 67, 71].

Ore observed the following.

Observation 2.1 (Ore [64]) *If G is a graph without isolated vertices then the complement of a minimal dominating set of G is also a dominating set of G.*

This implies that every such graph has two disjoint dominating sets and hence,

$$\gamma(G) \leq \frac{1}{2}|V(G)|.$$

This inequality is best-possible and all extremal graphs were characterized by Payan and Xuong [68]. Randerath and Volkmann [69] and Baogen, Cockayne, Haynes, Hedetniemi, and Shangchao [5] characterized all graphs G with $\gamma(G) = \left\lfloor \frac{1}{2}|V(G)| \right\rfloor$. For graphs G with $\delta(G) \geq 2$, Blank [9] and McCuaig and Shepherd [62] proved that

$$\gamma(G) \leq \frac{2}{5}|V(G)|$$

unless G is one of the seven graphs $H_1, H_2, \ldots, H_7$ in Figure 2.3. Also this inequality is best-possible and McCuaig and Shepherd [62] characterized all edge-minimal extremal graphs. While these two bounds are best-possible, Reed conjectured that his upper bound [71]

$$\gamma(G) \leq \frac{3}{8}|V(G)|$$

for graphs G with $\delta(G) \geq 3$ can be improved to $\gamma(G) \leq \lceil n(G)/3 \rceil$ for cubic graphs G. Kostochka and Stodolsky [44] described counterexamples to Reed's conjecture but improved [45] his upper bound to

$$\gamma(G) \leq \frac{4}{11}|V(G)|$$

for connected cubic graphs G with $n(G) > 8$. While Reed's conjecture is false in general it was verified for cubic graphs of large girth [42, 45, 47, 55, 70].

Several authors studied so-called *domatic partitions*, which are partitions of the vertex set of a graph into dominating sets. The maximum number of disjoint dominating sets into which a graph can be partitioned is known as the *domatic number* [15] (cf. Zelinka's contribution to [30]). Furthermore, graphs G having two disjoint minimum dominating sets [4] and also the minimum intersection of pairs of minimum dominating sets [12, 19, 25] were considered.

Recently several authors have studied pairs of disjoint dominating sets. Kulli and Sigarkanti [48] introduced the *inverse domination number* $\gamma^{-1}(G)$ of a graph G as the minimum cardinality of a dominating set whose complement contains a minimum dominating set of G. A false proof for the inequality $\gamma^{-1}(G) \leq \alpha(G)$ that appeared in [48] motivated several authors [18, 21] to study this parameter.

Motivated by the inverse domination number, Hedetniemi, Hedetniemi, Laskar, Markus, and Slater [32] defined and studied the disjoint domination number $\gamma\gamma(G)$ of a graph G. By definition, $\gamma\gamma(G) \geq 2\gamma(G)$ for any graph G but, as shown by a star, no upper bound of the form $\gamma\gamma(G) \leq c \cdot \gamma(G)$, where c is a constant, exists. Observation 2.1 implies,

$$\gamma\gamma(G) \leq |V(G)|$$

for every graph G without isolated vertices and Hedetniemi, Hedetniemi, Laskar, Markus, and Slater [32] characterized all extremal graphs for this bound. These are C_4 and all graphs with satisfy the property that each vertex of the graph contains at least one vertex of degree 1 in its closed neighborhood. They also proved that it is NP-hard to determine $\gamma\gamma(G)$ even for chordal graphs G. In Chapter 5 we show that the calculation of $\gamma\gamma(G)$ is NP-hard even when restricted to bipartite graphs G, which answers a question posted in [32].

It is a natural question to ask why to devote special attention to the case of two disjoint dominating sets rather than k disjoint dominating sets for general k. The reason is that, by Observation 2.1, the trivial necessary minimum degree condition is also sufficient for the existence of two disjoint dominating sets. For all fixed $k \geq 3$, it is NP-complete [22] to decide the existence of k disjoint dominating sets and no minimum degree condition is sufficient for the existence of three disjoint dominating sets. As a simple example attributed to Zelinka [77] consider a bipartite graph G with one partite set A containing $3\delta - 2$ vertices and a second partite set B containing $\binom{3\delta-2}{\delta}$ vertices each of which is adjacent to a different set of δ vertices from A. Clearly, this graph has minimum degree δ. If $D_1 \cup D_2 \cup D_3$ is a partition of $A \cup B$ such that $|D_1 \cap A| \geq |D_2 \cap A| \geq |D_3 \cap A|$, then $|D_1 \cap A| \geq \delta$. Hence, there is a vertex $v \in B$ such that $N_G(v) \subseteq D_1$ and so G does not contain three disjoint dominating sets.

Imposing lower as well as upper bounds on the vertex degrees implies the existence of many disjoint dominating sets. Feige, Halldórsson, Kortsarz, and Srinivasan [20] (cf. also [16]) proved that every graph G can be partitioned into

$$(1 - o(1)) \frac{\delta(G) + 1}{\ln \Delta(G)}$$

dominating sets where the $o(1)$-term tends to 0 as $\Delta(G)$ tends to infinity. Considering the smallest k of these sets implies that every graph G has k disjoint dominating sets whose total cardinality is

$$(1 + o(1))\, \frac{k \ln \Delta(G)}{\delta(G) + 1} |V|. \tag{2.1}$$

In Section 2.1 we prove an upper bound on the disjoint domination number of graphs of minimum degree at least 2 together with the characterization of the seven exceptional graphs (Theorem 2.2). This result is inspired by McCuaig and Shepherd's [62] work and their seven exceptional graphs $H_1, H_2, \ldots, H_7$ play an important role. We close that section with a conjecture, which would improve Theorem 2.2. In Section 2.2 we present an asymptotically best-possible upper bound on the disjoint domination number of graphs of minimum degree at least 5 (Theorem 2.9). This result improves (2.1) for $k = 2$ and relies on a beautiful probabilistic argument used by Alon and Spencer [2] to prove the asymptotically best-possible bound

$$\gamma(G) \leq \frac{1 + \ln(\delta(G) + 1)}{\delta(G) + 1} |V(G)|. \tag{2.2}$$

In the last section of this chapter (Section 2.3) we prove an upper bound on the disjoint domination number of cubic graphs (Theorem 2.10). Our approach relies on Reed's [71] and Kostochka and Stodolsky's work [45]. Again, we close that section with a conjecture, which would improve Theorem 2.10. The results of Section 2.1 and Section 2.2 are based on [56] and the results of Section 2.3 are based on [57].

2.1 Graphs of Minimum Degree at Least 2

As our main result in this section we prove the following.

Theorem 2.2 *If G is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G *connected, and*

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) < \frac{6}{7}|V(G)|$.

Before we start with the proof we need some more terminology. For a graph G and some $i \in \mathbb{N}$, let $V_i(G) = \{u \in V(G) \mid d_G(u) = i\}$ and $V_{\geq i}(G) = \{u \in V(G) \mid d_G(u) \geq i\}$. A *multigraph* G is a triple $(V(G), E(G), \Psi)$, where $V(G)$ and $E(G)$ are finite sets and $\psi : E(G) \to \{X \subseteq V(G) : |X| = 2\}$. A *directed multigraph* G is a triple $(V(G), E(G), \Psi)$, where $V(G)$ and $E(G)$ are finite sets and $\psi : E(G) \to \{(v, w) \in V(G) \times V(G) : v \neq w\}$.

Unless we explicitly say so, we use the same terminology for multigraphs and directed multigraphs as for graphs.

If nothing is defined different, we use the same terminology in connection with multigraphs and directed multigraphs as in connection with graphs. We first prove the desired bound for graphs that arise by suitably subdividing the edges of some multigraph.

Theorem 2.3 *Let G^* be a multigraph that may contain multiple edges but no loops such that every vertex is incident with at least 3 edges. Let $E_1^* \cup E_2^* \cup E_3^*$ be a partition of the edge set $E(G^*)$ of G^*.*

If the graph G arises from G^* by subdividing every edge in E_i^* exactly i times for $1 \leq i \leq 3$, then G has a dominating pair (D_1, D_2) such that $V_{\geq 3}(G) = V(G^*) \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V(G)|$.

A path of length $i+1$ whose endvertices are of degree at least 3 and whose i internal vertices are all of degree 2 is called an *open i-ear*. A cycle of length $i+1$ that contains i vertices of degree 2 and one vertex of degree at least 3 is called a *closed i-ear*.

Proof: Let G^* and G be as in the statement of the result. We will prove the desired statement by explicitly describing the construction of a suitable dominating pair (D_1, D_2) for G. Initially, let $(D_1, D_2) = (\emptyset, \emptyset)$.

Note that the edges in E_i^* correspond exactly to the open i-ears of G. Let $p_i = |E_i^*|$ for $1 \leq i \leq 3$. Furthermore, let $n_i = |V_i(G)|$ and $n_{\geq i} = |V_{\geq i}(G)|$ for $i \in \mathbb{N}$. Clearly, counting the vertices of G and the edges of G^* we obtain

$$|V(G)| = n_{\geq 3} + p_1 + 2p_2 + 3p_3 \text{ and} \tag{2.3}$$
$$|E(G^*)| = p_1 + p_2 + p_3 \geq \frac{3}{2}n_3 + 2n_{\geq 4}. \tag{2.4}$$

As a first step, we add all vertices in $V_{\geq 3}(G) = V(G^*)$ to either D_1 or D_2.

If $u, v \in V_{\geq 3}(G)$ are the endvertices of an open i-ear P, then we call P *good*, if either $i \in \{1, 3\}$ and u and v do not both lie in one of the two sets D_1 and D_2, or $i = 2$ and u and v both lie in one of the two sets D_1 and D_2, i.e.

$$\begin{array}{rcl} \text{either} & i \in \{1,3\} & \text{and } |\{u,v\} \cap D_1| = |\{u,v\} \cap D_2| = 1, \\ \text{or} & i = 2 & \text{and } \{|\{u,v\} \cap D_1|, |\{u,v\} \cap D_2|\} = \{0, 2\}. \end{array}$$

We call open i-ears *bad*, if they are not good and denote the number of bad open i-ears by b_i for $1 \leq i \leq 3$.

We assume that the vertices in $V_{\geq 3}(G) = V(G^*)$ are added to either D_1 or D_2 in such a way that the total number of bad open i-ears is as small as possible, i.e.

$$(b_1 + b_2 + b_3) \quad \to \quad \min. \tag{2.5}$$

Next, for every good open i-ear, we add $i-1$ of the internal vertices to either D_1 or D_2 and for every bad open i-ear, we add all i internal vertices to either D_1 or D_2 in such a way that (D_1, D_2) dominates all vertices of degree 2 and as many vertices of degree at least 3 as possible, i.e. if $\dot{V}_i(G)$ and $\dot{V}_{\geq i}(G)$ denote the sets of vertices in $V_i(G)$ and $V_{\geq i}(G)$ that are not — yet — dominated by (D_1, D_2), $\dot{n}_i = |\dot{V}_i(G)|$, and $\dot{n}_{\geq i} = |\dot{V}_{\geq i}(G)|$, then

$$\dot{n}_{\geq 3} \quad \rightarrow \quad \min. \tag{2.6}$$

Clearly, we may assume that the internal vertices of all open i-ears are added to either D_1 or D_2 as indicated in Figure 2.1 where all vertices within squares belong to one of the two sets D_1 or D_2 and all vertices within cycles belong to the other set.

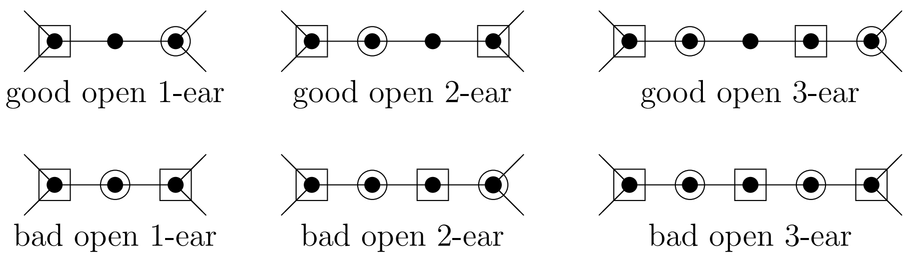


Figure 2.1

Let $\ddot{V}_j(G)$ and $\ddot{V}_{\geq j}(G)$ denote the set of vertices in $V_j(G)$ and $V_{\geq j}(G)$ that do not belong to a bad open i-ear or a good open 3-ear. Let $\ddot{n}_j = |\ddot{V}_j(G)|$ and $\ddot{n}_{\geq j} = |\ddot{V}_{\geq j}(G)|$. Since all vertices in $V_{\geq 3}(G)$ that lie on a bad open i-ear or a good open 3-ear are already dominated by (D_1, D_2), we have

$$\dot{n}_3 \leq n_3 \tag{2.7}$$

and

$$\dot{n}_{\geq 3} \leq \ddot{n}_{\geq 3}. \tag{2.8}$$

Claim 1

$$(b_1 + b_2 + b_3) \leq \frac{1}{2}(p_1 + p_2 + p_3) - \frac{1}{4}n_3 - \ddot{n}_{\geq 4} - \frac{1}{2}\ddot{n}_3 \tag{2.9}$$

Proof of Claim 1: It follows by the handshaking lemma that

$$2(p_1 + p_2 + p_3) = \sum_{i \geq 3} i n_i.$$

Furthermore, by (2.5), every vertex in $V_{\geq 3}(G)$ belongs to at least as many good open i-ears as bad open i-ears. Therefore, another application of the handshaking lemma yields

$$\begin{aligned} 2\left(\sum_{i=1}^{3} p_i - \sum_{i=1}^{3} b_i\right) &\geq \sum_{i \geq 3} i\ddot{n}_i + \sum_{i \geq 3} \left\lceil \frac{i}{2} \right\rceil (n_i - \ddot{n}_i) \\ &= \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i + \sum_{i \geq 3} \left\lceil \frac{i}{2} \right\rceil n_i. \end{aligned}$$

Combining these two observations, we obtain

$$\begin{aligned} 2(b_1+b_2+b_3) &\leq 2(p_1+p_2+p_3) - \sum_{i\geq 3}\left\lfloor\frac{i}{2}\right\rfloor \ddot{n}_i - \sum_{i\geq 3}\left\lceil\frac{i}{2}\right\rceil n_i \\ &= (p_1+p_2+p_3) + \sum_{i\geq 3}\frac{i}{2}n_i - \sum_{i\geq 3}\left\lfloor\frac{i}{2}\right\rfloor \ddot{n}_i - \sum_{i\geq 3}\left\lceil\frac{i}{2}\right\rceil n_i \\ &\leq (p_1+p_2+p_3) - \frac{1}{2}n_3 - 2\ddot{n}_{\geq 4} - \ddot{n}_3, \end{aligned}$$

which is equivalent to the statement of the claim. $\square$

We consider a directed graph $\vec{G}^*$ with vertex set $V(\vec{G}^*) = V_{\geq 3}(G)$ that contains a directed edge (u,v) from u to v for every good open 2-ear $P = uxyv$ in G such that $y \in D_1 \cup D_2$, i.e. a directed edge "(u,v)" indicates that v is already properly dominated by the vertices on P. (Note that $\vec{G}^*$ can contain multiple directed edges.)

For a vertex $u \in \dot{V}_{\geq 3}(G)$, let T_u denote the set of vertices $v \in V_{\geq 3}(G)$ such that $\vec{G}^*$ contains a directed path from u to v.

Claim 2 *If $v \in T_u$ for some $u \in \dot{V}_{\geq 3}(G)$, then v is not contained in a bad open i-ear or a good open 3-ear in G and v is not the endvertex of two directed edges in $\vec{G}^*$.*

Proof of Claim 2: For contradiction, we assume that vertices u and v as stated in the claim exist.

Let $P = u_0u_1\dots u_l$ be a directed path in $\vec{G}^*$ from $u = u_0$ to $v = u_l$. By definition, every directed edge (u_{r-1}, u_r) for some $1 \leq r \leq l$, corresponds to a good open 2-ear $P_r = u_{r-1}x_ry_ru_r$ with $y_r \in D_s$ for some fixed $s \in \{1,2\}$. If we replace the vertex y_r in D_s with x_r for $1 \leq r \leq l$, then, by the assumption, all vertices that were dominated by (D_1, D_2) — in particular v — are still dominated by the new pair and the total number of bad open i-ear remains unchanged. Since u is dominated by the new pair, $\dot{n}_{\geq 3}$ is reduced by 1, which is a contradiction to (2.6). $\square$

By Claim 2, the sets T_u for $u \in \dot{V}_{\geq 3}(G)$ induce disjoint rooted tree $\vec{T}_u$ within $\vec{G}^*$ with root u. Furthermore, again by Claim 2, every leaf of $\vec{T}_u$ that is different from u is the endvertex of at least two good open 1-ears. Clearly, the sum of the number of good open 1-ears that contain u and the number of leaves in $\vec{T}_u$ is at least $d_G(u) \geq 3$. Therefore, we can associate 3 good open 1-ears to every vertex in $\dot{V}_{\geq 3}(G)$ such that every good open 1-ear is associated at most twice to vertices in $\dot{V}_{\geq 3}(G)$. By double counting, we obtain

$$\dot{n}_{\geq 3} \leq \frac{2}{3}(p_1 - b_1) \leq \frac{2}{3}p_1. \tag{2.10}$$

We now turn (D_1, D_2) into a dominating pair of G by adding at most $\dot{n}_{\geq 3}$ vertices to the two sets and possibly moving some vertices from D_s to D_{3-s}, if all their neighbors belong to D_s.

We are ready to estimate the cardinality of (D_1, D_2).

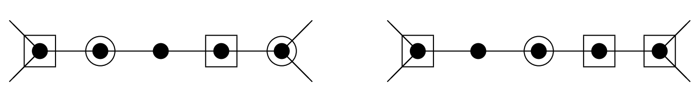
$$\begin{aligned}
|D_1 \cup D_2| &\leq n_{\geq 3} + b_1 + p_2 + b_2 + 2p_3 + b_3 + \dot{n}_{\geq 3} \\
&\overset{(2.9)}{\leq} n_{\geq 3} + \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 - \frac{1}{4}n_3 - \ddot{n}_{\geq 4} - \frac{1}{2}\ddot{n}_3 + \dot{n}_{\geq 3} \\
&= \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 + (\dot{n}_{\geq 4} - \ddot{n}_{\geq 4}) + \frac{1}{2}\left(\dot{n}_3 - \ddot{n}_3\right) \\
&\overset{(2.8)}{\leq} \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 \\
&\overset{(2.10)}{\leq} \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 + \left(\frac{1}{4}p_1 - \frac{3}{8}\dot{n}_3\right) \\
&\overset{(2.7)}{\leq} \frac{3}{4}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{7}{8}n_3 + n_{\geq 4} \\
&\overset{(2.4)}{\leq} \frac{3}{4}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{7}{8}n_3 + n_{\geq 4} + \left(\frac{1}{14}\left(p_1 + p_2 + p_3\right) - \frac{3}{28}n_3 - \frac{1}{7}n_{\geq 4}\right) \\
&= \frac{23}{28}p_1 + 2 \cdot \frac{11}{14}p_2 + 3 \cdot \frac{6}{7}p_3 + \frac{43}{56}n_3 + \frac{6}{7}n_{\geq 4} \\
&\overset{(2.3)}{\leq} \frac{6}{7}|V|,
\end{aligned}$$

where equality is only possible if $p_1 = p_2 = n_3 = 0$, i.e. every vertex in G belongs to an open 3-ear and no vertex has degree exactly 3.

In this case

$$|V(G)| = 3p_3 + n_{\geq 4}, \tag{2.11}$$
$$p_3 \geq 2n_{\geq 4} \tag{2.12}$$

and we construct a dominating pair (D_1, D_2) for G in the following way: First, we add all vertices in $V_{\geq 4}(G)$ to either D_1 or D_2 in such a way that the number of bad open 3-ears is minimum as in (2.5). Clearly, every vertex in $V_{\geq 4}(G)$ belongs to a good open 3-ear. Therefore, we can turn (D_1, D_2) to a dominating pair of G by adding exactly two internal vertices of every open 3-ear to either D_1 or D_2 as indicated in Figure 2.2.



good open 3-ear bad open 3-ear

Figure 2.2

Now

$$\begin{aligned}
|D_1 \cup D_2| &\leq n_{\geq 4} + 2p_3 \\
&\overset{(2.12)}{\leq} n_{\geq 4} + 2p_3 + \left(\frac{1}{7}p_3 - \frac{2}{7}n_{\geq 4}\right) \\
&= \frac{5}{7}n_{\geq 3} + \frac{15}{7}p_3 \\
&\overset{(2.11)}{\leq} \frac{5}{7}|V(G)| \\
&< \frac{6}{7}|V(G)|,
\end{aligned}$$

and the proof is complete. $\square$

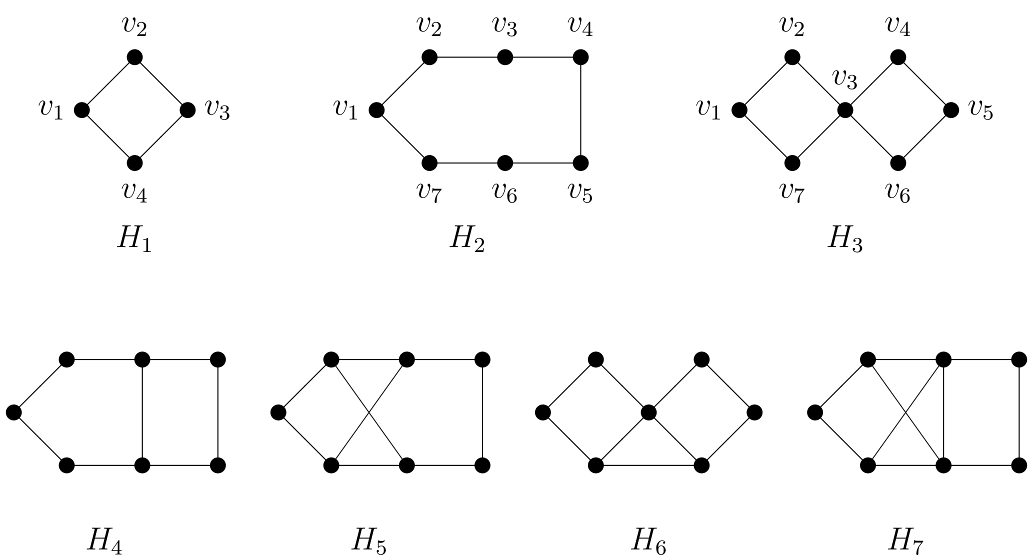


Figure 2.3: The seven exceptional graphs for Theorem 2.2.

Lemma 2.4 (a) $\gamma\gamma(H_1) = 4$, $\gamma\gamma(H_2) = \ldots = \gamma\gamma(H_7) = 6$.

(b) *If $G \in \{H_1, H_2, H_3\}$ and $v \in V(G)$, then G has a minimum dominating pair (D_1, D_2) such that $v \in D_1$.*

(c) *If $G \in \{H_1, H_2, H_3\}$ and $v \in V(G)$, then there is a pair (D_1, D_2) of disjoint sets of vertices of G such that $|D_1 \cup D_2| = \gamma\gamma(G) - 1$, $v \in D_1$, D_1 is a dominating set, and $V(G) \setminus \{v\} \subseteq N_G[D_2]$.*

r	s	D_1	D_2
3	1	$\{v_2, v_4\}$	$\{v_3\}$
3	3	$\{v_3, v_6\}$	$\{v_2, v_5\}$
3	4	$\{v_2, v_4, v_7\}$	$\{v_3, v_6\}$
3	5	$\{v_3, v_5, v_8\}$	$\{v_2, v_4, v_7\}$
4	1	$\{v_2, v_5\}$	$\{v_3, v_4\}$
4	3	$\{v_2, v_4, v_7\}$	$\{v_1, v_3, v_6\}$
4	4	$\{v_2, v_5, v_8\}$	$\{v_3, v_4, v_7\}$
4	5	$\{v_3, v_4, v_6, v_9\}$	$\{v_2, v_5, v_8\}$
5	1	$\{v_2, v_4, v_6\}$	$\{v_3, v_5\}$
5	3	$\{v_3, v_5, v_8\}$	$\{v_2, v_4, v_7\}$
5	4	$\{v_2, v_4, v_6, v_9\}$	$\{v_3, v_5, v_8\}$
5	5	$\{v_3, v_5, v_7, v_{10}\}$	$\{v_2, v_4, v_6, v_9\}$

Table 2.1

(d) *If G arises from a path $P = v_1v_2 \ldots v_rv_{r+1} \ldots v_{r+s}$ by adding the edge v_1v_r such that $r \in \{3, 4, 5\}$ and $s \in \{1, 3, 4, 5\}$, then G has a minimum dominating pair (D_1, D_2) with $v_{r+s} \in D_1$, $v_{r+s-1} \in D_2$, and $v_r \subseteq D_1 \cup D_2$. Furthermore, $\gamma\gamma(G) \leq \frac{6}{7}|V(G)|$ with equality if and only if $(r, s) = (4, 3)$.*

Proof: Since (a) is easily verified, we proceed to (b).

Clearly, $(\{v_1, v_3\}, \{v_2, v_4\})$ is a dominating pair of H_1, $(\{v_1, v_3, v_6\}, \{v_2, v_4, v_7\})$ is a dominating pair of H_2, and $(\{v_1, v_5, v_6\}, \{v_3, v_4, v_7\})$ is a dominating pair of H_3. By symmetry - considering suitable automorphisms of the graphs, (b) follows.

If $G = H_1$, then let $(D_1, D_2) = (\{v_1, v_2\}, \{v_3\})$, and, if $G = H_2$, then let $(D_1, D_2) = (\{v_1, v_4, v_5\}, \{v_3, v_6\})$. In both cases $v_1 \in D_1$, D_1 is dominating, and $V(G) \setminus \{v_1\} \subseteq N_G[D_2]$ which, by symmetry, implies (iii) for $G \in \{H_1, H_2\}$.

If $G = H_3$ and $(D_1, D_2) = (\{v_1, v_4, v_6\}, \{v_3, v_5\})$, then $v_1 \in D_1$, D_1 is dominating and $V(G) \setminus \{v_1\} \subseteq N_G[D_2]$. If $G = H_3$ and $(D_1, D_2) = (\{v_2, v_3, v_6\}, \{v_5, v_7\})$, then $v_2 \in D_1$, D_1 is dominating and $V(G) \setminus \{v_2\} \subseteq N_G[D_2]$. Finally, if $G = H_3$ and $(D_1, D_2) = (\{v_3, v_6, v_7\}, \{v_1, v_5\})$, then $v_3 \in D_1$, D_1 is dominating and $V(G) \setminus \{v_3\} \subseteq N_G[D_2]$. By symmetry, the above observations imply (c) for $G = H_3$.

Now let G be as in (d). It is easy to verify that the Table 2.1 defines suitable minimum dominating pairs for G which completes the proof. $\square$

Lemma 2.5 *If G is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G is connected,

(iii) $V_{\geq 3}(G)$ *is independent, and*

(iv) $G \not\in \{H_1, H_2, H_3\}$,

then G *has a dominating pair* (D_1, D_2) *with* $V_{\geq 3}(G) \subseteq D_1 \cup D_2$ *and* $|D_1 \cup D_2| < \frac{6}{7}|V(G)|$.

Proof: For contradiction, we assume that G is a counterexample of minimum order. It is easy to check that $|V(G)| \geq 5$.

Claim 1 *There is no path* $P = v_1v_2v_3v_4v_5$ *in* G *such that the vertices* v_1, v_2, v_3, *and* v_4 *are of degree* 2 *and* $v_1v_5 \not\in E(G)$.

Proof of Claim 1: For contradiction, we assume that a path P as described in the claim exists. The graph

$$G' = G - \{v_2, v_3, v_4\} + v_1v_5$$

satisfies (i)-(iii) of the hypothesis.

If $G' \in \{H_1, H_2, H_3\}$, then G is either H_2, or a cycle of length 10 or arises from H_3 by subdividing one edge three times. In all three cases the desired result follows easily. Hence, we may assume that $G' \not\in \{H_1, H_2, H_3\}$.

By the choice of G, this implies the existence of a dominating pair (D_1', D_2') of G' with $V_{\geq 3}(G) = V_{\geq 3}(G') \subseteq D_1' \cup D_2'$ and $|D_1' \cup D_2'| < \frac{6}{7}(|V(G)| - 3)$. Since $d_{G'}(v_1) = 2$, either v_1 or v_5 belong to $D_1' \cup D_2'$.

If $v_1 \not\in D_1' \cup D_2'$ and $v_5 \in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_3\}, D_2' \cup \{v_2\})$, if $v_1 \in D_1'$ and $v_5 \in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_4\}, D_2' \cup \{v_2\})$, and if $v_1 \in D_1'$ and $v_5 \not\in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_4\}, D_2' \cup \{v_3\})$. In all three cases (D_1, D_2) is a dominating pair of G with

$$|D_1 \cup D_2| = |D_1' \cup D_2'| + 2 < \frac{6}{7}(|V(G)| - 3) + 2 < \frac{6}{7}|V(G)|,$$

which is a contradiction. By symmetry, this completes the proof. □

Claim 2 *There is no cycle* $C = v_1v_2v_3v_4v_1$ *in* G *such that* $d_G(v_1) + d_G(v_3) \geq 7$, $d_G(v_2) = d_G(v_4) = 2$ *and* $G - \{v_2, v_4\}$ *has two components with vertex sets* $\{v_1\} \cup U_1$ *and* $\{v_3\} \cup U_3$ *such that* $v_1 \not\in U_1$ *and* $v_3 \not\in U_3$. *(Note that one of the two sets* U_1 *and* U_3 *may be empty.)*

Proof of Claim 2: For contradiction, we assume that a cycle C as described in the claim exists. The graph G' that arises by contracting the cycle C to a single vertex v (see Figure 2.4) satisfies (i)-(iii) of the hypothesis. Since $d_{G'}(v) \geq 3$, the graph G' is different from H_1. Therefore, by Lemma 2.4 (a) and the choice of G, G' has a dominating pair (D_1', D_2') such that $v \in D_1'$ and $|D_1' \cup D_2'| \leq \frac{6}{7}(|V(G)| - 3)$. By symmetry, we may assume that v has a neighbor v' in $D_2' \cap U_1$. Now (D_1, D_2) with

$$\begin{aligned} D_1 &= \{v_1, v_2\} \cup (D_1' \cap U_1) \cup (D_2' \cap U_3) \text{ and} \\ D_2 &= \{v_3\} \cup (D_2' \cap U_1) \cup (D_1' \cap U_3) \end{aligned}$$

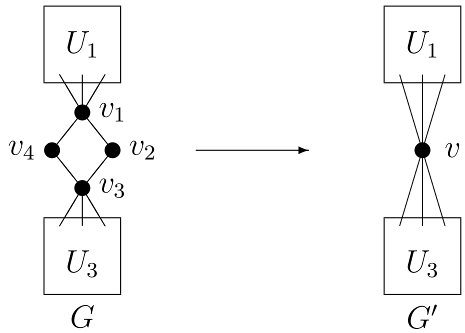


Figure 2.4

is a dominating pair of G with

$$|D_1 \cup D_2| = |(D_1' \setminus \{v\}) \cup D_2'| + 3 \leq \left(\frac{6}{7}(|V(G)| - 3) - 1\right) + 2 < \frac{6}{7}|V(G)|,$$

which is a contradiction. $\Box$

Claim 3 *There are no six vertices $v_1, v_2, v_3, v_4, v_5, v_6 \in V(G)$ such that*

$$v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_2v_6, v_4v_6 \in E(G),$$

v_1, v_3, v_5, and v_6 are of degree 2, v_2 and v_4 are of degree 3, $G[V(G) \setminus \{v_2\}]$ is not connected.

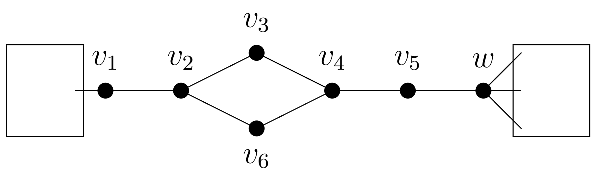


Figure 2.5

Proof of Claim 3: For contradiction, we assume that six vertices $v_1, v_2 \ldots, v_6$ as described in the claim exist. Let w be the neighbor of v_5 different from v_4 (see Figure 2.5). The graph

$$G' = G - \{v_2, v_3, v_4, v_5, v_6\} + v_1w$$

satisfies (i)-(iii) of the hypothesis.

Since the edge v_1w is a bridge of G', $G' \not\in \{H_1, H_2, H_3\}$. By the choice of G, this implies the existence of a dominating pair (D_1', D_2') of G' with $V_{\geq 3}(G) \setminus \{v_2, v_4\} \subseteq D_1' \cup D_2'$ and $|D_1' \cup D_2'| < \frac{6}{7}(|V(G)| - 5)$. Since $d_{G'}(v_1) = 2$, either $v_1 \in D_1' \cup D_2'$ or $w \in D_1' \cup D_2'$.

If $v_1 \not\in D_1' \cup D_2'$ and $w \in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_4, v_6\}, D_2' \cup \{v_2, v_3\})$, if $v_1 \in D_1'$ and $w \not\in D_1' \cup D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_2, v_5\}, D_2' \cup \{v_3, v_4\})$, if $v_1 \in D_1'$ and

$w \in D_1'$, then let $(D_1, D_2) = (D_1' \cup \{v_4\}, D_2' \cup \{v_2, v_5\})$, and if $v_1 \in D_1'$ and $w \in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_4, v_5\}, D_2' \cup \{v_2, v_3\})$. In all four cases (D_1, D_2) is a dominating pair of G with

$$|D_1 \cup D_2| \leq |D_1' \cup D_2'| + 4 \leq \frac{6}{7}(|V(G)| - 5) + 4 < \frac{6}{7}|V(G)|,$$

which is a contradiction. By symmetry, this completes the proof. □

By Claim 1, for every open i-ear in G, we have $i \in \{1, 2, 3\}$ and for every closed i-ear in G, we have $i \in \{2, 3, 4\}$.

If G has no closed i-ear, then the desired result follows from Theorem 2.3. Hence, we may assume that

$$C = v_1 v_2 \dots v_r v_1$$

with $r \in \{3, 4, 5\}$ is a closed $(r-1)$-ear and $d_G(v_r) \geq 3$. If $d_G(v_r) = 3$, then there is an open $(s-1)$-ear

$$P = v_r v_{r+1} \dots v_{r+s}$$

in G with $s \in \{2, 3, 4\}$, $v_{r+1} \notin \{v_1, v_{r-1}\}$, and $d_G(v_{r+s}) \geq 3$. If $d_G(v_r) \geq 4$, then let $s = 0$, i.e. $s \in \{0, 2, 3, 4\}$.

Claim 4 *$d_G(v_r) \leq 4$ and, if $d_G(v_r) = 3$, then $d_G(v_{r+s}) = 3$.*

Proof of Claim 4: For contradiction, we assume that $d_G(v_r) \geq 5$ or that $d_G(v_r) = 3$ and $d_G(v_{r+s}) \geq 4$. The graph $G' = G[V(G) \setminus \{v_1, v_2, \dots, v_{r+s-1}\}]$ satisfies (i)-(iii) of the hypothesis and is different from H_1 and H_2. Therefore, by Lemma 2.4 (a) and the choice of G, G' has a dominating pair (D_1', D_2') such that $v_{r+s} \in D_1'$ and $|D_1' \cup D_2'| \leq \frac{6}{7}(|V(G)| - (r+s-1))$.

Table 2.2 summarizes how to construct a suitable dominating pair (D_1, D_2) for G, which yields a contradiction and completes the proof of the claim. □

By Claim 4, v_{r+s} has exactly two neighbors $x, y \notin \{v_1, v_2, \dots, v_{r+s-1}\}$. By (iii), $d_G(x) = d_G(y) = 2$.

If $xy \in E(G)$, then $V(G) = \{v_1, v_2, \dots, v_{r+s}, x, y\}$ and the result follows easily using Lemma 2.4 (d). Therefore, the unique neighbor z of y different from v_{r+s} is different from x.

If $xz \in E(G)$, then Claim 2 and Claim 3 imply that $V(G) = \{v_1, v_2, \dots, v_{r+s}, x, y, z\}$ and the result follows easily. Therefore, $xz \notin E(G)$.

The graph

$$G' = G - \{v_1, v_2, \dots, v_{r+s}, y\} + xz$$

satisfies (i)-(iii) of the hypothesis.

If $G' \in \{H_1, H_2, H_3\}$, then the desired result follows easily by combining Lemma 2.4 (c) and (d). Hence, we may assume that $G' \notin \{H_1, H_2, H_3\}$. This implies, by the choice of G, that G' has a dominating pair (D_1', D_2') with $V_{\geq 3}(G') \subseteq D_1' \cup D_2'$ and $|D_1' \cup D_2'| < \frac{6}{7}|V(G')|$. In this case, Lemma 2.5 (iv) easily implies that G has a dominating pair (D_1, D_2) with $V_{\geq 3}(G) \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V(G)|$, which is a contradiction and completes the proof. □

r	s	$D_1 \setminus D_1'$	$D_2 \setminus D_2'$
3	0	$\emptyset$	$\{v_1\}$
3	2	$\{v_2\}$	$\{v_3\}$
3	3	$\{v_3\}$	$\{v_2, v_4\}$
3	4	$\{v_2, v_4\}$	$\{v_1, v_5\}$
4	0	$\{v_3\}$	$\{v_2\}$
4	2	$\{v_1, v_3\}$	$\{v_2, v_4\}$
4	3	$\{v_3, v_4\}$	$\{v_2, v_5\}$
4	4	$\{v_2, v_5\}$	$\{v_1, v_3, v_6\}$
5	0	$\{v_3\}$	$\{v_1, v_4\}$
5	2	$\{v_2, v_4\}$	$\{v_3, v_5\}$
5	3	$\{v_3, v_5\}$	$\{v_2, v_4, v_6\}$
5	4	$\{v_2, v_4, v_6\}$	$\{v_1, v_3, v_7\}$

Table 2.2

Lemma 2.6 *If G is a graph such that*

(i) $\delta(G) \geq 2$*,*

(ii) G *connected,*

(iii) G *is edge-minimal with respect to (i)-(ii), and*

(iv) $G \notin \{H_1, H_2, H_3\}$*,*

then $\gamma\gamma(G) < \frac{6}{7}|V(G)|$*.*

Proof: Let $c(G)$ denote the number of closed 3-ears of G with exactly one vertex of degree 3. For contradiction, we assume that G is a counterexample such that $|V(G)| + c(G)$ is minimum. Clearly, we may assume again that $|V(G)| \geq 5$.

In view of Lemma 2.5, we may assume that $V_{\geq 3}(G)$ is not independent, i.e. $v'v'' \in E(G)$ for some $v', v'' \in V_{\geq 3}(G)$. By (iii) of the hypothesis, the edge $v'v''$ must be a bridge, i.e. G arises from the disjoint union of two graphs G' and G'' by adding the bridge $v'v''$ where $v' \in V(G')$ and $v'' \in V(G'')$. Note that G' and G'' satisfy (i)-(iii) of the hypothesis.

First, we assume that $G', G'' \in \{H_1, H_2, H_3\}$. In this case let (D_1', D_2') and (D_1'', D_2'') be as in Lemma 2.4 (c) with $v' \in D_1'$ and $v'' \in D_1''$. Clearly, $(D_1' \cup D_2'', D_1'' \cup D_2')$ is a dominating pair of G and $|D_1' \cup D_2'' \cup D_1'' \cup D_1'| < \frac{6}{7}|V(G)|$, which is a contradiction.

Next, we assume that $G' \notin \{H_1, H_2, H_3\}$ and $G'' \neq H_1$. Since $c(G'), c(G'') \leq c(G) + 1$ and $|V(G')|, |V(G'')| \geq 3$, we obtain, by the choice of G, $\gamma\gamma(G') < \frac{6}{7}|V(G')|$ and $\gamma\gamma(G'') \leq \frac{6}{7}|V(G'')|$. If (D_1', D_2') and (D_1'', D_2'') are minimum dominating pairs of G' and G'', then

$(D_1, D_2) = (D_1' \cup D_1'', D_2' \cup D_2'')$ is a dominating pair of G with $|D_1 \cup D_2| < \frac{6}{7}|V(G)|$, which is a contradiction.

Therefore, we may assume that $G' \notin \{H_1, H_2, H_3\}$ and $G'' = H_1$, i.e. G'' is a closed 3-ear of G with exactly one vertex of degree 3. Let

$$G'' = (\{v'' = v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\})$$

and let

$$G''' = G - v_1v_4 + v'v_4 = (V(G), (E(G) \setminus \{v_1v_4\}) \cup \{v'v_4\}).$$

Clearly, G''' satisfies (i)-(iii) of the hypothesis, $G''' \notin \{H_1, H_2, H_3\}$ and $c(G''') < c(G)$. Therefore, by the choice of G, we obtain that $\gamma\gamma(G''') < \frac{6}{7}|V(G)|$.

Let (D_1''', D_2''') be a minimum dominating pair of G'''. Note that

$$|(D_1''' \cup D_2''') \cap \{v', v_1, v_2, v_3, v_4\}| \geq 4$$

and that we may assume $v' \in D_1'''$. Now, (D_1, D_2) with

$$\begin{aligned} D_1 &= (D_1''' \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_3\} \text{ and} \\ D_2 &= (D_2''' \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_1, v_2\} \end{aligned}$$

is a dominating pair of G with $|D_1 \cup D_2| < \frac{6}{7}|V(G)|$, which is a contradiction.

This completes the proof. $\square$

With the help of the following lemma for small graphs, we can finally prove Theorem 2.2.

Lemma 2.7 (McCuaig and Sherpherd, cf. Lemma 2 in [62]) *If G is a connected graph with $|V(G)| \leq 7$, $\delta(G) \geq 2$, and $\gamma(G) > \frac{2}{5}|V(G)|$, then*

$$G \in \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}.$$

Recall the statement of Theorem 2.2.

Theorem 2.2 *If G is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G *connected, and*

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) < \frac{6}{7}|V(G)|$.

Proof: Let G' be a graph with $V(G') = V(G)$ and $E(G') \subseteq E(G)$ such that

(i) $\delta(G') \geq 2$,

(ii) G' connected, and

(iii) G' is edge-minimal with respect to (i)-(ii).

Clearly, $\gamma\gamma(G') \geq \gamma\gamma(G)$, and thus, by Lemma 2.6, the statement of the theorem is true, if $G' \notin \{H_1, H_2, H_3\}$.

If $G' = H_1$, then it is straightforward to check that $\gamma\gamma(G) \leq \frac{3}{4}|V(G)|$, because $G \neq H_1$. Therefore, we may assume that $G' \in \{H_2, H_3\}$.

If G has a hamiltonian cycle and $\gamma(G) \leq 2$, then $\gamma\gamma(G) \leq 5$, because for any 2 vertices $v_i, v_j \in V(G)$, there exists a dominating set of G of cardinality 3 that does not contain v_i or v_j. Thus, if $G' = H_2$, then, by Lemma 2.7, $\gamma\gamma(G) \leq \frac{5}{7}|V(G)|$, because $G \notin \{H_2, H_4, H_5, H_6, H_7\}$.

Hence, we may assume that G has no hamiltonian cycle and $G' = H_3$. If G'' is a graph that arises from H_3 by adding an edge $e \in E(G) \setminus E(G')$, then $\gamma\gamma(G'') \geq \gamma\gamma(G)$. By symmetry, $e \in \{v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_2v_7\}$ (cf. Figure 2.3). Thus, $\gamma\gamma(G'') \leq \frac{5}{7}|V(G'')|$ or $G'' = H_6$ in which case G has a hamiltonian cycle — a contradiction. This completes the proof. □

We believe that the following considerable strengthening is possible.

Conjecture 2.8 *If G is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G connected, and

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) \leq \frac{4}{5}|V(G)|$.

By the results of McCuaig and Shepherd [62], there would be infinitely many extremal graphs for Conjecture 2.8. In fact, we believe that the edge-minimal extremal graphs for the bound in Conjecture 2.8 are the same as those described in [62] for the bound $\gamma(G) \leq \frac{2}{5}|V(G)|$.

2.2 Graphs with Large Minimum Degree

In this section we prove an upper bound on $\gamma\gamma(G)$ for graphs G using the probabilistic method.

The proof builds on an elegant probabilistic argument given by Alon and Spencer [2]. The result is asymptotically best-possible, because (2.2) is so too (see [1,2]). Several times during the proof we will use Observation 2.1. We denote the expected value of a random variable X by $\mathbf{E}[X]$ and we denote the probability of an event A by $\mathbf{P}[A]$.

Theorem 2.9 *If G is a graph of minimum degree $\delta(G) \geq 5$, then*

$$\gamma\gamma(G) \leq 2\frac{1 + \ln(\delta(G) + 1)}{\delta(G) + 1} n(G).$$

Proof: Let $p = \frac{\ln(\delta(G)+1)}{\delta(G)+1}$. Note that $p \leq \frac{1}{2}$. We construct a partition of $V(G)$ into three sets

$$V(G) = D_1^0 \cup D_2^0 \cup Y$$

by assigning every vertex independently at random to the set D_1^0 with probability p, to the set D_2^0 with probability p, and to the set Y with probability $(1-2p)$. Clearly, $\mathbf{E}[|D_1^0|] = \mathbf{E}[|D_2^0|] = n(G)p$. Let

$$Z^1 = \left\{v \in V(G) \mid N_G[v] \cap (D_1^0 \cup D_2^0) = \emptyset\right\}.$$

For a fixed vertex $v \in V(G)$, we have

$$\mathbf{P}\left[v \in Z^1\right] = \mathbf{P}\left[N_G[v] \subseteq Y\right] = (1-2p)^{d_G(v)+1}.$$

Let D_1^1 be a minimal dominating set of $G[Z^1]$ and let D_2^1 be the union of $Z^1 \setminus D_1^1$ and a minimal set of vertices of G such that each isolated vertex in $G[Z^1]$ has a neighbor in D_2^1. Clearly, $D_2^1 \subseteq Y \setminus D_1^1$ and, by Observation 2.1, (D_1^1, D_2^1) dominates every vertex in Z^1. Note that $|D_1^1| + |D_2^1| \leq 2|Z^1|$ and thus,

$$\mathbf{E}\left[|D_1^1| + |D_2^1|\right] \leq 2 \sum_{v \in V(G)} (1-2p)^{d_G(v)+1}.$$

Let

$$Z_1^2 = \left\{v \in V(G) \mid N_G[v] \cap (D_1^0 \cup D_1^1) = \emptyset\right\}.$$

Note that $|N_G[v] \cap D_2^0| \geq 1$ for each $v \in Z_1^2$, since otherwise $v \in Z^1$ and thus, $|N_G[v] \cap D_1^1| \geq 1$, which would be a contradiction to $v \in Z_1^2$. For a fixed vertex $v \in V(G)$,

$$\begin{aligned}
\mathbf{P}\left[v \in Z_1^2\right] &= \mathbf{P}\left[N_G[v] \cap (D_1^0 \cup D_1^1) = \emptyset\right] \\
&\leq \mathbf{P}\left[(N_G[v] \cap D_1^0 = \emptyset) \wedge (N_G[v] \cap D_2^0 \neq \emptyset)\right] \\
&= \mathbf{P}\left[N_G[v] \cap D_1^0 = \emptyset\right] - \mathbf{P}\left[N_G[v] \cap (D_1^0 \cup D_2^0) = \emptyset\right] \\
&= (1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}.
\end{aligned}$$

Let D_1^2 be a minimal set of vertices in $V(G) \setminus (D_2^0 \cup D_2^1)$ such that each vertex $v \in Z_1^2$ that satisfies

$$|N_G[v] \cap (D_2^0 \cup D_2^1)| < d_G(v) + 1$$

is dominated by D_1^2. Note that $|D_1^2| \leq |Z_1^2|$ and thus,

$$\mathbf{E}\left[|D_1^2|\right] \leq \sum_{v \in V(G)} \left((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}\right).$$

Let

$$Z_2^2 = \left\{v \in V(G) \mid N_G[v] \cap (D_2^0 \cup D_2^1) = \emptyset\right\}.$$

Note that $|N_G[v] \cap D_1^0| \geq 1$ for each $v \in Z_2^2$, since otherwise $v \in Z^1$ and thus, $|N_G[v] \cap D_2^1| \geq 1$, which would be a contradiction to $v \in Z_2^2$. For a fixed vertex $v \in V(G)$,

$$\begin{aligned}
\mathbf{P}\left[v \in Z_2^2\right] &= \mathbf{P}\left[N_G[v] \cap (D_2^0 \cup D_2^1) = \emptyset\right] \\
&\leq \mathbf{P}\left[(N_G[v] \cap D_2^0 = \emptyset) \wedge (N_G[v] \cap D_1^0 \neq \emptyset)\right] \\
&= \mathbf{P}\left[N_G[v] \cap D_2^0 = \emptyset\right] - \mathbf{P}\left[N_G[v] \cap (D_2^0 \cap D_1^0) = \emptyset\right] \\
&= (1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}.
\end{aligned}$$

Let D_2^2 be a minimal set of vertices in $V(G) \setminus (D_1^0 \cup D_1^1 \cup D_1^2)$ such that each vertex $v \in Z_2^2$ that satisfies

$$|N_G[v] \cap (D_1^0 \cup D_1^1 \cup D_1^2)| < d_G(v) + 1$$

is dominated by D_2^2. Note that $|D_2^2| \leq |Z_2^2|$ and thus,

$$\mathbf{E}\left[|D_2^2|\right] \leq \sum_{v \in V(G)} \left((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}\right).$$

For $i \in \{1, 2\}$, let

$$D_i' = D_i^0 \cup D_i^1 \cup D_i^2.$$

Clearly, $D_1' \cap D_2' = \emptyset$. For $i \in \{1, 2\}$, let

$$X_i = \{v \in V(G) \mid N_G[v] \subseteq D_i'\}.$$

Let D_i^3 be a minimal dominating set of $G[X_{3-i}]$ for $i \in \{1, 2\}$. Let

$$\begin{aligned}
D_1 &= (D_1' \setminus D_2^3) \cup D_1^3 \text{ and} \\
D_2 &= (D_2' \setminus D_1^3) \cup D_2^3.
\end{aligned}$$

Clearly, by Observation 2.1, (D_1, D_2) is a dominating pair of G and, by the first moment method [2], we obtain

$$\begin{aligned}
\gamma\gamma(G) &\leq \mathbf{E}\left[|D_1| + |D_2|\right] \\
&= \mathbf{E}\left[|(D_1' \setminus D_2^3) \cup D_1^3|\right] + \mathbf{E}\left[|(D_2' \setminus D_1^3) \cup D_2^3|\right] \\
&= \mathbf{E}\left[|D_1'|\right] + \mathbf{E}\left[|D_2'|\right] \\
&= \mathbf{E}\left[|D_1^0 \cup D_1^1 \cup D_1^2|\right] + \mathbf{E}\left[|D_1^0 \cup D_1^1 \cup D_1^2|\right] \\
&\leq 2n(G)p + 2\sum_{v \in V(G)} (1-2p)^{d_G(v)+1} + 2\sum_{v \in V(G)} \left((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}\right) \\
&= 2n(G)p + 2\sum_{v \in V(G)} (1-p)^{d_G(v)+1} \\
&\leq 2n(G)p + 2n(G)(1-p)^{\delta(G)+1} \\
&\leq 2n(G)p + 2n(G)e^{-p(\delta(G)+1)} \\
&= 2n(G)\frac{1 + \ln(\delta(G)+1)}{\delta(G)+1},
\end{aligned}$$

which completes the proof. $\square$

The extension of Alon and Spencer's proof from one dominating set to two disjoint dominating sets was not too difficult. Nevertheless, an extension to three disjoint dominating sets is not possible. If we consider the proof of Theorem 2.9, then by [77], we can't guarantee the existence of three disjoint sets of vertices, such that each of the three sets dominates a set that corresponds to Z_1.

2.3 Connected Cubic Graphs

After considering bounds on the $\gamma\gamma(G)$ for graphs G with $\delta(G) \geq 2$, respectively large minimum degree, we consider connected cubic graphs next. As our main result in this section we prove the following.

Theorem 2.10 *If G is a connected cubic graph, then*

$$\gamma\gamma(G) \leq \frac{157}{198}n(G) + \frac{8}{9} \approx 0.793n(G) + \frac{8}{9}.$$

In Subsection 2.3.1, we prove Theorem 2.10 and, in Subsection 2.3.2, we prove a technical lemma used in Subsection 2.3.1. This lemma is an extension of results in [45, 71].

2.3.1 Proof of Theorem 2.10

Following Reed [71], we consider suitable path covers and introduce some more terminology. Let G be a graph. If P is a path with $n(P) \equiv i \bmod 3$, then P is called an *i-mod-3-path*. If an endvertex u of P has a neighbor outside of $V(P)$, then u is an *out-endvertex of P*. A *vdp-cover* of G is a collection of vertex-disjoint paths such that all vertices of G are contained in one of these paths. For a vdp-cover S, let S_i denote the set of i-mod-3-paths in S for $i \in \{0, 1, 2\}$. A vdp-cover S of G is *optimal* if

(R1) $|S|$ is minimized.

(R2) Subject to (R1), $|S_1|$ is minimized.

(R3) Subject to (R1) and (R2), if for some $P \in S_1$, the graph $G[V(P)]$ has a hamiltonian path with an out-endvertex, then P is one such path.

The next lemma collects some properties of optimal vdp-covers and corresponds to observations in [71] and Lemma 1 in [45]. For the sake of completeness, we include the proof based on simple exchange arguments.

Lemma 2.11 *Let S be an optimal vdp-cover of a graph G. Let x be an out-endvertex of a 1-mod-3-path P in S and let y be a neighbor of x on a path Q in $S \setminus \{P\}$. If $Q = Q'yQ''$, then*

(a) *Q is not a 1-mod-3-path.*

(b) *If Q is a 2-mod-3-path, then both Q' and Q'' are 2-mod-3-paths.*

(c) *If Q is a 0-mod-3-path, then both Q' and Q'' are 1-mod-3-paths.*

Proof: First, we assume that Q is a 1-mod-3-path. At least one of Q' and Q'', say Q', is no 1-mod-3-path. Since the vdp-cover $S' = (S \setminus \{P, Q\}) \cup \{Q', PyQ''\}$ is obtained from S by replacing two 1-mod-3-paths by a 2-mod-3-path and a 0-mod-3-path, we obtain a contradiction to (R2), which proves (a).

Next, we assume that Q is a 2-mod-3-path and Q' is not a 2-mod-3-path. One of Q' and Q'', say Q', is a 0-mod-3-path. Let S' be as above. Since S' is obtained from S by replacing a 1-mod-3-path and a 2-mod-3-path by two 0-mod-3-paths, we obtain a contradiction to (R2), which proves (b).

Finally, we assume that Q is a 0-mod-3-path and Q' is not a 1-mod-3-path. One of Q' and Q'', say Q', is a 2-mod-3-path. Let S' be as above. Since S' is obtained from S by replacing a 1-mod-3-path and a 0-mod-3-path by two 2-mod-3-paths, we obtain a contradiction to (R2), which proves (c). $\Box$

The following technical lemma is an extension of Lemma 2 in [45], which in turn extended Fact 9 in [71]. Its proof is postponed to Section 2.3.2.

Lemma 2.12 *If H is a subcubic graph such that H has a hamiltonian path, $n(H) \leq 19$, $n(H) \equiv 1$ mod 3, and every endvertex of every hamiltonian path of H has degree 3, then $\gamma\gamma(H) \leq \frac{2n(H)+1}{3}$.*

Our final ingredient is the following.

Theorem 2.13 (Reed [71]) *Every connected cubic graph G has a vdp-cover S with*

$$|S| \leq \left\lceil \frac{n(G)}{9} \right\rceil .$$

We proceed to the

Proof of Theorem 2.10: Let G be a connected cubic graph. Let S be an optimal vdp-cover of G. For each 1-mod-3-path $P \in S$ that has an out-endvertex, we select one such out-endvertex x_P. Furthermore, we choose a neighbor $y_P \notin V(P)$ of x_P and call the path in S containing y_P *accepting.*

Now, we construct a dominating pair (D_1, D_2) of G starting with $(D_1, D_2) = (\emptyset, \emptyset)$.

- For each 1-mod-3-path P that has an out-endvertex, say $P = v_1 \dots v_{n(P)}$ and $v_1 = x_P$, we include every third vertex of P to D_1 starting with v_3 and we include every third vertex of P to D_2 starting with v_1. Clearly, $|\,(D_1 \cup D_2) \cap V(P)| = \frac{2|V(P)|+1}{3}$.

- For each 1-mod-3-path P that has no out-endvertex, the path P has order at least 4. Let (D_1^P, D_2^P) be a dominating pair of $G[P]$ of minimum cardinality. We include the vertices of D_1^P to D_1 and the vertices of D_2^P to D_2. Clearly, $|\,(D_1 \cup D_2) \cap V(P)| \leq \frac{2|V(P)|+4}{3}$. If $V(P) < 22$, then, by Lemma 2.12, $|\,(D_1 \cup D_2) \cap V(P)| \leq \frac{2|V(P)|+1}{3}$.
- For each 0-mod-3-path $P = v_1 \dots v_{n(P)}$, we include every third vertex of P to D_1 starting with v_2 and we include v_1 as well as every third vertex of P to D_2 starting with v_3. Clearly, $|\,(D_1 \cup D_2) \cap V(P)| = \frac{2|V(P)|+3}{3}$.
- For each accepting 2-mod-3-path $P = v_1 \dots v_{n(P)}$, Lemma 2.11 implies that P has order at least 5. We include v_2 and v_k as well as every third vertex of P to D_1 starting with v_3 and we include every third vertex of P to D_2 starting with v_1. Clearly, $|\,(D_1 \cup D_2) \cap V(P)| = \frac{2|V(P)|+5}{3}$.
- For each non-accepting 2-mod-3-path $P = v_1 \dots v_{n(P)}$, we include every third vertex of P to D_1 starting with v_1 and we include every third vertex of P to D_2 starting with v_2. Clearly, $|\,(D_1 \cup D_2) \cap V(P)| = \frac{2|V(P)|+2}{3}$.

By construction, each of the two sets D_1 and D_2 dominates all vertices that lie either on a 1-mod-3-path in S that has no out-endvertex or on a 0-mod-3-path in S or on a 2-mod-3-path in S. Similarly, D_2 dominates all vertices that lie on a 1-mod-3-path in S that has an out-endvertex. Furthermore, if P is a 1-mod-3-path in S that has an out-endvertex, then D_1 dominates all vertices of P distinct from x_P. Finally, for every 1-mod-3-path P in S that has an out-endvertex, by Lemma 2.11 and the above construction, the neighbor y_P of the selected out-endvertex x_P belongs to D_1. Altogether, (D_1, D_2) is a dominating pair of G.

Let S_0, S_1, and S_2 denote the set of 0-mod-3-paths, 1-mod-3-paths, and 2-mod-3-paths in S, respectively. Let $S_2^{\rm acc}$ denote the set of accepting 2-mod-3-paths in S and let $S_2^{\neg \rm acc} = S_2 \setminus S_2^{\rm acc}$. Furthermore, let $S_1^{1/3}$ denote the set of 1-mod-3-paths P in S with $|\,(D_1 \cup D_2) \cap V(P)| \leq \frac{2|V(P)|+1}{3}$. Note that this includes all 1-mod-3-paths that have an out-endvertex. Hence, $|S_2^{\rm acc}| \leq |S_1^{1/3}|$. Finally, let $S_1^{4/3} = S_1 \setminus S_1^{1/3}$ denote the set of 1-mod-3-paths P in S with $|\,(D_1 \cup D_2) \cap V(P)| = \frac{2|V(P)|+4}{3}$. By Lemma 2.12, the vertex set $V\left(S_1^{4/3}\right)$ of the union of all paths in $S_1^{4/3}$ has order at least $22\left|S_1^{4/3}\right|$.

By Theorem 2.13 and (R1) in the definition of optimal vdp-covers, we have $|S| \leq \frac{n(G)+8}{9}$ and estimate $\gamma\gamma(G)$ as follows.

$$\begin{aligned}
\gamma\gamma(G) &\leq |D_1| + |D_2| \\
&\leq \sum_{P \in S_0} \frac{2|V(P)|+3}{3} + \sum_{P \in S_1^{1/3}} \frac{2|V(P)|+1}{3} + \sum_{P \in S_1^{4/3}} \frac{2|V(P)|+4}{3} \\
&\quad + \sum_{P \in S_2^{\neg \rm acc}} \frac{2|V(P)|+2}{3} + \sum_{P \in S_2^{\rm acc}} \frac{2|V(P)|+5}{3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}n(G) + |S_0| + \frac{1}{3}\left|S_1^{1/3}\right| + \frac{4}{3}\left|S_1^{4/3}\right| + \frac{2}{3}\left|S_2^{\neg\mathrm{acc}}\right| + \frac{5}{3}\left|S_2^{\mathrm{acc}}\right| \\
&= \frac{2}{3}n(G) + |S_0| + \left(\frac{1}{3}\left|S_1^{1/3}\right| + \frac{2}{3}\left|S_2^{\mathrm{acc}}\right|\right) + \frac{4}{3}\left|S_1^{4/3}\right| + \frac{2}{3}\left|S_2^{\neg\mathrm{acc}}\right| + \left|S_2^{\mathrm{acc}}\right| \\
&\leq \frac{2}{3}n(G) + |S_0| + \left|S_1^{1/3}\right| + \left|S_2^{\neg\mathrm{acc}}\right| + \left|S_2^{\mathrm{acc}}\right| + \frac{4}{3}\left|S_1^{4/3}\right| \\
&\leq \frac{2}{3}n(G) + |S_0| + \left|S_1^{1/3}\right| + \left|S_2^{\neg\mathrm{acc}}\right| + \left|S_2^{\mathrm{acc}}\right| + \left|S_1^{4/3}\right| + \frac{1}{66}\left|V\left(S_1^{4/3}\right)\right| \\
&\leq \frac{15}{22}n(G) + |S| \\
&\leq \frac{15}{22}n(G) + \frac{n(G)+8}{9} \\
&= \frac{157}{198}n(G) + \frac{8}{9}.
\end{aligned}
$$

This concludes the proof. □

2.3.2 Proof of Lemma 2.12

In order to complete the proof of Theorem 2.10, it remains to prove Lemma 2.12, which is obtained by combining the statements of Lemmas 2.23, 2.24, and 2.25 below. We follow the general approach used in [45]. Unfortunately, for the Lemmas 2.23, 2.24, and 2.25, we practically have to reiterate the proofs given in [45].

We introduce some more terminology. If a subcubic graph H has a hamiltonian path with endvertices u and v, then u is called *v-distant*. If D_1, D_2, V_1, and V_2 are sets of vertices of a graph G such that D_1 and D_2 are disjoint, D_1 dominates every vertex in V_1, and D_2 dominates every vertex in V_2, then (D_1, D_2) is a *(V_1, V_2)-dominating pair* of G.

Lemma 2.14 *If H is a subcubic graph, $P = v_1v_2v_3v_4v_5$ is a hamiltonian path of H, and all v_5-distant vertices have degree* 3*, then there is a $(V(H)\setminus\{v_5\}, V(H))$-dominating pair of cardinality at most* 3*.*

Proof: If $v_1v_3 \in E(H)$, then $(\{v_3\}, \{v_2, v_4\})$ is a $(V(H)\setminus\{v_5\}, V(H))$-dominating pair of cardinality 3. Hence, we may assume that $v_1v_4, v_1v_5 \in E(H)$, which implies that $v_2v_3v_4v_1v_5$ and $v_3v_2v_1v_4v_5$ are hamiltonian paths, i.e. v_2 and v_3 are v_5-distant. By symmetry with v_1, this implies $v_2v_5, v_3v_5 \in E(H)$, which contradicts the assumption that H is subcubic. □

Lemma 2.15 *If a graph H of order $3k+1$ for $k \in \mathbb{N}$ has a hamiltonian path $P = v_1 \dots v_{3k+1}$ and an edge of the form v_iv_{i+3j-1} where $i, j \in \mathbb{N}$ and i is not divisible by* 3*, then $\gamma\gamma(H) \leq 2k+1$.*

Proof: If $i \equiv 1 \bmod 3$, then let $D_1 = \{v_2, v_5, \dots, v_{i-2}, v_{i+2}, v_{i+5}, \dots, v_{3k}\}$. Since $v_{i+3j-1} \in D_1$, D_1 dominates $V(H)$. If $i \equiv 2 \bmod 3$, then let $D_1 = \{v_2, v_5, \dots, v_{i+3j-3}, v_{i+3j+1}, v_{i+3j+4}, \dots, v_{3k}\}$. Since $v_i \in D_1$, D_1 dominates $V(H)$. In both cases (D_1, D_2) with $D_2 = \{v_1, v_4, \dots, v_{3k+1}\}$ is a dominating pair of H of cardinality $2k+1$. □

Lemma 2.15 immediately implies the next result.

Lemma 2.16 *If a graph H of order $3k+1$ for $k \in \mathbb{N}$ has a hamiltonian cycle $C = v_1 \ldots v_{3k+1}v_1$ and an edge of the form v_iv_j where $i, j \in \mathbb{N}$, $i < j$, and $j-i+1$ is divisible by 3, then $\gamma\gamma(H) \leq 2k+1$.*

Lemma 2.17 *If H is a subcubic graph, $C = v_1v_2 \ldots v_7v_1$ is a hamiltonian cycle of H, $d_H(v_7) = 2$, and and all v_7-distant vertices have degree 3, then there is a dominating pair (D_1, D_2) of H of cardinality at most 5 such that $v_7 \in D_1 \cup D_2$.*

Proof: By Lemma 9 in [45], H has a dominating set D_1 of cardinality 2. The graph $H - D_1$ has order 5 and a vdp-cover consisting of at most 2 paths. Since v_7 has degree 2, v_7 either belongs to D_1 or is an endvertex of a path in the vdp-cover of $H - D_1$. This easily implies that H has a dominating set D_2 that is disjoint from D_1 and has cardinality 3 such that $v_7 \in D_1 \cup D_2$. By symmetry, Figure 2.6 illustrates all relevant cases. In this figure, the vertices in D_1 are indicated by empty circles, the vertices in $V(H) \setminus D_1$ are indicated by filled circles, and the vertices of D_2 are indicated by encircled filled circles. The desired statement follows. $\square$

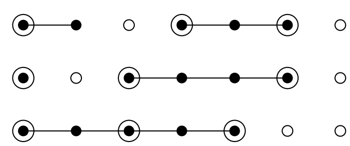


Figure 2.6

Lemma 2.18 *If H is a subcubic graph, $C = v_1v_2 \ldots v_8v_1$ is a hamiltonian cycle of H, $d_H(v_8) = 2$, and all v_8-distant vertices have degree 3, then there is a $(V(H) \setminus \{v_8\}, V(H))$-dominating pair of cardinality at most 5.*

Proof: By Lemma 10 in [45], there is a set D_1 of cardinality 2 dominating $V(H) \setminus \{v_8\}$. The graph $H - D_1$ has order 6 and a vdp-cover consisting of at most 2 paths. This easily implies that H has a dominating set D_2 that is disjoint of D_1 and has cardinality 3 (cf. Figure 2.7). The desired statement follows. $\square$

Lemma 2.19 *If H is a subcubic graph, $C = v_1v_2 \ldots v_{10}v_1$ is a hamiltonian cycle of H, $d_H(v_{10}) = 2$, and all v_{10}-distant vertices have degree 3, then there is a dominating pair (D_1, D_2) of H of cardinality at most 7 such that $v_{10} \in D_1 \cup D_2$.*

Proof: By Lemma 11 in [45], H has a dominating set D_1 of cardinality 3. Since 3 consecutive vertices of C dominate at most 8 vertices, D_1 does not consist of 3 consecutive vertices. This implies that the graph $H - D_1$ has order 7 and a vdp-cover consisting of

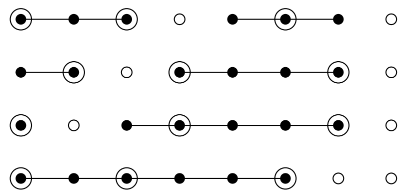


Figure 2.7

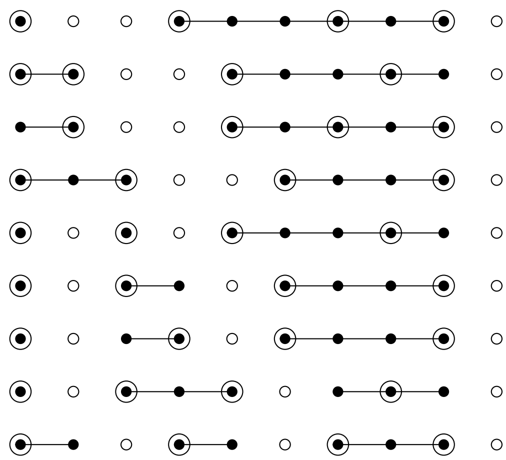


Figure 2.8

either 2 or 3 paths. Since v_{10} has degree 2, v_{10} either belongs to D_1 or is an endvertex of a path in the vdp-cover of $H - D_1$. This easily implies that H has a dominating set D_2 that is disjoint from D_1 and has cardinality 4 such that $v_{10} \in D_1 \cup D_2$ (cf. Figure 2.8). The desired statement follows. □

Lemma 2.20 *If H is a subcubic graph, $C = v_1v_2 \dots v_{11}v_1$ is a hamiltonian cycle of H, $d_H(v_{11}) = 2$, and all v_{11}-distant vertices have degree 3, then there is a $(V(H)\setminus\{v_{11}\}, V(H))$-dominating pair of cardinality at most 7.*

Proof: By Lemma 12 in [45], there is a set D_1 of cardinality 3 dominating $V(H) \setminus \{v_{11}\}$. As in the proof of Lemma 2.19, we may assume that D_1 does not consist of 3 consecutive vertices. This implies that the graph $H - D_1$ has order 8 and a vdp-cover consisting of either 2 or 3 paths. This easily implies that H has a dominating set D_2 that is disjoint of D_1 and has cardinality 4 (cf. Figure 2.9). The desired statement follows. □

Lemma 2.21 *If H is a cubic hamiltonian graph of order 10, then $\gamma\gamma(G') \leq 7$.*

Proof: Let $v_1v_2 \dots v_{10}v_1$ denote a hamiltonian cycle of H. By Lemma 2.16 and symmetry, we may assume that either $v_1v_4 \in E(H)$ or $v_1v_5 \in E(H)$.

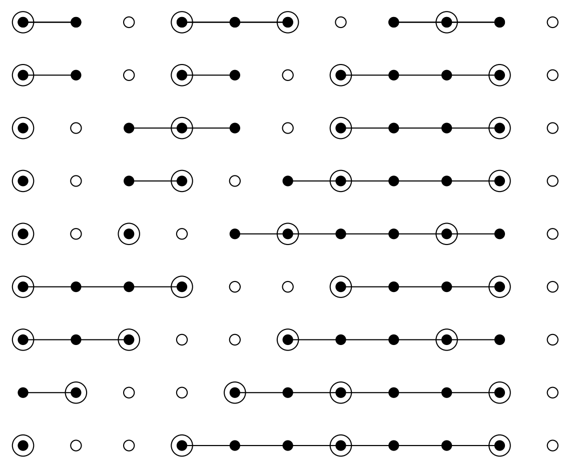


Figure 2.9

First, we assume $v_1v_5 \in E(H)$. By Lemma 2.16 and symmetry, we may assume that either $v_3v_6 \in E(H)$ or $v_3v_7 \in E(H)$. If $v_3v_6 \in E(H)$, then $(\{v_1, v_3, v_8\}, \{v_2, v_5, v_7, v_{10}\})$ is a dominating pair of H. If $v_3v_7 \in E(H)$, then $(\{v_3, v_5, v_9\}, \{v_2, v_4, v_7, v_{10}\})$ is a dominating pair of H. By symmetry, we may assume that H contains no edge of the form v_iv_{i+4} or v_iv_{i+6}.

Next, we assume $v_1v_4 \in E(H)$. By Lemma 2.16 and symmetry, we may assume that either $v_3v_6 \in E(H)$ or $v_3v_{10} \in E(H)$. If $v_3v_6 \in E(H)$, then $(\{v_1, v_6, v_8\}, \{v_2, v_5, v_7, v_{10}\})$ is a dominating pair of H. If $v_3v_{10} \in E(H)$, then, by Lemma 2.16 and symmetry, we may assume that $v_2v_9 \in E(H)$ and $(\{v_4, v_6, v_9\}, \{v_2, v_5, v_7, v_{10}\})$ is a dominating pair of H. This completes the proof. □

Lemma 2.22 *If H is a cubic hamiltonian graph of order 16, then $\gamma\gamma(G') \leq 11$.*

Proof: Let $v_1v_2 \ldots v_{16}v_1$ denote a hamiltonian cycle of H. For contradiction, we assume that $\gamma\gamma(G') > 11$. By Lemma 2.16,

$$H \text{ contains no edge of the form } v_iv_{i+2}, v_iv_{i+5}, v_iv_{i+8}, v_iv_{i+11}, \text{ or } v_iv_{i+14}. \tag{2.13}$$

where indices are identified modulo 16. By symmetry, we may assume that v_{16} has a neighbor in $\{v_3, v_4, v_6, v_7\}$.

Claim 1 *H contains no edge of the form v_iv_{i+4} or v_iv_{i+12}.*

Proof of Claim 1: For contradiction, we assume, by symmetry, that $v_4v_{16} \in E(H)$. By symmetry and (2.13), v_2 has a neighbor in $\{v_5, v_6, v_8, v_9\}$. If $v_2v_5 \in E(H)$, then $(\{v_2, v_7, v_{10}, v_{13}, v_{16}\}, \{v_1, v_4, v_6, v_9, v_{12}, v_{15}\})$ is dominating pair of H. If $v_2v_6 \in E(H)$, then $(\{v_2, v_4, v_8, v_{11}, v_{14}\}, \{v_1, v_3, v_6, v_9, v_{12}, v_{15}\})$ is dominating pair of H. If $v_2v_8 \in E(H)$, then $(\{v_2, v_6, v_{10}, v_{13}, v_{16}\}, \{v_1, v_4, v_7, v_9, v_{12}, v_{15}\})$ is dominating pair of H. If $v_2v_9 \in E(H)$, then $(\{v_2, v_4, v_7, v_{11}, v_{14}\}, \{v_1, v_3, v_6, v_9, v_{12}, v_{15}\})$ is dominating pair of H. These contradictions complete the proof of the claim. □

Claim 2 *If $v_3v_{16} \in E(H)$, then $v_1v_{10}, v_2v_9 \in E(H)$.*

Proof of Claim 2: Let $v_3v_{16} \in E(H)$. If v_2 has a neighbor in $\{v_5, v_8, v_{11}, v_{14}\}$, then $(\{v_5, v_8, v_{11}, v_{14}, v_{16}\}, \{v_1, v_4, v_6, v_9, v_{12}, v_{15}\})$ is dominating pair of H. Thus, by (2.13) and Claim 1, v_2 has a neighbor in $\{v_9, v_{12}, v_{15}\}$. By the symmetry between v_1 and v_2, v_1 has a neighbor in $\{v_4, v_7, v_{10}\}$. If $v_2v_9 \notin E(H)$, then $(\{v_4, v_7, v_{10}, v_{12}, v_{15}\}, \{v_2, v_5, v_8, v_{11}, v_{14}, v_{16}\})$ is dominating pair of H. Thus, $v_2v_9 \in E(H)$, and by symmetry, $v_1v_{10} \in E(H)$. This completes the proof of the claim. □

Claim 3 *If $v_7v_{16} \in E(H)$, then $v_2v_5 \in E(H)$.*

Proof of Claim 3: Let $v_7v_{16} \in E(H)$. By (2.13) and Claim 1, v_5 has a neighbor in $\{v_2, v_8, v_{11}, v_{12}, v_{14}, v_{15}\}$. If $v_5v_8 \in E(H)$, then $(\{v_2, v_5, v_{10}, v_{13}, v_{16}\}, \{v_1, v_4, v_7, v_8, v_{11}, v_{14}\})$ is dominating pair of H. If $v_5v_{11} \in E(H)$, then $(\{v_2, v_5, v_9, v_{13}, v_{16}\}, \{v_1, v_4, v_7, v_8, v_{11}, v_{14}\})$ is dominating pair of H. If $v_5v_{12} \in E(H)$, then $(\{v_2, v_5, v_7, v_{10}, v_{14}\}, \{v_1, v_3, v_6, v_9, v_{12}, v_{15}\})$ is dominating pair of H. If $v_5v_{14} \in E(H)$, then $(\{v_2, v_5, v_9, v_{12}, v_{16}\}, \{v_1, v_4, v_7, v_8, v_{11}, v_{14}\})$ is dominating pair of H. If $v_5v_{15} \in E(H)$, then $(\{v_2, v_5, v_7, v_{10}, v_{13}\}, \{v_1, v_3, v_6, v_9, v_{12}, v_{15}\})$ is dominating pair of H. These contradictions imply $v_5v_2 \in E(H)$, which completes the proof of the claim. □

If $v_3v_{16} \in E(H)$, then Claim 2 implies $v_1v_{10}, v_2v_9 \in E(H)$. By symmetry between v_1v_{10} and v_7v_{16}, Claim 3 implies $v_{12}v_{15} \in E(H)$. By symmetry between $v_{12}v_{15}$ and v_3v_{16}, Claim 2 implies $v_6v_{13}, v_5v_{14} \in E(H)$. Now $(\{v_1, v_2, v_5, v_8, v_{12}\}, \{v_4, v_7, v_9, v_{10}, v_{13}, v_{16}\})$ is dominating pair of H. Hence, by symmetry,

$$H \text{ contains no edge of the form } v_iv_{i+3} \text{ or } v_iv_{i+13}. \tag{2.14}$$

If $v_7v_{16} \in E(H)$, then Claim 2 implies $v_2v_5 \in E(H)$, which contradicts (2.14). Hence, by symmetry,

$$H \text{ contains no edge of the form } v_iv_{i+7} \text{ or } v_iv_{i+9}. \tag{2.15}$$

We may assume $v_6v_{16} \in E(H)$. By (2.13), Claim 1, (2.14), and (2.15), either $v_4v_{10} \in E(H)$ or $v_4v_{14} \in E(H)$. If $v_4v_{14} \in E(H)$, then $(\{v_2, v_6, v_9, v_{11}, v_{14}\}, \{v_1, v_4, v_7, v_{10}, v_{13}, v_{16}\})$ is dominating pair of H. Hence, we may assume $v_4v_{10} \in E(H)$ and, by symmetry, $v_2v_{12} \in E(H)$. Now $(\{v_4, v_8, v_{12}, v_{14}, v_{16}\}, \{v_2, v_5, v_7, v_{10}, v_{13}, v_{16}\})$ is dominating pair of H. This final contradiction completes the proof. □

The graph that arises from a path $v_1v_2 \dots v_n$ by adding the edge v_1v_r is called a (v_n, n, r)-*lasso* and $v_1v_2 \dots v_rv_1$ is called the *cycle of the lasso*.

Lemma 2.23 *If H is a subcubic graph of order 19 with a hamiltonian path such that every endvertex of every hamiltonian path of H has degree 3, then $\gamma\gamma(H) \leq 13$.*

Proof: For contradiction, we assume $\gamma\gamma(H) > 13$. Let a hamiltonian path $P = v_1v_2 \dots v_{19}$ of H and an edge $v_1v_r \in E(H)$ be chosen such that r is largest possible, i.e. P together with v_1v_r forms a $(v_{19}, 19, r)$-lasso with the longest possible cycle $C = v_1v_2 \dots v_rv_1$. If $r = 19$, then C is a hamiltonian cycle and every vertex is the endvertex of some hamiltonian path. This implies that H is cubic, which is impossible, because the order of H is odd. By Lemma 2.15, r is not divisible by 3. We consider different cases.

Case 1 $r = 17$.

Clearly, $d_H(v_{19}) = 3$. This implies that also v_{18} is an endvertex of a hamiltonian path and hence $d_H(v_{18}) = 3$. By Lemma 2.15 and since H has no hamiltonian cycle, the distance on C of neighbors of v_{18} and v_{19} on C is larger than 2 and not equivalent to 0 or 2 modulo 3. Since v_{18} is adjacent to v_{17}, v_{19} has two neighbors in $\{v_4, v_7, v_{10}, v_{13}\}$. If v_4 and v_{13} are the neighbors of v_{19}, then Lemma 2.15 and the maximality of r imply a contradiction to $d_H(v_{18}) = 3$. By symmetry, this yields two possible cases.

(1.1) *v_{18} is adjacent to v_{17} and v_3 and v_{19} is adjacent to v_7 and v_{13}.*

(1.2) *v_{18} is adjacent to v_{17} and v_3 and v_{19} is adjacent to v_7 and v_{10}.*

In both cases, v_1 is an endvertex of a hamiltonian path in H and has a third neighbor on C. By Lemma 2.15, v_1 is not adjacent to v_6, v_9, v_{12}, or v_{15}. By Lemma 2.15 applied to the path $v_2v_1v_{17}v_{16} \dots v_3v_{18}v_{19}$, v_1 is not adjacent to v_4, v_{10}, v_{13}, or v_{16}. By symmetry between v_1 and v_2, the third neighbor of v_2 on C is in $\{v_6, v_9, v_{12}, v_{15}\}$. If $v_1v_8 \in E(H)$, then $v_1v_2 \dots v_7v_{19}v_{18}v_{17} \dots v_8v_1$ is a hamiltonian cycle, which is a contradiction. Altogether, the third neighbor of v_1 on C is in $\{v_5, v_{11}, v_{14}\}$.

Now we consider the two cases identified above. First, we consider Case (1.1). If $v_1v_{14} \in E(H)$, then $v_1v_2 \dots v_{13}v_{19}v_{18}v_{17} \dots v_{14}v_1$ is a hamiltonian cycle, which is a contradiction. Hence, v_1 has a neighbor in $\{v_5, v_{11}\}$ and, by symmetry, v_2 has a neighbor in $\{v_9, v_{15}\}$. Now $(\{v_{18}, v_{19}, v_5, v_9, v_{11}, v_{15}\}, \{v_3, v_4, v_7, v_{10}, v_{13}, v_{16}, v_{17}\})$ is a dominating pair in H, which is a contradiction.

Next, we consider Case (1.2). If $v_1v_{11} \in E(H)$, then $v_1v_2 \dots v_{10}v_{19}v_{18}v_{17} \dots v_{11}v_1$ is a hamiltonian cycle, which is a contradiction. Hence, v_1 is adjacent to either v_5 or v_{14}. By the symmetry between v_1 and v_9, v_9 is adjacent to either v_5 or v_{13}. If $v_2v_6 \in E(H)$, then $v_2v_1v_{17}v_{16} \dots v_7v_{19}v_{18}v_3v_4v_5v_6v_2$ is a hamiltonian cycle, which is a contradiction. Since v_2 is not adjacent to v_9, v_2 is adjacent to either v_{12} or v_{15}. By the symmetry between v_2 and v_8, v_8 is adjacent to either v_{12} or v_{15}. If $v_1v_5 \in E(H)$, then $(\{v_{19}, v_{18}, v_9, v_5, v_{12}, v_{15}\}, \{v_3, v_4, v_7, v_{10}, v_{13}, v_{16}, v_{17}\})$ is a dominating pair in H. Thus, we may assume that $v_1v_{14} \in E(H)$. By symmetry, we also may assume that $v_9v_{13} \in E(H)$. Now $v_1v_2 \dots v_9v_{13}v_{12}v_{11}v_{10}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_1$ is a hamiltonian cycle, which is a contradiction. This concludes Case 1.

Case 2 $r = 16$.

By the maximality of r, v_{19} is not adjacent to v_1, v_2, v_3, v_{15}, v_{14}, or v_{13}. By Lemma 2.15, v_{19} is not adjacent to v_{17}, v_{11}, v_8, or v_5. By the maximality of r, v_{19} is adjacent to two non-consecutive vertices on C. Since v_{19} has neighbors on C, v_{17} is the endvertex of a hamiltonian path and hence $d_H(v_{17}) = 3$. Let v_x be the neighbor of v_{17} on C distinct from v_{16}. By symmetry, we may assume that $8 \leq x \leq 14$.

If $x = 12$, then, by Lemma 2.15, v_{19} is not adjacent to v_{12}, v_{10}, v_7, or v_4, thus $v_{19}v_9 \in E(H)$. Now H contains a $(v_{10}, 19, 17)$-lasso with the cycle $v_1v_2 \dots v_9v_{19}v_{18}v_{17}v_{12} \dots v_{16}v_1$ contradicting the maximality of r. Hence $x \neq 12$.

If the two neighbors of v_{19} on C are v_6 and v_{10}, then, by Lemma 2.15, either $v_{17}v_9 \in E(H)$ or $v_{17}v_{13} \in E(H)$. If $v_{17}v_9 \in E(H)$, then H has hamiltonian cycle, and if $v_{17}v_{13} \in E(H)$, then has a $(v_{12}, 19, 17)$-lasso contradicting the maximality of r. Hence, the set of the two neighbors of v_{19} on C is not $\{v_6, v_{10}\}$.

Claim 4 *The distance on C between some neighbor of v_{19} on C and some neighbor of v_{17} on C equals 4.*

Proof of Claim 4: For contradiction, we assume that v_4 and v_{12} are not neighbors of v_{19}. Now some neighbor v_y of v_{19} belongs to $\{v_9, v_{10}\}$. By the maximality of r, $x \geq 13$. In order to avoid distance 4 from v_y on C, we need $x = 14$ and $y = 9$. But this contradicts Lemma 2.15 applied to the path $v_{19}v_{18}v_{17}v_{14}v_{13} \dots v_1v_{16}v_{15}$. Thus the claim holds. □

By Claim 4, we may assume that v_{19} is adjacent to v_4. By the maximality of r, the distance on C between any neighbor of v_{19} on C and any neighbor of v_{17} on C is at least 4. By Lemma 2.15, this distance is not equivalent to 2 modulo 3. By these properties and by symmetry, it is sufficient to consider the following cases.

(2.1) *v_{17} is adjacent to v_{16} and v_8, and v_{19} is adjacent to v_4 and v_{12}.*

(2.2) *v_{17} is adjacent to v_{16} and v_{13}, and v_{19} is adjacent to v_4 and v_6.*

(2.3) *v_{17} is adjacent to v_{16} and v_{10}, and v_{19} is adjacent to v_4 and v_6.*

(2.4) *v_{17} is adjacent to v_{16} and v_{13}, and v_{19} is adjacent to v_4 and v_7.*

(2.5) *v_{17} is adjacent to v_{16} and v_{11}, and v_{19} is adjacent to v_4 and v_7.*

In Case (2.1), $(\{v_2, v_6, v_{10}, v_{14}, v_{17}, v_{19}\}, \{v_1, v_4, v_7, v_9, v_{12}, v_{15}, v_{18}\})$ is a dominating pair of H, which is a contradiction.

In Case (2.2), we consider the $(v_3, 19, 16)$-lasso with cycle $v_{16}v_{15} \dots v_4v_{19}v_{18}v_{17}v_{16}$. (Here the vertices v_1, v_2, and v_3 play the roles of v_{17}, v_{18}, and v_{19}, respectively.) By Lemma 2.15 and the maximality of r, v_3 is adjacent to one of v_7, v_9, v_{10}, or v_{12}. If $v_3v_{12} \in E(H)$, then, by the maximality of r, v_1 must be adjacent to v_8 and we obtain Case 2.1. If $v_3v_{10} \in E(H)$, then H contains a $(v_{11}, 19, 17)$-lasso with cycle $v_{17}v_{18}v_{19}v_4v_5 \dots v_{10}v_3v_2v_1v_{16}v_{15} \dots v_{13}v_{17}$ contradicting the maximality of r. If $v_3v_9 \in E(H)$, then by Lemma 2.15 and the maximality of r, v_1 has no possible third neighbor. If $v_3v_7 \in E(H)$, then H contains a $(v_1, 19, 17)$-lasso with cycle $v_7v_8 \dots v_{16}v_{17}v_{18}v_{19}v_6v_5v_4v_3v_7$ contradicting the maximality of r.

In Case (2.3), we consider the $(v_7, 19, 16)$-lasso with cycle $v_{10}v_{11} \dots v_{16}v_1v_2 \dots v_6v_{19}v_{18}v_{17}v_{10}$. The vertex v_7 is adjacent to v_6 and, by Lemma 2.15 and the maximality of r, v_7 is adjacent to one of v_1, v_3, or v_{14}. If $v_7v_{14} \in E(H)$, then, by the maximality of r, v_9 must be adjacent to v_2 and we obtain Case 2.1. If $v_7v_1 \in E(H)$, then $v_{11}v_{12} \dots v_{19}v_6v_5 \dots v_1v_7v_8 \dots v_{11}$ is a hamiltonian cycle, which is a contradiction. If $v_7v_3 \in E(H)$, then H contains a $(v_2, 19, 17)$-lasso with cycle $v_{11}v_{12} \dots v_{19}v_6v_5 \dots v_3v_7v_8 \dots v_{11}$ contradicting the maximality of r.

In Case (2.4), we consider the $(v_3, 19, 16)$-lasso with cycle $v_{16}v_{15} \dots v_4v_{19}v_{18}v_{17}v_{16}$. Now, by Lemma 2.15 and the maximality of r, v_3 is adjacent to one of v_6, v_9, v_{10}, or v_{12}. If v_3 is adjacent to one of v_9, v_{10}, or v_{12}, we argue as in Case (2.2). If $v_3v_6 \in E(H)$, then H contains a $(v_{17}, 19, 17)$-lasso with cycle $v_6v_5v_4v_{19}v_7v_8 \dots v_{16}v_1v_2v_3v_6$ contradicting the maximality of r.

In Case (2.5), we consider the $(v_3, 19, 16)$-lasso with cycle $v_{16}v_{15} \dots v_4v_{19}v_{18}v_{17}v_{16}$. Now, by Lemma 2.15 and the maximality of r, v_3 is adjacent to one of v_6, v_9, v_{10}, or v_{12}. If $v_3v_{12} \in E(H)$, then by the maximality of r, v_1 must be adjacent to v_8 and we obtain Case 2.1. If $v_3v_{10} \in E(H)$, then $v_{10}v_9 \dots v_4v_{19}v_{18}v_{17}v_{11}v_{12} \dots v_{16}v_1v_2v_3v_{10}$ is a hamiltonian cycle, which is a contradiction. If $v_3v_9 \in E(H)$, then H contains a $(v_{10}, 19, 18)$-lasso with cycle $v_9v_8 \dots v_4v_{19}v_{18}v_{17}v_{11}v_{12} \dots v_{16}v_1v_2v_3v_9$ contradicting the maximality of r. If $v_3v_6 \in E(H)$, then H contains a $(v_{17}, 19, 17)$-lasso with cycle $v_6v_5v_4v_{19}v_7v_8 \dots v_{16}v_1v_2v_3v_6$ contradicting the maximality of r. This concludes Case 2.

Case 3 $r = 14$.

Let $H' = H[\{v_{15}, v_{16}, \dots, v_{19}\}]$. If H' has a $(V(H') - v_{15}, V(H'))$-dominating pair (D_1', D_2') of cardinality at most 3, then $(D_1' \cup \{v_2, v_5, v_8, v_{11}, v_{14}\}, D_2' \cup \{v_1, v_4, v_7, v_{10}, v_{13}\})$ is a dominating pair of H, which is a contradiction. Hence, by Lemma 2.14 applied H', we may assume that v_{19} has a neighbor y in $V(C)$. Since G' has no lasso with a cycle of order more than 14, $y \in \{v_6, v_7, v_8\}$. By Lemma 2.15, $y = v_7$. Since $d_G(v_{19}) = 3$, H' must contain a neighbor v_i of v_{19} distinct from v_{18}. Now v_{i+1} is v_{15}-distant in H', which, by symmetry with v_{19}, implies the contradiction $v_{i+1}v_7 \in E(H)$. This concludes Case 3.

Case 4 $r = 13$.

Again, let $H' = H[\{v_{15}, v_{16}, \dots, v_{19}\}]$. If H' has a $(V(H') - v_{15}, V(H'))$-dominating pair (D_1', D_2') of cardinality at most 3, then $(D_1' \cup \{v_2, v_5, v_8, v_{11}, v_{14}\}, D_2' \cup \{v_1, v_4, v_7, v_{10}, v_{13}\})$ is a dominating pair of H, which is a contradiction. Hence, by Lemma 2.14 applied H', we may assume that v_{19} has a neighbor y not in $V(H')$. By Lemma 2.15, $y \neq v_{14}$, i.e. $y \in V(C)$. This implies a contradiction to the maximality of r, which concludes Case 4.

Case 5 $4 \leq r \leq 11$.

Recall that r is not divisible by 3. Let H' be the subgraph of H induced by the set $\{v_1, v_2, \ldots, v_r\}$. By the maximality of r, no v_r-distant vertex of H' has a neighbor outside of $V(H')$. If $r = 4$, then $v_1v_3 \in E(H)$, which contradicts Lemma 2.15. The remaining cases $r = 11$, 10, 8, 7, and 5, follow easily from Lemmas 2.20, 2.19, 2.18, 2.17, and 2.14, respectively. This concludes the proof of Lemma 2.23. $\Box$

Lemma 2.24 *If H is a subcubic graph of order 16 with a hamiltonian path such that every endvertex of every hamiltonian path of H has degree 3, then $\gamma\gamma(H) \leq 11$.*

Proof: For contradiction, we assume $\gamma\gamma(H) > 11$. Let a hamiltonian path $P = v_1v_2 \ldots v_{16}$ of H and an edge $v_1v_r \in E(H)$ be chosen such that r is largest possible, i.e. P together with v_1v_r forms a $(v_{16}, 16, r)$-lasso with the longest possible cycle $C = v_1v_2 \ldots v_rv_1$.

By Lemma 2.22, if H has a hamiltonian cycle, then some vertex has degree less than 3. Since in this case every vertex is the endvertex of some hamiltonian path, this contradicts the assumption. Hence $r \leq 15$. By Lemma 2.15, r is not divisible by 3. We consider different cases.

Case 1 $r = 14$.

As endvertices of hamiltonian paths, both of v_{16} and v_{15} have two neighbors in $V(C)$. By Lemma 2.15 and the maximality of r, v_{16} is adjacent to two vertices among v_4, v_7, and v_{10}. If v_{16} is adjacent to v_4 and v_{10}, then the second neighbor of v_{15} implies a contradiction to Lemma 2.15 or the maximality of r. Hence, we may assume that v_{16} is adjacent to v_7 and v_{10}. By Lemma 2.15 and the maximality of r, this implies that v_{15} is adjacent to v_3 and v_{14}.

By Lemma 2.15 applied to P and the hamiltonian path $v_2v_1v_{14}v_{13} \ldots v_3v_{15}v_{16}$, v_1 is adjacent to a vertex among v_5, v_8, and v_{11}. If $v_1v_8 \in E(H)$, then $v_1v_2 \ldots v_7v_{16}v_{15} \ldots v_8v_1$ is a hamiltonian cycle, which is a contradiction. If $v_1v_{11} \in E(H)$, $v_1v_2 \ldots v_{10}v_{16}v_{15} \ldots v_{11}v_1$ is a hamiltonian cycle, which is a contradiction. Hence $v_1v_5 \in E(H)$. By the symmetry between v_1 and v_9, we obtain $v_9v_5 \in E(H)$, which contradicts the assumption that H is subcubic. This concludes Case 1.

Case 2 $r = 13$.

As endvertices of hamiltonian paths, both of v_{16} and v_{14} have two neighbors in $V(C)$. By Lemma 2.15 and the maximality of r, v_{16} is adjacent to two vertices among v_4, v_6, v_7, and v_9. By the maximality of r, v_{16} is not adjacent to v_6 and v_7. By symmetry, we may assume that v_{14} is adjacent to v_x for some $7 \leq x \leq 11$. This implies, by the maximality of r, that v_{16} is not adjacent to v_9 and v_{16} is adjacent to v_4. If $v_{16}v_7 \in E(H)$, then, by the maximality of r, v_{14} is adjacent to v_{11}. If $v_{14}v_{11} \in E(H)$, then, by Lemma 2.15, v_{16} is not adjacent to v_6. Altogether, if v_{16} is adjacent to v_6, then v_{14} is adjacent to v_{10}, and if v_{16} is adjacent to v_7, then v_{14} is adjacent to v_{11}. By symmetry, it suffices to consider the case that v_{16} is adjacent to v_4 and v_7, and v_{14} is adjacent to v_{11} and v_{13}.

We consider the $(v_3, 16, 13)$-lasso with cycle $v_{13}v_{12}\dots v_4v_{16}v_{15}v_{14}v_{13}$. Now, by Lemma 2.15 and the maximality of r, v_3 is adjacent to one of v_6 or v_9. If v_3 is adjacent to v_6, then H contains a $(v_{15}, 16, 14)$-lasso with cycle $v_{13}v_{12}\dots v_7v_{16}v_4v_5v_6v_3v_2v_1v_{13}$, which is a contradiction. If v_3 is adjacent to v_9, then H contains a $(v_{10}, 16, 15)$-lasso with cycle $v_{11}v_{14}v_{15}v_{16}v_4v_5\dots v_9v_3v_2v_1v_{13}v_{12}v_{11}$, which is a contradiction. This concludes Case 2.

Case 3 $4 \le r \le 11$.

The proof repeats the argument of Case 5 of Lemma 2.23. □

Lemma 2.25 *If H is a subcubic graph such that H has a hamiltonian path, $n(H) \le 13$, $n(H) \equiv 1$ mod 3, and every endvertex of every hamiltonian path of H has degree 3, then $\gamma\gamma(H) \le \frac{2n(H)+1}{3}$.*

Proof: For contradiction, we assume $\gamma\gamma(H) > \frac{2n(H)+1}{3}$. If $n(H) = 4$, then every vertex of H is an endvertex of some hamiltonian path of H. Hence, H is complete, which implies the contradiction $\gamma\gamma(H) = 2$. Hence $n(H) \ne 4$. If $n(H) = 7$, then Lemma 2.15 implies that v_1 is adjacent to two vertices among v_4, v_5, and v_7. If $v_1v_5 \in E(H)$, then Lemma 2.14 easily implies a contradiction. Hence $v_1v_7 \in E(H)$ and, by symmetry, $v_4v_7 \in E(H)$, which is a contradiction. Hence $n(H) \ne 7$. Let a hamiltonian path $P = v_1v_2\dots v_{n(H)}$ of H and an edge $v_1v_r \in E(H)$ be chosen such that r is largest possible, i.e. P together with v_1v_r forms a $(v_{n(H)}, n(H), r)$-lasso with the longest possible cycle $C = v_1v_2\dots v_rv_1$.

First, we assume that $n(H) = 13$. Since H has no hamiltonian cycle, this implies $r \le 12$. By Lemma 2.15, r is not divisible by 3. If $r = 4$, then $v_1v_3 \in E(H)$, which contradicts Lemma 2.15. The remaining cases $r = 11$, 10, 8, 7, and 5 follow from Lemmas 2.20, 2.19, 2.18, 2.17, and 2.14, respectively.

Finally, we assume $n(H) = 10$. By Lemma 2.21, H has no hamiltonian cycle, which implies $r \le 9$. Again, r is not divisible by 3 and $r \ne 4$. The remaining cases 8, 7, and 5 follow from Lemmas 2.18, 2.17, and 2.14, respectively. This completes the proof. □

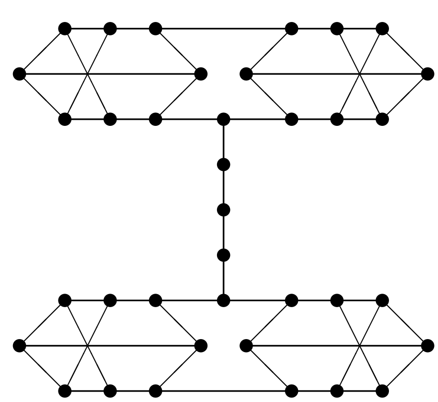


Figure 2.10

In view of possible improvements of Theorem 2.10 it is natural to ask whether Lemma 2.12 is valid for larger orders. The graph in Figure 2.10 shows that there are limits to such improvements. The illustrated graph G is subcubic, has order $n(G) = 37 \equiv 1 \mod 3$, every endvertex of every hamiltonian path of G has degree 3, but $\gamma\gamma(G) > 25$ (Claim 3 in [44] easily implies $\gamma(G) \geq 13$).

We close this chapter with the following bold conjecture.

Conjecture 2.26 *If*

$$\begin{aligned}\gamma &:= \limsup_{n\to\infty} \left\{ \frac{\gamma(G)}{n(G)} \mid G \textit{ is a connected cubic graph of order } n(G) \geq n \right\} \textit{ and} \\ \gamma\gamma &:= \limsup_{n\to\infty} \left\{ \frac{\gamma\gamma(G)}{n(G)} \mid G \textit{ is a connected cubic graph of order } n(G) \geq n \right\},\end{aligned}$$

then $\gamma\gamma = 2\gamma$.

State of the art is $\frac{7}{20} \leq \gamma \leq \frac{4}{11}$ [43, 45].

Chapter 3

Partition Problems Related to $\gamma\gamma_t(G)$

As mentioned in Chapter 2 the simple yet fundamental Observation 2.1 implies that every graph of minimum degree at least one contains two disjoint dominating sets, i.e., the trivial necessary minimum degree condition for the existence of two disjoint dominating sets is also sufficient. In contrast to that, Zelinka [76] observed that no minimum degree condition is sufficient for the existence of two disjoint total dominating sets. For that consider a bipartite graph G with one partite set A containing $2\delta - 1$ vertices and a second partite set B containing $\binom{2\delta-1}{\delta}$ vertices each of which is adjacent to a different set of δ vertices from A. Clearly, this graph has minimum degree δ. If $T_1 \cup T_2$ is a partition of $A \cup B$ such that $|T_1 \cap A| \geq |T_2 \cap A|$, then $|T_1 \cap A| \geq \delta$. Hence, there is a vertex $v \in B$ such that $N_G(v) \subseteq T_1$ and so G does not contain two disjoint total dominating sets.

Clearly, if the domatic number [77] of a graph G is at least $2k$, then, by definition, G contains $2k$ disjoint dominating sets and hence also k disjoint total dominating sets. Therefore, the results of Calkin and Dankelmann [10] and Feige, Halldórsson, Kortsarz, and Srinivasan [20] imply that a sufficiently large minimum degree and a sufficiently small maximum degree together imply the existence of arbitrarily many disjoint (total) dominating sets.

In [37] Henning and Southey give an elegant exchange argument for the following result, which is somehow located between Ore's positive and Zelinka's negative observation. By a C_5-component we mean a component that is a C_5.

Theorem 3.1 (Henning and Southey [37]) *If G is a graph of minimum degree at least 2 with no C_5-component, then $V(G)$ can be partitioned into a dominating set D and a total dominating set T.*

A characterization of graphs with disjoint dominating and total dominating sets is given in [38].

A DT-pair (D, T) of G is *exhaustive* if $|D| + |T| = |V(G)|$. Thus, a DT-pair (D, T) of G is non-exhaustive if $|D| + |T| < |V(G)|$. Note that Theorem 3.1 implies that every graph with minimum degree at least 2 and with no C_5-component has an exhaustive DT-pair.

We call a DT-pair (D, T) whose union $D \cup T$ has cardinality $\gamma\gamma_t(G)$ a $\gamma\gamma_t(G)$-pair. By Theorem 3.1, $\gamma\gamma_t(G)$ exists for every graph G with minimum degree at least 2 and with no C_5-component. Hence, we have the following immediate consequence of Theorem 3.1.

Corollary 3.2 *If G is a graph with minimum degree at least 2 with no C_5-component, then $\gamma\gamma_t(G) \leq |V(G)|$.*

In Chapter 5 we show that it is NP-complete to decide for a given graph G and a given integer k, whether $\gamma\gamma_t(G) \leq k$. In this chapter, we study graphs that achieve equality in the upper bound in Corollary 3.2. A characterization of such graphs seems difficult to obtain, since there are several families each containing infinitely many graphs that achieve equality in Corollary 3.2. For example, consider the following three families of connected graphs with minimum degree at least 2 for which every DT-pair is exhaustive.

- **The Family $\mathcal{D}_1$:** For $k \geq 0$, we define $\mathcal{D}_1(k)$ to be the connected graph obtained from two disjoint 5-cycles by joining a vertex from one of the cycles to a vertex in the other and subdividing the resulting edge k times. Let $\mathcal{D}_1 = \{\mathcal{D}_1(k) : k \geq 0\}$. The family $\mathcal{D}_1$ is depicted in Figure 3.1(a). We remark that a graph in the family $\mathcal{D}_1$ is called a *dumb-bell* in the literature.

- **The Family $\mathcal{D}_2$:** For $k \geq 0$ and $\ell \geq 0$ with $k + \ell \geq 2$, let $\mathcal{D}_2(k, \ell)$ be the connected graph that is constructed from $k+\ell$ disjoint 5-cycles by identifying a set of k vertices, one from each of k cycles, into one vertex u and joining a vertex from each of the remaining ℓ cycles by a path of length 2 to u. Let $\mathcal{D}_2 = \{\mathcal{D}_2(k, \ell) : k, l \geq 0 \text{ and } k + \ell \geq 2\}$. The family $\mathcal{D}_2$ is depicted in Figure 3.1(b).

- **The Family $\mathcal{D}_3$:** For $k \geq 1$ and $\ell \geq 1$, let $\mathcal{D}_3(k, \ell)$ be the connected graph that is constructed from $k + \ell$ disjoint 5-cycles by identifying a set of k vertices, one from each of k cycles, into one vertex u and identifying a set of ℓ vertices, one from each of the remaining ℓ cycles, into one vertex v and then adding a path of length 2 joining u and v. Let $\mathcal{D}_3 = \{\mathcal{D}_3(k) : k \geq 1 \text{ and } \ell \geq 1\}$. The family $\mathcal{D}_3$ is depicted in Figure 3.1(c).

It is a routine exercise to check that if $G \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, then $\gamma\gamma_t(G) = |V(G)|$. However, all these graphs G contain induced 5-cycles. Several further graphs G that contain induced 5-cycles and satisfy $\gamma\gamma_t(G) = |V(G)|$ can readily be constructed. These families suggest that a full characterization of all graphs that achieve equality in Corollary 3.2 seems difficult to obtain. In Section 3.1, we therefore restrict our attention to graphs with no induced cycle on five vertices. The results in Section 3.1 are based on [36]. In Section 3.2, we restrict our attention to graphs of minimum degree at least 3, which may have induced cycles on five vertices. The results in Section 3.2 are based on [34].

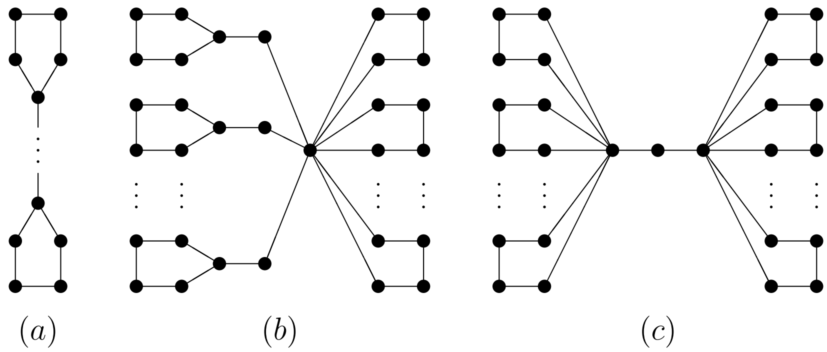


Figure 3.1: Graphs containing no non-exhaustive DT-pairs.

3.1 C_5-free Graphs with Minimum Degree at Least 2

We say that a graph is *F-free* if it does not contain F as an induced subgraph. For a graph G and some $i \in \mathbb{N}$, let $V_i(G) = \{u \in V(G) \mid d_G(u) = i\}$ and $V_{\geq i}(G) = \{u \in V(G) \mid d_G(u) \geq i\}$. The graph obtained from a complete graph K_n of order $n \geq 4$ by subdividing every edge once is denoted by K_n^*. Note that $|V(K_n^*)| = |V(K_n)| + |E(K_n)| = n + \binom{n}{2}$. We define the families $\mathcal{C}$ and $\mathcal{K}^*$ of particular cycles and subdivided complete graphs as follows:

$$\mathcal{C} \;=\; \{C_n :\, n \geq 3 \text{ and } n \neq 5\} \quad \text{and} \quad \mathcal{K}^* = \{K_n^* :\, n \geq 4\}.$$

As our main result in this section we prove the following.

Theorem 3.3 *If G is a connected C_5-free graph with $\delta(G) \geq 2$, then $\gamma\gamma_t(G) = |V(G)|$ if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

We will refer to a graph G as an $n(G)$-*minimal graph* if G is edge-minimal with respect to satisfying the following three conditions:

(i) $\delta(G) \geq 2$,

(ii) G is connected, and

(iii) $\gamma\gamma_t(G) = n(G)$.

Note that if G is an $n(G)$-minimal graph and H is a graph with $\delta(H) \geq 2$ and no C_5-component that arises from G by deleting edges, then, by Corollary 3.2, $n(G) = \gamma\gamma_t(G) \leq \gamma\gamma_t(H) \leq n(H) = n(G)$, i.e. $\gamma\gamma_t(H) = n(G)$.

The following result characterizes n-minimal C_5-free graphs and is a main step towards the proof of Theorem 3.3.

Theorem 3.4 *If G is a C_5-free graph, then G is $n(G)$-minimal if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

We note that every graph $G \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ is an $n(G)$-minimal graph but, as remarked earlier, such graphs are not C_5-free. We shall proceed as follows. First, we prove a number of useful preliminary results in Subsection 3.1.1. Then, we prove Theorem 3.4 in Subsection 3.1.2 and Theorem 3.3 in Subsection 3.1.3.

3.1.1 Preliminary Results

In this subsection we present several useful preliminary results

Lemma 3.5 *If G is a graph, (D,T) is a DT-pair of G, and u is a vertex in G such that all neighbors of u are of degree at most 2, then $u \in D \cup T$.*

In particular, $\gamma\gamma_t(C_n) = n$ for $n \neq 5$.

Proof: Let G, (D,T), and u be as in the statement. For contradiction, we assume $u \notin D \cup T$. Let v be a neighbor of u with $v \in T$. Since v has degree at most 2, it has either no neighbor in D or no neighbor in T, which is a contradiction and implies the desired statement. □

Lemma 3.6 *If $G \in \mathcal{K}^*$ and (D,T) is a DT-pair of G, then $|D| + |T| = |V(G)|$.*

Proof: Let $G \in \mathcal{K}^*$. By definition, G may be obtained from the complete graph K_ℓ, for some $\ell \geq 4$, by subdividing every edge exactly once. By Theorem 3.1, there exists a DT-pair (D,T) of G. If there are two vertices in $V_{\geq 3}(G)$ that do not belong to T, then the vertex in $V_2(G)$ with these two vertices as its neighbors is not totally dominated by T, a contradiction. Hence, T contains all vertices in $V_{\geq 3}(G)$, except possibly one. If $V_{\geq 3}(G) \subseteq T$, then, since every vertex of degree 2 is dominated by D, we have that $V_2(G) \subseteq D$. But then no vertex in $V_{\geq 3}(G)$ is totally dominated by T, a contradiction. Hence, exactly one vertex, v say, in $V_{\geq 3}(G)$ is not in T. Since every vertex in $V_2(G) \setminus N_G(v)$ has both its neighbors in T, and since $V_2(G) \setminus N_G(v)$ is dominated by D, we have that $V_2(G) \setminus N_G(v) \subseteq D$. Furthermore, in order for T to totally dominate $V_{\geq 3}(G) \setminus \{v\}$ we have that $N_G(v) \subseteq T$. But then $v \in D$ in order for the set D to dominate $N_G(v)$. Thus, $D = (V_2(G) \setminus N_G(v)) \cup \{v\}$ and $T = (V_{\geq 3}(G) \setminus \{v\}) \cup N_G(v)$, and so $|D| + |T| = |V_2(G)| + |V_{\geq 3}(G)| = |V(G)|$, as desired. □

The following observation follows from the proofs of Lemmas 3.5 and 3.6.

Observation 3.7 *If $G \in \mathcal{C} \cup \mathcal{K}^*$ and $v \in V(G)$, then G has the following properties.*

(a) *There exist DT-pairs (D_1,T_1) and (D_2,T_2) with $v \in D_1$ and with $v \in T_2$.*

(b) *If $G \in \mathcal{C}$ and $uv \in E(G)$ then there exist DT-pairs (D_1,T_1) and (D_2,T_2) with $\{u,v\} \subseteq T_1$ and with $u \in D_2$ and $v \in T_2$.*

(c) *If $G \in \mathcal{K}^*$ and $v \in V_{\geq 3}(G)$, then there exists a DT-pair (D,T) with $v \in D$ and $N_G(v) \subseteq T$. Furthermore, every vertex in $V_{\geq 3}(G) \setminus \{v\}$ belongs to T and has exactly one neighbor in T with the remaining neighbors all in D.*

Lemma 3.8 *If $G = C_n$, where $n \neq 5$, and $v \in V(G)$, then there exists a pair (D, T) of disjoint sets of vertices in G such that $|D| + |T| < n$, $v \in T$, and*

(i) either D dominates $V(G)$ and T totally dominates $V(G) \setminus \{v\}$,

(ii) or D dominates $V(G) \setminus \{v\}$ and T totally dominates $V(G)$.

Proof: Let G be the cycle $v_1v_2 \dots v_nv_1$, where $n \neq 5$ and $v = v_1$. If $n = 3$, let $D = \{v_2\}$ and $T = \{v_1\}$, while if $n = 4$, let $D = \{v_3\}$ and $T = \{v_1, v_2\}$. If $n \geq 6$ and $n \equiv 0 \pmod 3$, let $v_i \in D$ if $i \equiv 0 \pmod 3$ and let $v_i \in T$ if $i \equiv 1, 2 \pmod 3$ and $i \neq 2$. If $n \geq 6$ and $n \equiv 1 \pmod 3$, let $v_i \in D$ if $i \equiv 0 \pmod 3$ and let $v_i \in T$ if $i \equiv 1, 2 \pmod 3$ and $i \notin \{2\}$. If $n \geq 6$ and $n \equiv 2 \pmod 3$, let $v_i \in D$ if $i \equiv 0 \pmod 3$ and let $v_i \in T$ if $i \equiv 1, 2 \pmod 3$ and $i \notin \{2, n-1\}$, and let $v_{n-1} \in D$. In all cases, the pair (D, T) satisfies the requirements of the lemma. □

Lemma 3.9 *Let $F \neq C_5$ be a connected graph with $\delta(F) \geq 2$ and let G be obtained from F by subdividing an edge of F three times. If $\gamma\gamma_t(G) = |V(G)|$, then $\gamma\gamma_t(F) = |V(F)|$.*

Proof: We use a proof by contrapositive. Suppose that $\gamma\gamma_t(F) < |V(F)|$. We show that $\gamma\gamma_t(G) < |V(G)|$. Let (D_F, T_F) be a $\gamma\gamma_t(F)$-pair of F. We have $|D_F| + |T_F| = \gamma\gamma_t(F) < |V(F)|$. Let $e = uv$ be the edge of F that is subdivided three times to produce the path $uv_1v_2v_3v$ in G. Note that u and v are not adjacent in G.

Suppose that $T_F \cap \{u, v\} \neq \emptyset$. Renaming vertices, if necessary, we may assume that $u \in T_F$. If $v \in T_F$, let $D = D_F \cup \{v_2\}$ and let $T = T_F \cup \{v_1, v_3\}$. If $v \in D_F$, let $D = D_F \cup \{v_1\}$ and let $T = T_F \cup \{v_2, v_3\}$. If $v \notin D_F \cup T_F$, let $D = D_F \cup \{v_2\}$ and let $T = T_F \cup \{v, v_3\}$. Then, (D, T) is a DT-pair of G with $|D| + |T| = |D_F| + |T_F| + 3 < |V(F)| + 3 = |V(G)|$. Hence, $\gamma\gamma_t(G) < |V(G)|$, as desired. Thus, we may assume that $T_F \cap \{u, v\} = \emptyset$.

Suppose that $D_F \cap \{u, v\} \neq \emptyset$. Renaming vertices, if necessary, we may assume that $u \in D_F$. In this case, let $D = D_F \cup \{v_3\}$ and let $T = T_F \cup \{v_1, v_2\}$, and once again (D, T) is a DT-pair of G with $|D| + |T| < |V(G)|$.

Thus, we may assume that $D_F \cap \{u, v\} = \emptyset$. Now, $|D_F| + |T_F| \leq |V(F)| - 2$. We note that each of u and v is adjacent to a vertex in D_F and to a vertex in T_F. We now let $D = D_F \cup \{v, v_1\}$ and let $T = T_F \cup \{v_2, v_3\}$. Then, (D, T) is a DT-pair of G with $|D| + |T| = |D_F| + |T_F| + 4 \leq |V(F)| + 2 < |V(G)|$. Hence, $\gamma\gamma_t(G) < |V(G)|$. □

We remark that the converse of Lemma 3.9 is not necessarily true (cf. for instance the graphs in Figure 3.1).

Lemma 3.10 *Let G be the graph obtained from $k \geq 2$ disjoint cycles $F_1, F_2, \dots, F_k$ of lengths $n_1, n_2, \dots, n_k$, respectively, by identifying a set of k vertices, one from each cycle, into one vertex called v. If $n_i \neq 5$ for $i = 1, 2, \dots, k$, then G has a non-exhaustive DT-pair.*

Proof: Let G be the graph defined in the statement of the lemma. For $i \in \{1, 2, \ldots, k\}$, let v_i be the vertex of F_i that was identified into the vertex v. Let (D_1, T_1) be a pair of disjoint sets of vertices in F_1 that satisfies the requirements of Lemma 3.8 for the graph F_1 with v_1 as the specified vertex in the cycle. Then, $v_1 \in T_1$, $|D_1| + |T_1| < n_1$, and either (i) D_1 dominates $V(F_1)$ and T_1 totally dominates $V(F_1) \setminus \{v_1\}$ or (ii) D_1 dominates $V(F_1) \setminus \{v_1\}$ and T_1 totally dominates $V(F_1)$. For each $i \in \{2, \ldots, k\}$, $F_i \in \mathcal{C}$ and hence, by Observation 3.7(a), there exists a DT-pair (D_i, T_i) in F_i such that $v_i \in T_i$. Let

$$D = \bigcup_{i=1}^{k} D_i \quad \text{and} \quad T = \left(\bigcup_{i=1}^{k} (T_i \setminus \{v_i\}) \right) \cup \{v\}.$$

Then, (D, T) is a non-exhaustive DT-pair of G. $\square$

Lemma 3.11 *Let G be a connected C_5-free graph with $\delta(G) \geq 2$ and $\gamma\gamma_t(G) = n(G)$. If G is not $n(G)$-minimal, then G contains an $n(G)$-minimal spanning C_5-free subgraph.*

Proof: Let G be as in the statement of the lemma such that G is not $n(G)$-minimal. By removing edges from G, we can obtain an $n(G)$-minimal spanning subgraph of G. Among all such subgraphs, choose F so that the number of induced 5-cycles in F is minimized. For the sake of contradiction, suppose that F contains the induced 5-cycle $C = v_1v_2v_3v_4v_5v_1$. If $n = 5$, then, since G is C_5-free, we may assume, relabeling vertices if necessary, that $v_1v_3 \in E(G)$. But then $(\{v_3, v_4\}, \{v_1, v_5\})$ is a non-exhaustive DT-pair in G, a contradiction. Hence, $n \neq 5$ and since F is connected, we may assume $d_F(v_1) \geq 3$. By the minimality of F, $d_F(v_2) = d_F(v_5) = 2$.

For the sake of contradiction, suppose that $d_F(v_3) \geq 3$. Then by the minimality of F, $d_F(v_4) = 2$. If $v_2v_4 \in E(G)$, then the graph obtained from F by adding this edge and removing the edge v_1v_2 is an $n(G)$-minimal spanning subgraph of G containing fewer induced 5-cycles than F, contradicting the choice of F. Hence, $v_2v_4 \notin E(G)$. Similarly, $v_2v_5 \notin E(G)$. If $v_1v_4 \in E(G)$, then the graph obtained from F by adding this edge and removing the edge v_3v_4 is an $n(G)$-minimal spanning subgraph of G with fewer induced 5-cycles than F, contradicting the choice of F. Hence, $v_1v_4 \notin E(G)$ and, by a similar argument, $v_3v_5 \notin E(G)$. If $v_1v_3 \in E(G)$, let $F' = F + v_1v_3$. By Theorem 3.1, there exists a DT-pair (D', T') in F'. To totally dominate v_2 we may assume, without loss of generality, that $v_1 \in T'$. If $v_3 \in D'$, then $((D' \setminus \{v_2, v_5\}) \cup \{v_4\}, (T' \setminus \{v_2, v_4\}) \cup \{v_5\})$ is a non-exhaustive DT-pair of F' and hence in G, a contradiction. Hence, $v_3 \in T'$. To dominate v_2, we therefore have that $v_2 \in D'$. But then $((D' \setminus \{v_4\}) \cup \{v_5\}, T' \setminus \{v_4, v_5\})$ is a non-exhaustive DT-pair of F' and hence in G, again a contradiction. Thus, $v_1v_3 \notin E$. Hence, C is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Therefore, $d_F(v_3) = 2$. Similarly, $d_F(v_4) = 2$.

If $v_2v_i \in E(G)$ for some $i \in \{4, 5\}$, then the graph obtained from F by adding this edge and removing the edge v_1v_2 is an $n(G)$-minimal spanning subgraph of G containing fewer induced 5-cycles than F, contradicting the choice of F. Hence, $v_2v_5 \notin E(G)$ and $v_2v_4 \notin E(G)$. By a similar argument, $v_3v_5 \notin E(G)$. If $v_1v_3 \in E(G)$, let $F' = F + v_1v_3$.

By Theorem 3.1, there exists a DT-pair (D', T') in F'. If $v_1 \in T'$, then $((D' \setminus \{v_2, v_5\}) \cup \{v_3, v_4\}, (T' \setminus \{v_2, v_3, v_4\}) \cup \{v_5\})$ is a non-exhaustive DT-pair of F' and hence in G, a contradiction. Hence, $v_1 \in D'$. But then $((D' \setminus \{v_2, v_3, v_4\}) \cup \{v_5\}, (T' \setminus \{v_2, v_5\}) \cup \{v_3, v_4\})$ is a non-exhaustive DT-pair of F' and hence in G, again a contradiction. Hence, $v_1v_3 \notin E(G)$. Similarly, $v_1v_4 \notin E(G)$. Thus, C is an induced 5-cycle in G, contradicting the fact that G is C_5-free. □

Lemma 3.12 *If $G \neq C_n$ is a C_5-free hamiltonian graph, then $\gamma\gamma_t(G) < n(G)$.*

Proof: Let $G \neq C_n$ be a C_5-free hamiltonian graph and let C be a hamiltonian cycle in G. Thus, every edge in $E(G) \setminus E(C)$ is a chord of C in G. Among all chords of C, let uv be chosen so that $k = d_C(u, v)$ is minimized. Since a chord of C is not an edge of C, we note that $k \geq 2$. Let $P = u_0u_1 \ldots u_k$ be a shortest u-v path in C, where $u = u_0$ and $v = u_k$, and let C' be the cycle $u_0u_1 \ldots u_ku_0$. By our choice of uv, C' is an induced cycle in G. If $k = 4$, then C' is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Hence, $C' \in \mathcal{C}$.

Let $v_0v_1 \ldots v_\ell$ be the v-u path in C not containing u_1, where $v = v_0$ and $u = v_\ell$. Thus, C is the cycle $u_0u_1 \ldots u_kv_1v_2 \ldots v_\ell$ and $n(G) = k + \ell$. Since $k = d_C(u, v)$, we note that $\ell \geq k \geq 2$. We now apply Observation 3.7(b) to the cycle $C' \in \mathcal{C}$ as follows. If $\ell \equiv 0, 1 \pmod 3$, let (D', T') be a DT-pair of C' such that $\{u, v\} = \{u_0, u_k\} \subseteq T'$, while if $\ell \equiv 2 \pmod 3$, let (D', T') be a DT-pair of C' such that $u = u_0 \in D'$ and $v = u_k \in T'$. Let $D'' = \{v_i \mid i \equiv 2 \pmod 3 \text{ and } 1 < i < \ell\}$ and let $T'' = \{v_i \mid i \equiv 0, 1 \pmod 3 \text{ and } 1 < i < \ell\}$. Let $D = D' \cup D''$ and let $T = T' \cup T''$. We note that $v_1 \notin D \cup T$ and that (D, T) is a DT-pair of $C + uv$. Hence, (D, T) is a non-exhaustive DT-pair of $C + uv$ and therefore in G, and so $\gamma\gamma_t(G) < n(G)$. □

Lemma 3.13 *Let G be a connected C_5-free graph. If there exists a spanning proper subgraph F of G such that $F \in \mathcal{K}^*$, then $\gamma\gamma_t(G) < n(G)$.*

Proof: Let G be a connected C_5-free graph and suppose there exists a spanning proper subgraph F of G such that $F \in \mathcal{K}^*$. Among all edges in $E(G) \setminus E(F)$, let the edge uv be chosen so that $d_F(u) + d_F(v)$ is maximized and, subject to that, the number of common neighbors of u and v in F is maximized. Let $F' = F + uv$.

By definition of the family $\mathcal{K}^*$, we note that $V_{\geq 3}(F) \geq 4$. Suppose $\{u, v\} \subseteq V_{\geq 3}(F)$. Let $w \in V_{\geq 3}(F) \setminus \{u, v\}$. Let u' be the common neighbor of u and w in F, and let v' be the common neighbor of v and w in F. By Observation 3.7(c), there exists a DT-pair (D, T) in F such that $w \in D$, $\{u', v'\} \subseteq N_F(w) \subseteq T$ and $\{u, v\} \subseteq T$. Now $(D, T \setminus \{u'\})$ is a non-exhaustive DT-pair of F' and therefore in G, and so $\gamma\gamma_t(G) < n(G)$. Hence we may assume, without loss of generality, that $d_F(u) = 2$.

Suppose $v \in V_{\geq 3}(F)$. Since $uv \notin E(F)$, we note that $v \notin N_F(u)$. Let $w \in N_F(u)$. Then, $w \in V_{\geq 3}(F)$. Let v' be the common neighbor of v and w. By Observation 3.7(c), there exists a DT-pair (D, T) in F such that $w \in D$, $\{u, v'\} \subseteq N_F(w) \subseteq T$ and $v \in T$. Now $(D, T \setminus \{v'\})$ is a non-exhaustive DT-pair of F' and therefore in G, and so $\gamma\gamma_t(G) < n(G)$. Hence we may assume that $d_F(v) = 2$.

Let $N_F(u) = \{u_1, u_2\}$ and let $N_F(v) = \{v_1, v_2\}$. Then, $\{u_1, u_2\} \subseteq V_{\geq 3}(F)$ and $\{v_1, v_2\} \subseteq V_{\geq 3}(F)$. Suppose that u and v have no common neighbor in F. Then, $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$. Let w be the common neighbor of u_1 and v_1 in F. Then, $C' = uu_1wv_1vu$ is a 5-cycle in F' and hence in G. By our choice of the edge uv, the cycle C' is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Hence, u and v have a common neighbor in F and we may assume that $u_1 = v_1$. By Observation 3.7(c), there exists a DT-pair (D, T) in F such that $u_1 \in D$, $\{u, v\} \subseteq N_F(u_1) \subseteq T$ and $\{u_2, v_2\} \subseteq T$. Furthermore, we note that every neighbor of u_2 in F, different from u, is totally dominated by $T \setminus \{u_2\}$. Thus, $(D, T \setminus \{u_2\})$ is a non-exhaustive DT-pair of F' and therefore in G, and so $\gamma\gamma_t(G) < n(G)$. □

We now combine Lemma 3.12 and Lemma 3.13 into the following result.

Lemma 3.14 *Let G be a connected C_5-free graph. If there exists a spanning proper subgraph F of G such that $F \in \mathcal{C} \cup \mathcal{K}^*$, then $\gamma\gamma_t(G) < n(G)$.*

3.1.2 Proof of Theorem 3.4

We are now in a position to prove our key preliminary result, namely Theorem 3.4. Recall that a graph G is an $n(G)$-minimal graph if G is edge-minimal with respect to satisfying the following three conditions: (i) $\delta(G) \geq 2$, (ii) G is connected, (iii) $\gamma\gamma_t(G) = n(G)$. Recall the statement of Theorem 3.4.

Theorem 3.4 *If G is a C_5-free graph, then G is $n(G)$-minimal if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

Proof: If $G \in \mathcal{C} \cup \mathcal{K}^*$, then, by definition of the families $\mathcal{C}$ and $\mathcal{K}^*$, $\delta(G) \geq 2$ and G is connected. By Lemmas 3.5 and 3.6, $\gamma\gamma_t(G) = n(G)$. Furthermore, $\delta(G - e) = 1$ for any edge e in G, and so G is $n(G)$-minimal. This establishes the sufficiency.

To prove the necessity, we proceed by induction on the order $n(G)$ of an $n(G)$-minimal C_5-free graph G. If $n(G) \in \{3, 4\}$, then $G = C_{n(G)} \in \mathcal{C}$. Suppose $n(G) = 5$. Since $G \neq C_5$, either G contains a C_3, in which case either G arises by adding an edge to C_5 or G can be obtained from two disjoint 3-cycles by identifying a vertex from each cycle into one vertex, or G contains a C_4 but no C_3, in which case $G = K_{2,3}$. In both cases, there exists a non-exhaustive (D, T)-pair in G, contradicting the fact that G is $n(G)$-minimal. Hence, $n(G) \neq 5$. This establishes the base cases.

Let $n(G) \geq 6$ and assume that the result is true for all n'-minimal C_5-free graphs, where $3 \leq n' < n(G)$. Let G be an $n(G)$-minimal C_5-free graph. Before proceeding further, we present two observations that will be useful in what follows.

Observation 3.15 *If $e \in E$, then either e is a bridge of G or $\delta(G - e) = 1$.*

Proof: For contradiction, we assume that $\delta(G - e) \geq 2$. Since G is a connected C_5-free graph of order at least 6, $G - e$ has no C_5-component. Therefore, by Corollary 3.2, $n = \gamma\gamma_t(G) \leq \gamma\gamma_t(G - e) \leq n(G)$, which implies $\gamma\gamma_t(G - e) = n(G)$. Since G is $n(G)$-minimal, this implies that $G - e$ is not connected, which completes the proof. □

Observation 3.16 *If G' is a connected subgraph of G of order $n(G') < n(G)$ with $\delta(G') \geq 2$, then either $G' \in \mathcal{C} \cup \mathcal{K}^*$ or $\gamma\gamma_t(G') < n(G')$.*

Proof: Let G' be a connected subgraph of G of order $n(G') < n(G)$ with $\delta(G') \geq 2$. Clearly, G' is C_5-free. Suppose $\gamma\gamma_t(G') = n(G')$. Then, by Lemma 3.11, G' contains a spanning C_5-free subgraph G'' that is $n(G')$-minimal. By induction, $G'' \in \mathcal{C} \cup \mathcal{K}^*$. If G'' is a proper subgraph of G', then Lemma 3.14 implies a contradiction. Hence, $G' = G''$, and so $G' \in \mathcal{C} \cup \mathcal{K}^*$. □

If $|V_{\geq 3}(G)| = 0$, then $G = C_n$ and, since G is C_5-free, $G \in \mathcal{C}$ and we are done. Hence, we may assume that $|V_{\geq 3}(G)| \geq 1$. If $|V_{\geq 3}(G)| = 1$, then G satisfies the conditions of Lemma 3.10 and thus has a non-exhaustive DT-pair, contradicting the fact that G is $n(G)$-minimal. Hence, $|V_{\geq 3}(G)| \geq 2$. We prove the following claim about the set $V_{\geq 3}(G)$ of vertices of degree at least 3 in G.

Claim 1 *$V_{\geq 3}(G)$ is an independent set in G.*

Proof of Claim 1: For the sake of contradiction, suppose that $\{u, v\} \subseteq V_{\geq 3}(G)$ with $uv \in E(G)$. Then, by Observation 3.15, uv is a bridge of G. Let G_u and G_v denote the components of $G - uv$ containing u and v respectively. We note that $\gamma\gamma_t(G) \leq \gamma\gamma_t(G_u) + \gamma\gamma_t(G_v)$. If $\gamma\gamma_t(G_u) < |V(G_u)|$ or $\gamma\gamma_t(G_v) < |V(G_v)|$, then $\gamma\gamma_t(G) < n(G)$, a contradiction. Hence, $\gamma\gamma_t(G_u) = |V(G_u)|$ and $\gamma\gamma_t(G_v) = |V(G_v)|$. Therefore, by Observation 3.16, $\{G_u, G_v\} \subset \mathcal{C} \cup \mathcal{K}^*$. If $G_u \in \mathcal{C}$, then, by Lemma 3.8, there exists a pair (D_1, T_1) of disjoint sets of vertices in G_u such that $u \in T_1$, $|D_1| + |T_1| < |V(G_u)|$, and either (i) D_1 dominates $V(G_u)$ and T_1 totally dominates $V(G_u) \setminus \{u\}$ or (ii) D_1 dominates $V(G_u) \setminus \{u\}$ and T_1 totally dominates $V(G_u)$. Using Observation 3.7(a), let (D_2, T_2) be a DT-pair of G_v with $v \in T_2$ if (i) holds and $v \in D_2$ if (ii) holds. In both cases, $(D_1 \cup D_2, T_1 \cup T_2)$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $G_u \in \mathcal{K}^*$. Similarly, $G_v \in \mathcal{K}^*$.

Let u' be a neighbor of u in G_u. Since uu' is not a bridge in G_u, the edge uu' is not a bridge in G, and so, by Observation 3.15, $\delta(G - uu') = 1$. Since $d_G(u) \geq 3$, we note that $d_{G-uu'}(u) \geq 2$, implying that $d_G(u') = 2$ and so $d_{G_u}(u') = 2$. Let u'' be the neighbor of u' distinct from u. Since every edge in G_u is incident with a vertex of degree at least 3 and a vertex of degree at most 2, $d_{G_u}(u) \geq 3$ and $d_{G_u}(u'') \geq 3$. Therefore, by Observation 3.7(c), there exists a DT-pair (D_1, T_1) such that $u'' \in D_1$, $u' \in N_{G_u}(u'') \subseteq T_1$ and $u \in T_1$. By Observation 3.7(a), there exists a DT-pair (D_2, T_2) in G_v with $v \in T_2$. Thus, $(D_1 \cup D_2, T_1 \cup T_2 \setminus \{u'\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, we conclude that $V_{\geq 3}(G)$ is an independent set in G. □

Let R be any component of $G - V_{\geq 3}(G)$. Note that R is a path. If R has only one vertex, or has at least two vertices with the two ends of R adjacent in G to different vertices of degree at least 3, then we say that R is a *light path*. Otherwise we say that R is a *light handle*.

Claim 2 *Every light path in G contains at most two vertices.*

Proof of Claim 2: Let $P = v_1 \dots v_k$ be a longest light path in G and let v_0 and v_{k+1} be the vertices of degree at least 3 that are adjacent in G to v_1 and v_k, respectively. By definition of a light path, we note that $v_0 \neq v_{k+1}$. For the sake of contradiction, suppose that $k \geq 3$. Let F be the graph obtained from G by deleting the vertices v_1, v_2 and v_3 and adding the edge v_0v_4. Then G can be obtained from F by subdividing an edge of F three times. Since $V_{\geq 3}(G) = V_{\geq 3}(F)$ and $|V_{\geq 3}(G)| \geq 2$, we note that F is not a cycle. In particular, $F \neq C_5$. By construction, F is a connected graph with $\delta(F) \geq 2$. Hence, by Lemma 3.9, $\gamma\gamma_t(F) = |V(F)|$. We proceed further with a subclaim showing that F is C_5-free.

Subclaim 1 *F is C_5-free.*

Proof of Subclaim 1: Suppose that F contains an induced 5-cycle C. Since G is C_5-free, we note that C contains the edge v_0v_4 and therefore $k \in \{3, 4, 5\}$. Suppose that $k = 3$. Let C be the cycle $v_0w_1w_2w_3v_4v_0$. We note that either $w_1w_2w_3$ is a light path in G or $w_2 \in V_{\geq 3}(G)$. We now consider the graph $F' = F - v_0v_4$ and note that F' is a connected subgraph of G with $\delta(F') \geq 2$ and $V(F') = V(F)$. Further, $|V(F')| \geq \gamma\gamma_t(F') \geq \gamma\gamma_t(F) = |V(F)|$, and so $\gamma\gamma_t(F') = |V(F')|$. By Observation 3.16, $F' \in \mathcal{C} \cup \mathcal{K}^*$. We note that $v_0w_1w_2w_3v_4$ is a path in F'. If $F' \in \mathcal{C}$, then, by our choice of P, we have that $F' \in \{C_6, C_7, C_8\}$. In all three cases, we can find a DT-pair (D', T') in F' such that $\{v_0, v_4\} \subseteq T'$. If $F' \in \mathcal{K}^*$, then since w_1 and w_3 have degree 2 in both G and F', we note that $\{v_0, v_4, w_2\} \subseteq V_{\geq 3}(F')$ and by Observation 3.7(c), there exists a DT-pair (D', T') in F such that $w_2 \in D'$ and $\{v_0, v_4\} \subseteq T'$. But then $(D' \cup \{v_2\}, T' \cup \{v_1\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $k \in \{4, 5\}$.

Suppose that $k = 4$. Let C be the cycle $v_0w_1w_2v_5v_4v_0$. We note that, since $V_{\geq 3}(G)$ is an independent set, w_1w_2 is a light path in G. We now consider the graph $F' = F - v_4$ and note that F' is a connected subgraph of G with $\delta(F') \geq 2$. If $\gamma\gamma_t(F') < |V(F')|$, let (D', T') be a $\gamma\gamma_t(F')$-pair. But then $(D' \cup \{v_1, v_4\}, T' \cup \{v_2, v_3\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $\gamma\gamma_t(F') = |V(F')|$, and so by Observation 3.16, $F' \in \mathcal{C} \cup \mathcal{K}^*$. Since both ends of the edge $w_1w_2 \in E(F')$ are vertices of degree at most 2 in F', we note that $F' \notin \mathcal{K}^*$. Hence, $F' \in \mathcal{C}$. By Observation 3.7(b), there exists a DT-pair (D', T') in F' such that $\{v_0, w_1\} \subseteq T'$. Necessarily, $w_2 \in D'$. If $v_5 \in T'$, let $D = D' \cup \{v_2, v_3\}$ and $T = (T' \setminus \{w_1\}) \cup \{v_1, v_4\}$. If $v_5 \in D'$, let $D = D' \cup \{v_2\}$ and $T = T' \cup \{v_3, v_4\}$. In both cases, (D, T) is a non-exhaustive DT-pair of G, a contradiction. Hence, $k = 5$.

Let C be the cycle $v_0v_4v_5v_6v'v_0$. We note that, since $V_{\geq 3}(G)$ is an independent set, $v' \in V_2(G)$ and $N_G(v') = \{v_0, v_6\}$. We now consider the graph $F' = F - \{v_4, v_5\}$ and note that F' is a connected graph with $\delta(F') \geq 2$. Furthermore, F' is a subgraph of G and hence $F' \neq C_5$. Let (D', T') be a $\gamma\gamma_t(F')$-pair. In order to totally dominate the vertex v' in F', $|\{v_0, v_6\} \cap T'| \geq 0$. We may assume, without loss of generality, that $v_0 \in T'$. But then $(D' \cup \{v_2, v_5\}, T' \cup \{v_3, v_4\})$ is a non-exhaustive DT-pair of G, a contradiction. This completes the proof of Subclaim 1. □

We now return to the proof of Claim 2. By Subclaim 1, the graph F is a connected C_5-free graph with $\delta(F) \geq 2$ that satisfies $\gamma\gamma_t(F) = |V(F)|$. Let $n' = n - 3$, and so $|V(F)| = n'$. If

F is not n'-minimal, then by Lemma 3.11, F contains an n'-minimal spanning subgraph F' with no induced 5-cycle. But then, by the induction hypothesis, $F' \in \mathcal{C} \cup \mathcal{K}^*$ and therefore, by Lemma 3.14, $\gamma\gamma_t(F) < n' = |V(F)|$, a contradiction. Hence, F is n'-minimal, and by the induction hypothesis, $F \in \mathcal{C} \cup \mathcal{K}^*$. As observed earlier, F is not a cycle, and so $F \in \mathcal{K}^*$. Since $V_{\geq 3}(G) = V_{\geq 3}(F)$, we note that $v_0 \in V_{\geq 3}(F)$ and that $k = 4$. Let w be a vertex of degree at least 3 different from v_0 and v_5. Let v_0' be the common neighbor of v_0 and w in F, and let v_5' be the common neighbor of v_5 and w in F. By Observation 3.7(c), there exists a DT-pair (D', T') such that $w \in D'$, $\{v_0', v_5'\} \subseteq N_F(w) \subseteq T'$ and $\{v_0, v_5\} \subseteq T'$. But now $((D' \setminus \{v_4\}) \cup \{v_2, v_3\}, (T' \setminus \{v_0'\}) \cup \{v_1, v_4\})$ is a non-exhaustive DT-pair of G, a contradiction. $\square$

Claim 3 *Every light path in G contains exactly one vertex.*

Proof of Claim 3: Let $P = v_1 \dots v_k$ be a longest light path in G. By Claim 2, $k \leq 2$. For the sake of contradiction, suppose that $k = 2$. Let v_0 and v_3 be the vertices of degree at least 3 that are adjacent in G to v_1 and v_2, respectively. Let $F = G - \{v_1, v_2\}$.

Suppose that F is disconnected. Let F_1 and F_2 denote the components containing v_0 and v_3, respectively. Then, $F = F_1 \cup F_2$. We consider first the case where $\gamma\gamma_t(F_1) < |V(F_1)|$ and $\gamma\gamma_t(F_2) < |V(F_2)|$. Let (D_1, T_1) and (D_2, T_2) be non-exhaustive DT-pairs in F_1 and F_2, respectively. If $v_0 \notin D_1$ then $(D_1 \cup D_2 \cup \{v_2\}, T_1 \cup T_2 \cup \{v_0, v_1\})$ is a non-exhaustive DT-pair of G, a contradiction. Therefore, $v_0 \in D_1$. Similarly, $v_3 \in D_2$. But then $(D_1 \cup D_2, T_1 \cup T_2 \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair in G, again a contradiction. Hence, without loss of generality, we may assume that $\gamma\gamma_t(F_1) = |V(F_1)|$. By Observation 3.16, $F_1 \in \mathcal{C} \cup \mathcal{K}^*$ and therefore, by Observation 3.7(a), there is a DT-pair (D_1, T_1) in F_1 with $v_0 \in T_1$. If $\gamma\gamma_t(F_2) = |V(F_2)|$, then, similarly, $F_2 \in \mathcal{C} \cup \mathcal{K}^*$ and there is a DT-pair (D_2, T_2) in F_2 with $v_3 \in T_2$. But then $(D_1 \cup D_2 \cup \{v_1\}, T_1 \cup T_2)$ is a non-exhaustive DT-pair of G, a contradiction. Thus, $\gamma\gamma_t(F_2) < |V(F_2)|$. As before, let (D_2, T_2) be a non-exhaustive DT-pair of F_2. But then $(D_1 \cup D_2 \cup \{v_2\}, T_1 \cup T_2 \cup \{v_1\})$ is a non-exhaustive DT-pair of G, again a contradiction. Hence, F is connected.

Suppose $\gamma\gamma_t(F) < |V(F)|$. Let (D, T) be a $\gamma\gamma_t(F)$-pair. If $v_0 \in T$, then $(D \cup \{v_2\}, T \cup \{v_1\})$ is a non-exhaustive DT-pair of G, a contradiction. Therefore, $v_0 \notin T$. Similarly, $v_3 \notin T$. By Claim 1 and Lemma 3.5, $v_0, v_3 \in D$ and $(D, T \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $\gamma\gamma_t(F) = |V(F)|$.

By Observation 3.16, $F \in \mathcal{C} \cup \mathcal{K}^*$. Suppose $F \in \mathcal{K}^*$. Since every neighbor of v_0 is a vertex of degree 2 in G and hence in F, we note that $v_0 \in V_{\geq 3}(F)$. Similarly, $v_3 \in V_{\geq 3}(F)$. We note that v_0v_3 is not an edge of F. Let v' be the common neighbor of v_0 and v_3 in F. But then $v_0v_1v_2v_3v'v_0$ is an induced 5-cycle in G, contradicting the fact that G is C_5-free. Hence, $F \notin \mathcal{K}^*$, and so $F \in \mathcal{C}$. Since G is C_5-free, we note that v_0 and v_3 have no common neighbor in F. Hence, by the choice of P, we note that $F = C_6$ and that $d_F(v_0, v_3) = 3$. Let F be the cycle $w_0w_1 \dots w_5w_0$ where $w_0 = v_0$ and $w_3 = v_3$. Then, $(\{w_1, w_4, v_1\}, \{w_0, w_2, w_3, w_5\})$ is a non-exhaustive DT-pair in G, a contradiction. $\square$

Claim 4 *There is no light handle in G.*

Proof of Claim 4: For the sake of contradiction, suppose that there is a light handle in G. Among all light handles in G, let $P = v_1v_2 \dots v_k$ have maximum length. Let v be the common neighbor of v_1 and v_k. We note that $v \in V_{\geq 3}(G)$. Further, we note that $k \geq 2$ and since G is C_5-free, $k \neq 4$. Let C be the cycle $vv_1v_2 \dots v_kv$ and let v' be a neighbor of v not on C. Since $V_{\geq 3}(G)$ is an independent set in G, $d_G(v') = 2$.

Suppose $d_G(v) \geq 4$. Let $F = G - V(P)$. Then, F is a connected C_5-free graph with $\delta(F) = 2$. If $\gamma\gamma_t(F) < |V(F)|$, let (D_1, T_1) be a $\gamma\gamma_t(F)$-pair. If $\gamma\gamma_t(F) = |V(F)|$, then by Observation 3.16, $F \in \mathcal{C} \cup \mathcal{K}^*$ and let (D_1, T_1) be a DT-pair of F such that v in T_1. We note that such a pair exists by Observation 3.7(a). If $v \in D_1$, let (D_2, T_2) be a DT-pair of C such that $v \in D_2$. Once again, such a pair exists by Observation 3.7(a). If $v \in T_1$, let (D_2, T_2) be a pair of disjoint sets of vertices in C such that $|D_2| + |T_2| < |V(C)|$, $v \in T_2$, and either (i) D_2 dominates $V(C)$ and T_2 totally dominates $V(C) \setminus \{v\}$, or (ii) D_2 dominates $V(C) \setminus \{v\}$ and T_2 totally dominates $V(C)$. In all cases, $(D_1 \cup D_2, T_1 \cup T_2)$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(v) = 3$, and so $N_G(v) = \{v_1, v_k, v'\}$.

We note that, since vv' is a bridge in G, the vertex of degree 2 v' belongs to a light path in G. Let $N_G(v') = \{v, w\}$. By Claim 3, $w \in V_{\geq 3}(G)$. Let $F = G - (V(C) \cup \{v'\})$. Then, F is a connected C_5-free graph with $\delta(F) = 2$. Let (D_1, T_1) be a $\gamma\gamma_t(F)$-pair. If $w \in D_1$, let (D_2, T_2) be a DT-pair in C such that $v \in T_2$. If $w \in T_1$, let (D_2, T_2) be a DT-pair of C such that $v \in D_2$. In both cases, we note that such a pair exists by Observation 3.7(a). Furthermore, in both cases, $(D_1 \cup D_2, T_1 \cup T_2)$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $w \notin D_1 \cup T_1$ and (D_1, T_1) is a non-exhaustive DT-pair of F. We now let (D_2, T_2) be a DT-pair of C such that $v \in T_2$. Then, $(D_1 \cup D_2 \cup \{v'\}, T_1 \cup T_2)$ is a non-exhaustive DT-pair of G, a contradiction. □

The following result is an immediate consequence of Claims 3 and 4.

Claim 5 *The graph G is a bipartite graph with partite sets $V_{\geq 3}(G)$ and $V_2(G)$.*

We show next that a common neighbor of two vertices of degree at least 3 is unique.

Claim 6 *Every two vertices in $V_{\geq 3}(G)$ have at most one common neighbor.*

Proof of Claim 6: For the sake of contradiction, suppose that $\{u, v\} \subseteq V_{\geq 3}(G)$ and w and w' are distinct common neighbors of u and v. Let $F = G - w'$. Then, F is a connected C_5-free graph with $\delta(F) = 2$. Suppose $\gamma\gamma_t(F) < |V(F)|$. Let (D, T) be a $\gamma\gamma_t(F)$-pair. Since T totally dominates w, $\{u, v\} \cap T \neq \emptyset$. But then $(D \cup \{w'\}, T)$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $\gamma\gamma_t(F) = |V(F)|$, and so, by Observation 3.16, $F \in \mathcal{C} \cup \mathcal{K}^*$. If $F \in \mathcal{K}^*$ then, since $d_F(w) = 2$, we have that $\{u, v\} \subseteq V_{\geq 3}(F)$. Therefore, by Observation 3.7(c), there exists a DT-pair (D,T) in F such that $u \in D$ and $v \in T$. But then (D, T) is a non-exhaustive DT-pair of G, a contradiction. Hence, $F \notin \mathcal{K}^*$, and so $F \in \mathcal{C}$. But then $F = C_4$, and so $n = 5$, a contradiction. □

Claim 7 *Every two vertices in $V_{\geq 3}(G)$ have exactly one common neighbor.*

Proof of Claim 7: By Claim 6, every two vertices in $V_{\geq 3}(G)$ have at most one common neighbor. Hence it suffices to show that every two vertices in $V_{\geq 3}(G)$ have a common neighbor. For the sake of contradiction, suppose that $\{u, v\} \subseteq V_{\geq 3}(G)$ and that u and v have no common neighbor. Let $N_G(u) = \{u_1, u_2, \ldots, u_r\}$, and so $d_G(u) = r$. By Claim 5, we note that $N_G(u) \subseteq V_2(G)$. For $i = 1, 2, \ldots, r$, let $N_G(u_i) = \{u, v_i\}$. By Claim 5, we note that $v_i \in V_{\geq 3}(G)$ for each such i. By Claim 6, $v_i \neq v_j$ for $1 \leq i < j \leq r$. Let $F = G - N_G[u]$. Then, F is a C_5-free graph with $\delta(F) = 2$. We note that F is possibly disconnected.

Suppose $\gamma\gamma_t(F) < |V(F)|$. Let (D, T) be a $\gamma\gamma_t(F)$-pair. For $i = 1, 2, \ldots, r$, let w_i be a neighbor of v_i in T. By Claim 5, $w_i \in V_2(G)$. Hence, since D dominates and T totally dominates w_i, we note that $v_i \in D \cup T$. If $v_i \in D$ for some i, $1 \leq i \leq r$, then $(D \cup (N_G(u) \setminus \{u_i\}), T \cup \{u, u_i\})$ is a non-exhaustive DT-pair of G, a contradiction. Therefore, $\{v_1, v_2, \ldots, v_r\} \subseteq T$. But then $(D \cup \{u\}, T \cup \{u_1\})$ is a non-exhaustive DT-pair of G, again a contradiction. Hence, $\gamma\gamma_t(F) = |V(F)|$.

Suppose F is disconnected. Let $F_1, F_2, \ldots, F_t$ be the components in F. By assumption, $t \geq 2$. Since $\gamma\gamma_t(F) = |V(F)|$, we note that $\gamma\gamma_t(F_i) = |V(F_i)|$ for all $i = 1, 2, \ldots, t$. Hence, by Observation 3.16, $F_i \in \mathcal{C} \cup \mathcal{K}^*$. Switching indices if necessary, we may assume that $v_i \in F_i$ for $i = 1, 2, \ldots, t$. For each such i, let (D_i, T_i) be a DT-pair of F_i such that $v_i \in D_i$. We note that such pairs exist by Observation 3.7(a). Let $D = \bigcup_{i=1}^{t} D_i$ and let $T = \bigcup_{i=1}^{t} T_i$. Then, (D, T) is a DT-pair of F and $(D \cup (N_G(u) \setminus \{u_1, u_2\}), T \cup \{u, u_1\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, F is connected.

By Observation 3.16, $F \in \mathcal{C} \cup \mathcal{K}^*$. Since $d_F(v) = d_G(v) \geq 3$, F is not a cycle and therefore $F \in \mathcal{K}^*$. By Claim 5, the set $V_{\geq 3}(G) \setminus \{u\} = V_{\geq 3}(F)$. In particular, each vertex $v_i \in V_{\geq 3}(F)$ for $i = 1, 2, \ldots, r$. By Observation 3.7(c), there exists a DT-pair (D, T) in F such that $v \in D$ and $\{v_1, v_2, \ldots, v_r\} \subseteq T$. But then $(D \cup \{u\}, T \cup \{u_1\})$ is a non-exhaustive DT-pair of G, a contradiction. □

We now return to the proof of Theorem 3.4. By Claims 5 and 7, the graph G is a bipartite graph with partite sets $V_{\geq 3}(G)$ and $V_2(G)$ where every two vertices in $V_{\geq 3}(G)$ have exactly one common neighbor. Hence, $G \in \mathcal{K}^*$. This completes the necessity and the proof of Theorem 3.4. □

3.1.3 Proof of Theorem 3.3

We are now in a position to present a proof of Theorem 3.3. Recall the statement of Theorem 3.3.

Theorem 3.3 *If G is a connected C_5-free graph with $\delta(G) \geq 2$, then $\gamma\gamma_t(G) = |V(G)|$ if and only if $G \in \mathcal{C} \cup \mathcal{K}^*$.*

Proof: The sufficiency follows from Lemmas 3.5 and 3.6. To prove the necessity, let G be a connected C_5-free graph with $\delta(G) \geq 2$ such that $\gamma\gamma_t(G) = n(G)$. Suppose that $G \notin \mathcal{C} \cup \mathcal{K}^*$. Then, by Theorem 3.4, G is not an $n(G)$-minimal graph. Hence, by

Lemma 3.11, G contains an $n(G)$-minimal spanning subgraph F with no induced 5-cycle. By Theorem 3.4, $F \in \mathcal{C} \cup \mathcal{K}^*$. Therefore, by Lemma 3.14, $\gamma\gamma_t(G) < n$, a contradiction. Hence, $G \in \mathcal{C} \cup \mathcal{K}^*$. $\square$

3.2 Graphs with Minimum Degree at Least 3

While in the previous section C_5-free graphs were studied, we consider graphs that may have induced cycles on five vertices in this section. We increase the minimum degree condition to at least 3. As our main result in this section we prove the following.

Theorem 3.17 *If G is a graph of minimum degree at least 3 with at least one component different from the Petersen graph, then G contains a dominating set D and a total dominating set T that are disjoint and satisfy $|D| + |T| < |V(G)|$.*

We first prove the result for graphs G, for which, additionally, the set of vertices of degree at least 4 is independent. But before that we collect some useful observations about the Petersen graph.

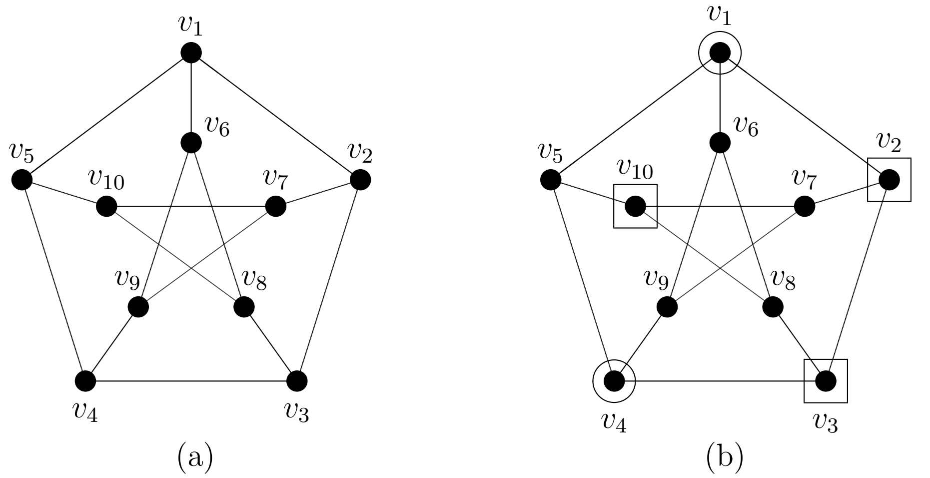


Figure 3.2: The encircled vertices belong to D and the framed vertices belong to T.

Lemma 3.18 *The following properties hold for the Petersen graph.*

(a) *If G is the union of disjoint Petersen graphs, then every DT-pair in G is exhaustive.*

(b) *If G arises from the Petersen graph by adding an edge between two non-adjacent vertices, then G has a non-exhaustive DT-pair.*

(c) *If G arises from the union of two disjoint Petersen graphs by adding an edge between the two Petersen graphs, then G has a non-exhaustive DT-pair.*

Proof: In order to reduce the number of cases that we have to consider, we will use the known facts that the Petersen graph is 3-*arc transitive*, *distance-transitive*, and *vertex-transitive* (see Sections 4.4 and 4.5 of [23]).

Let P denote the Petersen graph where (see Figure 3.2(a))

$$\begin{aligned} V(P) &= \{v_1, v_2, \ldots, v_{10}\} \\ E(P) &= \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\} \cup \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\} \\ &\quad \cup \{v_6v_8, v_8v_{10}, v_{10}v_7, v_7v_9, v_9v_6\}. \end{aligned}$$

Let (D, T) be an DT-pair of P. Since P is 3-arc transitive, we may assume, by symmetry, that $v_2, v_3 \in T$ and $v_1, v_4 \in D$. Since $|N_P(v_5) \cap T| \geq 1$, $v_{10} \in T$ (see Figure 3.2(b)). Suppose no vertex in $\{v_7, v_8\}$ belongs to $D \cup T$. Then, $v_5 \in T$ to totally dominate v_{10}, while $\{v_6, v_9\} \subseteq D$ to dominate $\{v_7, v_8\}$. But then no vertex of T totally dominates v_6 or v_9. Hence, at least one vertex in $\{v_7, v_8\}$ belongs to $D \cup T$. We may assume, by symmetry, that $v_7 \in D \cup T$.

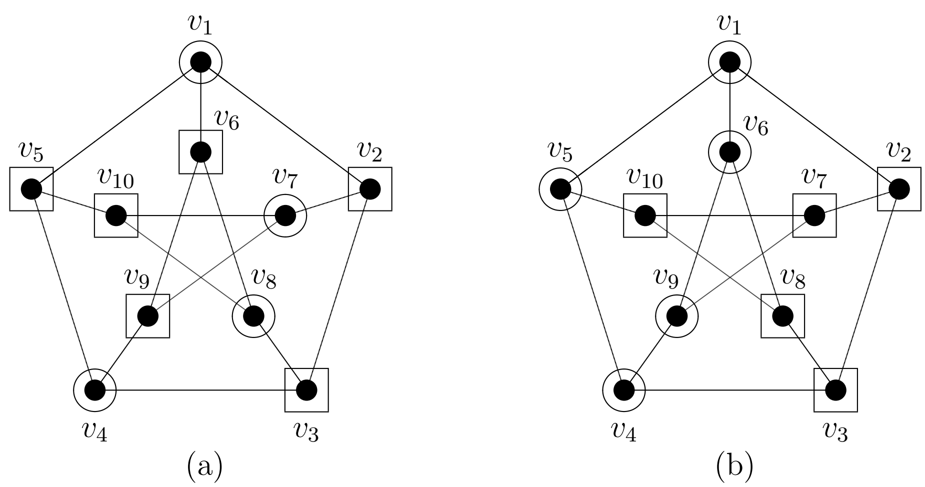


Figure 3.3: The encircled vertices belong to D and the framed vertices belong to T.

First, we assume $v_7 \in D$. Since $|N_P(v_9) \cap T| \geq 1$, $v_6 \in T$. Since $|N_P[v_8] \cap D| \geq 1$, $v_8 \in D$. Since $|N_P(v_6) \cap T| \geq 1$, $v_9 \in T$. Since $|N_P(v_{10}) \cap T| \geq 1$, $v_5 \in T$ (see Figure 3.3(a)). Now, $|D| + |T| = |V(P)|$.

Next, we assume $v_7 \in T$. Since $|N_P[v_7] \cap D| \geq 1$, $v_9 \in D$. Since $|N_P(v_6) \cap T| \geq 1$, $v_8 \in T$. Since $|N_P[v_8] \cap D| \geq 1$, $v_6 \in D$. Since $|N_P[v_{10}] \cap D| \geq 1$, $v_5 \in D$ (see Figure 3.3(b)). Again, $|D| + |T| = |V(P)|$.

Since in both cases (D, T) is exhaustive, the proof of (a) is complete.

Since the Petersen graph is distance-transitive, Figure 3.4(a) proves (b).

Finally, since the Petersen graph is vertex-transitive, Figure 3.4(b) proves (c). $\square$

The next lemma contains the core of our argument.

Lemma 3.19 *If G is a graph such that*

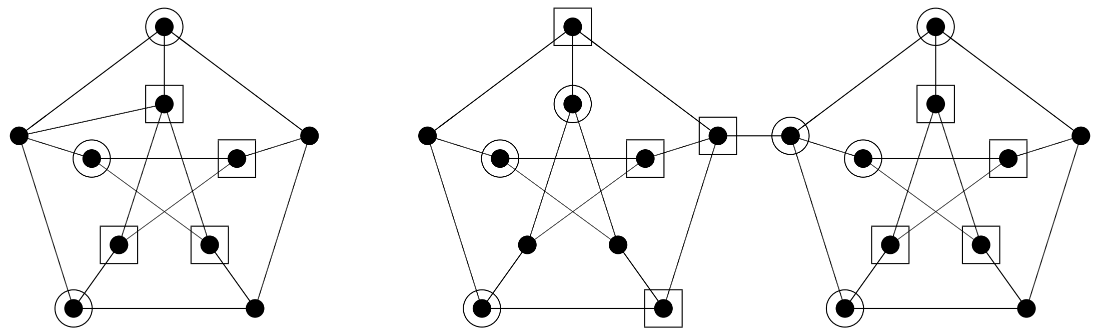

(a) (b)

Figure 3.4: The encircled vertices constitute a dominating set and the framed vertices constitute a total dominating set.

(i) *the minimum degree of G is at least 3,*

(ii) *G is not the union of disjoint Petersen graphs, and*

(iii) *the set of vertices of degree at least 4 is independent,*

then G has a non-exhaustive DT-pair.

Proof: For sake of contradiction, we assume that G is a counterexample of minimum order. Hence, G satisfies condition (i), (ii) and (iii), but G has no non-exhaustive DT-pair.

By (i) and Theorem 3.1, G has a non-exhaustive DT-pair if and only if some component of G has a non-exhaustive DT-pair. Hence, by the minimality of G, the graph G is connected.

We establish a series of claims concerning G.

Claim 1 *For $u \in V(G)$, the subgraph $G - u$ of G has no C_5-component.*

Proof of Claim 1: For contradiction, we assume that for some vertex u of G, the graph $G' = G - u$ has at least one C_5-component. Let V_5 denote the set of vertices of all C_5-components of G'. By the minimum degree condition (i) in G, we note that u is adjacent to every vertex of V_5 in G. If $V_5 \cup \{u\} = V(G)$, then letting $v \in V_5$, we have that $(D,T) = (\{u\}, V_5 \setminus \{v\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $V_5 \cup \{u\} \neq V(G)$. Let $G'' = G - (\{u\} \cup V_5)$. Then, G'' has no C_5-component and has minimum degree at least 2. Thus, by Theorem 3.1, G'' has an exhaustive DT-pair (D'', T''). If $v \in V_5$, then $(D,T) = (D'' \cup \{u\}, T'' \cup (V_5 \setminus \{v\}))$ is a non-exhaustive DT-pair of G, a contradiction. □

Claim 2 *For a 5-cycle C in G, the graph $G - V(C)$ either has a C_5-component or is of minimum degree less than 2.*

Proof of Claim 2: For contradiction, we assume that $C = v_1v_2v_3v_4v_5v_1$ is a 5-cycle in G such that $G' = G - V(C)$ has minimum degree at least 2 and no C_5-component. By Theorem 3.1, G' has an exhaustive DT-pair (D', T'). If a vertex in T' is adjacent to a vertex of C, say to v_1, then $(D, T) = (D' \cup \{v_2, v_5\}, T' \cup \{v_3, v_4\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, by condition (i), every vertex of C has a neighbor in D'. But then $(D, T) = (D', T' \cup \{v_1, v_2, v_3\})$ is a non-exhaustive DT-pair of G, once again producing a contradiction. □

Claim 3 *G contains no 3-cycle.*

Proof of Claim 3: For contradiction, we assume that $C = v_1v_2v_3v_1$ is a 3-cycle in G. First, we assume that there is a vertex $v_4 \in V(G) \setminus V(C)$ that is adjacent to at least two vertices of C, say to v_1 and to v_2. By (iii), at least one of the vertices v_1 and v_2 has degree exactly 3, say v_2. Now the graph $G' = G - v_1$ has minimum degree at least 2 and, by Claim 1, has no C_5-component. Thus, by Theorem 3.1, G' has an exhaustive DT-pair (D', T'). Since $d_{G'}(v_2) = 2$, $|D' \cup \{v_2, v_3, v_4\}| > 0$ and $|T' \cup \{v_3, v_4\}| > 0$. Thus, $(D, T) = (D', T')$ is a non-exhaustive DT-pair of G, a contradiction. Hence, every vertex in $V(G) \setminus V(C)$ is adjacent to at most one vertex of C. Thus, the graph $G' = G - V(C)$ has minimum degree at least 2. If G' has a C_5-component G_5, then $G - V(G_5)$ has no C_5-component and is of minimum degree at least 2, which contradicts Claim 2. Hence, G' has no C_5-component. Applying Theorem 3.1 to G', the graph G' has an exhaustive DT-pair (D', T'). If a vertex in T' is adjacent to a vertex of C, say to v_1, then $(D, T) = (D' \cup \{v_3\}, T' \cup \{v_1\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, every vertex of C has a neighbor in D'. But then $(D, T) = (D', T' \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair of G, once again producing a contradiction. □

Claim 4 *G contains no $K_{3,3}$ as a subgraph.*

Proof of Claim 4: For contradiction, we assume that G contains a $K_{3,3}$-subgraph with partite sets $V_v = \{v_1, v_2, v_3\}$ and $V_w = \{w_1, w_2, w_3\}$. Note that, by Claim 3, every $K_{3,3}$-subgraph of G is induced. By (iii), we may assume that all vertices in V_v have degree exactly 3. Since $K_{3,3}$ has a non-exhaustive DT-pair, we may assume that w_1 has degree more than 3. Now the graph $G' = G - w_1$ is of minimum degree at least 2 and, by Claim 1, has no C_5-component. By Theorem 3.1, G' has an exhaustive DT-pair (D', T'). Since $|N_{G'}(v_1) \cap T'| \geq 1$, $|D' \cap \{w_2, w_3\}|$ is either 0 or 1. If $|D' \cap \{w_2, w_3\}| = 0$, then $\{v_1, v_2, v_3\} \subseteq D'$, $\{w_2, w_3\} \subseteq T'$, and $(D, T) = ((D' \setminus \{v_1, v_2\}) \cup \{w_1\}, T' \cup \{v_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $|D' \cap \{w_2, w_3\}| = 1$. But then $(D, T) = ((D' \setminus V_v) \cup \{v_1\}, (T' \setminus V_v) \cup \{v_2\})$ is a non-exhaustive DT-pair of G, once again producing a contradiction. □

Claim 5 *G contains no $K_{3,3} - e$ as a subgraph.*

Proof of Claim 5: For contradiction, we assume that G contains a $(K_{3,3} - e)$-subgraph, i.e., there is a subset $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ of vertices in G such that $\{v_1w_1, v_1w_2, v_1w_3,$

$v_2w_1, v_2w_2, v_2w_3, v_3w_1, v_3w_2\} \subseteq E(G)$ and $v_3w_3 \notin E(G)$. By Claim 3, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are independent sets.

If $d_G(v_3) > 3$ and $d_G(w_3) > 3$, then, by (iii), $d_G(v_1) = d_G(w_1) = d_G(v_2) = d_G(w_2) = 3$. The graph $G' = G - \{v_1, v_2, w_1, w_2\}$ has minimum degree at least 2. Since $d_{G'}(u) \geq 3$ for all $u \in V(G') \setminus \{v_3, w_3\}$, G' contains no C_5-component. Therefore, by Theorem 3.1, G' has an exhaustive DT-pair (D', T'). If $v_3 \in D'$, let $(D,T) = (D' \cup \{w_1\}, T' \cup \{v_2, w_2\})$. If $v_3 \in T'$, let $(D,T) = (D' \cup \{v_1, w_1\}, T' \cup \{w_2\})$. In both cases, (D,T) is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(v_3) = 3$ or $d_G(w_3) = 3$. By symmetry and (iii), we may assume that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 3$.

Suppose that $d_G(w_3) > 3$. If at least one vertex in $\{w_1, w_2\}$ is of degree more than 3, say w_2, then $G' = G - \{v_1, v_2, w_1\}$ has minimum degree at least 2. By Claim 3, at most two neighbors of w_1 can belong to a possible C_5-component of G'. Since w_2, w_3, and the three neighbors of w_1 are the only vertices that can have degree exactly 2 in G', G' contains no C_5-component. Thus, by Theorem 3.1, G' has an exhaustive DT-pair (D', T'). If $\{v_3, w_2\} \subseteq D'$, let $(D,T) = (D', T' \cup \{v_1, w_1\})$. If $\{v_3, w_2\} \subseteq T'$, let $(D,T) = (D' \cup \{v_1, w_1\}, T')$. If $v_3 \in D'$ and $w_2 \in T'$, let $(D,T) = (D' \cup \{w_1\}, T' \cup \{v_1\})$. If $v_3 \in T'$ and $w_2 \in D'$, let $(D,T) = (D' \cup \{v_1\}, T' \cup \{w_1\})$. In all cases, (D,T) is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(w_1) = d_G(w_2) = 3$. Thus, $G' = G - \{v_1, v_2, v_3, w_1, w_2\}$ has minimum degree at least 2. Let $N_G(v_3) = \{w_1, w_2, v_3'\}$. Since $d_{G'}(u) \geq 3$ for all $u \in V(G') \setminus \{w_3, v_3'\}$, G' contains no C_5-component. Thus, by Theorem 3.1, G' has an exhaustive DT-pair (D', T'). Now, $(D,T) = (D' \cup \{v_1, w_1\}, T' \cup \{v_2, w_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(w_3) = 3$.

Suppose that at least one vertex in $\{w_1, w_2\}$ is of degree more than 3, say w_2. Then, $G' = G - \{v_2, v_3, w_1\}$ has minimum degree at least 2. Let $N_G(v_3) = \{w_1, w_2, v_3'\}$ and let $w_2' \in V(G) \setminus \{v_1, v_2, v_3\}$ be a neighbor of w_2. By Claim 3, $v_3' \neq w_2'$.

First, we assume that G' contains a C_5-component C. By Claim 3, at most two neighbors of w_1 can belong to C. Since w_2 and w_3 are the only neighbors of v_1 in G', either $|V(C) \cap \{w_2, v_1, w_3\}| = 0$ or $|V(C) \cap \{w_2, v_1, w_3\}| = 3$. Since w_2, w_3, v_3', and the neighbors of w_1 are the only vertices that can have degree exactly 2 in G', we have that $V(C) = \{v_1, v_3', w_2, w_2', w_3\}$ implying that $d_G(v_3') = d_G(w_2') = 3$, $d_G(w_2) = 4$, and $\{w_1w_2', v_3'w_3, v_3'w_2'\} \subset E(G)$. Thus, the graph F shown in Figure 3.5 is a subgraph of G. We note that the degree of every vertex in the subgraph F, except possibly for the vertex w_1, is the same as its degree in the graph G; that is, $d_F(v) = d_G(v)$ for all $v \in V(F) \setminus \{w_1\}$.

If $G = F$, then $(D,T) = (\{v_1, w_1, w_2'\}, \{v_2, v_3', w_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $G \neq F$. We now consider the graph $G'' = G - V(F)$. Every vertex in G'' has degree at least 3, except possibly for vertices in $N_G(w_1) \setminus V(F)$ that have degree at least 2 in G''. By Claim 1, the graph G'' has no C_5-component. Thus, by Theorem 3.1, G'' has an exhaustive DT-pair (D'', T''). Now, $(D,T) = (D'' \cup \{v_2, w_2, w_2'\}, T'' \cup \{v_3, v_3', w_3\})$ is a non-exhaustive DT-pair of G, a contradiction. We deduce, therefore, that G' has no C_5-component.

By Theorem 3.1, G' has an exhaustive DT-pair (D', T'). If $w_2 \in T'$, let $(D,T) = (D' \cup \{w_1\}, T' \cup \{v_2\})$. If $\{v_1, w_2\} \subseteq D'$, let $(D,T) = (D', T' \cup \{v_2, w_1\})$. If $w_2 \in D'$ and $v_1 \in T'$, let $(D,T) = (D' \cup \{v_2\}, T' \cup \{w_1\})$. In all cases, (D,T) is a non-exhaustive DT-

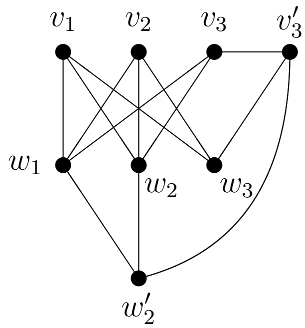


Figure 3.5

pair of G, a contradiction. We deduce, therefore, that the vertices $v_1, v_2, v_3, w_1, w_2, w_3$ are all of degree 3 in G.

Let $N_G(v_3) = \{w_1, w_2, v_3'\}$. We now consider the graph G' obtained from $G - \{v_2, v_3, w_1\}$ by adding the edge w_2v_3'. Then, G' has minimum degree at least 2. Since $d_{G'}(u) \geq 3$ for all $u \in V(G') \setminus \{v_1, w_2, w_3\}$, the graph G' contains no C_5-component. Thus, by Theorem 3.1, G' has an exhaustive DT-pair (D', T').

If $\{v_1, w_2\} \subseteq D'$, then $\{w_3, v_3'\} \subseteq T'$, and let $(D, T) = (D' \cup \{v_3\}, T' \cup \{v_2\})$. If $v_1 \in D'$ and $w_2 \in T'$, then $v_3' \in T'$ and let $(D, T) = (D' \cup \{w_1\}, T' \cup \{v_3\})$. If $v_1 \in T'$ and $w_2 \in D'$, then $w_3 \in T'$ and let $(D, T) = (D' \cup \{v_3\}, T' \cup \{w_1\})$. Finally, if $\{v_1, w_2\} \subseteq T'$, then $\{w_3, v_3'\} \subseteq D'$, and let $(D, T) = (D' \cup \{v_2\}, T' \cup \{v_3\})$. In all cases, (D, T) is a non-exhaustive DT-pair of G, a contradiction, which completes the proof of the claim. □

Claim 6 *G contains no $K_{2,3}$ as a subgraph.*

Proof of Claim 6: For contradiction, we assume that G contains a $K_{2,3}$-subgraph, i.e., there are two vertices v_1 and v_2 that have $\ell \geq 3$ common neighbors $w_1, w_2, \ldots, w_\ell$. By Claim 3, $\{v_1, v_2\}$ and $\{w_1, w_2, \ldots, w_l\}$ are independent sets. We now consider the graph $G' = G - \{v_1, v_2, w_1, w_2, \ldots, w_\ell\}$. By Claims 3, 4 and 5, every vertex in $V(G')$ is adjacent in G to at most one vertex in $V(G) \setminus V(G')$. Hence, G' has minimum degree at least 2. By Claim 2, G' therefore has no C_5-component. Hence, by Theorem 3.1, G' has an exhaustive DT-pair (D', T'). Now, $(D, T) = (D' \cup \{v_1, w_1\}, T' \cup \{v_2, w_2\})$ is a non-exhaustive DT-pair of G, a contradiction. □

Claim 7 *G contains no 4-cycle.*

Proof of Claim 7: For contradiction, we assume that $C = v_1v_2v_3v_4v_1$ is a 4-cycle in G. Let $G' = G - V(C)$. By Claim 3 and 6, every vertex in $V(G')$ is adjacent in G to at most one vertex in $V(G) \setminus V(G')$. Hence, G' has minimum degree at least 2. By Claim 2, G' therefore has no C_5-component. Hence, by Theorem 3.1, G' has an exhaustive DT-pair (D', T'). If a vertex in D' is adjacent to a vertex of C, say to v_1, then $(D, T) = (D' \cup \{v_3\}, T' \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, no vertex in D' is adjacent to a vertex of C'. Thus, every vertex of C has a neighbor in T'. But then $(D, T) = (D' \cup \{v_1, v_2\}, T')$ is a non-exhaustive DT-pair of G, a contradiction. □

Claim 8 *G contains no* 5*-cycle.*

Proof of Claim 8: For contradiction, we assume that $C = v_1v_2v_3v_4v_5v_1$ is a 5-cycle in G. Let $G' = G - V(C)$. By Claim 3 and G, every vertex in $V(G')$ is adjacent in G to at most one vertex in $V(G) \setminus V(G')$. Hence, G' has minimum degree at least 2. By Claim 2, G' therefore has a C_5-component $C' = v_6v_8v_{10}v_7v_9v_6$ and, again by Claim 2, $V(G) = V(C) \cup V(C')$. We may assume that $v_1v_6 \in E(G)$. By (i), symmetry, and Claims 3 and 7, we may assume that $v_2v_7 \in E(G)$ and $v_3v_8 \in E(G)$. Now Claims 3 and 7 imply $v_5v_{10} \in E(G)$, $v_2v_7 \in E(G)$, and $v_4v_9 \in E(G)$, i.e., G is the Petersen graph, a contradiction. □

By Claims 3, 7, and 8, the graph G contains no 3-cycle, 4-cycle, or 5-cycle. Let $P = v_1v_2v_3v_4$ be a path in G and let $v_1' \in V(G) \setminus \{v_1, v_3\}$ be a neighbor of v_2. Let $G' = G - \{v_1, v_2, v_3, v_4, v_1'\}$. Since G has girth at least 6, the graph G' has minimum degree at least 2 and contains no C_5-component. Hence, by Theorem 3.1, G' has an exhaustive DT-pair (D', T').

If a vertex in D' is adjacent to a vertex in $\{v_1, v_1'\}$, say to v_1', let $(D, T) = (D' \cup \{v_1, v_4\}, T' \cup \{v_2, v_3\})$. If every vertex in $\{v_1, v_4, v_1'\}$ has a neighbor in T', let $(D, T) = (D' \cup \{v_2, v_3\}, T' \cup \{v_1, v_4\})$. If every vertex of $\{v_1, v_1'\}$ has a neighbor in T' and v_4 has a neighbor in D', then $(D, T) = (D' \cup \{v_2\}, T' \cup \{v_3, v_4\})$. In all cases, (D, T) is a non-exhaustive DT-pair of G, a contradiction, which completes the proof of the lemma. □

With the help of Lemma 3.19, the proof of Theorem 3.17 follows readily. Recall the statement of Theorem 3.17

Theorem 3.17. *If G is a graph of minimum degree at least* 3 *with at least one component different from the Petersen graph, then G contains a dominating set D and a total dominating set T that are disjoint and satisfy $|D| + |T| < |V(G)|$.*

Proof: We prove the result by induction on the number of edges between vertices of degree at least 4. If there is no such edge, then the result follows immediately from Lemma 3.19. Hence, we assume that $e \in E(G)$ is such an edge. If e is a bridge, then deleting e results in two components G_1 and G_2. If both of G_1 and G_2 are the Petersen graph, then the result follows from Lemma 3.18(c). If at least one of G_1 or G_2 is not the Petersen graph, then the result follows by induction. Hence, we may assume that e is no bridge. If $G' = G - e$ is the Petersen graph, then the result follows from Lemma 3.18(b). If G' is not the Petersen graph, then the result follows by induction. This completes the proof of the theorem. □

Chapter 4

Results for and with Trees

Not only domination but also independence in graphs is a fundamental and well-studied topic [60]. The problem of partitioning the vertex set into dominating sets [15, 16, 20] and even more so the problem of partitioning the vertex set into independent sets, i.e. vertex colourings [40], have been studied extensively. Several authors paid attention to the characterization of trees that have two disjoint vertex sets with special properties. As early as 1978 Bange, Barkauskas, and Slater [4] and Slater [74] characterized trees that have two disjoint minimum dominating sets and two disjoint maximum independent sets, respectively. In [31] trees with two disjoint minimum independent dominating sets are characterized. In Section 4.1 we contribute to this line of research. We characterize the trees with the smallest possible size of two disjoint dominating sets (Theorem 4.2), we exhibit a tree that does not have two disjoint minimum dominating sets even though no single vertex is in all minimum dominating sets (Observation 4.3), and we show that every tree has a minimum dominating set whose complement contains an independent dominating set (Theorem 4.4). All these results answer problems mentioned [49] in [32] and are based on [33].

Though our proof of Theorem 4.4 is short and very economical, Johnson, Prier, and Walsh [41] proved the result once more with a clunky algorithmic proof that is much longer than our proof. Their motivation to do so was to illuminate the following conjecture.

Conjecture 4.1 (Johnson, Prier, and Walsh [41]) *If T is a tree of order at least 2 and D is a minimum dominating set of T containing at most one leaf of T, then there is an independent dominating set I of T that is disjoint from D.*

As pointed out in [41], Conjecture 4.1, if true, would be best-possible. This may be seen by considering a path $P = v_1v_2v_3\dots v_{3k+1}$ on $3k+1 \geq 4$ vertices and the dominating set $D = \{v_1, v_4, \dots, v_{3k+1}\}$ of P. Note that D is minimum and that P has no independent dominating that is disjoint from D.

The motivation of Johnson, Prier, and Walsh [41] for posing their conjecture is a related conjecture concerning the inverse domination number of graphs. As mentioned in Chapter 2 the *inverse domination number* $\gamma^{-1}(G)$ of a graph G is the minimum cardinality of a dominating set whose complement contains a minimum dominating set of G. Inverse domination in graphs was introduced by Kulli and Sigarkanti [48]. In their original paper in 1991, they include a proof that for all graphs with no isolated vertex, the inverse domination number is at most the independence number. However, this proof contained an error and in 2004, Domke, Dunbar, and Markus [18] formally posed this "result" of Kulli and Sigarkant as a conjecture. This conjecture still remains open and has been proved for many special families of graphs, including claw-free graphs, bipartite graphs, split graphs, very well covered graphs, chordal graphs and cactus graphs (see [21]). We prove Conjecture 4.1 in Section 4.1 (Theorem 4.5). This result is based on [35].

Still in Section 4.1 we consider graphs that have a maximum independent set and a minimum dominating set that are disjoint, thus graphs with an (α, γ)-pair. Intuitively, two independent sets or two dominating sets compete for similar types of vertices, while an independent set and a dominating set seem easier to reconcile. While the decision problem whether a given graph has an (α, γ)-pair is NP-hard (see Theorem 5.3), we give a constructive characterization of trees with an (α, γ)-pair (Theorem 4.10) in Section 4.1. This result is based on [58].

In the remaining sections of this chapter we leave the area of domination in graphs. Section 4.2 is devoted to independence in graphs. In view of its computational hardness (see e.g. [22]), various bounds on the independence number have been proposed. Caro [11] and Wei [75] proved

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{d_G(u)+1} \tag{4.1}$$

for every graph G. Since the only graphs for which (4.1) is best-possible are the disjoint unions of cliques, additional structural assumptions excluding these graphs allow improvements. Natural candidates for such assumptions are triangle-freeness or — more generally — odd girth conditions as well as connectivity.

For triangle-free graphs G, Shearer [72] proved

$$\alpha(G) \geq \sum_{u \in V(G)} f_{\mathrm{Sh}}(d_G(u))$$

where $f_{\mathrm{Sh}}(0) = 1$ and $f_{\mathrm{Sh}}(d) = \frac{1+(d^2-d)f_{\mathrm{Sh}}(d-1)}{d^2+1}$ for $d \in \mathbb{N}$. The function f_{Sh} has the best-possible order of magnitude $f_{\mathrm{Sh}}(d) = \Omega\left(\frac{\log d}{d}\right)$. For graphs with a specified odd girth, Denley [17] and Shearer [73] gave bounds in terms of the vertex degrees.

For connected graphs G, Harant and Rautenbach [27] proved the existence of a positive integer $k \in \mathbb{N}$ and a function $f : V(G) \to \mathbb{N}_0$ with non-negative integer values such that $f(u) \leq d_G(u)$ for $u \in V(G)$,

$$\alpha(G) \geq k \geq \sum_{u \in V(G)} \frac{1}{d_G(u)+1-f(u)}, \text{ and } \sum_{u \in V(G)} f(u) \geq 2(k-1).$$

Their result is a best-possible improvement of an earlier result due to Harant and Schiermeyer [28].

The calculation of $\alpha(T)$ is polynomial for trees T. We can easily construct a maximum independent set I for T as follows. Start with $I = \emptyset$. While T has a vertex v of degree at most 1, add v to I and delete $N_T[v]$ in T. In Section 4.2 we prove lower bounds of $\alpha(G)$ for graphs G that are close to trees, i.e. connected graphs with a small average degree. First, we prove a lower bound on the independence number of connected graphs of specified odd girth (Corollary 4.13). This result relies on a very simple argument but is best-possible for small average degrees. Furthermore, we give an improvement for arbitrary odd girth and larger average degrees (Theorem 4.14). Both results are based on [54]. To prove these results, we define $\alpha\alpha(G)$ as the maximum cardinality of two disjoint independent sets of a graph G. Clearly, $\alpha\alpha(G)$ is also the maximum order of an induced bipartite subgraph of G. Zhu [78] studied the so-called *bipartite ratio* $b^*(G) = n'(G)/n(G)$ of graphs G. In Chapter 5 we show that it is NP-complete to decide for a given graph G and a given integer k, whether $\alpha\alpha(G) \leq k$ and that it is NP-hard to decide for a given graph G, whether the obvious inequality $\alpha\alpha(G) \leq 2\alpha(G)$ is satisfied with equality.

Section 4.3 is devoted to spanning tree congestion. Tree congestion $t(G)$ and spanning tree congestion $s(G)$ of graphs are two special examples of the numerous graph embedding and layout problems, which have been considered in connection with applications to networking and circuit design. Restricting trees to paths, $t(G)$ corresponds exactly to the very well studied *cutwidth* [14]. Several other host graphs instead of trees, such as cycles [13], grids [7], and binary trees [8] have been considered. In [39, 46, 50, 66] the exact values of $t(G)$ and $s(G)$ are determined for several special graphs. In Chapter 5 we show that it is NP-complete to decide for a given graph G and a given integer k, whether $s(G) \leq k$.

In [65] Ostrovskii proves that $t(G)$ always equals the maximum number of edge-disjoint paths connecting two vertices of G; this is also a consequence of the existence of Gomory-Hu trees [24]. Furthermore, he studies the growth rate of the maximum possible value $\mu(n) = \max\{s(G) \mid G \text{ is a connected graph of order } n\}$ of the spanning tree congestion for connected graphs of order n. Ostrovskii proves that $\mu(n) < \left\lfloor \frac{n^2}{4} \right\rfloor$ for $n \geq 6$. For odd $k \in \mathbb{N}$, he constructs a connected graph G_k of order $3k^2 - 2k$ with $s(G_k) \geq \frac{1}{4}k^3$; thus $\mu(n) = \Omega\left(n^{\frac{3}{2}}\right)$. As the main open problem in [65], Ostrovskii asks for more precise estimates on the growth rate of $\mu(n)$. In the Section 4.3 we prove that $\mu(n) \leq n(G)^{\frac{3}{2}}$. In view of the graphs G_k, this determines the growth rate of $\mu(n)$ quite accurately up to constants and terms of lower order. Furthermore, we prove that $s(G) \leq nt(G)$ for connected graphs G. Both results are based on [59].

Before we start with the results we need some more terminology concerning trees. A tree T is called *rooted in* r, if a vertex $r \in V(T)$ is specified as the *root* of T. Let T be a tree that is rooted in r. The *parent* of a vertex $v \in V(T) \setminus \{r\}$ is the neighbor of v on the unique v-r-path. A *child* of a vertex $v \in V(T)$ is a vertex of which v is the parent. An *ancestor* of a vertex $v \in V(T) \setminus \{r\}$ is a vertex $w \neq v$ on the unique v-r-path. A *descendant* of a vertex $v \in V(T)$ is a vertex of which v is an ancestor.

4.1 Structural Results for Trees

In [32] it is shown that $\gamma\gamma(T) \geq 2(n(T)+1)/3$ for all trees T of order $n(T) \geq 2$. We characterize the trees achieving equality in this bound. This answers a problem posed in [32].

Theorem 4.2 *If T is a tree, then $\gamma\gamma(T) \geq 2(n(T)+1)/3$ with equality if and only if $V(T)$ can be partitioned into two sets D and R such that D induces a perfect matching and R is an independent set all vertices of which have degree 2 in T.*

Proof: Let T be a tree of order n and let D_1 and D_2 be two disjoint dominating sets of T such that $\gamma\gamma(T) = |D_1| + |D_2|$. We assume that $|D_1| \geq |D_2|$. Let $D = D_1 \cup D_2$ and let $R = V(T) \setminus D$. Since every vertex in R has a neighbor in D_1 and a neighbor in D_2 and every vertex in D_1 has a neighbor in D_2, counting the edges of T yields

$$n(T) - 1 \geq 2|R| + |D_1| \geq 2|R| + |D|/2 = 2(n(T) - \gamma\gamma(T)) + \gamma\gamma(T)/2,$$

which implies $\gamma\gamma(T) \geq 2(n(T)+1)/3$.

If $\gamma\gamma(T) = 2(n(T)+1)/3$, then equality holds throughout the above inequality chain. This implies that $|D_1| = |D_2|$, every vertex in R has exactly one neighbor in D_1 and one neighbor in D_2, every vertex from D_1 has exactly one neighbor in D_2 and the three sets D_1, D_2 and R are independent. Since every vertex of D_2 has at least one neighbor in D_1, the set D induces a perfect matching and the structure of T is as described in the statement of the result.

Conversely, we assume now that $V(T)$ can be partitioned into two sets D and R such that D induces a perfect matching and R is an independent set all vertices of which have degree 2 in T. We will prove by induction on $n(T)$ that $\gamma\gamma(T) = 2(n(T)+1)/3$. More specifically, we prove that D can be partitioned into two independents sets D_1 and D_2 which are both dominating. Note that, by the assumptions, such sets D_1 and D_2 satisfy $|D_1| + |D_2| = 2(n(T)+1)/3$. If $n(T) = 2$, then the statement is trivial. Hence, we may assume that $n(T) \geq 3$. Let uv be an edge which corresponds to a leaf of the tree that arises from T by contracting all edges of the perfect matching induced by D. Note that after these contractions all vertices in R are still of degree 2. This implies that we may assume that u is a leaf of T and v has degree 2 in T. Let w be the neighbor of v different from u. Clearly, $w \in R$. The vertex set $V(T) \setminus \{u, v, w\}$ of the tree $T' = T - \{u, v, w\}$ can be partitioned into two sets $D' = D \setminus \{u, v\}$ and $R' = R \setminus \{w\}$ such that D' induces a perfect matching and R' is an independent set all vertices of which have degree 2 in T'. Hence, by induction, D' can be partitioned into two independent sets D'_1 and D'_2 both of which are dominating in T'. We may assume that the neighbor of w different from v belongs to D'_1. Now the two sets $D_1 = D'_1 \cup \{u\}$ and $D_2 = D'_2 \cup \{v\}$ are independent and dominating in T and partition D, which completes the proof. $\square$

For a tree T to satisfy $\gamma\gamma(T) = 2\gamma(T)$, it is an obvious necessary condition that no vertex of T belongs to every minimum dominating set of T. We describe an example showing that this condition is not sufficient. This disproves a conjecture of Hedetniemi, Hedetniemi, Laskar, Markus, and Slater [32].

Observation 4.3 *There are trees T for which no vertex belongs to every minimum dominating set of T and which do not have two disjoint minimum dominating sets, i.e., $\gamma\gamma(T) > 2\gamma(T)$.*

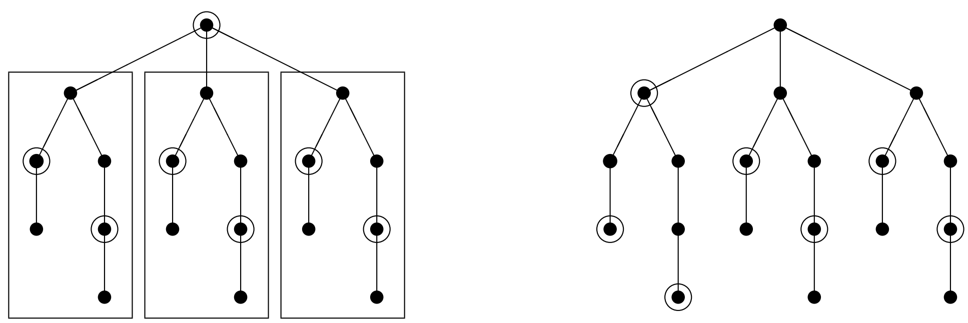


Figure 4.1

Proof: The tree two copies of which are shown in Figure 4.1 has domination number 7 and the two indicated minimum dominating sets show that no vertex belongs to every minimum dominating set of T. On the other hand it is easy to see that the union of every two disjoint dominating sets of T contains at least five vertices in each of the indicated rectangular boxes, which implies that one of the sets cannot be minimum. $\Box$

Before we proceed to our next result, we introduce some terminology. Given a rooted tree T, a set D of vertices of T and a vertex $v \in D$, we define an *external D-private child of v in T* to be a child of v in $N_T(v) \setminus N_T[D \setminus \{v\}]$. Hence, u is an external D-private child of v in T if and only if $u \notin D$, u is a child of v in T, and $N_T(u) \cap D = \{v\}$.

Theorem 4.4 answers a problem posed in [32].

Theorem 4.4 *Every tree of order at least two has a minimum dominating set and an independent dominating set that are disjoint.*

Proof: Let u be a leaf of T. Let D be a minimum dominating set containing the neighbor r of u such that

$$f(D) := \sum_{v \in D} \mathrm{dist}_T(v, r)$$

is minimum. Root T at r. Note that u is an external D-private child of r in T. If some vertex $v \in D \setminus \{r\}$ has no external D-private child in T, then the parent w of v is not in

D. Because the set $D' = (D \setminus \{v\}) \cup \{w\}$ is a minimum dominating set of T containing r with $f(D') = f(D) - 1$, which is a contradiction. Hence, all vertices in D have external D-private children in T. Clearly, a set I containing exactly one external D-private child of every vertex in D is an independent set and a maximal independent subset of $V(T) \setminus D$ that contains I is a dominating set of T. This completes the proof. □

Note that 4.4 is not true for arbitrary graphs. The graph in Figure 4.2 has no independent dominating set in the complement of the unique minimum dominating set.

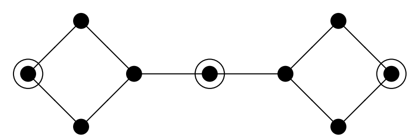


Figure 4.2: The encircled vertices belong to the unique minimum dominating set.

Now we prove Conjecture 4.1. Note that Theorem 4.5 implies Theorem 4.4. We included the proof for Theorem 4.4 because of its simplicity.

Theorem 4.5 *Conjecture 4.1 is true.*

Before we proceed to the proof, we explain our general strategy. Given T and D as in the statement of the conjecture, it suffices to determine an independent set J of vertices that is disjoint from D and contains a neighbor of every vertex in D, because a maximal independent set I that contains J but is disjoint from D is clearly a dominating set of T. A simple strategy to select the elements of J is to root T in some vertex r in D and to select a child of every vertex in D that itself is not contained in D. Since T has order at least 2 and D contains at most one leaf of T, choosing the root r of T as a leaf, if possible, every vertex in D has at least one child. If this strategy succeeds, then the selected vertices will clearly form an independent set. Nevertheless, this strategy fails in the presence of vertices u in D all children of which are also in D. For such a vertex, we necessarily have to choose its parent. Since J has to be independent, this choice affects the choosability of the children of ancestors of u in D. Working out the consequences of this reasoning, leads to the algorithm Select (cf. Algorithm 1 below).

We proceed to the

Proof of Theorem 4.5: In view of the above remarks it suffices to argue that Select successfully determines an independent set J of T such that $D \cap J = \emptyset$ and $D \subseteq N_T(J)$. Note that, since D contains at most one leaf and by the choice of r in line 3, every vertex in D has at least one child.

Claim *The vertex u in line 5 has a parent that does not belong to D.*

Input: A tree T of order at least 2 and a minimum dominating set D of T containing at most one leaf of T.

Output: An independent dominating set I of T that is disjoint from D.

1 **begin**
2 Choose a vertex $r \in D$ of minimum degree $d_T(r) = \min\{d_T(u) \mid u \in D\}$;
3 Root T in r;
4 $J \leftarrow \emptyset$;
5 **while** $\exists\ u \in D$ *such that* $u \notin N_T(J)$ *and all children of* u *lie in* $D \cup N_T(J)$ **do**
6 Let v be the parent of u;
7 $J \leftarrow J \cup \{v\}$;
8 $\text{partner}(u) \leftarrow v$;
9 **end**
10 **while** $\exists\ u \in D$ *such that* $u \notin N_T(J)$ **do**
11 Choose a child v of u such that $v \notin D \cup N_T(J)$;
12 $J \leftarrow J \cup \{v\}$;
13 **end**
14 Let I be a maximal independent set of T with $J \subseteq I$ and $D \cap I = \emptyset$;
15 **end**

Algorithm 1: Select

Proof of the Claim: For contradiction, we consider the first execution of the **while**-loop in line 5 for which the vertex u has no parent that does not belong to D, i.e. either u is the root r of T or the parent of u belongs to D.

Let D' denote the set of vertices u' from D that can be reached from u on a path P of the form

$$P = u_0 w_1 v_1 u_1 w_2 v_2 u_2 \ldots w_l v_l u_l \tag{4.2}$$

with $u_0 = u$, $u_l = u'$, $l \in \mathbb{N}$, $w_i \notin D$, and $\text{partner}(u_i) = v_i$ for $1 \leq i \leq l$. Note that w_1 is a child of u. Let the set D'' contain the parent of the parent of u' — the grandparent of u' — for every vertex u' in D'. Let $\tilde{D} = (D \setminus (D' \cup \{u\})) \cup D''$. Note that $|\tilde{D}| < |D|$.

Let w'' be a child of u. Clearly, $w'' \notin J$. If $w'' \in D$, then $w'' \in \tilde{D}$. If $w'' \notin D$, then w'' has a child v'' that belongs to J, and v'' has a child u'' that belongs to D such that $\text{partner}(u'') = v''$. Since $uw''v''u''$ is a path as in (4.2), we obtain, by the definition of D', that $u'' \in D'$. This implies $w'' \in D''$, and hence $w'' \in \tilde{D}$. Therefore, in both cases, $u, w'' \in N_T[\tilde{D}]$ and all vertices that were dominated by u in D are still dominated by vertices in $\tilde{D}$.

Let $u' \in D'$. Let P be as in (4.2) with $u' = u_l$. Since $w_l \in \tilde{D}$, we have $v_l \in N_T[\tilde{D}]$. If w'' is a child of u', then exactly the same argument as above implies that $w'' \in \tilde{D}$. Hence, again all vertices that were dominated by u' in D are still dominated by vertices in $\tilde{D}$.

Altogether, we obtain that $\tilde{D}$ is a dominating set of T, which contradicts the assumption that D is a minimum dominating set. $\square$

By the claim, the **while**-loop in line 5 successfully adds to the set J the parents of vertices in D that do not belong to D. By the condition for the **while**-loop in line 5, just before the first execution of the **while**-loop in line 10, the set J is independent and every vertex $u \in D$ with $u \notin N_T(J)$ has at least one child that does not belong to D and is non-adjacent to the vertices in J. Since during the executions of the **while**-loop in line 10 only children of vertices in D are added to J, this property is maintained throughout the remaining execution of Select. Hence, the **while**-loop in line 10 successfully adds to the set J the children of vertices in D that do not belong to D such that after the last execution of the **while**-loop in line 10, the set J is independent, disjoint from D and $D \subseteq N_T(J)$.

By the above remarks, the set I defined in line 14 is an independent dominating set of T, which completes the proof. $\square$

4.1.1 Trees with an (α, γ)-pair

In this subsection we describe a constructive characterization of trees that have an (α, γ)-pair. We describe suitable reductions and explain how these reductions yield a constructive characterization of trees with an (α, γ)-pair. The results of this subsection imply the existence of an polynomial algorithm that decides whether a given tree has an (α, γ)-pair (Corollary 5.4).

The first lemma deals with some small trees.

Lemma 4.6 (a) *For $2 \le n \le 6$, the path $P_n = u_1 u_2 \dots u_n$ has the following (α, γ)-pair (I_n, D_n):*

$$\begin{aligned}
(I_2, D_2) &= (\{u_1\}, \{u_2\}) \\
(I_3, D_3) &= (\{u_1, u_3\}, \{u_2\}) \\
(I_4, D_4) &= (\{u_1, u_4\}, \{u_2, u_3\}) \\
(I_5, D_5) &= (\{u_1, u_3, u_5\}, \{u_2, u_4\}) \\
(I_6, D_6) &= (\{u_1, u_3, u_6\}, \{u_2, u_5\}).
\end{aligned}$$

(b) *The tree T^* with*

$$\begin{aligned}
V(T^*) &= \{u_0, u_1, v_0, v_1, v_2, w_0, w_1, w_2, w_3, x\} \\
E(T^*) &= \{u_0u_1, u_1x, v_0v_1, v_1v_2, v_2x, w_0w_1, w_1w_2, w_2w_3, w_3x\}
\end{aligned}$$

has the (α, γ)-pair

$$(\{u_0, v_0, w_0, v_2, w_2\}, \{u_1, v_1, w_1, x\}).$$

Proof: It is very easy to check that the given sets are maximum independent sets and minimum dominating sets that are disjoint. $\square$

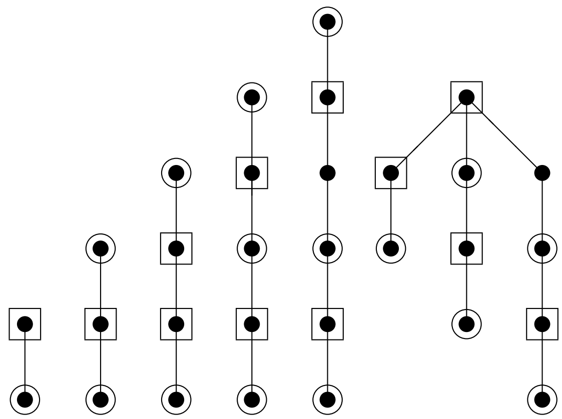


Figure 4.3: The trees $P_2, P_3, \ldots, P_6$ and T^*.

Lemma 4.7 *Let T contain a path $P = u_0u_1 \ldots u_5$ such that $d_T(u_0) = 1$ and $d_T(u_1) = d_T(u_2) = d_T(u_3) = d_T(u_4) = 2$.*

(a) $\alpha(T') + 2 \leq \alpha(T) \leq \alpha(T') + 3$ *for* $T' = T - \{u_0, u_1, \ldots, u_4\}$.

(b) *If $\alpha(T) = \alpha(T') + 3$, then T has an (α, γ)-pair if and only if $T'' = T - \{u_0, u_1\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T'') + 1$, and $\gamma(T) = \gamma(T'') + 1$.*

(c) *If $\alpha(T) = \alpha(T') + 2$, then T has an (α, γ)-pair if and only if $T''' = T - \{u_0, u_1, u_2\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T''') + 1$, and $\gamma(T) = \gamma(T''') + 1$.*

Proof: (a) The first inequality follows, since for every independent set I' of T', the set $I' \cup \{u_0, u_2\}$ is an independent set of T. The second inequality follows, since every independent set I of T contains at most three of the vertices in $\{u_0, u_1, \ldots, u_4\}$ and $I \setminus \{u_0, u_1, \ldots, u_4\}$ is an independent set of T'.

(b) Let $\alpha(T) = \alpha(T') + 3$. Note that this implies that every maximum independent set of T contains u_0, u_2 and u_4. Therefore, if T has an (α, γ)-pair (I, D), then $u_0, u_2, u_4 \in I$ and hence $u_1, u_3 \in D$. Clearly, $\alpha(T'') \leq \alpha(T') + 2$. Since $I \setminus \{u_0\}$ is an independent set in T'', we have $\alpha(T'') \geq \alpha(T) - 1 = \alpha(T') + 2$ and thus $\alpha(T) = \alpha(T') + 3 = \alpha(T'') + 1$. Clearly, $\gamma(T) \leq \gamma(T'') + 1$. Since $D \setminus \{u_1\}$ is a dominating set in T'', we have $\gamma(T'') \leq \gamma(T) - 1$ and thus $\gamma(T) = \gamma(T'') + 1$. Now $(I \setminus \{u_0\}, D \setminus \{u_1\})$ is an (α, γ)-pair of T''.

Conversely, if T'' has an (α, γ)-pair (I'', D''), $\alpha(T) = \alpha(T'') + 1$ and $\gamma(T) = \gamma(T'') + 1$, then $(I'' \cup \{u_0\}, D'' \cup \{u_1\})$ is an (α, γ)-pair of T.

(c) Let $\alpha(T) = \alpha(T') + 2$. If T has an (α, γ)-pair (I, D), then we may assume without loss of generality that $u_0, u_3 \in I$ and $u_1, u_4 \in D$. Clearly, $\alpha(T''') \leq \alpha(T') + 1$. Since $I \setminus \{u_0\}$ is an independent set in T''', we have $\alpha(T''') \geq \alpha(T) - 1 = \alpha(T') + 1$ and thus $\alpha(T) = \alpha(T') + 2 = \alpha(T''') + 1$. Clearly, $\gamma(T) \leq \gamma(T''') + 1$. Since $D \setminus \{u_1\}$ is a dominating

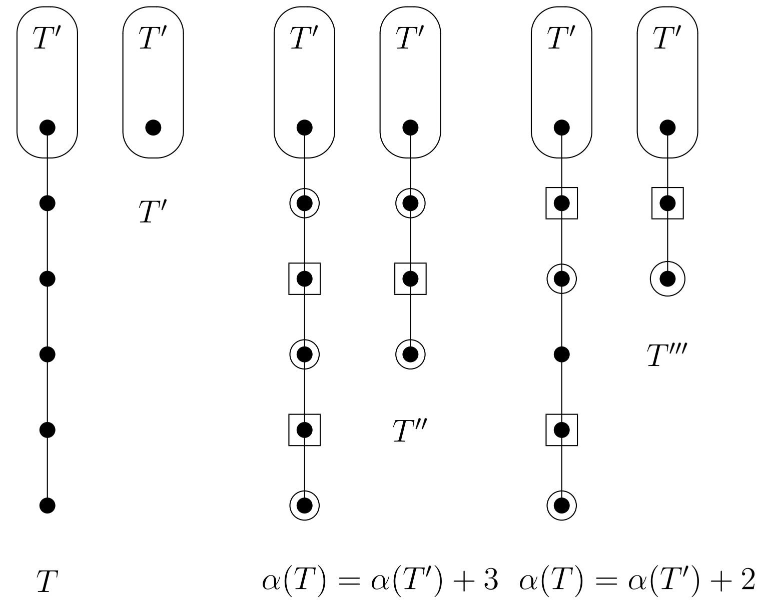


Figure 4.4: The trees T, T', T'' and T'''.

set in T''', we have $\gamma(T''') \leq \gamma(T) - 1$ and thus $\gamma(T) = \gamma(T''') + 1$. Now $(I \setminus \{u_0\}, D \setminus \{u_1\})$ is an (α, γ)-pair of T'''.

Conversely, if T''' has an (α, γ)-pair (I''', D'''), $\alpha(T) = \alpha(T''') + 1$ and $\gamma(T) = \gamma(T''') + 1$, then $(I''' \cup \{u_0\}, D''' \cup \{u_1\})$ is an (α, γ)-pair of T. $\Box$

Combining Lemma 4.6 (a) with Lemma 4.7 it is easy to check that the only paths P_n with an (α, γ)-pair satisfy $n \in \{2, 3, 4, 5, 6, 7, 8, 10\}$.

Lemma 4.8 *Let T contain a path $P = u_0 u_1 \dots u_r w v_s v_{s-1} \dots v_0$ with $r, s \geq 0$ such that $d_T(u_0) = d_T(v_0) = 1$, $d_T(u_i) = 2$ for $1 \leq i \leq r$ and $d_T(v_j) = 2$ for $1 \leq j \leq s$.*

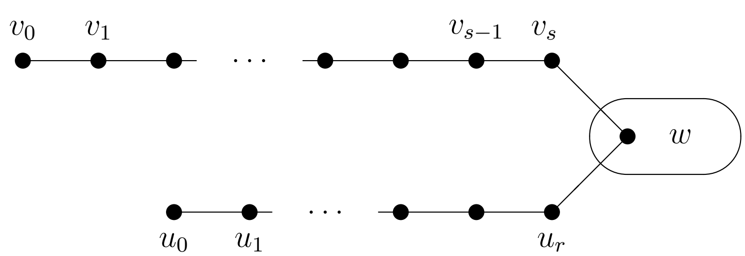


Figure 4.5: The path $P = u_0 u_1 \dots u_r w v_s v_{s-1} \dots v_0$.

(a) *If $r = 2k$ and $s = 2l$ for some $0 \leq k, l \leq 1$ with $k \geq l$, then T has an (α, γ)-pair if and only if $T' = T - \{u_i \mid 0 \leq i \leq 2k\}$ has an (α, γ)-pair, $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + k$.*

(b) *If* $r = 2k+1$ *and* $s = 0$ *for some* $0 \leq k \leq 1$*, then* T *has an* (α,γ)*-pair if and only if* $T' = T - \{u_i \mid 0 \leq i \leq 2k+1\}$ *has an* (α,γ)*-pair,* $\alpha(T) = \alpha(T') + k + 1$ *and* $\gamma(T) = \gamma(T') + 1$.

(c) *If* $r = s = 1$*, then* T *has an* (α,γ)*-pair if and only if* $T' = T - \{u_0, u_1\}$ *has an* (α,γ)*-pair.*

(d) *If* $r = s = 3$*, then* T *has an* (α,γ)*-pair if and only if* $T' = T - \{u_0, u_1, u_2, v_0, v_1, v_2\}$ *has an* (α,γ)*-pair and* $\alpha(T) = \alpha(T') + 2$.

(e) *If* $r = 1$*,* $s = 2$ *and* $d_T(w) = 3$*, then* T *has an* (α,γ)*-pair if and only if* $T' = T - V(P)$ *has an* (α,γ)*-pair.*

(f) *If* $r = 1$*,* $s = 3$ *and* $d_T(w) = 3$*, then* T *has an* (α,γ)*-pair if and only if* $T' = T - \{u_0, u_1\}$ *has an* (α,γ)*-pair.*

(g) *If* $r = 2$*,* $s = 3$ *and* $d_T(w) = 3$*, then* T *has an* (α,γ)*-pair if and only if* $T' = T - \{u_0, u_1, v_0, v_1, v_2, v_3\}$ *has an* (α,γ)*-pair.*

Proof: (a) Note that every maximum independent set I of T satisfies $I \cap V(P) = \{u_{2i} \mid 0 \leq i \leq k\} \cup \{v_{2j} \mid 0 \leq j \leq l\}$.

Therefore, if T has an (α,γ)-pair (I, D), then $u_{2i} \in I$ for $0 \leq i \leq k$, $v_{2j} \in I$ for $0 \leq j \leq l$, $u_{2i+1} \in D$ for $0 \leq i \leq k-1$ and $v_{2j+1} \in D$ for $0 \leq j \leq l-1$. Clearly, $\alpha(T) \leq \alpha(T') + k + 1$. Since $I \setminus \{u_{2i} \mid 0 \leq i \leq k\}$ is an independent set in T', we have $\alpha(T') \leq \alpha(T) - (k+1)$ and thus $\alpha(T) = \alpha(T') + k + 1$. Clearly, $\gamma(T) \leq \gamma(T') + k$ — note that $k = 0$ implies $l = 0$ and $w \in D$. Since $D \setminus \{u_{2i+1} \mid 0 \leq i \leq k-1\}$ is a dominating set in T', we have $\gamma(T') \leq \gamma(T) - k$ and thus $\gamma(T) = \gamma(T') + k$. Now $(I \setminus \{u_{2i} \mid 0 \leq i \leq k\}, D \setminus \{u_{2i+1} \mid 0 \leq i \leq k-1\})$ is an (α,γ)-pair of T'.

Conversely, if T' has an (α,γ)-pair (I', D'), $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + k$, then in view of $l \leq 1$ we may assume that $v_{2l} \in I'$. Hence, $w \notin I'$ and $(I' \cup \{u_{2i} \mid 0 \leq i \leq k\}, D' \cup \{u_{2i+1} \mid 0 \leq i \leq k-1\})$ is an (α,γ)-pair of T.

(b) If T has an (α,γ)-pair, then it has an (α,γ)-pair (I, D) such that $v_0 \in I$, $w \in D$, $|I \cap \{u_i \mid 0 \leq i \leq 2k+1\}| = k+1$ and $|D \cap \{u_i \mid 0 \leq i \leq 2k+1\}| = 1$. Similarly, if T' has an (α,γ)-pair, then it has an (α,γ)-pair (I', D') such that $v_0 \in I$ and $w \in D$. This easily implies that $\alpha(T) = \alpha(T') + k + 1$, $\gamma(T) = \gamma(T') + 1$ and that T has an (α,γ)-pair if and only if T' has an (α,γ)-pair.

(c) If T has an (α,γ)-pair, then it has an (α,γ)-pair (I, D) such that $v_0 \in I$ and $v_1 \in D$. Similarly, if T' has an (α,γ)-pair, then it has an (α,γ)-pair (I', D') such that $v_0 \in I$ and $v_1 \in D$. This easily implies that $\alpha(T) = \alpha(T') + 1$, $\gamma(T) = \gamma(T') + 1$ and that T has an (α,γ)-pair if and only if T' has an (α,γ)-pair.

(d) Note that every minimum dominating set of T contains w, u_1 and v_1. Similarly every minimum dominating set of T' contains w. This easily implies that $\alpha(T) = \alpha(T') + 2$, $\gamma(T) = \gamma(T') + 2$ and that T has an (α,γ)-pair if and only if T' has an (α,γ)-pair.

(e) It is easy to see that $\alpha(T) = \alpha(T') + 3$ and $\gamma(T) = \gamma(T') + 2$. If T has an (α, γ)-pair, then it has an (α, γ)-pair (I, D) such that $u_0, v_0, v_2 \in I$ and $u_1, v_1 \in D$. This easily implies that T has an (α, γ)-pair if and only if T' has an (α, γ)-pair.

(f) It is easy to see that $\alpha(T) = \alpha(T') + 1$. Similarly, since T' has a minimum dominating set containing w, we have $\gamma(T) = \gamma(T') + 1$, which again implies the desired result.

(g) Note that T has a maximum independent set containing u_2 and a minimum dominating set containing w. This easily implies that $\alpha(T) = \alpha(T') + 3$ and $\gamma(T) = \gamma(T') + 2$, which again implies the desired result. □

Lemma 4.9 *Let T contain three internally vertex-disjoint paths $P = u_0u_1x$, $Q = v_0v_1v_2x$ and $R = w_0w_1w_2w_3x$ such that $d_T(u_0) = d_T(v_0) = d_T(w_0) = 1$, $d_T(u_1) = d_T(v_1) = d_T(v_2) = d_T(w_1) = d_T(w_2) = d_T(w_3) = 2$ and $d_T(x) = 4$, then T has an (α, γ)-pair if and only if*

$$T' = T - \{u_0, u_1, v_0, v_1, w_0, w_1, w_2, w_3\}$$

has an (α, γ)-pair.

Proof: Note that T has a maximum independent set I such that $I \cap (V(P) \cup V(Q) \cup V(R)) = \{u_0, v_0, w_0, v_2, w_2\}$ and a minimum dominating set D such that $D \cap (V(P) \cup V(Q) \cup V(R)) = \{u_1, v_1, w_1, x\}$. This easily implies that $\alpha(T) = \alpha(T') + 4$ and $\gamma(T) = \gamma(T') + 3$.

If T has an (α, γ)-pair, then T has an (α, γ)-pair (I, D) such that $I \cap (V(P) \cup V(Q) \cup V(R)) = \{u_0, v_0, w_0, v_2, w_2\}$ and $D \cap (V(P) \cup V(Q) \cup V(R)) = \{u_1, v_1, w_1, x\}$. In this case $(I \setminus \{u_0, v_0, w_0, w_2\}, D \setminus \{u_1, v_1, w_1\})$ is an (α, γ)-pair of T'. Conversely, if T' has an (α, γ)-pair, then T' has an (α, γ)-pair (I', D') such that $v_2 \in I'$ and $x \in D'$. In this case $(I' \cup \{u_0, v_0, w_0, w_2\}, D' \cup \{u_1, v_1, w_1\})$ is an (α, γ)-pair of T, which completes the proof. □

For integers $k \geq 1$ and $d_1 \geq d_2 \geq \ldots \geq d_k \geq 1$, a tree T is said to have a $(d_1, d_2, \ldots, d_k)$-*tinsel* $(P_1, P_2, \ldots, P_k)$ *pending on* v if $P_1, P_2, \ldots, P_k$ are k internally vertex-disjoint paths in T such that

$$P_i = u_{i,0}u_{i,1} \ldots u_{i,d_i-1}v,$$

$d_T(u_{i,0}) = 1$ and $d_T(u_{i,j}) = 2$ for $1 \leq i \leq k$ and $1 \leq j \leq d_i - 1$ and $d_T(v) = k + 1$. For integers $\partial d_1, \partial d_2, \ldots, \partial d_k$ with $0 \leq \partial d_i \leq d_i$ for $1 \leq i \leq k$, the tree

$$T - \bigcup_{i=1}^{k} \bigcup_{j=0}^{\partial d_i - 1} \{u_{i,j}\}$$

is said to arise from the tree T by

$(\partial d_1, \partial d_2, \ldots, \partial d_k)$-*cutting* the $(d_1, d_2, \ldots, d_k)$-tinsel $(P_1, P_2, \ldots, P_k)$.

Note that a tree T that is not a path and is rooted at an endvertex of a longest path has a tinsel $(P_1, P_2, \ldots, P_k)$ pending on some vertex v such that $k \geq 2$ and all vertices of the paths P_i are either v or descendants of v.

The next result summarizes the reductions captured by Lemmas 4.7 to 4.9 and yields a constructive characterization of trees having an (α, γ)-pair.

Theorem 4.10 *Let $T = (V, E)$ be a tree that is not a path and different from the tree T^*. Let $(P_1, P_2, \ldots, P_k)$ be a $(d_1, d_2, \ldots, d_k)$-tinsel pending on v with $k \geq 2$.*

The tree T has an (α, γ)-pair if and only if the tree T' that arises from the tree T by

$$(\partial d_1, \partial d_2, \ldots, \partial d_k)\text{-cutting the } (d_1, d_2, \ldots, d_k)\text{-tinsel } (P_1, P_2, \ldots, P_k)$$

has an (α, γ)-pair and $(\alpha(T) - \alpha(T'), \gamma(T) - \gamma(T')) = (\partial\alpha, \partial\gamma)$ where

(a) *if $d_1 \geq 5$ and $\alpha(T) = \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) + 3$, then*

$$(\partial d_1, \partial d_2, \ldots, \partial d_k) = (2, 0, \ldots, 0)$$

and $(\partial\alpha, \partial\gamma) = (1, 1)$.

(b) *if $d_1 \geq 5$ and $\alpha(T) = \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) + 2$, then*

$$(\partial d_1, \partial d_2, \ldots, \partial d_k) = (3, 0, \ldots, 0)$$

and $(\partial\alpha, \partial\gamma) = (1, 1)$.

(c) *if there are two indices $1 \leq i < j \leq k$ such that $d_i, d_j \in \{1, 3\}$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial\alpha, \partial\gamma) = \left(\frac{d_i+1}{2}, \frac{d_i-1}{2}\right)$.*

(d) *if $d_k = 1$ and there is an index $1 \leq i < k$ such that $d_i \in \{2, 4\}$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial\alpha, \partial\gamma) = \left(\frac{d_i}{2}, 1\right)$.*

(e) *if there are two indices $1 \leq i < j \leq k$ such that $d_i = d_j = 2$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial\alpha, \partial\gamma) = (1, 1)$.*

(f) *if there are two indices $1 \leq i < j \leq k$ such that $d_i = d_j = 4$, then $\partial d_i = \partial d_j = 3$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \notin \{i, j\}$ and $\partial\alpha = 2$.*

(g) *if $k = 2$ and $(d_1, d_2) = (3, 2)$, then $T' = T - (V(P_1) \cup V(P_2))$.*

(h) *if $k = 2$ and $(d_1, d_2) = (4, 2)$, then $(\partial d_1, \partial d_2) = (0, 2)$.*

(i) *if $k = 2$ and $(d_1, d_2) = (4, 3)$, then $(\partial d_1, \partial d_2) = (4, 2)$.*

(j) *if $k = 3$ and $(d_1, d_2) = (4, 3, 2)$, then $(\partial d_1, \partial d_2, \partial d_3) = (4, 2, 2)$.*

Furthermore, one of the cases (a)-(j) occurs.

Proof: If $d_1 \geq 5$, then, by Lemma 4.7 (a),

$$2 \leq \alpha(T) - \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) \leq 3.$$

Now, by Lemma 4.7 (b) and (c), either (a) or (b) occurs. Hence, we may assume that $d_1 \leq 4$, i.e. all d_i are at most 4. If there are two odd d_i's, then, by Lemma 4.8 (a), the case (c) occurs. Hence, we may assume that at most one of the d_i is odd. If $d_k = 1$, then, by Lemma 4.8 (b), the case (d) occurs. Hence, we may assume that all d_i are either 2, 3 or 4. If there are two d_i's equal to 2, then, by Lemma 4.8 (c), the case (e) occurs. Hence, we may assume that at most one of the d_i is 2. If there are two d_i's equal to 4, then, by Lemma 4.8 (d), the case (f) occurs. Hence, we may assume that at most one of the d_i is 4. If $k \geq 3$, then $k = 3$, $(d_1, d_2, d_3) = (4, 3, 2)$ and, by Lemma 4.9, the case (j) occurs. Hence, we may assume $k = 2$ and, by Lemma 4.8 (e) through (g), one of the cases (g) through (i) occurs. This completes the proof. □

4.2 Independence in Connected Graphs with Specified Odd Girth

We need the following two notions related to the independence number. If G is a graph and H is a spanning bipartite subgraph of G with a fixed bipartition $V(G) = A \cup B$, then let

$$\begin{aligned}
\alpha\alpha(G) &= \max\{|I_1| + |I_2| \ \mid \ I_1 \text{ and } I_2 \text{ are disjoint independent sets in } G\}, \text{ and} \\
\alpha\alpha(G, H) &= \max\left\{ |I_1| + |I_2| \ \middle| \ \begin{array}{l} (I_1 \text{ and } I_2 \text{ are independent sets in } G) \\ \wedge \ (I_1 \subseteq A) \ \wedge \ (I_2 \subseteq B) \end{array} \right\}.
\end{aligned}$$

Clearly,

$$2\alpha(G) \geq \alpha\alpha(G) \geq \alpha\alpha(G, H).$$

The basic idea of our approach in this section is captured by the following very simple lemma.

Lemma 4.11 *If G is a graph and H is a spanning bipartite subgraph of G, then*

$$\alpha\alpha(G, H) \geq n(G) - |E(G) \setminus E(H)|.$$

Proof: Starting with $(I_1, I_2) = (A, B)$ where $V(G) = A \cup B$ is the fixed bipartition of H and adding the edges of $E(G) \setminus E(H)$ one by one to H, we have to remove at most one vertex from either I_1 or I_2 for every added edge. Therefore, after adding all edges from $E(G) \setminus E(H)$ into H, we obtain two disjoint independent sets of G respecting the bipartition of H that are of total cardinality at least $n(G) - |E(G) \setminus E(H)|$. □

The next result is a first application of this idea.

Proposition 4.12 *If G is a connected graph and T is a spanning tree of G, then the following statements hold.*

(a)

$$\alpha\alpha(G, T) \geq 2n(G) - m(G) - 1 \tag{4.3}$$

with equality if and only if $E(G) \setminus E(T)$ is a matching and $T + e$ has an odd cycle for every edge $e \in E(G) \setminus E(T)$.

(b)

$$\alpha\alpha(G) \geq 2n(G) - m(G) - 1 \tag{4.4}$$

with equality if and only if all cycles of G are odd and vertex-disjoint.

Proof: The lower bounds in (a) and (b) follow immediately from Lemma 4.11. It remains to characterize the extremal graphs for (4.3) and (4.4).

(a) Let $V(G) = A \cup B$ denote the bipartition of T. If $E(G) \setminus E(T)$ is a matching and $T + e$ has an odd cycle for every edge $e \in E(G) \setminus E(T)$, then $G' = G - E(T)$ is the union of complete graphs of orders 1 and 2. Since $\alpha\alpha(G,T) = \alpha(G')$, this easily implies equality in (4.3).

Conversely, we assume that equality holds in (4.3). If $T + e$ has no odd cycle for some edge $e \in E(G) \setminus E(T)$, then

$$\begin{aligned} \alpha\alpha(G,T) &= \alpha\alpha(G-e,T) \\ &\geq 2n(G) - (m(G)-1) - 1 \\ &= 2n(G) - m(G), \end{aligned}$$

which is a contradiction. Hence, $T + e$ has an odd cycle for every edge $e \in E(G) \setminus E(T)$.

If $E(G) \setminus E(T)$ contains two distinct edges e and f that are both incident with a common vertex u, then T is a spanning tree of $G' = G - \{e, f\}$. For every pair (I_1', I_2') of disjoint independent sets of G' with $I_1' \subseteq A$ and $I_2' \subseteq B$, $(I_1' \setminus \{u\}, I_2' \setminus \{u\})$ is a pair of disjoint independent sets of G with $I_1 \subseteq A$ and $I_2 \subseteq B$, which implies the contradiction

$$\begin{aligned} \alpha\alpha(G,T) &\geq \alpha\alpha(G',T) - 1 \\ &\geq 2n(G') - m(G') - 1 - 1 \\ &= 2n(G) - m(G). \end{aligned} \tag{4.5}$$

This completes the proof of (a).

(b) Let G be a connected graph such that all cycles of G are odd and vertex-disjoint. If G contains a vertex u of degree 1, then, by an inductive argument,

$$\begin{aligned} \alpha\alpha(G) &= \alpha\alpha(G-u) + 1 \\ &= 2n(G-u) - m(G-u) - 1 + 1 \\ &= 2(n(G)-1) - (m(G)-1) - 1 + 1 \\ &= 2n(G) - m(G) - 1. \end{aligned}$$

Hence, we may assume that G has an endblock that is an odd cycle C. Clearly, for every pair (I_1, I_2) of disjoint independent sets of G, the set $I_1 \cup I_2$ contains at most $n(C) - 1$

many vertices of C. Let $G' = G - V(C)$. If G' is empty, then G is an odd cycle and equality in (4.4) is trivial. Otherwise, by an inductive argument,

$$\begin{aligned}\alpha\alpha(G) &\leq \alpha\alpha(G') + n(C) - 1 \\ &= 2n(G') - m(G') - 1 + n(C) - 1 \\ &= 2(n(G) - n(C)) - (m(G) - (n(C) + 1)) - 1 + n(C) - 1 \\ &= 2n(G) - m(G) - 1,\end{aligned}$$

i.e. equality in (4.4) holds.

Conversely, let G be a connected graph with equality in (4.4). If G contains two incident edges whose removal does not disconnect the graph, then we obtain a similar contradiction as in (4.5). Therefore, removing any pair of incident edges disconnects G, which immediately implies that all cycles of G are vertex-disjoint. In view of this restricted structure of G, the assumption of the existence of an even cycle easily leads to the contradiction $\alpha\alpha(G) \geq 2n(G) - m(G)$, which completes the proof. $\Box$

Proposition 4.12 immediately implies the following.

Corollary 4.13 *If G is a connected graph, then*

$$\alpha(G) \geq n(G) - \frac{m(G)}{2} - \frac{1}{2} \tag{4.6}$$

with equality only if all cycles of G are odd and vertex-disjoint.

In view of the extremal graphs, the estimates (4.4) and (4.6) are best-possible for graphs G if and only if their size is at most $\frac{(g_{odd}(G)+1)n(G)}{g_{odd}(G)} - 1$. Intuitively speaking, up to this maximum possible size, the "*price*" of an additional edge is 1 for $\alpha\alpha(G)$ and 1/2 for $\alpha(G)$. Our next result shows that beyond this maximum possible size, additional edges are at least "50% *off*".

If T is a tree and e is an edge such that $T+e$ is not bipartite, then e is called *T-unfaithful*.

Theorem 4.14 *Let G be a connected graph. If $m(G) \geq \left\lfloor \frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)} \right\rfloor - 1$, then*

$$\begin{aligned}2\alpha(G) &\geq \alpha\alpha(G) \\ &\geq \left\lceil \frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)} \right\rceil - \frac{1}{2}\left(m(G) - \left(\left\lfloor \frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)} \right\rfloor - 1\right)\right).\end{aligned}$$

Proof: We consider a finite sequence

$$G_0, G_1, \ldots, G_k$$

of connected graphs defined as follows. Let $G_0 = G$. If for some $i \in \mathbb{N}_0$, the graph G_i is defined, then let T_i be a spanning tree of G_i. Let m_i denote the number of T_i-unfaithful edges of G_i. Note that all cycles created in T_i by adding a T_i-unfaithful edge of G_i have length at least $g_{\rm odd}(G)$. If $m_i \leq \left\lfloor \frac{n(G)}{g_{\rm odd}(G)} \right\rfloor$, then set $k = i$ and terminate the sequence. If $m_i > \left\lfloor \frac{n(G)}{g_{\rm odd}(G)} \right\rfloor$, then there are two T_i-unfaithful edges of G_i such that the two cycles created in T_i by adding these edges intersect. Clearly, this implies the existence of two incident edges e_i and f_i of G_i such that $G_i - \{e_i, f_i\}$ is connected. Let $G_{i+1} = G_i - \{e_i, f_i\}$. Since, for $i \geq 0$, the graph G_{i+1} arises from G_i by deleting two edges, this process necessarily terminates. By the choice of k, we have $m_{k-1} \geq \left\lfloor \frac{n(G)}{g_{\rm odd}(G)} \right\rfloor + 1$. Furthermore, since G_{k-1} has exactly $m(G) - 2(k-1)$ edges, we have $m_{k-1} \leq m(G) - (n(G) - 1) - 2(k-1)$. Combining these two estimates yields

$$k \leq \frac{1}{2}\left(m(G) - \left(\left\lfloor \frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)} \right\rfloor - 1\right)\right) + \frac{1}{2}$$

with equality if and only if

$$\begin{aligned}
\left\lfloor \frac{n(G)}{g_{\rm odd}(G)} \right\rfloor + 1 &= m_{k-1} \\
&= |E(G_{k-1}) \setminus E(T_{k-1})| \\
&= m(G) - (n(G)-1) - 2(k-1), \qquad (4.7)
\end{aligned}$$

which implies that all edges in $E(G_{k-1}) \setminus E(T_{k-1})$ are T_{k-1}-unfaithful.

Let G'_k arise from G_k by deleting all non-T_k-unfaithful edges of G_k that do not belong to T_k.

By definition,

$$\alpha\alpha(G_k, T_k) = \alpha\alpha(G'_k, T_k).$$

By Lemma 4.11,

$$\begin{aligned}
\alpha\alpha(G'_k, T_k) &\geq n(G) - |E(G'_k) \setminus E(T_k)| \\
&= n(G) - m_k \\
&\geq n(G) - \left\lfloor \frac{n(G)}{g_{\rm odd}(G)} \right\rfloor \\
&= \left\lceil \frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)} \right\rceil. \qquad (4.8)
\end{aligned}$$

Since, for $0 \leq i \leq k-1$, the graph G_{i+1} arises from G_i by deleting two incident edges, we have $\alpha\alpha(G_i) \geq \alpha\alpha(G_{i+1}) - 1$, which implies

$$\alpha\alpha(G) = \alpha\alpha(G_0) \geq \alpha\alpha(G_k) - k \geq \alpha\alpha(G_k, T_k) - k = \alpha\alpha(G'_k, T_k) - k. \qquad (4.9)$$

If $k < \frac{1}{2}\left(m(G) - \left(\left\lfloor \frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)} \right\rfloor - 1\right)\right) + \frac{1}{2}$, then, by (4.8) and (4.9),

$$\begin{aligned}
\alpha\alpha(G) &\geq \alpha\alpha(G'_k, T_k) - k \\
&\geq \left\lceil \frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)} \right\rceil - \frac{1}{2}\left(m(G) - \left(\left\lfloor \frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)} \right\rfloor - 1\right)\right).
\end{aligned}$$

If $k=\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)+\frac{1}{2}$, then

$$\begin{aligned}
\alpha\alpha(G) &\overset{(4.9)}{\geq} \alpha\alpha(G'_k,T_k)-k\\
&\overset{(4.8)}{\geq} n(G)-m_k-k\\
&= n(G)-(m_{k-1}-2)-k\\
&\overset{(4.7)}{=} n(G)-\left(\left\lfloor\frac{n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\\
&\quad -\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)-\frac{1}{2}\\
&> \left\lceil\frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)}\right\rceil-\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right),
\end{aligned}$$

which completes the proof. □

4.3 On Spanning Tree Congestion

Before we proceed to the main result in this section, we recall a beautiful theorem due to Győri [26] and Lovász [52] concerning highly connected graphs. A *vertex cut* C of a connected graph G is a subset of the vertex set $V(G)$ of G such that $G-C$ is not connected. A graph is *k-connected* if it has no vertex cut of cardinality less than k.

Theorem 4.15 (Győri [26], Lovász [52]) *For $k\in\mathbb{N}$ with $k\geq 2$, let G be a k-connected graph. If $v_1,v_2,\ldots,v_k$ are k distinct vertices of G and the integers $n_1,n_2,\ldots,n_k\in\mathbb{N}$ are such that $n_1+n_2+\ldots+n_k=n(G)$, then there exists a partition of the vertex set of G into $V_1,V_2,\ldots,V_k$ such that v_i lies in V_i, $|V_i|=n_i$, and $G[V_i]$ is connected for all $1\leq i\leq k$.*

With this tool at hand, we can proceed to our main result.

Theorem 4.16 *If G is a connected graph of order n, then $s(G)\leq n(G)^{\frac{3}{2}}$.*

Proof: If G has a vertex of degree at least $n(G)-2$, then G has a spanning tree T that arises by subdividing at most one edge of a star. In this case $c(G,T)\leq\max\{n(G)-1,2(n(G)-2)\}\leq n(G)^{\frac{3}{2}}$. Hence, we may assume that G has no such vertex, which implies that G has at most $\frac{n(G)(n(G)-3)}{2}$ edges. Since $c(G,T)\leq m(G)$ for every tree T, and since $\frac{n(G)(n(G)-3)}{2}\leq n(G)^{\frac{3}{2}}$ for $n(G)\leq 9$, the result holds for $n(G)\leq 9$. We may assume that $n(G)\geq 10$ and prove the result by an inductive argument considering two cases.

Case 1 *G has a vertex cut of cardinality at most $\sqrt{n(G)}$.*

Let Y be a vertex cut of minimum cardinality, and let Z denote the vertex set of a smallest component of $G-Y$. Let $X=V(G)\setminus(Y\cup Z)$, $x=|X|$, $y=|Y|$, and $z=|Z|$. Note that

$x \geq z$ and $y \leq \sqrt{n(G)}$. The subgraph $G[X \cup Y]$ is connected, and there is no edge joining X and Z.

Let $T(X \cup Y)$ be a spanning tree of the subgraph $G[X \cup Y]$ with

$$c(G[X \cup Y], T(X \cup Y)) \leq (x+y)^{\frac{3}{2}},$$

and let $T(Z)$ be a spanning tree of $G[Z]$ with

$$c(G[Z], T(Z)) \leq z^{\frac{3}{2}}.$$

Let $uv \in E(G)$ with $u \in Y$ and $v \in Z$, and let

$$T = (V(G), E(T(X \cup Y)) \cup \{uv\} \cup E(T(Z)))\,.$$

Note that there are at most yz edges joining $X \cup Y$ and Z. This implies that, if $e \in E_{T(X \cup Y)}$, then

$$\begin{aligned} c(e,(G,T)) &\leq (x+y)^{\frac{3}{2}} + yz \\ &= (n(G)-z)^{\frac{1}{2}} \cdot (n(G)-z) + yz \\ &\leq \sqrt{n(G)} \cdot (n(G)-z) + \sqrt{n(G)} \cdot z \\ &= n(G)^{\frac{3}{2}}. \end{aligned}$$

Furthermore, if $e \in E(T(Z))$, then

$$\begin{aligned} c(e,(G,T)) &\leq z^{\frac{3}{2}} + yz \\ &= z \cdot \left(\sqrt{z} + y\right) \\ &\leq \frac{1}{2} n(G) \cdot \left(\sqrt{n(G)} + \sqrt{n(G)}\right) \\ &= n(G)^{\frac{3}{2}}. \end{aligned}$$

Finally, if $e = uv$, then $c(e,(G,T)) \leq yz < n(G)^{\frac{3}{2}}$. Altogether, $c(G,T) \leq n(G)^{\frac{3}{2}}$, which completes the proof in this case.

Case 2 *G has no vertex cut of cardinality at most $\sqrt{n(G)}$.*

Let u be a vertex of degree at least $d = \lfloor \sqrt{n(G)} \rfloor + 1$, and let $v_1, v_2, \ldots, v_d$ be d neighbors of u. If $b = (-n(G)) \bmod (\lfloor \sqrt{n(G)} \rfloor + 1)$ and $a = (n(G)+b)/(\lfloor \sqrt{n(G)} \rfloor + 1)$, then $0 \leq b \leq \lfloor \sqrt{n(G)} \rfloor$, $n(G) = a \cdot (\lfloor \sqrt{n(G)} \rfloor + 1) - b$, and

$$\begin{aligned} a &= \frac{n(G)}{\lfloor \sqrt{n(G)} \rfloor + 1} + \frac{b}{\lfloor \sqrt{n(G)} \rfloor + 1} \\ &< (\lfloor \sqrt{n(G)} \rfloor + 1) + 1 \\ &= \lfloor \sqrt{n(G)} \rfloor + 2, \end{aligned}$$

which implies $a \leq \sqrt{n(G)}+1$. This implies that, if $n(G) = n_1+n_2+\ldots+n_d$ and $|n_i-n_j| \leq 1$ for $1 \leq i < j \leq d$, then $n_i \leq \sqrt{n(G)} + 1$.

By Theorem 4.15, there is a partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_d$ such that $v_i \in V_i$ and $G[V_i]$ is connected for $1 \leq i \leq d$. We may assume that $u \in V_1$. For $1 \leq i \leq d$, let T_i be an arbitrary spanning tree of $G[V_i]$, and let

$$T = (V(G), E(T)) = \left(V(G), E(T_1) \cup \bigcup_{i=2}^{d} \{uv_i\} \cup E(T_i) \right).$$

Since for every edge $e \in E(T)$, one component of $T - e = (V(G), E(T) \setminus \{e\})$ has at most $\sqrt{n(G)} + 1$ many vertices and $n(G) \geq 10$, we obtain

$$\begin{aligned} c(G,T) &\leq \max_{1 \leq x \leq \sqrt{n(G)}+1} x(n(G) - x) \\ &= \left(\sqrt{n(G)} + 1\right) \left(n(G) - \sqrt{n(G)} - 1\right) \\ &< n(G)^{\frac{3}{2}}, \end{aligned}$$

which completes the proof. □

In view of the estimates for $s(G)$ in terms of the expanding constant (also known as the Cheeger constant [61]), see Theorem 1 (b) in [65], and the existence of families of expanders, there exist infinite families of graphs for which $\frac{s(G)}{t(G)}$ is at least linear in $n(G)$. Our next result shows that there is a linear estimate from above.

Proposition 4.17 *If G is a connected graph, then $s(G) \leq n(G)t(G)$.*

Proof: We prove the result by induction on the order of G. For $n(G) \leq 2$, the result is trivial. Hence, let $n(G) \geq 3$. Let $V_1 \cup V_2$ be a partition of $V(G)$ such that $E(V_1, V_2) = \{uv \in E(G) \mid u \in V_1, v \in V_2\}$ is a minimum edge cut of G. Since G is connected, the choice of $V_1 \cup V_2$ implies that $G_i = G[V_i]$ is connected for $i = 1, 2$. Let T_i be a spanning tree of G_i with $c(G_i, T_i) \leq |V_i|t(G_i)$. If $uv \in E(V_1, V_2)$ and T is a tree with vertex set $V(G)$ and edge set $E_{T_1} \cup E_{T_2} \cup \{uv\}$, then

$$\begin{aligned} c(G,T) &\leq \max\{c(G_1,T_1), c(G_2,T_2)\} + |E(V_1,V_2)| \\ &\leq (n(G) - 1)t(G) + t(G) \\ &= n(G)t(G), \end{aligned}$$

which completes the proof. □

Chapter 5

Decision Problems

In the previous chapters we presented "positive" results, such as bounds for graph parameters or characterizations of graphs that have a special property. By contrast, in this chapter we present "negative" results, i.e. we prove for some decision problems that they have no polynomial-time algorithm, unless $P = NP$. Hence, restrictions to simpler graph classes and the bounds in the previous chapters are motivated.

In [32] Hedetniemi, Hedetniemi, Laskar, Markus, and Slater initiate the study of $\gamma\gamma(G)$, $\gamma i(G)$, and $ii(G)$ for graphs G. Since a maximal independent set is a minimal dominating set, Ore's observation implies that $\gamma i(G)$ exists for every graph G without isolated vertices. However, Hedetniemi et al. [32] proved that it is NP-complete to decide whether $ii(G)$ exists for a given graph G. Various graph theoretic and algorithmic properties of these parameters are presented in [32].

In the first two sections of this chapter we present hardness results concerning the above concepts. We prove that deciding equality of $\gamma\gamma(G)$ and $\gamma i(G)$, or $\gamma i(G)$ and $ii(G)$ is NP-hard (Theorem 5.1). This implies in both cases that there is most likely no algorithmically efficient characterization of such graphs. For bipartite graphs G, we prove that it is NP-complete to decide whether $\gamma\gamma(G) \leq k$, $\gamma i(G) \leq k$, and $ii(G) \leq k$, respectively (Theorem 5.2). These results solve problems posed in [32] and are based on [33].

In Section 5.3 we describe a polynomial algorithm that decides whether a given tree has an (α, γ)-pair (Corollary 5.4). Furthermore, we show that it is NP-hard to decide whether a given graph has an (α, γ)-pair (Theorem 5.3). The results of Section 5.3 are based on [58].

In Chapter 2 we considered whether $\gamma\gamma_t(G) \leq k$ for $k = n(G)-1$ and graphs G that are C_5-free or satisfy $\delta(G) \geq 3$, respectively. We prove in Section 5.4 that the corresponding decision problem is NP-complete even when restricted to C_5-free graphs G with $\delta(G) \geq 3$.

The graph parameter $\alpha\alpha(G)$ is defined in Section 4.2. By definition $\alpha\alpha(G) \leq \alpha(G)$. A natural question is when equality holds for the above inequality. In Section 5.5 we prove that it is NP-complete to decide for a given graph G and a given integer k, whether $\alpha\alpha(G) \leq k$ and that it is NP-hard to decide for a given graph G, whether $\alpha\alpha(G) = 2\alpha(G)$.

Not only Section 4.3 is devoted to spanning tree congestion but also Section 5.6. As the last result of this thesis, we show that it is NP-complete to decide for a given graph G

and a given integer k, whether $s(G) \leq k$. The result of Section 5.6 is based on [53].

For basic notation and terminology concerning algorithmic complexity, we refer to [22].

5.1 $\gamma\gamma(G) = \gamma i(G)$ and $\gamma i(G) = ii(G)$

Deciding equality of $\gamma\gamma(G)$ and $\gamma i(G)$, or $\gamma i(G)$ and $ii(G)$ is hard. This answers a problem posed in [32].

Theorem 5.1 *Given a graph G the following two problems are NP-hard.*

(a) *Decide whether G satisfies $\gamma\gamma(G) = \gamma i(G)$.*

(b) *Decide whether G satisfies $\gamma i(G) = ii(G)$.*

We prove the result by reducing the well-known NP-complete 3Sat problem [22] to the considered decision problems.

Proof: Given a 3Sat instance $\mathcal{C}$, we construct two graphs G and G' whose order is polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if $\gamma\gamma(G) = \gamma i(G)$ if and only if $\gamma i(G') = ii(G')$.

For the construction of G, we proceed as follows. For every boolean variable x occurring in $\mathcal{C}$, we introduce a copy G_x of the gadget shown in the left part of Figure 5.1, which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$, we introduce a copy G_C of the gadget shown in the middle part of Figure 5.1, which contains one specified vertex C.

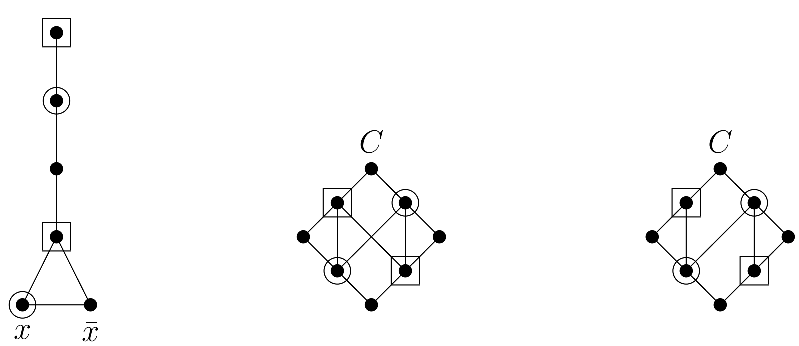


Figure 5.1: The gadgets G_x, G_C and G'_C.

If the literal $x^* \in \{x, \bar{x}\}$ occurs in clause C we connect the specified vertex x^* in G_x with the specified vertex C in G_C. (For an example see Figure 5.2 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.)

For the graph G', we proceed exactly as above using the gadget G'_C shown in the right part of Figure 5.1 instead of G_C.

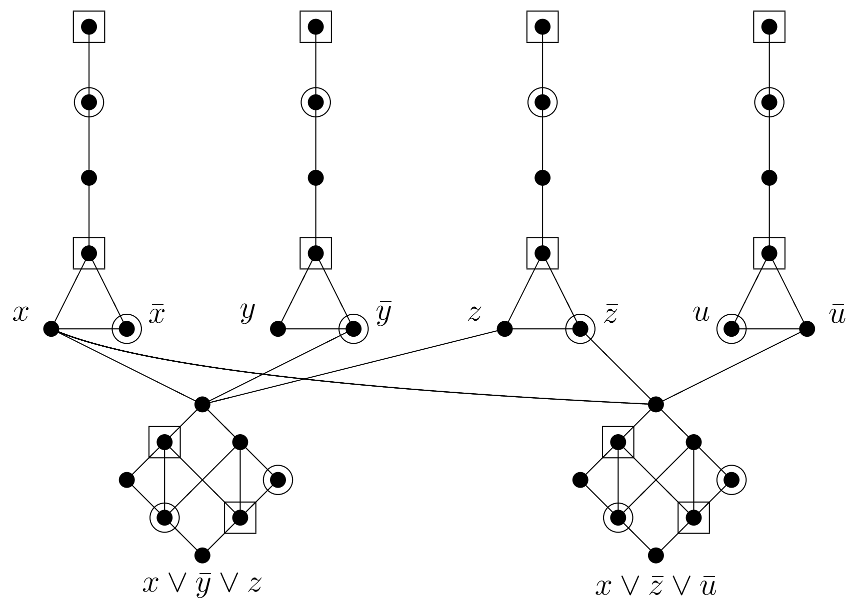


Figure 5.2: The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.

Let $\mathcal{C}$ use p boolean variables and contain q clauses. Note that the orders of G and G' are $6p + 8q$. Every dominating set of G contains at least two vertices from every gadget G_x and at least two vertices from every gadget G_C. Conversely, choosing in every gadget the vertices as indicated in Figure 5.1 yields two disjoint minimum dominating sets, i.e., $\gamma\gamma(G) = 2\gamma(G) = 4p + 4q$. Similarly, $\gamma i(G') = 2\gamma(G') = 4p + 4q$.

If $\mathcal{C}$ is satisfiable, then we consider a satisfying truth assignment for $\mathcal{C}$. We choose the two disjoint minimum dominating sets described above such that from every gadget G_x the vertex corresponding to the true literal is in one of the two sets. Furthermore, in every gadget G_C, we choose vertices as indicated in Figure 5.2. This yields two disjoint minimum dominating sets one of which is independent, i.e., $\gamma\gamma(G) = \gamma i(G)$. Similar arguments yield $\gamma i(G') = ii(G')$.

Conversely, we assume now that G satisfies $\gamma\gamma(G) = \gamma i(G)$. Let D_1 and I_2 be two disjoint dominating sets such that I_2 is independent and $|D_1| + |I_2| = \gamma\gamma(G) = \gamma i(G) = 2\gamma(G)$, i.e., D_1 and I_2 are both minimum dominating. By the above reasoning, each of D_1 and I_2 contains exactly two vertices from each gadget G_C. This easily implies that in every gadget G_C the specified vertex C is dominated within one of D_1 and I_2 by a vertex not contained in G_C. Furthermore, for every gadget G_x, the set $D_1 \cup I_2$ contains at most one of the two specified vertices x and $\bar{x}$. Therefore, the vertices in $D_1 \cup I_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$. (The two minimum dominating sets indicated in Figure 5.2 correspond to setting x, y and z *false* and u *true*.) Again, if we assume that G' satisfies $\gamma i(G') = ii(G')$, then the same train of thought implies that $\mathcal{C}$ is satisfiable. This completes the proof. $\square$

5.2 $\gamma\gamma(G), \gamma i(G)$, and $ii(G)$ in Bipartite Graphs

In [32] it is shown that the calculation of $\gamma\gamma(G)$ is NP-hard even when restricted to chordal graphs. In our next result we prove that it is NP-hard to determine $\gamma\gamma(G), \gamma i(G)$, and $ii(G)$ even for bipartite graphs G. This answers problems posed in [32].

Theorem 5.2 *For a given bipartite graph G without isolated vertices and an integer k, the following problems are NP-complete.*

(a) *Decide whether $\gamma\gamma(G) \leq k$.*

(b) *Decide whether $\gamma i(G) \leq k$.*

(c) *Decide whether G has two disjoint independent dominating sets D_1 and D_2 with $|D_1| + |D_2| \leq k$.*

Proof: The three decision problems are clearly in NP. Given a 3Sat instance $\mathcal{C}$, we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ and specify an integer k also polynomially bounded in the size of $\mathcal{C}$ such that if $\mathcal{C}$ is satisfiable, then $ii(G) \leq k$ and if $\gamma\gamma(G) \leq k$, then $\mathcal{C}$ is satisfiable. This clearly implies the desired statement.

For every boolean variable x occurring in $\mathcal{C}$, we introduce a copy G_x of the gadget shown in the left part of Figure 5.3, which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$, we introduce a copy G_C of the gadget shown in the right part of Figure 5.3, which contains two specified vertices C and $\bar{C}$.

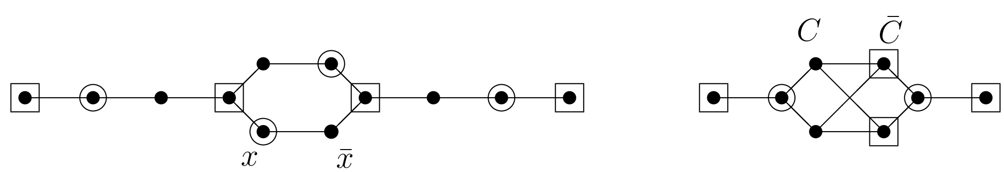


Figure 5.3: The gadgets G_x and G_C.

If the (unnegated) variable x occurs in clause C, then we connect the specified vertex x in G_x with the specified vertex C in G_C. Similarly, if the negated variable $\bar{x}$ occurs in clause C, we connect the specified vertex $\bar{x}$ in G_x with the specified vertex $\bar{C}$ in G_C. Note that this way of adding edges to the disjoint union of the bipartite gadgets results in a bipartite graph. (For an example see Figure 5.4 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.) Let G denote the resulting graph.

Let $\mathcal{C}$ use p boolean variables and contain q clauses. Note that the order of G is $12p+8q$. Let $k = 8p + 5q$.

First, we assume that $\mathcal{C}$ is satisfiable and describe how to obtain two disjoint independent dominating sets D_1 and D_2 of G with $|D_1| + |D_2| \leq k$. Consider a satisfying truth

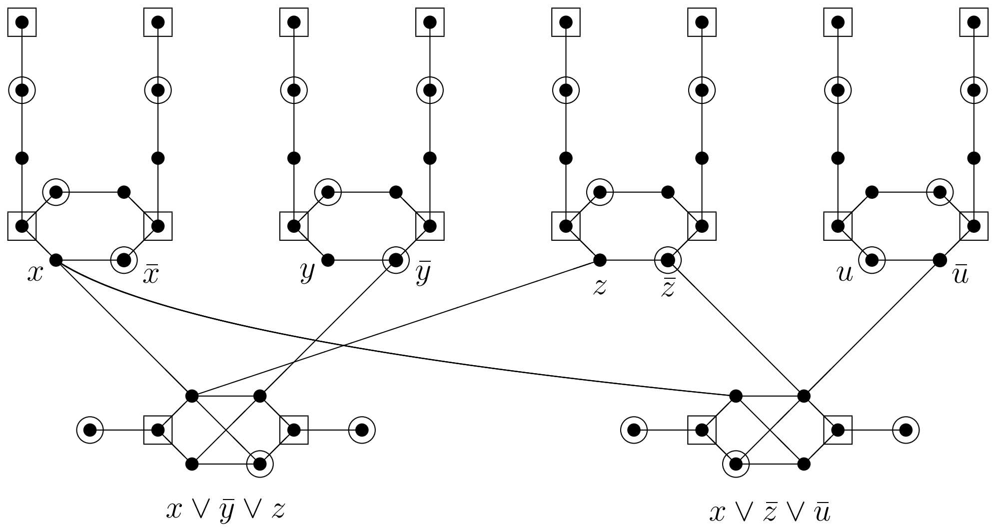


Figure 5.4: The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.

assignment for $\mathcal{C}$. We choose in every gadget G_x the vertices for the sets D_1 and D_2 as indicated in the left part of Figure 5.3 or its mirror image such that D_1 contains the vertex corresponding to the true literal among x or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices C or $\bar{C}$ in every gadget G_C is dominated in D_1 by a vertex not contained in $V(G_C)$. This implies that the two sets D_1 and D_2 can be extended as indicated in Figure 5.4 using a total of five vertices in each of the gadgets G_C. Hence, $|D_1| + |D_2| = k$. Note that D_1 and D_2 are independent by construction.

Next, we assume that G has two disjoint dominating sets D_1 and D_2 such that $|D_1| + |D_2| \leq k$. In every gadget G_x, the set $V(G_x) \cap (D_1 \cup D_2)$ contains at least eight vertices in order to dominate the ten vertices on the path $G_x - \{x, \bar{x}\}$. Furthermore, if $V(G_x) \cap (D_1 \cup D_2)$ contains exactly eight vertices, then at least one of x and $\bar{x}$ is not contained in $D_1 \cup D_2$.

If for some gadget G_C, neither C nor $\bar{C}$ are dominated by a vertex in $D_1 \cup D_2$ not contained in $V(G_C)$, then $V(G_C) \cap (D_1 \cup D_2)$ contains at least six vertices. (One possible configuration is shown in the right part of Figure 5.3.) Furthermore, if for some gadget G_C, one or both of C and $\bar{C}$ are dominated by vertices in $D_1 \cup D_2$ not contained in $V(G_C)$, then $V(G_C) \cap (D_1 \cup D_2)$ contains at least five vertices.

Since $|D_1| + |D_2| \leq 8p + 5q$, we obtain that for every gadget G_x, at most one of x and $\bar{x}$ is contained in $D_1 \cup D_2$ and for every gadget G_C, one of C and $\bar{C}$ is dominated by a vertex in $D_1 \cup D_2$ not contained in $V(G_C)$. This implies that the vertices contained in $D_1 \cup D_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$ and the proof is complete. $\Box$

5.3 Existence of an (α, γ)-Pair

Theorem 5.3 *It is NP-hard to decide whether a given graph has an (α, γ)-pair.*

Proof: Given a 3Sat instance $\mathcal{C}$, we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if G has an (α, γ)-pair. This clearly implies the desired statement.

For every boolean variable x occurring in $\mathcal{C}$, we introduce a copy G_x of the gadget shown in the left part of Figure 5.5, which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$, we introduce a copy G_C of the gadget shown in the right part of Figure 5.5, which contains two specified vertices C and C'.

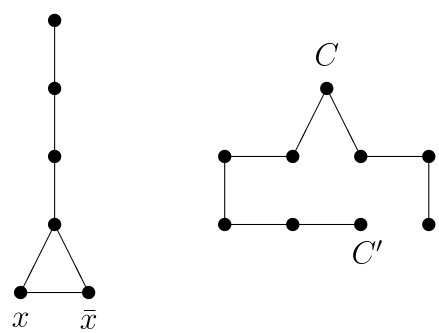


Figure 5.5: The gadgets G_x and G_C.

If the literal $x^* \in \{x, \bar{x}\}$ occurs in clause C, then we connect the specified vertex x^* in G_x with the specified vertex C in G_C. (For an example see Figure 5.6 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.) Let G denote the resulting graph.

Let $\mathcal{C}$ use p boolean variables and contain q clauses. Note that the order of G is $6p+9q$.

Clearly, every independent set of G contains at most three vertices from each of the gadgets G_x and at most five vertices from every of the gadgets G_C, i.e. $\alpha(G) \leq 3p + 5q$. Since choosing three independent vertices from each of the gadgets G_x and the vertices at distance one, three, and five from C from each of the gadgets G_C yields an independent set of order $3p + 5q$, we have $\alpha(G) = 3p + 5q$.

Clearly, every dominating set of G contains at least two vertices from every of the gadgets G_x and at least three vertices from each of the gadgets G_C. Hence $\gamma(G) \geq 2p+3q$. Furthermore, since choosing x and the neighbor of the endvertex from each of the gadgets G_x and the vertices at distance one, four, and seven from C' from each of the gadgets G_C yields a dominating set of order $2p + 3q$, we have $\gamma(G) = 2p + 3q$.

If $\mathcal{C}$ has a satisfying truth assignment, then choosing three independent vertices containing the false literal among x and $\bar{x}$ from every of the gadgets G_x and the vertices at distance one, three, and five from C from every of the paths G_C yields a maximum independent set I. Furthermore, choosing the true literal among x and $\bar{x}$ and the neighbor

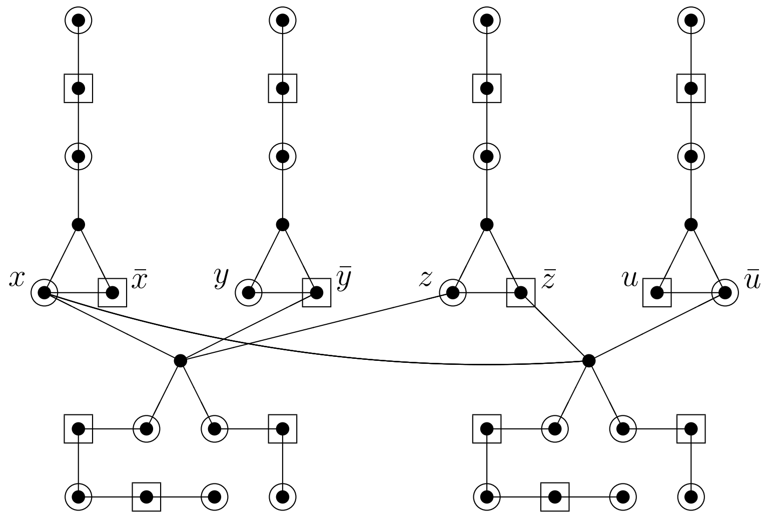


Figure 5.6: The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$

.

of the endvertex from every of the gadgets G_x and the vertices at distance one, three, and seven from C' from every of the paths G_C yields a minimum dominating set D that is disjoint from I. Hence, (I, D) is an (α, γ)-pair of G. (For an example see Figure 5.6. The encircled vertices form a maximum independent set and the framed vertices form a minimum dominating set.)

Conversely, if G has an (α, γ)-pair (I, D), then we may assume that D contains exactly one of the two vertices x and $\bar{x}$ from every of the gadgets G_x. If one of the vertices C from some gadget G_C is not dominated by a vertex from one of the gadgets G_x, then D must contain the vertex at distance four from C', because D is a minimum dominating set. But also I must contain this vertex, because I is a maximum independent set. This implies that the vertices contained in D corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$ and the proof is complete. $\Box$

The next result actually follows from far more general results concerning efficiently solvable problems for graphs of bounded treewidth. Nevertheless, we include its simple proof based on our characterization given in Subsection 4.1.1.

Corollary 5.4 *It is possible to decide in polynomial time whether a given tree of order at least* 2 *has an* (α, γ)*-pair.*

Proof: If T is a path of order at most 6 or the tree T^* shown in Figure 4.3, then, by Lemma 4.6, T has an (α, γ)-pair. If T is a path of order at least 7, then Lemma 4.7 allows to reduce the decision problem to a smaller tree in polynomial time. If T is neither a path nor the tree T^*, then Theorem 4.10 allows to reduce the decision problem to a smaller tree in polynomial time. $\Box$

5.4 $\gamma\gamma_t(G) \leq k$

In our next result we prove that it is NP-complete to decide for C_5-free graphs G with $\delta(G) \geq 3$ whether $\gamma\gamma_t(G) \leq k$.

Theorem 5.5 *It is NP-complete to decide for a given C_5-free graph G with $\delta(G) \geq 3$ and a given integer k, whether $\gamma\gamma_t(G) \leq k$.*

Proof: The decision problem is clearly in NP. Given a 3Sat instance $\mathcal{C}$, we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ and specify an integer k also polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if $\gamma\gamma_t(G) \leq k$. This clearly implies the desired statement.

For every boolean variable x occurring in $\mathcal{C}$, we introduce a copy G_x of the gadget shown in the left part of Figure 5.7, which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$, we introduce a copy G_C of the gadget shown in the right part of Figure 5.7, which contains one specified vertex C.

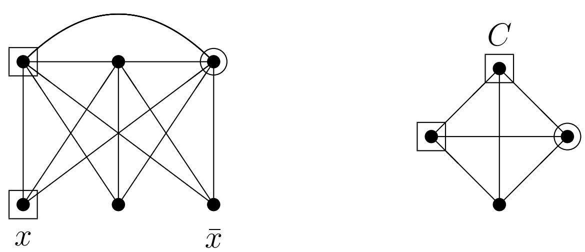


Figure 5.7: The gadgets G_x and G_C.

If the literal $x^* \in \{x, \bar{x}\}$ occurs in clause C, then we connect the specified vertex x^* in G_x with the specified vertex C in G_C. Note that this way of adding edges to the disjoint union of the gadgets results in a C_5-free graph with minimum degree 3. (For an example see Figure 5.8 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$. The encircled vertices form a dominating set and the framed vertices form a total dominating set.) Let G denote the resulting graph.

Let $\mathcal{C}$ use p boolean variables and contain q clauses. Note that the order of G is $6p+4q$. Let $k = 3p + 2q$.

First, we assume that $\mathcal{C}$ is satisfiable and describe how to obtain a DT-pair (D, T) of G with $|D| + |T| \leq k$. Consider a satisfying truth assignment for $\mathcal{C}$. We choose in every gadget G_x the vertices for the sets D and T as indicated in the left part of Figure 5.7 or its mirror image such that T contains the vertex corresponding to the true literal among x or $\bar{x}$. (The encircled vertices belong to D and the framed vertices belong to T.) Since

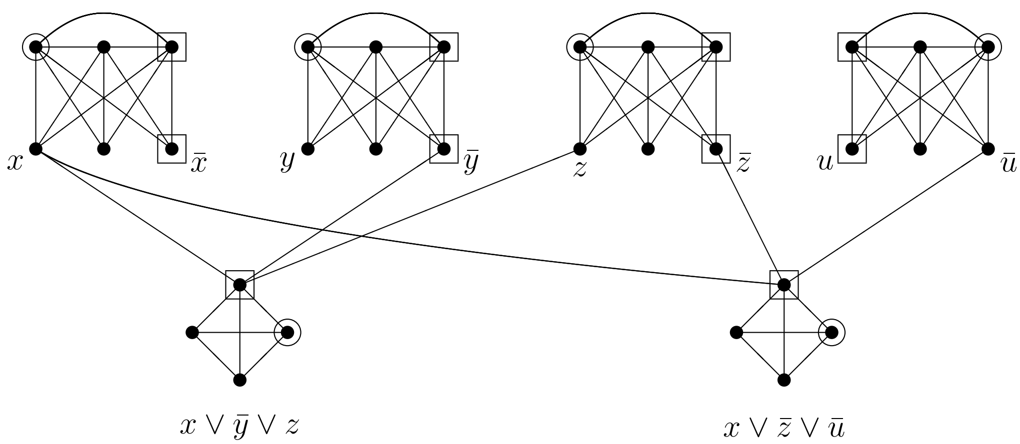


Figure 5.8: The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.

the truth assignment is satisfying, the vertex C in every gadget G_C has a neighbor in $T \setminus V(G_C)$. This implies that the two sets D and T can be extended as indicated in Figure 5.8 using a total of two vertices in each of the gadgets G_C. Hence, $|D| + |T| = k$.

Conversely, we assume that G has a DT-pair (D, T) such that $|D| + |T| \leq k$. In every gadget G_x, the set $V(G_x) \cap D$ contains at least one vertex in order to dominate the four vertices $G_x - \{x, \bar{x}\}$. Since each vertex of $G_x - \{x, \bar{x}\}$ has a neighbor in T holds that $V(G_x) \cap T$ contains at least two vertices. Furthermore, if $V(G_x) \cap (D \cup T)$ contains exactly three vertices, then at most one of x and $\bar{x}$ is contained in T.

If for some gadget G_C, the vertex C has no neighbor in $T \setminus V(G_C)$, then $V(G_C) \cap (D \cup T)$ contains at least three vertices. Furthermore, if for some gadget G_C, the vertex C has a neighbor in $T \setminus V(G_C)$, then $V(G_C) \cap (D \cup T)$ contains at least two vertices.

Since $|D| + |T| \leq 3p + 2q$, we obtain that for every gadget G_x, at most one of x and $\bar{x}$ is contained in $D \cup T$ and for every gadget G_C, the vertex C has a neighbor in $T \setminus V(G_C)$. This implies that the vertices contained in T corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$ and the proof is complete. $\Box$

5.5 $\alpha\alpha(G) = 2\alpha(G)$ and $\alpha\alpha(G) \geq k$

Theorem 5.6 (a) *It is NP-hard to decide for a given graph G, whether $\alpha\alpha(G) = 2\alpha(G)$.*

(b) *It is NP-complete to decide for a given graph G and a given integer k, whether $\alpha\alpha(G) \geq k$.*

Since $\alpha\alpha(G)$ is also the maximum order of an induced bipartite subgraph of G, (b) is already proved by Lewis and Yannakakis [51]. However, we can prove (b) without additional effort.

Proof: The decision problem (b) is clearly in NP. Given a 3Sat instance $\mathcal{C}$ we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ and specify an integer k also polynomially bounded in the size of $\mathcal{C}$ such that $\alpha\alpha(G) = 2\alpha(G)$ if and only if $\alpha\alpha(G) \geq k$ if and only if $\mathcal{C}$ is satisfiable. This clearly implies the desired statement.

Let H be a graph that has one vertex for each instance of each literal in $\mathcal{C}$. Two vertices in $V(H)$ are adjacent if they either correspond to literals in the same clause, or to a variable and its inverse. (For an example see Figure 5.9 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}, \bar{x} \vee y \vee \bar{u}, \bar{x} \vee \bar{y} \vee \bar{u}\}$.)

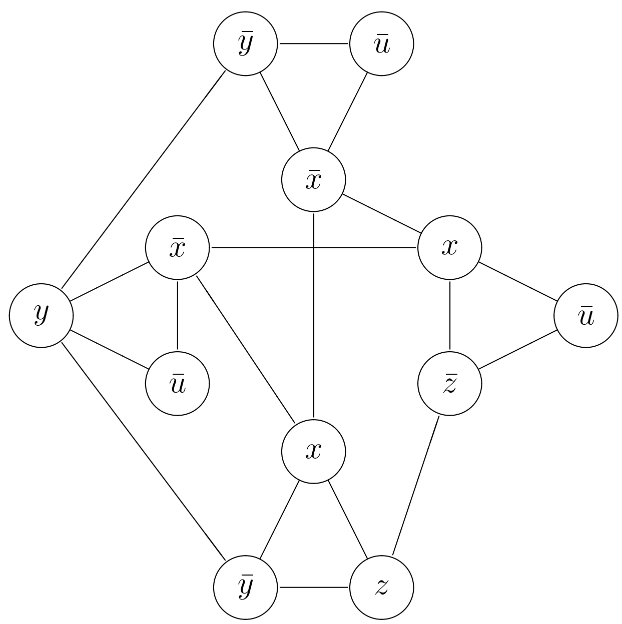


Figure 5.9: The graph H for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}, \bar{x} \vee y \vee \bar{u}, \bar{x} \vee \bar{y} \vee \bar{u}\}$.

Let $\mathcal{C}$ contain q clauses. Note that the order of H is $3q$. Let G be the graph that arises from H by adding q vertices $v_1, \ldots, v_q$ and adding all possible edges joining a vertex in $V(H)$ with a vertex in $\{v_1, \ldots, v_q\}$. (For an example see Figure 5.10 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.) Note that the order of G is $4q$.

Let $k = 2q$. Clearly, the vertices $v_1, \ldots, v_q$ form a maximum independent set of G. Hence, $\alpha(G) = q$ and so $\alpha\alpha(G) = 2\alpha(G)$ if and only if $\alpha\alpha(G) \geq k$.

First, we assume that $\mathcal{C}$ is satisfiable and describe how to obtain two disjoint independent sets I_1 and I_2 with $|I_1| + |I_2| \geq k$. Consider a satisfying truth assignment for $\mathcal{C}$. For I_1 we choose one vertex in each clause that corresponds to a literal that is *true* and for I_2 we choose the vertices $v_1, \ldots, v_q$.

Conversely, we assume that G has two disjoint independent sets I_1 and I_2 with $|I_1| + |I_2| \geq k$. In this case one independent set is the vertex set $\{v_1, \ldots, v_q\}$ and the other

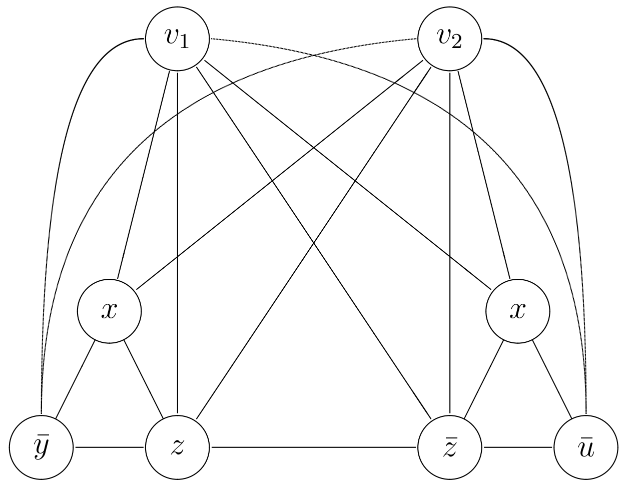


Figure 5.10: The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee \bar{z} \vee \bar{u}\}$.

independent set must contain one vertex of each clause. To obtain a satisfying truth assignment, we assign the value *true* to each literal that corresponds to a vertex in the second independent set. Since two vertices are adjacent that correspond to a variable and its inverse, the assignment is consistent. There may be variables that have no literal in the independent set. We can set these to any value we like. This completes the proof. □

5.6 $s(G) \leq k$

Theorem 5.7 *It is NP-complete to decide for a given graph G and a given integer k, whether $s(G) \leq k$.*

Proof: The decision problem is clearly in NP. Given a 3Sat instance $\mathcal{C}$, we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ and specify an integer k also polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if $s(G) \leq k$. This clearly implies the desired statement.

Let $\mathcal{C}$ use the $p \geq 3$ boolean variables $v_1, \ldots, v_p$ and contain the $q \geq 1$ clauses $C_1, \ldots, C_q$. We may assume that no clause contains a boolean variable as well as its negation. We construct G as follows starting with the empty graph. For every boolean variable v_i, we add a clique K^i on $4q^2 + 11q + 1$ vertices

$$v_i, \bar{v}_i, v_{i,1}, v_{i,2}, \ldots, v_{i,4q^2+11q-1}.$$

The vertices v_i and $\bar{v}_i$ are called the *literal vertices* of the clique K^i for $1 \leq i \leq p$. We add a vertex w to G, which is of degree $p(2q^2+6q+1)$. The vertex w is adjacent to the vertices

$$v_i, \bar{v}_i, v_{i,1}, v_{i,2}, \ldots, v_{i,2q^2+6q-1}$$

for $1 \leq i \leq p$. Furthermore, for every clause C_j, we add a *clause vertex* C_j of degree $3q+9$. A clause vertex C_j is adjacent to every literal vertex that corresponds to a literal contained in C_j. Additionally, a clause vertex C_j is adjacent to the vertices

$$v_{i,(j-1)(q+2)+1}, v_{i,(j-1)(q+2)+2}, \ldots, v_{i,j(q+2)-1}, v_{i,j(q+2)}$$

for every boolean variable v_i that is contained in C_j. Clearly, except for the literal vertices, every vertex has at most one neighbor in $\bigcup_{1 \leq j \leq q}\{C_j\}$ in G. Note that $N_G(C_j) \subseteq N_G(w)$ for $1 \leq j \leq q$, because $q(q+2) \leq 2q^2 + 6q - 1$. The graph G contains no further vertices or edges. Note that G has order $p(4q^2 + 11q + 1) + q + 1$. Let $k = 4q^2 + 12q + 1$.

First, we assume that $\mathcal{C}$ is satisfiable and describe how to obtain a spanning tree T of G with $c(G,T) \leq k$. Consider a satisfying truth assignment for $\mathcal{C}$. T contains all edges of the form xy where $x \in V(K^i)$ is a literal vertex corresponding to a true literal and $y \in (V(K^i) \cup \{w\}) \setminus \{x\}$. Furthermore, for every clause C_j, we add to T exactly one edge of the form C_jx where x is a true literal. Thus, T is a spanning tree of G. Note that all edges of T are either incident to w or incident to a leaf in T. The degree of a vertex in $V(G) \setminus \{w\}$ is at most $4q^2 + 12q + 1 = k$ in G. Hence, $c(e,(G,T)) \leq k$ for an edge e that is incident to a leaf in T. For some $1 \leq i \leq p$, let e_i be the edge of T that joins w with the true literal vertex of K^i. If a clause vertex C_j is adjacent to the true literal vertex of K^i in T, then C_j is adjacent to $q+3$ vertices of K^i in G. Hence, the edge e_i is contained in $2q+6$ paths in T that correspond to edges of the form C_jx in G where $x \notin V(K^i)$. If a clause vertex C_j is not adjacent to the true literal vertex of K^i in T, then the edge e_i is contained in at most $q+3$ paths in T that correspond to edges of the form C_jx in G where $x \in V(K^i)$. Since w is adjacent to $2q^2 + 6q + 1$ vertices of K^i in G, the edge e_i is contained in $2q^2 + 6q + 1$ paths in T that correspond to edges of the form wx in G where $x \in V(K^i)$. Hence,

$$c(e_i,(G,T)) \leq (2q^2 + 6q + 1) + q(2q+6) = 4q^2 + 12q + 1 = k.$$

Thus, $c(G,T) \leq k$ and hence $c(G) \leq k$.

Conversely, we assume that G has a spanning tree T such that $c(G,T) \leq k$. For $1 \leq i \leq p$, let

$$T^i = T[V(K^i) \cup \{w\}].$$

If for some $1 \leq i \leq p$, there is a vertex x such that for all pairs of distinct vertices $y,z \in V(T^i) \setminus \{x\}$, the unique y-z-path in T contains x, then we call x a *central vertex* of T^i.

Claim 1 *T^i has exactly one central vertex for every $1 \leq i \leq p$. Furthermore, that central vertex is contained in $V(K^i)$.*

Proof of Claim 1: First, we prove that T^i has at least one central vertex. Note that $|V(T^i)| \geq 4$ for every $1 \leq i \leq p$. For contradiction, we assume that there is some $1 \leq i \leq p$ such that T^i has no central vertex. Since subtrees of a tree have the Helly property (see [6]), this implies the existence of four distinct vertices $y_1, y_2, y_3, y_4 \in V(T^i)$ such that, if P denotes the unique y_1-y_2-path in T and Q denotes the unique y_3-y_4-path in T, then $V(P) \cap V(Q) = \emptyset$. Let u denote the last vertex on the unique y_2-y_3-path in T, which is contained in $V(P)$ and let u' be the neighbor of u on the y_2-y_3-path in T that is not contained in $V(P)$. Let A, B denote the two components of $T - uu'$. Furthermore, let $n_a = |V(K^i) \cap A|$ and $n_b = |V(K^i) \cap B|$. Note that both A and B contain 2 vertices from T^i. Clearly, $n_a + n_b = 4q^2 + 11q + 1$. If $n_a \geq 2$ and $n_b \geq 2$, then

$$c(uu', (G,T)) \geq n_a \cdot n_b \geq 2 \cdot (4q^2 + 11q - 1) > 4q^2 + 12q + 1 = k,$$

which is a contradiction to our assumption $c(G,T) \leq k$. Otherwise, if one of n_a or n_b, say n_a, is less than 2, then w and a vertex of K^i are in A and thus,

$$c(uu', (G,T)) \geq (2q^2 + 6q) + (4q^2 + 11q) > 4q^2 + 12q + 1 = k.$$

Again, we have a contradiction to our assumption $c(G,T) \leq k$. Hence, T^i has at least one central vertex for every $1 \leq i \leq p$.

Next, we prove that for every $1 \leq i \leq p$, T^i has at most one central vertex. For contradiction, we assume that for some $1 \leq i \leq p$, x_1, x_2 are two central vertices of T^i. Since $|V(T^i)| \geq 3$, there exists another vertex $y \in V(T^i) \setminus \{x_1, x_2\}$. Now, either x_2 is not on the unique y-x_1-path in T or x_1 is not on the unique y-x_2-path in T. Hence, at least one of x_1 and x_2 is not a central vertex of T^i, which is a contradiction.

Since for every $1 \leq i \leq p$, the vertex $v_{i,4q^2+11q-1}$ has only neighbors in K^i in G, $v_{i,4q^2+11q-1}$ or a neighbor of $v_{i,4q^2+11q-1}$ in G is the central vertex of T^i. Hence, the central vertex of T^i is in $V(K^i)$, which completes the proof of the claim. $\Box$

Claim 2 *For every $1 \leq i \leq p$, no vertex of $V(K^i)$ is isolated in T^i.*

Proof of Claim 2: For contradiction, we assume that for some $1 \leq i \leq p$, there is a vertex $y \in V(K^i)$ that is isolated in T^i. If y is the central vertex of T^i, then at least one of the vertices $v_{i,4q^2+11q-2}$ or $v_{i,4q^2+11q-1}$ is isolated in T, a contradiction to the connectivity of T. Hence, y is not the central vertex of T^i. Let x denote the central vertex of T^i and let P denote the unique y-x-path in T. Let c denote the neighbor of y on P. Thus $c \in \{C_1, \ldots, C_q\}$. Furthermore, let x' denote the neighbor of x on P and let A denote the component of $T - xx'$ that contains y. By Claim 1, $V(A) \cap V(T^i) = \{y\}$. Since y has $4q^2 + 11q$ neighbors in $V(K^i)$ in G and c has $q + 2$ neighbors in $V(K^i) \setminus \{y\}$ in G,

$$c(xx', (G,T)) \geq (4q^2 + 11q) + (q + 2) > 4q^2 + 12q + 1 = k.$$

Hence, we have a contradiction to our assumption $c(G,T) \leq k$. $\Box$

Claim 3 *For every $1 \leq i \leq p$, the vertex w is not isolated in T^i.*

Proof of Claim 3: For contradiction, we assume that for some $1 \leq i \leq p$, the vertex w is isolated in T^i. Let x denote the central vertex of T^i and let P denote the unique w-x-path in T. Let y denote the neighbor of w on P and let j such that $y \in V(T^j)$. Note that $j \neq i$. By Claim 1, w is a leaf in T^j and y is the central vertex of T^j. Let A denote the component of $T - wy$ that contains w. By Claim 1, $V(A) \cap V(T^i) = \{w\}$ and $V(A) \cap V(T^j) = \{w\}$. Since w has $2q^2 + 6q + 1$ neighbors in $V(K^i)$ in G and $2q^2 + 6q + 1$ neighbors in $V(K^j)$ in G,

$$c(wy, (G, T)) \geq 2 \cdot (2q^2 + 6q + 1) > 4q^2 + 12q + 1 = k,$$

which is a contradiction to our assumption $c(G, T) \leq k$. $\square$

Claim 4 *For every $1 \leq j \leq q$, the vertex C_j has exactly one neighbor y in T. Furthermore, y is a central vertex.*

Proof of Claim 4: By Claims 1 to 3, T^i forms a star for every $1 \leq i \leq p$, and hence, the graph

$$T^* = \bigcup_{1 \leq i \leq p} T^i$$

is connected. Thus, since for every $1 \leq j \leq q$, C_j has only neighbors in $V(T^*)$ in G, C_j is a leaf in T. Let y be the neighbor of C_j in T. For contradiction, we assume that y is not a central vertex. Let i such that $y \in V(T^i)$ and let x denote the central vertex of T^i. Let A denote the component of $T - xy$ that contains y. By Claim 1, $V(A) \cap V(T^i) = \{y\}$. Since C_j has $q + 2$ neighbors in $V(K^i) \setminus \{y\}$ in G and y has $4q^2 + 11q + 1$ neighbors in $V(K^i) \cup \{w\}$ in G,

$$c(yx, (G, T)) \geq q + 2 + (4q^2 + 11q + 1) > 4q^2 + 12q + 1 = k.$$

Hence, we have a contradiction to our assumption $c(G, T) \leq k$. $\square$

Claim 4 implies that for every $1 \leq i \leq p$, there is at most one vertex (namely the central vertex) in T^i, which has neighbors among the clause vertices. Clearly, since $c(G, T) \leq k$ and by the definition of G, there is a spanning tree T^* of G such that $c(G, T^*) \leq k$ and every central vertex of T^* is a literal vertex of G. Hence, the set of central vertices of T^* define a satisfying truth assignment for C and the proof of the theorem is complete. $\square$

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