Norman Hopfe

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Abstract

The thesis deals with the control of linear single input, single output and multi input, multi output systems with unknown but bounded relative degree and linear multi input, multi output Volterra-Stieltjes systems. The following two control strategies are considered: adaptive high-gain output derivative feedback control and funnel control. Each control strategy requires the structural properties of strict relative degree, stable zero dynamics and positive high-frequency gain.

For many control applications their parameters are not precisely known. In general, it cannot be expected to have complete information about a system, but instead only structural properties are known. One possible control strategy is an adaptive controller. The aim of Chapter 2 is a universal adaptive controller which learns from the behaviour of the system and achieves a prespecified control objective. Possible objectives are stabilization of the system and λ-tracking.
For example, for systems with relative degree one the λ-tracking controller means that the output of the system should stay close to a given reference signal, where a prespecified small tracking error of size $\lambda > 0$ is tolerated. For systems with higher relative degree the λ-tracking controller uses the output and its derivatives. The drawback of the derivatives can be solved if an observer is used which estimates the output of the system and its first derivatives. It has to be noted that this controller stabilize or track any system if the relative degree is known, provided the system has stable zero dynamics and the high-frequency gain matrix is positive definite.
In the thesis the adaptive λ-tracking controller is extended to systems with unknown relative degree, where an upper bound of the relative degree is known. This is achieved if a high-gain output derivative feedback is allowed. It is proven λ-tracking and stabilization are guaranteed. An advantage of the proposed controller is its relative simple structure which is helpful for the implementation and the understanding how the controller works. In this thesis, the adaptive λ-tracking controller is applied to a serially connected mass-spring damper system with unknown relative degree 1, 2 or 3.

The main drawback is that the gain $k(\cdot)$ increases. In Chapter 3, the well known concept of funnel control for systems with relative degree one is introduced which

overcomes this drawback. It is shown that the classical funnel controller applied to linear multi input, multi output systems achieves in presence of input saturation the control objectives of funnel control. The presence of explicit input constraints is a distinguishing feature of this thesis. A feasibility relationship is derived under which the efficiency of funnel control in the presence of input saturation is established. The drawback is that sufficient a priori system information is required in order to check the feasibility condition.
Chapter 5 generalizes the classical results of funnel control and the new results of input constraints to linear multi input, multi output Volterra-Stieltjes systems with relative degree one.

It has to be noted that the system in Chapter 3 has strict relative degree one which is important. The aim of Chapter 4 is to generalize the results for funnel control to linear single input, single output systems with relative degree two. It is known that the funnel controller can be extended to systems with higher relative degree, where the controller involves a filter, the feedback strategy dynamic and a backstepping construction of the feedback strategy. A drawback is that the controller is no longer simple.
It is shown that the simplicity of the control strategy can be preserved if derivative feedback is allowed. The thesis designs a simple feedback structure which relies on two funnels; one for the output and the other one for its derivative. This new funnel controller is robust for systems of unknown relative degree, i.e. the new funnel controller can be achieved to linear single input, single output systems with unknown relative degree one or two.
If the system has relative degree one, then the application of the new funnel controller yields a closed-loop system which is an implicit differential equation. An existence and uniqueness result for a maximal solution of an implicit ordinary differential equation is proven. Moreover, the results of Chapter 3 are generalized to systems with relative degree two.

Chapter 5 considers time-varying and time-invariant linear multi input, multi output Volterra-Stieltjes systems with regard to positivity, various stability concepts, zero dynamics and funnel control.
Positive systems, i.e., loosely speaking, for any non-negative input and any non-negative initial condition, the corresponding solution of the system is also non-negative, are of great practical importance which occurs quite often in numerous applications and in nature. An existence and uniqueness result for Volterra-Stieltjes systems is proven and in this case the concept of positivity is characterized. Thereafter, various stability concepts for linear time-invariant systems are generalized to time-invariant

Volterra-Stieltjes systems and the differences are presented. Explicit criteria for various stability concepts are derived for positive Volterra-Stieltjes equations. In view of the concept of (stable) zero dynamics, Byrnes-Isidori form and Appendix 1.1, these concepts are generalized to time-invariant Volterra-Stieltjes systems. It is proven that positive Volterra-Stieltjes systems with stable zero dynamics and a special structure of the input output matrices (in particular, relative degree one) are high-gain stabilizable while preserving positivity.
These results are exploited to generalize funnel controller to Volterra-Stieltjes systems in this thesis. In case of stable zero dynamics and suitable assumptions on the high-frequency gain matrix funnel control is guaranteed and also positivity of the trajectory of the closed-loop system. Under a suitable feasibility assumption, funnel control is possible in the presence of input constraints which generalizes the results of Chapter 3 to Volterra-Stieltjes systems. A further modification of the proposed funnel controller is presented which guarantees non-negative input. These results are applied to a control problem in anesthesia. The control objective is to keep the concentration of anesthetic gas close to a target value chosen by the anesthetist.

Zusammenfassung

Diese Dissertation behandelt die Regelung von linearen Systemen mit mehrdimensionalen Eingängen und Ausgängen und unbekanntem, aber beschränktem, Relativgrad und lineare Volterra-Stieltjes Systeme mit mehreren Eingängen und Ausgängen. Die vorgelegte Arbeit behandelt die folgenden zwei Regler: adaptive Rückführung des Ausgangssignals und dessen Ableitung und Funnel Regelung. Für alle Regler werden bestimmte strukturelle Voraussetzungen an die Systeme gestellt, auf die der Regler angewendet werden soll.

Für viele Regelungsanwendungen sind keine guten Modelle vorhanden oder die Modelle sind nur ungenau bekannt. Gewöhnlich kann nicht erwartet werden, dass vollständige Informationen eines Systems vorhanden sind. Stattdessen sind nur strukturelle Eigenschaften (z.B. stabile Nulldynamik, Relativgrad) bekannt. Bei der Regelung solcher Systeme kann ein adaptiver Regler angewendet werden. Das Ziel von Kapitel 2 ist, einen universellen adaptiven Regler zu entwerfen, der vom Systemverhalten lernt und ein vorab festgelegtes Regelungsziel gewährleistet. Mögliche Zielsetzungen sind Stabilisierung des Systems und λ-tracking.
Die meisten Regler, die λ-tracking benutzen, können nur für Systeme mit Relativgrad eins angewendet werden. Das bedeutet, dass der Ausgang des Systems einem gegebenen Referenzsignal folgen soll, wobei ein Fehler von zuvor festgelegter Größe $\lambda > 0$ toleriert wird. Der λ-tracking Regler kann auf Systeme mit höherem Relativgrad $\varrho \geq 1$ erweitert werden, indem der Regler einen Beobachter enthält, der den Systemausgang und dessen ersten $\varrho - 1$ Ableitungen schätzt. Es ist wichtig zu erwähnen, dass dieser Regler jedes System stabilisiert oder verfolgt, dessen Relativgrad bekannt ist, vorausgesetzt das System hat stabile Nulldynamik und die Hochverstärkungsmatrix ist positiv definit.
In dieser Arbeit wird der λ-tracking Regler auf Systeme mit unbekanntem Relativgrad erweitert für die eine obere Schranke bekannt ist. Dies wird dadurch erreicht, dass der Regler eine Rückführung des Ausgangsignals und dessen Ableitungen benutzt. Es wird gezeigt, dass bei Nutzung dieses Reglers λ-tracking und Stabilisierung der betrachteten Systeme erreicht wird. Ein Vorteil des vorgestellten Reglers ist seine relativ einfache Struktur, die es erleichtert, das Funktionieren des Reglers zu verstehen und ihn zu implementieren. Der vorgestellte Regler wird auf ein in Reihe

geschaltetes Feder-Masse-Dämpfer System mit unbekanntem Relativgrad angewendet. Es ist wichtig zu erwähnen, dass der Relativgrad des Systems unbekannt und nur eine obere Schranke bekannt ist.

Hauptnachteil des λ-tracking Reglers ist, dass die Regelgüte nicht direkt in den Entwurf eingeht und die Verstärkung $k(\cdot)$ groß werden kann. Einerseits erhöht eine große Verstärkung die Sensitivität gegenüber Messrauschen als auch die Möglichkeit, dass Zustände kurzfristig sehr groß werden. Andererseits kann bei kleiner Verstärkung $k(\cdot)$ der Fehler für lange Zeit relativ groß sein.
In Kapitel 3 wird das bekannte Konzept der Funnel Regelung für Systeme mit Relativgrad eins eingeführt, das diese Nachteile beseitigt. Es wird gezeigt, dass der klassische Funnel Regler, angewendet auf lineare Systeme mit mehreren Eingängen und Ausgängen und Eingangsbeschränkungen, die Regelungsziele des Funnel Reglers gewährleistet. Es ist hervorzuheben, dass die Gegenwart von Eingangsbeschränkungen ein wichtiges Merkmal dieser Arbeit ist. Eine Bedingung wird abgeleitet, die die Anwendbarkeit des Funnel Reglers unter Eingangsbeschränkungen sicherstellt. Der Nachteil ist, dass genügend Systeminformationen vorher bekannt sein müssen, um diese Bedingung zu überprüfen.
Kapitel 5 verallgemeinert die klassischen Ergebnisse des Funnel Reglers und die neuen Ergebnisse im Zusammenhang mit Eingangsbeschränkungen auf lineare Volterra-Stieltjes Systeme mit mehreren Eingängen und Ausgängen.

Die Systeme in Kapitel 3 müssen strikten Relativgrad eins haben. Das Ziel in Kapitel 4 ist es, die Ergebnisse des Funnel Reglers auf lineare Systeme mit einem Eingang und Ausgang mit Relativgrad zwei zu verallgemeinern. Es ist bekannt, dass der Funnel Regler auf System mit höherem Relativgrad erweitert werden kann, wobei der Regler einen Filter, die Rückführungsdynamik und einen Backstepping Algorithmus enthält. Ein Nachteil ist, dass der Regler nicht mehr einfach ist.
Die Einfachheit der Kontrollstrategie kann erhalten werden, wenn die Rückführung der Ableitung erlaubt ist. Die vorgelegte Arbeit entwirft eine einfache Rückführung, die von zwei Funneln abhängt; einer für den Ausgang des Systems und der andere für dessen Ableitung. Dieser neue Funnel Regler ist robust gegenüber Systemen mit unbekanntem Relativgrad, d.h. der neue Funnel Regler kann auf lineare Systeme mit einem Eingang und Ausgang mit Relativgrad eins oder zwei angewendet werden.
Bei Systemen mit Relativgrad eins führt die Anwendung des neuen Funnel Reglers zu einem geschlossenen System, das eine implizite Differentialgleichung ist. In diesem Fall ist ein neues Existenz- und Eindeutigkeitsergebnis für eine implizite Differentialgleichung bewiesen. Die Ergebnisse von Kapitel 3 hinsichtlich Eingangsbeschränkungen werden auf Systeme mit Relativgrad zwei verallgemeinert.

In Kapitel 5 werden zeitvariante und zeitinvariante lineare Volterra-Stieltjes Systeme mit mehreren Eingängen und Ausgängen hinsichtlich Positivität, verschiedenen Stabilitätskonzepten, Nulldynamik und Funnel Regler betrachtet.
Positive Systeme, d.h. grob gesprochen, für jeden nichtnegative Eingang und jede nichtnegative Anfangsbedingung ist die zugehörige Lösung des Systems auch nichtnegativ, sind von großer praktischer Bedeutung, die ziemlich oft in zahlreichen Anwendungen und in der Natur auftreten. Ein Existenz- und Eindeutigkeitsergebnis für Volterra-Stieltjes Systeme wird bewiesen. In diesem Fall wird das Konzept der Positivität charakterisiert. Danach werden verschiedene Stabilitätskonzepte für lineare zeitinvariante Systeme auf zeitinvariante Volterra-Stieltjes Systeme verallgemeinert und die Unterschiede dargestellt. Explizite Kriterien für die verschiedenen Stabilitätskonzepte werden für positive Volterra-Stieltjes Gleichungen abgeleitet. Hinsichtlich des Konzepts (stabiler) Nulldynamik, Byrnes-Isidori Form und Anhang 1.1 werden diese Konzepte auf zeitinvariante Volterra-Stieltjes Systeme verallgemeinert. Es wird bewiesen, dass positive Volterra-Stieltjes Systeme mit stabiler Nulldynamik und einer speziellen Struktur der Eingangs- und Ausgangsmatrizen (insbesondere Relativgrad eins) stabilisierbar sind während die Positivität erhalten bleibt.
Diese Ergebnisse werden ausgenutzt, um in dieser Arbeit den Funnel Regler auf Volterra-Stieltjes Systeme zu verallgemeinern. Im Falle von stabiler Nulldynamik und geeigneten Voraussetzungen an die Verstärkungsmatrix ist der Funnel Regler anwendbar und die Positivität der Lösung des geschlossenen Systems ist sicher gestellt. Unter geeigneten Voraussetzungen ist der Funnel Regler im Zusammenhang mit Eingangsbeschränkungen anwendbar. Dies verallgemeinert die Ergebnisse aus Kapitel 3 auf Volterra-Stieltjes Systeme. Eine weitere Anpassung des vorgestellten Funnel Reglers stellt nichtnegativen Eingang sicher. Die vorgestellten Ergebnisse werden auf ein Regelungsproblem der Anästhesie angewendet. Das Regelziel ist, die Konzentration von Anästhesiegas nahe dem vom Anästhesisten gewählten Referenzwert zu halten.

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Contents

Nomenclature

Basic notation

$\mathbb{N}$	the set of natural numbers without zero
$\mathbb{R}$, $\mathbb{C}$	the sets of real/complex numbers
$\mathbb{R}^n$, $\mathbb{C}^n$	the set of real/complex vectors
$\langle\cdot,\cdot\rangle$	the scalar product in $\mathbb{R}^n$
rk A	the rank of a matrix $A \in \mathbb{R}^{n\times m}$
adj A	the adjoint of a matrix $A \in \mathbb{C}^{n\times n}$
$\Re z$	the real part of a complex vector $z \in \mathbb{C}^n$
$\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$	the set of real/complex matrices
$\mathbb{R}^{n\times m}_{\geq 0}$	$:= \left\{(a_{ij}) \in \mathbb{R}^{n\times m} \mid a_{ij} \geq 0 \; \forall\, i,\, j\right\}$
$\mathbb{R}^{n\times m}_{>0}$	$:= \left\{(a_{ij}) \in \mathbb{R}^{n\times m} \mid a_{ij} > 0 \; \forall\, i,\, j\right\}$
$\mathbb{R}^{n\times m}_{<0}$	$:= \left\{(a_{ij}) \in \mathbb{R}^{n\times m} \mid a_{ij} < 0 \; \forall\, i,\, j\right\}$
$\mathbb{C}_\lambda$	$:= \{s \in \mathbb{C} \mid \Re s \geq \lambda\},\ \lambda \in \mathbb{R}$
$\mathbb{C}_-$	$:= \{s \in \mathbb{C} \mid \Re s < 0\}$
$\mathcal{B}_\lambda(x)$	$:= \{v \in \mathbb{R}^n \mid \|v - x\| < \lambda\}$, $x \in \mathbb{R}^n$, $\lambda > 0$
$\mathbf{1}_n$	$:= (1, ..., 1)^\top \in \mathbb{R}^n$
$0_{n\times m} \in \mathbb{R}^{n\times m}$	the 0-matrix of dimension $n \times m$
$I_n \in \mathbb{R}^{n\times n}$	the identity matrix of dimension $n \times n$
diag $(a_1, ..., a_n) \in \mathbb{C}^{n\times n}$	a matrix with $a_i \in \mathbb{C}$, $i = 1, ..., n$, on the diagonal and zeroes else
$\|x\|$	$:= \sqrt{x^\top x}$, the Euclidian norm of $x \in \mathbb{R}^n$

$\|A\|$	$:= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} \|a_{ij}\|^2}$, the Frobenius norm of $A \in \mathbb{C}^{n\times m}$
$\|A\|_{\mathrm{Op}}$	$:= \max\{\|Ax\| \mid \|x\| = 1\}$, the operator norm of $A \in \mathbb{C}^{n\times m}$ induced by the 2-norm
spec (A)	$:= \{\lambda \in \mathbb{C} \mid \det(\lambda I_n - A) = 0\}$, the spectrum of $A \in \mathbb{C}^{n\times n}$
$\mu(A)$	$:= \max\{\Re\, s \mid s \in$ spec $(A)\}$, the spectral abscissa of $A \in \mathbb{C}^{n\times n}$
A Metzler	iff $A_{ij} \geq 0$ for all $i,\ j = 1, ..., n$ with $i \neq j$, $A =\in \mathbb{R}^{n\times n}$
$M^{\mathbb{N}}$	:= the set of all mappings from $\mathbb{N}$ to a set M
$f(\cdot) \leq g(\dot)$	$:\Leftrightarrow\ f_{ij}(\cdot) \leq g_{i,j}(\cdot)$ for all $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, $f,\ g : J \to \mathbb{R}^{n\times m}$, $J \subset \mathbb{R}$ an interval
$f(\cdot)$ is *non-decreasing*	iff $f(t_1) \leq f(t_2)$ for all $t_1, t_2 \in J$ with $t_1 < t_2$, $f : J \to \mathbb{R}^{n\times m}$, $J \subset \mathbb{R}$ an interval
$f(\cdot) \geq 0$	iff $f(t) \geq 0$ for all $t \in J$, $f(\cdot) \in \mathcal{C}(J, \mathbb{R}^{n\times m})$, $J \subset \mathbb{R}$ an interval, it is said that $f(\cdot)$ is *non-negative*
deg p	the degree of a polynomial $p(\cdot) \in \mathbb{R}[s]$
$\frac{1}{0}$	$:= \infty$
$d_\lambda : \mathbb{R} \to \mathbb{R}$	$s \mapsto d_\lambda(s) := \max\{\|s\| - \lambda,\ 0\}$, for $\lambda > 0$
$\mathrm{dist} : \mathbb{R} \to \mathbb{R}$	$s \mapsto \mathrm{dist}(s, J) := \inf_{j\in J} \|s - j\|$, the distance function from a nonempty interval $J \subset \mathbb{R}$
$\mathrm{sat}_{\hat{\mathrm{u}}} : \mathbb{R}^m \to \{w \in \mathbb{R}^m \mid \|w\| \leq \hat{u}\}$	$v \mapsto \mathrm{sat}_{\hat{\mathrm{u}}}(v) := \begin{cases} \frac{\hat{u}}{\|v\|}v, & \|v\| > \hat{u} \\ v, & \text{otherwise} \end{cases}$, the saturation function with constraint parameter $\hat{u} > 0$

$[(M,\eta) * f](t)$	$:= Mf(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, f(t-\theta)$ for almost all $t \geq 0$, $M \in \mathbb{R}^{n\times n}$, $\eta(\cdot) \in \mathcal{BV}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ and $f(\cdot) \in \mathcal{L}^1_{\text{loc}}(R_{\geq 0}, \mathbb{R}^n)$
$d_A : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$	$v \mapsto d_A(v) := \inf_{a\in A} \|v - a\|_2$, the Euclidean distance function for a nonempty set $A \subset \mathbb{R}^n$
$\text{pr}_j : A_1 \times ...A_n \to A_j$	$x := (x_1, ..., x_n) \mapsto \text{pr}_j(x) := x_j$, the canonical projection to the j-th component for $j \in \{1, ..., n\}$ and $A_i \subset \mathbb{R}^{n_i\times m_i}$, n_i, $m_i \in \mathbb{N}$, $i = 1, ..., n$
graph : $\mathcal{C}(J, \mathbb{R}^n) \to J \times \mathbb{R}^n$,	$f(\cdot) \mapsto \text{graph}(f) := \{(t, f(t)) \mid t \in J\}$, for $J \subset \mathbb{R}$ an interval
$\lambda(\cdot)$	denotes the Lebesgue measure

Function spaces

$\mathbb{R}[s]$	$:= \left\{ \sum_{i=0}^{N} a_i s^i \,\middle|\, N \in \mathbb{N},\ a_i \in \mathbb{R}\ \forall\, i = 0, ..., N \right\}$, the ring of polynomials with real coefficients
$\mathbb{R}(s)$	$:= \left\{ \frac{p}{q} \,\middle|\, p,\ q \in \mathbb{R}[s],\ q \not\equiv 0 \right\}$, the quotient field of $\mathbb{R}[s]$
$\mathbb{R}[s]^{n\times m}$	$:= \left\{ \sum_{i=0}^{N} M_i s^i \,\middle|\, N \in \mathbb{N},\ M_i \in \mathbb{R}^{n\times m}\ \forall\, i = 0, ..., N \right\}$
$\mathbb{R}(s)^{n\times m}$	$:= \left\{ M \in \mathbb{R}^{n\times m} \,\middle|\, \forall\, i \in \{1, ..., n\}\ \forall\, j \in \{1, ..., m\} :\ M_{ij} \in \mathbb{R}(s) \right\}$
$\mathcal{C}^r(J, \mathbb{R}^{n\times m})$	r-times continuously differentiable functions $f : J \to \mathbb{R}^{n\times m}$, $J \subset \mathbb{R}$ an interval, $r \in \mathbb{N} \cup \{0, \infty\}$
$\mathcal{AC}(J, \mathbb{R}^{n\times m})$	(locally) absolutely continuous functions $f : J \to \mathbb{R}^{n\times m}$, $J \subset \mathbb{R}$ an interval
$\mathcal{C}_{\text{pw}}(J, \mathbb{R}^m)$	the set of piecewise continuous functions from J to $\mathbb{R}^m$, where $J \subset \mathbb{R}$ is an interval

$\mathcal{BV}([\alpha,\beta],\mathbb{R}^{n\times m})$	functions $f:[\alpha,\beta]\to\mathbb{R}^{n\times m}$, $\alpha\le\beta$, of bounded variation and with norm
$\|f\|_{\mathrm{Var}(\alpha,\beta)} := \mathrm{Var}(f;\alpha,\beta)$	$:= \|f(\alpha)\| + \sup\sum_{i=1}^{k}\|f(t_i)-f(t_{i-1})\|$, the *variation* of $f:[\alpha,\beta]\to\mathbb{R}^{n\times m}$, where the supremum is taken over all finite $\alpha\le t_0\le\ldots\le t_k\le\beta$, $k\in\mathbb{N}$,
$\mathcal{BV}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times m})$	functions $f:\mathbb{R}_{\ge 0}\to\mathbb{R}^{n\times m}$ of total variation satisfying $f(0)=0$ and with total variation
$\int_0^\infty \lvert\,\mathrm{d}f(t)\,\rvert$	$:= \lim_{t\to\infty}\mathrm{Var}(f;0,t)<\infty$
$\mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times m})$	locally functions of bounded variation $f:\mathbb{R}_{\ge 0}\to\mathbb{R}^{n\times m}$, i.e. $f\vert_J(\cdot)\in\mathcal{BV}(J,\mathbb{R}^{n\times m})$ for any bounded interval $J\subset\mathbb{R}_{\ge 0}$, satisfying $f(0)=0$
$\mathcal{NBV}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times m}) := \left\{ f:\mathbb{R}_{\ge 0}\to\mathbb{R}^{n\times m} \;\middle\vert\; f(\cdot) \text{ is c.f.r.},\ f(0)=0,\ \int_0^\infty \lvert\,\mathrm{d}f(t)\,\rvert<\infty \right\}$	functions of bounded total variation that are continuous from right and vanish at 0
$\mathcal{L}^p(J,\mathbb{R}^{n\times m})$	p-integrable functions $f:J\to\mathbb{R}^{n\times m}$, $J\subset\mathbb{R}$ an interval, with $\int_J\|f(t)\|^p\,\mathrm{d}t<\infty$, $p\in[1,\infty)$, and norm
$\|f\|_{\mathcal{L}^p(J)}$	$:= \left(\int_J\|f(t)\|^p\,\mathrm{d}t\right)^{1/p}$
$\mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times m})$	locally $\mathcal{L}^p$-functions $f:\mathbb{R}_{\ge 0}\to\mathbb{R}^{n\times m}$, i.e. $f\vert_J\in\mathcal{L}^p(J,\mathbb{R}^{n\times m})$ for any bounded interval $J\subset\mathbb{R}$
$\mathcal{L}^\infty(J,\mathbb{R}^{n\times m})$	measurable and essentially bounded functions $f:J\to\mathbb{R}^{n\times m}$, $J\subset\mathbb{R}$ an interval, and norm
$\|f\|_{\mathcal{L}^\infty(J)}$	$:= \operatorname{ess-sup}_{t\in J}\|f(t)\|$
$\mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times m})$	locally $\mathcal{L}^\infty$-functions $f:\mathbb{R}_{\ge 0}\to\mathbb{R}^{n\times m}$, i.e. $f\vert_J\in\mathcal{L}^\infty(J,\mathbb{R}^{n\times m})$ for any bounded interval $J\subset\mathbb{R}$

$\mathcal{W}^{m,\infty}(J,\mathbb{R}^n)$	bounded locally absolutely continuous functions $f : J \to \mathbb{R}^n$, $J \subset \mathbb{R}$ an interval, with essentially bounded first m derivatives $\dot{f}(\cdot)$,...,$f^{(m)}(\cdot)$, and norm
$\\|f\\|_{\mathcal{W}^{m,\infty}}$	$:= \\|f\\|_{\mathcal{L}^\infty(J)} + \sum_{i=1}^{m} \\|f^{(i)}\\|_{\mathcal{L}^\infty(J)}$

1 Introduction

The topic of the thesis is feedback control for single input, single output and multi input, multi output linear systems. The thesis is divided into two parts – Chapter 2-4 and Chapter 5, respectively.

Chapters 2 and 3 consider time-invariant multi input, multi output linear systems

$$\left.\begin{array}{rcll} \dot{x}(t) & = & Ax(t) + Bu(t), & x(0) = x^0 \in \mathbb{R}^n \\ y(t) & = & Cx(t) & \end{array}\right\} \tag{1.1}$$

with $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, where $u(\cdot)$ denotes the input of the system and $y(\cdot)$ is the system output. In the simplest case, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is a locally integrable function and a solution is a functions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ which satisfies (1.1). Variation-of-Constants formula shows that the solution $x(\cdot) = x(\cdot; 0, x^0, u)$ and the output $y(\cdot)$ are given by

$$t \mapsto x(t; 0, x^0, u) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s\,,$$

$$t \mapsto y(t) = C\left[e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s\right].$$

Chapter 4 considers systems (1.1) which are single input, single output (i.e. $m = 1$). It has to be noted that a system (1.1) is called *single input, single output* if $m = 1$. The aim is to design a controller $u(\cdot)$ such that the output $y(\cdot)$ of a system (1.1) tracks a given reference signal. Many controllers base on information about the system matrices A, B and C (e.g. controllability or observability). In general the system matrices are unknown and only structural information are known. What is the design of the input $u(\cdot)$ such that the output $y(\cdot)$ has some special properties without knowing the model parameters precisely? One basic approach is a controller which uses the output $y(\cdot)$ and its derivatives. The knowledge of the structure of the model (e.g. strict relative degree or stable zero dynamics) are only required. The survey article [42] presents such control strategies.
One aim is a controller which is simple and not too complicated. The simplicity simplifies the implementation into an application and a controller with a clear structure is

easier to understand. This allows to tune the controller parameter in a straightforward manner and makes it better to understand its behaviour. Two control strategies are the adaptive λ-tracker and the funnel controller which are applied to systems (1.1) and presented in Chapter 2-4.

Two new adaptive controllers for adaptive stabilization and adaptive λ-tracking are proposed in Chapter 2. The chapter starts with a presentation of the concept of stabilization and λ-tracking. Thereafter, the system class and their assumptions are introduced. The new controllers incorporate a high-gain derivative feedback. For stabilization the input $u(\cdot)$ is designed as

$$\begin{aligned} u(t) &= -\sum_{i=0}^{\nu} k(t)^{\nu+1-i} p_i y^{(i)}(t) \\ \dot{k}(t) &= k(t)^{-2\gamma} \| y(t), k(t)^{-1}\dot{y}(t), ..., k(t)^{-(\nu-1)} y^{(\nu-1)}(t) \|^2, \qquad k(0) = k_0 > 0, \end{aligned}$$

where $p_0, ..., p_\nu \in \mathbb{R}$ are suitable parameters and $\nu \in \mathbb{N}$, $\gamma > 0$ known, and, for λ-tracking,

$$\begin{aligned} u(t) &= -\sum_{i=0}^{\nu} k(t)^{\nu+1-i} p_i e^{(i)}(t) + d_u(t), \qquad e(t) = (y(t) + d_y(t)) - y_{\mathrm{ref}}(t) \\ \dot{k}(t) &= k(t)^{-2\gamma} d_\lambda(\| e(t), k(t)^{-1}\dot{e}(t), ..., k(t)^{-(\nu-1)} e^{(\nu-1)}(t) \|)^2, \qquad k(0) = k_0 > 0, \end{aligned}$$

where $d_u(\cdot)$, $d_y(\cdot)$ and $y_{\mathrm{ref}}(\cdot)$ are suitable functions and $\lambda > 0$. The definition of the function $d_\lambda(\cdot)$ is given in the nomenclature. It is assumed that the system matrices of (1.1) are unknown. The system satisfies the following structural properties: the relative degree ϱ is unknown and an upper bound $\nu \in \mathbb{N}$ is known, it has stable zero dynamics and positive high-frequency gain. For this large class of linear single input, single output and multi input, multi output systems with unknown relative degree stabilization and λ-tracking of most relevant reference trajectories are shown. Moreover, if the control objective is λ-tracking, then input and output disturbances are allowed. One result to achieve stabilization for single input, single output systems of unknown relative degree is due to [57]. A counterexample to the main result of [57] is presented in [29, 30]. Section 2.3 contains the main theoretical results of this chapter. Theorems 2.3.4 and 2.3.8 contain the main theoretical results for stabilization and Theorems 2.3.7 and 2.3.10 generalize these results to λ-tracking. The controllers are applied to a serially connected mass-spring damper system. The results of Chapter 2 are new and important. But the λ-tracker has two drawbacks

- the tracking error will only be achieved asymptotically and
- though bounded, the gain $k(\cdot)$ is increasing.

To overcome this, funnel control is introduced.

Chapter 3 adopts the funnel control viewpoint

$$u(t) = -k(t)[y(t) - y_{\text{ref}}(t)], \qquad k(t) = \frac{1}{\psi(t) - \|y(t) - y_{\text{ref}}(t)\|},$$

where $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are suitable functions – but differs from its precursor [43] in an essential manner: input constraints are a distinguishing feature of the underlying system class. A feasibility relationship involving the system data, funnel data, reference signal data and the saturation bound is derived under which the efficacy of funnel control in the presence of input saturation is established. However, there is a price to pay: sufficient a priori plant information is required in order to check the feasibility condition. Theorems 3.4.2 and 3.4.6 present two kinds of input constraints – Euclidean saturation and componentwise saturation, respectively. It is essentially that the system has strict relative degree one.

The concept of funnel control is extended to single input, single output linear systems (1.1) with relative degree two and input saturation in Chapter 4. The controller gets the form

$$\begin{aligned} u(t) &= -k_0(t)^2 e(t) - k_1(t)\dot{e}(t), \qquad e(t) = y(t) - y_{\text{ref}}(t), \\ k_i(t) &= \tfrac{\varphi_i(t)}{1-\varphi_i(t)|e^{(i)}(t)|}, \ i = 0, 1, \end{aligned}$$

where $\varphi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = 0, 1$, and $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are suitable functions. The controller uses the system output and its derivative. It is shown that this controller is robust for systems with relative degree one which is a first generalization of the results of Chapter 2 with funnel control. The new funnel controller differs from its precursor of relative degree one – two performance funnels, within which the tracking error $e(\cdot)$ and its derivative $\dot{e}(\cdot)$ are required. Apart from this central result it is also shown that the same control objective can be achieved in the presence of input constraints provided a feasibility condition, formulated in terms of the system data, the saturation bounds, the funnel data, bounds on the reference signal, and the initial state, holds.
If the system has relative degree one, some care must be exercised in formulating the closed-loop initial value problem which is an implicit ordinary differential equation. A proof of existence and uniqueness of a maximal solution of such an initial value problem is given. The theoretical results are illustrated on a serially connected mass-spring damper system. Chapter 4 shows that the new funnel controller can be achieved to systems with unknown relative degree $\varrho \in \{1, 2\}$.

The concept of strict relative degree, zero dynamics, together a characterization, and Byrnes-Isidori form is introduced in Section 1.1. The second part, Chapter 5, considers time-varying multi input, multi output linear Volterra-Stieltjes systems

$$\left.\begin{array}{rcll} \dot{x}(t) &=& A(t)\,x(t) + \int\limits_0^t \mathrm{d}[\eta(\theta)]\,x(t-\theta) + B\,u(t), & x|_{[0,\sigma]} = \phi \\ y(t) &=& C\,x(t) & \text{for a.a. } t \geq \sigma, \end{array}\right\} \quad (1.2)$$

where

$$(A(\cdot), B, C, \eta(\cdot)) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}),$$

initial data $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$, $\sigma \geq 0$, and input function $u(\cdot) \in \mathcal{L}^1_{\mathrm{loc}}([\sigma,\infty), \mathbb{R}^m)$. The Riemann-Stieltjes integral in (1.2) is defined in Definition 5.1.1. The funnel control strategy of Chapter 3 is applied to time-invariant Volterra-Stieltjes systems.

In Section 5.2, positivity of time-varying linear Volterra-Stieltjes systems (1.2) is investigated; that is for any non-negative input $u(\cdot)$ and any non-negative initial condition, the corresponding solution of the system is also non-negative. This concept is characterized and thereafter, various stability concepts are presented in Section 5.3 – explicit criteria for uniform asymptotic stability and exponential asymptotic stability of positive linear time-invariant equations are derived. In Section 5.4, the standard system theoretic concept of (stable) zero dynamics is recalled. This concept coincides with minimum phase if (1.2) is time-invariant without Volterra term. The Byrnes-Isidori form (this form separates the direct influence of the input to the zero dynamics) is derived and exploited to characterize stable zero dynamics for time-invariant Volterra-Stieltjes systems (1.2). Finally, it is shown that positive systems with stable zero dynamics and special structure of the input output matrices (in particular, relative degree one) are high-gain stabilizable while preserving positivity.

The results of Sections 5.2-5.4 may be interesting in their own right, but these results are exploited in Section 5.5 to generalize the funnel controller of Chapter 3 to time-invariant Volterra-Stieltjes systems (1.2). In case of stable zero dynamics and suitable assumptions on the high-frequency gain matrix CB, Section 5.5 shows that funnel control is guaranteed and also positivity of the trajectory of the closed-loop system. Moreover, under a suitable feasibility assumption, funnel control is possible in the presence of input constraints which generalizes the results of Chapter 3 to Volterra-Stieltjes systems. Finally, in Section 5.6 the results of Section 5.5 are applied to control the depth of anesthesia of a three compartment mammillary patient's model.

1.1 Linear systems

Some well known results of ordinary differential equations and linear systems are presented in this section.

1.1.1 Solution

Definition 1.1.1. (Solution)
Consider, for $f : \mathcal{D} \to \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, the initial value problem

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x^0,\ (t_0, x^0) \in \mathcal{D}. \tag{1.3}$$

(i) A function $x : [t_0, \omega) \to \mathbb{R}^n$, $\omega \in (t_0, \max(\mathrm{pr}_1(\mathcal{D}))$, is said to be a *solution* of the initial value problem (1.3) if, and only if, $x(\cdot)$ is continuous differentiable on $[t_0, \omega)$, $x(\cdot)$ satisfies (1.3) for all $t \in [t_0, \omega)$ and $\mathrm{graph}(x) \subset \mathcal{D}$. It is denoted by $x(\cdot; t_0, x^0)$.

(ii) A function $x(\cdot)$ is called *maximal solution* if, and only if, $x(\cdot)$ satisfies (i) and there exists no right extension of $x(\cdot)$ which is a solution.

(iii) If $\mathrm{pr}_1(\mathcal{D}) = \mathbb{R}_{\geq 0}$, then $x(\cdot)$ is called a *global solution* if, and only if, $x(\cdot)$ satisfies (i) and $\omega = \infty$. $\diamond$

Existence and uniqueness of a solution of the initial value problem (1.3) provide the following theorem. The proof can be found in [88, Theorem III.10.VI] and is omitted here.

Theorem 1.1.2. ***(Existence and uniqueness)***
Let, for $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, $f : \mathcal{D} \to \mathbb{R}^n$ be continuous in $\mathcal{D}$. If $f(\cdot, \cdot)$ satisfies a local Lipschitz condition in the second argument, i.e.

$$\exists\, K > 0\ \forall\, (t, x),\ (t, \overline{x}) \in \mathcal{D} : \quad \|f(t, x) - f(t, \overline{x})\| \leq K\|x - \overline{x}\|,$$

then the initial value problem (1.3) *has exactly one solution. The solution can be extended to the right up to the boundary of $\mathcal{D}$.*

1.1.2 Stability

For any $A \in \mathbb{R}^{n\times n}$ and $t_0 \geq 0$, various stability concepts for linear, time invariant systems of the form

$$\dot{x}(t) = Ax(t), \quad x(t_0) = x^0 \in \mathbb{R}^n,\ t \geq t_0, \tag{1.4}$$

are investigated which are well known (see [28, 79, 88]).

Definition 1.1.3. (Stability concepts for linear systems (1.4))
A system (1.4) – or more precisely, its zero solution – is said to be

1. *stable* $:\Leftrightarrow\ \forall\, \varepsilon > 0\ \forall\, t_0 \geq 0\ \exists\, \delta > 0\ \forall\, x^0 \in \mathcal{B}_\delta(0)\ \forall\, t \geq t_0 :\ \|x(t;t_0,x^0)\| < \varepsilon$,
2. *uniformly stable* $:\Leftrightarrow$ stable and $\delta > 0$ can be chosen independently of t_0,
3. *attractive* $:\Leftrightarrow\ \forall\, t_0 \geq 0\ \forall\, x^0 \in \mathbb{R}^n :\ \lim_{t\to\infty} x(t;t_0,x^0) = 0$,
4. *asymptotically stable* $:\Leftrightarrow$ stable and attractive,
5. *uniformly asymptotically stable* $:\Leftrightarrow$ uniformly stable and
$$\exists\, \delta > 0\ \forall\, \varepsilon > 0\ \exists\, T > 0\ \forall\, t_0 \geq 0\ \forall\, x^0 \in \mathcal{B}_\delta(0)\ \forall\, t \geq t_0 + T :\ \|x(t;t_0,x^0)\| < \varepsilon,$$
6. *exponentially stable* $:\Leftrightarrow$
$$\forall\, t_0 \geq 0\ \exists\, M,\ \lambda > 0\ \forall\, x^0 \in \mathbb{R}^n\ \forall\, t \geq t_0 :\ \|x(t;t_0,x^0)\| \leq Me^{-\lambda(t-t_0)}\|x^0\|,$$
7. *uniformly exponentially stable* $:\Leftrightarrow$ exponentially stable and M, λ can be chosen independently of t_0,
8. $\mathcal{L}^p$*-stable*, $p \in [1,\infty]$ $:\Leftrightarrow\ X(\cdot) := e^{A\cdot} \in \mathcal{L}^p(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, where $X(\cdot)$ denotes the fundamental matrix of (1.4). ◇

Proposition 1.1.4. ***(Characterization of stability concepts for linear systems** (1.4)**)***
For the system (1.4) *the following stability concepts are equivalent.*

(i) *asymptotic stability*

(ii) $\mathcal{L}^1$*-stability*

(iii) *$\mathcal{L}^p$-stability for all* $p \in [1, \infty]$

(iv) $\forall\, s \in \mathbb{C}_0 : \quad \det(sI - A) \neq 0$

(v) *uniform asymptotic stability*

(vi) $\exists\, M,\ \lambda > 0\ \forall\, t \geq 0 : \quad \|X(t)\| \leq M e^{-\lambda t}$*, where* $X(\cdot)$ *denotes the fundamental matrix of* (1.4)

(vii) *exponential stability*

(viii) *uniform exponential stability.*

Proof:
As immediate conclusions of Definition 1.1.3 and [28, 79], it follows

$$\begin{array}{ccccccccc}
\text{(i)} & \stackrel{Def\ 1.1.3}{\Longleftarrow} & \text{(v)} & \stackrel{[28, Th.\ 3.3.8]}{\Longleftrightarrow} & \text{(viii)} & \stackrel{Def\ 1.1.3}{\Longrightarrow} & \text{(vii)} & \stackrel{Def\ 1.1.3}{\Longrightarrow} & \text{(vi)} \\
 & & & & & & \Updownarrow & [79, Th.\ 6.10] & \\
 & & & & \text{(viii)} & \stackrel{[79, Th.\ 8.7]}{\Longleftarrow} & \text{(iv)} & \stackrel{[28, Th.\ 3.3.20]}{\Longleftrightarrow} & \text{(i)} \\
 & & & & \text{(vi)} & \stackrel{Def\ 1.1.3}{\Longrightarrow} & \text{(ii)} & \stackrel{p=1}{\Longleftarrow} & \text{(iii)}
\end{array}$$

and it remains to show

$$\text{(i)} \Rightarrow \text{(iii)}, \qquad \text{(ii)} \Rightarrow \text{(iii)} \qquad \text{and} \qquad \text{(ii)} \Rightarrow \text{(iv)}.$$

STEP 1: *The conclusion "(i) ⇒ (iii)" is shown.*
Condition (i) implies (vi) (see above), i.e. $\|X(t)\| \leq M e^{-\lambda t}$ for some M, $\lambda > 0$ and all $t \geq t_0$, which implies (iii).

STEP 2: If $X(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n})$, then (1.4) implies that $\dot{X}(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n})$ which gives (see [36, Lem. 2.1.7])

$$\lim_{t \to \infty} X(t) = 0 \tag{1.5}$$

and thus,

$$X(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n}). \tag{1.6}$$

STEP 3: *The conclusion "(ii) ⇒ (iii)" is shown, i.e.*

$$X(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n}) \qquad \Rightarrow \qquad \forall\, p \in [1, \infty] :\ X(\cdot) \in \mathcal{L}^p(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n}).$$

If $p = 1$, then $X(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ is trivially satisfied by (ii). If $p = \infty$, then Step 2 and (1.6) give $X(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$. Moreover, Step 2 shows

$$\forall\, p \in (1, \infty): \quad \int_0^\infty \|X(t)\|^p \,\mathrm{d}t \;\leq \|X\|_\infty^{p-1} \int_0^\infty \|X(t)\| \,\mathrm{d}t \overset{(1.6)}{<} \infty,$$

which completes the conclusion.

STEP 4: *The implication "(ii) ⇒ (iv)" is shown.*
Condition (ii) and Step 2 give

$$\lim_{t\to\infty} x(t; t_0, x^0) = \lim_{t\to\infty} X(t - t_0)x^0 \overset{(1.5)}{=} 0$$

which is, by [28, Prop. 3.3.22], equivalent to spec $(A) \subset \mathbb{C}_-$, i.e. (iv). This completes the proof of the proposition. □

A matrix $A \in \mathbb{R}^{n\times n}$ is called *Hurwitz* if, and only if, all its eigenvalues have negative real part, i.e.

$$\text{spec}\, A \subset \{s \in \mathbb{C} \mid \Re s < 0\} \tag{1.7}$$

(see [84, Def. C.5.2]). It has to be noted that (1.7) is equivalent to

$$\forall s \in \mathbb{C}_0 : \;\det(sI - A) \neq 0.$$

Then, Proposition 1.1.4 gives that A is Hurwitz if, and only if, the system (1.4), i.e. $\dot{x}(t) = Ax(t)$, $x(t_0) = x^0$, is exponentially stable, i.e. for all solutions $x(\cdot)$ of (1.4) holds:

$$\exists \alpha,\; \beta > 0 \;\forall\, t \geq t_0 : \;\|x(t)\| = \left\| e^{A\,(t-t_0)}x^0 \right\| \leq \beta e^{-\alpha(t-t_0)}\|x^0\|. \tag{1.8}$$

1.1.3 Relative Degree, zero dynamics, Byrnes-Isidori form

The concepts relative degree, (stable) zero dynamic and Byrnes-Isidori form play an essential role for the design of the controllers in this thesis. These concepts will be introduced and discussed in this section. The definition for the relative degree of linear, time-invariant, multi input, multi output systems (1.1) is given. After them, for given relative degree, the Byrnes-Isidori form is introduced. The Byrnes-Isidori form allows characterization of the zero dynamics. The relative degree, Byrnes-Isidori form and characterization of the zero dynamics are utilized in Chapter 2-4.

The concept of the relative degree goes back to systems theory in the frequency domain, where a single input, single output system (1.1) may be described in the frequency domain by a transfer function

$$G(s) = \frac{p(s)}{q(s)} = C(sI_n - A)^{-1}B = \sum_{k=0}^{\infty} CA^k B s^{-(k+1)} \tag{1.9}$$

with polynomials $p(\cdot)$, $q(\cdot) \in \mathbb{R}[s]$. Then the difference $\varrho = \deg q - \deg p$ is called relative degree of the system (1.1) in the frequency domain. Now, together with (1.9), it is easy to see that the frequency domain definition of the relative degree $\varrho = \deg q - \deg p$ is equivalent to the following time domain definition. For example, the relative degree of systems (1.1) is studied in the textbooks [49, 81].

Definition 1.1.5. (Strict relative degree and positive high-frequency gain)
The system (1.1) has *strict relative degree* $\varrho \in \mathbb{N}$ if, and only if,

$$CA^iB = 0 \text{ for } i = 0, ..., \varrho - 2 \qquad \text{and} \qquad \det\left(CA^{\varrho-1}B\right) \neq 0. \tag{1.10}$$

The *positive high-frequency gain* of the system (1.1) with relative degree $\varrho \in \mathbb{N}$ is defined by $CA^{\varrho-1}B$ positive definite, i.e.

$$\exists\, \gamma > 0\; \forall\, v \in \mathbb{R}^m : \quad v^\top CA^{\varrho-1}Bv \geq \gamma\|v\|^2. \tag{1.11}$$

◇

In other words, the relative degree of (1.1) is the least number of times (if it is well defined) one has to differentiate the output to have the input appear explicitly.

It has to be noted that a matrix $A \in \mathbb{C}^{n\times n}$ is called *positive definite* if, and only if, its Hermitian part $\frac{1}{2}(A + A^*)$ is positive definite, i.e. $v^*(A + A^*)v > 0$ for all $v \in \mathbb{C}^n \setminus \{0\}$. Thus it is not necessarily assumed that A is Hermitian (or symmetric in case of real matrices).

The concept of (stable) zero dynamics of multi input, multi output systems (1.1) goes back to [13, 14]. The definition for zero dynamics of a linear system (1.1) which is given in Definition 1.1.6 is similar to [49]. The zero dynamics is the dynamic of the system (1.1) when $y(\cdot)$ is equal to zero.

Definition 1.1.6. (Zero dynamics)
The *zero dynamics* of the system (1.1) are defined as the real vector space of trajectories

$$\mathcal{ZD}(A,B,C) := \left\{ (x,u,y) \in \mathcal{C}^1(\mathbb{R}_{\geq 0},\mathbb{R}^n) \times \mathcal{C}_{\mathrm{pw}}(\mathbb{R}_{\geq 0},\mathbb{R}^m) \times \mathcal{C}^1(\mathbb{R}_{\geq 0},\mathbb{R}^m) \,\middle|\, (x,u,y) \text{ solves (1.1) with } y(\cdot) \equiv 0 \right\}.$$

The system (1.1) is said to have *stable zero dynamics* if, and only if,

$$\forall\, (x(\cdot),u(\cdot),y(\cdot)) \in \mathcal{ZD}(A,B,C): \quad \lim_{t\to\infty} x(t) = 0.$$

◇

It has to be noted that the zero dynamics is independent on the initial condition x^0.

The following proposition provides a state space form into which every system (1.1) with relative degree $\varrho \in \mathbb{N}$ can be converted, see e.g. [45, Lem. 3.5]. This so called *Byrnes-Isidori form* was introduced in [13] for linear single input, single output systems and is contained in [49]. Byrnes-Isidori form for systems (1.1) with strict relative degree is well known, see [45, 49]. It makes possible the separation of the inputs and outputs from the rest of the system states.

Proposition 1.1.7. ***(Byrnes-Isidori form)***
Consider the system (1.1) *with strict relative degree* $\varrho \in \mathbb{N}$. *For*

$$\mathcal{C} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\varrho-1} \end{bmatrix} \in \mathbb{R}^{\varrho m \times n} \qquad \textit{and} \qquad \mathcal{B} := \begin{bmatrix} B, & AB, ..., & A^{\varrho-1}B \end{bmatrix} \in \mathbb{R}^{n \times \varrho m}$$

set

$$\left.\begin{array}{l} V \in \mathbb{R}^{n\times(n-\varrho m)} \text{ such that } \ker \mathcal{C} = \operatorname{im} V \\ N := (V^\top V)^{-1} V^\top [I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}] \in \mathbb{R}^{(n-\varrho m)\times n}. \end{array}\right\} \tag{1.12}$$

Then $T := [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1},\ V]$ *has inverse* $T^{-1} = \begin{bmatrix} \mathcal{C} \\ N \end{bmatrix}$ *and* $(\xi^\top,\ z^\top)^\top := T^{-1}x$ *converts* *the initial value problem* (1.1) *into*

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi(t)\\ z(t)\end{pmatrix} & = & \left(\begin{array}{cccc|c} 0 & I_m & & & 0\\ & \ddots & \ddots & & \vdots\\ & & 0 & I_m & 0\\ R_1 & R_2 & \dots & R_\varrho & S\\ \hline P_1 & 0 & \dots & 0 & Q \end{array}\right)\begin{pmatrix}\xi(t)\\ z(t)\end{pmatrix} + \left(\begin{array}{c} 0\\ \vdots\\ 0\\ \Gamma\\ \hline 0\end{array}\right) u(t)\\ y(t) & = & \left(\begin{array}{cccc|c} I_m & 0 & \dots & 0 & 0\end{array}\right)\begin{pmatrix}\xi(t)\\ z(t)\end{pmatrix}\\ \begin{pmatrix}\xi(0)\\ z(0)\end{pmatrix} & = & \begin{pmatrix}\mathcal{C}x^0\\ Nx^0\end{pmatrix} \end{array}\right\} \qquad (1.13)$$

where $R_1, ..., R_\varrho \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times(n-m\varrho)}$ *with* $[R_1, ..., R_\varrho, S] := CA^\varrho T$ *and* $\Gamma := CA^{\varrho-1}B$, $P_1 := NA^\varrho B\Gamma^{-1} \in \mathbb{R}^{(n-m\varrho)\times m}$, $Q := NAV \in \mathbb{R}^{(n-m\varrho)\times(n-m\varrho)}$ *(wherein* 0 *denotes* $m \times m$ *zero matrix). Moreover,* (1.1) *has stable zero dynamics if, and only if, the matrix* Q *is Hurwitz.*

Remark 1.1.8. (Comments on Byrnes-Isidori form)

(i) If system (1.1) has relative degree $\varrho \in \mathbb{N}$, then (1.13) gives

$$\begin{array}{rclrcl} y(\cdot) & = & \xi_1(\cdot), & \xi_1(0) & = & C\,x^0\\ \dot{y}(\cdot) & = & \dot{\xi}_1(\cdot) = \xi_2(\cdot), & \xi_2(0) & = & C\,A\,x^0\\ & \vdots & & & & \\ y^{(\varrho-1)}(\cdot) & = & \dot{\xi}_{\varrho-1}(\cdot) = \xi_\varrho(\cdot), & \xi_{\varrho-1}(0) & = & C\,A^{\varrho-2}\,x^0\\ y^{(\varrho)}(\cdot) & = & \dot{\xi}_\varrho(\cdot) = \sum_{i=1}^{\varrho} R_i\xi_i(\cdot) + Sz(\cdot) + \Gamma\,u(t), & \xi_\varrho(0) & = & C\,A^{\varrho-1}\,x^0\\ \dot{z}(t) & = & P_1\xi_1(\cdot) + Qz(\cdot) = P_1 y(\cdot) + Qz(\cdot), & z(0) & = & N\,x^0. \end{array}$$

This shows that the input $u(\cdot)$ only affects the ϱ-th derivative of the output $y(\cdot)$ and the $z(\cdot)$-part can be separated which is only influenced by $\dot{y()}$.

(ii) In view of the system (1.13), (i) and Definition 1.1.6, the zero dynamics of (1.13) (which is equivalent to (1.1), respectively) will be described by the subsystem $\dot{z}(\cdot) = Qz(\cdot)$. Then the assumption of stable zero dynamics of (1.1) is equivalent that the matrix Q is Hurwitz. In this case, there exist positive constants $\alpha, \beta > 0$ such that $\left\|e^{Q\,t}\right\| \leq \beta e^{-\alpha t}$ for all $t \geq 0$ (see (1.8)).

(iii) It has to be noted that the Byrnes-Isidori form (1.13) is not uniquely defined. ⋄

The following proposition gives a characterization of stable zero dynamics. The proof follows immediately from Definition 1.1.6 and Proposition 1.1.7 and is omitted here.

Proposition 1.1.9. ***(Characterization of stable zero dynamics)***
Suppose (1.1) *has relative degree* $\varrho \in \mathbb{N}$. *Then the following statements are equivalent:*

(i) (1.1) *has stable zero dynamics.*

(ii)
$$\forall\, s \in \mathbb{C}_0 : \quad \det \begin{pmatrix} sI_n - A & B \\ C & 0 \end{pmatrix} \neq 0 \tag{1.14}$$

(iii) *The subsystem*
$$\dot{z}(t) = NAVz(t), \qquad t \geq 0 \tag{1.15}$$
is exponentially stable, where N *and* V *are as in* (1.12).

Moreover, if $(x(\cdot), u(\cdot), y(\cdot)) \in \mathcal{ZD}(A, B, C)$*, then* $\lim_{t\to\infty} u(t) = 0$.

A system (1.1) is called *minimum phase* if, and only if, the zero dynamics is exponentially stable, i.e.

$$\exists\, \alpha,\ \beta > 0\ \forall\, (x(\cdot), u(\cdot), y(\cdot)) \in \mathcal{ZD}(A, B, C)\ \forall\, t \geq 0 : \quad \|x(t)\| \leq \beta e^{-\alpha t}\|x(0)\|.$$

It has to be noted that in some literature minimum phase is often defined by (1.14), see [36, Sec. 2.1] or [74]. In view of Proposition 1.1.9 (ii), it is easy to see that a system is minimum phase if the associated transfer function has zeroes only in the open left half complex plane.

1.2 Previously published results and joint work

The first part of this dissertation, Chapter 2, is not published. The results of Section 2.3 are a joined work with Achim Ilchmann (Ilmenau University of Technology).
The input saturation results in Section 3.4 stems from a joined work with Achim Ilchmann and Eugene P. Ryan (University of Bath) which are published in [32]. The funnel control results for relative degree two systems in Section 4.2 are from a joined

work with Christoph Hackl (Munich University of Technology), Achim Ilchmann, Markus Mueller (Ilmenau University of Technology) and Stephan Trenn (University of Illinois) which is submitted for publication [22].
The last chapter, Chapter 5, stems from a joined work with Achim Ilchmann and Pham Huu Anh Ngoc (HUE University, Vietnam) which is unpublished.

2 Adaptive λ-tracking with derivative output feedback

This chapter presents the main results of adaptive λ-tracking controllers with derivative feedback. In this Chapter, an adaptive tracking and disturbance algorithm for single input, single output and multi input, multi output linear time invariant systems (1.1) with unknown bounded relative degree and stable zero dynamics is presented. The controller requires knowledge of the sign of the high-frequency gain. The adaptive controller guarantees that the tracking error converges to a λ-strip around a reference signal. Unlike model reference adaptive control methods, this control does not require knowledge of the system order and requires only an upper bound on the relative degree. Furthermore, this adaptive controller presented herein has only one adaptive parameter regardless of the system order. The adaptive controller presented in this chapter is applicable to handle bounded input and output disturbances.

Section 2.1 introduces the concept of stabilization and λ-tracking. The principal structural properties which are used in this chapter are discussed in Section 2.2. The main results for stabilization and λ-tracking are presented in Section 2.3. Finally, in Section 2.4 the results of the main section are presented on a practical application - a serially connected spring-mass damper system. For purposes of illustration, all proofs are deferred to Section 2.5.

2.1 Adaptive λ-tracking

An overview of adaptive stabilization and adaptive λ-tracking controllers is given in this section. Their different components are described. Adaptive stabilization and λ-tracking controllers are simple controllers that are robust against a large set of uncertainties, making them attractive for many practical applications. These controllers only require the knowledge of structural information of the system, not of specific system parameters.

A classical control objective is stabilization of a system (1.1). A piecewise continuous and locally Lipschitz controller

$$u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m \tag{2.1}$$

stabilizes a system (1.1), if for arbitrary initial values $x^0 \in \mathbb{R}^n$, the closed-loop system (1.1), (2.1) has a solution with the properties

(i) there exists a unique solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$

(ii) $x(\cdot)$, $u(\cdot)$ are bounded

(iii) $\lim_{t\to\infty} y(t) = 0$.

The concept of *tracking* is similar. Suppose a reference signal $y_{\text{ref}}(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ is given. It is desired that the error between the output $y(\cdot)$ of (1.1) and the reference signal $y_{\text{ref}}(\cdot)$

$$e(\cdot) := y(\cdot) - y_{\text{ref}}(\cdot)$$

is forced, via the feedback $u(\cdot)$, to zero, i.e. for every initial value $x^0 \in \mathbb{R}^n$, the closed-loop system (1.1), (2.1) has a solution with the properties

(i) there exists a unique solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$

(ii) $x(\cdot)$, $u(\cdot)$ blow up no faster than $y_{\text{ref}}(\cdot)$

(iii)

$$\lim_{t\to\infty} \left[y(t) - y_{\text{ref}}(t)\right]. \tag{2.2}$$

For example, in the context of measurement noise, the objective (2.2) is too restrictive in practical applications. A certain output error is often a better choice. *Asymptotic λ-tracking*, for a user-defined parameter $\lambda > 0$, is a suitable control objective for such applications. The output is not forced exactly to the reference signal $y_{\text{ref}}(\cdot)$ as in (2.2), but towards a ball around the reference signal of arbitrary small prespecified radius $\lambda > 0$, i.e.

$$\limsup_{t\to\infty} \operatorname{dist}\left(y(t) - y_{\text{ref}}(t), [-\lambda, \lambda]\right) = 0, \tag{2.3}$$

see also Figure 2.1. (2.3) is a weaker condition as (2.2). The concept of approximate tracking was introduced by [57]. The term λ-tracking was coined by [39].

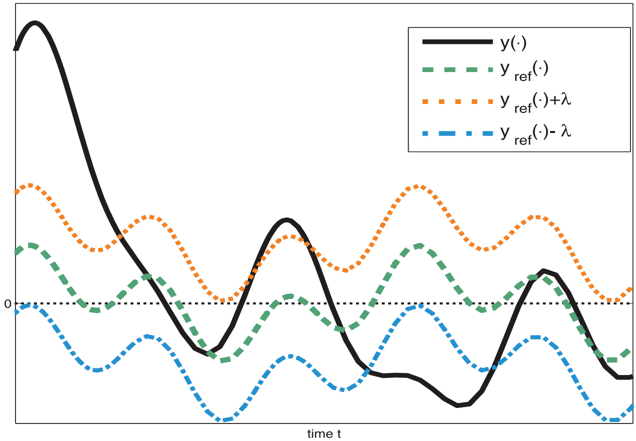


Figure 2.1: λ-tracking

In the following the principle of stabilization and adaptive λ-tracking are presented. For simplicity a linear first order scalar system of the form

$$\dot{y}(t) = a\, y(t) + b\, u(t), \qquad y(0) = y^0 \in \mathbb{R}, \quad (a, b) \in \mathbb{R} \times \mathbb{R}_{>0} \tag{2.4}$$

is considered. It has to be noted that (2.4) is the scalar prototype of a system (1.1).

(i) Note that the system (2.4) can be stabilized by the proportional high-gain feedback

$$u(t) = -ky(t),$$

if $k > \frac{|a|}{b}$. A sufficient large gain k can be found adaptively with the law $\dot{k}(t) = y(t)^2$, $k(0) = k_0 > 0$. In the general setup a systems (1.1) which has

- strict relative degree one, i.e. $\det CB \neq 0$,
- positive high-frequency gain, i.e. CB positive definite, and
- stable zero dynamics

can be stabilized by the adaptive controller

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = \|y(t)\|^2, \; k(0) = k^0 \in \mathbb{R} \tag{2.5}$$

(see [35]). The controller (2.5) was introduced by [57, 63, 89]. Note the simplicity of the controller (2.5). The controller (2.5) exploits the high-gain property of a system (1.1). The proportional feedback (2.5) is designed in such a way that $k(\cdot)$ is monotonically increasing, thereby $k(\cdot)$ becomes large if $\|y(\cdot)\|$ do not approach zero. If the control objective is tracking, the controller need an internal model which is a drawback.

(ii) Now consider the problem of λ-tracking. Let the reference signal be constant, i.e. $y_{\text{ref}}(\cdot) \equiv y_{\text{ref}} \in \mathbb{R}$. Consider the proportional output feedback controller

$$u(t) = -k\left(y(t) - y_{\text{ref}}\right), \qquad k \in \mathbb{R}. \tag{2.6}$$

The controller has only one parameter k and the closed-loop system (2.4), (2.6) is given by

$$\dot{y}(t) = (a - k\, b)\, y(t) + k\, b\, y_{\text{ref}}, \qquad y(0) = y^0. \tag{2.7}$$

If $k > \left|\frac{a}{b}\right|$, then the system matrix $a - kb$ of (2.7) is Hurwitz. If $\left|\frac{kb}{kb-a}\right| |y_{\text{ref}}| = \left|1 + \frac{a}{kb-a}\right| \; |y_{\text{ref}}| < \lambda$, then the system (2.7) has a global, unique solution $y(\cdot)$ which tends asymptotically to the λ-strip. The control objective of λ-tracking is achieved by choosing any fixed, sufficiently large gain k. Note that if a system (2.4) can be tracked by the controller (2.6) with $k = k^* > 0$, then the system (2.4) can be tracked with (2.6) for every $k > k^*$.
Such a controller is called *high-gain* controller (see [36, Chap. 2.2]). The main component of an adaptive λ-tracking controller also is a high-gain controller.
The disadvantage of the controller (2.6) is that the parameter k depends on the system data a, b and the reference signal y_{ref}.

Instead of fixed k, it is possible to adapt this parameter. For example, the adaption

$$\dot{k}(t) = d_\lambda(|y(t) - y_{\text{ref}}|)^2, \qquad k(0) = k_0 > 0 \tag{2.8}$$

can be used (see [57]). This adaption can be described in the following way: whenever the output is outside of the λ-strip around the reference signal, i.e. $|y(t) - y_{\text{ref}}| > \lambda$, the controller gain increases (see Figure 2.2).

Then the controller (2.6), together with the adaption (2.8), gets the form

$$u(t) = -k(t)\left(y(t) - y_{\text{ref}}\right), \qquad \dot{k}(t) = d_\lambda(|y(t) - y_{\text{ref}}|)^2,\; k(0) = k_0 > 0, \tag{2.9}$$

and achieves λ-tracking and boundedness of the state $y(\cdot)$ and the adaption parameter $k(\cdot)$. These objectives are also achieved for time-varying reference sig-

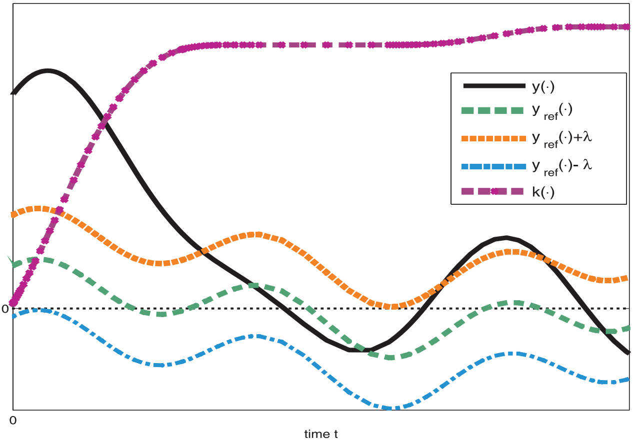


Figure 2.2: Gain adaptation

nals $y_{\text{ref}}(\cdot)$. Moreover, it is well known (see [40]) that the adaptive controller

$$\begin{aligned} u(t) &= -k(t)e(t), & e(t) &= y(t) - y_{\text{ref}} \\ \dot{k}(t) &= \|e(t)\|\, d_\lambda(\|e(t)\|), & k(0) &= k_0 \in \mathbb{R} \end{aligned}$$

applied to a system (1.1) which has

- strict relative degree one,
- positive high-frequency gain and
- stable zero dynamics

achieves:

- all states of the closed-loop system are bounded and
- $\|e(\cdot)\|$ is ultimately bounded by λ, where $\lambda > 0$ is prespecified and may be arbitrarily small.

The *control objective* is λ-tracking with an adaptive controller of the form (2.9), i.e. the error $\|y(\cdot) - y_{\text{ref}}(\cdot)\|$ asymptotically approaches a λ-strip, $\lambda > 0$, at zero such that all states of the closed-loop system, the input $u(\cdot)$ and the gain $k(\cdot)$ are bounded. The last point means that $\lim_{t\to\infty} k(t) =: k_\infty \in \mathbb{R}$. The controller has a simple form and only

one tuning parameter λ which is given by the designer. Furthermore the controller is robust with respect to model uncertainty. A sub-goal is stabilization.
There are several reasons to use λ-tracking instead of stabilization. As many specifications include a tolerance, it is a natural control objective. For example, if the output (or/and the input) is corrupted by noise, stabilization requires control energy which can be saved if the λ-strip is large enough to tolerate these measurement errors.

2.2 System class and control objective

In this chapter a time-invariant linear system (1.1) with unknown system data $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ and unknown relative degree $\varrho \in \mathbb{N}$ (i.e. (1.10)) are considered which has stable zero dynamics (i.e. (1.14)) and positive high-frequency gain (i.e. (1.11)). Suppose that an upper bound $\nu \in \mathbb{N}$ of the relative degree ϱ is known. The definitions of relative degree, stable zero dynamics and positive high-frequency gain can be found in Chapter 1.1.

More formally, for known $\nu \in \mathbb{N}$, this chapter considers systems of the form (1.1) which satisfy the following assumptions.

(A1) *(Known upper bound of the relative degree, positive high-frequency gain)*
$\exists\ \varrho \in \{1, ..., \nu - 1\}$: (1.10) and (1.11) hold.

(A1') *(Known upper bound of the relative degree, positive high-frequency gain)*
$\exists\ \varrho \in \{1, ..., \nu\}$: (1.10) and (1.11) hold.

(A2) *(Stable zero dynamics)*
$$\forall\ s \in \mathbb{C}_0 : \quad \det \begin{pmatrix} sI_n - A & B \\ C & 0 \end{pmatrix} \neq 0$$

It has to be noted that Assumption (A1') allows systems (1.1) with relative degree $\varrho = \nu$, instead of (A1) which only allows $\varrho < \nu$. The Assumption (A1) is important for multi input, multi output systems (see Remark 2.3.6, Theorem 2.3.7, Theorem 2.3.10). Suppose a system (1.1) which is single input, single output (i.e. $m = 1$). In this setup the Assumption (A1) can be relaxed to Assumption (A1').

An upper bound $\nu \in \mathbb{N}$ of the relative degree of a system (1.1) is used. If the dimension $n \in \mathbb{N}$ is known, then the relative degree satisfies $\varrho \leq n$ (invoking Cayley-Hamilton Theorem). In this case the choice $\nu = n + 1$ or $\nu = n$ is sufficient to satisfy (A1) or (A1'), respectively.

For a linear system (1.1) the classical assumptions (see [11, 40, 44]) of adaptive control are

(i) stable zero dynamics,

(ii) strict relative degree and

(iii) positive high-frequency gain.

Thus the assumptions (A1), (A1') and (A2) are not restrictive.

The control objectives of this chapter are

(i) adaptive stabilization: the output satisfies $\lim_{t\to\infty} y(t) = 0$ and all states and inputs are square integrable and the gain function is bounded, i.e. $(x(\cdot), u(\cdot)) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $k(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$.

(ii) adaptive λ-tracking of a reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{\nu,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ while tolerating a tracking error smaller than a user-defined λ, i.e. $\lim_{t\to\infty} \text{dist}(\|y(t) - y_{\text{ref}}(t)\|,\ [0,\lambda]) = 0$. All states, inputs and the gain function remain bounded, i.e. $(x(\cdot), u(\cdot), k(\cdot)) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \times \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \times \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$.

2.3 High-gain output derivative feedback with unknown relative degree

Derivative feedback controllers for linear systems (1.1) with higher relative degree are known in control theory (see [62]). For the design of the feedback law the system data are used.
In this section control strategies for stabilization and λ-tracking of linear systems (1.1) with unknown relative degree are presented which use the systems output and its derivative.

It has to be noted that the feedback laws require only structural information of the system – a known upper bound of the strict relative degree, stable zero dynamics and positive high-frequency gain. The explicit knowledge of the system data is not required.
This section is divided into three parts

(i) motivation for the derivative feedback controller,

(ii) stabilization of systems (1.1) and

(iii) λ-tracking of such systems (1.1).

The problem of λ-tracking includes input and output disturbances. For single input, single output systems and multi input, multi output systems the feedback controllers are identical.

2.3.1 Motivation

The motivation is restricted to the goal of stabilization; otherwise the argumentation for λ-tracking is similar. The control objective of stabilization is achieved by the controller

$$\boxed{\begin{aligned} u(t) &= -\sum_{i=0}^{\nu} k(t)^{\nu+1-i} p_i y^{(i)}(t) \\ \dot{k}(t) &= k(t)^{-2\gamma} \|y(t), k(t)^{-1}\dot{y}(t), ..., k(t)^{-(\nu-1)} y^{(\nu-1)}(t)\|^2, \qquad k(0) = k_0 \end{aligned}} \tag{2.10}$$

(see Section 2.3.2 and (2.28)), where the simple proportional feedback has a gain function $k(\cdot)$ which depends on the output $y(\cdot)$ and its derivatives and coefficients p_i of a Hurwitz polynomial.

The motivation for this controller comes from the static high-gain state feedback controller. In this subsection the high-gain derivative feedback controller of the form

$$u(t) = -\sum_{i=0}^{\nu} k^{\nu+1-i} p_i y^{(i)}(t), \qquad k \in \mathbb{R} \tag{2.11}$$

is considered, where $\nu \in \mathbb{N}$ known and $p_0, ..., p_\nu \in \mathbb{R}$ are suitable design parameters which are independent of the systems data. The controller (2.11) will be applied to linear systems (1.1) which are single input, single output systems (i.e. $m = 1$) with unknown relative degree $\varrho = 1, 2, 3$, known upper bound $\nu \in \mathbb{N}$, positive high-frequency gain and without zero dynamics. Assume without restriction of generality, that the considered system is in Byrnes-Isidori form (see Section 1.1); otherwise there exists a transformation and the transformed problem is considered.

Let $\nu = 3$.

(i) Consider the system

$$\dddot{y}(t) = R_1 y(t) + R_2 \dot{y}(t) + R_3 \ddot{y}(t) + \Gamma u(t), \tag{2.12}$$

where R_1, R_2, $R_3 \in \mathbb{R}$ and $\Gamma > 0$ which has strict relative degree $\varrho = 3$ and can be written as, for $\xi(\cdot) = [y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot)]^\top$,

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ R_1 & R_2 & R_3 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 0 \\ \Gamma \end{bmatrix} u(t), \qquad \xi(0) = \xi^0 \in \mathbb{R}^3. \tag{2.13}$$

Furthermore there exist positive constants q_0,...,q_3 such that the polynomial

$$(1+\Gamma q_3)s^3 + (\Gamma q_2 - R_3)s^2 + (\Gamma q_1 - R_2)s + (\Gamma q_0 - R_1) \in \mathbb{R}[s] \quad \text{is Hurwitz.} \tag{2.14}$$

If

$$q_3 > 0, \quad q_2 > \frac{|R_3|}{\Gamma}, \quad q_0 > \frac{|R_1|}{\Gamma}, \quad q_1 > \frac{1+\Gamma q_3}{\Gamma} \frac{\Gamma q_0 + |R_1|}{\Gamma q_2 - |R_3|} + \frac{|R_2|}{\Gamma},$$

then (2.14) is satisfied (see [28, Th. 3.4.71, Ex. 3.4.72]). It has to be noted that the high-frequency gain $\Gamma > 0$ must be known explicitly. Then (2.13), together with

$$u(t) = -\left[q_0 y(t) + q_1 \dot{y}(t) + q_2 \ddot{y}(t) + q_3 \dddot{y}(t)\right], \tag{2.15}$$

gives the closed-loop system (2.13), (2.15)

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-\Gamma q_0 + R_1}{1+\Gamma q_3} & \frac{-\Gamma q_1 + R_2}{1+\Gamma q_3} & \frac{-\Gamma q_2 + R_3}{1+\Gamma q_3} \end{bmatrix} \xi(t), \qquad \xi(0) = \xi^0.$$

The system matrix of the closed-loop system is Hurwitz if, and only if, the polynomial (2.14) is Hurwitz. Thus, the controller (2.15) stabilizes (2.13).

If the parameters q_i, $i = 0, ..., 3$ are chosen as $q_i = p_i k^{\varrho+1-i}$ with p_i, $k > 0$, then the controller (2.15) gets the form

$$u(t) = -\left[p_0 k^4 y(t) + p_1 k^3 \dot{y}(t) + p_2 k^2 \ddot{y}(t) + p_3 k \dddot{y}(t)\right] \tag{2.16}$$

which is parameterized by a parameter k. This parameter is the high-gain parameter for this state feedback controller. Then the closed-loop system (2.13), (2.16) gets the form

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-\Gamma p_0 k^4 + R_1}{1+\Gamma p_3 k} & \frac{-\Gamma p_1 k^3 + R_2}{1+\Gamma p_3 k} & \frac{-\Gamma p_2 k^2 + R_3}{1+\Gamma p_3 k} \end{bmatrix} \xi(t), \qquad \xi(0) = \xi^0,$$

where the characteristic polynomial of the system matrix is given by

$$s^3 + \frac{\Gamma p_0 k^4 - R_1}{1+\Gamma p_3 k} s^2 + \frac{\Gamma p_1 k^3 - R_2}{1+\Gamma p_3 k} s + \frac{\Gamma p_2 k^2 - R_3}{1+\Gamma p_3 k} \in \mathbb{R}[s]. \tag{2.17}$$

An easy calculation shows that, for every $k > 0$, the polynomial (2.17) is Hurwitz if

$$\Gamma^2 \left[p_1 p_2 - p_0 p_3\right] k^5 - \Gamma \left[p_0 k^4 + p_1 R_3 k^3 + p_2 R_2 k^2 - p_3 R_1 k\right] + R_1 + R_2 R_3 > 0$$
$$\wedge \ k^2 > \frac{|R_3|}{p_2 \Gamma} \ \wedge \ k^4 > \frac{|R_1|}{p_0 \Gamma}. \tag{2.18}$$

It has to be noted that the following implication holds:

$$\left[\ p_3 s^3 + p_2 s^2 + p_1 s + p_0 \in \mathbb{R}[s] \text{ is Hurwitz }\right]$$
$$\Rightarrow \left[\exists\, k^* > 0\ \forall\, k \geq k^* :\ \ R_1 + R_2 R_3 - \Gamma\left[p_0 k^4 + p_1 R_3 k^3 + p_2 R_2 k^2 - p_3 R_1 k\right] + \Gamma^2 \left[p_1 p_2 - p_0 p_3\right] k^5 > 0\right].$$

Hence there exists $k^* > 0$ such that for all $k \geq k^*$ and $\xi^0 \in \mathbb{R}^3$ holds:

(2.16) stabilizes (2.12)

if $p_3 s^3 + p_2 s^2 + p_1 s + p_0 \in \mathbb{R}[s]$ is Hurwitz.

It has to be noted that [49, Th. 9.3.1] shows that there exists parameter k, $p_1, ..., p_\nu$ such that the feedback (2.11) stabilizes single input, single output systems (2.12) with known relative degree $\varrho = \nu$ and known lower bound for the high-frequency gain (see [50, Th. 121.1.1] for multi input, multi output systems with strict relative degree $\varrho = \nu$).

In the present chapter it is assumed that the relative degree is unknown but only an upper bound ν is known. It has to be noted that $\varrho = \nu$ in (i). The controllers (2.15) or (2.16), respectively, require the exact knowledge of the relative

degree ϱ.
One may ask the question as to whether the controllers (2.15) or (2.16), respectively, which are designed for systems with known relative degree ϱ, also works for systems with unknown relative degree. The answer is affirmative.

(ii) Consider a system

$$\dot{y}(t) = R_1 y(t) + \Gamma u(t), \qquad y(0) = y^0 \in \mathbb{R} \tag{2.19}$$

which has the relative degree one. The closed-loop system (2.19), (2.15) gets the form

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-\Gamma q_0 + R_1}{\Gamma q_3} & \frac{-\Gamma q_1 - 1}{\Gamma q_3} & \frac{-\Gamma q_2}{\Gamma q_3} \end{bmatrix} \xi(t), \qquad \xi(0) = \begin{bmatrix} y^0, & \xi^1, & \xi^2 \end{bmatrix}^\top, \tag{2.20}$$

where $(\xi^1, \xi^2)^\top \in \mathbb{R}^2$ are arbitrarily. The controller (2.15) stabilizes (2.19) if, and only if, the polynomial

$$\Gamma q_3 s^3 + \Gamma q_2 s^2 + (\Gamma q_1 + 1)s + (\Gamma q_0 - R_1) \in \mathbb{R}[s] \quad \text{is Hurwitz}$$

which is satisfied for

$$q_3,\ q_2 > 0 \ \wedge\ q_0 > \frac{|R_1|}{\Gamma} \ \wedge\ q_1 > \max\left\{0, \frac{\Gamma q_0 + |R_1|}{q_2} q_3 - 1\right\}.$$

Stabilization of (2.19) with the controller (2.15) follows similarly to the calculation in (2.18): if $p_3 s^3 + p_2 s^2 + p_1 s + p_0 \in \mathbb{R}[s]$ is Hurwitz, then there exists $k^* > 0$ such that the controller (2.16) stabilizes (2.19) for all $k \geq k^*$ and $(y^0, \xi^1, \xi^2)^\top \in \mathbb{R}^3$.

(iii) Consider a relative degree two system

$$\ddot{y}(t) = R_1 y(t) + R_2 \dot{y}(t) + \Gamma u(t), \qquad \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} y^0 \\ y^1 \end{bmatrix} \in \mathbb{R}^2 \tag{2.21}$$

and the closed-loop system (2.21), (2.15)

$$\dot{\xi}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-\Gamma q_0 + R_1}{\Gamma q_3} & \frac{-\Gamma q_1 + R_2}{\Gamma q_3} & \frac{-\Gamma q_2 - 1}{\Gamma q_3} \end{bmatrix} \xi(t), \qquad \xi(0) = \begin{bmatrix} y^0 \\ y^1 \\ \xi^2 \end{bmatrix}, \tag{2.22}$$

where $\xi^2 \in \mathbb{R}$ is arbitrarily. The controller (2.15) stabilizes (2.21) if, and only if, the polynomial

$$\Gamma q_3 s^3 + (\Gamma q_2 + 1)s^2 + (\Gamma q_1 - R_2)s + (\Gamma q_0 - R_1) \in \mathbb{R}[s] \quad \text{is Hurwitz}$$

which is satisfied for

$$q_3 > 0 \ \wedge \ q_2 > \Gamma^{-1} \ \wedge \ q_0 > \frac{|R_1|}{\Gamma} \ \wedge \ q_1 > \frac{\Gamma q_0 + |R_1|}{\Gamma q_2 + 1} q_3 + \frac{|R_2|}{\Gamma}.$$

It follows similarly to (2.18) and (ii) that (2.16) stabilizes (2.21) if $p_3 s^3 + p_2 s^2 + p_1 s + p_0 \in \mathbb{R}[s]$ is Hurwitz.

The conclusion of the motivation is that if $p_3 s^3 + p_2 s^2 + p_1 s + p_0 \in \mathbb{R}[s]$ is Hurwitz, then there exists $k^* > 0$ such that (2.16) stabilizes (2.12), (2.21) and (2.19) for all $k \geq k^*$ and initial values. It has to be noted that the Hurwitz polynomial $p_3 s^3 + p_2 s^2 + p_1 s + p_0 \in \mathbb{R}[s]$ is given by the designer and hence known. It is obvious that $k^* > 0$ depends on the system data $R_1, ..., R_3$ and Γ. If this information is given, the controller (2.16) can be used for stabilization for large enough gain values k. If this information is not available, a good alternative is to adapt the parameter k. Therefore, the feedback (2.11) contains an adaptive law. Moreover, Subsection (2.3.3) shows that the controller (2.11) solves the λ-tracking problem.

Remark 2.3.1. (Controller (2.10) **and higher derivatives of the output** $y(\cdot)$**)**
As seen in the motivation, the main idea of the controller (2.10) is derivative feedback, i.e. the output $y(\cdot)$ and its higher derivatives are used. For a simple rotatory model for the standard position control problem, the output $y(\cdot)$ and its derivatives are available (see [22]). In general and in applications, it cannot be assumed that the higher derivatives are available. Therefore, the controllers and main results of this chapter are a provisional result or a prestage on the way to controllers which use only the output $y(\cdot)$ (e.g. high-gain observers). ◇

2.3.2 Feedback stabilization for SISO- and MIMO-systems

The case of stabilization is considered. With the parameter $\nu \in \mathbb{N}$ and the polynomial

$$p(s) := p_\nu s^\nu + p_{\nu-1} s^{\nu-1} + ... + p_1 s + p_0 \in \mathbb{R}[s] \quad \text{Hurwitz}, \tag{2.23}$$

the matrices

$$A_k := \begin{pmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ -k^\nu \frac{p_0}{p_\nu} I_m & \cdots & -k^2 \frac{p_{\nu-2}}{p_\nu} I_m & -k \frac{p_{\nu-1}}{p_\nu} I_m \end{pmatrix} \in \mathbb{R}[k]^{\nu m \times \nu m} \tag{2.24}$$

are introduced. Since A_1 is Hurwitz, there exists a unique positive definite, symmetric matrix $T_1 = T_1^\top := \int\limits_0^\infty e^{A_1^\top s} e^{A_1 s} \, \mathrm{d}s \in \mathbb{R}^{\nu m \times \nu m}$ (see [87, Th. 5.4.42]) which satisfies the Lyapunov equation

$$A_1^\top T_1 + T_1 A_1 = -I_{\nu m}. \tag{2.25}$$

Choose $\vartheta \in \mathbb{R}$ such that

$$\vartheta \geq \|T_1\| \, \|T_1^{-1}\| \, (\nu - 1) \tag{2.26}$$

which is well defined. It holds that ϑ is zero if, and only if, $\nu = 1$.
It has to be noted that the parameter $\nu \in \mathbb{N}$ is given and $p_0, ..., p_\nu \in \mathbb{R}$ are user-defined parameters and so the associate matrix A_1 is known which defines the matrix T_1.

The parameter ϑ (see (2.26)) depends on T_1^{-1}. The calculation of the inverse matrix T_1^{-1} can be difficult in many cases, e.g. by numerical calculations. The following proposition gives an upper bound of the norm for a regular invertible matrix.

Proposition 2.3.2. ***(Norm of an inverse matrix)***
Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ with $\det A \neq 0$. Then

$$\|A^{-1}\| \leq \frac{1}{n^{\frac{n-2}{2}} |\det A|} \|A\|^{n-1}. \tag{2.27}$$

Moreover, if $n \in \{1, 2\}$, then it is equality in (2.27).

The proof of Proposition 2.3.2 is in Section 2.5 on page 68.

Remark 2.3.3. (Comment on (2.26))
Proposition (2.3.2) gives an upper bound for the norm of the inverse matrix of T_1. If

$$\vartheta \geq \frac{1}{(\nu m)^{\frac{\nu m-2}{2}} |\det T_1|} \|T_1\|^{\nu m} (\nu - 1),$$

then (2.26) holds. It has to be noted that this inequality is a conservative bound. $\diamond$

Theorem 2.3.4. ***(Stabilization, SISO systems)***
For given $\nu \in \mathbb{N}$ suppose a system (1.1) *which is single input, single output (i.e. $m = 1$) satisfying (A1') and (A2). Let $p_0, ..., p_\nu \in \mathbb{R}$ be such that* (2.23) *holds and $\gamma \geq \vartheta$ with ϑ as defined in* (2.26)*. Then application of the adaptive controller*

$$\boxed{\begin{array}{rcl} u(t) & = & -\sum_{i=0}^{\nu} k(t)^{\nu+1-i} p_i y^{(i)}(t) \\ \dot{k}(t) & = & k(t)^{-2\gamma} \|y(t), k(t)^{-1}\dot{y}(t), ..., k(t)^{-(\nu-1)} y^{(\nu-1)}(t)\|^2, \qquad k(0) = k_0 \end{array}} \tag{2.28}$$

to (1.1) *yields, for any initial data $x^0 \in \mathbb{R}^n$ and $k_0 > 0$, a closed-loop initial value problem with the following properties.*

(i) *Precisely one maximal solution $(x, k) : [0, \omega) \to \mathbb{R}^n \times [k_0, \infty)$ exists and this solution is global , i.e. $\omega = \infty$.*

(ii) *The global solution $x(\cdot)$ and the input $u(\cdot)$ are square integrable, i.e.*

$$x(\cdot; 0, x^0) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \qquad \textit{and} \qquad u(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}).$$

(iii) $k(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, [k_0, \infty))$

(iv) $\lim_{t\to\infty} x(t; 0, x^0) = 0$

(v) $\forall\, i \in \{0, ..., \nu\} : \quad y^{(i)}(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ *and* $\lim_{t\to\infty} y^{(i)}(t) = 0.$

The proof of Theorem 2.3.4 is in Section 2.5 on page 69.

Remark 2.3.5. (Comments on Theorem 2.3.4)

(i) It has to be noted that the parameter $\gamma \geq \vartheta$. Hence γ depends on A_1, T_1 and ν.

(ii) In view of the controller (2.28), some care must be exercised in formulating the closed-loop initial value problem (1.1), (2.28). At first view, it seems like an implicit differential equation. The the closed-loop initial value problem can be formulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x(t)\\k(t)\end{bmatrix} = F(x(t),k(t)), \qquad \begin{bmatrix}x(0)\\k(0)\end{bmatrix} = \begin{bmatrix}x^0\\k_0\end{bmatrix},$$

for suitable $F : \mathcal{D} \to \mathbb{R}^n \times \mathbb{R}$ with appropriately defined domain $\mathcal{D}$ which is not an implicit differential equation. This is done in Step 2 of the proof of Theorem 2.3.4. $\diamond$

Theorem 2.3.4 considers systems (1.1) which are single input, single output (i.e. $m = 1$) with unknown relative degree $\varrho \in \{1, ..., \nu\}$. The main coordinate transformation of the closed-loop system (1.1), (2.28) is sketched in the following remark. One may ask the question as to whether the controller (2.28) which is designed for single input, single output systems works for multi input, multi output systems with unknown relative degree ϱ. The answer is that the controller (2.28) only works for multi input, multi output systems (1.1) with unknown relative degree $\varrho \in \{1, ..., \nu-1\}$. The problem of $\varrho = \nu$ is stated in the following remark.

Remark 2.3.6. (Theorem 2.3.4, generalization to MIMO)
Recall that, in view of Proposition 1.1.7, every systems (1.1) with relative degree $\varrho \in \mathbb{N}$ can be transformed into Byrnes-Isidori form (1.13), where the matrices $R_1, ..., R_\varrho \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times(n-m\varrho)}$, $P_1 \in \mathbb{R}^{(n-m\varrho)\times m}$ and $Q \in \mathbb{R}^{(n-m\varrho)\times(n-m\varrho)}$ can be presented explicitly in terms of the system matrices (A, B, C) (see Proposition 1.1.7).

(i) Theorem 2.3.4 considers systems (1.1) which are single input, single output (i.e. $m = 1$). Assume without of restriction of generality that the closed-loop system (1.1), (2.28) is in Byrnes-Isidori form (1.13). Let $\varrho \in \{1, ..., \nu\}$ be the relative degree of a system (1.1). Furthermore, the initial values are omitted. Then the closed-loop system gets the form

$$\begin{aligned} y^{(\varrho)}(t) &= R_1 y(t) + ... + R_\varrho y^{(\varrho-1)}(t) + Sz(t) - \Gamma\sum_{i=0}^{\nu-1} k(t)^{\nu+1-i} p_i y^{(i)}(t) \\ &\quad -\Gamma k(t) p_\nu y^{(\nu)}(t) \\ \dot{z}(t) &= P_1 e_1^\top y(t) + Qz(t) \end{aligned} \tag{2.29}$$

with suitable matrices (see Step 2 of the proof of Theorem 2.3.4 or Proposition 1.1.7). It has to be noted that $\Gamma > 0$ is a real value in the single input, single output case. The system (2.29) is equivalent to

$$k(t)p_\nu \Gamma y^{(\nu)}(t) = \sum_{i=1}^{\varrho} R_i y^{(i-1)}(t) + Sz(t) - \Gamma \sum_{i=0}^{\nu-1} k(t)^{\nu+1-i} p_i y^{(i)}(t) - y^{(\varrho)}(t) \quad , \text{ if } \varrho \in \{1, ..., \nu-1\} \tag{2.30}$$

$$(1 + k(t)p_\nu \Gamma) y^{(\nu)}(t) = \sum_{i=1}^{\varrho} R_i y^{(i-1)}(t) + Sz(t) - \Gamma \sum_{i=0}^{\nu-1} k(t)^{\nu+1-i} p_i y^{(i)}(t) \quad , \text{ if } \varrho = \nu \tag{2.31}$$

which can divided by $k(\cdot)p_\nu\Gamma$ or $(1 + k(\cdot)p_\nu\Gamma)$, respectively. This is done in Step 2 and (2.41) of the proof of Theorem 2.3.4. It has to be noted that the definition of $k(\cdot)$, p_ν and Γ gives that $k(\cdot)p_\nu\Gamma > 0$. The coordinate transformation $w(\cdot) := \text{diag}\,(1, k, ..., k^{\nu-1})^{-1} \left[y(\cdot), \dot{y}(\cdot), ..., y^{(\nu-1)}(\cdot) \right]$ applied to (2.30) and (2.31) gives

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} w \\ z \end{pmatrix} &= \begin{pmatrix} k\mathcal{A}_{1,k} + \widehat{\mathcal{R}}(k) - \frac{\dot{k}}{k}\Delta, & k^{-(\nu-1)}\mathcal{S}(k) \\ P_1 e_1^\top, & Q \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \\
\dot{k} &= k^{-2\gamma} \|w\|^2 \\
\begin{pmatrix} w(0) \\ z(0) \end{pmatrix} &= \begin{pmatrix} K(k_0)^{-1}\mathcal{Y}(0) \\ Nx^0 \end{pmatrix}, \quad k(0) = k_0
\end{aligned}$$

for suitable matrices (see Step 3 and (2.42), (2.43) of the proof of Theorem 2.3.4). $k\mathcal{A}_{1,k}$ is the "good matrix" which is used in the Ljapunov candidate $V(\cdot)$ (see Step 3 of the proof of Theorem 2.3.4). It has to be noted that $\mathcal{A}_{1,k}$ is Hurwitz.

(ii) Now the generalized multi input, multi output systems (1.1) are considered. In the multi input, multi output setup the equations (2.30) and (2.31) get the form

$$y^{(\nu)}(t) = -\frac{1}{p_\nu k(t)} \sum_{i=0}^{\nu-1} k(t)^{\nu+1-i} p_i y^{(i)}(t) + \frac{1}{p_\nu k(t)} \Gamma^{-1} \left[-y^{(\varrho)}(t) + \sum_{i=1}^{\varrho} R_i y^{(i-1)}(t) + Sz(t) \right] \tag{2.32}$$

if $\varrho \in \{1, ..., \nu - 1\}$ and

$$y^{(\nu)}(t) = -\left(I_m + p_\nu k(t)\Gamma\right)^{-1} \Gamma \sum_{i=0}^{\nu-1} k(t)^{\nu+1-i} p_i y^{(i)}(t) + \left(I_m + p_\nu k(t)\Gamma\right)^{-1} \left[\sum_{i=1}^{\varrho} R_i y^{(i-1)}(t) + Sz(t)\right] \quad (2.33)$$

if $\varrho = \nu$. It has to be noted that Γ is a positive definite matrix. Since I_m and Γ are positive definite matrices, $p_\nu > 0$ and $k(\cdot)$ a positive monotonic increasing function, Γ^{-1} and $\left(I_m + p_\nu k(t)\Gamma\right)^{-1}$ exist and are positive definite.

(1) If $\varrho \in \{1, ..., \nu-1\}$, then the proof for multi input, multi output systems (1.1) follows similarly to the proof of Theorem 2.3.4. The statement is presented in Theorem 2.3.7 without a detailed proof.

(2) Let $\varrho = \nu$. The coordinate transformation of (i) applied to (2.33) gives

$$\dot{w}(t) = \begin{bmatrix} 1 & & & \\ & k & & \\ & & \ddots & \\ & & & k^{\nu-1} \end{bmatrix}^{-1} \left(I_m + p_\nu k(t)\Gamma\right)^{-1} A_k \begin{bmatrix} 1 & & & \\ & k & & \\ & & \ddots & \\ & & & k^{\nu-1} \end{bmatrix} w(t) + Sz(t)$$
$$+ \begin{bmatrix} 1 & & & \\ & k & & \\ & & \ddots & \\ & & & k^{\nu-1} \end{bmatrix}^{-1} \left(I_m + p_\nu k(t)\Gamma\right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} [R_1, ..., R_\nu] \begin{bmatrix} 1 & & & \\ & k & & \\ & & \ddots & \\ & & & k^{\nu-1} \end{bmatrix} w(t).$$

The term $\operatorname{diag}(1, k, ..., k^{\nu-1})^{-1} \left(I_m + p_\nu k(t)\Gamma\right)^{-1} A_k \operatorname{diag}(1, k, ..., k^{\nu-1})$ does not give an expression of the form $\left(I_m + p_\nu k(t)\Gamma\right)^{-1} A_1$ which was essential for the Ljapunov candidate $V(\cdot)$ (see Step 3 of the proof of Theorem 2.3.4) are not generalizable to multi input, multi output systems 1.1 of $\varrho = \nu$. ◇

With a slight modification, the result of Theorem 2.3.4 can be generalized to multi input, multi output systems with unknown relative degree $\varrho \in \{1, ..., \nu - 1\}$ which is presented in the next theorem. It has to be noted that (A1') is replaced by (A1), i.e.

$$1 \leq \varrho \leq \nu - 1.$$

The problem of $\varrho = \nu$ is stated in Remark (2.3.6).

Theorem 2.3.7. ***(Stabilization, MIMO systems)***
For given $\nu \in \mathbb{N}$ suppose a system (1.1) *satisfying (A1) and (A2). Let $p_0, ..., p_\nu \in \mathbb{R}$ be such that* (2.23) *holds and $\gamma \geq \vartheta$ with ϑ as defined in* (2.26). *Then application of the adaptive controller* (2.28) *to* (1.1) *yields, for any initial data $x^0 \in \mathbb{R}^n$ and $k_0 > 0$, a closed-loop initial value problem which satisfies the properties (i)-(v) of Theorem 2.3.4.*

The proof of Theorem 2.3.7 is in Section 2.5 on page 74. The structure of the proof closely resembles that of Theorem 2.3.4. For brevity the details are omitted.

The difference between Theorem 2.3.4 and Theorem 2.3.7 is that (A1') is replaced by (A1). Theorem 2.3.7 only allows systems (1.1) with relative degree ϱ smaller than ν, i.e. $\varrho \in \{1, ..., \nu-1\}$. A discussion of the problem with $\varrho = \nu$ is given in Remark 2.3.6.

2.3.3 Adaptive λ-tracking for SISO- and MIMO-systems

Next, the attention is turned to the case of λ-tracking. Now the main result of this chapter is presented. It has to be noted that only the structural requirements (A1), (A2) or (A1'), (A2) are necessary to show that the application of the high-gain derivative feedback (2.34) to a linear system (1.1) tracks a suitable reference signal. Since $\nu \in \mathbb{N}$ is an upper bound of the relative degree, $\mathcal{W}^{\nu,\infty}$-disturbances for the input and output are allowed. Figure 2.3 illustrates the adaptive controller presented in the following theorem.

Theorem 2.3.8. ***(Tracking, SISO systems)***
For given $\nu \in \mathbb{N}$ suppose a system (1.1) *which is single input, single output (i.e. $m = 1$) satisfying (A1') and (A2). Let $p_0, ..., p_\nu \in \mathbb{R}$ be such that* (2.23) *holds and $\gamma \geq 2\vartheta$ with ϑ as defined in* (2.26). *Assume that the reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{\nu,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and the input and output disturbances $d_u(\cdot)$, $d_y(\cdot) \in \mathcal{W}^{\nu,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$. Then, for every $\lambda > 0$, application of the adaptive controller*

$$\boxed{\begin{array}{rcl} u(t) &=& -\sum\limits_{i=0}^{\nu} k(t)^{\nu+1-i} p_i e^{(i)}(t) + d_u(t), \quad e(t) = (y(t) + d_y(t)) - y_{\text{ref}}(t) \\ \dot{k}(t) &=& k(t)^{-2\gamma} d_\lambda(\|e(t), k(t)^{-1}\dot{e}(t), ..., k(t)^{-(\nu-1)} e^{(\nu-1)}(t)\|)^2, \quad k(0) = k_0 \end{array}} \tag{2.34}$$

to (1.1) *yields, for any initial data $x^0 \in \mathbb{R}^n$ and $k_0 > 0$, a closed-loop initial value problem with the following properties.*

(i) *Precisely one maximal solution $(x, k) : [0, \omega) \to \mathbb{R}^n \times [k_0, \infty)$ exists and this solution is global, i.e. $\omega = \infty$.*

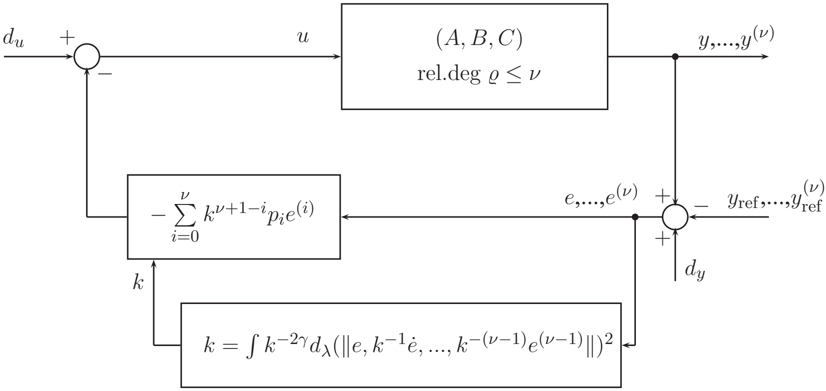


Figure 2.3: Closed-loop system (1.1), (2.34) with parameter $\nu \in \mathbb{N}$, γ, $\lambda > 0$ and disturbances $d_y(\cdot)$, $d_u(\cdot)$

(ii) *The global solution $x(\cdot)$ and the input $u(\cdot)$ are bounded, i.e.*

$$x(\cdot; 0, x^0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \qquad \textit{and} \qquad u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}).$$

(iii) $k(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, [k_0, \infty))$

(iv) $\lim_{t\to\infty} \text{dist}\,(\|e(t), k(t)^{-1}\dot{e}(t), ..., k(t)^{-(\nu-1)}e^{(\nu-1)}(t)\|, [0, \lambda]) = 0$

(v) $\lim_{t\to\infty} \text{dist}\,(|e(t)|, [0, \lambda]) = 0$

The proof of Theorem 2.3.8 is in Section 2.5 on page 75.

Remark 2.3.9. (Comment on the Theorem 2.3.8)

(i) The proof of Theorem 2.3.8 follows similarly to the proof of Theorem 2.3.4,

(ii) It has to be noted that the condition $\gamma \geq 2\vartheta$, instead of $\gamma \geq \vartheta$ as in Theorem 2.3.4, is important. Moreover, Theorem 2.3.8 allows input and output disturbances. ◇

With a slight modification, the result of Theorem 2.3.8 can be generalized to multi input, multi output systems with unknown relative degree $\varrho \in \{1, ..., \nu - 1\}$ which is presented in the following theorem. The problem of $\varrho = \nu$ is formulated in Remark 2.3.6 in the setup of stabilization which gives the same problem in the tracking setup.

Theorem 2.3.10. ***(Tracking, MIMO systems)***
For given $\nu \in \mathbb{N}$ suppose a system (1.1) *satisfying (A1) and (A2). Let $p_0, ..., p_\nu \in \mathbb{R}$ be such that* (2.23) *holds and $\gamma \geq 2\vartheta$ with ϑ as defined in* (2.26)*. Assume that the reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{\nu-1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and the input and output disturbances $d_u(\cdot)$, $d_y(\cdot) \in \mathcal{W}^{\nu-1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$. Then, for every $\lambda > 0$, application of the adaptive controller* (2.34) *to* (1.1) *yields, for any initial data $x^0 \in \mathbb{R}^n$ and $k_0 > 0$, a closed-loop initial value problem which satisfies the properties (i)-(v) of Theorem 2.3.8. More precisely, the statements (iv) and (v) get the form*

(iv) $\lim_{t\to\infty} \text{dist}\left(\|e(t), k(t)^{-1}\dot{e}(t), ..., k(t)^{-(\nu-1)}e^{(\nu-1)}(t)\|, [0, \lambda]\right) = 0$

(v) $\lim_{t\to\infty} \text{dist}\left(\|e(t)\|, [0, \lambda]\right) = 0.$

The proof of Theorem 2.3.10 is in Section 2.5 on page 82. Since the proof resembles the proof of Theorem 2.3.8, the details are omitted.

Remark 2.3.11. (Gain functions $k(\cdot)$)
In this chapter the two adaptive laws (2.28) and (2.34), i.e.

$$\begin{aligned}
\dot{k}(t) &= k(t)^{-2\gamma}\|y(t), k(t)^{-1}\dot{y}(t), ..., k(t)^{-(\nu-1)}y^{(\nu-1)}(t)\|^2, \\
&\qquad\qquad \text{with } \gamma \geq \|T_1\|\,\|T_1^{-1}\|\,(\nu-1) \geq 0 \\
\dot{k}(t) &= k(t)^{-2\gamma}d_\lambda(\|e(t), k(t)^{-1}\dot{e}(t), ..., k(t)^{-(\nu-1)}e^{(\nu-1)}(t)\|)^2, \\
&\qquad\qquad \text{with } \gamma \geq 2\|T_1\|\,\|T_1^{-1}\|\,(\nu-1) \geq 0
\end{aligned}$$

are presented. If $k(\cdot)$ and/or γ are large, the factor $k(\cdot)^{-2\gamma}$ slows down the adaption. The lower bound of γ depends on the upper bound ν of the relative degree ϱ and the choice of the Hurwitz polynomial $p(\cdot)$ (see (2.23)). For example consider the three-mass serially connected mass-spring damper system of Section 2.4 which gives $\gamma = 530$. Hence, $k(\cdot)$ increases slowly.
If the relative degree of the system is known and to be one, then $\gamma = 0$ is a valid choice and the adaptive laws get the well known form

$$\dot{k}(t) = \|y(t)\|^2 \qquad \text{or} \qquad \dot{k}(t) = d_\lambda(\|e(t)\|)^2$$

(see [40]).

The main drawback of (2.28) and (2.34) is the increasing gain function $k(\cdot)$. This drawback circumvents the controller of Chapter 3 and 4 which only allows relative degree one or unknown relative degree $\varrho \in \{1, 2\}$, respectively. ◇

Remark 2.3.12. (Negative definite high-frequency gain)
In view of Proposition 1.1.7 and Byrnes-Isidori form (1.13), the input $u(\cdot)$ of the system (1.13) or, equivalently (1.1), is multiplied by $\Gamma := CA^{\varrho-1}B$ which is positive definite (see (1.11)). Thus, if the control laws (2.28) or (2.34), respectively, stabilize and track, respectively, any linear system (1.1) which has stable zero dynamics with positive high-frequency gain, the controller (2.28) or (2.34), respectively, multiplied by -1, stabilize or track, respectively, any linear system (1.1) which has stable zero dynamics with negative high-frequency gain.
With other words, systems (1.1) with *negative high-frequency gain*, i.e., in view of (1.11),

$$\exists\, \gamma > 0\ \forall\, v \in \mathbb{R}^m : \quad v^\top CA^{\varrho-1}Bv \leq -\gamma\|v\|^2,$$

can be treated with the modified control laws

$$u(t) = \sum_{i=0}^{\nu} k(t)^{\nu+1-i} p_i y^{(i)}(t) \quad \text{or} \quad u(t) = \sum_{i=0}^{\nu} k(t)^{\nu+1-i} p_i e^{(i)}(t) + d_u(t), \text{ respectively.}$$

Thus without restriction of generality, the results in this chapter are restricted to systems with positive high-frequency gain (see (A1) or (A1'), respectively). ◇

The practical application and performance of the proposed controller (2.34) is shown in the following section.

2.4 Example

Consider the three-mass serially connected mass-spring damper system shown in Figure 2.4 in which adjacent masses are connected by springs and dashpots. The dynamics of the system is given by

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = bu(t), \qquad [q(0), \dot{q}(0)] \in \mathbb{R}^6 \tag{2.35}$$

with the output

$$y(t) = q_1(t) \quad \text{or} \quad y(t) = q_2(t) \quad \text{or} \quad y(t) = q_3(t), \tag{2.36}$$

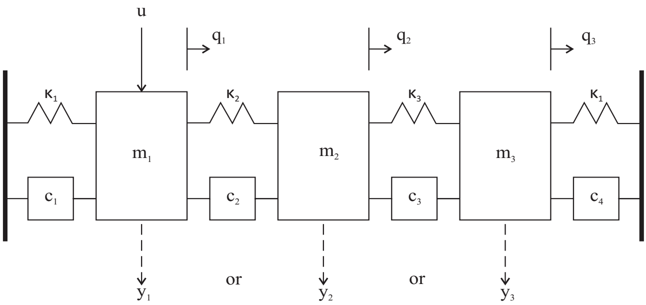


Figure 2.4: Three-mass serially connected mass-spring damper

where $q(\cdot) = [q_1(\cdot), q_2(\cdot), q_3(\cdot)]^\top$, $b = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ and

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix},$$

$$K = \begin{bmatrix} \kappa_1 + \kappa_2 & -\kappa_2 & 0 \\ -\kappa_2 & \kappa_2 + \kappa_3 & -\kappa_3 \\ 0 & -\kappa_3 & \kappa_3 + \kappa_4 \end{bmatrix}$$

and

$$\begin{aligned} q_i(t) &\;\widehat{=}\; \text{position of the mass } m_i \text{ at time } t,\ i = 1,2,3 \\ y(t) &\;\widehat{=}\; \text{output, position of the mass } m_i \text{ at time } t,\ i = 1,2,3 \\ m_i &\;\widehat{=}\; \text{mass } m_i \text{ in } kg,\ i = 1,2,3 \\ c_i &\;\widehat{=}\; \text{damping coefficient } c_i \text{ in } kg/s,\ i = 1,2,3,4 \\ \kappa_i &\;\widehat{=}\; \text{spring constant } \kappa_i \text{ in } kg/s^2,\ i = 1,2,3,4. \end{aligned}$$

It is assumed that all spring constants and damping coefficients are positive. Then the matrices C and K are positive definite (see [9, Fact 8.7.35]). The masses are $m_1 = 1 = m_3$, $m_2 = \frac{1}{2}$, the damping coefficients are $c_1 = c_2 = c_3 = c_4 = 2$ and the spring constants are $\kappa_1 = 2$, $\kappa_2 = 4$, $\kappa_3 = 1$, $\kappa_4 = 3$.

Define $x(\cdot) := [q(\cdot), \dot{q}(\cdot)]^\top$. Then the system (2.35), (2.36) can be written as a first order system of the form

$$\left.\begin{aligned} \dot{x}(t) &= \underbrace{\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -6 & 4 & 0 & -4 & 2 & 0 \\ 8 & -10 & 2 & 4 & -8 & 4 \\ 0 & 1 & -4 & 0 & 2 & -4 \end{array}\right]}_{=:A} x(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{array}\right] u(t), \quad x(0) = x^0 \\ y(t) &= c x(t), \end{aligned}\right\} \tag{2.37}$$

where $c \in \mathbb{R}^{1\times 6}$. In view of (2.36), the vector c is

$$c^\top = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \quad \text{or} \quad c^\top = \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \quad \text{or} \quad c^\top = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right].$$

It has to be noted that the system (2.37) has the relative degree 2 or 3 or 4.

The matrix A is Hurwitz (see [31, Lem. 3.2]). Then all single input, single output systems (2.37) of a serially connected structure with a Hurwitz system matrix A are minimum phase (see [31, Th. 9.1]). Hence the system (2.37) has stable zero dynamics. Moreover, [31, Th. 9.2] shows that the relative degree of a single input, single output system (2.37) of a serially connected structure is equal to the number of the intervening masses minus one plus two. For a three mass system (2.35), the relative degree is less than 4. Therefore, $\nu = 4$ is an upper bound on the relative degree for a three mass system (2.35) or equivalently, (2.37). Furthermore, [31, Th. 9.2] implies that the system (2.35) or equivalently, (2.37) has positive high-frequency gain.

It is assumed that the reference and disturbance signals are given by

$$t \mapsto y_{\text{ref}}(t) = 10, \qquad d_u(\cdot) \equiv 0 \equiv d_y(\cdot)$$

and consider the polynomial

$$p(s) = s^4 + 4s^3 + 6s^2 + 4s + 1 = (s+1)^4 \in \mathbb{R}[s]$$

which is Hurwitz and satisfies (2.23). An easy calculation gives

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} \frac{25}{8} & 4 & \frac{19}{8} & \frac{1}{2} \\ 4 & \frac{67}{8} & \frac{11}{2} & \frac{9}{8} \\ \frac{19}{8} & \frac{11}{2} & \frac{41}{8} & 1 \\ \frac{1}{2} & \frac{9}{8} & 1 & \frac{3}{8} \end{pmatrix}$$

(see (2.24) and (2.25)). Numerical calculations yield $\|T_1\| \leq 14.667$ and $\|T_1^{-1}\| \leq 6.015$ and the inequality (2.26) gets the form

$$\vartheta \geq \|T_1\| \, \|T_1^{-1}\| \, (\nu - 1) = 264.666.$$

To satisfy the assumption of Theorem 2.3.8 chose $\gamma = 530$. The mass-spring damper system (2.37) is simulated with the initial condition

$$x^0 = \left[-\frac{1}{2}, \frac{1}{4}, 1, \frac{1}{10}, -\frac{1}{5}, \frac{3}{10} \right]^\top .$$

Moreover, let $\lambda = \frac{1}{10}$ and $k_0 = 1$.

(1) Now suppose that the sensor measures the position of the mass m_1, i.e.

$$y(\cdot) = \begin{pmatrix} 1, & 0, & 0, & 0, & 0, & 0 \end{pmatrix} x(\cdot).$$

Then the system (2.37) has the relative degree $\varrho = 2$. In view of Proposition 1.1.7, the Byrnes-Isidori form of (2.37) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\begin{array}{c} y(t) \\ \dot{y}(t) \\ \hline z(t) \end{array} \right] = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & -4 & 4 & 0 & 2 & 0 \\ \hline 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -24 & 0 & -10 & 2 & -8 & 4 \\ 8 & 0 & 1 & -4 & 2 & -4 \end{array} \right] \left[\begin{array}{c} y(t) \\ \dot{y}(t) \\ \hline z(t) \end{array} \right] + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array} \right] u(t)$$

$$\left[\begin{array}{c} y(0) \\ \dot{y}(0) \\ \hline z(0) \end{array} \right] = \left[\begin{array}{cc|cccc} -\frac{1}{2} & 0.1 & \frac{1}{4} & 1 & 1.8 & 0.3 \end{array} \right].$$

An easy calculation shows that this system has stable zero dynamics. In view of the closed-loop system (2.38), (2.34), the arbitrary parameter $v \in \mathbb{R}^2$ of (2.46) is chosen as $v = [-10\ 10]^\top$.

The Figures 2.5a and 2.5b show $d_\lambda(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|)$.
As stated in Remark 2.3.11, dist$(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|, [0,\lambda])$ converge very slow to zero which shows that γ is very conservative.
Figure 2.5b depicts $d_\lambda(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|)$ for a long-run. In simulations, it can be seen that the term $k(\cdot)^{-2\gamma}$ in the controller improves significantly the control performance.
The short- and long-run of the error $e(\cdot)$ and its derivative $\dot{e}(\cdot)$ are shown in the Figures 2.5c and 2.5d. Both pictures show that $\dot{e}(\cdot)$ decreases fast and tracks perfectly $\dot{y}_{\text{ref}}(\cdot) \equiv 0$. For the error $e(\cdot)$, there is a slow convergence to $[0,\lambda]$. Figure 2.5d depicts the long-run.
Figure 2.5e shows the controller gain $k(\cdot)$. The slow convergence implies that the adaption $k(\cdot)$ reaches slowly its dead zone (see Figure 2.5b and Remark 2.3.11) and increases slowly. The input $u(\cdot)$ is depicted in Figure 2.5f.

(2) Now the sensor measures the position of the mass m_2, i.e.

$$y(\cdot) = \begin{pmatrix}0, & 1, & 0, & 0, & 0, & 0\end{pmatrix} x(\cdot).$$

Then the system (2.37) has the relative degree $\varrho = 3$ and can be transformed into the Byrnes-Isidori form

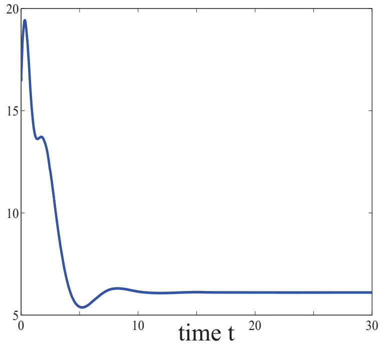
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ \hline z(t) \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -24 & -12 & -10 & 40 & 24 & 18 \\ \hline -\frac{5}{2} & 0 & 0 & 0 & 1 & 0 \\ 11 & 0 & 0 & -16 & -10 & -\frac{9}{2} \\ -7 & 0 & 0 & 16 & 8 & 4 \end{array}\right] \left[\begin{array}{c} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \\ \hline z(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 4 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] u(t) \quad (2.38)$$

with the initial value

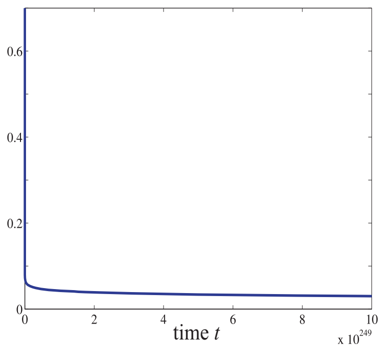
$$\left[\begin{array}{ccc|c} y(0) & \dot{y}(0) & \ddot{y}(0) & z(0)^\top \end{array}\right]^\top = \left[\begin{array}{ccc|ccc} \frac{1}{4} & -0.2 & -1.3 & -0.825 & 1.35 & -0.2 \end{array}\right]^\top.$$

With an easy calculation the system has stable zero dynamics and the parameter $v = 10$ (see (2.46)).

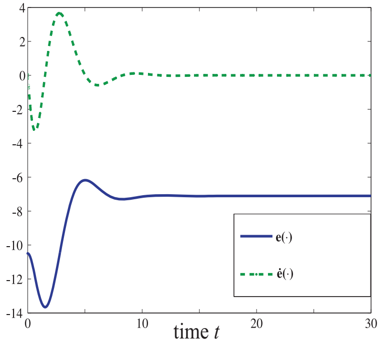
The Figures 2.6a and 2.6b depict $d_\lambda(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|)$ for short- and long-run. As in Figure 2.5a and 2.5b, the convergence is slow. The gain function $k(\cdot)$ and the error $e(\cdot)$, together with its derivatives $\dot{e}(\cdot)$ and $\ddot{e}(\cdot)$, are shown in Figure 2.6c and 2.6d. It can be seen in a long-run that the error $e(\cdot)$ and its derivatives $\dot{e}(\cdot)$, $\ddot{e}(\cdot)$ converge, but a long-run picture is omitted. Figure 2.6e depicts the input $u(\cdot)$ and the zero dynamics $z(\cdot)$ is shown in Figure 2.6f.



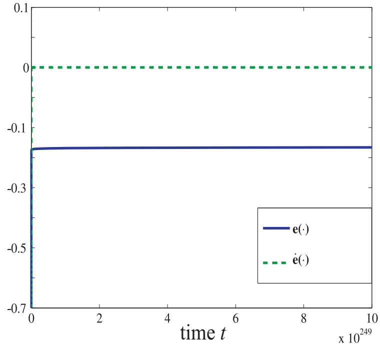
(a) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, short-run



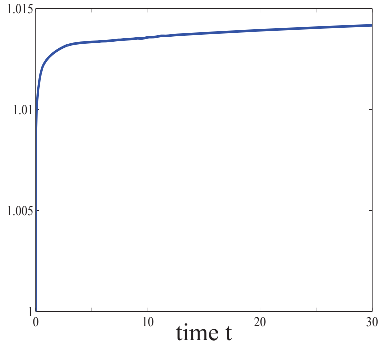
(b) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, long-run



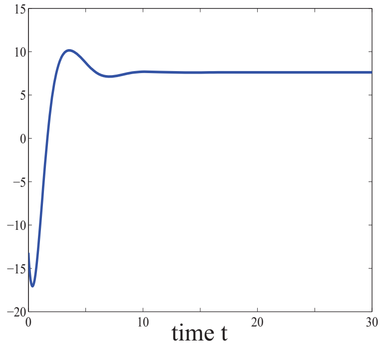
(c) $e(\cdot)$ and $\dot{e}(\cdot)$, short-run



(d) $e(\cdot)$ and $\dot{e}(\cdot)$, long-run

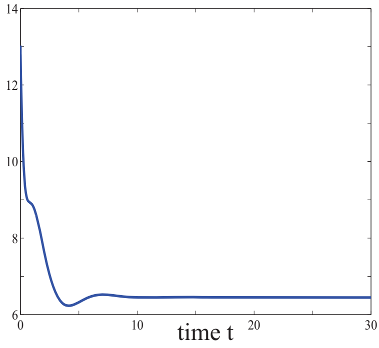


(e) Gain function $k(\cdot)$

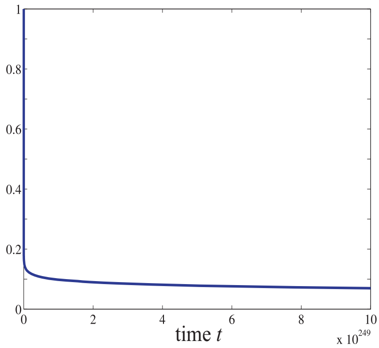


(f) Control $u(\cdot)$

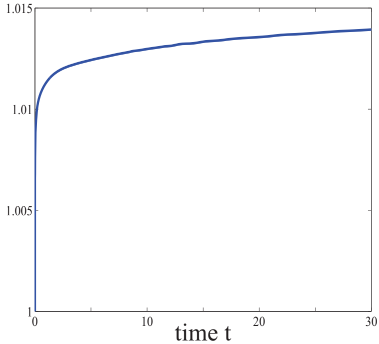
Figure 2.5: Closed-loop system (2.38), (2.34) with $\varrho = 2$



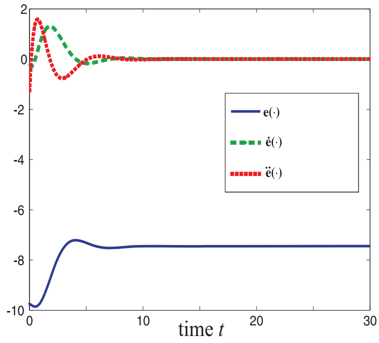
(a) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, short-run



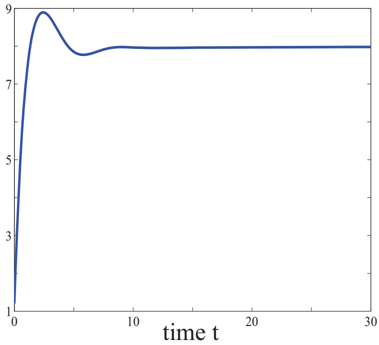
(b) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, long-run



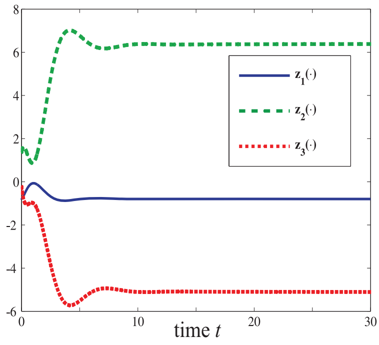
(c) Gain function $k(\cdot)$



(d) $e(\cdot)$, $\dot{e}(\cdot)$ and $\ddot{e}(\cdot)$



(e) Control $u(\cdot)$



(f) Functions $z_1(\cdot)$,$z_2(\cdot)$ and $z_3(\cdot)$

Figure 2.6: Closed-loop system (2.38), (2.34) with $\varrho = 3$

(3) Suppose that the sensor measures the position of the mass m_3, i.e.

$$y(\cdot) = \begin{pmatrix}0, & 0, & 1, & 0, & 0, & 0\end{pmatrix} x(\cdot).$$

Then the system (2.37) has the relative degree $\varrho = 4$. The Byrnes-Isidori form gets the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} y(t) \\ \dot y(t) \\ \ddot y(t) \\ y^{(3)}(t) \\ \hline z(t) \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{997}{16} & -\frac{699}{16} & -\frac{197}{4} & -\frac{27}{2} & -16 & -\frac{21}{2} \\ \hline \frac{171}{64} & 0 & 0 & 0 & -2 & -\frac{25}{8} \\ -\frac{9}{16} & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{array}\right] \left[\begin{array}{c} y(t) \\ \dot y(t) \\ \ddot y(t) \\ y^{(3)}(t) \\ \hline z(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 8 \\ \hline 0 \\ 0 \end{array}\right] u(t) \tag{2.39}$$

with the initial value

$$\left[\begin{array}{cccc|c} y(0) & \dot y(0) & \ddot y(0) & \dddot y(0) & z(0)^\top \end{array}\right]^\top = \left[\begin{array}{cccc|cc} 1 & 0.3 & -5.35 & 17.4 & -1.84375 & 0.825 \end{array}\right]^\top.$$

An easy calculation shows that this system has stable zero dynamics.

Figure 2.7 depicts the behaviour of the closed-loop system (2.39), (2.34). The long-run pictures follows similarly to the Figures 2.5 and 2.6 and are omitted.

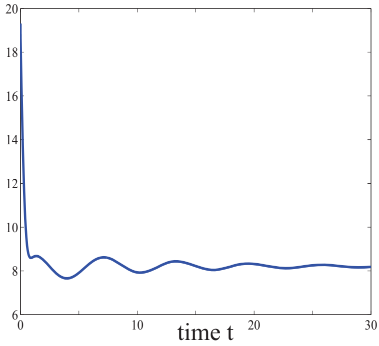
(4) In view of the physics and practical applications, it is typically to measure the position of the mass m_i, $i = 1, 2, 3$. Hence, a system (2.37) only has relative degree 2, 3, or 4. To demonstrate that the controller (2.34) is applicable for systems with relative degree one, it is assumed that the sensor measures the velocity of the mass m_1, i.e.

$$y(\cdot) = \begin{pmatrix}0, & 0, & 0, & 1, & 0, & 0\end{pmatrix} x(\cdot).$$

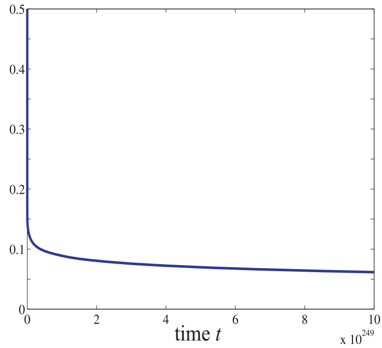
For practical applications it is not typically to measure the velocity. It is often too expensive or has large dispersions. The Byrnes-Isidori form of (2.37) has the form

$$\left[\begin{array}{c} \dot y(t) \\ \hline \dot z(t) \end{array}\right] = \left[\begin{array}{c|ccccc} -4 & -6 & 4 & 0 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 8 & -10 & 2 & -8 & 4 \\ 0 & 0 & 1 & -4 & 2 & -4 \end{array}\right] \left[\begin{array}{c} y(t) \\ \hline z(t) \end{array}\right] + \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] u(t)$$

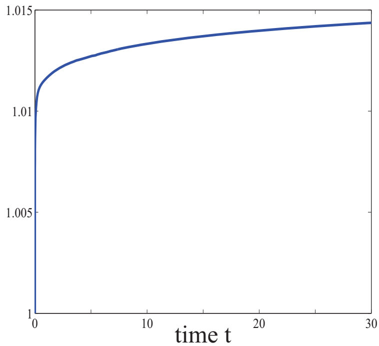
$$\left[\begin{array}{c} y(0) \\ \hline z(0) \end{array}\right] = \left[\begin{array}{c|ccccc} 0.1 & -\frac{1}{2} & \frac{1}{4} & 1 & -0.2 & 0.3 \end{array}\right].$$



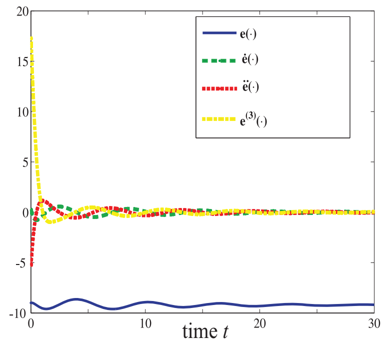
(a) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, short-run



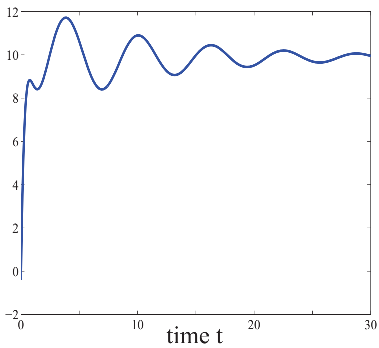
(b) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, long-run



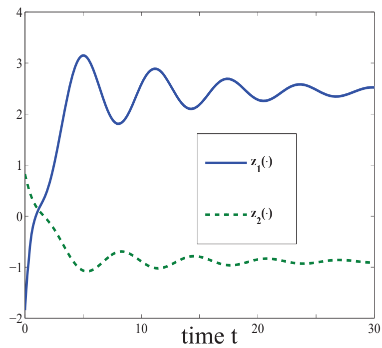
(c) Gain function $k(\cdot)$



(d) $e(\cdot)$, $\dot{e}(\cdot)$, $\ddot{e}(\cdot)$ and $\dddot{e}(\cdot)$



(e) Control $u(\cdot)$



(f) Functions $z_1(\cdot)$ and $z_2(\cdot)$

Figure 2.7: Closed-loop system (2.39), (2.34) with $\varrho = 4$

An easy calculation shows that the system (2.40) has stable zero dynamics and the parameter $v = [0\ -10\ 10]^\top$ (see (2.46)).

Figure 2.8 shows the long-run behaviour of the closed-loop system (2.40), (2.34) which confirms the results of Theorem 2.3.8. It can be seen that the error $e(\cdot)$, the input $u(\cdot)$ and $d_\lambda(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|)$ have large oscillations. As output, the velocity of mass m_1 is measured which explains these oscillation phenomena. Moreover, it can be seen in the simulations that the choice $\gamma \geq 2\vartheta$ is very conservative but important for the proofs. In this example the reference signal is constant which means that more information about the system is known and a better calculation of γ is possible.

The examples (1) - (4) show that the controller (2.34) is very conservative which can be seen by the slow convergence of dist$(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|, [0,\lambda])$ to zero (see Figure 2.5 (b) with $t \sim 10^{250}$ or Figure 2.8 (b) with $t \sim 10^5$) and the "explosion" of

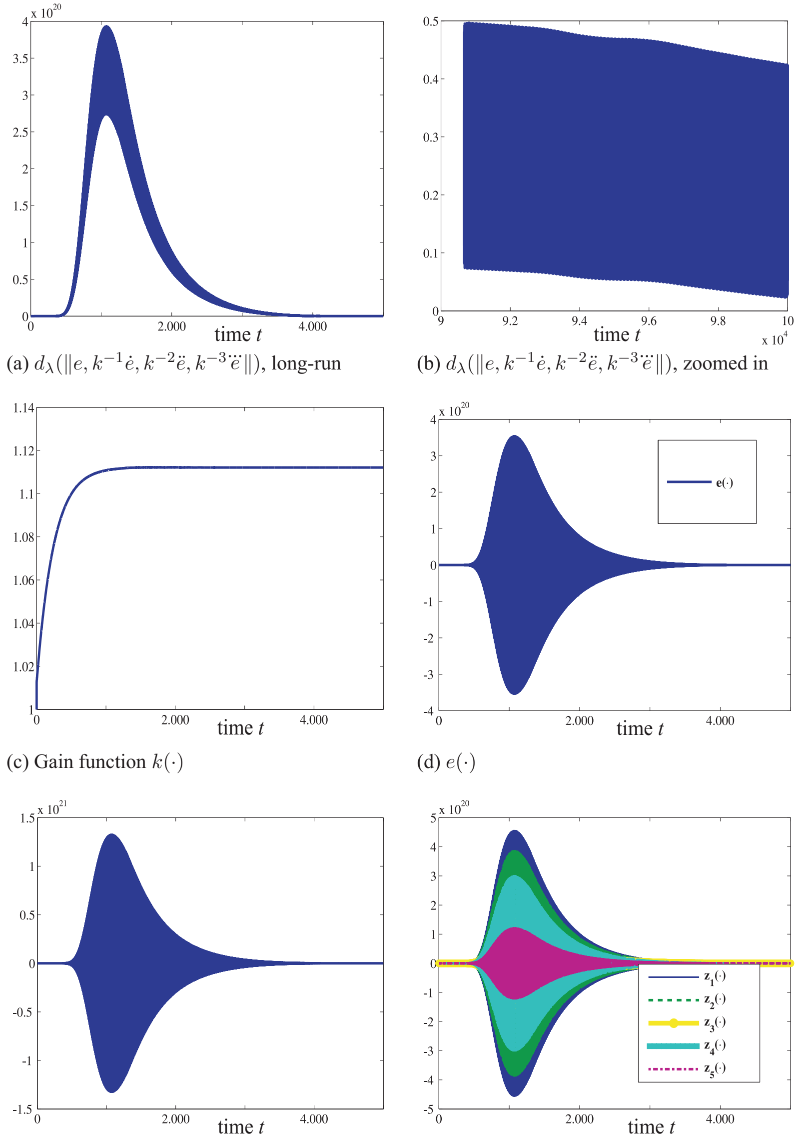
$$\text{dist}(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|, [0,\lambda]) \leq 4 \cdot 10^{20}$$

in Figure 2.8 (a). Therefore, the presented controller is not practicable for applications with unknown systems (1.1). The main reason for the conservativity of the controller (2.34) is that unknown systems (1.1) will be considered which only satisfy the structural assumptions (A1), (A2) or (A1'), (A2). In applications with further information, the controller may be useful.

Remark 2.4.1. (To the choice of $y_{\text{ref}}(\cdot)$, $d_u(\cdot)$, $d_y(\cdot)$)
As seen in the example, the convergence of

$$\text{dist}(\|e(\cdot), k(\cdot)^{-1}\dot{e}(\cdot), k(\cdot)^{-2}\ddot{e}(\cdot), k(\cdot)^{-3}\dddot{e}(\cdot)\|, [0,\lambda])$$

to zero is slow (see Figures 2.5 - 2.8). The simulations are generated with Matlab and Simulink. To confirm the results of Theorem 2.3.8, long-run simulations are necessary (see Figure 2.5b with $0 \leq t \leq 10^{250}$).
If the reference signal $y_{\text{ref}}(\cdot)$ is constant and the disturbances are zero, the Simulink solvers allow a large step size $\gg 1$ such that the long-run simulations are generated in a short time; otherwise the solvers only allow a small step size $\ll 1$ such that the long-run simulations generate a stack overflow for $t \in [10^5, 10^6]$.



(a) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, long-run

(b) $d_\lambda(\|e, k^{-1}\dot{e}, k^{-2}\ddot{e}, k^{-3}\dddot{e}\|)$, zoomed in

(c) Gain function $k(\cdot)$

(d) $e(\cdot)$

(e) Control $u(\cdot)$

(f) Functions $z_1(\cdot)$, ..., $z_5(\cdot)$

Figure 2.8: Closed-loop system (2.40), (2.34) with $\varrho = 1$

Introduce the Lorenz system

$$\begin{array}{rclcrcl} \dot{\xi}_1(t) & = & \xi_2(t) - \xi_1(t), & & \xi_1(0) & = & 1 \\ \dot{\xi}_2(t) & = & \frac{28}{10}\xi_1(t) - \frac{1}{10}\xi_2(t) - \xi_1(t)\xi_3(t), & & \xi_2(0) & = & 0 \\ \dot{\xi}_3(t) & = & \xi_1(t)\xi_2(t) - \frac{8}{30}\xi_3(t), & & \xi_3(t) & = & 3. \end{array}$$

It is shown in [85, App. C] that the Lorenz system is chaotic and bounded with bounded derivatives. A numerical computation yields

$$\|\xi_1\|_\infty \le \frac{9}{5}, \qquad \|\xi_2\|_\infty \le \frac{5}{2}, \qquad \|\dot{\xi}_1\|_\infty \le \frac{6}{5}, \qquad \|\dot{\xi}_2\|_\infty \le \frac{12}{5}.$$

For example, short-run simulations (i.e. $0 \le t \le 10^5$) are generated with

$$t \mapsto y_{\text{ref}}(t) = 5\sin(t) + 5, \qquad t \mapsto d_u(t) = \xi_1(t), \quad t \mapsto d_y(t) = 7\cos(2\,t) - 8,$$

where $\xi_1(\cdot)$ denotes the first component of the solution of the Lorenz system. ⋄

2.5 Proofs

Proof of Proposition 2.3.2:
If $n = 1$, then $|\det A| = |a_{11}|$ and $A^{-1} = \frac{1}{a_{11}}$ which shows (2.27). Moreover, there is equality in (2.27).

Let $n \ge 2$. Since A is regular, the inverse of A is given by $A^{-1} = \frac{\text{adj } A}{\det A}$ and thus

$$\|A^{-1}\| = \left\| \frac{\text{adj } A}{\det A} \right\| = \frac{1}{|\det A|} \|\text{adj } A\| \,.$$

Hence it remains to show that

$$\|\text{adj } A\| \le \frac{1}{n^{\frac{n-2}{2}}} \|A\|^{n-1}. \tag{2.40}$$

For $n = 2$ an easy calculation gives that $\text{adj } A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ which implies $\|\text{adj } A\| = \|A\|$ and thus,

$$\|\text{adj } A\| = \|A\| = \frac{1}{n^{\frac{n-2}{2}}} \|A\|^{n-1}$$

which shows (2.40) for $n = 2$; moreover, there is equality in (2.40).

Now it is assumed that $n \geq 3$. It has to be noted that $AA^* \in \mathbb{R}^{n \times n}$ is *Hermitian* (i.e. $AA^* = (AA^*)^*$) and thus, the eigenvalues of AA^* are non-negative (see [33, Lem. 2.5.7]). Denote $\lambda_1, ..., \lambda_n \geq 0$ the non-negative eigenvalues of AA^*. By [33, Cor. 7.3.3], there exists a *unitary* matrix $U \in \mathbb{C}^{n \times n}$ (i.e. $U^*U = I_n$) and a positive semi-definite matrix $P \in \mathbb{C}^{n \times n}$ such that

$$A = P\,U,$$

where $P = (AA^*)^{1/2}$. Note that P is Hermitian and thus, in view of [33, Lem. 2.5.7], the eigenvalues of P are given by $\sqrt{\lambda_1}, ..., \sqrt{\lambda_n} \geq 0$. [33, Ex. 2.5.2, Th. 2.5.4] implies that there exists a unitary matrix $V \in \mathbb{C}^{n \times n}$ such that the Hermitian matrix P can be written as

$$P = V \Lambda V^*,$$

where $\Lambda := \operatorname{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$. Define, for $k \in \mathbb{N}$, the k-th elementary function

$$c_k : \mathbb{C}^n \to \mathbb{C}, \qquad (s_1, ..., s_n) \mapsto c_k(s_1, ..., s_n) := \sum_{1 \leq j_1 < ... < j_k \leq n} s_{j_1} ... s_{j_k}.$$

It has to be noted that the adjunct of an unitary matrix is again unitary. Then it follows that

$$\|A\| = \|V \Lambda V^* U\| \overset{[28, p.\ 492]}{=} \|\Lambda\| = \sqrt{c_1(\lambda_1, ..., \lambda_n)}$$

and

$$\begin{aligned} \|\operatorname{adj} A\| &= \|\operatorname{adj}(V \Lambda V^* U)\| = \|\operatorname{adj} U \operatorname{adj} V^* \operatorname{adj} \Lambda \operatorname{adj} V\| \\ &= \|\operatorname{adj} \Lambda\| = \sqrt{c_{n-1}(\lambda_1, ..., \lambda_n)}. \end{aligned}$$

In view of [27, Th. 52], it follows that

$$c_{n-1}(\lambda_1, ..., \lambda_n) < \frac{1}{n^{n-2}} c_1(\lambda_1, ..., \lambda_n)^{n-1}$$

which shows (2.40). This completes the proof. □

Proof of Theorem 2.3.4:
The proof uses the notation of Proposition 1.1.7, (2.24) and (2.25).

STEP 1: *Some notation is introduced.*

Let $J := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{\nu\times\nu}$ and $e_i \in \mathbb{R}^\nu$, $i = 1, ..., \nu$, denotes the i-th unit vector of $\mathbb{R}^\nu$. It has to be noted that $\varrho \in \{1, ..., \nu\}$. Define

$$\mathcal{Y}(\cdot) := \left[y(\cdot), \dot{y}(\cdot), ..., y^{(\nu-1)}(\cdot)\right]^\top$$

and, for A_k as in (2.24),

$$\begin{aligned}
\mathcal{A}_{k,\tau} &:= \begin{cases} A_k & , \text{ if } \varrho \in \{1, ..., \nu-1\} \\ \frac{\Gamma p_\nu \tau}{1+\Gamma p_\nu \tau} A_k & , \text{ if } \varrho = \nu \end{cases} , \quad \text{for } \tau > 0 \\
\mathcal{R}(k) &:= \begin{cases} \frac{1}{\Gamma p_\nu k} e_\nu \left[R_1, ..., R_\varrho, -1, 0, ..., 0\right] & , \text{ if } \varrho \in \{1, ..., \nu-1\} \\ \frac{1}{1+\Gamma p_\nu k} \left(e_\nu \left[R_1, ..., R_\varrho\right] + J\right) & , \text{ if } \varrho = \nu \end{cases} \\
\mathcal{S}(k) &:= \begin{cases} \frac{1}{\Gamma p_\nu k} e_\nu S & , \text{ if } \varrho \in \{1, ..., \nu-1\} \\ \frac{1}{1+\Gamma p_\nu k} e_\nu S & , \text{ if } \varrho = \nu \end{cases}
\end{aligned}$$

with $\mathcal{A}_{k,\tau} \in \mathbb{R}[k]^{\nu\times\nu}$, $\mathcal{R}(k) \in \mathbb{R}(k)^{\nu\times\nu}$ and $\mathcal{S}(k) \in \mathbb{R}(k)^{\nu\times(n-\varrho)}$. Furthermore define

$$K(k) := \text{diag}\left(1, k, ..., k^{\nu-1}\right) \in \mathbb{R}[k]^{\nu\times\nu}, \qquad \Delta := \text{diag}\left(0, 1, ..., \nu-1\right) \in \mathbb{R}^{\nu\times\nu}$$

and

$$\begin{aligned}
\widehat{\mathcal{R}}(k) &:= K(k)^{-1}\mathcal{R}(k)K(k) \\
&= \begin{cases} \frac{k}{\Gamma p_\nu k} e_\nu \left[R_1 k^{-\nu}, ..., R_\varrho k^{-(\nu+1-\varrho)}, -k^{-(\nu-\varrho)}, 0, ..., 0\right] & , \text{ if } \varrho \in \{1, ..., \nu-1\} \\ \frac{k}{1+\Gamma p_\nu k} \left(e_\nu \left[R_1 k^{-\nu}, ..., R_\varrho k^{-1}\right] + J\right) & , \text{ if } \varrho = \nu, \end{cases}
\end{aligned}$$

where $\widehat{\mathcal{R}}(k) \in \mathbb{R}(k)^{\nu\times\nu}$.

STEP 2: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (2.28) *or, equivalently,* (1.13)*,* (2.28) *are shown.*
Define, for the relatively open set $\mathcal{D} := \mathbb{R}^\nu \times \mathbb{R}^{n-\varrho} \times [k_0, \infty)$, the function

$$f: \mathcal{D} \to \mathbb{R}^\nu \times \mathbb{R}^{n-\varrho} \times \mathbb{R}_{>0}, \qquad (\xi, \mu, \eta) \mapsto \begin{pmatrix} \left[\mathcal{A}_{\eta,\eta} + \mathcal{R}(\eta)\right]\xi + \mathcal{S}(\eta)\mu \\ P_1 e_1^\top \xi + Q\mu \\ \eta^{-2\gamma} \sum_{i=0}^{\nu-1} (\eta^{-i}\xi_{i+1})^2 \end{pmatrix}.$$

Then, the closed-loop initial value problem (1.13), (2.28) gets the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathcal{Y}(t)\\ z(t)\\ k(t)\end{pmatrix} = f\left(\mathcal{Y}(t), z(t), k(t)\right), \qquad \begin{pmatrix}\mathcal{Y}(0)\\ z(0)\\ k(0)\end{pmatrix} = \begin{pmatrix}\left[(\mathcal{C}x^0)^\top,\ v^\top\right]\\ Nx^0\\ k_0\end{pmatrix} \tag{2.41}$$

with arbitrary $v \in \mathbb{R}^{\nu-\varrho}$. It follows that the right-hand side of (2.41) is locally Lipschitz on the relatively open set $\mathcal{D}$ in the sense that, for all $(\overline{\xi}, \overline{\mu}, \overline{\eta}) \in \mathcal{D}$, there exists a neighborhood $\mathcal{O}$ of $(\overline{\xi}, \overline{\mu}, \overline{\eta})$ and a constant $L > 0$ such that

$$\forall\, (\xi, \mu, \eta) \in \mathcal{O}: \quad \|f(\xi, \mu, \eta) - f(\overline{\xi}, \overline{\mu}, \overline{\eta})\| \le L(\|\xi - \overline{\xi}\| + \|\mu - \overline{\mu}\| + \|\eta - \overline{\eta}\|).$$

Now standard theory of ordinary differential equations (see [88, Th. III.10.VI]) yields existence of a solution, i.e. a continuous differentiable function $(\mathcal{Y}, z, k) : [0, \omega) \to \mathbb{R}^\nu \times \mathbb{R}^{n-\varrho} \times \mathbb{R}_{>0}$, $0 < \omega \le \infty$, satisfying (2.41) and $(\mathcal{Y}(t), z(t), k(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$. Moreover, the solution is unique and ω can be chosen maximal, i.e. the solution is not completely contained in any compact subset of $\mathcal{D}$.

STEP 3: *Coordinate transformation and Ljapunov function candidate*
Introduce the coordinate transformation

$$w(\cdot) := K(k(\cdot))^{-1}\mathcal{Y}(\cdot). \tag{2.42}$$

The closed-loop initial value system (2.41) can be written as

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} w\\ z\end{pmatrix} &= \begin{pmatrix} k\mathcal{A}_{1,k} + \widehat{\mathcal{R}}(k) - \frac{\dot{k}}{k}\Delta, & k^{-(\nu-1)}\mathcal{S}(k)\\ P_1 e_1^\top, & Q\end{pmatrix}\begin{pmatrix} w\\ z\end{pmatrix}\\ \dot{k} &= k^{-2\gamma}\|w\|^2\\ \begin{pmatrix} w(0)\\ z(0)\end{pmatrix} &= \begin{pmatrix} K(k_0)^{-1}\mathcal{Y}(0)\\ Nx^0\end{pmatrix}, \quad k(0) = k_0. \end{aligned}\right\} \tag{2.43}$$

It has to be noted that the matrix Q is Hurwitz (see (A2) and Proposition 1.1.7). As the matrix A_1 (which is given by (2.24)) is Hurwitz, there exist unique positive definite, symmetric matrices $T_1 = T_1^\top \in \mathbb{R}^{\nu\times\nu}$ and $T_2 = T_2^\top \in \mathbb{R}^{(n-\varrho)\times(n-\varrho)}$ such that

$$A_1^\top T_1 + T_1 A_1 = -I_\nu \qquad \text{and} \qquad Q^\top T_2 + T_2 Q = -I_{n-\varrho}$$

and thus, for all $w \in \mathbb{R}^\nu$,

$$-\,w^\top T_1 \Delta w \le \|T_1\|\,\|\Delta\|\,\|w\|^2 \overset{[28, Prop.\ 4.3.4,\ (4.3.22)]}{\le} (\nu-1)\|T_1\|\,\|T_1^{-1}\| w^\top T_1 w \overset{(2.26)}{=} \vartheta w^\top T_1 w. \tag{2.44}$$

Note that T_1 is the same matrix as in (2.25) which defines ϑ (see (2.26)). Define

$$V:\ [0,\omega)\to\mathbb{R},\quad t\mapsto V(t):=k(t)^{-2\gamma}\left(w(t)^\top T_1 w(t)+z(t)^\top T_2 z(t)\right). \tag{2.45}$$

The derivative of $V(\cdot)$ along the trajectory of (2.43) yields, for all $t\in[0,\omega)$, (the argument t is omitted)

$$\begin{aligned}
\dot V &= 2k^{-2\gamma}\left(w^\top T_1\dot w+z^\top T_2\dot z-\gamma\frac{\dot k}{k}\left(w^\top T_1 w+z^\top T_2 z\right)\right)\\
&\le 2k^{-2\gamma}\left[w^\top T_1\left[\left(k\mathcal{A}_{1,k}+\widehat{\mathcal{R}}(k)-\frac{\dot k}{k}\Delta-\gamma\frac{\dot k}{k}\right)w+k^{-(\nu-1)}\mathcal{S}(k)z\right]\right.\\
&\quad\left.+z^\top T_2\left(P_1e_1^\top w+Qz\right)\right]\\
&\overset{(2.44)}{\le} k^{-2\gamma}\begin{cases}\left[-k+2\|T_1\|\,\|\widehat{\mathcal{R}}(k)\|+\left(2\frac{k^{-(\nu-1)}}{\Gamma p_\nu k}\|T_1\|\,\|S\|\right)^2+(2\|P_1\|\,\|T_2\|)^2\right]\|w\|^2\\ -\frac{1}{2}\|z\|^2+2\frac{\dot k}{k}(\vartheta-\gamma)w^\top T_1 w & ,\ \text{if }\varrho\in\{1,...,\nu-1\}\\ -\frac{1}{2}\|z\|^2+\left[-\frac{\Gamma p_\nu k^2}{1+\Gamma p_\nu k}+2\|T_1\|\,\|\widehat{\mathcal{R}}(k)\|+\left(2\frac{k^{-(\nu-1)}}{1+\Gamma p_\nu k}\|T_1\|\,\|S\|\right)^2+\right.\\ \left.(2\|P_1\|\,\|T_2\|)^2\right]\|w\|^2+2\frac{\dot k}{k}(\vartheta-\gamma)w^\top T_1 w & ,\ \text{if }\varrho=\nu.\end{cases}
\end{aligned}$$

STEP 4: *Assertions (i) and (iii) are shown.*
Seeking a contradiction to Assertion (iii), assume that $k(\cdot)$ is unbounded on $[0,\omega)$. Since $\gamma\ge\vartheta$, $\dot k(\cdot)\ge 0$ and $k(\cdot)>0$, it follows that $2\frac{\dot k(\cdot)}{k(\cdot)}(\vartheta-\gamma)w(\cdot)^\top T_1 w(\cdot)\le 0$ and Step 3 implies that (the argument t is omitted)

$$\begin{aligned}
\dot V &\overset{\exists\, t_1\in[0,\omega)\ \forall\, t\in[t_1,\omega)}{\le} -k^{-2\gamma}\begin{cases}\frac{k}{2}\|w\|^2+\frac{1}{2}\|z\|^2 & ,\ \text{if }\varrho\in\{1,...,\nu-1\}\\ \frac{1}{2}\frac{\Gamma p_\nu k^2}{1+\Gamma p_\nu k}\|w\|^2+\frac{1}{2}\|z\|^2 & ,\ \text{if }\varrho=\nu\end{cases}\\
&\overset{\exists\, t_2\in[t_1,\omega)\ \forall\, t\in[t_2,\omega)}{\le} -\frac{1}{2}k^{-2\gamma}\left(\|w\|^2+\|z\|^2\right)
\end{aligned}$$

which yields the contradiction

$$\begin{aligned}
\forall\, t\in[t_2,\omega):\quad -V(t_2)\le V(t)-V(t_2) &= \int_{t_2}^t \dot V(\tau)\,\mathrm{d}\tau\\
&\le -\frac{1}{2}\int_{t_2}^t k(\tau)^{-2\gamma}\|w(\tau)\|^2\,\mathrm{d}\tau\\
&= -\frac{1}{2}\int_{t_2}^t \dot k(\tau)\,\mathrm{d}\tau\ =-\frac{1}{2}\left[k(t)-k(t_2)\right].
\end{aligned}$$

Boundedness of $k(\cdot)$, together with (2.41), implies that there exists $m_1, m_2 > 0$ such that

$$\forall (\mathcal{Y}, z, k) \in \mathcal{D}: \quad \|f(\mathcal{Y}, z, k)\| \leq m_1 \|(\mathcal{Y}, z, k)\| + m_2$$

which gives $\omega = \infty$ (see [1, Prop. II.7.8]).

STEP 5: *Assertion (ii) and $y^{(i)}(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R})$, $i = 0, ..., \nu$, are shown.*
Denote $k_\infty := \lim_{t\to\infty} k(t) < \infty$. Then it follows that, for all $t \geq 0$,

$$k_\infty^{-2\gamma} \int_0^t \|w(\tau)^\top\|^2 \,\mathrm{d}\tau \leq \int_0^t k(\tau)^{-2\gamma} \|w(\tau)^\top\|^2 \,\mathrm{d}\tau = \int_0^t \dot{k}(\tau) \,\mathrm{d}\tau \leq k_\infty - k_0 < \infty$$

and therefore, in view of (2.42),

$$y(\cdot),\; k(\cdot)^{-1}\dot{y}(\cdot), ...,\; k(\cdot)^{-(\nu-1)} y^{(\nu-1)}(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R})$$

which yields, together with the boundedness and monotonicity of $k(\cdot)$, immediately that

$$y(\cdot),\; \dot{y}(\cdot), ...,\; y^{(\nu-1)}(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}).$$

Since the matrix Q is Hurwitz, the second subsystem of (2.41), i.e. $\dot{z}(t) = Qz(t) + P_1 y(t)$, implies that

$$z(\cdot; 0, Nx^0) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^{n-\varrho}) \quad \text{and} \quad \lim_{t\to\infty} z(t; 0, Nx^0) = 0$$

(see [37, Lem. 2.2], [18, Lem. 2.5.1]) which yields, in view of boundedness of $k(\cdot)$ and the first equation of (2.41),

$$y^{(\nu)}(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R})$$

and thus Assertion (ii) holds with Proposition 1.1.7.

STEP 6: *Assertions (iv) and (v) are shown.*
The first claim of Assertion (v) follows from Step 5. It remains to show the second statement. Applying [36, Lem. 2.1.7], it follows that $\lim_{t\to\infty} \mathcal{Y}(t) = 0$ and thus, in view of the first equation of (2.41),

$$\lim_{t\to\infty} \dot{\mathcal{Y}}(t) = 0$$

which shows the second claim of Assertion (v). The coordinate transformation $x(\cdot) = T\left(\mathcal{Y}_1(\cdot), z(\cdot)\right)^\top$ with T as in Proposition 1.1.7 shows Assertion (iv). This completes the proof. □

Proof of Theorem 2.3.7:
The proof uses the notation of Proposition 1.1.7, (2.24), (2.25) and of Step 1 of the proof of Theorem 2.3.4.

STEP 1: *Some notation is introduced.*
Let $J := \begin{pmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{\nu m \times \nu m}$ (wherein 0 denotes $m \times m$ zero matrix) end denote $E := \begin{bmatrix} 0, & ..., & 0, & I_m \end{bmatrix}^\top \in \mathbb{R}^{\nu m \times m}$. Note that $\varrho \in \{1, ..., \nu - 1\}$. Define $\mathcal{Y}(\cdot)$ as in Step 1 of the proof of Theorem 2.3.4 and

$$\mathcal{R}(k) := \frac{1}{p_\nu k}\Gamma^{-1}E\left[R_1, ..., R_\varrho, -I_m, 0, ..., 0\right], \qquad \mathcal{S}(k) := \frac{1}{p_\nu k}\Gamma^{-1}ES$$

with $\mathcal{R}(k) \in \mathbb{R}(k)^{\nu m \times \nu m}$ and $\mathcal{S}(k) \in \mathbb{R}(k)^{\nu m \times (n - \varrho m)}$. Furthermore define

$$\begin{aligned} K(k) &:= \operatorname{diag}\left(I_m, kI_m, ..., k^{\nu-1}I_m\right) \in \mathbb{R}[k]^{\nu m \times \nu m} \\ \Delta &:= \operatorname{diag}\left(0, I_m, ..., (\nu-1)I_m\right) \in \mathbb{R}^{\nu m \times \nu m} \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathcal{R}}(k) &:= K(k)^{-1}\mathcal{R}(k)K(k) \\ &= \frac{1}{p_\nu}\Gamma^{-1}E\left[k^{-\nu}R_1, ..., k^{-(\nu+1-\varrho)}R_\varrho, -k^{-(\nu-\varrho)}I_m, 0, ..., 0\right] \in \mathbb{R}(k)^{\nu m \times \nu m}. \end{aligned}$$

STEP 2: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (2.28) *or, equivalently,* (1.13)*,* (2.28) *are shown.*
Define, for the relatively open set $\mathcal{D} := \mathbb{R}^{\nu m} \times \mathbb{R}^{n-\varrho m} \times [k_0, \infty)$, the function

$$f:\ \mathcal{D} \to \mathbb{R}^{\nu m} \times \mathbb{R}^{n-\varrho m} \times \mathbb{R}_{>0}, \qquad (\xi, \mu, \eta) \mapsto \begin{pmatrix} \left[A_\eta + \mathcal{R}(\eta)\right]\xi & + & \mathcal{S}(\eta)\mu \\ P_1[I_m, 0, ..., 0]\xi & + & Q\mu \\ \multicolumn{3}{c}{\eta^{-2\gamma}\sum_{i=0}^{\nu-1}(\eta^{-i}\xi_{i+1})^2} \end{pmatrix}.$$

Existence and uniqueness of the solution of the closed-loop initial value problem (1.13), (2.28) which has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathcal{Y}(t) \\ z(t) \\ k(t) \end{pmatrix} = f\left(\mathcal{Y}(t), z(t), k(t)\right), \qquad \begin{pmatrix} \mathcal{Y}(0) \\ z(0) \\ k(0) \end{pmatrix} = \begin{pmatrix} \left[(\mathcal{C}x^0)^\top,\ v^\top\right] \\ Nx^0 \\ k_0 \end{pmatrix}$$

with arbitrary $v \in \mathbb{R}^{m\nu - \varrho m}$ follow similarly to Step 2 in the proof of Theorem 2.3.4.

STEP 3: *Assertions (i) and (iii) are shown.*
Consider the coordinate transformation (2.42) and the Ljapunov function candidate (2.45) of the proof of Theorem 2.3.4. It follows similarly, for all $t \in [0, \omega)$,

$$\dot V \le k^{-2\gamma}\Bigg(\Bigg[-k + 2\|T_1\|\,\|\widehat{\mathcal{R}}(k)\| + \left(2\frac{k^{-(\nu-1)}}{p_\nu k}\|\Gamma^{-1}\|\|T_1\|\,\|S\|\right)^2 + \left(2\|P_1\|\,\|T_2\|\right)^2\Bigg]\|w\|^2 - \frac{1}{2}\|z\|^2 + 2\frac{\dot k}{k}(\vartheta - \gamma)w^\top T_1 w\Bigg).$$

As $\gamma \ge \vartheta$ and T_1 positive definite, it follows that $\frac{\dot k(\cdot)}{k(\cdot)}(\vartheta - \gamma)w^\top(\cdot)T_1 w(\cdot) \le 0$. All inequalities derived in Step 4 of the proof of Theorem 2.3.4 hold true which shows the Assertions (i) and (iii). The details are omitted.

STEP 4: *Assertions (ii), (iv) and (v) are shown.*
With minor modifications the Step 5 and 6 of the proof of Theorem 2.3.4 go through and are omitted. This completes the proof. □

Proof of Theorem 2.3.8:
The proof uses the notation of Proposition 1.1.7, (2.24), (2.25) and of Step 1 of the proof of Theorem 2.3.4.

STEP 1: *Some notation is introduced.*
Note that $\varrho \in \{1, ..., \nu\}$. Define $\mathcal{E}(\cdot) := \mathcal{Y}(\cdot) + \mathcal{D}_{\mathcal{Y}}(\cdot) - \mathcal{Y}_{\text{ref}}(\cdot)$ with

$$\begin{aligned}
\mathcal{D}_{\mathcal{Y}}(\cdot) &:= \left[d_y(\cdot), \dot d_y(\cdot), ..., d_y^{(\nu-1)}(\cdot)\right] \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0}, \mathbb{R}^\nu)\\
\mathcal{Y}_{\text{ref}}(\cdot) &:= \left[y_{\text{ref}}(\cdot), \dot y_{\text{ref}}(\cdot), ..., y_{\text{ref}}^{(\nu-1)}(\cdot)\right] \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0}, \mathbb{R}^\nu)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}(k) &:= \begin{cases} \frac{1}{\Gamma p_\nu k} e_\nu [R_1, ..., R_\varrho, -1, 0, ..., 0] & \text{, if } \varrho \in \{1, ..., \nu - 1\} \\ \frac{1}{1 + \Gamma p_\nu k} e_\nu [R_1, ..., R_\varrho, -1] & \text{, if } \varrho = \nu \end{cases}\\
\mathcal{D}_{u,k}(\cdot) &:= \begin{cases} \frac{1}{p_\nu k} e_\nu d_u(\cdot) & \text{, if } \varrho \in \{1, ..., \nu - 1\} \\ \frac{\Gamma}{1 + \Gamma p_\nu k} e_\nu d_u(\cdot) & \text{, if } \varrho = \nu \end{cases}\\
\overline{\mathcal{Y}}_{\text{ref,d}}(\cdot) &:= \left[\mathcal{Y}_{\text{ref}}(\cdot)^\top, y_{\text{ref}}^{(\nu)}(\cdot)\right]^\top - \left[\mathcal{D}_{\mathcal{Y}}(\cdot)^\top, d_y^{(\nu)}(\cdot)\right]^\top \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0}, \mathbb{R}^{\nu+1})
\end{aligned}$$

with $\mathcal{B}(k) \in \mathbb{R}(k)^{\nu\times(\nu+1)}$.

STEP 2: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (2.34) *or, equivalently,* (1.13)*,* (2.34) *are shown.*
Define, for the relatively open set $\mathcal{D} := [0,\infty)\times\mathbb{R}^\nu\times\mathbb{R}^{n-\varrho}\times[k_0,\infty)$,

$$f:\ \mathcal{D}\to\mathbb{R}^\nu\times\mathbb{R}^{n-\varrho}\times\mathbb{R}_{>0},$$
$$(t,\xi,\mu,\eta)\mapsto\begin{pmatrix}[\mathcal{A}_{\eta,\eta}+\mathcal{R}(\eta)]\,\xi & + & \mathcal{S}(\eta)\mu & + & \mathcal{B}(\eta)\overline{\mathcal{Y}}_{\mathrm{ref,d}}(t)+\mathcal{D}_{u,k}(t)\\ P_1e_1^\top\xi & + & Q\mu & + & P_1e_1^\top(\mathcal{Y}_{\mathrm{ref}}(t)-\mathcal{D}_{\mathcal{Y}}(t))\\ \multicolumn{5}{c}{\eta^{-2\gamma}d_\lambda(\|\xi_1,\eta^{-1}\xi_2,\dots,\eta^{-(\nu-1)}\xi_\nu\|)^2}\end{pmatrix}.$$

Then, the initial value problem (1.13), (2.34) has the form

$$\left.\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathcal{E}(t)\\ z(t)\\ k(t)\end{pmatrix} &= f\left(t,\mathcal{E}(t),z(t),k(t)\right)\\ \begin{pmatrix}\mathcal{E}(0)\\ z(0)\\ k(0)\end{pmatrix} &= \begin{pmatrix}\left[(\mathcal{C}x^0+\mathcal{D}_{\mathcal{Y}}(0)-\mathcal{Y}_{\mathrm{ref}}(0))^\top,\ v^\top\right]\\ Nx^0\\ k_0\end{pmatrix}\end{aligned}\right\}\qquad(2.46)$$

with arbitrary $v\in\mathbb{R}^{\nu-\varrho}$. Existence and uniqueness of the solution $(\mathcal{E},z,k):\ [0,\omega)\to\mathbb{R}^\nu\times\mathbb{R}^{n-\varrho}\times\mathbb{R}_{>0}$, $0<\omega\le\infty$, follow similarly to Step 2 in the proof of Theorem 2.3.4.

STEP 3: *Coordinate transformation and Ljapunov function candidate*
For $\gamma\ge 2\vartheta$ and $\vartheta\ge 0$ given by (2.26), choose

$$\delta\in[2\vartheta,\gamma]\qquad\text{and}\qquad\theta\in\left(\delta+\frac{1}{2},\delta+\frac{3}{2}\right).\qquad(2.47)$$

Introduce the coordinates

$$w(\cdot):=k(\cdot)^{-\delta}K(k(\cdot))^{-1}\mathcal{E}(\cdot)\qquad\text{and}\qquad v(\cdot):=k(\cdot)^{-\theta}z(\cdot).\qquad(2.48)$$

The closed-loop initial value system (2.46) can be written as

$$\left.\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}w\\ v\end{pmatrix} &= \begin{pmatrix}k\mathcal{A}_{1,k}+\widehat{\mathcal{R}}(k)-\frac{\dot k}{k}(\Delta+\delta I_\nu), & k^{-\delta+\theta}k^{-(\nu-1)}\mathcal{S}(k)\\ k^{\delta-\theta}P_1e_1^\top, & Q-\theta\frac{\dot k}{k}I_{n-\varrho}\end{pmatrix}\begin{pmatrix}w\\ v\end{pmatrix}\\ &\qquad+\begin{pmatrix}k^{-\delta}k^{-(\nu-1)}\left[\mathcal{B}(k)\overline{\mathcal{Y}}_{\mathrm{ref,d}}+\mathcal{D}_{u,k}(t)\right]\\ k^{-\theta}P_1e_1^\top(\mathcal{Y}_{\mathrm{ref}}(t)-\mathcal{D}_{\mathcal{Y}}(t))\end{pmatrix}\\ \dot k &= k^{-2\gamma}d_\lambda(k^\delta\|w\|)^2\\ \begin{pmatrix}w(0)\\ v(0)\end{pmatrix} &= \begin{pmatrix}k_0^{-\delta}K(k_0)^{-1}\mathcal{E}(0)\\ k_0^{-\theta}Nx^0\end{pmatrix},\qquad k(0)=k_0.\end{aligned}\right\}\qquad(2.49)$$

Let $T_1 = T_1^\top \in \mathbb{R}^{\nu\times\nu}$ and $T_2 = T_2^\top \in \mathbb{R}^{(n-\varrho)\times(n-\varrho)}$ be defined as in Step 3 of the proof of Theorem 2.3.4. Define

$$V:\ [0,\omega) \to \mathbb{R}, \quad t \mapsto V(t) := w(t)^\top T_1 w(t) + v(t)^\top T_2 v(t). \tag{2.50}$$

The derivative of $V(\cdot)$ along the trajectory of (2.49), together with (2.44), yields, for all $t \in [0,\omega)$, (the argument t is omitted)

$$\begin{aligned}
\dot V &= 2w^\top T_1 \dot w + 2v^\top T_2 \dot v \\
&= 2w^\top T_1 \left[\left(k\mathcal{A}_{1,k} + \widehat{\mathcal{R}}(k) - \frac{\dot k}{k}(\Delta + \delta I_\nu)\right) w + k^{-\delta+\theta} k^{-(\nu-1)} \mathcal{S}(k) v\right] \\
&\quad + 2k^{-\theta} v^\top T_2 P_1 e_1^\top \left[\mathcal{Y}_{\text{ref}} - \mathcal{D}_{\mathcal{Y}}\right] + 2v^\top T_2 \left[k^{\delta-\theta} P_1 e_1^\top w + \left(Q - \theta\frac{\dot k}{k} I_{n-\varrho}\right) v\right] \\
&\quad + 2k^{-\delta} k^{-(\nu-1)} w^\top T_1 \left[\mathcal{B}(k)\overline{\mathcal{Y}}_{\text{ref,d}} + \mathcal{D}_{u,k}(t)\right] \\
&\le \begin{cases}
\Big[-k + 2\|T_1\|\,\|\widehat{\mathcal{R}}(k)\| + \left(2k^{-\delta+\theta}\frac{k^{-(\nu-1)}}{\Gamma p_\nu k}\|T_1\|\,\|S\|\right)^2 + \left(2k^{\delta-\theta}\|T_2\|\,\|P_1\|\right)^2 \\
\quad + \left(\frac{1}{\Gamma p_\nu}\|T_1\|\ \left[\|\mathcal{B}(1)\|\,\|\overline{\mathcal{Y}}_{\text{ref,d}}\| + \|d_u\|\right]\right)^2 \Big] \|w\|^2 + k^{-2(\theta-1/2)} + k^{-2(\delta+\nu)} \\
\quad + \left[-\frac{1}{2} + \left(k^{-1/2}\|T_2\|\,\|P_1\|\,\|\mathcal{Y}_{\text{ref}} - \mathcal{D}_{\mathcal{Y}}\|\right)^2 \right] \|v\|^2 \\
\quad + 2\frac{\dot k}{k}\left[(\vartheta - \delta) w^\top T_1 w - \theta v^\top T_2 v\right] & ,\ \text{if } \varrho \in \{1,\dots,\nu-1\} \\
\Big[-\frac{\Gamma p_\nu k^2}{1+\Gamma p_\nu k} + 2\|T_1\|\,\|\widehat{\mathcal{R}}(k)\| + \left(2k^{-\delta+\theta}\frac{k^{-(\nu-1)}}{1+\Gamma p_\nu k}\|T_1\|\,\|S\|\right)^2 + \left(2k^{\delta-\theta}\|T_2\|\,\|P_1\|\right)^2 \\
\quad + \left(\frac{k}{1+\Gamma p_\nu k}\|T_1\|\ \left[\|\mathcal{B}(1)\|\,\|\overline{\mathcal{Y}}_{\text{ref,d}}\| + \|d_u\|\right]\right)^2 \Big] \|w\|^2 + k^{-2(\theta-1/2)} + k^{-2(\delta+\nu)} \\
\quad \left[-\frac{1}{2} + \left(k^{-1/2}\|T_2\|\,\|P_1\|\,\|\mathcal{Y}_{\text{ref}} - \mathcal{D}_{\mathcal{Y}}\|\right)^2 \right] \|v\|^2 \\
\quad + 2\frac{\dot k}{k}\left[(\vartheta - \delta) w^\top T_1 w - \theta v^\top T_2 v\right] & ,\ \text{if } \varrho = \nu.
\end{cases}
\end{aligned}$$

STEP 4: *Assertions (i) and (iii) are shown.*
Seeking a contradiction to Assertion (iii), assume that $k(\cdot)$ is unbounded on $[0,\omega)$. In view of (2.47), the following inequalities hold

$$\delta \ge 0, \qquad \theta > 0, \qquad -\delta + \theta - \nu < \tfrac{1}{2}, \qquad \delta - \theta < \tfrac{1}{2}, \qquad \vartheta - \delta \le 0, \qquad \delta - \theta \le \theta$$

and thus, for $\mu := \frac{1}{4}\min\{\|T_1\|^{-1},\ \|T_2\|^{-1}\}$, Step 3 implies that (the argument t is omitted)

$$\dot V \overset{\exists\, t_1\in[0,\omega)\ \forall\, t\in[t_1,\omega)}{\le} \begin{cases} -\frac{k}{4}\|w\|^2 - \frac{1}{4}\|v\|^2 + 2\frac{\dot k}{k}(\vartheta-\delta)V + 2k^{-2(\theta-1/2)} \\ \qquad\qquad\qquad\qquad\qquad\qquad ,\ \text{if } \varrho \in \{1,...,\nu-1\} \\ -\frac{1}{4}\frac{\Gamma p_\nu k^2}{1+\Gamma p_\nu k}\|w\|^2 - \frac{1}{4}\|v\|^2 + 2\frac{\dot k}{k}(\vartheta-\delta)V + 2k^{-2(\theta-1/2)} \\ \qquad\qquad\qquad\qquad\qquad\qquad ,\ \text{if } \varrho = \nu \end{cases}$$

$$\overset{\exists\, t_2\in[t_1,\omega)\ \forall\, t\in[t_2,\omega)}{\le} \begin{cases} -\frac{1}{4}\|w\|^2 - \frac{1}{4}\|v\|^2 + 2\frac{\dot k}{k}(\vartheta-\delta)V + 2k^{-2(\theta-1/2)} \\ \qquad\qquad\qquad\qquad\qquad\qquad ,\ \text{if } \varrho \in \{1,...,\nu-1\} \\ -\frac{1}{4}\frac{p_\nu k}{p_\nu k}\|w\|^2 - \frac{1}{4}\|v\|^2 + 2\frac{\dot k}{k}(\vartheta-\delta)V + 2k^{-2(\theta-1/2)} \\ \qquad\qquad\qquad\qquad\qquad\qquad ,\ \text{if } \varrho = \nu \end{cases}$$

$$\le \quad -\mu V + 2\frac{\dot k}{k}(\vartheta-\delta)V + 2k^{-2(\theta-1/2)}. \tag{2.51}$$

For

$$\sigma : [0,\omega) \to \mathbb{R}_{\ge 0}, \quad t \mapsto \sigma(t) := \frac{1}{\sqrt{\|T_1^{-1}\|}}\frac{\lambda}{2}k(t)^{-\delta}$$

and

$$D_\sigma : [0,\omega) \to \mathbb{R}_{\ge 0}, \quad t \mapsto D_\sigma(t) := \max\{\sqrt{V(t)} - \sigma(t),\ 0\} \tag{2.52}$$

it follows that

$$\begin{aligned} \forall\, t\in[0,\omega):\quad k(t)^\delta\|w(t)\| &= k(t)^\delta\sqrt{\|w(t)\|^2} \le k(t)^\delta\sqrt{\|T_1^{-1}\|}\sqrt{V(t)} \\ &= k(t)^\delta\sqrt{\|T_1^{-1}\|}\sqrt{V(t)} + \frac{\lambda}{2} - \frac{k(t)^\delta\sqrt{\|T_1^{-1}\|}}{k(t)^\delta\sqrt{\|T_1^{-1}\|}}\frac{\lambda}{2} \\ &= k(t)^\delta\sqrt{\|T_1^{-1}\|}\left[\sqrt{V(t)} - \sigma(t)\right] + \frac{\lambda}{2} \\ &\le k(t)^\delta\sqrt{\|T_1^{-1}\|}D_\sigma(t) + \frac{\lambda}{2} \qquad (2.53) \\ &\le k(t)^\delta\sqrt{\|T_1^{-1}\|}D_\sigma(t) + \lambda. \qquad (2.54) \end{aligned}$$

It has to be noted that

- if $t\in[0,\omega)$ such that $\sqrt{V(t)} \le 2\sigma(t)$, then (2.53) implies that $k(t)^\delta\|w(t)\| \le \lambda$ which yields $\dot k(t) = k(t)^{-2\gamma}d_\lambda(k(t)^\delta\|w(t)\|)^2 = 0$.
- if $t\in[0,\omega)$ such that $\sqrt{V(t)} > 2\sigma(t)$, then it holds

$$\frac{D_\sigma(t)}{\sqrt{V(t)}}\frac{\dot k(t)}{k(t)}\left[(\vartheta-\delta)V(t) + \delta\sigma\sqrt{V(t)}\right] < \frac{\dot k(t)}{k(t)}D_\sigma(t)\sqrt{V(t)}\left[\vartheta - \frac{\delta}{2}\right] \le 0,$$

where the implication

$$(2.47) \qquad \Rightarrow \qquad \vartheta - \frac{\delta}{2} \le 0$$

is used.

Thus it follows that

$$\forall\, t \in [0,\omega): \quad \frac{D_\sigma(t)}{\sqrt{V(t)}} \frac{\dot{k}(t)}{k(t)} \left[(\vartheta - \delta) V(t) + \delta\sigma \sqrt{V(t)} \right] \le 0.$$

Moreover, (2.47) implies $\delta < \theta - \frac{1}{2}$. Since T_1 and T_2 are positive definite, together with (2.52), $D_\sigma(\cdot)^2$ is differentiable and thus, for

$$W : [0,\omega) \to \mathbb{R}, \quad t \mapsto W(t) := \frac{1}{2} D_\sigma(t)^2,$$

the derivative of $W(\cdot)$ along the trajectory of (2.49) yields (the argument t is omitted)

$$\begin{aligned}
\dot{W} \quad &= \quad \frac{D_\sigma}{2\sqrt{V}} \left[\dot{V} - 2\dot{\sigma}\sqrt{V} \right] \\
&\overset{(2.51)}{\underset{\forall\, t \in [t_2,\omega)}{\le}} \quad \frac{D_\sigma}{2\sqrt{V}} \left[-\mu V + 2k^{-2(\theta - 1/2)} \right] + \frac{D_\sigma}{\sqrt{V}} \frac{\dot{k}}{k} \left[(\vartheta - \delta) V + \delta\sigma\sqrt{V} \right] \\
&\le \quad \frac{D_\sigma}{2\sqrt{V}} \left[-\mu V + 2k^{-2(\theta - 1/2)} \right] \\
&\overset{\exists\, t_3 \in [t_2,\omega)\ \forall\, t \in [t_3,\omega)}{\le} \quad \frac{D_\sigma}{2\sqrt{V}} \left[-\mu V + \mu\sigma^2 \right] \\
&\le \quad -\mu \frac{1}{2} D_\sigma^2 = -\mu W
\end{aligned}$$

which yields

$$\forall\, t \in [t_3,\omega): \quad W(t) \le e^{-\mu(t-t_3)} W(t_3).$$

In view of (2.47) and (2.54), it follows that (the argument t is omitted)

$$\begin{aligned}
\dot{k} = k^{-2\gamma} d_\lambda(k^\delta \|w\|)^2 \le k^{-2\gamma} \left(k^\delta \|w\| - \lambda \right)^2 &\overset{(2.54)}{\le} k^{-2\gamma} \left(k^\delta \sqrt{\|T_1^{-1}\|} D_\sigma \right)^2 \\
= k^{-2(\gamma - \delta)} 2 \|T_1^{-1}\| W &\overset{\exists\, t_4 \in [t_3,\omega)\ \forall\, t \in [t_4,\omega)}{\le} 2\|T_1^{-1}\| W
\end{aligned}$$

which gives the contradiction, for all $t \in [t_4,\omega)$

$$k(t) - k(t_4) = \int_{t_4}^{t} \dot{k}(\tau)\, \mathrm{d}\tau \le 2\|T_1^{-1}\| \int_{t_4}^{t} W(\tau)\, \mathrm{d}\tau \le \frac{2}{\mu} \|T_1^{-1}\| e^{-\mu(t-t_4)} e^{\mu t_3} W(t_3) < \infty.$$

Boundedness of $k(\cdot)$, together with (2.46), implies that there exists $m_1(\cdot)$, $m_2(\cdot) \in (\mathcal{C} \cap \mathcal{L}^1)(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ such that

$$\forall\, (\mathcal{Y}, z, k) \in \mathcal{D}: \quad \|f(t, \mathcal{E}, z, k)\| \leq m_1(t)\|(\mathcal{E}, z, k)\| + m_2(t)$$

which gives $\omega = \infty$ (see [1, Prop. II.7.8]).

STEP 5: *$z(\cdot; 0, Nx^0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n-\varrho})$ is shown.*
Denote $k_\infty := \lim_{t\to\infty} k(t) < \infty$. Then it follows that

$$\begin{aligned}\forall\, t \geq 0: \quad k_\infty^{-2\gamma} \int_0^t d_\lambda(k(\tau)^\delta \|w(\tau)\|)^2 \, \mathrm{d}\tau \;&\leq \int_0^t k(\tau)^{-2\gamma} d_\lambda(k(\tau)^\delta \|w(\tau)\|)^2 \, \mathrm{d}\tau \\ &= \int_0^t \dot{k}(\tau) \, \mathrm{d}\tau \;\leq k_\infty - k_0 < \infty\end{aligned}$$

and therefore

$$d_\lambda(k(\cdot)^\delta \|w(\cdot)\|) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}). \tag{2.55}$$

Define

$$D_2 : \mathbb{R}^\nu \to \mathbb{R}^\nu, \quad v \mapsto D_2(v) := \begin{cases} d_\lambda(\|v\|)\frac{v}{\|v\|} & , \text{ if } \|v\| \geq \lambda \\ 0 & , \text{ if } \|v\| \leq \lambda \end{cases}$$

$$D_\infty : \mathbb{R}^\nu \to \mathbb{R}^\nu, \quad v \mapsto D_\infty(v) := \begin{cases} \left[1 - \frac{d_\lambda(\|v\|)}{\|v\|}\right] v & , \text{ if } \|v\| \geq \lambda \\ v & , \text{ if } \|v\| \leq \lambda \end{cases}$$

which yields

$$\begin{aligned}&\left[t \mapsto D_2(k(t)^\delta w(t))\right] \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^\nu), \qquad \left[t \mapsto D_\infty(k(t)^\delta w(t))\right] \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^\nu) \\ &k(\cdot)^\delta w(\cdot) = D_2(k(\cdot)^\delta w(\cdot)) + D_\infty(k(\cdot)^\delta w(\cdot)).\end{aligned}$$

Since the matrix Q is Hurwitz and $\mathcal{Y}_{\text{ref}}(\cdot)$ and $\mathcal{D}_{\mathcal{Y}}(\cdot)$ are bounded, it follows that (see (2.46))

$$\dot{z}(t) = Qz(t) + P_1 e_1^\top D_2(k(t)^\delta w(t)) + P_1 e_1^\top \left[D_\infty(k(t)^\delta w(t)) + \mathcal{Y}_{\text{ref}}(t) - \mathcal{D}_{\mathcal{Y}}(t)\right]$$

which yields

$$z(\cdot; 0, Nx^0) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n-\varrho})$$

(see [37, Lem. 2.2]).

STEP 6: *Assertion (ii) is shown.*

Define

$$\begin{aligned} r(k) &:= \begin{cases} \frac{1}{\Gamma p_\nu k} e_\nu \left[R_1, ..., R_\varrho, -1, 0, ..., 0\right] &, \text{ if } \varrho \in \{1, ..., \nu-1\} \\ \frac{1}{1+\Gamma p_\nu k} e_\nu \left[R_1, ..., R_\varrho\right] &, \text{ if } \varrho = \nu \end{cases} \\ \kappa(k) &:= \begin{cases} \frac{1}{\Gamma p_\nu k} &, \text{ if } \varrho \in \{1, ..., \nu-1\} \\ \frac{1}{1+\Gamma p_\nu k} &, \text{ if } \varrho = \nu \end{cases} \\ p &:= \left[p_0, ..., p_{\nu-1}\right]^\top \in \mathbb{R}^\nu \end{aligned}$$

with $r(k) \in \mathbb{R}(k)^{\nu\times\nu}$ and $\kappa(k) \in \mathbb{R}(k)$. It is obvious that

$$\text{rk} \begin{bmatrix} e_1^\top \\ e_1^\top (J + r(k_\infty)) \\ \vdots \\ e_1^\top (J + r(k_\infty))^{\nu-1} \end{bmatrix} = \text{rk } I_\nu = \nu$$

and therefore $\left(J + r(k_\infty), e_1^\top\right)$ is observable and thus, there exists $L \in \mathbb{R}^\nu$ such that $(J + r(k_\infty) - Le_1^\top) \in \mathbb{R}^{\nu\times\nu}$ is Hurwitz (see [78, p. 246, Th. 14.9]). Then, the first equation of (2.46) can be written as

$$\begin{aligned} \dot{\mathcal{E}}(t) &= (J + r(k(t)))\mathcal{E}(t) - \kappa(k(t))\Gamma k(t)^{\nu+1} k(t)^\delta p^\top w(t) e_\nu + \mathcal{S}(k(t))z(t) \\ &\quad + \mathcal{D}_{u,k} + \mathcal{B}(k(t))\overline{\mathcal{Y}}_{\text{ref,d}} \qquad (2.56) \\ &= (J + r(k_\infty) - Le_1^\top)\mathcal{E}(t) + (r(k(t)) \\ &\quad - r(k_\infty))\mathcal{E}(t) - \kappa(k(t))\Gamma k(t)^{\nu+1} p^\top D_2(k(t)^\delta w(t)) e_\nu + \Big[\mathcal{D}_{u,k} + \mathcal{B}(k(t))\overline{\mathcal{Y}}_{\text{ref,d}} \\ &\quad - \kappa(k(t))\Gamma k(t)^{\nu+1} p^\top D_\infty(k(t)^\delta w(t)) e_\nu + \mathcal{S}(k(t))z(t)\Big]. \end{aligned}$$

Since $\lim_{t\to\infty} (r(k(t)) - r(k_\infty)) = 0$, [37, Lem. 2.2] yields $\mathcal{E}(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^\nu)$ and, in view of (2.56), $\dot{\mathcal{E}}(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^\nu)$. Thus, Proposition 1.1.7 yields Assertion (ii).

STEP 7: *Assertions (iv) and (v) are shown.*
Since Assertion (v) follows from Assertion (iv), it remains to show Assertion (iv). Boundedness of $k(\cdot)$, $\mathcal{Y}(\cdot)$ (see Step 6) and $\mathcal{Y}_{\text{ref}}(\cdot)$ imply boundedness of $\dot{k}(\cdot)$ (see (2.34)) and $w(\cdot)$ which implies, by (2.49), boundedness of $\dot{w}(\cdot)$. As

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left[k(t)^{-\gamma} d_\lambda(k(t)^\delta \|w(t)\|)\right]^2 &= 2k(t)^{-2\gamma} d_\lambda(k(t)^\delta \|w(t)\|) \Bigg[k(t)^\delta \frac{\langle w(t), \dot{w}(t)\rangle}{\|w(t)\|} \\ &\quad + \delta k(t)^\delta \frac{\dot{k}(t)}{k(t)} \|w(t)\| - \gamma \frac{\dot{k}(t)}{k(t)} d_\lambda(k(t)^\delta \|w(t)\|)\Bigg], \end{aligned}$$

$\left[t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\left[k(t)^{-\gamma} d_\lambda(k(t)^\delta \|w(t)\|)\right]^2\right] \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ which implies that

$$k(\cdot)^{-2\gamma} d_\lambda(k(\cdot)^\delta \|w(\cdot)\|)^2$$

is uniformly continuous. By (2.55), $k(\cdot)^{-2\gamma} d_\lambda(k(\cdot)^\delta \|w(\cdot)\|)^2 \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ and Barbălat's Lemma (see [52, Lem. 8.2]) yields

$$\lim_{t\to\infty}\left[k(t)^{-2\gamma} d_\lambda(k(t)^\delta \|w(t)\|)^2\right] = 0$$

which shows Assertion (iv). This completes the proof. □

Proof of Theorem 2.3.10:
The proof uses the notation of Proposition 1.1.7, (2.24), (2.25) and of Step 1 of the proof of Theorem 2.3.7 and 2.3.8.

STEP 1: *Some notation is introduced.*
Note that $\varrho \in \{1, ..., \nu - 1\}$. Define $\mathcal{E}(\cdot)$ as in Step 1 of the proof of Theorem 2.3.8 and

$$\begin{aligned} \mathcal{B}(k) &:= \frac{1}{p_\nu k}\Gamma^{-1} E\,[R_1, ..., R_\varrho, -I_m, 0, ..., 0] \\ \mathcal{D}_{u,k}(\cdot) &:= \frac{1}{p_\nu k} E d_u(\cdot) \\ \overline{\mathcal{Y}}_{\text{ref,d}}(\cdot) &:= \mathcal{Y}_{\text{ref}}(\cdot) - \mathcal{D}_{\mathcal{Y}}(\cdot)^\top \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^\nu) \end{aligned}$$

with $\mathcal{B}(k) \in \mathbb{R}(k)^{\nu\times\nu}$.

STEP 2: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (2.34) *or, equivalently,* (1.13)*,* (2.34) *are shown.*
Define, for the relatively open set $\mathcal{D} := [0, \infty) \times \mathbb{R}^{\nu m} \times \mathbb{R}^{n-\varrho m} \times [k_0, \infty)$,

$$\begin{aligned} &f : \mathcal{D} \to \mathbb{R}^\nu \times \mathbb{R}^{n-\varrho} \times \mathbb{R}_{>0}, \\ &(t, \xi, \mu, \eta) \mapsto \begin{pmatrix} [\mathcal{A}_{\eta,\eta} + \mathcal{R}(\eta)]\,\xi + \mathcal{S}(\eta)\mu + \mathcal{B}(\eta)\overline{\mathcal{Y}}_{\text{ref,d}}(t) + \mathcal{D}_{u,k}(t) \\ P_1[I_m, 0, ..., 0]\xi + Q\mu + P_1[I_m, 0, ..., 0]\,(\mathcal{Y}_{\text{ref}}(t) - \mathcal{D}_{\mathcal{Y}}(t)) \\ \eta^{-2\gamma} d_\lambda(\|\xi_1, \eta^{-1}\xi_2, ..., \eta^{-(\nu-1)}\xi_\nu\|)^2 \end{pmatrix}. \end{aligned}$$

Then, the initial value problem (1.13), (2.34) has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathcal{E}(t)\\ z(t)\\ k(t)\end{pmatrix} = f\left(t, \mathcal{E}(t), z(t), k(t)\right), \qquad \begin{pmatrix}\mathcal{E}(0)\\ z(0)\\ k(0)\end{pmatrix} = \begin{pmatrix}\left[(\mathcal{C}x^0 + \mathcal{D}y(0) - \mathcal{Y}_{\text{ref}}(0))^\top,\ v^\top\right]\\ Nx^0\\ k_0\end{pmatrix}$$

with arbitrary $v \in \mathbb{R}^{m\nu-\varrho}$. Existence and uniqueness of the solution $(\mathcal{E}, z, k) : [0, \omega) \to \mathbb{R}^{m\nu} \times \mathbb{R}^{n-\varrho m} \times \mathbb{R}_{>0}$, $0 < \omega \leq \infty$, follow similarly to Step 2 in the proof of Theorem 2.3.4.

STEP 3: *Assertions (i) and (iii) are shown.*
Consider the coordinate transformation (2.48) and the Ljapunov function candidate (2.50) of the proof of Theorem 2.3.8. It follows similarly, for all $t \in [0, \omega)$,

$$\begin{aligned}
\dot{V} \ \leq\ & \Bigg[-k + 2\|T_1\|\,\|\widehat{\mathcal{R}}(k)\| + \left(2k^{-\delta+\theta}\frac{k^{-(\nu-1)}}{\Gamma p_\nu k}\|T_1\|\,\|S\|\right)^2 \\
& + \left(\frac{1}{\Gamma p_\nu}\|T_1\|\ \left[\|\mathcal{B}(1)\|\,\|\overline{\mathcal{Y}}_{\text{ref,d}}\| + \|d_u\|\right]\right)^2 + \left(2k^{\delta-\theta}\|T_2\|\,\|P_1\|\right)^2\Bigg]\|w\|^2 \\
& + \left[-\frac{1}{2} + \left(k^{-1/2}\|T_2\|\,\|P_1\|\,\|\mathcal{Y}_{\text{ref}} - \mathcal{D}y\|\right)^2\right]\|v\|^2 + \\
& 2\frac{\dot{k}}{k}\left[(\vartheta-\delta)w^\top T_1 w - \theta v^\top T_2 v\right] + k^{-2(\theta-1/2)} + k^{-2(\delta+\nu)}.
\end{aligned}$$

All inequalities derived in Step 4 of the proof of Theorem 2.3.8 hold true which shows the Assertions (i) and (iii). The details are omitted.

STEP 4: *Assertions (ii), (iv) and (v) are shown.*
With minor modifications the Step 5 - 7 of the proof of Theorem 2.3.8 go through and are omitted. This completes the proof. □

2.6 Notes and references

Many adaptive control methods rely on parameter estimation algorithms (see [6, 48]). Adaptive high-gain feedback can stabilize multi input, multi output systems (1.1) which have stable zero dynamics and strict relative degree one (see [15, 89]). Generally, high-gain methods can stabilize systems with relative degree one. However, [57] presents a high-gain dynamic compensation which guarantee output convergence of single input, single output systems with stable zero dynamics and arbitrary but known

relative degree. This approach is surprising since classical root locus is not high-gain stable for systems with relative degree exceeding two. Moreover, in [29, 30] it is shown that the approach of [57] can fail when the relative degree of the system exceeds four. Furthermore, in [29, 30] the Fibonacci series is used to construct a direct adaptive stabilization algorithm for systems with stable zero dynamics and unknown but bounded relative degree. Many high-gain adaptive methods are restricted to the stabilization problem. λ-tracking has been introduced in [39]. High-gain adaptive controllers that utilize high-gain observers have been used for λ-tracking in [10, 90], where the relative degree must be known. Knowledge of the relative degree is required for developing the adaption law (see [48]) and the observer (see [10, 90]). Model reference adaptive control methods with relaxed assumptions on the relative degree of the system are considered in [64, 86]. In [64] a model reference adaptive controller is proposed for systems with relative degree one or two. However, the method is restricted to stabilization. As in [86], a model reference adaptive controller is proposed for systems with upper and lower bounded relative degree. This controller requires that an adaptive parameter lies inside a known convex set and for large uncertainty in the relative degree, calculating the convex set can be difficult.
The attention of the present chapter lies on the unknown relative degree.
A first approach to achieve stabilization for single input, single output systems of unknown relative degree is due to [57], but a counterexample to the main approach is presented in [29, 30]. [29, 30] solve the problem of adaptive stabilization of linear systems with higher relative degree. Further results for systems with unknown relative degree can be found in [64, 86]. The main results of [29, 30, 64, 86] are proven in the frequency domain which is a problem if the high-frequency gain is generated adaptively. Especially, the adaptive results of [29, 30] are crucial – the proofs are a mix of frequency domain and time domain arguments (see [30, Th. 7.1]).
The results of Section 2.3 show that the λ-tracker can be achieved to systems with unknown but bounded relative degree. The main idea of the proofs of Theorem 2.3.4, 2.3.7, 2.3.8 and 2.3.10 bases on the proof of [10] which considers linear systems 1.1 with known relative degree. The presented proof of [10] is incomplete and has many mistakes. In this chapter, these errors are canceled and the results are generalized to linear systems 1.1 with unknown but bounded relative degree.
The proof of Proposition 2.3.2 is new. The result cannot be found in the literature.

Although important results, the λ-tracker has two drawbacks

- the tracking error will only be achieved asymptotically and
- though bounded, the gain $k(\cdot)$ increases which is easy to see since the right-hand side of the adaptive law $\dot{k}(\cdot)$ is nonnegative.

In the following chapters funnel control is introduced which overcomes both drawbacks. Moreover, Chapter 4 shows that the funnel controller can be achieved to systems with unknown relative degree $\varrho = 1, 2$.

3 Funnel control and saturation for systems with relative degree 1

Tracking – by the system output – of a reference signal (assumed bounded with essentially bounded derivative) is considered in the context of linear m-input, m-output systems (1.1) in the presence of input saturation. The system is assumed to have strict relative degree one with positive high-frequency gain and stable zero dynamics. Prespecified is a parameterized performance funnel. The tracking error, or alternatively each component, is required to evolve within the funnel: transient and asymptotic behaviour of the tracking error is influenced through choice of parameter values which define the funnel. The proposed control structure is a saturating error feedback wherein the gain function evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data (A, B, C) and the saturation bound, the funnel data, the reference signal $y_{\mathrm{ref}}(\cdot)$, and the initial state x^0) is presented under which the tracking objective is achieved, whilst maintaining boundedness of all signals.
In Section 3.3 the performance funnel is introduced with a parameter $\tau \geq 0$. This is required for the funnel control results in Chapter 5.5 by invoking the framework of positive Volterra-Stieltjes systems (1.2). In Section 3.4 new funnel control results for input saturation are presented.

3.1 Introduction

The prototypical example for a system class – rather than a single system – is that of linear m-input, m-output systems with relative degree one, positive high-frequency gain and stable zero dynamics, i.e. minimum phase. It is known that the proportional output feedback

$$u(t) = -k\, y(t) \tag{3.1}$$

applied to (1.1) yields a closed-loop system which is stable if $k > 0$ is sufficiently large (see [36, Lem. 2.2.7]). The high-gain property of the system class is used in adaptive control which means that the feedback (3.1) becomes time-varying: the simple output feedback

$$u(t) = -k(t)\, y(t) \tag{3.2}$$

stabilizes each system belonging to the above class provided $k(\cdot)$ is appropriately generated: e.g. by the differential equation

$$\dot{k}(t) = \|y(t)\|^2, \quad k(0) = k_0 \in \mathbb{R} \tag{3.3}$$

or variants thereof (see the survey [42] and references therein). If (3.2), (3.3) is applied to (1.1), then, for any initial data $(x^0, k_0) \in \mathbb{R}^n \times \mathbb{R}$, the closed-loop system satisfies

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R} \tag{3.4}$$

(see Chapter 2). It is obviously that the gain $k(\cdot)$ increases as long as $\|y(\cdot)\|$ is large until it is sufficiently large such that the closed-loop system is asymptotically stable. The two major drawbacks of the latter strategy (and its variants) are

(i) albeit bounded, the gain $k(\cdot)$ is monotonically increasing and

(ii) whilst asymptotic performance is guaranteed, transient behaviour is not generally taken into account (an exception being the contribution [59], wherein the issue of prescribed transient behaviour is successfully addressed).

A fundamentally different approach – so called ‘funnel control’ – was introduced in [43] in the context of output tracking: this control ensures prespecified transient behaviour of the tracking error, has a non-monotone gain, is simpler than the above adaptive controller (insofar as the gain is not dynamically generated), and does not invoke any internal model.

3.2 Motivation

As motivation, consider the simple scalar linear system

$$\dot{y}(t) = a\, y(t) + b\, v(t), \qquad a \in \mathbb{R}, \;\; b > 0, \quad y(0) = y^0 \in \mathbb{R}. \tag{3.5}$$

The *control objective* is output tracking, of a (suitably regular) reference signal $y_{\mathrm{ref}}(\cdot)$, with prescribed transient and asymptotic behavior in the sense that, for some given function $\psi : [0, \infty) \to [\lambda, \infty)$, $\lambda > 0$, the tracking error is bounded by $\psi(\cdot)$:

$$\forall\, t \geq 0 : \quad |y(t) - r(t)| < \psi(t).$$

For example, if $\psi(\cdot)$ is given by $\psi(t) = \max\{1 - \ell t,\ \lambda\}$ with $\ell > 0$ and $\lambda \in (0, 1)$ (and so $\psi(\cdot)$ is globally Lipschitz, with Lipschitz constant ℓ), then attainment of the tracking objective implies that a prescribed tracking accuracy, quantified by $\lambda > 0$, is achieved in prescribed time $t^* = \frac{1-\lambda}{\ell}$: specifically, $|y(t) - r(t)| < \lambda$ for all $t \geq t^*$.
In the general case, if $\psi(\cdot)$ is globally Lipschitz and bounded away from zero, and the reference signal $y_{\text{ref}}(\cdot)$ is a bounded absolutely continuous function with essentially bounded derivative, then it is known (see [43]) that the tracking objective is achieved by the following simple strategy

$$u(t) = -k(t)[y(t) - y_{\text{ref}}(t)], \;\; k(t) := \frac{1}{\psi(t) - |y(t) - y_{\text{ref}}(t)|} \tag{3.6}$$

if, and only if, the feasibility condition

$$|y^0 - y_{\text{ref}}(0)| < \psi(0) \tag{3.7}$$

holds. Moreover, the gain $k(\cdot)$, and hence the control $v(\cdot)$, is bounded. The controller ensures that all signals and states of the closed-loop system are bounded.

In many practical applications the input $v(\cdot)$ may be subject to certain bounds, i.e. there is some maximal bound $\widehat{u} > 0$ such that $|v(t)| \leq \widehat{u}$ is required for all $t \geq 0$. Consider again the scalar system (3.5), with the same control objective and the control strategy (3.6), but now with saturation in the input channel, i.e.

$$\dot{y}(t) = a\, y(t) + b\ \text{sat}_{\widehat{\text{u}}}(v(t)), \qquad \widehat{u} > 0, \quad y(0) = y^0. \tag{3.8}$$

In other words the funnel controller (3.6) is replaced by

$$u(t) = \text{sat}_{\widehat{\text{u}}}(-k(t)e(t))$$

with $e(\cdot)$ and $k(\cdot)$ as in (3.6).

In the scalar case the *saturation function* has the form

$$\text{sat}_{\widehat{\text{u}}} : \mathbb{R} \to [-\widehat{u}, \widehat{u}], \quad s \mapsto \text{sat}_{\widehat{\text{u}}}(s) := \begin{cases} -\widehat{u} & s, \ \leq -\widehat{u} \\ s & , \ |s| < \widehat{u} \\ \widehat{u} & , \ s \geq \widehat{u} \end{cases}$$

Again, $|y^0 - r(0)| < \psi(0)$ is a necessary condition for feasibility. However, a moment's reflection confirms that the latter condition is not sufficient: the question of feasibility of the tracking objective in the presence of input saturation is delicate and inevitably involves addressing the interplay between the plant data (a, b, y^0), the reference signal $y_{\text{ref}}(\cdot)$, the function $\psi(\cdot)$ and the saturation bound $\widehat{u}$.
For example, if $a > 0$, then it is readily seen that $b\widehat{u} \geq a|y^0|$ is a necessary condition for feasibility. Moreover, it is clear that, for feasibility, the saturation level $\widehat{u}$ should also be commensurate with the magnitude of the reference signal $y_{\text{ref}}(\cdot)$ and its derivative $\dot{y}_{\text{ref}}(\cdot)$.
To illustrate the interplay between $\widehat{u}$ and the function $\psi(\cdot)$, consider the case wherein $a = 0$ and $y_{\text{ref}}(\cdot) = 0$. Assume feasibility of the tracking objective. Then it follows that

$$1 - \lambda = \psi(0) - \psi(t^*) < \psi(0) - y(t^*) = 1 - y^0 + y^0 - y(t^*) \leq 1 - y^0 + t^* b\widehat{u}$$

which must hold for all $|y^0| < 1$ and hence, $1 - \lambda \leq t^* b\widehat{u}$. Therefore

$$b\widehat{u} \geq \lambda \tag{3.9}$$

is a necessary condition for feasibility. It has to be noted that ℓ is the Lipschitz constant of $\psi(\cdot)$.

The purpose of Subsection 3.4 is to extend the above investigations to a more general context of m-input $u(\cdot)$, m-output $y(\cdot)$, n-dimensional linear systems (1.1) subject to input saturation. The system (1.1) is assumed

(i) to have strict relative degree one with positive high-frequency gain (i.e., in view of Definition 1.1.5, $CB > 0$) and

(ii) to satisfy a minimum-phase condition (i.e. (A2)).

Two scenarios are investigated: first, the saturation constraint is *Euclidean* in the sense that, for some $\widehat{u} > 0$ the input $u(\cdot)$ is required to satisfy the constraint

$$\forall\, t \geq 0: \quad \|u(t)\| \leq \widehat{u}. \tag{3.10}$$

Second, the saturation constraint is imposed *componentwise* in the sense that, for some $\widehat{u} = (\widehat{u}_1, ..., \widehat{u}_m)$, $\widehat{u}_i > 0$, the input $u(\cdot) = (u_1(\cdot), ..., u_m(\cdot))$ is required to satisfy

$$\forall\, t \geq 0\, \forall\, i \in \{1, ..., m\}: \quad |u_i(t)| \leq \widehat{u}_i. \tag{3.11}$$

Restrict momentarily to the single input, single output case (in which case (3.10) and (3.11) are equivalent).

The proposed control structure is a saturating error feedback of the form $u(t) = -\operatorname{sat}_{\widehat{u}}(k(t)e(t))$ wherein the gain function $t \mapsto k(t) := \frac{1}{\psi(t)-|e(t)|}$ evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data (A, B, C) and $\widehat{u}$, the funnel data $\psi(\cdot)$, the reference signal $y_{\mathrm{ref}}(\cdot)$, and the initial state x^0) is presented under which the tracking objective is achieved, whilst maintaining boundedness of the state $x(\cdot)$ and gain function $k(\cdot)$.

In the context of the motivating scalar system (3.8), the main result of the Subsection 3.4 translates into the following: if

$$|y^0 - r(0)| < \psi(0) \qquad \text{and} \qquad b\,\widehat{u} \geq |a|\,[\|\psi\|_\infty + \|r\|_\infty] + \|\dot{r}\|_\infty + \ell, \tag{3.12}$$

then the simple control strategy

$$u(t) = \operatorname{sat}_{\widehat{u}}(v(t)) = -\operatorname{sat}_{\widehat{u}}(k(t)e(t)), \;\; k(t) = \frac{1}{\psi(t) - |e(t)|}, \;\; e(t) = y(t) - r(t),$$

ensures attainment of the tracking objective (and, moreover, the gain function $k(\cdot)$ is bounded). Furthermore, if the first inequality in (3.12) is replaced by

$$|y^0 - r(0)| < \psi(0)\left(\frac{\widehat{u}}{1+\widehat{u}}\right),$$

then input saturation does not occur and so the control strategy coincides with (3.6).

Restrict momentarily to the single input, single output case (in which case (3.10) and (3.11) are equivalent). The *control objective* is output tracking: determine a feedback structure which ensures that, for a given reference signal $y_{\mathrm{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, the output tracking error $e(\cdot) = y(\cdot) - y_{\mathrm{ref}}(\cdot)$ evolve within the funnel (i.e. graph$(e) \subset \{(t,\xi) \mid |\xi| < \psi(t)\}$) determined by some suitable function $\psi(\cdot)$. Transient and asymptotic behaviour of the tracking error is influenced through the choice of the function $\psi(\cdot)$ which defines the funnel.

3.3 Performance funnel, funnel controller and control objective

In the following section funnel control is introduced. As for λ-tracking in the previous chapter, multi input, multi output systems (1.1) satisfying the classical assumptions of adaptive control, i.e. having strict relative degree one, positive high-frequency gain and stable zero dynamics (i.e. (A2)), are considered. In Chapter 5, the system class (1.1) is generalized to the class (1.2).

A central ingredient of the approach is the concept of a performance funnel within which the tracking error $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$, where $y_{\text{ref}}(\cdot)$ is a reference signal, is required to evolve. The concept of funnel control was originally introduced by [43] not only for the class of linear systems (1.1) with CB positive definite, i.e. for systems (1.1) with strict relative degree one. It had been applied successfully in experiments on electric drive systems (see [47]).

3.3.1 Funnel controller

Consider first only output stabilization, i.e. $y_{\text{ref}}(\cdot) \equiv 0$. The simple form (3.2) is preserved and the adaptive gain (3.3) is replaced by

$$k(t) = \frac{1}{\psi(t) - \|y(t)\|}, \tag{3.13}$$

for some prespecified function $\psi : \mathbb{R}_{\geq 0} \to [\lambda, \infty)$, $\lambda > 0$ (see Figure 3.1). If (3.2), (3.13) is applied to (1.1), then for any initial data $x^0 \in \mathbb{R}^n$ such that the initial output is in the funnel, i.e. $\|y(0)\| < \psi(0)$, the closed-loop system has a unique solution on $\mathbb{R}_{\geq 0}$, the gain $k(\cdot)$ is bounded and the output $y(\cdot)$ evolves within the funnel, i.e. $\|y(t)\| < \psi(t)$ for all $t \geq 0$ (see [43]).
Loosely speaking, funnel control exploits the high-gain property of a system (1.1) designing a proportional feedback $u(t) = -k(t)e(t)$ in such a way that $k(t)$ (as in (3.13)) becomes large if $\|e(t)\|$ approaches a prespecified performance funnel boundary $\psi(\cdot)$, thereby precluding contact with the funnel boundary.
As seen in (3.4), the output tends to zero as $t \to \infty$ in adaptive control. The funnel controller only guarantees that

$$\limsup_{t\to\infty} \|y(t)\| < \lambda,$$

where $\lambda > 0$ is prespecified and can be chosen arbitrarily small.

3.3.2 Performance funnels

In the context of [43] the family of functions, determined by $\tau \geq 0$,

$$\mathcal{S}_1(\tau) := \left\{ \varphi : [\tau, \infty) \to \mathbb{R} \;\middle|\; \begin{array}{l} (1)\ \varphi(\cdot) \text{ is locally absolutely continuous}, \forall\, t > \tau : \varphi(t) > 0 \\ (2)\ \forall\, \varepsilon > 0 : \left[\exists\, \lambda > 0 : \dfrac{1}{\varphi(\cdot)} \in \mathcal{L}^\infty([\tau+\varepsilon, \infty), [\lambda, \infty)) \right] \\ \qquad \left[\exists\, \ell \geq 0 \text{ for a.a. } t \geq \tau + \varepsilon : \left| \dfrac{\dot{\varphi}(t)}{\varphi(t)^2} \right| \leq \ell \right] \end{array} \right\}$$

is introduced. In [43] it is not assumed that $\varphi(\cdot)$ satisfies a Lipschitz condition as given in $\mathcal{S}_1(\tau)$. This is a weak assumption and introduced for technical reasons.
The family $\mathcal{S}_1(0)$ with $\tau = 0$ was originally introduced in [43]. By invoking the framework of positive Volterra-Stieltjes systems (1.2) in Chapter 5, the parameter τ is important and the class $\mathcal{S}_1(0)$ is generalized to the class $\mathcal{S}_1(\tau)$ which can be interpreted as a shift of $\mathcal{S}_1(0)$.

It has to be noted that the class $\mathcal{S}_1(\tau)$ allows $\varphi(\tau) = 0$. If $\varphi(\cdot) \in \mathcal{S}_1(\tau)$ with $\varphi(\tau) \neq 0$, then the function $\left(t \mapsto \psi(t) := \frac{1}{\varphi(t)}\right) \in \mathcal{L}^\infty([\tau, \infty), \mathbb{R})$ is well defined and satisfies

$$\exists\, \ell \geq 0 \;\text{ for a.a. } t \geq \tau : \; |\dot{\psi}(t)| \leq \ell.$$

The funnel is given by

$$\mathcal{F}(\tau, \varphi) := \{(t, \eta) \in [\tau, \infty) \times \mathbb{R} \mid \varphi(t)\, |\eta| < 1\}, \tag{3.14}$$

and determined by a function $\varphi(\cdot) \in \mathcal{S}_1(\tau)$.

It has to be noted that the funnel boundary is given by $\varphi(\cdot)^{-1}$. This gives, for $\varphi(\tau) = 0$,

$$0 = \varphi(\tau)|e(\tau)| = \varphi(\tau)|cx^0 - y_{\text{ref}}(\tau)| < 1$$

and puts no restrictions on the initial value. Hence this is noted as *infinite* funnel and proves global results ($(\tau, \mathbb{R}) \subset \mathcal{F}(\tau, \varphi)$).

In the presence of input saturation, arbitrary initial values cannot be allowed, hence the subclass of *finite* funnels

$$\mathcal{G}_1(\tau) := \{\varphi(\cdot) \in \mathcal{S}_1(\tau) \mid \varphi(\tau) \neq 0\}$$

is considered. With other words, $\mathcal{G}_1(\tau)$ contains all functions $\varphi(\cdot) \in \mathcal{S}_1(\tau)$ such that $\psi := \frac{1}{\varphi} : [\tau, \infty) \to [\lambda, \infty)$, $\lambda > 0$, is well defined and globally bounded with global Lipschitz constant ℓ. It has to be noted that $\mathcal{G}_1(\tau)$ is the set of all functions of $\mathcal{S}_1(\tau)$ which satisfy the condition (2) of $\mathcal{S}_1(\tau)$ for all $\varepsilon \geq 0$.

Two typical funnels are illustrated in Figure 3.1.

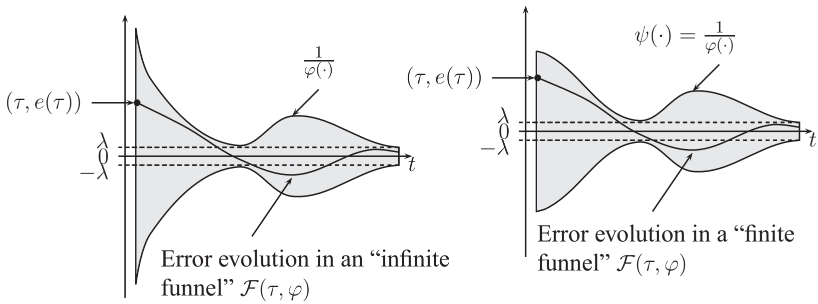


Figure 3.1: Error evolution in a prescribed performance funnel $\mathcal{F}(\tau, \varphi)$ – left: infinite funnel with $\varphi(\cdot) \in \mathcal{S}_1(\tau)$; right: finite funnel with $\varphi(\cdot) \in \mathcal{G}_1(\tau)$.

Remark 3.3.1. (On the function classes $\mathcal{S}_1(\tau)$ and $\mathcal{G}_1(\tau)$)

(a) If $\varphi(\cdot) \in \mathcal{S}_1(\tau)$, then the *funnel boundary* is given by $t \mapsto \psi(t) := \frac{1}{\varphi(t)}$ for $t > \tau$. Then condition (2) of $\mathcal{S}_1(\tau)$ can be interpreted as condition on the funnel boundary:

$$\forall\, \varepsilon > 0\; \exists\, \lambda > 0 : \quad \psi(\cdot) \in \mathcal{L}^\infty([\tau + \varepsilon, \infty), [\lambda, \infty)) \tag{3.15}$$

$$\exists\, \ell \geq 0 \text{ for a.a. } t \geq \tau + \varepsilon : \quad \left|\dot{\psi}(t)\right| \leq \ell. \tag{3.16}$$

Condition (3.16) means that $\psi(\cdot)$ is Lipschitz on $[\tau+\varepsilon, \infty)$ with Lipschitz constant ℓ. If $\varphi(\cdot) \in \mathcal{G}_1(\tau)$, i.e. $\varphi(\tau) \neq 0$, then the above properties (3.15), (3.16) hold for $\varepsilon = 0$, too. It has to be noted that then the function $\psi(\cdot) := \frac{1}{\varphi(\cdot)}$ is well defined for all $t \geq \tau$.

(b) Another important property of the funnel class $\mathcal{S}_1(\tau)$ (or $\mathcal{G}_1(\tau)$, respectively) is that each funnel $\mathcal{F}(\tau, \varphi)$ with $\varphi(\cdot) \in \mathcal{S}_1(\tau)$ (or $\varphi(\cdot) \in \mathcal{G}_1(\tau)$, respectively) is bounded away from zero, i.e. there exists $\lambda > 0$ such that $\frac{1}{\varphi(t)} \geq \lambda$ for all $t \geq \tau$. This condition is equivalent to the assumption that $\varphi(\cdot)$ is bounded. $\diamond$

A variety of funnels are possible. The following gives some examples for functions of the two classes $\mathcal{S}_1(\tau)$ and $\mathcal{G}_1(\tau)$. It has to be noted that the funnel boundaries need not be monotone.

Example 3.3.2. (On the function classes $\mathcal{S}_1(\tau)$ and $\mathcal{G}_1(\tau)$)
For simplicity let $\tau = 0$.

(i) Choose $a,\ \lambda > 0$. Then the function $t \mapsto \varphi_1(t) = \min\left\{\frac{t}{a\,\lambda},\ \frac{1}{\lambda}\right\}$ is in $\mathcal{S}_1(0)$ which satisfies $\varphi_1(0) = 0$. In view of Remark 3.3.1, the corresponding function $\psi_1(\cdot)$ is given by $t \mapsto \psi_1(t) = \max\left\{\frac{\lambda\,a}{t},\ \lambda\right\}$ on $(0, \infty)$. Evolution within the associated funnel ensures a prescribed exponential decay in the transient phase $[0, T]$, $T = a\lambda^2$, and tracking accuracy $\lambda > 0$ thereafter.

(ii) Let $\ell \geq 0$ and $\lambda > 0$ be given. Choose $a,\ b > 0$ such that $a > \lambda$ and $ab \leq \ell$, then the function $t \mapsto \varphi_2(t) = \min\{a^{-1}e^{b\,t},\ \lambda^{-1}\}$ is in $\mathcal{G}_1(0)$ which satisfies $\varphi_2(0) \neq 0$. In view of Remark 3.3.1, the function $\psi_3(\cdot)$ is given by $t \mapsto \psi_3(t) = \max\{ae^{-bt},\ \lambda\}$ which is globally bounded with global Lipschitz constant ℓ. Evolution within the associated funnel ensures a prescribed exponential decay in the transient phase $[0, T]$, $T = \frac{\ln(a/\lambda)}{b}$, and tracking accuracy $\lambda > 0$ thereafter.

(iii) Choose $a \in (0, 1)$ and $\lambda > 0$, then the function

$$\left(t \mapsto \varphi_3(t) = \begin{cases} \frac{1}{1-a\,t} & ,\ t \in \left[0,\ \frac{1-\lambda}{a}\right] \\ \frac{1}{\lambda} & ,\ t \geq \frac{1-\lambda}{a} \end{cases} \right) \in \mathcal{G}_1(0).$$

It has to be noted that, in view of Remark 3.3.1, the function $\psi_3(\cdot)$ is, for all $t \geq 0$, defined as

$$t \mapsto \psi_3(t) = \max\{1 - a\,t,\ \lambda\}$$

which is globally bounded with global Lipschitz constant a.

As in Figure 3.1, the funnel boundary needs not to be monotone. For example, non-monotone functions are:

(iv) Let $a > 0$ be given. Then the function $t \mapsto \varphi_4(t) = \min\left\{a\,t,\ \frac{1}{\max\{3/5\ \cos(t/3),\ 0.2\}}\right\}$ is in $\mathcal{S}_1(0)$ which satisfies $\varphi_4(0) = 0$ and is non-monotone.

(v) The function

$$t \mapsto \varphi_5(t) = \begin{cases} \min\left\{\frac{1}{\cos(t/5)},\ 5\right\} & ,\ t \in [0,10] \\ \min\left\{\frac{5}{3\cos(t/3)},\ 5\right\} & ,\ \text{else} \end{cases}$$

is in $\mathcal{G}_1(0)$ which satisfies $\varphi_5(0) \neq 0$ and is non-monotone. In view of Remark 3.3.1, the function $\psi_5(\cdot)$ is given by

$$t \mapsto \psi_5(t) = \begin{cases} \max\{\cos(t/5),\ 0.2\} & ,\ t \in [0,10] \\ \max\{3/5\ \cos(t/3),\ 0.2\} & ,\ \text{else} \end{cases}$$

which is globally bounded and Lipschitz. ◇

As indicated in Figure 3.1, it is not assumed that the funnel boundary decreases monotonically; whilst in most situation the control designer will choose a monotone funnel, there are situations where widening the funnel at some later time might be beneficial: e.g., when it is known that the reference signal changes strongly or the system is perturbed by some calibration so that a large error would enforces a large control action.

3.3.3 Funnel control for systems (1.1) with relative degree one

As mentioned in the Motivation (see Section 3.2, two kinds of input saturation will be considered in the main results of Chapter 3.4: *Euclidean saturation* (see (3.10) or Chapter 3.2) and *componentwise saturation* (see (3.11) or Chapter 3.2).
In the case of Euclidean saturation, the definition of the funnel (3.14) is modified to

$$\mathcal{F}(\tau,\varphi) := \{(t,\eta) \in [\tau,\infty) \times \mathbb{R}^m \mid \varphi(t)\|\eta\| < 1\}\,, \tag{3.17}$$

and determined by $\varphi(\cdot) \in \mathcal{G}_1(\tau)$ (see Figure 3.2). It has to be noted that the difference of both definitions is that (3.17) deals with vectors $\eta \in \mathbb{R}^m$ (see Figure 3.2).
In the case of componentwise saturation the control objective is to keep every component of the error signal within some funnel, i.e. the error $e(\cdot)$ evolves within

$$\times_{i=1}^{m} \mathcal{F}(\tau,\varphi_i) \tag{3.18}$$

for some family $\varphi(\cdot) = (\varphi_1(\cdot),...,\varphi_m(\cdot))$ of functions $\varphi_i(\cdot) \in \mathcal{G}_1(\tau)$, $i = 1,...,m$,

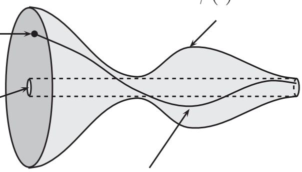


Figure 3.2: Prescribed performance funnels $\mathcal{F}(\tau, \varphi)$.

$m \in \mathbb{N}$.

In each scenario, the *control objective* is a feedback structure which – given a reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and under appropriate feasibility conditions – ensures that the closed-loop system has unique global bounded solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and the tracking error $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$ evolves within the corresponding performance funnel. Moreover, the designed controller should not depend on the actual system data $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times n}$.
This is achieved, for systems (1.1) which are single input, single output (i.e. $m = 1$) with $\tau = 0$, high-frequency gain $CB > 0$ and stable zero dynamics (see [43]), by the proportional time-varying feedback

$$u(t) = -k(t)\, e(t), \quad k(t) = \frac{\varphi(t)}{1 - \varphi(t)\,|e(t)|}, \quad e(t) = y(t) - r(t). \tag{3.19}$$

As noted in Remark 2.3.11, one drawback of the stabilizing- and tracking-controller (2.28) and (2.34) is the increasing gain function $k(\cdot)$. The idea behind the funnel controller (3.19) is that the feedback structure essentially exploits an intrinsic high-gain property of the system to ensure that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact. The control strategy (3.19) uses $k(\cdot)$ which is large if necessary and small else.

It has to be noted that the funnel will regulate the output error quite severely, but requires that the funnel is bounded away from 0 by some $\lambda > 0$. However, this λ may be chosen arbitrarily small, and so practically the difference to asymptotic tracking is negligible.

The function $\varphi(\cdot) \in \mathcal{S}_1(\tau)$ may be chosen arbitrarily. So one can set $\varphi(\cdot)$ such that

$$\lim_{t\to\infty} \frac{1}{\varphi(t)} = \lambda,$$

for arbitrarily small $\lambda > 0$. Then, as by the λ-tracker, it can also arrive at

$$\limsup_{t\to\infty} \text{dist}(|e(t)|, [0, \lambda]) = 0.$$

Furthermore, one can choose $\varphi(\cdot)$ such that $\varphi(t)^{-1} = \lambda$ for all $t \geq T$, where $T > \tau$ is some designer specified time. Then it follows

$$\forall\, t \geq T: \quad \text{dist}(|e(t)|, [0, \lambda]) = 0$$

which is harder than the λ-tracking result. Several examples of functions $\varphi(\cdot)$ are given in Example 3.3.2.

Remark 3.3.3. (Comparison to other high-gain controllers)
In high-gain adaptive control or λ-tracking, where the gain is tuned by $\dot{k}(t) = \|e(t)\|^2$ or $\dot{k}(t) = \max\{0, \|e(t)\| - \lambda\}^2$, resp., the gain $k(\cdot)$ converges but there are two drawbacks:

(i) the gain $k(\cdot)$ is monotonically increasing, albeit bounded, which might lead to a very large gain which is disadvantages since it amplified measurement noise and

(ii) whilst asymptotic performance is guaranteed, transient behavior is not generally taken into account.

It has to be noted that the controller (3.19) is not an adaptive controller as in Chapter 2. The present output error feedback (3.19) is a simple time-varying proportional feedback which circumvents the above two drawbacks. In this setup the gain $k(\cdot)$ is not monotone and may actually decrease, transient behavior of the tracking error is prespecified and is simpler than the above adaptive controller insofar as the gain is not dynamically generated (therefore it is not called an adaptive controller) and does not invoke any internal model. $\diamond$

The motivation (Section 3.2) shows that the necessary feasibility condition depends directly on the function $\psi(\cdot)$ (see (3.9) or, in a more general context, (3.12)) and its global Lipschitz constant. Therefore, the saturation analysis considers the restricted class of funnel boundary functions $\mathcal{G}_1(\tau)$ which only allows to apply the funnel controller to systems (1.1) with bounded initial values. Again, the last fact is seen by the second feasibility condition (3.7), i.e. $|y^0 - y_{\text{ref}}(\tau)| < \psi(\tau)$, which is equivalent to $y^0 \in (y_{\text{ref}}(\tau), -\psi(\tau),\ \psi(\tau) - y_{\text{ref}}(\tau))$.

3.3.4 Funnel control for systems (1.1) with relative degree two

It will be seen in Chapter 4 that the funnel controller for systems (1.1) with relative degree two gets the form

$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t),$$

where

$$k_0(t) = \left(\frac{\varphi_0(t)}{1 - \varphi_0(t)|e(t)|}\right)^2 \qquad \text{and} \qquad k_1(t) = \frac{\varphi_1(t)}{1 - \varphi_1(t)|\dot{e}(t)|}$$

with funnel boundaries $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$, respectively. The introduced funnel controller takes *derivative feedback* to achieve output tracking of relative degree two systems where a funnel for each output error and its derivative is prespecified to shape the transient behaviour.

This derivative funnel might originate in physical bounds on the derivative of the error or could be seen as a controller design parameter. If the error evolves within the funnel $\mathcal{F}(0,\varphi)$ for some $\varphi(\cdot) \in \mathcal{S}_1(0)$, then the derivative of the error eventually has to fulfill

$$\dot{e}(t) < \mathrm{d}t \left(\frac{1}{\varphi}\right)(t) \qquad \text{or} \qquad \dot{e}(t) > -\,\mathrm{d}t \left(\frac{1}{\varphi}\right)(t), \tag{3.20}$$

i.e. at some time the error must decrease faster than the upper funnel boundary gets smaller or the error must increase faster than the lower funnel boundary grows. This implies that the derivative funnel must be large enough to allow the error to follow the funnel boundaries. Therefore, the following family of tuples $(\varphi_0(\cdot), \varphi_1(\cdot))$

$$\mathcal{S}_2 := \left\{ (\varphi_0, \varphi_1) : \mathbb{R}_{\geq 0} \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} (1)\ \varphi_0(\cdot),\ \varphi_1(\cdot) \text{ are absolutely continuous,} \\ (2)\ \forall\, i \in \{0,1\}\ \forall\, \varepsilon > 0 : \\ \qquad \left[\exists\, \lambda_i > 0 : \dfrac{1}{\varphi_i(\cdot)} \in \mathcal{L}^\infty([\varepsilon,\infty),[\lambda_i,\infty))\right] \\ \qquad \left[\exists\, \ell_i \geq 0 \text{ for a.a. } t \geq \varepsilon : \left|\dfrac{\dot\varphi_i(t)}{\varphi_i(t)^2}\right| \leq \ell_i\right] \\ (3)\ \exists\, \delta > 0 \text{ for a.a. } t \geq \varepsilon : \frac{1}{\varphi_1(t)} \geq \delta + \frac{\dot\varphi_0(t)}{\varphi_0(t)^2} \end{array} \right\}$$

with corresponding funnel $\mathcal{F}(0, \varphi_0)$ for the error and $\mathcal{F}(0, \varphi_1)$ for the derivative of the error is considered. It has to be noted that the class $\mathcal{S}_2$ allows $\varphi_i(0) = 0$ for some $i \in \{1, 2\}$. In the presence of input saturation, arbitrary initial values cannot be allowed, hence the subclass of *finite* funnels

$$\mathcal{G}_2 := \{(\varphi_0, \varphi_1) \in \mathcal{S}_2 \ \mid \varphi_0(0) \neq 0,\ \varphi_1(0) \neq 0\}\,.$$

is considered. With other words, $\mathcal{G}_2$ contains all functions $(\varphi_0, \varphi_1)(\cdot) \in \mathcal{S}_2$ such that $\psi_i := \frac{1}{\varphi_i} : \mathbb{R}_{\geq 0} \to [\lambda_i, \infty)$, $\lambda > 0$, $i = 1, 2$, is well defined and globally bounded with global Lipschitz constant ℓ_i. It has to be noted that $\mathcal{G}_2$ is the set of all functions of $\mathcal{S}_2$ which satisfy the condition (2) of $\mathcal{S}_2$ for all $\varepsilon \geq 0$.

Remark 3.3.4. (On the function classes $\mathcal{S}_2$ and $\mathcal{G}_2$)

(i) If $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$, then the *funnel boundaries* are given by $t \mapsto \psi_i(t) := 1/\varphi_i(t)$ for $t > 0$, $i \in \{0, 1\}$. Then condition (2) and (3) of $\mathcal{S}_2$ can be interpreted as condition on the funnel boundaries:

$$\forall\, i \in \{0,1\}\ \forall\, \varepsilon > 0\ \exists\, \lambda_i > 0 : \quad \psi_i(\cdot) \in \mathcal{L}^\infty([\varepsilon,\infty),[\lambda_i,\infty)) \tag{3.21}$$

$$\forall\, i \in \{0,1\}\ \exists\, \ell_i \geq 0 \text{ for a.a. } t \geq \varepsilon : \quad |\dot\psi_i(t)| \leq \ell_i \tag{3.22}$$

$$\exists\, \delta > 0 \text{ for a.a. } t \geq \varepsilon : \quad \psi_1(t) \geq \delta - \dot\psi_0(t). \tag{3.23}$$

Condition (3.22) means that $\psi_i(\cdot)$, $i = 0, 1$, is Lipschitz on $[\varepsilon, \infty)$ with Lipschitz constant ℓ_i. If $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{G}_2$, i.e. $\varphi_0(0) \neq 0$ and $\varphi_1(0) \neq 0$, then the above properties (3.21) - (3.23) hold for $\varepsilon = 0$, too. It has to be noted that then the functions $\psi_i(\cdot) := \frac{1}{\varphi_i(\cdot)}$, $i = 1, 2$, are well defined for all $t \geq 0$.

(ii) Condition (3.23) of the funnel class $\mathcal{S}_2$ (or $\mathcal{G}_2$, respectively) guarantees that on every interval $[\varepsilon, \infty)$ the funnel boundary for the error derivative $\dot e(\cdot)$ is larger than the derivative of the funnel boundary for the error $e(\cdot)$. This condition is important for the design of the funnel controller for relative degree two systems to ensure (3.20). $\diamond$

A variety of funnels are possible. The following gives some examples for functions of the two classes $\mathcal{S}_2$ and $\mathcal{G}_2$.

Example 3.3.5. (On the function classes $\mathcal{S}_2$ and $\mathcal{G}_2$)
The notation of Remark 3.3.4 is used.

(i) Consider the function $\varphi_1(\cdot)$ which is defined in Example 3.3.2 (i) with the corresponding function $\psi_1(\cdot)$. For $\varepsilon > 0$ and $(\lambda_1, \ell_1) = \left(\lambda, \frac{1}{a\varepsilon^2}\right)$ and $\delta \in (0, b]$ the conditions (2) and (3) of $\mathcal{S}_2$ are satisfied for $(\varphi_1(\cdot), \varphi_1(\cdot))$ and thus $(\varphi_1(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$.

(ii) Let the function $\varphi_3(\cdot)$ of Example 3.3.2 (iii) be given. If $\lambda > a$, then, for $(\lambda_3, \ell_3) = (\lambda, a)$ and $\delta \in (0, \lambda - a]$, the conditions (2) and (3) of $\mathcal{S}_2$ are satisfied for $(\varphi_2(\cdot), \varphi_2(\cdot))$. Moreover, since $\varphi_2(0) \neq 0$, $(\varphi_2(\cdot), \varphi_2(\cdot)) \in \mathcal{G}_2$.

As remarked in Example 3.3.2, the funnel boundaries need not to be monotone.

(iii) Consider the pair $(\varphi_3(\cdot), \varphi_1(\cdot))$. For $\varepsilon > 0$, $(\lambda_1, \ell_1) = \left(\lambda, \frac{1}{a\varepsilon^2}\right)$, $(\lambda_3, \ell_3) = (\lambda, a)$ and $\delta \in (0, \lambda]$ the conditions (2) and (3) of $\mathcal{S}_2$ are satisfied for $(\varphi_3(\cdot), \varphi_1(\cdot))$ and thus $(\varphi_3(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$.

(iv) Let the functions $\varphi_4(\cdot)$ and $\varphi_5(\cdot)$ of Example 3.3.2 (iv) and (v) be given. For $\varepsilon > 0$, $(\lambda_4, \ell_4) = \left(\frac{1}{5}, \frac{1}{a\varepsilon^2}\right)$, $\lambda_5 = \frac{1}{5} = \ell_5$ and $\delta \in \left(0, \frac{4}{25}\right]$ the conditions (2) and (3) of $\mathcal{S}_2$ are satisfied for $(\varphi_4(\cdot), \varphi_5(\cdot))$ and thus $(\varphi_4(\cdot), \varphi_5(\cdot)) \in \mathcal{S}_2$. $\diamond$

3.4 Input saturation

In this section m-input, m-output linear system (1.1) are considered which are minimum phase (i.e. (A2)), has relative degree one and positive high-frequency gain (i.e. $CB > 0$). It is shown that the funnel controller (3.19) applied to any system (1.1) achieves in presence of input saturation the control objectives of funnel control.

Since the connection between Proposition 1.1.7, Proposition 1.1.9 and systems (1.1) with relative degree one is used in the main results, this is summarized in the following remark.

Remark 3.4.1. (Connection of Proposition 1.1.7, 1.1.9 and relative degree one systems)
Consider system (1.1) which has strict relative degree $\varrho = 1$, that is, in view of Definition 1.1.5,

$$\det(CB) \neq 0. \tag{3.24}$$

It is immediate, that if (3.24) holds, then, in view of Proposition 1.1.7, for any $V \in \mathbb{R}^{n\times(n-m)}$ such that

$$\operatorname{im} V = \ker C \quad \text{and} \quad N := (V^\top V)^{-1} V^\top [I_n - B(CB)^{-1}C],$$

the similarity transformation $T = \big(B(CB)^{-1},\, V\big)$ has inverse $T^{-1} := (C^\top, N^\top)^\top$ and takes system (1.1) into system (1.13) which gets the form

$$\left.\begin{array}{ll} \dot y(t) = R_1 y(t) + S z(t) + CB\, u(t), & y(0) = Cx^0 \\ \dot z(t) = P_1 y(t) + Q z(t), & z(0) = Nx^0, \end{array}\right\} \tag{3.25}$$

where

$$R_1 := CAB(CB)^{-1}, \;\; S := CAV, \;\; P_1 := NAB(CB)^{-1}, \;\; Q = NAV.$$

Moreover, if system (1.1) has stable zero dynamics which, in view of Proposition 1.1.9, is equivalent to (1.14), then, by Proposition 1.1.7, Q is a Hurwitz matrix, in which case, together with (1.8), the following holds:

$$\exists\, \alpha,\ \beta > 0\ \forall\, t \geq 0 : \quad \left\| e^{Q\,t} \right\| \leq \beta e^{-\alpha t}.$$

Now for any solution $(y(\cdot), z(\cdot), u(\cdot))$ of (3.25) on some interval $[0, \omega) \subset \mathbb{R}_{\geq 0}$ it follows, together with the Variations-of-Constants formula ,

$$\forall\, t \in [0, \omega) : \quad z(t) = e^{Qt} N x^0 + \int_0^t e^{Q(t-s)} P_1 y(s)\, \mathrm{d}s$$

and so, by (1.8),

$$\forall\, t \in [0, \omega) : \quad \|z(t)\| \leq \beta \|Nx^0\| + \frac{\beta}{\alpha} \|P_1\|\, \|y\|_{\mathcal{L}^\infty(0,t)}. \tag{3.26}$$

◇

The main contribution of Chapter 3 is summarized in the following two subsections with proofs in Section 3.6.

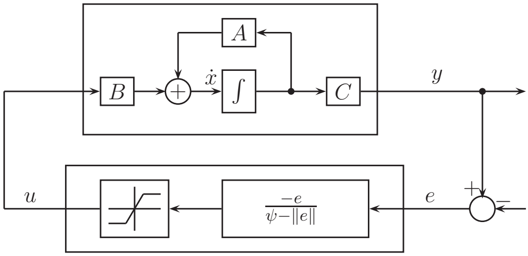


Figure 3.3: Feedback with Euclidean saturation.

3.4.1 Euclidean saturation

In the precursor [43], the efficacy of *unconstrained* funnel control was established for the class of systems (1.1) which are minimum phase (i.e. (A2)), have relative degree one and positive high-frequency gain (i.e. $CB > 0$). The same class of systems is considered, but now subject to input saturation. This section is following previous investigations on scalar funnel control systems and gives extend results in multi input, multi output systems. The control strategy (3.30) follows the formal (3.6) closely.

First, the *Euclidean* saturation constraint case is considered wherein the input is subject to the constraint (3.10) for some $\widehat{u} > 0$. In addition to the hypotheses of its precursor [43], the presence of input saturation in the present Section necessitates an additional assumption on the system, namely, the feasibility assumption (3.28) of Theorem 3.4.2 below. The closed-loop system (1.1), (3.30) is depicted in Figure 3.3.

Theorem 3.4.2. ***(Euclidean saturation)***
Suppose a system (1.1) *satisfying (A2), has strict relative degree one (i.e.* (1.10)*) and positive high-frequency gain (i.e.* (1.11)*). For $\varphi(\cdot) \in \mathcal{G}_1(0)$ adopt the notation of Remark 3.3.1 with $\varepsilon = 0$ and define the performance funnel $\mathcal{F}(0, \varphi)$ as in* (3.17)*. Assume that the initial data $x^0 \in \mathbb{R}^n$ and reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ are such that*

$$\varphi(0)\|Cx^0 - r(0)\| < 1. \tag{3.27}$$

The notation of (1.8) *and Remark 3.4.1 is adopted. Assume that $\widehat{u} > 0$ is such that the feasibility assumption*

$$\gamma\widehat{u} - (L + \ell) =: \Delta > 0 \tag{3.28}$$

holds with

$$L := \left[\|R_1\| + \|S\|\,\|P_1\|\frac{\beta}{\alpha}\right]\left[\|\psi\|_\infty + \|y_{\text{ref}}\|_\infty\right] + \beta\,\|S\|\,\|Nx^0\| + \|\dot{y}_{\text{ref}}\|_\infty, \tag{3.29}$$

where $\psi(\cdot) := \frac{1}{\varphi(\cdot)}$ *and* γ *is given by* (1.11). *Then application of the feedback strategy*

$$\boxed{u(t) = -\operatorname{sat}_{\widehat{u}}(k(t)e(t)), \;\; k(t) = \frac{\varphi(t)}{1-\varphi(t)\|e(t)\|}, \;\; e(t) = Cx(t) - y_{\text{ref}}(t)} \tag{3.30}$$

to (1.1) *yields a closed-loop initial value problem with the following properties.*

(i) Precisely one maximal solution $x : [0,\omega) \to \mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega = \infty$*).*

(ii) The global solution $x(\cdot; 0, x^0)$ *is bounded and the tracking error* $e(\cdot) = Cx(t) - y_{\text{ref}}(t)$ *evolves within the performance funnel* $\mathcal{F}(0,\psi)$*; more precisely,*

$$\forall\, t \geq 0: \quad \psi(t) - \|e(t)\| \geq \varepsilon_0 := \min\left\{\frac{\lambda}{2}, \frac{\lambda}{2\widehat{u}}, \psi(0) - \|e(0)\|\right\}. \tag{3.31}$$

(iii) The gain function $k(\cdot)$ *is bounded, with* $\|k\|_\infty \leq \frac{1}{\varepsilon_0}$.

(iv) The input is unsaturated at some time; i.e. there exists $\tau \geq 0$ *such that* $\|u(\tau)\| < \widehat{u}$.
If the input is unsaturated at time τ*, then it remains unsaturated thereafter, i.e.*

$$\tau \geq 0,\; \|u(\tau)\| < \widehat{u} \quad \Rightarrow \quad \forall\, t \geq \tau: \; \|u(t)\| < \widehat{u}.$$

The input is globally unsaturated (i.e. $\|u(t)\| < \widehat{u}$ *for all* $t \geq 0$*) if, and only if,*

$$\|e(0)\| < \psi(0)\frac{\widehat{u}}{1+\widehat{u}}. \tag{3.32}$$

(In this case, the first of equations (3.30) *takes the simple form* $u(t) = -k(t)e(t)$*).*

The proof of Theorem 3.4.2 is in Section 3.6 on page 118.

Remark 3.4.3. (Comments on Theorem 3.4.2)

(a) Hypothesis (1.11) is simply the assumption that CB is positive definite. Symmetry of CB is not required.

(b) (3.27) is equivalent to $\|Cx^0 - r(0)\| < \psi(0)$ with $\psi(\cdot) := \frac{1}{\varphi(\cdot)}$ (see Remark 3.3.1). Then the gain $k(\cdot)$ of (3.30) gets the form $k(t) = \frac{1}{\psi(t) - \|e(t)\|}$.

(c) As seen in the feasibility condition (3.28), information of the reference trajectory and of its derivative must be known a priori.

(d) It has to be noted that $\frac{1}{\varepsilon_0}$ of statement (iii) is well defined.

(e) In view of the potential singularity in (3.30), some care must be exercised in formulating the closed-loop initial value problem (1.1), (3.30). This is done in Step 1 of the proof, wherein the closed-loop initial value problem is formulated as

$$\dot{x}(t) = F(t, x(t)), \qquad x(0) = x^0, \qquad (0, x^0) \in \mathcal{D}, \tag{3.33}$$

for suitable $F : \mathcal{D} \to \mathbb{R}^n$ with appropriately defined domain $\mathcal{D} := \{(t, \eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n) \mid (t, C\eta - r(t)) \in \mathcal{F}(0, \varphi)\}$. By a solution of (3.33) it is meant a continuously differentiable function $x : [0, \omega) \to \mathbb{R}^n$ which satisfies (3.33) and has graph in $\mathcal{D}$; $x(\cdot)$ is maximal if it has no right extension that is also a solution; $x(\cdot)$ is global if $\omega = \infty$. Assertion (i) of the theorem confirms the existence of precisely one maximal solution $x(\cdot)$ of (3.33) and, moreover, this solution is global. It has to be noted that the requirement that graph(x) is in $\mathcal{D}$ implies that the graph of the tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ is in $\mathcal{F}(0, \varphi)$: this, together with boundedness of $x(\cdot)$, is the content of Assertion (ii). Boundedness of the control gain function $k(\cdot) = \kappa(\cdot, e(\cdot))$, with $\kappa(\cdot, \cdot)$ as in Step 1 of the proof of Theorem 3.4.2, is established in Assertion (iii). Assertion (iv) implies that the control input cannot remain saturated for all $t \geq 0$ and, when it becomes unsaturated, then it remains so thereafter; furthermore, if the control input is initially unsaturated (i.e. if $\|u(0)\| < \widehat{u}$), then the saturation bound is never attained.

(f) The first feasibility condition (3.27) is a necessary condition for attainment of the control objective and is equivalent to the requirement that $(0, x^0) \in \mathcal{D}$.

(g) In conjunction with the other hypotheses, the second feasibility condition (3.28) is a sufficient condition for attainment of the control objective. It quantifies and exhibits the interplay between the saturation bound (sufficiently large to ensure performance) and bounds on the plant data, funnel data, initial data and reference signal data.

(h) It has to be noted that the feasibility condition (3.28) incorporates bounds of the zero dynamics as specified in (3.26). The nature of the dependence of the saturation bound on these data is not surprising. For example,

(i) the minimum-phase condition (A2) ensures exponential stability of the zero dynamics of the linear triple (A, B, C) – this translates into the condition (1.8) on the matrix Q in Proposition 1.1.7 – the parameter α quantifies the exponential decay rate of the zero dynamics and is inversely related to the saturation bound;

(ii) it is to be expected that tracking of "large and rapidly varying" reference signals $y_{\text{ref}}(\cdot)$ would require control inputs capable of taking sufficiently large values – this is reflected in the dependence of the saturation bound on both $\|r\|_\infty$ and $\|\dot{r}\|_\infty$;

(iii) transient and asymptotic behaviour of the tracking error is influenced by the choice of funnel $\mathcal{F}(0, \varphi)$ determined by the globally Lipschitz function $\psi(\cdot) = \frac{1}{\varphi(\cdot)}$ which is given, in view of Remark 3.3.1, by a function $\varphi(\cdot) \in \mathcal{G}_1(0)$ – a stringent requirement that transient behaviour decays rapidly would be reflected in a large Lipschitz constant ℓ which, not unexpectedly, appears as an additive term in the saturation bound.

(i) In other words: (3.28) is a conservative assumption on $\widehat{u}$. ⋄

Remark 3.4.4. (*Infinite* funnel)
As seen in (3.28) and (3.29), the input constraint $\widehat{u}$ depends on $\|\psi\|_\infty$ and $\ell \geq \|\dot{\psi}\|_\infty$. If the infinite funnel is allowed, i.e. $\varphi(\cdot) \in \mathcal{S}_1(0)$ with $\varphi(0) = 0$, then $\|\psi\|_\infty = \|1/\varphi\|_\infty = \infty$ and it is not possible to fulfill (3.28) for some finite $\widehat{u} > 0$. In this case, loosely speaking, $\widehat{u} = \infty$ which can be interpreted as unsaturated input and so (3.30) gets the unconstrained funnel control strategy (see (3.6), (3.19) or [43]). ⋄

Remark 3.4.5. (Drawbacks)
The input saturation requires an additional assumption on the system, in other words more knowledge of the system, namely that the feasibility assumption (3.28) holds; in view of the example considered in the motivation (see Section 3.2), this additional assumption is not surprising. It has to be noted that this form of saturation allows for a simple feedback law (3.30) invoking only one scalar time-varying output error feedback to ensure that the norm of the error $\|e(t)\|$ evolves within the funnel. This simplicity has the drawback that the gain $k(t)$ does not depend on the individual errors $|e_i(t)|$ but on $\|e(t)\|$ and its distance to the funnel boundary.
The theorem consider an input constraint of the form $\|u(t)\| \leq \widehat{u}$ for all $t \geq 0$. This means that in the multi input, multi output case, the inputs are not constrained elementwise but in the sense of the norm which also has the consequence that the

saturation function changes the length of the vector $u(\cdot)$, but not its direction. This simplifies the problem, but each component of $u(\cdot)$ must have the same bound which might be a drawback in real cases. Moreover, since the tracking error $e(\cdot)$ is a vector valued function, a scalar gain $k(\cdot)$ is conservative in general.
Now a saturation in each input channel $u_i(t)$ is considered and ensures that each component $e_i(t)$ of the error evolves within a prespecified funnel $\psi_i(\cdot)$. Moreover, for each input channel $u_i(\cdot)$ a gain function $k_i(\cdot)$ is considered which only depends on the error $|e_i(\cdot)|$. $\diamond$

3.4.2 Componentwise saturation

Second, the attention is turned to the case in which the saturation constraint is imposed *componentwise* in the sense that, for some $\widehat{u} = (\widehat{u}_1, ..., \widehat{u}_m)$, $\widehat{u}_i > 0$, the input $u(\cdot) = (u_1(\cdot), ..., u_m(\cdot))$ is required to satisfy (3.11). To conform with this componentwise structure, it is imposed a componentwise performance funnel, as in (3.18). In particular, for prescribed functions $\psi_i(\cdot)$, $i = 1, ..., m$, which are given, in view of Remark 3.3.1, by $\varphi_i(\cdot) \in \mathcal{G}_1(0)$, it is seeked a control structure which ensures that for any given reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, the output $y(\cdot)$ is such that the tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ evolves componentwise (components $e_i(\cdot)$, $i = 1, ..., m$) in the funnel, that is,

$$\forall\, i \in \{1, ..., m\} : \quad \text{graph}(e_i) \subset \mathcal{F}(0, \varphi_i).$$

Suppose a m-input, m-output system (1.1) with initial data $x^0 \in \mathbb{R}^n$ and a given reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$.
It is imposed that

$$\det(CB) \neq 0,\;\; [CB]_{ii} > 0,\; i \in \{1, ..., m\},\;\; [CB]_{ij} \leq 0,\; i, j \in \{1, ..., m\},\; i \neq j. \tag{3.34}$$

It is also required a type of “diagonal dominance” condition, viz.

$$\forall\, i \in \{1, ..., m\} : \quad \sum_{j=1}^{m} [CB]_{ij}\, \widehat{u}_j - (L + \ell_i) =: \Delta_i > 0 \tag{3.35}$$

with L given by (3.29) (wherein $\psi(\cdot) = (\psi_1(\cdot), ..., \psi_m(\cdot)) = \left(\frac{1}{\varphi_1(\cdot)}, ..., \frac{1}{\varphi_m(\cdot)}\right)$). The generalization of Theorem 3.4.2 to saturation in each input is now ready state.

Theorem 3.4.6. ***(Componentwise saturation)***
Suppose a system (1.1) *satisfying (A2), has strict relative degree one (i.e.* (1.10)*)*

and positive high-frequency gain (i.e. (1.11)*) and is such that* (3.34) *holds. For* $\varphi_i(\cdot) \in \mathcal{G}_1(0)$*,* $i = 1, ..., m$*, adopt the notation of Remark 3.3.1 with* $\varepsilon = 0$ *and define the performance funnels* $\mathcal{F}(0, \varphi_i)$*,* $i = 1, ..., m$*, as in* (3.14)*. Assume that the initial data* $x^0 \in \mathbb{R}^n$ *and reference signal* $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *are such that the initial error* $e(0) = Cx^0 - y_{\text{ref}}(0)$ *satisfies the componentwise inequalities*

$$\forall\, i \in \{1, .., m\}: \quad \varphi_i(0)|e_i(0)| < 1. \tag{3.36}$$

The notation of (1.8) *and Remark 3.4.1 is adopted. Assume that the componentwise saturation constraint* $\widehat{u} \in \mathbb{R}^m_{>0}$ *is such that the feasibility assumption* (3.35) *holds. Then, for any input disturbance* $d_u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$*, the application of the feedback strategy*

$$\boxed{\begin{array}{ll} u(t) = [u_1(t), ..., u_m(t)]^\top, & e_i(t) = y_i(t) - y_{\text{ref},i}(t) \\ u_i(t) = -\mathit{sat}_{\widehat{u}_i}(k_i(t)e_i(t) - d_{u,i}(t)), & k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)|e_i(t)|} \end{array}} \tag{3.37}$$

to (1.1) *yields a closed-loop initial value problem with the following properties.*

(i) Precisely one maximal solution $x : [0, \omega) \to \mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega = \infty$*).*

(ii) The global solution $x(\cdot; 0, x^0)$ *is bounded and, for each* $i \in \{1, ..., m\}$*, the tracking error component* $e_i(\cdot)$ *evolves within the performance funnel* $\mathcal{F}(0, \psi_i)$*; more precisely,*

$$\begin{aligned} &\forall\, t \geq 0\; \forall\, i \in \{1, ..., m\}: \\ &\quad \psi_i(t) - |e_i(t)| \geq \varepsilon_{0,i} := \min\left\{\frac{\lambda_i}{2}, \frac{\lambda_i}{2\,(\widehat{u}_i + \|d_{u,i}\|_\infty)}, \psi_i(0) - |e_i(0)|\right\}. \end{aligned} \tag{3.38}$$

with $\psi_i(\cdot) = \frac{1}{\varphi_i(\cdot)}$*,* $i = 1, ..., m$*.*

(iii) The gain functions $k_i(\cdot)$ *are bounded, with* $\|k_i\|_\infty \leq \frac{1}{\varepsilon_{0,i}}$*,* $i = 1, ...m$*.*

(iv) Each input is unsaturated at some time $\tau_i \geq 0$*; i.e. there exists* $\tau_i \geq 0$ *such that* $|u_i(\tau_i)| < \widehat{u}_i$*.*

(v) The input is unsaturated at some time; i.e. there exists $\tau \geq 0$ *such that* $\|u(\tau)\| < \widehat{u}$*.*
If the input is unsaturated at time τ*, then it remains unsaturated thereafter, i.e.*

$$\tau \geq 0,\ \|u(\tau)\| < \widehat{u} \quad \Rightarrow \quad \forall\, t \geq \tau: \ \|u(t)\| < \widehat{u}.$$

The input is globally unsaturated (i.e. $\|u(t)\| < \widehat{u}$ *for all* $t \geq 0$*) if, and only if,*

$$\|e(0)\| < \psi(0)\frac{\widehat{u}}{1+\widehat{u}}. \tag{3.39}$$

(In this case, the first of equations (3.30) *takes the simple form* $u(t) = -k(t)e(t)$*).*

The proof of Theorem 3.4.6 is in Section 3.6 on page 122.

Remark 3.4.7. (Comments on Theorem 3.4.6 and input disturbances)

(a) If the high-frequency gain CB is positive definite and *diagonal*, then the problem (1.1), (3.37) essentially decomposes into m single input, single output subproblems, to each of which Theorem 3.4.2 (specialized to the single input, single output case) may be applied. Of more interest is the case in which CB has non-zero off-diagonal entries.

(b) If the input disturbance $d_u(\cdot)$ vanishes, then Assertion (v) says that if an input component becomes unsaturated, then it remains so thereafter; furthermore, if this input is initially unsaturated, then the saturation bound is never attained. The assumption that $d_u(\cdot)$ vanishes is essentially in Assertion (v).

(c) The arguments used in establishing Theorem 3.4.2 are readily modified to conclude Theorem 3.4.6. The structure of the proof of Theorem 3.4.6 closely resembles that of Theorem 3.4.2. For simplicity, let $d_u(\cdot) \equiv 0$. A careful injection of the proof of Theorem 3.4.6 shows that the essential difference in the two cases is that, in the proof of Theorem 3.4.6, one argues *componentwise*: a key feature is the following counterpart of (3.44), the derivation of which invokes (3.34) and (3.35)

$$\begin{aligned}&\forall\, i \in \{1, ..., m\}: \ \ \text{sign } e_i(t)\dot{e}_i(t)\\ &\quad < -\ell_i + (CB)_{ii}\,[\widehat{u}_i - \ \text{sign } e_i(t)\text{sat}_{\widehat{u}_i}(k_i(t)e_i(t))] \quad \text{for almost all } t \in [0, \omega).\end{aligned}$$

(d) The high-frequency gain matrix CB satisfies (3.34) and it is required a type of "diagonal dominance" condition (3.35), viz.

$$\sum_{j=1}^{m}[CB]_{ij}\,\widehat{u}_j - (L + \ell_i) > 0, \qquad i = 1, ..., m.$$

It has to be noted that if CB is a M-matrix, i.e.

$$(C\,B)_{ij} \leq 0\,\forall\, i \neq j \quad \wedge \quad \det(C\,B) \neq 0 \quad \wedge \quad [(C\,B)^{-1}]_{ij} \geq 0\,\forall\, i, j \in \{1, ..., m\},$$

then, in view of [8, Th. 6.2.3],

$$\forall\, i \in \{1, ..., m\} : \quad [CB]_{ii} > 0$$

and so, CB satisfies (3.34). Moreover, the M-matrix properties imply that there exists $\widehat{u} \in \mathbb{R}^m_{>0}$ such that (3.35) holds.

(e) The input disturbance $d_u(\cdot)$ does not have influence on the saturation bound (see (3.35)). The disturbance only influences $\varepsilon_{0,i}$, $i = 1, ..., m$.

(f) Theorem 3.4.6 is formulated with an input disturbance $d_u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, but, in view of Theorem 3.4.2, Theorem 3.4.2 is written without input disturbance. The key feature of the proof of Theorem 3.4.2 is STEP 3. The idea behind STEP 3 is that the following implication holds:

$$\left[\forall\, t \in [t_0, t_1] : \ \mathrm{sat}_{\widehat{\mathrm{u}}}(k(t)e(t)) = \frac{\widehat{u}}{\|e(t)\|} e(t)\right]$$
$$\Rightarrow \left[\begin{array}{l} \langle e(t), CB\ \mathrm{sat}_{\widehat{\mathrm{u}}}(k(t)e(t))\rangle \geq \gamma\widehat{u}\|e(t)\|\ \forall\, t \in [t_0, t_1] \quad \text{and} \\ \langle e(t), \dot{e}(t)\rangle \leq -\ell\|e(t)\| \text{ for a.a. } t \in [t_0, t_1] \end{array}\right].$$

In the context of input disturbance in Theorem 3.4.2, i.e. the controller (3.30) gets the form $u(t) = -\,\mathrm{sat}_{\widehat{\mathrm{u}}}\,(k(t)e(t) - d_u(t))$, the assumption of the above implication has the form

$$\forall\, t \in [t_0, t_1] : \quad \mathrm{sat}_{\widehat{\mathrm{u}}}(k(t)e(t) - d_u(t)) = \frac{\widehat{u}}{\|k(t)e(t) - d_u(t)\|}(k(t)e(t) - d_u(t)).$$

If $d_u(\cdot)$ is not identically zero, the above implication does not hold, but these implication is essential for the proof of Theorem 3.4.2. Thus, Theorem 3.4.2 cannot be formulated with any input disturbance. $\diamond$

Remark 3.4.8. (Measurement noise)
The proposed controllers (3.30) and (3.37) tolerate output measurement disturbance $d_y(\cdot)$, provided that the disturbance belongs to the same function class as the reference signals. With reference to (3.30) and (3.37), the disturbed error signal is then $e(\cdot) = (y(\cdot) + d_y(\cdot)) - y_{\mathrm{ref}}(\cdot) = y(\cdot) - (y_{\mathrm{ref}}(\cdot) - d_y(\cdot))$. Therefore, from an analytical viewpoint, in the presence of output disturbance, the disturbance-free analysis is immediately applicable on replacing the reference signal $y_{\mathrm{ref}}(\cdot)$ by the signal $y_{\mathrm{ref}}(\cdot) - d_y(\cdot)$. With a slight modification, the proof is the same as the proofs of Theorem 3.4.2 and Theorem 3.4.6. Moreover, from a practical viewpoint, it is to expect

that the disturbance $d_y(\cdot)$ is "small" - if an a priori bound on the magnitude of the disturbance is available, then the asymptotic radius of the funnel should be chosen to be commensurate with that bound.
Hence the real error remains in the bigger funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the control.

In the presence of input noise, the funnel controller (3.37) copes with input noise $d_u(\cdot)$. ◇

Remark 3.4.9. (Different funnel in the gains and output noise)
In many applications more information about the system are known such that non-symmetric funnels in the gain functions (3.30) or (3.37), respectively, make sense. For $\varphi^\ell(\cdot),\ \varphi^u(\cdot) \in \mathcal{G}_1(0)$ or $\varphi_i^\ell(\cdot),\ \varphi_i^u(\cdot) \in \mathcal{G}_1(0)$, $i = 1, ..., m$, resp., this, together with Remark 3.3.1 and $\varepsilon = 0$, can be captured by a control strategy of the form

$$u(t) = -\operatorname{sat}_{\hat{\mathrm{u}}}(k(t)e(t)), \qquad k(t) = \frac{1}{\min\{\psi^\ell(t) - \|e(t)\|,\, \psi^u(t) - \|e(t)\|\}}$$

or

$$u(t) = [u_1(t), ..., u_m(t)]^\top, \quad u_i(t) = -\mathbf{sat}_{\hat{u}_i}(k_i(t)e_i(t)), \qquad k_i(t) = \frac{1}{\min\{\psi_i^\ell(t) - e_i(t),\, \psi_i^u(t) + e_i(t)\}},$$

resp. However, this is omitted since the proofs of Theorem 3.4.2 and Theorem 3.4.6 become unnecessary technically involved. Then the funnels (3.17) and (3.18) are given by , for $\tau \geq 0$,

$$\mathcal{F}(\tau, \varphi) := \left\{ (t, \eta) \in [\tau, \infty) \times \mathbb{R}^m \;\middle|\; \frac{1}{\varphi^\ell(t)} - \|\eta\| > 0, \frac{1}{\varphi^u(t)} - \|\eta\| > 0 \right\}, \tag{3.40}$$

determined by $\varphi(\cdot) := (\varphi^\ell(\cdot), \varphi^u(\cdot))$, or

$$\mathcal{F}(\tau, \varphi_i) := \left\{ (t, \eta) \in [\tau, \infty) \times \mathbb{R} \mid -\varphi_i^u(t) < \varphi_i^\ell \varphi_i^u \eta < \varphi_i^\ell(t) \right\}, \tag{3.41}$$

determined by $\varphi_i(\cdot) = (\varphi_i^\ell(\cdot), \varphi_i^u(\cdot))$, respectively. ◇

3.5 Example

The applicability and performance of the feedback strategies (3.30) and (3.37) are illustrated in this section. It has to be noted that the choice of the funnel boundary $\psi(\cdot)$, given by a function $\varphi(\dot{)} \in \mathcal{G}_1(0)$, is completely free (see Example 3.5.1) and given by the designer. The control law guarantees the tracking error evolves in a performance funnel (see Section 3.3. As seen in Example 3.3.2, this funnel can be defined in such a way that the tracking error decay exponentially until it remains in a certain radius around zero.

Consider the three-mass serially connected mass-spring damper system (2.40) which has relative degree $\varrho = 1$. As remarked in Example 2.4 (4), it is not typically to measure the velocity in the physics. Another drawback of this example is that system (2.40) is a single input, single output system. Thus, the simulation of the closed-loop system (2.40), (3.30) is omitted and an example is introduced which is multi input, multi output.

Example 3.5.1. (Illustration of Theorem 3.4.2)
For example, consider the following multi input, multi output system in Byrnes-Isidori form (1.13):

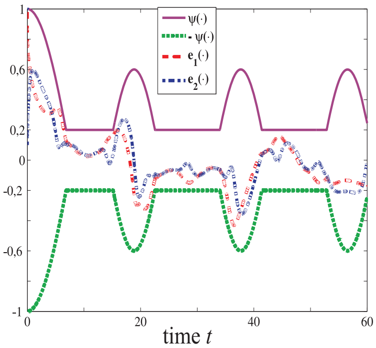
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1/2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(t), \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix}, \quad (3.42)$$

where y^0, $z^0 \in \mathbb{R}^2$. In this case, $\|R_1\| = 1$, $\|S\| = 2$, $\|P_1\| = \sqrt{2}$, and (1.8) holds with $\alpha = \frac{5-\sqrt{2}}{2}$ and $\beta = 1$.
The control input is subject to the *Euclidean* saturation with the constraint

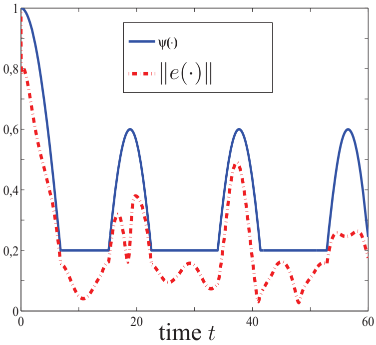
$$\forall\, t \geq 0 : \quad \|u(t)\| \leq \widehat{u} := 30.$$

Now, the first and second component of the solution of the Lorenz system which is introduced in Remark 2.4.1 are chosen as reference signal $y_{\mathrm{ref}}(\cdot) = [\xi_1(\cdot),\ -\xi_2(\cdot)]^\top$. It has to be noted that $y_{\mathrm{ref}}(0) = [1,\ 0]^\top$ and so, $\|y_{\mathrm{ref}}\|_\infty \leq \frac{\sqrt{949}}{10}$ and $\|\dot{y}_{\mathrm{ref}}\|_\infty \leq \frac{6}{\sqrt{5}}$ (see Remark 2.4.1).

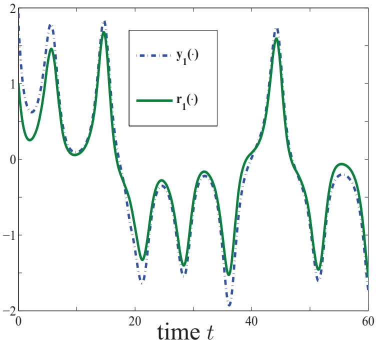
The notation of Remark 3.3.1 is used. The function $\psi(\cdot)$ which determines the funnel



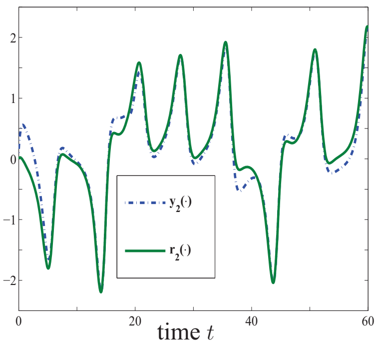
Funnel and tracking errors $e_1(\cdot),\ e_2(\cdot)$



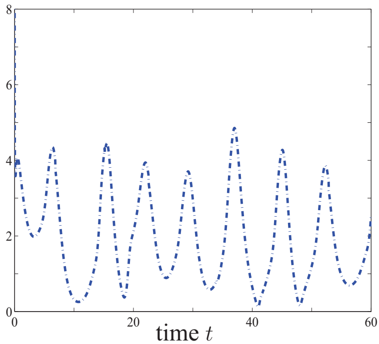
Funnel and $\|e(\cdot)\|$



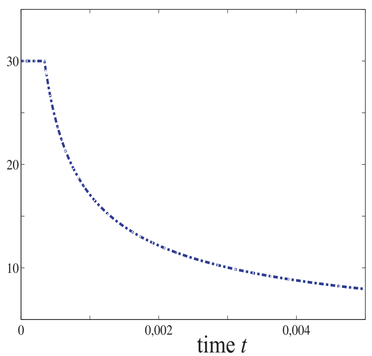
Reference signal $r_1(\cdot)$ and output $y_1(\cdot)$



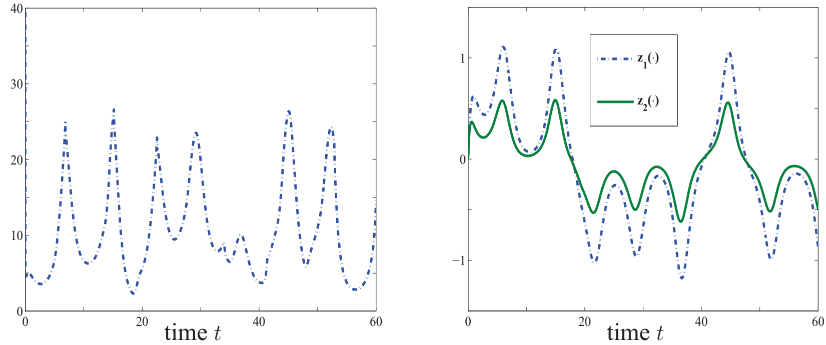
Reference signal $r_2(\cdot)$ and output $y_2(\cdot)$



Control $\|u(\cdot)\|$



Control $\|u(\cdot)\|$ - zoomed in



Gain function $k(\cdot)$ Functions $z_1(\cdot)$ and $z_2(\cdot)$

Figure 3.4: Behaviour of the closed-loop system (3.42), (3.30)

$\mathcal{F}\left(0, \frac{1}{\psi}\right)$ is given by

$$\psi(t) := \begin{cases} \max\{\cos(t/5),\ 0.2\} & ,\ t \in [0, 10] \\ \max\{3/5\ \cos(t/3),\ 0.2\} & ,\ \text{otherwise} \end{cases}$$

whence, $\lambda = 0.2 = \ell$ and $\|\psi\|_\infty = 1$. It has to be noted that $1/\psi(\cdot)$ is well defined and $1/\psi(\cdot) \in \mathcal{G}_1(0)$ (see Example 3.3.2 (v)).

Let $z^0 = 0$. An easy computation gives $L < 14$, where L is defined by (3.29). It has to be noted that $[CB]_{11} = 1 = [CB]_{22}$, $[CB]_{21} = -\frac{1}{2}$ and $[CB]_{12} = 0$. The Young inequality (see [2, Th. IV.2.15]) gives

$$\forall\, v \in \mathbb{R}^2 \setminus \{0\} : \quad v^\top CB\, v = v_1^2 + v_2^2 - \frac{1}{2} v_1 v_2 \geq \frac{1}{2} v_1^2 + \frac{1}{2} v_2^2 = \frac{1}{2}\|v\|^2.$$

Set $\gamma = \frac{1}{2}$. Then (1.11) and

$$\gamma \widehat{u} - (L + \ell) > 0$$

are satisfied and so, (3.28) holds.
If $y^0 \in \mathcal{B}_1((1,0))$, then the condition $\|e(0)\| < \psi(0)$ holds. Then all assumptions of Theorem 3.4.2 are satisfied.
It has to be noted that inequality (3.32) is fulfilled if, and only if, $y^0 \in \mathcal{B}_{\frac{30}{31}}((1,0))$.

To illustrate the occurrence of saturation of the control input in the simulations, let $y^0 = [1.98,\ 0.1]^\top$. Then, the inequality (3.32) fails to hold which implies that the input $u(\cdot)$ is not globally unsaturated (in this case, there exists $\tau > 0$ such that the control $u(\cdot)$ is saturated on $[0, \tau)$).

Figure 3.4 depicts the behaviour of the closed-loop system (3.42), (3.30). The simulations confirm the results of Theorem 3.4.2: the norm of the tracking error $e(\cdot)$ and the tracking errors $e_1(\cdot)$, $e_2(\cdot)$ remain uniformly bounded away from the funnel boundary. Non-monotonicity of the gain function $k(\cdot)$ is also evident: it increases when the error approach the funnel boundary and decreases when the error recedes from the boundary. The third row of pictures confirms that the input is initially saturated. The final picture shows the zero dynamics. $\diamond$

Example 3.5.2. (Illustration of Theorem 3.4.6, Example 3.5.1 revisited)
Again, consider the system (3.42). The control inputs are subject to the *component-wise* saturation with the constraints

$$\forall\, t \geq 0: \ [|u_1(t)| \leq \widehat{u}_1 := 20 \qquad \text{and} \qquad |u_2(t)| \leq \widehat{u}_2 := 30].$$

The funnel boundary $\psi(\cdot) = (\psi_1(\cdot), \psi_2(\cdot))$ is given by

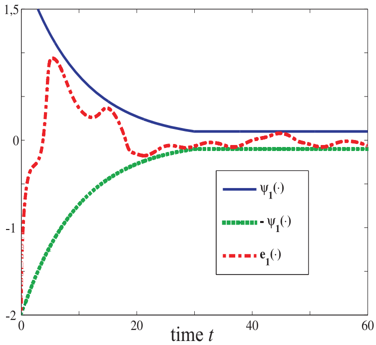
$$\psi_1(t) := \max\{2e^{-0.1\,t},\ 0.1\}, \quad \psi_2(t) := \begin{cases} \max\{\cos(t/5),\ 0.2\} & ,\ t \in [0, 10] \\ \max\{3/5\ \cos(t/3),\ 0.2\} & ,\ \text{otherwise} \end{cases}$$

whence, $(\lambda_1,\ \lambda_2) = (0.1,\ 0.2)$, $(\ell_1,\ \ell_2) = (0.2,\ 0.2)$ and $\|\psi\|_\infty = \sqrt{5}$. It has to be noted that, together with the notation of Remark 3.3.1, $1/\psi_i(\cdot)$ is well defined and $1/\psi_i(\cdot) \in \mathcal{G}_1(0)$ (see Example 3.3.2). Moreover, $\psi_1(\cdot)$ prescribes exponential (exponent 0.1) decay of the tracking error $e_1(\cdot)$ in the transient phase $[0, T]$, where $T = 10\ \ln 20 \approx 30$, and a tracking accuracy quantified by $\lambda_1 = 0.1$ thereafter.

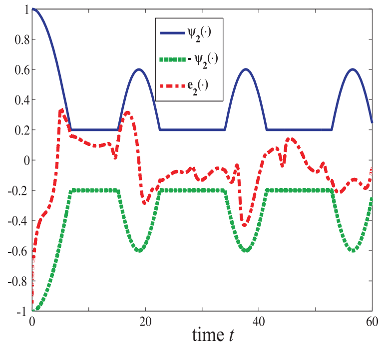
Let $z^0 = 0$ and a straightforward calculation gives $L < 17$. Since $[CB]_{11} = 1 = [CB]_{22}$, $[CB]_{21} = -\frac{1}{2}$ and $[CB]_{12} = 0$, (3.34) holds and

$$\sum_{j=1}^{2} [CB]_{ij} \widehat{u}_j - (L + \ell_i) = 20 - (L + \ell_i) > 2.5 > 0, \quad i = 1, 2,$$

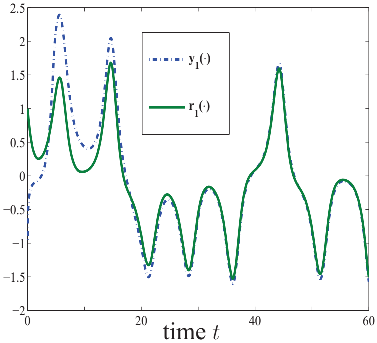
shows (3.35).
If $y^0 \in (-1, 3) \times (-1, 1)$, then $|e_i(0)| < \psi_i(0)$, $i = 1, 2$, holds and then, all assumptions of Theorem 3.4.6 are satisfied. To illustrate the occurrence of saturation of the control input in the simulations, let $y^0 = [-0.95,\ -0.95]^\top$. Then, the inequality (3.32)



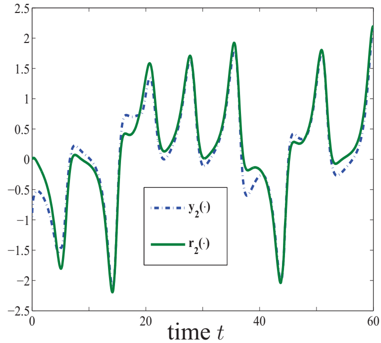
Funnel and tracking error $e_1(\cdot)$



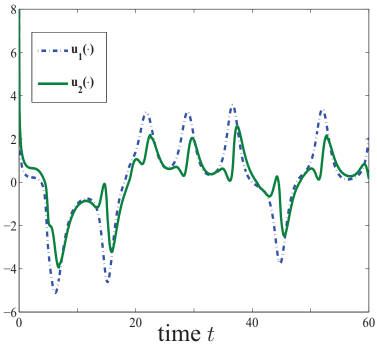
Funnel and tracking error $e_2(\cdot)$



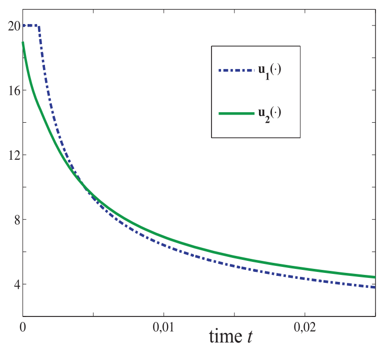
Reference signal $r_1(\cdot)$ and output $y_1(\cdot)$



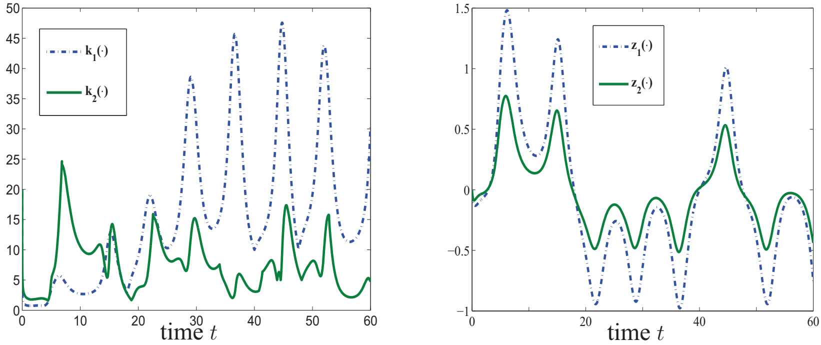
Reference signal $r_2(\cdot)$ and output $y_2(\cdot)$



Control $u_1(\cdot),\ u_2(\cdot)$



Control $u_1(\cdot),\ u_2(\cdot)$ - zoomed in



Gain functions $k_1(\cdot)$ and $k_2(\cdot)$ Functions $z_1(\cdot)$ and $z_2(\cdot)$

Figure 3.5: Behaviour of the closed-loop system (3.42), (3.37)

fails to hold for $i = 1$ (in this case, there exists $\tau_1 > 0$ such that the control $u_1(\cdot)$ is saturated on $[0, \tau_1)$).

Figure 3.5 depicts the behaviour of the closed-loop system (3.42), (3.37). The simulations confirm the results of Theorem 3.4.6: the tracking errors $e_1(\cdot)$, $e_2(\cdot)$ remain uniformly bounded away from the funnel boundaries. Non-monotonicity of gain functions $k_1(\cdot)$, $k_2(\cdot)$ are also evident. The third row of pictures confirms that the first input is initially saturated and the second input is globally unsaturated. The final picture shows the zero dynamics. ◇

Remark 3.5.3. (Comment on the above)
The initial value $y^0 \in (-1, 3) \times (-1, 1)$ is chosen in such a way that saturation of the first input $u_1(\cdot)$ is guaranteed (see inequality (3.32)) on the small interval $[0, \tau_1) \approx [0, 0.005)$ and, in view of Assertion (iv) of Theorem 3.4.6 and (3.32), the second input $u_2(\cdot)$ is globally unsaturated. To get a saturation in the second input, inequality (3.39) fails to hold for $i = 2$ which is reachable for $y^0 = [-0.95,\ -0.991]^\top$. The second value looks a little artificially chosen. ◇

3.6 Proofs

Proof of Theorem 3.4.2:
The proof uses the notation of Proposition 1.1.7, Remark 3.3.1 and (1.8).

STEP 1: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (3.30) *(or, equivalently,* (1.13)*,* (3.30)*) are shown.*
Some care must be exercised in formulating the initial value problem (1.1), (3.30) (or, equivalently, (1.13), (3.30)). Define

$$\begin{aligned}
&\kappa : \mathcal{F}(0,\varphi) \to \mathbb{R}, \quad (t,\eta) \mapsto \kappa(t,\eta) := \tfrac{1}{\psi(t)-\|\eta\|},\\
&\mathcal{D} := \big\{(t,\eta,\zeta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \mid (t,\eta - r(t)) \in \mathcal{F}(0,\varphi)\big\},\\
&f : \mathcal{D} \to \mathbb{R}^m, \quad (t,\eta,\zeta) \mapsto f(t,\eta,\zeta) := R_1\eta + S\zeta - CB\ \mathrm{sat}_{\hat{\mathrm{u}}}(\kappa(t,\eta - r(t))(\eta - r(t))).
\end{aligned}$$

Then the initial value problem (1.13), (3.30) may be expressed in the form

$$\left.\begin{array}{rclcrcl}
\dot{y}(t) &=& f(t,y(t),z(t)), && y(0) &=& Cx^0\\
\dot{z}(t) &=& P_1 y(t) + Qz(t), && z(0) &=& Nx^0.
\end{array}\right\} \tag{3.43}$$

Clearly, $(y,z) : [0,\omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ is a (maximal/global) solution of (3.43) if, and only if, $x = B(CB)^{-1}y(\cdot) + Vz : [0,\omega) \to \mathbb{R}^n$ is a (maximal/global) solution of (3.33). Now, it is readily verified that $(t,\mu,\zeta) \mapsto F(t,\mu,\zeta) := \big(f(t,\mu,\zeta),\ P_1\mu + Q\zeta\big)$ satisfies a local Lipschitz condition on the relatively open domain $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^{n-1}$, in the sense that, for each $(t,\mu,\zeta) \in \mathcal{D}$, there exists an open neighbourhood $\mathcal{O}$ of (t,μ,ζ) and a positive constant K such that

$$\forall\ (t,y,z) \in \mathcal{O} : \quad \|F(t,y,z) - F(t,\mu,\zeta)\| \leq K\big(\|y-\mu\| + \|z-\zeta\|\big).$$

By the standard theory of ordinary differential equations (see, e.g. Theorem 1.1.2 or [88, Theorem III.10.VI]), the initial value problem (3.43) has a unique maximal solution $(y,z) : [0,\omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$, $0 < \omega \leq \infty$; moreover, $\mathrm{graph}((y,z)) \subset \mathcal{D}$ does not have compact closure in $\mathcal{D}$.

STEP 2: *It is shown that the absolutely continuous tracking error* $e(\cdot)$ *satisfies*

$$\langle e(t), \dot{e}(t)\rangle \leq -L\|e(t)\| - \langle e(t), CB\,\mathrm{sat}_{\hat{\mathrm{u}}}(k(t)e(t))\rangle \quad \text{for almost all } t \in [0,\omega). \tag{3.44}$$

Since graph$((y,z))$ is in $\mathcal{D}$, it follows that graph(e) is in $\mathcal{F}(0,\varphi)$ and so

$$\forall\ t \in [0,\omega) : \quad \|e(t)\| < \psi(t) \leq \|\psi\|_\infty. \tag{3.45}$$

In view of (3.26), it follows that

$$\forall\, t \in [0, \omega): \quad \|z(t)\| \le M := \beta\|Nx^0\| + \frac{\beta}{\alpha}\|P_1\|\big[\|\psi\|_\infty + \|r\|_\infty\big]. \tag{3.46}$$

The conjunction of (3.29), (3.45) and (3.46) give

$$\|R_1\|\,\|e(t)\| + \|S\|\,\|z(t)\| \le L - \|R_1\|\,\|r\|_\infty - \|\dot r\|_\infty \quad \text{for almost all } t \in [0, \omega). \tag{3.47}$$

By absolute continuity of $e(\cdot)$ and the first subsystem in (3.43), it yields

$$\dot e(t) = R_1\big(e(t) + r(t)\big) + Sz(t) + CBu(t) - \dot r(t) \quad \text{for almost all } t \in [0, \omega),$$

from which, on invoking (3.47), it follows that

$$\langle e(t), \dot e(t)\rangle \le L\|e(t)\| - \langle e(t), CB\,\mathrm{sat}_{\widehat{u}}(k(t)e(t))\rangle \quad \text{for almost all } t \in [0, \omega),$$

which yields (3.44).

STEP 3: *It is shown that, for ε_0 as in* (3.31)*,*

$$\forall\, t \in [0, \omega): \quad \psi(t) - \|e(t)\| \ge \varepsilon_0. \tag{3.48}$$

Seeking a contradiction, suppose there exists $t_1 \in [0, \omega)$ such that $\psi(t_1) - \|e(t_1)\| < \varepsilon_0$. Since $\psi(0) - \|e(0)\| \ge \varepsilon_0$,

$$t_0 := \max\{t \in [0, t_1) \mid \psi(t) - \|e(t)\| = \varepsilon_0\} \in (0, t_1)$$

is well defined. Moreover,

$$\forall\, t \in [t_0, t_1]: \quad \|e(t)\| \ge \psi(t) - \varepsilon_0 \ge \lambda - \varepsilon_0 \ge \frac{\lambda}{2}$$

and so

$$\forall\, t \in [t_0, t_1]: \quad k(t)\|e(t)\| = \frac{\|e(t)\|}{\psi(t) - \|e(t)\|} \ge \frac{\lambda}{2\varepsilon_0} \ge \widehat{u}.$$

Therefore,

$$\forall\, t \in [t_0, t_1]: \quad \mathrm{sat}_{\widehat{u}}(k(t)e(t)) = \frac{\widehat{u}}{\|e(t)\|}e(t)$$

which, together with (1.11), implies that

$$\forall\, t \in [t_0, t_1]: \quad \langle e(t), CB\;\mathrm{sat}_{\widehat{u}}(k(t)e(t))\rangle \ge \gamma\widehat{u}\|e(t)\|,$$

and so, in view of (3.28) and (3.44), it follows that

$$\langle e(t), \dot{e}(t) \rangle \leq -\ell \|e(t)\| \quad \text{for almost all } t \in [t_0, t_1].$$

Integration, together with the Lipschitz property of $\psi(\cdot)$, yields

$$\|e(t_1)\| - \|e(t_0)\| \leq -\ell[t_1 - t_0] \leq -|\psi(t_1) - \psi(t_0)| \leq \psi(t_1) - \psi(t_0),$$

whence the contradiction:

$$\varepsilon_0 = \psi(t_0) - \|e(t_0)\| \leq \psi(t_1) - \|e(t_1)\| < \varepsilon_0.$$

Therefore, (3.48) holds.

STEP 4: *Assertions (i)-(iii) are shown.*
It immediately follows that the function $k(\cdot)$ is bounded, with $k(t) \leq \frac{1}{\varepsilon_0}$ for all $t \in [0, \omega)$. Moreover, in view of (3.45) and (3.46) and boundedness of $y_{\mathrm{ref}}(\cdot)$, boundedness of the solution

$$x : [0, \omega) \to \mathbb{R}^n, \quad t \mapsto x(t) = B(CB)^{-1} y(t) + V z(t)$$

can be conclude. To establish Assertions (i)-(iii), it remains to show that $\omega = \infty$. Suppose that $\omega < \infty$ and define

$$\mathcal{C} := \big\{ (t, \eta, \zeta) \in [0, \omega] \times \mathbb{R}^m \times \mathbb{R}^{n-m} \mid \psi(t) - \|\eta\| \geq \varepsilon,\ \|\eta\| \leq \|\psi\|_\infty,\ \|\zeta\| \leq M \big\}.$$

Then, in view of (3.45), (3.46) and (3.48), it follows that $\mathcal{C}$ is a compact set which contains graph$((e, z)) = \{(t, e(t), z(t)) \mid t \in [0, \omega)\}$, thereby contradicting the fact that the closure of the latter is not a compact subset of $\mathcal{D}$. Therefore, $\omega = \infty$.

STEP 5: *Finally, Assertion (iv) is established.*

STEP 5A: *Existence of $\tau \geq 0$ such that $\|u(\tau)\| < \widehat{u}$ is established.*
Seeking a contradiction, suppose that $\|u(t)\| = \widehat{u}$ for all $t \geq 0$, i.e.

$$\forall\, t \geq 0: \quad k(t)\|e(t)\| \geq \widehat{u}.$$

Then $\mathrm{sat}_{\widehat{\mathrm{u}}}(k(t)e(t)) = \frac{\widehat{u}}{\|e(t)\|} e(t)$ for all $t \geq 0$ and so, by (1.11),

$$\forall\, t \geq 0: \quad \langle e(t), CB\ \mathrm{sat}_{\widehat{\mathrm{u}}}(k(t)e(t)) \rangle \geq \gamma \widehat{u} \|e(t)\|$$

which, in conjunction with (3.44), yields

$$\forall\, t \geq 0: \quad \langle e(t), \dot{e}(t)\rangle \leq -\Delta\, \|e(t)\|$$

Integration gives

$$\forall\, t \geq 0: \quad 0 \leq \|e(t)\| \leq \|e(0)\| - \Delta\, t$$

which contradicts the fact that, by (3.28), $\Delta > 0$.

STEP 5B: *It is shown that if the input $u(\cdot)$ is unsaturated at some time $\tau \geq 0$, then it remains unsaturated for all $t \geq \tau$.*

Assume $\|u(\tau)\| < \widehat{u}$ for some $\tau \geq 0$. Suppose that there exists $t_1 > \tau$ such that $\|u(t_1)\| = \widehat{u}$. In view of (3.28), choose $\delta > 0$ sufficiently small so that

$$\|u(\tau)\| \leq (1-\delta)\widehat{u} \quad \text{and} \quad \delta\gamma\widehat{u} \leq \frac{\Delta}{2}.$$

Define

$$t_0 := \sup\{t \in [\tau, t_1] \mid \|u(t)\| = (1-\delta)\widehat{u}\}.$$

Then, invoking (3.28), it follows that

$$\forall\, t \in [t_0, t_1]: \quad \gamma\widehat{u} \geq \gamma\|u(t)\| \geq (1-\delta)\gamma\widehat{u} \geq L + \ell + \frac{\Delta}{2}.$$

Inequality (3.44) implies that

$$\langle e(t), \dot{e}(t)\rangle \leq -\ell\|e(t)\| \quad \text{for almost all } t \in [t_0, t_1],$$

which, on integration and invoking the Lipschitz property of $\psi(\cdot)$, yields

$$\|e(t_1)\| - \|e(t_0\| \leq -\ell\big[t_1 - t_0\big] \leq -|\psi(t_1) - \psi(t_0)| \leq \psi(t_1) - \psi(t_0),$$

whence the contradiction

$$\widehat{u} = \|u(t_1)\| = k(t_1)\|e(t_1)\| = \frac{\|e(t_1)\|}{\psi(t_1) - \|e(t_1)\|} < \frac{\|e(t_0)\|}{\psi(t_0) - \|e(t_0)\|} = k(t_0)\|e(t_0)\| = \|u(t_0)\| < \widehat{u}.$$

STEP 5C: *Finally, it is turned to the last claim in Assertion (iv).*
Note that

$$\|u(0)\| = \frac{\|e(0)\|}{\psi(0) - \|e(0)\|} < \widehat{u} \qquad \Leftrightarrow \qquad \|e(0)\| < \frac{\psi(0)\, \widehat{u}}{1+\widehat{u}}$$

and so the claim follows from Step 5B and setting $\tau = 0$. This completes the proof. □

Proof of Theorem 3.4.6:
The proof uses the notation of Proposition 1.1.7, Remark 3.3.1 and (1.8).

STEP 1: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (3.37) *(or, equivalently,* (1.13)*,* (3.37)*) is shown.*
The initial value problem (1.13), (3.37) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} e(t) \\ z(t) \end{pmatrix} = f(t, e(t), z(t)), \qquad \begin{pmatrix} e(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} Cx^0 - r(0) \\ Nx^0 \end{pmatrix}, \tag{3.49}$$

where

$$f : \mathcal{D} \to \mathbb{R}^n,\ (t, e, z) \mapsto \begin{pmatrix} R_1 e + Sz + R_1 r(t) - \dot{r}(t) \\ -CB \begin{pmatrix} \text{sat}_{\widehat{u}_1}\left(\frac{e_1}{\psi_1(t) - |e_1|} - d_{u,1}(t)\right) \\ \vdots \\ \text{sat}_{\widehat{u}_m}\left(\frac{e_m}{\psi_m(t) - |e_m|} - d_{u,m}(t)\right) \end{pmatrix} \\ P_1 e + Qz + P_1 r(t) \end{pmatrix}$$

is locally Lipschitz on the open set

$$\mathcal{D} := \left\{(t, e, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \mid \psi_i(t) - |e_i| > 0 \text{ for all } i = 1, .., m\right\}.$$

Now standard theory of ordinary differential equations (see, for example, Theorem 1.1.2 or [88, Th. II.6.IV]), yields existence of a solution, i.e. a differentiable function $(e, z) : [0, \omega) \to \mathbb{R}^n$, $0 < \omega \leq \infty$, satisfying (3.49) and $(t, e(t), z(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$. Moreover, the solution is unique and ω may be chosen maximal, i.e. the solution can be extended to the right up to the boundary of $\mathcal{D}$.

STEP 2: *It is shown that the absolutely continuous tracking error component* $e_i(\cdot)$ *satisfies*

$$\operatorname{sgn} e_i(t)\dot{e}_i(t) < -\ell_i + (CB)_{ii}\left[\widehat{u}_i - \operatorname{sgn} e_i(t)\text{sat}_{\widehat{u}_i}(k_i(t)e_i(t) - d_{u,i}(t))\right]$$

for almost all $t \in [0,\omega)$. (3.50)

Since graph$((e,z))$ is in $\mathcal{D}$, it follows that graph(e_i) is in $\mathcal{F}(0,\varphi_i)$ and so

$$\forall\, t \in [0,\omega): \quad |e_i(t)| < \psi_i(t) \le \|\psi\|_\infty. \tag{3.51}$$

In view of (3.26), (3.46) follows and the conjunction of (3.29), (3.51) and (3.46) gives (3.47). It make use of the assumption (3.34). The first equation of (3.49), together with the assumption (3.34), yields, for almost all $t \in [0,\omega)$,

$$\begin{aligned}
&\operatorname{sgn} e_i(t)\ \dot e_i(t) \\
&\quad\overset{(3.49)}{=} \operatorname{sgn} e_i(t)\ \left[R_1 e(t) + S z(t) + R_1 r(t) + \dot r(t)\right]_i \\
&\qquad\quad - \operatorname{sgn} e_i(t)\ \textstyle\sum_{j=1}^m (C\,B)_{ij} \mathsf{sat}_{\widehat{u}_j}(k_j(t)e_j(t) - d_{u,j}(t)) \\
&\quad\overset{(3.47)}{\le} L - \operatorname{sgn} e_i(t)\ \textstyle\sum_{j=1}^m (C\,B)_{ij} \mathsf{sat}_{\widehat{u}_j}(k_j(t)e_j(t) - d_{u,j}(t)) \\
&\quad\underset{(3.34)}{\overset{(3.35)}{<}} -\ell_i + (CB)_{ii}\ \widehat{u}_i - \textstyle\sum_{j=1,j\ne i}^m |(CB)_{ij}| \widehat{u}_j \\
&\qquad\quad - \operatorname{sgn} e_i(t)\ \textstyle\sum_{j=1}^m (C\,B)_{ij} \mathsf{sat}_{\widehat{u}_j}(k_j(t)e_j(t) - d_{u,j}(t)) \\
&\quad\le -\ell_i + (CB)_{ii}\ \widehat{u}_i - (C\,B)_{ii}\ \operatorname{sgn} e_i(t)\ \mathsf{sat}_{\widehat{u}_i}(k_i(t)e_i(t) - d_{u,i}(t)),
\end{aligned}$$

and so, (3.50) follows.

STEP 3: *It is shown that, for* $\varepsilon_{0,i}$ *as in* (3.38)*,*

$$\forall\, i \in \{1,...,m\}\ \forall\, t \in [0,\omega): \quad \psi_i(t) - |e_i(t)| \ge \varepsilon_{0,i}. \tag{3.52}$$

Seeking a contradiction, suppose there exists $i_0 \in \{1,...,m\}$ and $t_1 \in [0,\omega)$ such that $\psi_{i_0}(t_1) - |e_{i_0}(t_1)| < \varepsilon_{0,i_0}$. Since $\psi_{i_0}(0) - |e_{i_0}(0)| \ge \varepsilon_{0,i_0}$,

$$t_0 := \max\{t \in [0,t_1) \mid \psi_{i_0}(t) - |e_{i_0}(t)| = \varepsilon_{0,i_0}\} \in (0,t_1)$$

is well defined. Moreover,

$$\forall\, t \in [t_0,t_1]: \quad |e_{i_0}(t)| \ge \psi_{i_0}(t) - \varepsilon_{0,i_0} \ge \lambda_{i_0} - \varepsilon_{0,i_0} \ge \frac{\lambda_{i_0}}{2}$$

and so

$$\begin{aligned}
\forall\, t \in [t_0,t_1]: \quad |k_{i_0}(t)e_{i_0}(t) - d_{u,i_0}(t)| &\ge k_{i_0}(t)|e_{i_0}(t)| - \|d_{u,i_0}\|_\infty \\
&= \frac{|e_{i_0}(t)|}{\psi_{i_0}(t) - |e_{i_0}(t)|} - \|d_{u,i_0}\|_\infty \ge \frac{\lambda_{i_0}}{2\varepsilon_{0,i_0}} - \|d_{u,i_0}\|_\infty \ge \widehat{u}_{i_0}.
\end{aligned}$$

It has to be noted that $k_{i_0}(t)|e_{i_0}(t)| - \|d_{u,i_0}\|_\infty \geq \widehat{u}_{i_0}$ for all $t \in [t_0, t_1]$. Since $\operatorname{sgn} e_{i_0}(\cdot)$ is constant on $[t_0, t_1]$, it holds

$$\operatorname{sgn} e_{i_0}(\cdot) = \operatorname{sgn}(k_{i_0}(\cdot)e_{i_0}(\cdot) - d_{u,i_0}(\cdot)) \qquad \text{on } [t_0, t_1],$$

whence

$$-\operatorname{sgn} e_{i_0}(t)\ \mathrm{sat}_{\widehat{u}_{i_0}}(k_{i_0}(t)e_{i_0}(t) - d_{u,i_0}(t)) = -\widehat{u}_{i_0} \quad \text{for almost all } t \in [t_0, t_1],$$

and so, in view of (3.50), it follows that

$$\operatorname{sign}\, e_{i_0}(t)\ \dot{e}_{i_0}(t) \ < \ -\ell_{i_0} \quad \text{for almost all } t \in [t_0, t_1].$$

Integration gives

$$|e_{i_0}(t_1)| - |e_{i_0}(t_0)| = \int_{t_0}^{t_1} \operatorname{sign}\, e_{i_0}(\tau)\ \dot{e}_{i_0}(\tau)\,\mathrm{d}\tau \ < -\ell_{i_0}(t_1 - t_0)$$

whence, together with the Lipschitz property of $\psi_i(\cdot)$, the contradiction

$$\begin{aligned} 0 < \psi_{i_0}(t_0) - |e_{i_0}(t_0)| - [\psi_{i_0}(t_1) - |e_{i_0}(t_1)|] &= \psi_{i_0}(t_0) - \psi_{i_0}(t_1) \\ &+ [|e_{i_0}(t_1)| - |e_{i_0}(t_0)|] < \ell_{i_0}(t_1 - t_0) - \ell_{i_0}(t_1 - t_0) = 0. \end{aligned}$$

Therefore, (3.52) holds.

STEP 4: Assertions (i)-(iii) follow in the same way as in Step 4 of the proof of Theorem 3.4.2 and are omitted for brevity.

STEP 5: *Assertion (iv) is established.*
Seeking a contradiction to each input $u_i(\cdot)$ is unsaturated at some time $\tau_i \geq 0$, suppose that there exist $i_0 \in \{1, ..., m\}$ such that $|u_i(t)| = \widehat{u}_i$ for all $t \geq 0$, i.e.

$$\exists\ i_0 \in \{1, ..., m\}\ \forall\, t \geq 0 : \quad |k_{i_0}(t)e_{i_0}(t) - d_{u,i_0}(t)| \geq \widehat{u}_{i_0}.$$

Positivity of $\widehat{u}_{i_0}$ implies that $\operatorname{sgn}(k_{i_0}(\cdot)e_{i_0}(\cdot) - d_{u,i_0}(\cdot))$ is constant on $[0, \infty)$. Only the case $\operatorname{sgn}(k_{i_0}(\cdot)e_{i_0}(\cdot) - d_{u,i_0}(\cdot)) \equiv 1$ is considered, the other case follows analogously.

By the same argumentation as in STEP 2, it follows, for almost all $t \in [0, \infty)$,

$$
\begin{aligned}
&\dot{e}_{i_0}(t) \\
&\quad \underset{(3.47)}{\overset{(3.49)}{\leq}} L - \sum_{j=1}^{m} (C\,B)_{i_0,j} \mathrm{sat}_{\widehat{u}_j}(k_j(t) e_j(t) - d_{u,j}(t)) \\
&\quad \underset{(3.35)}{\overset{(3.34)}{<}} -\ell_{i_0} + (CB)_{i_0,i_0}\, \widehat{u}_{i_0} - \sum_{j=1, j\neq i_0}^{m} |(CB)_{i_0,j}| \widehat{u}_j \\
&\qquad - \sum_{j=1}^{m} (C\,B)_{i_0,j} \mathrm{sat}_{\widehat{u}_j}(k_j(t) e_j(t) - d_{u,j}(t)) \\
&\quad = -\ell_{i_0} + (CB)_{i_0,i_0}\, \widehat{u}_{i_0} - \sum_{j=1, j\neq i_0}^{m} |(CB)_{i_0,j}| \widehat{u}_j \\
&\qquad -(CB)_{i_0,i_0} \mathrm{sat}_{\widehat{u}_{i_0}}(k_{i_0}(t) e_{i_0}(t) - d_{u,i_0}(t)) + \sum_{j=1, j\neq i_0}^{m} |(CB)_{i_0,j}| \widehat{u}_j \\
&\quad = -\ell_{i_0}
\end{aligned}
$$

which, by integration, yields the contradiction

$$
\forall\, t > 0: \quad -\|\psi_{i_0}\|_\infty \leq -\psi_{i_0}(t) \leq e_{i_0}(t) < e_{i_0}(0) - \ell_{i_0}\, t.
$$

STEP 6: *Finally, Assertion (v) is established.*
Let the input disturbance $d_u(\cdot) \equiv 0$.

STEP 6A: *It is shown that the tracking error components $e_i(\cdot)$ satisfies*

$$
\begin{aligned}
\forall\, i \in \{1, ..., m\}: \quad & \Big[|e_i(t)| - |e_i(t_0)| < -\ell_i(t - t_0) \quad \text{for some } t > t_0 \geq 0 \Big] \\
& \qquad \Rightarrow \Big[k_i(t)|e_i(t)| - k_i(t_0)|e_i(t_0)| < -\ell_i k_i(t)(t - t_0) \Big]. \qquad (3.53)
\end{aligned}
$$

If

$$
|e_i(t)| - |e_i(t_0)| < -\ell_i(t - t_0) \quad \text{for some } t > t_0 \geq 0,
$$

then

$$
\begin{aligned}
k_i(t)|e_i(t)| - k_i(t_0)|e_i(t_0)| &= k_i(t) k_i(t_0) \left[|e_i(t)| \psi_i(t_0) - |e_i(t_0)| \psi_i(t) \right] \\
&< k_i(t) k_i(t_0) \left[(|e_i(t_0)| - \ell_i(t - t_0)) \psi_i(t_0) - |e_i(t_0)| \psi_i(t) \right] \\
&= k_i(t) k_i(t_0) \left[((\psi_i(t_0) - \psi_i(t)) |e_i(t_0)| - \ell_i(t - t_0) \psi_i(t_0) \right] \\
&\leq k_i(t) k_i(t_0) \left[\ell_i(t - t_0) |e_i(t_0)| - \ell_i(t - t_0) \psi_i(t_0) \right] \\
&= -\ell_i k_i(t) [t - t_0],
\end{aligned}
$$

and so, (3.53) holds.

STEP 6B: *It is shown that if an input $u_i(\cdot)$ is unsaturated at some time $\tau_i \geq 0$, then it remains unsaturated thereafter.*
Assume $|u_i(\tau_i)| < \widehat{u}_i$ for some $\tau_i \geq 0$, $i \in \{1, ..., m\}$. Seeking a contradiction, suppose there exist $i_0 \in \{1, ..., m\}$ and $t_1 > \tau_{i_0}$ such that $|u_{i_0}(t_1)| = \widehat{u}_{i_0}$. Since $\widehat{u}_{i_0} - |u_{i_0}(\tau_{i_0})| > 0$ for some $\tau_{i_0} \geq 0$, there exists $\delta \in \left(0, \frac{\widehat{u}_{i_0} - |u_{i_0}(\tau_{i_0})|}{\widehat{u}_{i_0}}\right)$ such that

$$(CB\,\widehat{u})_{i_0} - \delta (CB)_{i_0 i_0} \widehat{u}_{i_0} \geq L + \ell_i. \tag{3.54}$$

It has to be noted that $|u_{i_0}(\tau_{i_0})| = k_{i_0}(\tau_{i_0})|e_i(\tau_{i_0})| < (1-\delta)\,\widehat{u}_{i_0}$. Define

$$t_0 := \sup\{t \in [\tau_{i_0}, t_1] \mid |u_{i_0}(t)| = (1-\delta)\widehat{u}_{i_0}\}.$$

Then, it holds

$$\begin{aligned} \forall\, t \in [t_0, t_1):\ \widehat{u}_{i_0} > |u_{i_0}(t)| &\geq (1-\delta)\widehat{u}_{i_0} \\ &\overset{(3.34)}{\underset{(3.54)}{\geq}} \frac{1}{(CB)_{i_0 i_0}} \left[L + \ell_{i_0} + \sum_{j=1, j\neq i_0}^{m} |(CB)_{i_0 j}| \widehat{u}_j \right]. \end{aligned} \tag{3.55}$$

$(1-\delta)\widehat{u}_{i_0} > 0$ implies that $\operatorname{sign} e_{i_0}(\cdot)$ is constant on $[t_0, t_1]$ and thus $\operatorname{sign} e_{i_0}(t) u_{i_0}(t) = |u_{i_0}(t)|$ for all $t \in [t_0, t_1]$. By invoking (3.55), it follows

$$\begin{aligned} \operatorname{sign} e_{i_0}(t) \dot{e}_{i_0}(t) &\overset{(3.49)}{\underset{(3.46)}{\leq}} L - \operatorname{sign} e_{i_0}(t) \sum_{j=1}^{m} (CB)_{i_0 j} u_j(t) \\ &\overset{(3.34)}{\leq} L - (CB)_{i_0 i_0} |u_{i_0}(t)| + \sum_{j=1, j\neq i_0}^{m} |(CB)_{i_0 j}| \widehat{u}_j \\ &\overset{(3.55)}{\leq} -\ell_{i_0} \qquad \text{for almost all } t \in [t_0, t_1] \end{aligned}$$

which, on integration, yields

$$|e_{i_0}(t_1)| - |e_{i_0}(t_0)| < -\ell_{i_0}(t_1 - t_0),$$

whence, by (3.53), the contradiction

$$\widehat{u}_{i_0} = |u_{i_0}(t_1)| = k_{i_0}(t_1)|e_{i_0}(t_1)| < k_{i_0}(t_0)|e_{i_0}(t_0)| - \ell_{i_0} k_{i_0}(t_1)(t_1 - t_0) < \widehat{u}_{i_0}.$$

STEP 6C: *Finally, it is turned to the last claim in Assertion (v).*
It has to be noted that

$$\forall\, i \in \{1, ..., m\} : \quad \left[|u_i(0)| = \frac{|e_i(0)|}{\psi_i(0) - |e_i(0)|} < \widehat{u}_i \qquad \Leftrightarrow \qquad |e_i(0)| < \frac{\psi_i(0)\, \widehat{u}_i}{1 + \widehat{u}_i} \right]$$

and so the claim follows from STEP 6B and setting $\tau = 0$. This completes the proof. □

3.7 Notes and references

In the early 1980s, a novel feature in classical adaptive control was introduced: adaptive strategies which do not require identification of the particular system being controlled. Pioneering contributions to the area include [15, 57, 58, 63, 89] (see, also, the survey [42] and references therein). The concept of funnel control was originally introduced by [43] in the context of output tracking. Not only the class of linear systems (1.1) with CB positive definite, i.e. for systems (1.1) with strict relative degree one, was considered, but a rather general system class including nonlinear systems, nonlinear delay systems, systems with hysteresis and infinite-dimensional regular linear systems. Further results of funnel control can be found in [41, 43, 44, 45, 46, 47, 80]. This control concept has been successfully applied in experiments controlling the speed of electric devices (see [47]); see [42] for further applications.
In [44] linear systems with higher strict relative degree are considered. To apply the funnel controller (3.6) to linear systems with strict relative degree $\varrho \geq 2$ an additional filter is necessary.

The results of the present chapter are published in [32]. I developed a feasibility condition under which the tracking objective of the funnel control – in the presence of input saturation – is achieved and a proof is presented. The idea of this new proof simplifies the proof of the classical funnel control result without saturation (see [65]) which is more elementary than the proofs of [41, 43, 44, 45, 46, 80].
Moreover, this idea is required to introduce the concept of funnel control (together with input saturation) in Chapter 5 by invoking the framework of positive Volterra-Stieltjes systems (1.2).

4 Funnel control for systems with unknown relative degree 1 or 2

Chapter 3 deals with the tracking problem of a reference signal (assumed bounded with essentially bounded derivative) in the context of linear multi input, multi output systems (1.1) in the presence of input saturation. It is assumed that the system has strict relative degree one with positive high-frequency gain and stable zero dynamics. In the following chapter, tracking – by the system output and its derivative – of a reference signal and its derivative (both assumed essentially bounded) is considered in the context of systems (1.1) which are singe input, single output (i.e. $m = 1$).
The system is assumed to have strict relative degree two with stable zero dynamics. Prespecified are two parameterized performance funnels, within which the tracking error $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$ and its derivative $\dot{e}(\cdot) = \dot{y}(\cdot) - \dot{y}_{\text{ref}}(\cdot)$ are required to evolve: transient and asymptotic behaviour of $e(\cdot)$ and $\dot{e}(\cdot)$ are influenced through choice of parameter values which define the funnels. The proposed simple controller is

$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t), \tag{4.1}$$

where the feedback has two gain functions $k_0(\cdot)$ and $k_1(\cdot)$ designed in such a way to preclude contact of $e(\cdot)$ and $\dot{e}(\cdot)$ with the funnel boundaries, respectively. The controller also ensures boundedness of all signals.
Apart from this central result of Chapter 4.2.1, it is also showed that the same controller can be achieved

(i) to systems (1.1) with relative degree one, stable zero dynamics and positive high-frequency gain (see Chapter 4.2.2) and

(ii) in the presence of input constraints provided a feasibility condition (formulated in terms of the system data, the saturation bounds, the funnel data, bounds on the reference signal, and the initial state) holds (see Chapter 4.2.3).

In other words: the simple funnel controller $u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t)$ is applicable to single input, single output systems (1.1) with unknown relative degree one or two. Finally, the theoretical results are illustrated by some serially connected mass-spring damper.

4.1 Motivation and Introduction

The concept of high-gain proportional output feedback (3.1) and adaptive feedback (3.2), for multi input, multi output systems (1.1), was presented in the introduction of Chapter 3 (see Section 3.1). This means that, for scalar single input, single output systems

$$\dot{y}(t) + a_0 y(t) = u(t), \qquad a_0 \in \mathbb{R},\ y(0) = y^0 \in \mathbb{R}$$

(which has relative degree one, positive high-frequency gain one and no zero dynamics), the proportional feedback (3.1) yields a closed-loop system which is stable if $k > 0$ is sufficiently large. In the context of adaptive control this feedback law becomes time-varying, i.e. $u(t) = -k(t)y(t)$, $\dot{k}(t) = y(t)^2$, $k(0) = k_0 \in \mathbb{R}$, and the closed-loop system satisfies

$$\lim_{t\to\infty} y(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$$

(see Section 3.1).
The different approach of funnel control was introduced in Chapter 3. The simplicity of the feedback (3.2) is preserved and the adaption gain (3.3) is replaced by

$$k(t) = \frac{1}{\psi(t) - |y(t)|}, \tag{4.2}$$

where $\psi : [0, \infty) \to [\lambda, \infty)$, $\lambda > 0$, is a bounded differentiable function which represents the funnel boundary (see Section 3.3). The funnel controller guarantees that

$$\limsup_{t\to\infty} |y(t)| < \lambda.$$

It has to be noted that $\lambda > 0$ is prespecified and can be arbitrarily small.

If (3.1) or (3.2), together with (3.2), is applied to a relative degree two system, take for example the simple scalar system

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = u(t), \quad (y(0), \dot{y}(0))^\top = (y^0, y^1) \in \mathbb{R}^2, \tag{4.3}$$

where $a_0,\ a_1 \in \mathbb{R}$, then the closed-loop system is not asymptotically stable.
As motivation and for simplicity, the proportional feedback is considered.

(i) If

$$u(t) = -k_0 y(t), \quad k_0 \in \mathbb{R}$$

is applied to (4.3), then the closed-loop system is given by

$$\ddot{y}(t) + a_1\dot{y}(t) + [a_0 + k_0]\, y(t) = 0, \quad (y(0), \dot{y}(0))^\top = (y^0, y^1) \in \mathbb{R}^2$$

which is uniformly asymptotically stable if, and only if,

$$a_1 > 0 \quad \wedge \quad a_0 + k_0 > 0.$$

For $k_0 > 0$ large enough, the second condition is satisfied. The further information $a_1 > 0$ about the system (4.3) is used which is typically unknown.

(ii) As in (i), a similar calculation shows that the feedback

$$u(t) = -k_0^2 y(t), \quad k_0 \in \mathbb{R},$$

applied to (4.3), yields the closed-loop system

$$\ddot{y}(t) + a_1\dot{y}(t) + \left[a_0 + k_0^2\right] y(t) = 0, \quad (y(0), \dot{y}(0))^\top = (y^0, y^1) \in \mathbb{R}^2$$

which is uniformly asymptotically stable if, and only if, $a_1 > 0$ and $a_0 + k_0^2 > 0$. This gives the same problem as in (i): the information $a_1 > 0$ is important.

Now the proportional feedback

$$u(t) = -k_0 y(t) - k_1\dot{y}(t), \quad k_0,\ k_1 \in \mathbb{R}, \tag{4.4}$$

applied to (4.3), yields a closed-loop system which is given by

$$\ddot{y}(t) + [a_1 + k_1]\, \dot{y}(t) + [a_0 + k_0]\, y(t) = 0, \quad (y(0), \dot{y}(0))^\top = (y^0, y^1) \in \mathbb{R}^2$$

and uniformly asymptotically stable if, and only if,

$$a_1 + k_1 > 0 \quad \wedge \quad a_0 + k_0^2 > 0.$$

It has to be noted that a_0 and a_1 are typically unknown. If $k_1 = k$, $k_0 = k^2$ and k is sufficiently large, then the closed-loop system (4.3), (4.4) is stable. The idea is to tune k_i, $i = 0, 1$, time-varying, i.e.

$$\ddot{y}(t) + [a_1 + k_1(t)]\dot{y}(t) + [a_0 + k_0(t)^2]y(t) = 0, \quad (y(0), \dot{y}(0))^\top = (y^0, y^1) \in \mathbb{R}^2.$$

Note the different orders of k_0 and k_1 (see (4.1)).

In [44, 45], the concept of funnel controller has been extended to systems of higher relative degree. However, this controller involves a filter and the feedback strategy is dynamic. Already for linear minimum phase systems with relative degree 2, the feedback strategy is no longer simple: the feedback strategy is given by

$$\left.\begin{array}{rcl} u(t) &=& -k(t)e(t) - \left[|e(t)|^2 + k(t)^2\right] k(t)^4 \left[1 + |\xi(t)|^2\right] (\xi(t) + k(t)e(t)) \\ \dot{\xi}(t) &=& -\xi(t) + u(t), \quad \xi(0) = \xi^0 \in \mathbb{R} \end{array}\right\} \tag{4.5}$$

(see [44, Rem. 4 (ii), (iii)]) and the gain occurs with $k(\cdot)^6$.
The purpose of the present chapter is to show that simplicity of the control strategy $u(t) = -k(t)e(t)$ of Chapter 3 can be preserved, if derivative feedback is allowed. A simple feedback structure is designed which relies on two funnels (one for the output error, the other one for its derivative) and so, as a by product, also shape the output derivative. The funnel controller for systems with relative degree two is simply

$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t),$$

where $k_0(\cdot)$ and $k_1(\cdot)$ are defined analogously as in (4.2) with funnel boundaries $\psi_0(\cdot)$ and $\psi_1(\dot{})$, respectively. The definition of the performance funnel and the class of funnel boundaries can be found in Chapter 3, Section 3.3.3.

4.2 Funnel control results

Linear systems (1.1) which are single input, single output (i.e. $m = 1$) are considered in this section. It is assumed that the system has relative degree two, positive high-frequency gain (i.e. $CAB > 0$) and is minimum phase (i.e. (A2)).

Since the connection between Proposition 1.1.7, Proposition 1.1.9 and systems (1.1) with relative degree two is used in the main results, this is summarized in the following remark.

Remark 4.2.1. (Connection of Proposition 1.1.7, 1.1.9 and relative degree two systems)
Consider a system (1.1) which is single input, single output (i.e. $m = 1$) and has strict relative degree $\varrho = 2$, that is, in view of Definition 1.1.5,

$$CB = 0 \quad \text{and} \quad \det(CAB) \neq 0. \tag{4.6}$$

It is immediate, that if (4.6) holds, then, in view of Proposition 1.1.7, for

$$\mathcal{C} := \begin{bmatrix} C \\ CA \end{bmatrix} \in \mathbb{R}^{2\times n}, \qquad \mathcal{B} := \begin{bmatrix} B & AB \end{bmatrix} \in \mathbb{R}^{n\times 2}$$

and any $V \in \mathbb{R}^{n\times(n-2)}$ such that

$$\text{im}\, V = \ker \mathcal{C} \quad \text{and} \quad N := (V^\top V)^{-1} V^\top [I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}],$$

the similarity transformation $T = \big(\mathcal{B}(\mathcal{C}\mathcal{B})^{-1},\ V\big)$ has inverse $T^{-1} := (\mathcal{C}^\top, N^\top)^\top$ and takes system (1.1) into system (1.13) which gets the form

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \dot y(t) \end{pmatrix} &= \begin{bmatrix} 0 & 1 \\ R_0 & R_1 \end{bmatrix}\begin{pmatrix} y(t) \\ \dot y(t) \end{pmatrix} + \begin{bmatrix} 0 \\ S \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} u(t) \\ \dot z(t) &= \begin{bmatrix} P_0 & 0 \end{bmatrix}\begin{pmatrix} y(t) \\ \dot y(t) \end{pmatrix} + Q z(t) \\ \begin{pmatrix} y(0) \\ \dot y(0) \end{pmatrix} &= \mathcal{C}x^0 = \begin{pmatrix} Cx^0 \\ CAx^0 \end{pmatrix}, \qquad z(0) = N x^0, \end{aligned}\right\} \tag{4.7}$$

where

$$\begin{bmatrix} R_0, & R_1, & S \end{bmatrix} := CA^2 T, \quad P_0 := NA^2 B(CAB)^{-1}, \quad Q := NAV, \quad \Gamma := CAB.$$

Moreover, if system (1.1) has stable zero dynamics which, in view of Proposition 1.1.9, is equivalent to (1.14), then, by Proposition 1.1.7, Q is a Hurwitz matrix, in which case, together with (1.8), the following holds:

$$\exists\, \alpha,\ \beta > 0\ \forall\, t \ge 0 : \quad \left\| e^{Q\,t} \right\| \le \beta e^{-\alpha t}.$$

Now for any solution $(y(\cdot), z(\cdot), u(\cdot))$ of (4.7) on some interval $[0,\omega) \subset \mathbb{R}_{\ge 0}$ it follows, together with the Variations-of-Constants formula,

$$\forall\, t \in [0,\omega) : \quad \|z(t)\| \le \beta \|Nx^0\| + \frac{\beta}{\alpha}\|P_1\|\ \|y\|_{\mathcal{L}^\infty(0,t)}.$$

◇

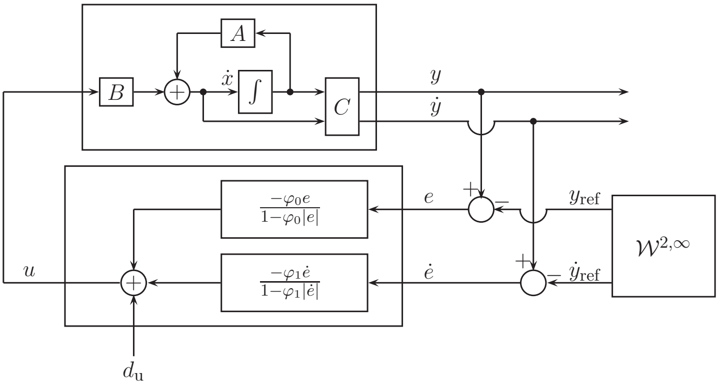


Figure 4.1: Closed-loop system (1.1), (4.9)

4.2.1 Funnel control for systems with relative degree two

As seen in the motivation, the control objective is tracking of the error $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$ and its derivative $\dot{e}(\cdot)$ within two prespecified performance funnels. This is achieved by the controller

$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t),$$

where the simplicity of the funnel controller is preserved and the gains $k_0(\cdot)$, $k_1(\cdot)$ are defined analogously as in Chapter 3 with funnel boundaries $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$.
The following theorem shows funnel control for systems with relative degree two and stable zero dynamics. This result is fundamental for the generalizations in Section 4.2.2 and 4.2.3.

Theorem 4.2.2. ***(Funnel control for systems with relative degree two)***
Suppose a system (1.1) *which is single input, single output (i.e.* $m = 1$*), satisfying (A2), has relative degree two (i.e.* (1.10)*) and positive high-frequency gain (i.e.* (1.11)*). Let* $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$ *define the pair of funnels* $(\mathcal{F}(0, \varphi_0), \mathcal{F}(0, \varphi_1))$ *as in* (3.14)*. Assume that the initial data* $x^0 \in \mathbb{R}^n$ *and the reference signal* $y_{\text{ref}}(\cdot) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ *are such that*

$$\varphi_0(0)|Cx^0 - y_{\text{ref}}(0)| < 1 \qquad \textit{and} \qquad \varphi_1(0)|CAx^0 - \dot{y}_{\text{ref}}(0)| < 1. \tag{4.8}$$

Then, for any input disturbance signal $d_u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$*, application of the feed-*

back strategy

$$\boxed{\begin{array}{rcll} u(t) & = & -k_0(t)^2 e(t) - k_1(t)\dot{e}(t) + d_u(t), & e(t) = y(t) - y_{\mathrm{ref}}(t) \\ k_i(t) & = & \frac{\varphi_i(t)}{1-\varphi_i(t)|e^{(i)}(t)|}, & i = 0,1 \end{array}} \tag{4.9}$$

to (1.1) *yields a closed-loop initial value problem with the following properties.*

(i) Precisely one maximal solution $x : [0,\omega) \to \mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega = \infty$*).*

(ii) The global solution $x(\cdot; 0, x^0)$ *is bounded and the tracking error* $e(\cdot) = Cx(t) - y_{\mathrm{ref}}(t)$ *and its derivative* $\dot{e}(\cdot) = C\dot{x}(t) - \dot{y}_{\mathrm{ref}}(t)$ *evolve within the corresponding funnels* $(\mathcal{F}(0,\varphi_0), \mathcal{F}(0,\varphi_1))$*; more precisely,*

$$\forall\, i \in \{0,1\}\ \exists\, \varepsilon_i > 0\ \forall\, t \geq 0 : \quad 1 - \varphi_i(t)|e^{(i)}(t)| \geq \varepsilon_i \varphi_i(t) \tag{4.10}$$

($\forall\, t \geq 0 :\ (t, e(t)) \in \mathcal{F}(0,\varphi_0)$ *and* $(t, \dot{e}(t)) \in \mathcal{F}(0,\varphi_1)$*).*

(iii) The input $u(\cdot)$ *and the gain functions* $k_i(\cdot)$*,* $i = 0, 1$*, are bounded.*

The proof of Theorem 4.2.2 is in Section 4.4 on page 151. The main idea of the proof is presented in the following remark.

Remark 4.2.3. (Main idea of the proof of Theorem 4.2.2)
In view of Remark 4.2.1, it can be assumed that the system (1.1) is in Byrnes-Isidori form (1.13). The theory of ordinary differential equations shows existence and uniqueness of the solution $(y(\cdot), \dot{y}(\cdot), z(\cdot))$ of the closed-loop system (1.13), (4.9) on $[0,\omega)$ for some maximal $\omega \in (0,\infty]$ (see Step 1 of the proof of Theorem 4.2.2). Hence the error $e(\cdot)$ and its derivative $\dot{e}(\cdot)$ are well defined on $[0,\omega)$. Since $e(\cdot)$ and $\dot{e}(\cdot)$ evolve within the funnels, the condition (A2) yields that $z(\cdot)$ is bounded (see Remark 4.2.1) and so

$$\exists\, M > 0\ \forall\, t \in [0,\omega) :\ \ddot{e}(t) \leq M + \Gamma u(t). \tag{4.11}$$

For simplicity, it is assumed that the derivatives of absolutely continuous functions are defined everywhere and that the error $e(\cdot)$ is positive.
Now if $u(t) < 0$ and $|u(t)|$ large enough, then $\ddot{e}(t) < 0$ and it will be shown in two steps that $e(\cdot)$ and $\dot{e}(\cdot)$ are bounded away from the funnels:

(i) If it is known that $k_0(\cdot)^2 e(\cdot)$ is bounded, then it follows from (4.11) that $\dot{e}(\cdot)$ remains bounded away from the funnel boundary. This is done in Step 4 of the proof of Theorem 4.2.2. Hence it remains to show that $k_0(\cdot)$ is bounded or, equivalently, that $e(\cdot)$ is bounded away from the funnel.

(ii) This is the key feature of the proof and done in Step 3 of the proof of Theorem 4.2.2. The major steps are two show that

- $\exists\, \alpha = \alpha(\varphi_0, \varphi_1) > 0 : \ \ddot{e}(t) < -\frac{\alpha}{\varepsilon_0}$ on an interval $[t_0, t_1]$ for some "small" $\varepsilon_0 > 0$.
- Twice integration yields the parabola

$$e(t) \leq e(t_0) + \dot{e}(t_0)(t - t_0) - \frac{\alpha}{2\varepsilon_0}(t - t_0)^2$$

which attains its maximum in $[t_0, t_1]$ with the maximum value ε_0.

Then, together with some other inequalities and the parameter $\delta > 0$ (see the definition of the class $\mathcal{S}_2$ in Section 3.3.4), it follows that the error $e(\cdot)$ is bounded away from the funnel boundary, i.e.

$$\exists\, \varepsilon_0 > 0\, \forall\, t \geq 0 : \ 1 - \varphi_0(t)e(t) \leq \varepsilon_0 \varphi_0(t).$$

◇

Remark 4.2.4. (Measurement noise)
The proposed controller (4.9) tolerates output measurement disturbance $d_y(\cdot)$, provided that the disturbance belongs to the same function class as the reference signals (see Remark 3.4.8). The funnel controller (4.9) copes with input noise $d_u(\cdot)$.
Even though the reference signal $y_{\text{ref}}(\cdot)$ and disturbance signal $d_y(\cdot)$ are assumed to be of the same class, one might reasonably expect that the disturbance $d_y(\cdot)$ is "small" - if an a priori bound on the magnitude of the disturbance is available, then the asymptotic radius of the funnel should be chosen to be commensurate with that bound, e.g., in view of Remark 3.3.4, $\lambda_i > \|n^{(i)}\|_\infty$, $i = 0, 1$. ◇

4.2.2 Funnel control for systems with relative degree one

The funnel controller (4.9) is designed for systems (1.1) with relative degree two which are single input, single output. Works this controller for minimum-phase systems with relative degree one? The answer gives the following theorem.

Theorem 4.2.5. ***(Controller*** (4.9) ***and relative degree one)***
Suppose a system (1.1) *which is single input, single output (i.e.* $m = 1$*) satisfying (A2), has relative degree one (i.e.* (1.10)*) and positive high-frequency gain (i.e.* (1.11)*). Let* $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2 \cap \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ *which satisfies* $\varphi_1(0) = 0$ *define the pair of funnels* $(\mathcal{F}(0, \varphi_0), \mathcal{F}(0, \varphi_1))$ *as in* (3.14)*. Assume that the initial data* $x^0 \in \mathbb{R}^n$ *and the reference signal* $y_{\rm ref}(\cdot) \in \mathcal{C}^2(\mathbb{R}_{\geq 0}, \mathbb{R}) \cap \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ *are such that* (4.8) *holds. Then, for any input disturbance signal* $d_u(\cdot) \in \left(\mathcal{L}^\infty \cap \mathcal{C}^1\right)(\mathbb{R}_{\geq 0}, \mathbb{R})$*, application of the feedback strategy* (4.9) *to* (1.1) *yields a closed-loop initial value problem which satisfies the properties (i)-(iii) of Theorem 4.2.2.*

The proof of Theorem 4.2.5 is in Section 4.4 on page 162.

Remark 4.2.6. (Comments on Theorem 4.2.5)
Suppose a system (1.1) which is single input, single output (i.e. $m = 1$), and has relative degree one (i.e. (1.10)). In view of the controller (4.9), some care must be exercised in formulating the closed-loop initial value problem (1.1), (4.9). This is done in Step 1 of the proof, wherein the closed-loop initial value problem is formulated in Byrnes-Isidori form as (see (4.45) and Remark 3.4.1)

$$\left.\begin{array}{rcll} \dot{e}(t) & = & f_1(t, e(t), \dot{e}(t), z(t)), & e(0) = e^0 \\ \dot{z}(t) & = & f_2(t, e(t), z(t)), & z(0) = z^0, \quad (0, e^0, z^0) \in \mathcal{D}, \end{array}\right\} \qquad (4.12)$$

for suitable $f_1 : \mathcal{D} \to \mathbb{R}$, $f_2 : \mathcal{D} \to \mathbb{R}^{n-1}$ with appropriately defined domain $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$. A solution of (4.12) is a continuously differentiable function $(e, z) : [0, \omega) \to \mathbb{R}^n$ which satisfies (4.12) and has graph in $\mathcal{D}$; $(e, z)(\cdot)$ is global if $\omega = \infty$.

Now, (4.12) is an implicit ordinary differential equation of first order. A general proof of existence and uniqueness of a maximal solution of the initial value problem (4.12) is not available in the literature. The essence of Step 1 of the proof of Theorem 4.2.5 is that it may now apply the following Proposition 4.2.7 to (4.12) and conclude that there exists a unique maximal solution $(e, z) : [0, \omega) \to \mathbb{R}^n$ of the initial value (4.12) for some $\omega > 0$.

To utilize the Implicit Function Theorem to prove existence and uniqueness of solutions, the class of allowed funnels and reference signals is slightly restricted: $\varphi_0(\cdot)$, $\varphi_1(\cdot)$ and $y_{\rm ref}(\cdot)$, $\dot{y}_{\rm ref}(\cdot)$ are assumed to be continuously differentiable instead of just being absolutely continuous.
The assumption $\varphi_1(0)$ is important (see Step 1A of the proof of Theorem 4.2.5).

(i) If $\varphi_1(0) = 0$, then $u(0)$ does not depend on $\dot{e}(0)$, hence the implicit ordinary differential equation is explicit for $\dot{e}(0)$ at $t = 0$, which yields existence and uniqueness of a local solution starting at $t = 0$. And the second inequality of (4.8) is trivially satisfied.

(ii) If $\varphi_1(0) > 0$, then $\dot{e}(0)$ has to fulfill $\varphi_1(0)|\dot{e}(0)| < 1$ (see (4.8)) which might contradict the implicit differential equation. $\diamond$

The following proposition shows existence, uniqueness and maximality of an implicit initial value problem which has a special structure and properties.

Proposition 4.2.7. ***(Existence of a unique maximal solution of an implicit ODE)***
Let $(e^0, p^0, z^0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ *and let* $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$ *be a non-empty and relatively open set with* $(t_0, e^0, p^0, z^0) \in \mathcal{D}$. *Consider the implicit initial value problem*

$$\left.\begin{array}{rcll} 0 &=& F(t, e(t), \dot{e}(t), z(t)), & e(t_0) = e^0 \\ \dot{z}(t) &=& P_0 e(t) + Q z(t) + f(t), & z(t_0) = z^0, \end{array}\right\} \tag{4.13}$$

where $P_0 \in \mathbb{R}^{n-1}$, $Q \in \mathbb{R}^{(n-1)\times(n-1)}$, $f(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ *and* $F : \mathcal{D} \to \mathbb{R}$ *continuously differentiable. Assume that the function* $F(\cdot,\cdot,\cdot,\cdot)$ *is such that*

$$F(t_0, e^0, p^0, z^0) = 0 \tag{4.14}$$

and

$$\forall\, (t, e, p, z) \in \mathcal{D} : \quad \frac{\partial F}{\partial p}(t, e, p, z) \neq 0. \tag{4.15}$$

Then the following holds.

(i) There exists a unique solution $(e, z) : [t_0, \omega) \to \mathbb{R}^n$, $\omega \in (t_0, T]$, *of* (4.13), *where* $T := \sup pr_1(\mathcal{D}) \in (0, \infty]$. *Moreover, this solution can be maximally extended.*

(ii) If $(e, z)(\cdot; t_0, e^0, z^0)$ *is a bounded maximal solution of* (4.13), *then* $\omega = T$.

The proof of Proposition 4.2.7 is in Section 4.4 on page 159.

Remark 4.2.8. (Comment to Proposition 4.2.7)
A careful inspection of Step 1, 2 of the proof of Proposition 4.2.7 shows that

$$\left[F(t_0, e^0, p^0, z^0) = 0, \; \frac{\partial F}{\partial p}(t_0, e^0, p^0, z^0) \neq 0\right]$$

$$\Rightarrow\ \exists \text{ maximal solution } (e,z) : [t_0,\omega) \to \mathbb{R}^n,\ \omega \in (t_0,T].$$

This is proven with the implicit function theorem and Hausdorff maximal principle. To show that this solution $(e,z)(\cdot)$ is unique, the assumption (4.15) is important (see Step 3 of the proof of Proposition 4.2.7). And then, Assertion (i) holds. ◇

4.2.3 Input saturation for systems with strict relative degree two

In many applications, the input may be subject to certain bounds, i.e. there is some maximal bound $\widehat{u} > 0$ such that $u8t)| \leq \widehat{u}$ is required for all $t \geq 0$. In this section two different results are presented in which the funnel controller had to replaced by

$$u(t) = -\operatorname{sat}_{\widehat{\mathrm{u}}}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t)) \qquad \text{(see Theorem 4.2.9)}$$

or

$$u(t) = -\mathrm{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) - \mathrm{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t)) \qquad \text{(see Theorem 4.2.11)}$$

with $k_0(\cdot)$, $k_1(\cdot)$ as in (4.9).
It has to be noted that in the single input, single output case the saturation function gets the form

$$\operatorname{sat}_{\widehat{\mathrm{u}}} : \mathbb{R} \to \{w \in \mathbb{R} \mid |w| \leq \widehat{u}\}, \qquad v \mapsto \operatorname{sat}_{\widehat{\mathrm{u}}} := \begin{cases} \widehat{u} \operatorname{sgn} v, & |v| > \widehat{u} \\ v, & \text{otherwise.} \end{cases}$$

The theorem shows that funnel control is feasible in the presence of input saturation provided the saturation satisfies a feasibility condition.

Theorem 4.2.9. ***(Funnel control with saturation, result 1)***
Suppose a system (1.1) *which is single input, single output (i.e.* $m = 1$*), satisfying (A2), has relative degree two (i.e.* (1.10)*) and positive high-frequency gain (i.e.* (1.11)*). Let* $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{G}_2$ *define the pair of funnels* $(\mathcal{F}(0,\varphi_0), \mathcal{F}(0,\varphi_1))$ *as in* (3.14) *and adopt the notation of Remark 3.3.4 with* $\varepsilon = 0$*. Let* $d_u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ *be an input disturbance. Assume that the initial data* $x^0 \in \mathbb{R}^n$ *and the reference signal* $y_{\mathrm{ref}}(\cdot) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ *are such that* (4.8) *holds. Furthermore assume that* $\widehat{u} > 0$ *is such that the feasibility assumption*

$$\Gamma \widehat{u} - (L + M) > 0 \tag{4.16}$$

holds, with $\Gamma > 0$ *as in* (1.13) *and*

$$
\begin{aligned}
L &:= \max\left\{\frac{3}{2}\ell_1,\ \frac{2(\|\psi_1\|_\infty+\ell_0)^2}{\lambda_0},\ \frac{(\|\psi_1\|_\infty+\ell_0)^2}{\psi_0(0)-|e(0)|},\ \frac{(\|\psi_1\|_\infty+\ell_0)^2}{\Gamma\lambda_0}\left[(\|\psi_1\|_\infty+\ell_0)^2\right.\right. \\
&\qquad \left.\left. +\sqrt{(\|\psi_1\|_\infty+\ell_0)^4+2\Gamma\lambda_0 M+2\Gamma^2\lambda_0\|d_u\|_\infty+\frac{4\Gamma^2\lambda_0\|\psi_1\|_\infty}{\delta}}\ \right]\right\}, \qquad (4.17)\\
M &:= |R_1|\left[\|\psi_0\|_\infty+\|y_{\mathrm{ref}}\|_\infty\right]+|R_2|\left[\|\psi_1\|_\infty+\|\dot y_{\mathrm{ref}}\|_\infty\right]+\|\ddot y_{\mathrm{ref}}\|_\infty \\
&\qquad +\|S\|\left(\beta\|Nx^0\|+\frac{\beta}{\alpha}\|P_1\|\left[\|\psi_0\|_\infty+\|y_{\mathrm{ref}}\|_\infty\right]\right),
\end{aligned}
$$

where $(\psi_0(\cdot),\psi_1(\cdot)) = \left(\frac{1}{\varphi_0(\cdot)},\frac{1}{\varphi_1(\cdot)}\right)$. *Then application of the feedback strategy*

$$
\boxed{
\begin{array}{rcll}
u(t) &=& -\operatorname{sat}_{\hat u}\left(k_0(t)^2 e(t)+k_1(t)\dot e(t)-d_u(t)\right), & e(t)=y(t)-y_{\mathrm{ref}}(t)\\
k_i(t) &=& \frac{\varphi_i(t)}{1-\varphi_i(t)|e^{(i)}(t)|}, & i=0,1
\end{array}}
\qquad (4.18)
$$

to (1.1) *yields a closed-loop initial value problem which satisfies the properties (i)-(iii) of Theorem 4.2.2:*

(i) Precisely one maximal solution $x:[0,\omega)\to\mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega=\infty$*).*

(ii) The global solution $x(\cdot;0,x^0)$ *is bounded and the tracking error* $e(\cdot)=Cx(t)-y_{\mathrm{ref}}(t)$ *and its derivative* $\dot e(\cdot)=C\dot x(t)-\dot y_{\mathrm{ref}}(t)$ *evolve within the performance funnels* $(\mathcal{F}(0,\varphi_0),\mathcal{F}(0,\varphi_1))$*; more precisely, for all* $t\in[0,\infty)$,

$$
\psi_0(t)-|e(t)|\ge\varepsilon_0 \qquad \textit{and} \qquad \psi_1(t)-|\dot e(t)|\ge\varepsilon_1
$$

with

$$
\left.
\begin{aligned}
\varepsilon_0 &:= \min\left\{\frac{\psi_0(0)-|e(0)|}{2},\ \frac{\lambda_0}{4},\ \frac{(\|\psi_1\|_\infty+\ell_0)^2}{3\ell_1},\ \sqrt{\frac{\lambda_0\delta}{8[\delta(\hat u+\|d_u\|_\infty)+2\|\psi_1\|_\infty]}}\right\}\\
\varepsilon_1 &:= \min\left\{\psi_1(0)-|\dot e(0)|,\ \frac{\lambda_1}{2},\ \frac{\lambda_1\varepsilon_0^2}{2[\varepsilon_0^2(\hat u+\|d_u\|_\infty)+\|\psi_0\|_\infty]}\right\}.
\end{aligned}
\right\}. \qquad (4.19)
$$

(iii) The input $u(\cdot)$ *and the gain functions* $k_i(\cdot)$*,* $i=0,1$*, are bounded, with* $\|k\|_\infty\le\frac{1}{\varepsilon_i}$.

(iv) Furthermore, the input is unsaturated at some time τ*, i.e.*

$$
\exists\,\tau\ge 0:\quad |u(\tau)|<\hat u.
$$

The proof of Theorem 4.2.9 is in Section 4.4 on page 167.

Section 4.2.2 shows that the controller (4.9) is applicable to systems (1.1) with relative degree one. One may ask the question as to wether the controller (4.18), which is designed for input saturation for systems with relative degree two, also works for systems with relative degree one. The answer gives the following remark and is no, i.e. the controller (4.18) is not applicable to systems (1.1) with relative degree one.

Remark 4.2.10. (Controller (4.19)**, saturation and relative degree one)**
Theorem 4.2.5 shows that the controller (4.9) can be applied to systems (1.1) which are single input, single output with relative degree one. The key feature of the proof of Proposition 4.2.7 is the application of the implicit function theorem (see Step 1 of the proof of proposition 4.2.7) which gives existence and uniqueness of a maximal solution of the closed-loop system (1.1), (4.9). Therefore, the assumptions (4.14) and (4.15) are important. It has to be noted that, in view of the funnel controller, it is essential that the error and its derivative start in the funnel. Thus, in view of the application of Proposition 4.2.7 in the proof of Theorem 4.2.5, Step 1A of the proof of Theorem 4.2.5 is the key step; more precisely,

$$\exists \text{ unique } p^0 \in \mathbb{R}: \quad \left[1 + \Gamma\kappa_1(0,p^0)\right] p^0 + \left[\Gamma\kappa_0(0,e^0)^2 - R_1\right] e^0 - Sz^0 - R_1 r(0) + \dot r(0) - \Gamma d_u(0) = 0 \quad (4.20)$$

and

$$(0,e^0) \in \mathcal{F}(0,\varphi_0) \qquad \text{and} \qquad (0,p^0) \in \mathcal{F}(0,\varphi_1)$$

hold, where $(t,\eta) \mapsto \kappa_i(t,\eta) := \frac{\varphi_i(t)}{1-\varphi_i(t)|\eta|}$, $i = 0,1$, is defined as in the proof of Theorem 4.2.5. Now the assumption $\varphi_1(0) = 0$ is important which implies that $\kappa_1(0,p^0) = 0$ for all $p^0 \in \mathbb{R}$ and thus, (4.20) can be dissolved to p^0 and therefore the error derivative $\dot e(0)$ is in the funnel. This means that the error and its derivative start in the funnel.
In view of input saturation and the feasibility assumption, the finite funnel, i.e. $\varphi_0(0) \neq 0$ and $\varphi_1(0) \neq 0$, is important. It has to be noted that $\varphi_1(0) = 0$ which implies that $\kappa_1(0,p^0) = 0$ for all $p^0 \in \mathbb{R}$. Now it is not clear that there exists $p^0 \in \mathbb{R}$ such that

$$(0,e^0) \in \mathcal{F}(0,\varphi_0) \qquad \text{and} \qquad (0,p^0) \in \mathcal{F}(0,\varphi_1) \qquad \text{and} \qquad (4.20) \text{ holds}$$

which is important for the application of the funnel controller. ◇

Theorem 4.2.11. ***(Funnel control with saturation, result 2)***
Suppose a system (1.1) *which is single input, single output (i.e.* $m = 1$*), satisfying (A2), has relative degree two (i.e.* (1.10)*) and positive high-frequency gain (i.e.* (1.11)*). Let* $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{G}_2$ *define the pair of funnels* $(\mathcal{F}(0,\varphi_0), \mathcal{F}(0,\varphi_1))$ *as in* (3.14) *and adopt the notation of Remark 3.3.4 with* $\varepsilon = 0$*. Let* $d_{u_0}(\cdot)$*,* $d_{u_1}(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ *be input disturbances. Assume that the initial data* $x^0 \in \mathbb{R}^n$ *and the reference signal* $y_{\text{ref}}(\cdot) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ *are such that* (4.8) *holds. Furthermore assume that* $\widehat{u}_0$*,* $\widehat{u}_1 > 0$ *are such that the feasibility assumptions*

$$\Gamma\widehat{u}_0 - \left(L_0 + M + \frac{\Gamma(\delta\|d_{u_1}\|_\infty + 2\|\psi_1\|_\infty)}{\delta}\right) > 0 \tag{4.21}$$

$$\Gamma\widehat{u}_1 - \max\left\{\frac{\Gamma(\delta\|d_{u_1}\|_\infty + 2\|\psi_1\|_\infty)}{\delta},\ \Gamma\,\widehat{u}_0 + M + \ell_1\right\} > 0 \tag{4.22}$$

hold, with $\Gamma > 0$ *as in* (1.13) *and*

$$L_0 := \max\left\{\frac{3}{2}\ell_1,\ \frac{2(\|\psi_1\|_\infty + \ell_0)^2}{\lambda_0},\ \frac{(\|\psi_1\|_\infty + \ell_0)^2}{\psi_0(0) - |e(0)|},\ \frac{(\|\psi_1\|_\infty + \ell_0)^2}{\Gamma\lambda_0}\left[(\|\psi_1\|_\infty + \ell_0)^2 \right.\right.$$
$$\left.\left. + \sqrt{(\|\psi_1\|_\infty + \ell_0)^4 + 2\Gamma\lambda_0 M + \frac{2\Gamma^2\lambda_0\left[\delta(\|d_{u_0}\|_\infty + \|d_{u_1}\|_\infty) + 2\|\psi_1\|_\infty\right]}{\delta}}\right]\right\} \tag{4.23}$$

where $(\psi_0(\cdot), \psi_1(\cdot)) = \left(\frac{1}{\varphi_0(\cdot)}, \frac{1}{\varphi_1(\cdot)}\right)$*. Then application of the feedback strategy*

$$\boxed{\begin{aligned} u(t) &= -\mathit{sat}_{\widehat{u}_0}\left(k_0(t)^2 e(t) - d_{u_0}(t)\right) - \mathit{sat}_{\widehat{u}_1}\left(k_1(t)\dot{e}(t) - d_{u_1}(t)\right) \\ k_i(t) &= \tfrac{\varphi_i(t)}{1-\varphi_i(t)|e^{(i)}(t)|},\ i = 0,1, \qquad e(t) = y(t) - y_{\text{ref}}(t) \end{aligned}} \tag{4.24}$$

to (1.1) *yields a closed-loop initial value problem which satisfies the properties (i)-(iii) of Theorem 4.2.9, where* (4.19) *had to be replaced, for all* $t \in [0,\infty)$*, by*

$$\left.\begin{aligned} \psi_0(t) - |e(t)| &\geq \varepsilon_0 := \min\left\{\tfrac{\psi_0(0)-|e(0)|}{2}, \tfrac{\lambda_0}{4}, \tfrac{(\|\psi_1\|_\infty+\ell_0)^2}{3\ell_1}, \sqrt{\tfrac{\lambda_0}{8[\widehat{u}_0+\|d_{u_0}\|_\infty]}}\right\} \\ \psi_1(t) - |\dot{e}(t)| &\geq \varepsilon_1 := \min\left\{\psi_1(0) - |\dot{e}(0)|, \tfrac{\lambda_1}{2}, \tfrac{\lambda_1}{2[\widehat{u}_1+\|d_{u_1}\|_\infty]}\right\}. \end{aligned}\right\} \tag{4.25}$$

Furthermore,

(iv) the input components of $u(\cdot)$ *are unsaturated at some time, i.e.*

$$\forall\, i \in \{0,1\}\ \exists\ \tau_0 \geq 0: \quad |\mathit{sat}_{\widehat{u}_i}\left(k_i(\tau_i)^{2-i} e^{(i)}(\tau_i) - d_{u_i}(\tau_i)\right)| < \widehat{u}_i.$$

(v) *Let $d_{u_1}(\cdot)$ be zero, i.e. $d_{u_1}(\cdot) \equiv 0$. If the second input component $k_1(\cdot)\dot{e}(\cdot)$ is unsaturated at time τ, then it remains unsaturated thereafter, i.e.*

$$[\exists\, \tau \geq 0 : \; |sat_{\widehat{u}_1}(k_1(\tau)\dot{e}(\tau))| < \widehat{u}_1] \quad \Rightarrow \quad [\forall\, t \geq \tau : \; |sat_{\widehat{u}_1}(k_1(t)\dot{e}(t))| < \widehat{u}_1] \,.$$

The proof of Theorem 4.2.11 is in Section 4.4 on page 172.

Remark 4.2.12. (Comments to Theorem 4.2.9 and Theorem 4.2.11)

(a) As shown in Theorem 4.2.2, the input of the closed-loop system (1.1), (4.9) is bounded. Theorem 4.2.9 and 4.2.11 state that a saturated input yields the same results if the saturation bound is large enough. These feasibility bounds (see (4.16), (4.21), (4.22)) depend on all parameters involved the closed-loop system. The calculated feasibility condition may be very conservative.

(b) The first feasibility condition (4.8) is a necessary condition for attainment of the control objective and is equivalent to the requirement that $(0, Cx^0 - y_{\text{ref}}(0)) \in \mathcal{F}(0, \varphi_0)$ and $(0, CAx^0 - \dot{y}_{\text{ref}}(0)) \in \mathcal{F}(0, \varphi_1)$.

(c) In conjunction with the other hypotheses, the second feasibility condition (4.16) or (4.21), (4.22), resp., is a sufficient condition for attainment of the control objective. It quantifies and exhibits the interplay between the saturation bound (sufficiently large to ensure performance) and bounds on the plant data, funnel data, initial data and reference signal data. The nature of the dependence of the saturation bound on these data is not surprising:

   (i) the minimum-phase condition (A2) ensures exponential stability of the zero dynamics of the linear triple (A, B, C) – this translates into the condition (1.8) on the matrix Q in (1.1.7) – the parameter α quantifies the exponential decay rate of the zero dynamics and is inversely related to the saturation bound;

   (ii) it is to be expected that tracking of large and rapidly varying reference signals $y_{\text{ref}}(\cdot)$ and its derivative would require control inputs capable of taking sufficiently large values, this is reflected in the dependence of the saturation bound on both $\|y_{\text{ref}}\|_\infty$ and $\|\dot{y}_{\text{ref}}\|_\infty$;

   (iii) transient and asymptotic behavior of the tracking error and its derivative is influenced by the choice of funnels determined by the globally Lipschitz functions $\psi_i(\cdot)$, $i = 0, 1$. The rapid decay of the transient behavior would be reflected in large Lipschitz constants ℓ_i, appears as an additive term in the saturation bound.

(d) It has to be noted that the right-hand side of (4.16) or (4.21), (4.22), resp., depends on $\|\psi\|_\infty$ and so, in view of Remark 3.3.4, the assumption $\varphi_i(0) \neq 0$, $i = 0, 1$, is important.

(e) The assumption $\varphi_i(0) \neq 0$, $i = 0, 1$, together with Remark 3.3.4, implies that

$$\forall\, t \geq 0: \quad k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|} = \frac{1}{\varphi_i(t)^{-1} - |e^{(i)}(t)|} = \frac{1}{\psi_i(t) - |e^{(i)}(t)|}.$$

(f) It has to be noted that L and L_0 given by (4.17) and (4.23), resp., are quite similar. The difference is that L depends on $2\Gamma^4\lambda_0\|d_u\|_\infty$ in contrast to L_0 which depends on $2\Gamma^2\lambda_0\,(\|d_{u_0}\|_\infty + \|d_{u_1}\|_\infty)$.

(g) In the relative degree one case, i.e. Section 3.4, the input disturbance $d_u(\cdot)$ does not have influence on the saturation bound $\widehat{u}$, see (3.28) and (3.35). The difference in Theorem 4.2.9 and 4.2.11 is that the input disturbances $d_u(\cdot)$ and $d_{u_0}(\cdot)$, $d_{u_1}(\cdot)$, resp., have influence on the saturation bounds $\widehat{u}$ and $\widehat{u}_0$, $\widehat{u}_1$, resp.

(h) Funnel control and input saturation for linear systems (1.1) with relative degree one, i.e. $\det CB \neq 0$, are considered in Section 3.4 and it is shown that the input is unsaturated at some time $\tau \geq 0$ and it remains unsaturated thereafter, see Theorem 3.4.2 (iv) and Theorem 3.4.6 (v). In the context of Theorem 4.2.9 and 4.2.11, it is only showed that the input is unsaturated at some time $\tau \geq 0$ but not that it remains unsaturated thereafter.
Moreover, Theorem 4.2.11 shows that if the input disturbance $d_{u_1}(\cdot)$ vanishes, then the error derivative remains unsaturated thereafter.

4.3 Example

The feedback strategies (4.9), (4.18) and (4.24) are illustrated.

For purposes of illustration the results of Theorem 4.2.2, 4.2.5, 4.2.9 and 4.2.11, the three-mass serially connected mass-spring damper system (2.35) is revisited. As in Example 2.4 shown, this system is equivalent to (2.37). It suffices to consider the system (2.37). Furthermore, assume that the reference signal $y_{\text{ref}}(\cdot)$ and disturbances $d_u(\cdot)$, $d_{u_0}(\cdot)$, $d_{u_1}(\cdot)$ are given by

$$t \mapsto y_{\text{ref}}(t) = \frac{1}{2}\sin(2\,t) + \frac{1}{4}, \qquad t \mapsto d_u(t) := d_{u_0}(t) = \frac{1}{2}\cos(8\,t) - \frac{1}{2}, \qquad d_{u_1}(\cdot) \equiv 0$$

and so,

$$\|y_{\rm ref}\|_\infty = \frac{3}{4}, \quad \|\dot y_{\rm ref}\|_\infty = 1, \quad \|\ddot y_{\rm ref}\|_\infty = 2, \quad \|d_u\|_\infty = \|d_{u_0}\|_\infty = 1, \quad \|d_{u_1}\|_\infty = 0.$$

The funnels $(\mathcal{F}(0,\varphi_0), \mathcal{F}(0,\varphi_1))$ are determined by $(\varphi_0(\cdot), \varphi_1(\cdot))$ with

$$t \mapsto \varphi_0(t) = \frac{1}{0.2 + e^{-t/20}} \quad \text{and} \quad t \mapsto \varphi_1(t) = \frac{2.5\,t}{5 + 0.5\,t}$$

to illustrate the results of Theorem 4.2.2 and Theorem 4.2.5. Then, in view of Remark 3.3.4, the functions

$$t \mapsto \psi_0(t) = \frac{1}{5} + e^{-\frac{t}{20}} \quad \text{and} \quad t \mapsto \psi_1(t) = \frac{2}{t} + \frac{1}{5},\ t > 0$$

are well defined which, for $\varepsilon > 0$, $(\lambda_0, \ell_0) = \left(\frac{1}{5}, \frac{1}{20}\right)$, $(\lambda_1, \ell_1) = \left(\frac{1}{5}, \frac{2}{\varepsilon^2}\right)$ and $\delta \in \left(0, \frac{3}{20}\right]$, satisfy the conditions (2) and (3) of $\mathcal{S}_2$ and thus, $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$. In view of Theorem 4.2.5, the assumption $\varphi_1(0) = 0$ is important.

(a) First consider the system (2.38) which has relative degree two.

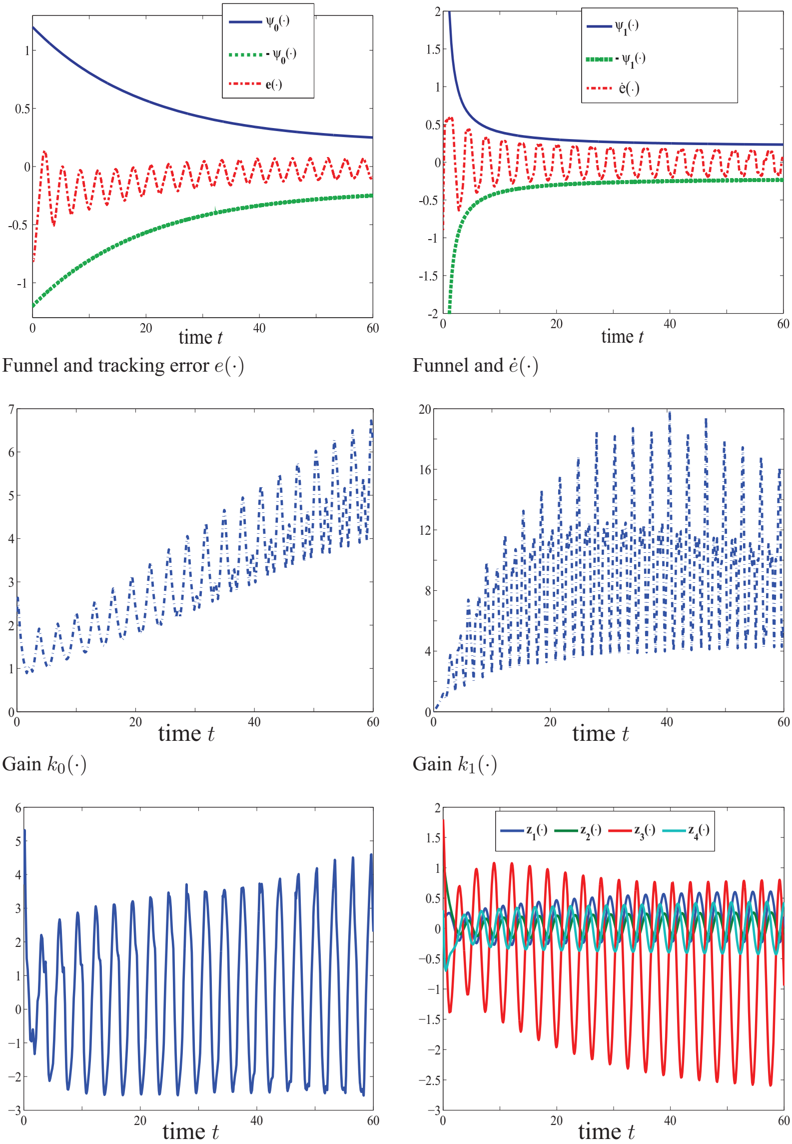
Figure 4.2 depicts the behaviour of the closed-loop system (2.38), (4.9). The simulations confirm the results of Theorem 4.2.2. The tracking error $e(\cdot)$ and its derivative $\dot e(\cdot)$ remain bounded away from the funnel boundary. Non-monotonicity of the gain functions $k_0(\cdot)$ and $k_1(\cdot)$ is also evident. It has to be noted that $\|k_0\| \approx 6.8$ and $\|k_1\| \approx 20$.

(b) Now consider the system (2.40) which has relative degree one.

Figure 4.3 depicts the behaviour of the closed-loop system (2.40), (4.9) which confirm the results of Theorem 4.2.5. The "oscillation" in the gains, speed and input are acceptable and depend on the oscillated reference signal.

(c) At last the controller (4.5) of [44] applied to (2.40) is illustrated. It is chosen $\xi^0 = 2$.

Figure 4.4 offers the simulation results of the closed-loop system (2.38), (4.5). In simulations, it can be seen that the results in Figure 4.2 have a better performance as the results in Figure 4.4, see especially the input $u(\cdot)$. Thus, the controller (4.9) yields better results as the controller (4.5) and is quite simpler. A careful inspection of Figure 4.4 shows that there is obviously a peak in the control $u(\cdot)$. This phenomenon is acceptable and depends on the filter in (4.5) and the reference signal.

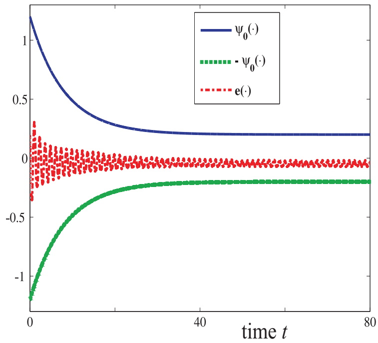


Funnel and tracking error $e(\cdot)$ — Funnel and $\dot{e}(\cdot)$

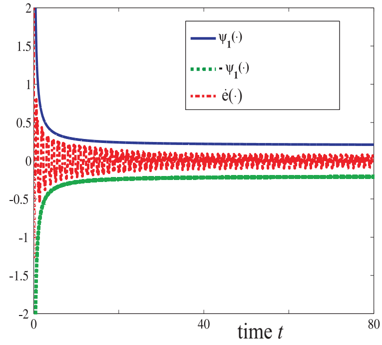
Gain $k_0(\cdot)$ — Gain $k_1(\cdot)$

Control $u(\cdot)$ — Functions $z_1(\cdot)$, ..., $z_4(\cdot)$

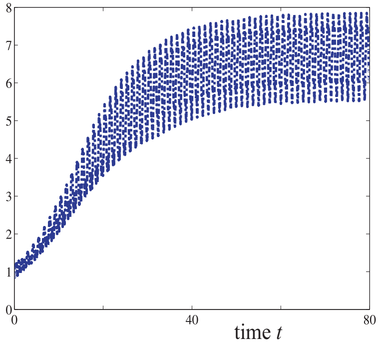
Figure 4.2: Closed-loop system (2.38), (4.9)



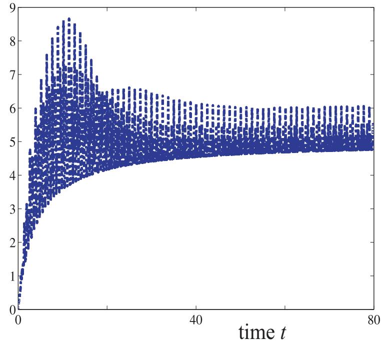
Funnel and tracking error $e(\cdot)$



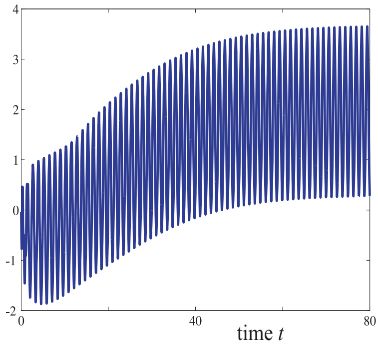
Funnel and $\dot{e}(\cdot)$



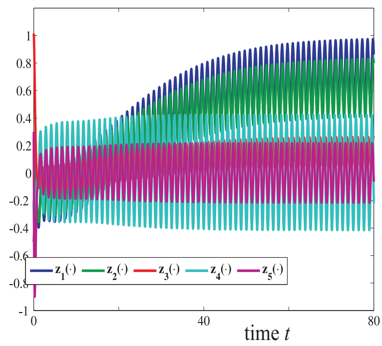
Gain $k_0(\cdot)$



Gain $k_1(\cdot)$

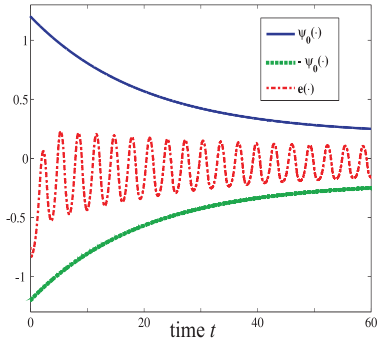


Control $u(\cdot)$

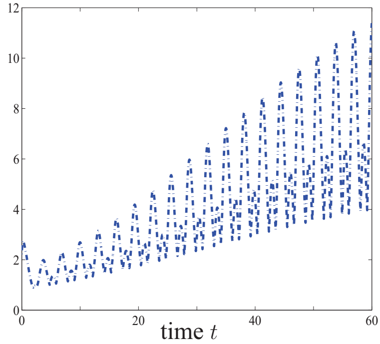


Functions $z_1(\cdot)$, ..., $z_5(\cdot)$

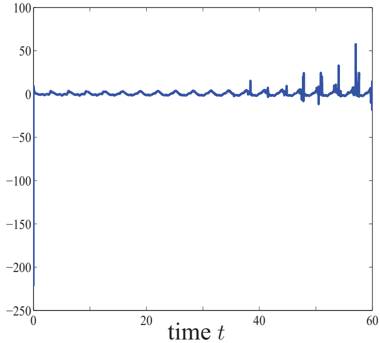
Figure 4.3: Closed-loop system (2.40), (4.9)



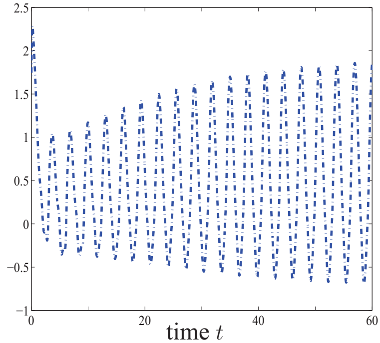
Funnel and tracking error $e(\cdot)$



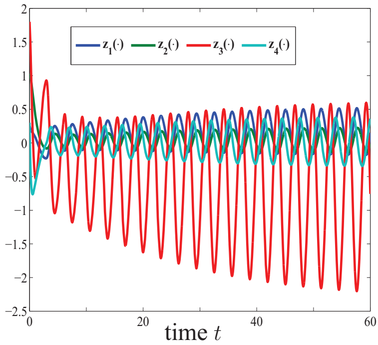
Gain $k_0(\cdot)$



Control $u(\cdot)$



Filter $\xi(\cdot)$



Functions $z_1(\cdot), \ldots, z_4(\cdot)$

Figure 4.4: Closed-loop system (2.38), (4.5)

For purposes of illustration the results of Theorem 4.2.9 and 4.2.11, the funnels $(\mathcal{F}(0,\varphi_0),\mathcal{F}(0,\varphi_0)$ are determined by $(\varphi_0(\cdot),\varphi_0(\cdot))$. An easy calculation shows that $(\varphi_0(\cdot),\varphi_0(\cdot))$ satisfies the conditions (2) and (3) of $\mathcal{G}_2$ and whence, $\|\psi_0\|_\infty = \frac{6}{5}$ and δ is chosen as $\delta = \frac{1}{5}$.

(d) Consider the closed-loop system (2.38), (4.18). In view of (2.38), it follows that

$$|R_1| = 2, \quad |R_2| = 4, \quad \|S\| = \sqrt{20}, \quad \|P_1\| = 656, \quad \|z^0\| = 4.3925,$$

and (1.8) holds with $\alpha = 1$ and $\beta = 2.8$. Moreover, (4.17) gives

$$M = 16087.82, \qquad L = 1303.71.$$

The control input is constraint with

$$\forall\, t \geq 0: \quad |u(t)| \leq \widehat{u} := 17400$$

and thus, (4.16) holds. In view of the large value $\widehat{u}$, it is not to expect that the input saturates. This computed bound $\widehat{u}$ is very large and unrealistic, compared to the actually required maximal input of 7.0 (see Figure 4.5 $u(\cdot)$). It demonstrates how conservative the feasibility bound of Theorem 4.2.9 and 4.2.11 can be.

Figure 4.5 depicts the behaviour of the closed-loop system (2.38), (4.18). The simulations confirm the results of Theorem 4.2.9. Note that $\|k_1\|_\infty \approx 14$ is smaller as in Figure 4.2.

(e) Now consider the closed-loop system (2.38), (4.24) and the control input is constraint with

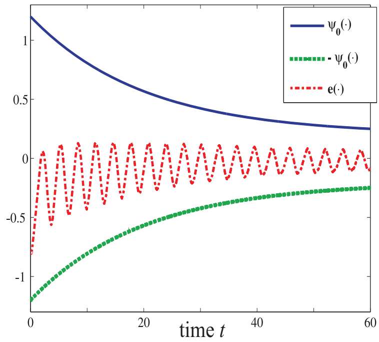
$$\widehat{u}_0 := 17405 \qquad \widehat{u}_1 := 33500.$$

An easy calculation (4.23) gives $L_0 = 1303.94$ and thus, (4.21) and (4.22) hold.

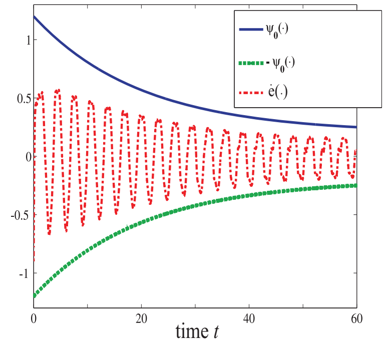
Now, it is to expect that the simulation results of the closed-loop systems (2.38), (4.24) have the same look as these of (2.38), (4.18) and thus, the pictures are omitted.

To illustrate the occurrence of saturation of the control input in the simulations, the initial value of the system (2.38) is replaced by

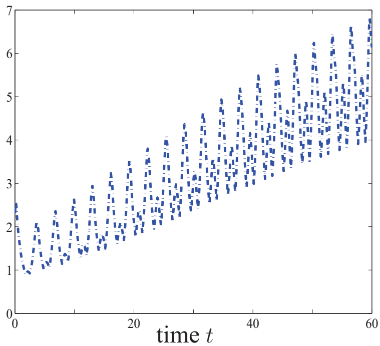
$$\left[\begin{array}{cc|c} y(0) & \dot{y}(0) & z(0)^\top \end{array}\right]^\top = \left[\begin{array}{cc|cccc} -0.94 & 0.1 & \frac{1}{4} & 1 & 1.8 & 0.3 \end{array}\right]^\top.$$



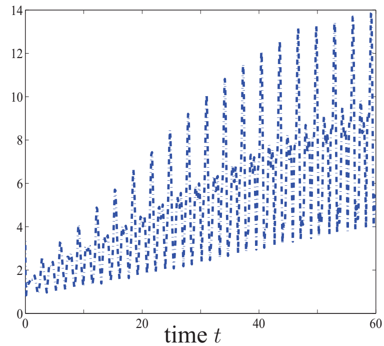
Funnel and tracking error $e(\cdot)$



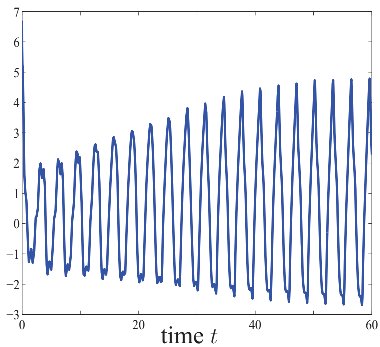
Funnel and $\dot{e}(\cdot)$



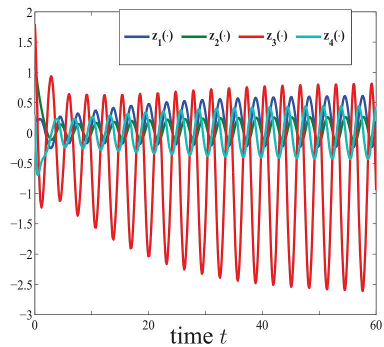
Gain $k_0(\cdot)$



Gain $k_1(\cdot)$



Control $u(\cdot)$



Functions $z_1(\cdot)$, ..., $z_4(\cdot)$

Figure 4.5: Closed-loop system (2.38), (4.18)

It has to be noted that only the initial value of $y(\cdot)$ is changed.

Figure 4.6 depicts the behaviour of the closed-loop system (2.38), (4.24) which confirms the results of Theorem 4.2.11. The error $e(\cdot)$ and its derivative $\dot{e}(\cdot)$ are shown in the long-run and the figures of the control input and the gains are zoomed in. It is highlighted that the results of Chapter 3 say either the input is initially saturated and becomes unsaturated and it remains so thereafter or the control input is initially unsaturated and the saturation bound is never attained (see Theorem 3.4.2 and 3.4.6). In contrast, the new statement of Theorem 4.2.11 is that both input parts are constraint and moreover, $k_1(\cdot)\dot{e}(\cdot) - d_{u_1}(\cdot)$ is saturated but not initially.

4.4 Proofs

Proof of Theorem 4.2.2:
The proof uses the notation of Proposition 1.1.7 and (1.8).

STEP 1: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (4.9) *or, equivalently,* (1.13)*,* (4.9) *is shown.*
It suffices to consider the system in Byrnes-Isidori form (1.13). For $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$, define the open set

$$\mathcal{D} := \left\{(t, \mu_0, \mu_1, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \,\middle|\, (t, \mu_0) \in \mathcal{F}(0, \varphi_0), (t, \mu_1) \in \mathcal{F}(0, \varphi_1)\right\}$$

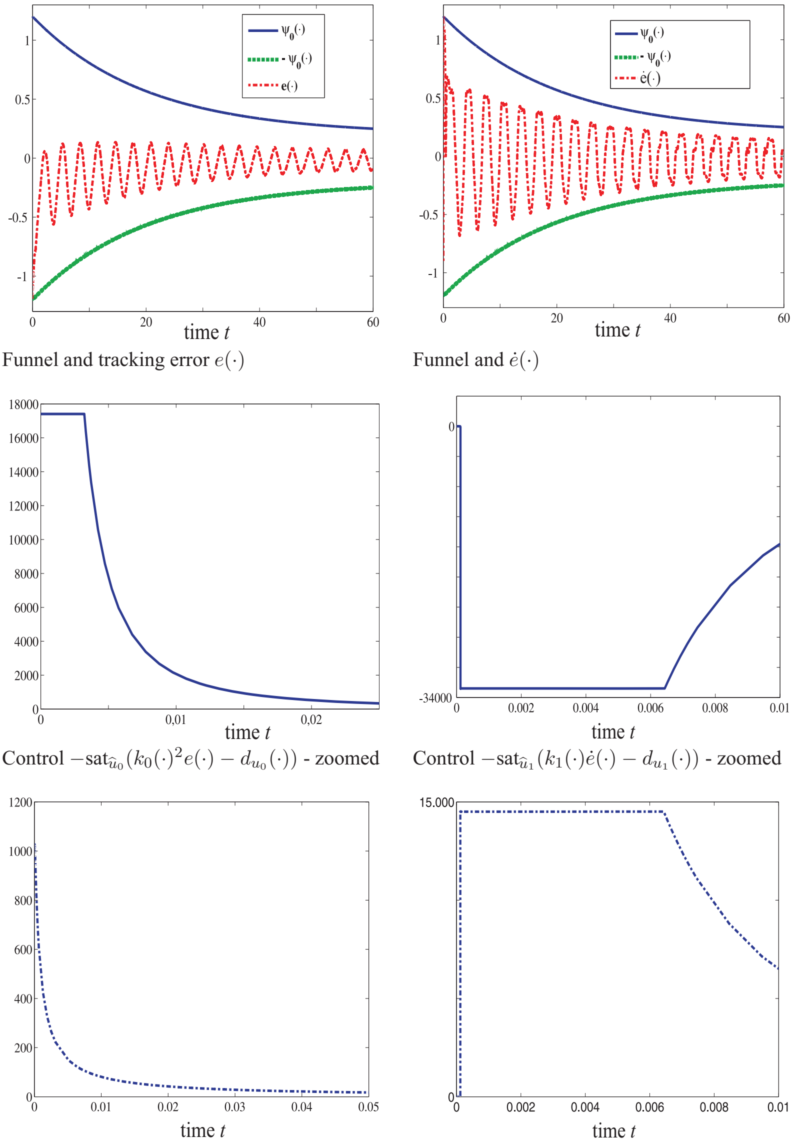
and

$$f : \mathcal{D} \to \mathbb{R}^n,$$

$$(t, \mu_0, \mu_1, \xi) \mapsto \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ R_1 & R_2 \end{bmatrix} \begin{pmatrix} \mu_0 + y_{\text{ref}}(t) \\ \mu_1 + \dot{y}_{\text{ref}}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ S \end{bmatrix} \xi - \begin{pmatrix} \dot{y}_{\text{ref}}(t) \\ \ddot{y}_{\text{ref}}(t) \end{pmatrix} + \Gamma \begin{bmatrix} 0 \\ d_u(t) \end{bmatrix} \\ -\Gamma \begin{bmatrix} 0 \\ \frac{\varphi_0(t)^2 \mu_0}{(1-\varphi_0(t)|\mu_0|)^2} + \frac{\varphi_1(t)\mu_1}{1-\varphi_1(t)|\mu_1|} \end{bmatrix} \\ [P_1, \ 0] \begin{pmatrix} \mu_0 + y_{\text{ref}}(t) \\ \mu_1 + \dot{y}_{\text{ref}}(t) \end{pmatrix} + Q\xi \end{pmatrix}.$$

Then, the initial value problem (1.13), (4.9) gets the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} e(t) \\ \dot{e}(t) \\ z(t) \end{pmatrix} = f(t, e(t), \dot{e}(t), z(t)), \qquad \begin{pmatrix} e(0) \\ \dot{e}(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} Cx^0 - y_{\text{ref}}(0) \\ CAx^0 - \dot{y}_{\text{ref}}(0) \\ Nx^0 \end{pmatrix}. \tag{4.26}$$



Funnel and tracking error $e(\cdot)$

Funnel and $\dot{e}(\cdot)$

Control $-\mathrm{sat}_{\widehat{u}_0}(k_0(\cdot)^2 e(\cdot) - d_{u_0}(\cdot))$ - zoomed

Control $-\mathrm{sat}_{\widehat{u}_1}(k_1(\cdot)\dot{e}(\cdot) - d_{u_1}(\cdot))$ - zoomed

Gain $k_0(\cdot)$ - zoomed in

Gain $k_1(\cdot)$ - zoomed in

Figure 4.6: Closed-loop system (2.38), (4.24)

Since $\varphi_i|_{[\varepsilon,\infty)}(\cdot)^{-1}$, $i \in \{0,1\}$, is globally Lipschitz for every $\varepsilon > 0$, together with (4.8), it follows that the right-hand side of (4.26) is locally Lipschitz in $\mathcal{D}$ in the sense: for all $(\tau, \chi_0, \chi_1, \eta) \in \mathcal{D}$, there exists a neighbourhood $\mathcal{O}$ of $(\tau, \chi_0, \chi_1, \eta)$ and a constant $L > 0$ such that

$$\forall\, (t, \mu_0, \mu_1, \xi) \in \mathcal{O}: \quad \|f(t, \mu_0, \mu_1, \xi) - f(\tau, \chi_0, \chi_1, \eta)\| \\ \leq L(\|(\mu_0, \mu_1)^\top - (\chi_0, \chi_1)^\top\| + \|\xi - \eta\|).$$

Now standard theory of ordinary differential equations (see Theorem 1.1.2 or [88, Th. II.10.VI]) yields existence of a solution, i.e. a continuous differentiable function $(e, \dot{e}, z) : [0, \omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, $0 < \omega \leq \infty$, satisfying (4.26) and $(t, e(t), \dot{e}(t9, z(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$. Moreover, the solution is unique and ω may be chosen maximal, i.e. the solution is not completely contained in any compact subset of $\mathcal{D}$.
In the following, let $e : [0, \omega) \to \mathbb{R}$ be the unique and maximally extended solution of the closed-loop initial value problem.

STEP 2: *It is shown that there exists $M > 0$ such that*

$$\text{for a.a. } t \in [\varepsilon, \omega): \quad -M - \Gamma[k_0(t)^2 e(t) + k_1(t)\dot{e}(t)] \\ \leq \ddot{e}(t) \leq M - \Gamma[k_0(t)^2 e(t) + k_1(t)\dot{e}(t)]. \quad (4.27)$$

Continuous differentiability of $e(\cdot)$ and $\dot{e}(\cdot)$, (4.8) and the conditions (1) and (2) of $\mathcal{S}_2$ imply

$$\exists\, \varepsilon \in [0, \min\{1, \omega\})\ \forall\, t \in [0, \varepsilon]\ \forall\, i \in \{0,1\}: \\ |e^{(i)}(t)| \leq |e^{(i)}(0)| + 1 \quad \wedge \quad 1 - \varphi_i(t)|e^{(i)}(t)| \geq \frac{1 - \varphi_i(0)|e^{(i)}(0)|}{2} > 0. \quad (4.28)$$

This ensures in particular that $e(\cdot)$ and $\dot{e}(\cdot)$ evolve in the funnel for all $t \in [0, \varepsilon]$. Moreover, (4.28) gives

$$\forall\, i \in \{0,1\}\ \forall\, t \in [0, \varepsilon]: \quad k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)|e^{(i)}(t)|} \leq \frac{2 \max_{s \in [0,\varepsilon]} |\varphi_i(s)|}{1 - \varphi_i(0)|e^{(i)}(0)|}, \quad (4.29)$$

thus $k_0(\cdot)$, $k_1(\cdot)$ are also uniformly bounded on $[0, \varepsilon]$.
For ease of notation, the funnel boundaries are denoted by the functions

$$\psi_i(\cdot) := \varphi_i|_{[\varepsilon,\infty)}(\cdot)^{-1}, \qquad i \in \{0,1\}.$$

It has to be noted that Remark 3.3.4 (i) holds. Applying Variation-of-Constants formula to the third subsystem in (4.26) and taking norm inequalities, it is obtained, in view of (1.8) and (4.28),

$$\forall\, t \in [0, \varepsilon] : \quad \|z(t)\| \leq \beta \|N x^0\| + \beta \|P_1\| \left[\|y_{\mathrm{ref}}\|_\infty + |e(0)| + 1\right] =: C_\varepsilon, \tag{4.30}$$

and so the solution $(e(\cdot), \dot{e}(\cdot), z(\cdot))$ of (4.26) is uniformly bounded on $[0, \varepsilon]$, i.e.

$$\|e^{(i)}\|_{L^\infty([0,\varepsilon])} \leq |e^{(i)}(0)| + 1,\ i \in \{0, 1\}, \quad \text{and} \quad \|z\|_{L^\infty([0,\varepsilon])} \leq C_\varepsilon. \tag{4.31}$$

Hence it remains to consider the interval $[\varepsilon, \omega)$. Since $e(\cdot)$ and $\dot{e}(\cdot)$ evolve within the funnel, resp., it follows

$$\forall\, i \in \{0, 1\}\ \forall\, t \in [\varepsilon, \omega) : \quad |e^{(i)}(t)| < \psi_i(t) \leq \|\psi_i\|_\infty. \tag{4.32}$$

Applying Variation-of-Constants formula to the third subsystem in (4.26) yields

$$\forall\, t \in [\varepsilon, \omega) : \quad z(t) = e^{Q(t-\varepsilon)} z(\varepsilon) + \int_\varepsilon^t e^{Q(t-s)} P_1 \left[e(s) + r(s)\right] \,\mathrm{d}s\,,$$

and thus, in view of (1.8) and (4.32), it follows that

$$\forall\, t \in [0, \omega) : \quad \|z(t)\| \leq M_z, \tag{4.33}$$

where

$$M_z := \beta C_\varepsilon + \frac{\beta}{\alpha} \|P_1\| \left[\|\psi_0\|_\infty + \|y_{\mathrm{ref}}\|_\infty\right]. \tag{4.34}$$

Now (4.26) yields

$$\begin{aligned} \forall\, t \in [\varepsilon, \omega) :\ \ddot{e}(t) = R_1 \left[e(t) + y_{\mathrm{ref}}(t)\right] &+ R_2 \left[\dot{e}(t) + \dot{r} y_{\mathrm{ref}}(t)\right] \\ &+ S z(t) - \ddot{y}_{\mathrm{ref}}(t) + \Gamma d_u(t) - \Gamma[k_0(t)^2 e(t) + k_1(t) \dot{e}(t)] \end{aligned}$$

and so (4.27) follows with

$$M := |R_1| \left[\|\psi_0\|_\infty + \|y_{\mathrm{ref}}\|_\infty\right] + |R_2| \left[\|\psi_1\|_\infty + \|\dot{y}_{\mathrm{ref}}\|_\infty\right] + \|S\| M_z + \|\ddot{y}_{\mathrm{ref}}\|_\infty + \Gamma \|d_u\|_\infty.$$

STEP 3: *It is shown that $|e(\cdot)|$ is bounded away from the funnel boundary $\psi_0(\cdot)$ on $[\varepsilon, \omega)$, more precisely:*

$$\exists\, \varepsilon_0 > 0\ \forall\, t \in [\varepsilon, \omega) : \quad \psi_0(t) - |e(t)| \geq \varepsilon_0.$$

STEP 3A: *It is shown that for $\varepsilon_0 \in \left(0, \frac{\lambda_0}{2}\right)$ the following implication holds on any* *interval* $[t_0, t_1] \subset [\varepsilon, \omega)$*:*

$$\left[\psi_0(t_0) - |e(t_0)| = 2\varepsilon_0 \;\wedge\; \text{for a.a. } t \in [t_0, t_1]:\; \ddot{e}(t) \operatorname{sgn} e(t) \le -\frac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)^2}{2\varepsilon_0}\right]$$
$$\implies \quad \forall\, t \in [t_0, t_1]:\; \psi_0(t) - |e(t)| \ge \varepsilon_0. \quad (4.35)$$

First the case

$$\forall\, t \in [t_0, t_1]: \quad \psi_0(t) - |e(t)| \le 2\varepsilon_0 \tag{4.36}$$

is considered. $\lambda_0 > 2\varepsilon_0$ implies that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. Consider the case $\operatorname{sgn} e(\cdot) = 1$, the case $\operatorname{sgn} e(\cdot) = -1$ follows analogously.

Integrating the inequality $\ddot{e}(\cdot) \le -\frac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}+\ell_0)^2}{2\varepsilon_0}$ twice yields

$$\forall\, t \in [t_0, t_1]: \quad e(t) \le e(t_0) - \tfrac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}+\ell_0)^2}{4\varepsilon_0}(t-t_0)^2 + \underbrace{\dot{e}(t_0)}_{\le \|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}} (t-t_0)$$

The inequality $|\psi_0(t) - \psi_0(t_0)| \le \ell_0(t - t_0)$, see Remark 3.3.4 (i), this implies that

$$\forall\, t \in [t_0, t_1]:\; \psi_0(t) - e(t) \ge \underbrace{\psi_0(t_0) - e(t_0)}_{=2\varepsilon_0} - \Big((\ell_0 + \|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))})(t-t_0)$$
$$- \tfrac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}+\ell_0)^2}{4\varepsilon_0}(t-t_0)^2 \Big).$$

The parabola $t \mapsto (\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)(t - t_0) - \frac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}+\ell_0)^2}{4\varepsilon_0}(t-t_0)^2$ attains its maximum at $t - t_0 = \frac{2\varepsilon_0}{\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}+\ell_0}$ with the maximum value ε_0, hence

$$\forall\, t \in [t_0, t_1]: \quad \psi_0(t) - e(t) \ge \varepsilon_0.$$

This proves Step 3A in the case of (4.36).

It remains to consider the case

$$\exists\, t \in [t_0, t_1]: \quad \psi_0(t) - |e(t)| > 2\varepsilon_0.$$

Now either

$$\forall\, t \in [t_0, t_1]: \quad \psi_0(t) - |e(t)| \ge 2\varepsilon_0,$$

in which case the claim of Step 3A is proved, or

$$\exists\, t_0 \in [t_0, t_1]: \quad \psi_0(t_0) - |e(t_0)| < 2\varepsilon_0.$$

Then there exists an interval $[\hat{t}_0, \hat{t}_1] \subset [t_0, t_1]$ such that (4.36) holds for $[t_0, t_1]$ replaced by $[\hat{t}_0, \hat{t}_1]$. Now the contradiction follows as in the first case which completes the proof of Step 3A.

STEP 3B: *It is shown that, for positive*

$$\varepsilon_0 \le \min\left\{ \frac{\lambda_0}{4}, \frac{\sqrt{\delta^2(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)^4 + 2\lambda_0\delta\Gamma(M\delta + 2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))})}}{4(M\delta + 2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))})} + \frac{-\delta(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)^2}{4(M\delta + 2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))})} \right\}, \quad (4.37)$$

the following implication holds on any interval $[t_0, t_1] \subset [\varepsilon, \omega)$*:*

$$\left[\forall\, t \in [t_0, t_1]:\ \dot{e}(t)\operatorname{sgn} e(t) \ge \frac{\delta}{2} - \psi_1(t)\ \wedge\ \psi_0(t) - \big|e(t)\big| \le 2\varepsilon_0\right]$$
$$\Longrightarrow \quad \text{for a.a. } t \in [t_0, t_1]:\ \ddot{e}(t)\operatorname{sgn} e(t \le -\frac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)^2}{2\varepsilon_0}. \quad (4.38)$$

(4.37) gives $2\varepsilon_0 \le \frac{\lambda_0}{2}$. The condition $\psi_0(t) - \big|e(t)\big| \le 2\varepsilon_0$ on $[t_0, t_1]$ implies that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. Consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, $\operatorname{sgn} e(\cdot) \equiv -1$ follows analogously. The condition $\dot{e}(t) \ge \frac{\delta}{2} - \psi_1(t)$ on $[t_0, t_1]$ implies that

$$\forall\, t \in [t_0, t_1]: \quad -k_1(t)\dot{e}(t) = \frac{-\dot{e}(t)}{\psi_1(t) + \dot{e}(t)} \le \frac{2\dot{e}(t)}{\delta} \le \frac{2\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta}.$$

From $\psi_0(t) - e(t) \le 2\varepsilon_0$ and $2\varepsilon_0 \le \frac{\lambda_0}{2}$, it follows that $e(t) \ge \frac{\lambda_0}{2}$ on $[t_0, t_1]$ and hence

$$\forall\, t \in [t_0, t_1]: \quad -k_0(t)^2 e(t) \le -\frac{\lambda_0}{8\varepsilon_0^2}.$$

Inserting these inequalities into (4.27) and invoking (4.32) yields

$$\text{for a.a. } t \in [t_0, t_1]: \quad \ddot{e}(t) \le M - \Gamma\frac{\lambda_0}{8\varepsilon_0^2} + \Gamma\frac{2\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta} \overset{(4.37)}{\le} -\frac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)^2}{2\varepsilon_0},$$

whence (4.38).

STEP 3C: *It is shown that the following implication holds for almost all* $t \in [\varepsilon, \omega)$*:*

$$\left[\dot{e}(t)\,\mathrm{sgn}\,e(t) = -\psi_1(t) + \frac{\delta}{2} \;\wedge\; \exists\,\hat{\varepsilon} > 0\;\forall\,\tau \in (t-\hat{\varepsilon}, t):\; \dot{e}(\tau)\,\mathrm{sgn}\,e(\tau) < -\psi_1(\tau) + \frac{\delta}{2}\right]$$
$$\implies \quad \ddot{e}(t)\,\mathrm{sgn}\,e(t) \geq -\ell_1. \quad (4.39)$$

Assume that $\mathrm{sgn}\,e(\tau)$ is constant on $(t-\hat{\varepsilon}, t]$. Only the case $\mathrm{sgn}\,e(\cdot) \equiv 1$ is considered, the other case follows analogously. Then for almost all $t \in [\varepsilon, \omega)$ satisfying the supposition it follows

$$\ddot{e}(t) = \lim_{h\to 0+} \frac{\dot{e}(t) - \dot{e}(t-h)}{h} \geq \lim_{h\to 0+} \frac{-\psi_1(t) + \psi_1(t-h)}{h} \geq -\ell_1.$$

STEP 3D: *It is shown that the following implication for any* $(t_0, t_1] \subset [\varepsilon, \omega)$*:*

$$\left[\forall\, t \in (t_0, t_1]:\; \dot{e}(t) < \frac{\delta}{2} - \psi_1(t) \;\wedge\; \mathrm{sgn}\,e(t) = 1\right]$$
$$\implies \quad t \mapsto \psi_0(t) - e(t) \text{ is monotonically increasing on } (t_0, t_1]. \quad (4.40)$$

Consider the case $\mathrm{sgn}\,e(\cdot) \equiv 1$, the other case follows analogously. By Remark 3.3.4 (i) it follows that

$$\text{for a.a. } t \in [t_0, t_1]: \quad \dot{\psi}_0(t) - \dot{e}(t) > \dot{\psi}_0(t) + \psi_1(t) - \frac{\delta}{2} \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

which yields (4.40).

STEP 3E: *Finally, it is shown that the claim of Step 3 holds true for* $\varepsilon_0 > 0$ *sufficiently small so that*

$$(4.37) \text{ holds} \quad \wedge \quad \frac{(\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \ell_0)^2}{2\varepsilon_0} > \ell_1 \quad \wedge \quad \psi_0(\varepsilon) - |e(\varepsilon)| \geq 2\varepsilon_0.$$

Seeking a contradiction, assume there exists $t' \in [\varepsilon, \omega)$ such that $\psi_0(t') - |e(t')| < \varepsilon_0$. Continuity of $t \mapsto \psi_0(t) - |e(t)|$ implies that the number

$$t_0 := \max\left\{t \in [\varepsilon, t') \mid \psi_0(t) - |e(t)| = 2\varepsilon_0\right\}$$

is well defined. Then it follows that

$$\forall\, t \in [t_0, t']: \quad \psi_0(t) - |e(t)| \leq 2\varepsilon_0,$$

hence, by $\varepsilon_0 \leq \frac{\lambda_0}{4}$, it holds that $\mathrm{sgn}\,e(\cdot)$ is constant on $[t_0, t']$. Consider the case $\mathrm{sgn}\,e(\cdot) \equiv 1$, the other case follows analogously.

If $\dot{e}(t) \geq -\psi_1(t) + \frac{\delta}{2}$ for all $t \in [t_0, t']$, then (4.38), together with (4.35), yields $\psi_0(t) - |e(t)| \geq \varepsilon_0$ for all $t \in [t_0, t']$ which contradicts the assumption.

Therefore, assume that there exists $s \in [t_0, t']$ such that $\dot{e}(s) < -\psi_1(s) + \frac{\delta}{2}$. Since $\dot{e}(\cdot)$ is continuous, there exists $s \in (t_0, t)$ such that $\dot{e}(\cdot)$ is differentiable at s. By the choice of ε_0, either the right hand side of implication (4.38) or the right hand side of implication (4.39) holds for almost all $\tau \in [t_0, t']$. Thus it follows that $\dot{e}(\tau) < -\psi_1(\tau) + \frac{\delta}{2}$ for all $\tau \in [s, t']$; otherwise (4.35) would again yield a contradiction to the assumption.

In view of $\psi_0(t_0) - e(t_0) = 2\varepsilon_0$ and the implication (4.40), it follows that there exists a unique $t_1 \in [t_0, t']$ such that

$$\forall\, \tau \in [t_0, t_1] :\ \dot{e}(\tau) \geq -\psi_1(\tau) + \frac{\delta}{2} \qquad \text{and} \qquad \forall\, \tau \in (t_1, t] :\ \dot{e}(\tau) < -\psi_1(\tau) + \frac{\delta}{2}.$$

Implication (4.38), together with implication (4.35), yields $\psi_0(t_1) - e(t_1) \geq \varepsilon_0$. Hence the absolutely continuous function $\tau \mapsto \psi_0(\tau) - e(\tau)$ is monotonically increasing on $(t_1, t']$ by (4.40) and thus $\psi_0(t) - e(t) \geq \varepsilon_0$. This contradicts the assumption $\psi_0(t) - e(t) < \varepsilon_0$.

STEP 4: *It is shown that $\psi_1(t) - |\dot{e}(t)| \geq \varepsilon_1$ for all $t \in [\varepsilon, \omega)$ and positive*

$$\varepsilon_1 < \min\left\{\frac{\lambda_1}{2}, \psi_1(\varepsilon) - |\dot{e}(\varepsilon)|, \frac{\Gamma\lambda_1\varepsilon_0^2}{2(\ell_1\varepsilon_0^2 + \Gamma\|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))} + \varepsilon_0^2 M)}\right\}, \tag{4.41}$$

with M as in (4.27) *and ε_0 as in Step 3E.*

It follows, for $\varepsilon_0 > 0$ as in Step 3E, that $k_0(t)^2 \leq \frac{1}{\varepsilon_0^2}$ for all $t \in [\varepsilon, \omega)$ which together with (4.32) yields

$$\forall\, t \in [\varepsilon, \omega) : \quad k_0(t)^2 |e(t)| \leq \frac{\|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2}.$$

Seeking a contradiction, suppose there exists $t_1 \in [\varepsilon, \omega)$ such that $\psi_1(t_1) - |\dot{e}(t)| < \varepsilon_1$. Since $\psi_1(\varepsilon) - |\dot{e}(\varepsilon)| \geq \varepsilon_1$, the following is well defined

$$t_0 := \max\left\{t \in [\varepsilon, t_1) \mid \psi_1(t) - |\dot{e}(t)| = \varepsilon_1\right\} \in (\varepsilon, t_1).$$

Moreover,

$$\forall\, t \in [t_0, t_1] : \quad |\dot{e}(t)| \geq \psi_1(t) - \varepsilon_1 \geq \lambda_1 - \varepsilon_1 > \frac{\lambda_1}{2}$$

whence

$$\forall\, t \in [t_0, t_1]: \quad k_1(t)|\dot e(t)| \geq \frac{\lambda_1}{2\varepsilon_1},$$

and so, in view of (4.27), it follows that, for almost all $t \in [t_0, t_1]$,

$$\begin{aligned}\ddot e(t) \operatorname{sgn} \dot e(t) &\overset{(4.27)}{\leq} \operatorname{sgn} \dot e(t) \left[M - \Gamma k_0(t)^2 e(t) - \Gamma k_1(t) \dot e(t)\right] \\ &< M + \Gamma \frac{\|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2} - \Gamma \frac{\lambda_1}{2\varepsilon_1} \overset{(4.41)}{<} -\ell_1.\end{aligned}$$

Integration gives

$$|\dot e(t_1)| - |\dot e(t_0)| = \int_{t_0}^{t_1} \ddot e(\tau) \operatorname{sgn} \dot e(\tau)\, \mathrm{d}\tau \; < -\ell_1\, (t_1 - t_0)$$

whence, together with the Lipschitz property of $\psi_1(\cdot)$ on $[\varepsilon, \omega)$, the contradiction

$$\begin{aligned}0 < \psi_1(t_0) - |\dot e(t_0)| - [\psi_1(t_1) - |\dot e(t_1)|] &= \psi_1(t_0) - \psi_1(t_1) \\ &+ [|\dot e(t_1)| - |\dot e(t_0)|] < \ell_1\, (t_1 - t_0) - \ell_1\, (t_1 - t_0) = 0.\end{aligned}$$

Hence Step 4 is proved.

STEP 5: *Assertions (i)–(iv) are shown.*
Boundedness of $e(\cdot)$, $\dot e(\cdot)$, $z(\cdot)$, $k_0(\cdot)$ and $k_1(\cdot)$ on $[0, \omega)$ follows from continuity of the functions and (4.28)-(4.32), (4.33), Step 3 and Step 4, resp. The inequality (4.10) holds on $[0, \omega)$ by Step 3 and Step 4 and since $\varphi_i(\cdot)$ is uniformly bounded from below, $i = 0, 1$. Therefore, Assertion (i)–(iv) hold if $\omega = \infty$. Let, for ε_0 as in Step 3E and ε_1 as in Step 4 with the additional restrictions $\varepsilon_i \leq \frac{1-\varphi_i(0)|e^{(i)}(0)|}{4 \max_{s\in[0,\varepsilon]} |\varphi_i(s)|}$ and M_z as in Step 2,

$$\mathcal{C} := \left\{(t, e_0, e_1, z) \in [0, \omega] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \mid \varphi_i(t)|e_i| \leq 1 - \varepsilon_i,\ \|z\| \leq M_z\right\}.$$

Let $\mathcal{D}$ be as in Step 1. If $\omega < \infty$ then $\mathcal{C} \subset \mathcal{D}$ is a compact subset of $\mathcal{D}$ which contains the whole graph of the solution $t \mapsto (e(t), \dot e(t), z(t))$, which contradicts the maximality of the solution. Hence $\omega = \infty$. This completes the proof. □

Proof of Proposition 4.2.7:
The partial derivatives of $F(\cdot,\cdot,\cdot,\cdot)$ are denoted by

$$(t, e, p, z) \mapsto F_{(t,e,z)}(t, e, p, z) := \frac{\partial F}{\partial(t, e, z)}(t, e, p, z) \quad \text{and}$$

$$(t, e, p, z) \mapsto F_p(t, e, p, z) := \frac{\partial F}{\partial p}(t, e, p, z).$$

STEP 1: *Existence and uniqueness of a local solution of the initial value problem* (4.13) *are shown.*
Since $F(\cdot,\cdot,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{D}, \mathbb{R})$ and, in view of (4.14) and (4.15), $F(t_0, e^0, p^0, z^0) = 0$, $F_p(t_0, e^0, p^0, z^0) \neq 0$, the Implicit Function Theorem (see [3, Th. VII.8.2]) gives that there exists a relatively open neighbourhood $\mathcal{O} \subset \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^{n-1}$ of (t_0, e^0, z^0), an open neighbourhood $\mathcal{V} \subset \mathbb{R}$ of p^0 and a unique function $g(\cdot,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{O}, \mathcal{V})$ such that $g(t_0, e^0, z^0) = p^0$ and $F(t, e, g(t, e, z), z) = 0$ for all $(t, e, z) \in \mathcal{O}$; moreover,

$$\forall\, (t, e, z) \in \mathcal{O} : \quad [F(t, e, p, z) = 0 \ \wedge \ p \in \mathcal{V}] \quad \Leftrightarrow \quad p = g(t, e, z).$$

Moreover (see [3, Th. VII.8.2]) $g(\cdot,\cdot,\cdot)$ satisfies

$$(t, e, z) \mapsto \frac{\mathrm{d}g}{\mathrm{d}(t, e, z)}(t, e, z) = -F_p(t, e, g(t, e, z), z)^{-1}\, F_{(t,e,z)}(t, e, g(t, e, z), z) \quad (4.42)$$

on $\mathcal{O}$. Then, the implicit initial value problem 4.13 is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} e(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} g(t, e(t), z(t)) \\ P_0 e(t) + Q z(t) + f(t) \end{bmatrix}, \quad \begin{bmatrix} e(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} e^0 \\ z^0 \end{bmatrix} \quad (4.43)$$

on $\mathcal{O}$ which is an explicit ordinary differential equation. Since $F(\cdot,\cdot,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{D}, \mathbb{R})$, together with (4.42), $\frac{\partial}{\partial(e,z)} g(\cdot,\cdot,\cdot)$ is continuous on $\mathcal{O}$ and thus, by [88, II.6.IV], the right-hand side of (4.43) is locally Lipschitz on the relatively open set $\mathcal{O}$. $\frac{\partial}{\partial(e,z)} f(\cdot,\cdot,\cdot)$ is continuous on $\mathcal{O}(0, e^0, z^0)$ and thus, by [88, II.6.IV], the right-hand side of (4.43) is locally Lipschitz on the open set $\mathcal{O}(0, e^0, z^0)$. Now standard theory of ordinary differential equations (see Theorem 1.1.2 or [88, Th. II.10.VI]) yields existence of a solution, i.e. a continuous differentiable function $(e, z) : [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $t_0 < \omega \leq \sup \mathrm{pr}_1(\mathcal{O})$, satisfying (4.43) and $(t, e(t), z(t)) \in \mathcal{O}$ for all $t \in [t_0, \omega)$. Moreover, the solution is unique and ω may be chosen maximal, i.e. the solution is not completely contained in any compact subset of $\mathcal{O}$. Thus, Assertion (i) holds. It has to be noted that $(e, z)(\cdot)$ is a unique maximal (local) solution of (4.43) but not necessarily of (4.13), i.e. $(e, z)(\cdot)$ is a unique (local) solution of (4.13).

STEP 2: *Every solution of* (4.13) *can be maximally extended.*
Let $(e, z) : [t_0, \omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$, $t_0 < \omega \leq T$, with $(e, z)([t_0, \omega)) \subset \mathcal{D}$ be a solution of (4.13). Define

$$\mathcal{A} := \Big\{ (\sigma, \xi(\cdot)) \Big| \sigma \in [\omega, T],\ \xi : [t_0, \sigma) \to \mathbb{R}^n,\ \xi([t_0,\sigma)) \subset \mathcal{D} \text{ is solution of (4.13) on } [t_0,\sigma),\ \xi|_{[t_0,\omega)} = (e,z) \Big\}$$

which is a non-empty set. This set comprising the solution $(e,z)(\cdot)$ and all proper right extensions of $(e,z)(\cdot)$ that are also solutions. Let a partial order on $\mathcal{A}$ be given by

$$(\sigma_1, \xi_1(\cdot)) \le (\sigma_2, \xi_2(\cdot)) \quad :\Leftrightarrow \quad \sigma_1 \le \sigma_2 \text{ and } \xi_2|_{[t_0,\sigma_1]} = \xi_1(\cdot).$$

Let $\mathcal{A}_1$ be a totally ordered subset of $\mathcal{A}$. Set

$$\sigma_{\max} := \sup\{\sigma \in [\omega, T] \mid (\sigma, \xi(\cdot)) \in \mathcal{A}_1\}.$$

For every $t \in [t_0, \sigma_{\max})$ there exists $(\sigma, \xi(\cdot)) \in \mathcal{A}_1$ such that $t \in [t_0, \sigma)$ and it is assigned $\xi_{\max}(\cdot) := \xi(\cdot)$. It has to be noted that $\xi_{\max}(\cdot)$ is independent on choosing $(\sigma, \xi(\cdot)) \in \mathcal{A}_1$ because $\mathcal{A}_1$ is a totally ordered subset of $\mathcal{A}$. Then $\mathcal{A}$ contains one maximal element by the Hausdorff maximal principle, see [51, Th. 0.24]. Hence there exists a maximal solution $\xi_{\max} : [t_0, \omega_*) \to \mathbb{R}^n$, $\omega_* \in (t_0, T]$, of the initial value problem (4.13).

STEP 3: *Uniqueness of a solution of the initial value problem* (4.13) *is shown, i.e. if* $(e,z)(\cdot)$ *and* $(\widetilde{e}, \widetilde{z})(\cdot)$ *are two solutions of the initial value problem* (4.13) *with* $(e,z)(t_0) = (e^0, z^0) = (\widetilde{e}, \widetilde{z})(t_0)$ *and if* $[t_0, \tau) \subset [t_0, T)$*,* $\tau \in (t_0, T]$*, is a common interval of existence of both solutions, then* $(e,z)(\cdot) = (\widetilde{e}, \widetilde{z})(\cdot)$ *in* $[t_0, \tau)$.
Seeking a contradiction, suppose that there exists $t_1 \in [t_0, \tau)$ such that $(e,z)(t_1) \neq (\widetilde{e}, \widetilde{z})(t_1)$. Then there exists a first time $\tau_0 \in [t_0, \tau)$ where the two solutions separate, more precisely,

$$\tau_0 := \max\left\{ t \in [0, t_1) \mid (e,z)(\cdot)|_{[t_0,t]} = (\widetilde{e}, \widetilde{z})(\cdot)|_{[t_0,t]} \right\} < t_1.$$

It has to be noted that $F(\tau_0, e(\tau_0), \dot{e}(\tau_0), z(\tau_0)) = 0$ and $\tau_0 = t_0$ is not excluded. Moreover, (4.15) implies that $F_p(\tau_0, e(\tau_0), \dot{e}(\tau_0), z(\tau_0)) > 0$. Consider, for $(e^1, z^1) := (e(\tau_0), z(\tau_0)) = (\widetilde{e}(\tau_0), \widetilde{z}(\tau_0))$, the initial value problem

$$\left.\begin{array}{rcll} 0 &=& F(t, e(t), \dot{e}(t), z(t)), & e(t_1) = e^1 \\ \dot{z}(t) &=& P_0 e(t) + Q z(t) + f(t), & z(t_1) = z^1. \end{array}\right\} \tag{4.44}$$

Nor for (4.44) instead of (4.13), the same arguments as in Step 1 yield the existence of a unique solution of (4.44) on some interval $[t_1, t_1 + \delta)$ for some $\delta > 0$. This is a contradiction to the assumption about τ_0.

STEP 4: *Assertion (ii) is shown.*
Assume that $(e,z)(\cdot) \in \mathcal{C}^1([t_0,\omega),\mathbb{R}^n)$ is a bounded maximal solution of (4.13). Seeking a contradiction, suppose $\omega < T$. By the boundedness of $(e,z)(\cdot)$, together with $F(\cdot,\cdot,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{D},\mathbb{R})$, it follows that $(\dot{e},\dot{z})(\cdot)$ is essentially bounded and therefore, $(e,z)(\cdot)$ is uniformly continuous and so extends to a continuous function $(e,z) : [t_0,\omega] \to \mathbb{R}^n$.
Now, by Assertion (i), the initial value problem

$$\begin{aligned} 0 &= F(t, w_1(t), \dot{w}_1(t), w_2(t)), & w_1(\omega) &= e(\omega) \\ \dot{w}_2(t) &= P_0 w_1(t) + Q w_2(t) + f(t), & w_2(\omega) &= z(\omega) \end{aligned}$$

has a solution $w : [\omega, \omega+\tau) \to \mathbb{R}^n$, $0 < \tau \le T - \omega$. Then it follows that

$$(\widetilde{e},\widetilde{z}) : [t_0,\omega+\tau) \to \mathbb{R}^n, \quad t \mapsto \begin{cases} (e,z)(t) & , \ t \in [t_0,\omega] \\ w(t) & , \ t \in [\omega,\omega+\tau) \end{cases}$$

is a solution of the initial value problem (4.13) and is a proper right extension of the solution $(e,z)(\cdot)$ which contradicts the maximality of $(e,z)(\cdot)$. Therefore, $\omega = T$. This completes the proof. □

Proof of Theorem 4.2.5:
The proof uses the notation of Proposition 1.1.7, (1.8) and Proposition 4.2.7. The proof is based on the existence and uniqueness result of the solution of an implicit ordinary differential equation (see Proposition 4.2.7).

STEP 1: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (4.9) *or, equivalently,* (1.13)*,* (4.9) *is shown.*
It suffices to consider the system in Byrnes-Isidori form (1.13). Then, the initial value problem (1.13), (4.9) can be written as

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} e(t) \\ z(t) \end{pmatrix} &= \begin{pmatrix} R_1(e(t)+y_{\mathrm{ref}}(t)) + Sz(t) - \dot{y}_{\mathrm{ref}}(t) + \Gamma d_u(t) \\ \qquad - \Gamma\left[\frac{\varphi_0(t)^2 e(t)}{(1-\varphi_0(t)|e(t)|)^2} + \frac{\varphi_1(t)\dot{e}(t)}{1-\varphi_1(t)|\dot{e}(t)|}\right] \\ P_1(e(t)+y_{\mathrm{ref}}(t)) + Qz(t) \end{pmatrix} \\ \begin{pmatrix} e(0) \\ z(0) \end{pmatrix} &= \begin{pmatrix} Cx^0 - y_{\mathrm{ref}}(0) \\ Nx^0 \end{pmatrix} \end{aligned}\right\} \quad (4.45)$$

(see Remark 4.2.1). (4.45) is an implicit ordinary differential equation of first order.

A proof of existence and uniqueness of a maximal solution of the initial value problem (4.45) is not available in the literature. The essence of the following Steps is the applying of Proposition 4.2.7 to (4.45) and so, there exists a unique maximal solution $(e, z) : [0, \omega) \to \mathbb{R}^n$ of the initial value (4.45) for some $\omega > 0$.

Define, for $(\varphi_0(\cdot), \varphi_1(\cdot)) \in \mathcal{S}_2$, the relatively open set

$$\mathcal{D} := \{(t, \mu_0, \mu_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \mid (t, \mu_0) \in \mathcal{F}(0, \varphi_0),\ (t, \mu_1) \in \mathcal{F}(0, \varphi_1)\} \times \mathbb{R}^{n-1}$$

and

$$\begin{aligned} \kappa_0 : F(0, \varphi_0) \to \mathbb{R}, \quad & (t, \mu_0) \mapsto \kappa_0(t, \mu_0) := \frac{\varphi_0(t)}{1 - \varphi_0(t)|\mu_0|} \\ \kappa_1 : F(0, \varphi_1) \to \mathbb{R}, \quad & (t, \mu_1) \mapsto \kappa_1(t, \mu_1) := \frac{\varphi_1(t)}{1 - \varphi_1(t)|\mu_1|}. \end{aligned}$$

Let $(e^0, z^0) = (Cx^0 - r(0), Nx^0)$ and define

$$\begin{aligned} F : \mathcal{D} \to \mathbb{R},\ (t, \mu_0, \mu_1, \xi) \mapsto [1 + \Gamma\kappa_1(t, \mu_1)]\, \mu_1 + \left[\Gamma\kappa_0(t, \mu_0)^2 - R_1\right] \mu_0 - S\xi \\ - R_1 y_{\text{ref}}(t) + \dot{y}_{\text{ref}}(t) - \Gamma d_u(t). \end{aligned}$$

STEP 1A: *It is shown that*

$$\exists \text{ unique } p^0 \in \mathbb{R} : \quad (0, e^0, p^0, z^0) \in \mathcal{D} \quad \wedge \quad F(0, e^0, p^0, z^0) = 0. \tag{4.46}$$

Let $p^0 := \left[R_1 - \Gamma\kappa_0(0, e^0)^2\right] e^0 + Sz^0 + R_1 r(0) - \dot{r}(0) + \Gamma d_u(0)$. It has to be noted that $\varphi_1(0) = 0$. Then it follows that $(0, e^0, p^0, z^0) \in \mathcal{D}$ and

$$F(0, e^0, z^0, p^0) = \left[\Gamma\kappa_0(0, e^0)^2 - R_1\right] e^0 + p^0 - Sz^0 - R_1 r(0) + \dot{r}(0) - \Gamma d_u(0) = 0.$$

which shows (4.46). It has to be noted that uniqueness of p^0 is obviously.

STEP 1B: *It is shown that* $F(\cdot, \cdot, \cdot, \cdot) \in \mathcal{C}^1(\mathcal{D}, \mathbb{R})$ *and* $\frac{\partial F}{\partial p}(t, e, p, z) > 0$ *for all* $(t, e, p, z) \in \mathcal{D}$.
Continuity of $d_u(\cdot)$ and $\kappa_i(\cdot, \cdot)$, $i = 0, 1$, implies continuity of $F(\cdot, \cdot, \cdot, \cdot)$ on $\mathcal{D}$. Since $y_{\text{ref}}(\cdot)$, $\dot{y}_{\text{ref}}(\cdot)$, $d_u(\cdot)$ and $\varphi_i(\cdot)$, $i = 0, 1$, are continuous differentiable, it follows that $F(\cdot, \cdot, \cdot, \cdot)$ is differentiable on $\mathcal{D}$ with

$$(t, e, p, z) \mapsto \frac{\mathrm{d}\, F}{\mathrm{d}(t, e, p, z)}(t, e, p, z)$$

$$= \begin{bmatrix} 2\Gamma\kappa_0(t,e)\frac{\dot\varphi_0(t)\,e}{(1-\varphi_0(t)|e|)^2} + \Gamma\frac{\dot\varphi_1(t)\,p}{(1-\varphi_1(t)|p|)^2} - R_1\dot r(t) + \ddot r(t) - \Gamma\dot d_u(t) \\ -R_1 + \Gamma\kappa_0(t,e)^2 + 2\Gamma\kappa_0(t,e)^3|e| \\ 1 + \Gamma\kappa_1(t,p) + \Gamma\kappa_1(t,p)^2|p| \\ -S \end{bmatrix}^\top \tag{4.47}$$

which is continuous on $\mathcal{D}$. Moreover it holds

$$\forall\, (t,e,p,z) \in \mathcal{D}: \quad \frac{\partial F}{\partial p}(t,e,p,z) = 1 + \Gamma\kappa_1(t,p) + \Gamma\kappa_1(t,p)^2|p| \geq 1 \tag{4.48}$$

which completes the claim.

STEP 1C: Applying Proposition 4.2.7 (i) to (4.45) yields that there exists a solution $(e,z) : [0,\omega) \to \mathbb{R}^n$, $\omega \in (0,\infty]$, of the initial value (4.45). Moreover, this solution is unique and ω can be chosen maximal which completes Step 1.

STEP 2: *It is shown that there exists $M > 0$ such that*

$$\forall\, t \in [\varepsilon,\omega): \quad -M - \Gamma[k_0(t)^2 e(t) + k_1(t)\dot e(t)] \\ \leq \dot e(t) \leq M - \Gamma[k_0(t)^2 e(t) + k_1(t)\dot e(t)]. \tag{4.49}$$

Since $e(\cdot)$ and $\dot e(\cdot)$ are continuous, (4.28) holds. (4.28) holds. This ensures in particular that $e(t)$ and $\dot e(t)$ evolve in the funnel for all $t \in [0,\varepsilon]$ with $\varepsilon > 0$ as in (4.28). It has to be noted that $\varphi_1(0) = 0$ implies $\varepsilon > 0$. Inequality (4.30) follows similarly to Step 2 in the proof of Theorem 4.2.2. Hence, (4.31) and (4.29) hold. Thus $k_0(\cdot)$, $k_1(\cdot)$ are also uniformly bounded on $[0,\varepsilon]$.
It is now turned to the interval $[\varepsilon,\omega)$. Since $e(\cdot)$ and $\dot e(\cdot)$ evolve within the funnel, resp., (4.32) follows. Now, the original system (4.45) is considered. Applying Variation-of-Constants formula to the second subsystem in (4.45) yields (4.33). Now (4.45) gives

$$\forall\, t \in [\varepsilon,\omega): \ \dot e(t) = R_1\,[e(t) + r(t)] + Sz(t) - \dot r(t) + \Gamma d_u(t) - \Gamma[k_0(t)^2 e(t) + k_1(t)\dot e(t)]$$

and so (4.49) follows with

$$M := |R_1|\,[\|\psi_0\|_\infty + \|y_{\text{ref}}\|_\infty] + \|S\| M_z + \|\dot y_{\text{ref}}\|_\infty + \Gamma\|d_u\|_\infty,$$

where M_z is given in (4.34).

STEP 3: *It is shown that $|e(\cdot)|$ is bounded away from the funnel boundary $\psi_0(\cdot)$ on*

$[\varepsilon, \omega)$, *more precisely:*

$$\exists\, \varepsilon_0 > 0\; \forall\, t \in [\varepsilon, \omega): \quad \psi_0(t) - |e(t)| \geq \varepsilon_0.$$

STEP 3A: *It is shown that for positive*

$$\varepsilon_0 < \min\left\{ \frac{\lambda_0}{2},\ \psi_0(\varepsilon) - |e(\varepsilon)|,\ \sqrt{\frac{\Gamma\lambda_0\delta}{\left|2\delta M + [4\Gamma + 2\delta]\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} - \delta^2\right|}} \right\} \tag{4.50}$$

the following implication holds on any interval $[t_0, t_1] \subset [\varepsilon, \omega)$*:*

$$[\forall\, t \in [t_0, t_1]:\ \psi_0(t) - |e(t)| \leq \varepsilon_0] \ \Rightarrow\ \left[\forall\, t \in [t_0, t_1]:\ \dot{e}(t) < \frac{\delta}{2} - \psi_1(t)\right]. \tag{4.51}$$

(4.50) implies $\varepsilon_0 \leq \frac{\lambda_0}{2}$. The condition $\psi_0(t) - \left|e(t)\right| \leq \varepsilon_0$ on $[t_0, t_1]$ implies that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. Consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, $\operatorname{sgn} e(\cdot) \equiv -1$ follows analogously. Then it follows, together with (4.50), for all $t \in [t_0, t_1]$,

$$\begin{aligned}
M + &\frac{2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta} + \psi_1(t) - \frac{\Gamma\lambda_0}{2\varepsilon_0^2} \\
\overset{(4.50)}{<}\ & M + \frac{2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta} + \psi_1(t) - \frac{\Gamma\lambda_0}{2}\frac{2}{\Gamma\lambda_0}\frac{\left|2\delta M + [4\Gamma + 2\delta]\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} - \delta^2\right|}{2\delta} \\
\leq\ & M + \frac{2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta} + \psi_1(t) - \left[M + \frac{2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta} + \|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))} - \frac{\delta}{2}\right] \\
=\ & \frac{\delta}{2}.
\end{aligned} \tag{4.52}$$

Seeking a contradiction, assume there exists $\tau \in [t_0, t_1]$ such that $\dot{e}(\tau) \geq \frac{\delta}{2} - \psi_1(\tau)$ which implies that

$$-k_1(\tau)\dot{e}(\tau) = \frac{-\dot{e}(\tau)}{\psi_1(\tau) + \dot{e}(\tau)} \leq \frac{2\dot{e}(\tau)}{\delta} \leq \frac{2\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta}.$$

From $\psi_0(t) - e(t) \leq \varepsilon_0$ and $\varepsilon_0 \leq \frac{\lambda_0}{2}$, it follows that $e(t) \geq \frac{\lambda_0}{2}$ on $[t_0, t_1]$ and hence

$$\forall\, t \in [t_0, t_1]: \quad -k_0(t)^2 e(t) \leq -\frac{\lambda_0}{2\varepsilon_0^2}.$$

Inserting these inequalities into (4.49), invoking (4.32) and (4.52), yields

$$\frac{\delta}{2} - \psi_1(\tau) \leq \dot{e}(\tau) \overset{(4.49)}{\leq} M + \frac{2\Gamma\|\psi_1\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\delta} - \frac{\Gamma\lambda_0}{2\varepsilon_0^2} \overset{(4.52)}{<} \frac{\delta}{2} - \psi_1(\tau)$$

which contradicts the assumption $\dot{e}(\tau) \geq \frac{\delta}{2} - \psi_1(\tau)$ and whence (4.51) holds.

STEP 3B: *Finally, it is shown that the claim of Step 3 holds true for $\varepsilon_0 > 0$ as in* (4.50).
Seeking a contradiction, assume there exists $t_1 \in [\varepsilon, \omega)$ such that $\psi_0(t_1) - |e(t_1)| < \varepsilon_0$. Continuity of $t \mapsto \psi_0(t) - |e(t)|$, gives that the number

$$t_0 := \max\{t \in [\varepsilon, t_1) \mid \psi_0(t) - |e(t)| = \varepsilon_0\}$$

is well defined. Then it follows that

$$\forall\, t \in [t_0, t_1] : \quad \psi_0(t) - |e(t)| \leq \varepsilon_0,$$

hence, by $\varepsilon_0 \leq \frac{\lambda_0}{2}$, it holds that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. Consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, the other case follows analogously.

In view of (4.51), it follows that $\dot{e}(t) < \frac{\delta}{2} - \psi_1(t)$ on $[t_0, t_1]$. By Remark 3.3.4 (i) it follows that

$$\forall\, t \in [t_0, t_1] : \quad \dot{\psi}_0(t) - \dot{e}(t) > \dot{\psi}_0(t) + \psi_1(t) - \frac{\delta}{2} \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0,$$

which yields that $t \mapsto \psi_0(t) - e(t)$ is strictly monotonically increasing on $(t_0, t_1]$ and whence the contradiction

$$0 < [\psi_0(t_0) - e(t_0)] - [\psi_0(t_1) - e(t_1)] = \varepsilon_0 - [\psi_0(t_1) - e(t_1)] < \varepsilon_0 - \varepsilon_0 = 0.$$

STEP 4: *It is shown that $\psi_1(t) - |\dot{e}(t)| \geq \varepsilon_1$ for all $t \in [\varepsilon, \omega)$ and positive*

$$\varepsilon_1 = \min\left\{\frac{\lambda_1}{2}, \psi_1(\varepsilon) - |\dot{e}(\varepsilon)|, \frac{1}{2}\left[\sqrt{\left(M + \frac{\Gamma\|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2}\right)^2 + 2\Gamma\lambda_1} - M - \frac{\Gamma\|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2}\right]\right\} \tag{4.53}$$

with M as in (4.49) *and ε_0 as in Step 3*A.

It follows, for $\varepsilon_0 > 0$ as in Step 3A, that $k_0(t)^2 \leq \frac{1}{\varepsilon_0^2}$ for all $t \in [\varepsilon, \omega)$ which together with (4.32) yields

$$\forall\, t \in [\varepsilon, \omega) : \quad k_0(t)^2 |e(t)| \leq \frac{\|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2}.$$

Seeking a contradiction, suppose there exists $t_1 \in [\varepsilon, \omega)$ such that $\psi_1(t_1) - |\dot{e}(t)| < \varepsilon_1$. Since $\psi_1(\varepsilon) - |\dot{e}(\varepsilon)| \geq \varepsilon_1$, the following is well defined

$$t_0 := \max \{ t \in [\varepsilon, t_1) \mid \psi_1(t) - |\dot{e}(t)| = \varepsilon_1 \} \in (\varepsilon, t_1).$$

By $\varepsilon_1 \leq \frac{\lambda_1}{2}$, it holds that $\operatorname{sgn} e(\cdot)$ is constant on $[t_0, t_1]$. Consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, the other case follows analogously. Moreover,

$$\forall\, t \in [t_0, t_1] : \quad \dot{e}(t) \geq \psi_1(t) - \varepsilon_1 \geq \frac{\lambda_1}{2}$$

whence

$$\forall\, t \in [t_0, t_1] : \quad k_1(t)\dot{e}(t) \geq \frac{\lambda_1}{2\varepsilon_1}$$

which, together with (4.49), yields the contradiction

$$\begin{aligned} 0 < [\psi_1(t_0) - \dot{e}(t_0)] - [\psi_1(t_1) - \dot{e}(t_1)] &\overset{(4.49)}{\leq} \varepsilon_1 - \psi_1(t_1) + M + \frac{\Gamma \|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2} - \frac{\Gamma\lambda_1}{2\varepsilon_1} \\ &< \varepsilon_1 + M + \frac{\Gamma \|\psi_0\|_{\mathcal{L}^\infty([\varepsilon,\infty))}}{\varepsilon_0^2} - \frac{\Gamma\lambda_1}{2\varepsilon_1} \overset{(4.53)}{\leq} 0. \end{aligned}$$

STEP 5: *Assertions (i)–(iv) are shown.*
Boundedness of $e(\cdot)$, $\dot{e}(\cdot)$, $z(\cdot)$, $k_0(\cdot)$ and $k_1(\cdot)$ on $[0, \omega)$ follows from continuity of the functions and (4.28)-(4.32), (4.33), Step 3 and Step 4, resp. The inequality (4.10) holds on $[0, \omega)$ by Step 3 and Step 4 and since $\varphi_i(\cdot)$ is uniformly bounded from below, $i = 0, 1$. Moreover (4.45) implies boundedness of $\dot{z}(\cdot)$.
Since $F(\cdot,\cdot,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{D}, \mathbb{R})$, see Step 1, and $(e, z)(\cdot)$ is a maximal solution of (4.45), see Step 1, Proposition 4.2.7 (ii) implies $\omega = \infty$ which shows Assertion (i)–(iv) and completes the proof. □

Proof of Theorem 4.2.9:
The proof uses the notation of Proposition 1.1.7 and (1.8). The structure of the proof closely resembles that of Theorem 4.2.2. For brevity, it is not included a full proof. Instead, the essential differences in the two cases are presented.

STEP 1: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (4.18) *or, equivalently,* (1.13)*,* (4.18) *is shown.*
It suffices to consider the system in Byrnes-Isidori form (1.13). For $(\varphi_0(\cdot), \varphi_1(\cdot)) \in$

$\mathcal{S}_2$, define the relatively open set

$$\mathcal{D} := \left\{(t,\mu_0,\mu_1,\xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \,\middle|\, (t,\mu_0) \in \mathcal{F}(0,\varphi_0), (t,\mu_1) \in \mathcal{F}(0,\varphi_1)\right\}$$

and

$$f : \mathcal{D} \to \mathbb{R}^n,$$

$$(t,\mu_0,\mu_1,\xi) \mapsto \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ R_1 & R_2 \end{bmatrix} \begin{pmatrix} \mu_0 + y_{\text{ref}}(t) \\ \mu_1 + \dot{y}_{\text{ref}}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ S \end{bmatrix} \xi - \begin{pmatrix} \dot{y}_{\text{ref}}(t) \\ \ddot{y}_{\text{ref}}(t) \end{pmatrix} \\ -\Gamma \begin{bmatrix} 0 \\ \operatorname{sat}_{\widehat{u}}\left(\frac{\varphi_0(t)^2 \mu_0}{(1-\varphi_0(t)|\mu_0|)^2} + \frac{\varphi_1(t)\mu_1}{1-\varphi_1(t)|\mu_1|} - d_u(t)\right) \end{bmatrix} \\ \begin{bmatrix} P_1, & 0 \end{bmatrix} \begin{pmatrix} \mu_0 + y_{\text{ref}}(t) \\ \mu_1 + \dot{y}_{\text{ref}}(t) \end{pmatrix} + Q\xi \end{pmatrix}.$$

Then, the initial value problem (1.13), (4.18) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} e(t) \\ \dot{e}(t) \\ z(t) \end{pmatrix} = f(t, e(t), \dot{e}(t), z(t)), \qquad \begin{pmatrix} e(0) \\ \dot{e}(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} cx^0 - y_{\text{ref}}(0) \\ cAx^0 - \dot{y}_{\text{ref}}(0) \\ Nx^0 \end{pmatrix}. \tag{4.54}$$

Existence of a unique maximal solution of the closed-loop system (1.13), (4.18) follows similarly to Step 1 in the proof of Theorem 4.2.2. The details are omitted. It has to be noted that, in view of Remark 3.3.4 (i), the condition $\varphi_i(0) \neq 0$, $i = 0, 1$, implies that $\varepsilon = 0$.

STEP 2: *It is shown that there exists $M > 0$ such that, for almost all $t \in [0, \omega)$,*

$$\begin{aligned} &-M - \Gamma \operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t)) \\ &\qquad \leq \ddot{e}(t) \leq M - \Gamma \operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t)). \end{aligned} \tag{4.55}$$

For ease of notation, the funnel boundaries are denoted by the functions

$$\psi_i(\cdot) := \varphi_i(\cdot)^{-1}, \qquad i \in \{0, 1\},$$

and so Remark 3.3.4 holds with $\varepsilon = 0$. Since $e(\cdot)$ and $\dot{e}(\cdot)$ evolve in the funnel, it follows that

$$\forall\, i \in \{0,1\}\ \forall\, t \in [\varepsilon, \omega): \quad |e^{(i)}(t)| < \psi_i(t) \leq \|\psi_i\|_\infty. \tag{4.56}$$

Applying Variation-of-Constants formula to the third subsystem in (4.26) yields

$$\forall\, t \in [\varepsilon, \omega): \quad z(t) = e^{Q(t-\varepsilon)} z(\varepsilon) + \int_\varepsilon^t e^{Q(t-s)} P_1 \left[e(s) + y_{\text{ref}}(s)\right] \mathrm{d}s\,,$$

and thus, in view of (1.8) and (4.56), it follows that

$$\forall\, t \in [0,\omega): \quad \|z(t)\| \le M_z \tag{4.57}$$

with

$$M_z := \frac{\beta}{\alpha}\|P_1\|\left[\|\psi_0\|_\infty + \|y_{\rm ref}\|_\infty\right].$$

Now (4.54) yields

$$\begin{aligned}\forall\, t \in [\varepsilon,\omega): \ \ddot e(t) = R_1\left[e(t)+r(t)\right] + R_2\left[\dot e(t)+\dot r(t)\right] + Sz(t) - \ddot r(t)\\ - \Gamma\,\mathrm{sat}_{\hat{\mathrm{u}}}(k_0(t)^2 e(t) + k_1(t)\dot e(t) - d_u(t))\end{aligned}$$

and so (4.55) follows with

$$M := |R_1|\left[\|\psi_0\|_\infty + \|y_{\rm ref}\|_\infty\right] + |R_2|\left[\|\psi_1\|_\infty + \|\dot y_{\rm ref}\|_\infty\right] + \|S\|M_z + \|\ddot y_{\rm ref}\|_\infty.$$

STEP 3: *It is shown that* $|e(\cdot)|$ *is bounded away from the funnel boundary* $\psi_0(\cdot)$ *on* $[0,\varepsilon_0)$ *with* ε_0 *as in* (4.19)*, more precisely:*

$$\forall\, t \in [0,\omega): \quad \psi_0(t) - |e(t)| \ge \varepsilon_0.$$

STEP 3A: Since $\varepsilon_0 \in \left(0, \frac{\lambda_0}{2}\right)$, it follows, on any interval $[t_0,t_1] \subset [0,\omega)$, the implication (4.35) in a similar way as in Step 3A of the proof of Theorem 4.2.2, i.e.

$$\begin{aligned}\left[\psi_0(t_0) - |e(t_0)| = 2\varepsilon_0 \ \wedge \ \text{for a.a. } t \in [t_0,t_1]: \ \ddot e(t)\,\mathrm{sgn}\, e(t) \le -\frac{(\|\psi_1\|_\infty + \ell_0)^2}{2\varepsilon_0}\right]\\ \Longrightarrow \quad \forall\, t \in [t_0,t_1]: \ \psi_0(t) - |e(t)| \ge \varepsilon_0.\end{aligned}$$

STEP 3B: *It is shown that, for* ε_0 *as in* (4.19)*, the implication* (4.38) *holds on any interval* $[t_0,t_1] \subset [0,\omega)$*, i.e.*

$$\begin{aligned}\left[\forall\, t \in [t_0,t_1]: \ \dot e(t)\,\mathrm{sgn}\, e(t) \ge \frac{\delta}{2} - \psi_1(t) \ \wedge \ \psi_0(t) - \big|e(t)\big| \le 2\varepsilon_0\right]\\ \Longrightarrow \quad \text{for a.a. } t \in [t_0,t_1]: \ \ddot e(t)\,\mathrm{sgn}\, e(t) \le -\frac{(\|\psi_1\|_\infty + \ell_0)^2}{2\varepsilon_0}.\end{aligned} \tag{4.58}$$

(4.19) gives $2\varepsilon_0 \le \frac{\lambda_0}{2}$. The condition $\psi_0(t) - \big|e(t)\big| \le 2\varepsilon_0$ on $[t_0,t_1]$ implies that $\mathrm{sgn}\, e(\cdot)$ is constant on $[t_0,t_1]$. Consider the case $\mathrm{sgn}\, e(\cdot) \equiv 1$, $\mathrm{sgn}\, e(\cdot) \equiv -1$ follows analogously. The condition $\dot{e}(t) \geq \frac{\delta}{2} - \psi_1(t)$ on $[t_0, t_1]$ implies that

$$\forall\, t \in [t_0, t_1]: \quad k_1(t)\dot{e}(t) = \frac{\dot{e}(t)}{\psi_1(t) + \dot{e}(t)} \leq \frac{2\dot{e}(t)}{\delta} \leq \frac{2\|\psi_1\|_\infty}{\delta}.$$

From $\psi_0(t) - e(t) \leq 2\varepsilon_0$ and $2\varepsilon_0 \leq \frac{\lambda_0}{2}$, it follows that $e(t) \geq \frac{\lambda_0}{2}$ on $[t_0, t_1]$ which implies, by $\varepsilon_0^2 \leq \frac{\lambda_0 \delta}{8[\delta(\widehat{u}+\|d_u\|_\infty)+2\|\psi_1\|_\infty]}$,

$$\forall\, t \in [t_0, t_1]: \quad k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t) \geq \frac{\lambda_0}{8\varepsilon_0^2} - \frac{2\|\psi_1\|_\infty}{\delta} - \|d_u\|_\infty \geq \widehat{u}$$

and hence

$$\forall\, t \in [t_0, t_1]: \quad \operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t)) = \widehat{u}. \tag{4.59}$$

It has to be noted that (4.16) implies $\Gamma\widehat{u} - M > 0$ and a simple calculation shows, together with (4.16) and (4.19),

$$\frac{(\|\psi_1\|_\infty + \ell_0)^2}{2(\Gamma\widehat{u} - M)} \leq \varepsilon_0$$

with

$$\varepsilon_0 = \min\left\{ \frac{\lambda_0}{4}, \frac{\psi_0(0) - |e(0)|}{2}, \frac{(\|\psi_1\|_\infty + \ell_0)^2}{3\ell_1}, \sqrt{\frac{\lambda_0 \delta}{8[\delta(\widehat{u} + \|d_u\|_\infty) + 2\|\psi_1\|_\infty]}} \right\}.$$

Inserting (4.59) into (4.55) yields

$$\forall\, t \in [t_0, t_1]: \quad \ddot{e}(t) \leq M - \Gamma \operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t)) = M - \Gamma\widehat{u}$$

whence (4.58).

STEP 3C: As in Step 3C and Step 3D of the proof of Theorem 4.2.2, it follows that the implication (4.39) holds for almost all $t \in [0, \omega)$ and implication (4.40) holds for any $(t_0, t_1] \subset [0, \omega)$ and so it is omitted for brevity. It has to be noted that the ε in the proof of Theorem 4.2.2 is, in view of Remark 3.3.4, zero in the context of Theorem 4.2.9.

STEP 3D: *Finally, it is shown that the claim of Step 3 holds true for $\varepsilon_0 > 0$ as in* (4.19).
If $\varepsilon_0 > 0$ in Step 3E of the proof of Theorem 4.2.2 is replaced by ε_0 as in (4.19), then the same proof as in Step 3E of the proof of Theorem 4.2.2 shows that the assumption

$\psi_0(t') - |e(t')| < \varepsilon_0$ for some $t' \in [0,\omega)$ leads to a contradiction. It has to be noted that $\varepsilon_0 \leq \frac{(\|\psi_1\|_\infty + \ell_0)^2}{3\ell_1}$ implies $\frac{(\|\psi_1\|_\infty+\ell_0)^2}{2\varepsilon_0} > \ell_1$. Hence Step 3 is proved.

STEP 4: *It is shown that* $\psi_1(t) - |\dot e(t)| \geq \varepsilon_1$ *for all* $t \in [0,\omega)$ *and* ε_1 *as in* (4.19).
It follows, for $\varepsilon_0 > 0$ as in (4.19), that $k_0(t)^2 \leq \frac{1}{\varepsilon_0^2}$ for all $t \in [\varepsilon, \omega)$ which together with (4.56) yields

$$\forall\, t \in [0,\omega): \quad k_0(t)^2 |e(t)| \leq \frac{\|\psi_0\|_\infty}{\varepsilon_0^2}.$$

Seeking a contradiction, suppose there exists $t_1 \in [0,\omega)$ such that $\psi_1(t_1) - |\dot e(t_1)| < \varepsilon_1$. Since $\psi_1(0) - |\dot e(0)| \geq \varepsilon_1$, the following is well defined

$$t_0 := \max\{t \in [0,t_1) \mid \psi_1(t) - |\dot e(t)| = \varepsilon_1\} \in (0,t_1).$$

Moreover,

$$\forall\, t \in [t_0,t_1]: \quad |\dot e(t)| \geq \psi_1(t) - \varepsilon_1 \geq \lambda_1 - \varepsilon_1 > \frac{\lambda_1}{2},$$

hence, it holds that $\operatorname{sgn} \dot e(\cdot)$ is constant on $[t_0,t_1]$. Consider the case $\operatorname{sgn} e(\cdot) \equiv 1$, the other case follows analogously. In view of (4.19), it follows that

$$\forall\, t \in [t_0,t_1]: \quad k_1(t)\dot e(t) \geq \frac{\lambda_1}{2\varepsilon_1} \overset{(4.19)}{\geq} \widehat{u} + \frac{\|\psi_0\|_\infty}{\varepsilon_0^2} + \|d_u\|_\infty$$

whence, for all $t \in [t_0,t_1]$,

$$\begin{aligned}\left|k_0(t)^2 e(t) + k_1(t)\dot e(t) - d_u(t)\right| &\geq k_0(t)^2 e(t) + k_1(t)\dot e(t) - d_u(t)\\ &\geq k_1(t)\dot e(t) - k_0(t)^2|e(t)| - \|d_u\|_\infty \geq \widehat{u},\end{aligned}$$

and so,

$$\forall\, t \in [t_0,t_1]: \quad u(t) = -\operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot e(t) - d_u(t)) = -\widehat{u}.$$

In view of (4.55), it follows that, for almost all $t \in [t_0,t_1]$,

$$\ddot e(t) \overset{(4.55)}{\leq} M - \Gamma \operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot e(t) - d_u(t)) = M - \Gamma\widehat{u} \overset{(4.16)}{<} -L \overset{(4.17)}{<} -\ell_1.$$

Integration gives

$$\dot e(t_1) - \dot e(t_0) = \int_{t_0}^{t_1} \ddot e(\tau)\,\mathrm{d}\tau < -\ell_1\,(t_1 - t_0)$$

whence, together with the Lipschitz property of $\psi_1(\cdot)$ on $[0,\omega)$, the contradiction

$$0 < \psi_1(t_0) - \dot{e}(t_0) - [\psi_1(t_1) - \dot{e}(t_1)] = \psi_1(t_0) - \psi_1(t_1) \\ + [\dot{e}(t_1) - \dot{e}(t_0)] < \ell_1\,(t_1 - t_0) - \ell_1\,(t_1 - t_0) = 0.$$

Hence Step 4 is proved.

STEP 5: The Assertions (i)-(iii) follow in the same way as in Step 5 of the proof of Theorem 4.2.2 and are omitted for brevity.

STEP 6: *Assertion (iv) is established, i.e. the existence of* $\tau \geq 0$ *such that* $|u(\tau)| < \widehat{u}$ *is shown.*
Seeking a contradiction, suppose

$$\forall\, t \geq 0: \quad |\operatorname{sat}_{\widehat{u}}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t))| = \widehat{u}.$$

Positivity of $\widehat{u}$ implies that $\operatorname{sgn}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t))$ is constant on $[0,\infty)$. Consider the case $\operatorname{sgn}(k_0(t)^2 e(t) + k_1(t)\dot{e}(t) - d_u(t)) \equiv 1$, the other case follows analogously. The condition $\Gamma\widehat{u} > M + \frac{2\|\psi_1\|_\infty^2}{\lambda_0}$, see (4.16), implies

$$\forall\, t \geq 0: \quad \ddot{e}(t) \leq M + \Gamma u(t) = M - \Gamma\widehat{u} < -\frac{2\|\psi_1\|_\infty^2}{\lambda_0} < 0$$

which gives, by integration, the contradiction

$$\forall\, t > 0: \quad -\|\psi_1\|_\infty \leq -\psi_1(t) \leq \dot{e}(t) < \dot{e}(0) - \frac{2\|\psi_1\|_\infty^2}{\lambda_0}\, t.$$

This completes the proof. □

Proof of Theorem 4.2.11:
The proof uses the notation of Proposition 1.1.7 and (1.8). The structure of the proof closely resembles that of Theorem 4.2.2 and Theorem 4.2.9. For brevity, it is not included a full proof. Instead, the essential differences are presented.

STEP 1: *Existence and uniqueness of a maximal solution of the initial value problem* (1.1)*,* (4.24) *or, equivalently,* (1.13)*,* (4.24) *is shown.*
In the case of Theorem 4.2.9 the function $f(\cdot,\cdot,\cdot,\cdot)$ of Step 1 of the proof of Theo-

rem 4.2.2 had to be replaced by

$$f : \mathcal{D} \to \mathbb{R}^n,$$
$$(t, \mu_0, \mu_1, \xi) \mapsto \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ R_1 & R_2 \end{bmatrix} \begin{pmatrix} \mu_0 + y_{\text{ref}}(t) \\ \mu_1 + \dot{y}_{\text{ref}}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ S \end{bmatrix} \xi - \begin{pmatrix} \dot{y}_{\text{ref}}(t) \\ \ddot{y}_{\text{ref}}(t) \end{pmatrix} \\ -\Gamma \begin{bmatrix} 0 \\ \text{sat}_{\widehat{u}_0}\left(\frac{\varphi_0(t)^2 \mu_0}{(1-\varphi_0(t)|\mu_0|)^2} - d_{u_0}(t) \right) + \text{sat}_{\widehat{u}_1}\left(\frac{\varphi_1(t)\mu_1}{1-\varphi_1(t)|\mu_1|} - d_{u_1}(t) \right) \end{bmatrix} \\ \begin{bmatrix} P_1, & 0 \end{bmatrix} \begin{pmatrix} \mu_0 + y_{\text{ref}}(t) \\ \mu_1 + \dot{y}_{\text{ref}}(t) \end{pmatrix} + Q\xi \end{pmatrix}.$$

The same argumentation as in Step 1 of the proof of Theorem 4.2.2 shows that the closed-loop system (1.13), (4.24) has a unique maximal solution. The details are omitted.

STEP 2: *It is shown that there exists $M > 0$ such that, for almost all $t \in [0, \omega)$,*

$$\begin{aligned} -M - \Gamma\left[\text{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) + \text{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t))\right] &\le \ddot{e}(t) \\ \le M - \Gamma\left[\text{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) + \text{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t))\right]&. \end{aligned} \tag{4.60}$$

Inequality (4.60) hold true with the analogous argumentation as in Step 2 of the proof of Theorem 4.2.9. So it is omitted.

STEP 3: *It is shown that $|e(\cdot)|$ is bounded away from the funnel boundary $\psi_0(\cdot)$ on $[0, \omega)$ with ε_0 as in* (4.25)*, more precisely:*

$$\forall\, t \in [0, \omega) : \quad \psi_0(t) - |e(t)| \ge \varepsilon_0.$$

Since $\varepsilon_0 \in \left(0, \frac{\lambda_0}{2}\right)$, the implication (4.35) holds true.

STEP 3A: *It is shown that, for ε_0 as in* (4.25)*, the implication* (4.58) *holds on any interval $[t_0, t_1] \subset [0, \omega)$.*
It has to be noted that (4.22) and $\widehat{u}_1 > \Gamma \frac{\delta \|d_{u_1}\|_\infty + 2\|\Psi_1\|_\infty}{\delta}$ give

$$\forall\, t \in [t_0, t_1] : \quad |k_1(t)\dot{e}(t) - d_{u_1}(t)| \le \frac{2\|\psi_1\|_\infty}{\delta} + \|d_{u_1}\|_\infty < \widehat{u}_1.$$

From $\psi_0(t) - e(t) \le 2\varepsilon_0$ and $2\varepsilon_0 \le \frac{\lambda_0}{2}$, it follows that $e(t) \ge \frac{\lambda_0}{2}$ on $[t_0, t_1]$ which implies, by $\varepsilon_0^2 \leq \frac{\lambda_0}{8[\widehat{u}_0+\|d_{u_0}\|_\infty]}$,

$$\forall\, t \in [t_0, t_1]: \quad k_0(t)^2 e(t) - d_{u_0}(t) \geq \frac{\lambda_0}{8\varepsilon_0^2} - \|d_{u_0}\|_\infty \geq \widehat{u}_0$$

and hence

$$\forall\, t \in [t_0, t_1]: \quad \mathrm{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) = \widehat{u}_0. \tag{4.61}$$

It has to be noted that (4.22) implies

$$\Gamma\delta\widehat{u}_0 - \delta M - \Gamma(\delta\|d_{u_1}\|_\infty + 2\|\Psi_1\|_\infty) > 0$$

and a simple calculation shows, together with (4.22) and (4.25),

$$\begin{aligned} &\frac{\delta(\|\psi_1\|_\infty + \ell_0)^2}{2(\Gamma\delta\widehat{u}_0 - \delta M - \Gamma(\delta\|d_{u_1}\|_\infty + 2\|\Psi_1\|_\infty))} \\ &\qquad \leq \varepsilon_0 = \min\left\{\frac{\lambda_0}{4}, \frac{\psi_0(0) - |e(0)|}{2}, \frac{(\|\psi_1\|_\infty + \ell_0)^2}{3\ell_1}, \sqrt{\frac{\lambda_0}{8[\widehat{u}_0 + \|u_{d_0}\|_\infty]}}\right\}. \end{aligned}$$

Inserting (4.61) into (4.60) yields

$$\begin{aligned} \forall\, t \in [t_0, t_1]: \quad \ddot{e}(t) &\leq M - \Gamma\mathrm{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) - \Gamma\mathrm{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t)) \\ &\leq M + \Gamma\frac{\delta\|d_{u_1}\|_\infty + 2\|\psi_1\|_\infty}{\delta} - \Gamma\widehat{u}_0 \end{aligned}$$

whence (4.58).

STEP 3B: Now the same arguments as in Step 3C and D of Theorem 4.2.9 complete the proof of Step 3.

STEP 4: *It is shown that* $\psi_1(t) - |\dot{e}(t)| \geq \varepsilon_1$ *for all* $t \in [0, \omega)$ *and* ε_1 *as in* (4.25). Seeking a contradiction and let t_0, t_1 be defined as in Step 4 of the proof of Theorem 4.2.9. With minor modifications Step 4 of the proof of Theorem 4.2.9 goes through. It has to be noted that, in view of (4.25), the inequality

$$\forall\, t \in [t_0, t_1]: \quad k_1(t)\dot{e}(t) \geq \frac{\lambda_1}{2\varepsilon_1} \overset{(4.25)}{\geq} \widehat{u}_1 + \|d_{u_1}\|_\infty$$

holds, whence whence

$$\forall\, t \in [t_0, t_1]: \quad |k_1(t)\dot{e}(t) - d_{u_1}(t)| \geq k_1(t)\dot{e}(t) - d_{u_1}(t) \geq k_1(t)\dot{e}(t) - -\|d_{u_1}\|_\infty \geq \widehat{u}_1,$$

and so,

$$\forall\, t \in [t_0, t_1]: \quad \mathrm{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t)) = \widehat{u}_1.$$

Now it follows that, for almost all $t \in [t_0, t_1]$,

$$\begin{aligned}\ddot{e}(t) &\overset{(4.60)}{\leq} M - \Gamma\mathrm{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) - \Gamma\mathrm{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t)) \\ &\leq M + \Gamma\widehat{u}_0 - \Gamma\widehat{u}_1 \overset{(4.22)}{<} -\ell_1\end{aligned}$$

and then the claim of Step 4 follows analogously as in the proof of Theorem 4.2.9 which is omitted for brevity.

STEP 5: The Assertions (i)-(iii) follow similarly to Step 5 in the proof of Theorem 4.2.2. The details are omitted.

STEP 6: *Assertion (iv) is established.*

STEP 6A: *It is shown that the following implication holds:*

$$\begin{aligned}&[\forall\, t \geq 0: \ |k_1(t)\dot{e}(t) - d_{u_1}(t)| \geq \widehat{u}_1] \\ &\qquad\qquad \Rightarrow \quad [\text{for a.a. } t \geq 0: \ \ddot{e}(t)\,\mathrm{sgn}(k_1(t)\dot{e}(t) - d_{u_1}(t)) \leq -\ell_1]. \qquad (4.62)\end{aligned}$$

Positivity of $\widehat{u}_1$ implies that $\mathrm{sgn}(k_1(t)\dot{e}(t) - d_{u_1}(t))$ is constant on $[0,\infty)$. Consider the case $\mathrm{sgn}(k_1(t)\dot{e}(t) - d_{u_1}(t)) \equiv 1$, the other case follows analogously. The assumption $k_1(t)\dot{e}(t) - d_{u_1}(t) \geq \widehat{u}_1$ on $[0,\infty)$ implies that $\mathrm{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t) - d_{u_1}(t)) = \widehat{u}_1$ on $[0,\infty)$. The condition $\Gamma\widehat{u}_1 > \Gamma\widehat{u}_0 + M + \ell_1 > 0$ (see (4.22)) implies

$$\text{for a.a. } t \geq 0: \quad \ddot{e}(t) \leq M + \Gamma u_0 - \Gamma u_1 < -\ell_1$$

whence (4.62).

STEP 6B: *It is shown that the following implication holds:*

$$\begin{aligned}&\begin{bmatrix} \forall\, t \geq 0: & |k_0(t)^2 e(t) - d_{u_0}(t)| \geq \widehat{u}_0 \\ \exists\, s \geq 0: & \dot{e}(s)\,\mathrm{sgn}(k_0(s)^2 e(s) - d_{u_0}(s)) \geq \frac{\delta}{2} - \psi_1(s) \end{bmatrix} \\ &\qquad \Rightarrow \quad \ddot{e}(s)\,\mathrm{sgn}(k_0(s)^2 e(s) - d_{u_0}(s)) < -\max\left\{\frac{3}{2}\ell_1,\ \frac{\|\psi_1\|_\infty^2}{\lambda_0}\right\} < 0. \qquad (4.63)\end{aligned}$$

It has to be noted that $\dot{e}(\cdot)$ is continuous which allows to assume that $\dot{e}(\cdot)$ is differentiable at s. Positivity of $\widehat{u}_0$ implies that $\mathrm{sgn}(k_0(t)^2 e(t) - d_{u_0}(t))$ is constant on $[0,\infty)$.

Consider the case $\operatorname{sgn}(k_0(t)^2 e(t) - d_{u_0}(t)) \equiv 1$, the other case follows analogously. The assumption $k_0(t)^2 e(t) - d_{u_0}(t) \geq \widehat{u}_0$ on $[0, \infty)$ implies that $\operatorname{sat}_{\widehat{u}_0}(k_0(t)^2 e(t) - d_{u_0}(t)) = \widehat{u}_0$ on $[0, \infty)$. The condition $\dot{e}(s) \geq \frac{\delta}{s} - \psi_1(s)$ implies, together with (4.22), that

$$|k_1(s)\dot{e}(s) - d_{u_1}(s)| \leq \frac{2\|\psi_1\|_\infty}{\delta} + \|d_{u_1}\|_\infty < \widehat{u}_1.$$

Inserting this into (4.60) and invoking (4.25) yields

$$\ddot{e}(s) \leq M + \Gamma \frac{\delta \|d_{u_1}\|_\infty + 2\|\psi_1\|_\infty}{\delta} - \Gamma \widehat{u}_0 < -\max\left\{\frac{3}{2}\ell_1, \ \frac{\|\psi_1\|_\infty^2}{\lambda_0}\right\} < 0$$

whence (4.63).

STEP 6C: *It is shown that the following implication holds:*

$$\left[\exists\, s \geq 0: \ \dot{e}(s) \operatorname{sgn}(k_0(s)^2 e(s) - d_{u_0}(s)) < \frac{\delta}{2} - \psi_1(s)\right]$$
$$\Rightarrow \quad \left[\forall\, t \geq s: \ \dot{e}(t) \operatorname{sgn}(k_0(t)^2 e(t) - d_{u_0}(t)) < \frac{\delta}{2} - \psi_1(t)\right]. \quad (4.64)$$

Seeking a contradiction, suppose that

$$\exists\, \tau > s: \quad \dot{e}(\tau) \operatorname{sgn}(k_0(\tau)^2 e(\tau) - d_{u_0}(\tau)) < \frac{\delta}{2} - \psi_1(\tau)$$

which implies that

$$\forall\, t \in [s, \tau): \ \dot{e}(t) \operatorname{sgn}(k_0(t)^2 e(t) - d_{u_0}(t)) < \frac{\delta}{2} - \psi_1(t)$$

and hence $\operatorname{sgn}(k_0(\cdot)^2 e(\cdot) - d_{u_0}(\cdot))$ is constant on $[s, \tau]$. Consider the case $\operatorname{sgn}(k_0(\cdot)^2 e(\cdot) - d_{u_0}(\cdot)) \equiv 1$, the other case follows analogously. It has to be noted that $\dot{e}(\cdot)$ is continuous which allows to assume that $\dot{e}(\cdot)$ is differentiable at τ. Now it holds

$$\ddot{e}(t) = \lim_{h \to 0+} \frac{\dot{e}(t) - \dot{e}(t-h)}{h} \geq \lim_{h \to 0+} \frac{-\psi_1(t) + \psi_1(t-h)}{h} \geq -\ell_1$$

which gives, together with (4.63), the contradiction

$$-\ell_1 \leq \ddot{e}(\tau) < -\frac{3}{2}\ell_1,$$

and so, (4.64) holds.

STEP 6D: *It is shown that Assertion (iv) fails to hold leads to a contradiction.* Seeking a contradiction, suppose that

$$\exists\, i \in \{0,1\}\ \forall\, t \geq 0: \quad |k_i(t)^{2-i} e^{(i)}(t) - d_{u_i}(t)| \geq \widehat{u}_i. \tag{4.65}$$

Positivity of $\widehat{u}_i$ implies that $\operatorname{sgn}(k_i(\cdot)^{2-i} e^{(i)}(\cdot) - d_{u_i}(\cdot))$ is constant on $[0,\infty)$. Consider the case $\operatorname{sgn}(k_i(\cdot)^{2-i} e^{(i)}(\cdot) - d_{u_i}(\cdot)) \equiv 1$, the other case follows analogously.
First if $i = 1$, then implication (4.62) gives, by integration, the contradiction

$$\forall\, t > 0: \quad -\|\psi_1\|_\infty \leq -\psi_1(t) \leq \dot{e}(t) < \dot{e}(0) - \ell_1\, t$$

which proves Step 6 in the case of $i = 1$.
It remains to consider the case $i = 0$. If

$$\forall\, t \geq 0: \quad \dot{e}(t) \geq \frac{\delta}{2} - \psi_1(t), \tag{4.66}$$

then (4.63) implies that

$$\forall\, t \geq 0: \quad \ddot{e}(t) \leq -\frac{\|\psi_1\|_\infty^2}{\lambda_0}$$

which gives, by integration, the contradiction

$$\forall\, t > 0: \quad -\|\psi_0\|_\infty \leq -\psi_0(t) \leq e(t) < e(0) + \dot{e}(0)\, t - \frac{1}{2}\frac{\|\psi_1\|_\infty^2}{\lambda_0} t^2$$

which proves Step 6 in case of $i = 0$ and (4.66). Therefore it remains to consider the case $i = 0$ and

$$\exists\, s \geq 0: \quad \dot{e}(s) < \frac{\delta}{2} - \psi_1(s). \tag{4.67}$$

By Remark 3.3.4 (i) and (4.64), it follows that

$$\text{for a.a. } t \geq s: \quad \dot{\psi}_0(t) - \dot{e}(t) \geq \dot{\psi}_0(t) + \psi_1(t) - \frac{\delta}{2} \geq \frac{\delta}{2}$$

which gives, by integration, the contradiction

$$\forall\, t \geq s: \quad 2\|\psi_0\|_\infty \geq \psi_0(t) - e(t) \geq \psi_0(s) - e(s) + \frac{\delta}{2}(t - s).$$

This proves Step 6 in case of $i = 0$ and (4.67) which completes the proof of Step 6.

STEP 7: *Assertion (v) is shown.*

It has to be noted that $d_{u_1}(\cdot) \equiv 0$.

STEP 7A: *It is shown that* $\dot{e}(\cdot)$ *satisfies*

$$\Big[|\dot{e}(t)| - |\dot{e}(t_0)| < -\ell_1(t-t_0) \quad \text{for some } t > t_0 \geq 0\Big]$$
$$\Rightarrow \Big[k_1(t)|\dot{e}(t)| - k_1(t_0)|\dot{e}(t_0)| < -\ell_1 k_1(t)(t-t_0)\Big]. \quad (4.68)$$

If

$$|\dot{e}(t)| - |\dot{e}(t_0)| < -\ell_1(t-t_0) \quad \text{for some } t > t_0 \geq 0,$$

then

$$\begin{aligned} k_1(t)|\dot{e}(t)| - k_1(t_0)|\dot{e}(t_0)| &= k_1(t)k_1(t_0)\left[|\dot{e}(t)|\psi_1(t_0) - |\dot{e}(t_0)|\psi_1(t)\right] \\ &< k_1(t)k_1(t_0)\left[(|\dot{e}(t_0)| - \ell_1(t-t_0))\psi_1(t_0) - |\dot{e}(t_0)|\psi_1(t)\right] \\ &= k_1(t)k_1(t_0)\left[((\psi_1(t_0) - \psi_1(t))|\dot{e}(t_0)| - \ell_1(t-t_0)\psi_1(t_0)\right] \\ &\leq k_1(t)k_1(t_0)\left[\ell_1(t-t_0)|\dot{e}(t_0)| - \ell_1(t-t_0)\psi_1(t_0)\right] \\ &= -\ell_1 k_1(t)[t-t_0], \end{aligned}$$

and so, (4.68) holds.

STEP 7B: *It is shown that Assertion (v) fails to hold leads to a contradiction.* Assume that $\tau \geq 0$ is such that $|\mathrm{sat}_{\widehat{u}_1}(k_1(\tau)\dot{e}(\tau))| < \widehat{u}_1$. Seeking a contradiction, suppose that

$$\exists\, t_1 > \tau : \quad |\mathrm{sat}_{\widehat{u}_1}(k_1(t_1)\dot{e}(t_1))| = \widehat{u}_1.$$

Positivity of $\widehat{u}_1 - |\mathrm{sat}_{\widehat{u}_1}(k_1(\tau)\dot{e}(\tau))|$, together with (4.22), yields that there exists $\widetilde{\varepsilon} \in \left(0, 1 - \frac{\Gamma\widehat{u}_0 + M + \ell_1}{\Gamma\widehat{u}_1}\right)$ such that

$$|\mathrm{sat}_{\widehat{u}_1}(k_1(\tau)\dot{e}(\tau))| \leq (1-\widetilde{\varepsilon})\widehat{u}_1$$

and

$$\Gamma(1-\widetilde{\varepsilon})\widehat{u}_1 > \Gamma\widehat{u}_0 + M + \ell_1. \quad (4.69)$$

Define

$$t_0 := \sup\left\{t \in [\tau, t_1] \mid |\mathrm{sat}_{\widehat{u}_1}(k_1(t)\dot{e}(t))| = (1-\widetilde{\varepsilon})\widehat{u}_1\right\}.$$

Then positivity of $\widehat{u}_1$ implies that $\mathrm{sgn}\,\dot{e}(\cdot)$ is constant on $[t_0, t_1]$. Consider the case $\mathrm{sgn}\,\dot{e}(\cdot) \equiv 1$, the other case follows analogously. Invoking (4.60) and (4.69), it follows that

$$\forall\, t \in [t_0, t_1]: \quad \ddot{e}(t) \overset{(4.60)}{\leq} M + \Gamma u(t) \leq M + \Gamma \widehat{u}_0 - \Gamma(1-\widehat{\varepsilon})\widehat{u}_1 \overset{(4.69)}{<} -\ell_1$$

which, on integration, yields

$$\dot{e}(t_1) - \dot{e}(t_0) < -\ell_1(t_1 - t_0),$$

whence, by (4.68), the contradiction

$$\widehat{u}_1 = |\mathrm{sat}_{\widehat{u}_1}(k_1(t_1)\dot{e}(t_1))| = k_1(t_1)|\dot{e}(t_1)| < k_1(t_0)|\dot{e}(t_0)| - \ell_1 k_1(t_1)(t_1 - t_0) < \widehat{u}_1.$$

This completes the proof of the theorem. □

4.5 Notes and references

The concept of funnel control was originally introduced by [43] for the class of linear systems (1.1) with CB positive definite, i.e. for systems (1.1) with strict relative degree one. It had been applied successfully in experiments on electric drive systems (see [47]). Further results of funnel control can be found in the discussion of Section 3.7.

Chapter 3 generalized these results for input constraints. In [44, 45] the concept of funnel control has been extended to systems of higher relative degree, where the controller involves a filter and the dynamic feedback strategy (4.5) which is no longer simple. Moreover [44, 45] use a backstepping construction of the feedback strategy which follows the ideas of [90].
This chapter introduced a simple funnel controller for single input, single output systems with relative degree one or two. Moreover, input constraints and disturbances are allowed. The results of Theorem 4.2.9 and 4.2.11 are a generalization of the relative degree one results of [43] and Chapter 3.

Results for implicit ODE can be found in [16] and the reference therein or in [34]. The main result of [16] is a Peano Theorem for the class of implicit ODE of the form

$$G_1(t, x(t), \dot{x}(t), ..., x^{(m)}(t)) = 0, \quad x^{(i)}(0) = x^i \in \mathbb{R},\ i = 0, ..., m-1, \tag{4.70}$$

for some suitable function $G_1(\cdot)$. Under some restrictions on $G_1(\cdot)$, [16, Th. 1.1] gets the well known Peano Theorem as for explicit ODE. It has to be noted that (4.70) is a scalar implicit ODE and a careful inspection of the proof of [16, Th. 1.1] offers the importance of this fact. But (4.13) is not a scalar system, it can be written as

$$G_2(t,(e,z)(t),(\dot{e},\dot{z})(t)) = 0, \quad (e,z)(0) = (e^0, z^0)$$

for some suitable function $G_2(\cdot)$ which cannot be handled by [16]. On the other hand, [34] consider first order implicit ODE higher dimension of the form

$$\dot{x}(t) = G_3(t, x(t), \dot{x}(t)), \quad x(0) = x^0 \in \mathbb{R}^n$$

for some suitable function $G_3(\cdot)$. With some hard restrictions on $G_3(\cdot)$, [34] offers existence of a local solution. Questions like maximal solution, finite escape time or uniqueness are unanswered. Proposition 4.2.7 proves existence and uniqueness of the solution of an implicit ordinary differential equation which is based on the Implicit Function Theorem.

5 Volterra-Stieltjes systems

Linear time-varying Volterra-Stieltjes equations are considered. First, existence and uniqueness of a solution is presented and positivity is characterized in terms of the system entries. Explicit criteria for uniform asymptotic stability and exponential asymptotic stability of positive linear time-invariant equations are given. A generalization of the concept of zero dynamics for Volterra-Stieltjes equations is shown and a characterization for positive equations is presented. Funnel control of Chapter 3 is considered for multi input, multi output Volterra-Stieltjes equations. The system is assumed to have stable zero dynamics and the high-frequency gain matrix must have a certain structure, the latter implies that it is of strict relative degree one. Moreover, the funnel controller is applicable for input constraints provided a feasibility condition (formulated in terms of the system data, the saturation bounds, the funnel data, bounds on the reference signal and the initial state) holds which generalizes the results of Chapter 3 to Volterra-Stieltjes systems. The funnel controller guarantees, for positive (Volterra-Stieltjes) equations and positive reference trajectory, that the systems states and the output remain non-negative. If a system satisfies further conditions, i.e. the matrix A of (1.2) is Hurwitz, then the funnel controller achieves non-negativity of the input. Finally, the theoretical results are illustrated by general anesthesia.

5.1 Introduction

To get an impression of Volterra-Stieltjes systems consider the integral equation

$$x(t) = F(t) + \int_0^t G(t, s, x(s))\, \mathrm{d}s \tag{5.1}$$

for suitable functions $F : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $G : \mathcal{D} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\mathcal{D} := \{(t, s) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mid 0 \leq s \leq t\}$. At the moment the properties of $F(\cdot)$ and $G(\cdot, \cdot, \cdot)$ are not specified. If differentiability of (5.1) is assumed, then the *integro-differential* *equation*

$$\dot{x}(t) = \dot{F}(t) + G(t,t,x(t)) + \int_0^t \frac{\partial G}{\partial t}(t,s,x(s))\,\mathrm{d}s \tag{5.2}$$

is obtained. If $G(\cdot,\cdot,\cdot)$ is independent of t, then (5.2) is an ordinary differential equation of the form

$$\dot{x}(t) = H(t,x(t))$$

for a suitable function $H(\cdot,\cdot)$. An overview of many results for integral equations (5.1) and integro-differential equations (5.2) can be found in [26, 61] with the detailed assumptions on $F(\cdot)$ and $G(\cdot,\cdot,\cdot)$. Integral equations (5.1) and integro-differential equations (5.2) arise by the modeling of physical processes (see [12]).
Equations of the form (5.2) can be regarded as an example of the fairly large Volterra-Stieltjes systems

$$\dot{x}(t) = \dot{F}(t) + \int_0^t \mathrm{d}[\eta(s)]\,x(t-s)$$

for some suitable function $\eta(\cdot)$ (see [21]). Time-invariant linear Volterra-Stieltjes systems, together with many examples, are well understood (see [21]). Hence, Section 5.2 deals with generalized time-varying linear Volterra-Stieltjes systems (1.2).

The integral term of (1.2) will be explained.

Definition 5.1.1. (Integral term of (1.2)**)**
For any $\eta(\cdot) \in \mathcal{BV}([\alpha,\beta],\mathbb{R}^{m\times n})$ and $\phi(\cdot) \in \mathcal{C}([\alpha,\beta],\mathbb{R}^n)$, where $-\infty < \alpha < \beta < \infty$, the integral $\int_\alpha^\beta \mathrm{d}[\eta(\theta)]\,\phi(\theta)$ exists and is defined as the limit of $S(P) := \sum_{k=1}^{p} [\eta(\theta_k) - \eta(\theta_{k-1})]\phi(\zeta_k)$ as $d(P) := \max_k |\theta_k - \theta_{k-1}| \to 0$, where $P = \{\theta_1 = \alpha \le \theta_2 \le \cdots \le \theta_p = \beta\}$, $p \in \mathbb{N}$, is any finite partition of the interval $[\alpha,\beta]$ and $\zeta_k \in [\theta_{k-1},\theta_k]$. $\diamond$

In Section 5.2, positivity of (1.2) is investigated; that is, loosely speaking, for any non-negative input $u(\cdot)$ and any non-negative initial condition, the corresponding solution of the system is also non-negative. This concept is characterized. Thereafter, various stability concepts are presented in Section 5.3 – explicit criteria for uniform asymptotic stability and exponential asymptotic stability of positive linear time-invariant equations are derived. In Section 5.4, the standard system theoretic concept of (stable) zero dynamics is recalled: these are, loosely speaking, those dynamics of the system which have an identically zero output or, in other words, those dynamics which are not "visible" at the output, the zero dynamics are stable if they tend to zero when t tends to ∞. This concept coincides with minimum phase if (1.2) is time-invariant without Volterra term (see Appendix, Section 1.1.3). The Byrnes-Isidori form (this form separates the direct influence of the input to the zero dynamics) is derived and exploited to characterize stable zero dynamics for time-invariant Volterra-Stieltjes systems (1.2) (i.e. $A(\cdot) \equiv A \in \mathbb{R}^{n\times n}$). Finally, it is shown that positive systems with stable zero dynamics and a special structure of the input output matrices (in particular relative degree one) are high-gain stabilizable while preserving positivity.

The results of the Sections 5.2-5.4 may be interesting in their own right, but these results are exploited in Section 5.5 to generalize the funnel controller (3.19) to time-invariant Volterra-Stieltjes systems (5.13) which are a special case of the class (1.2). As in Chapter 3, the controller (3.19) applied to systems (5.13) yields a closed-loop system which variables remain bounded if stable zero dynamics and suitable assumptions on the high-frequency gain matrix CB are assumed. Moreover, funnel controller guarantees also positivity of the trajectory of the closed-loop system. Under a suitable feasibility assumption, funnel control is possible in the presence of input constraints which generalizes the results of Chapter 3 to Volterra-Stieltjes systems. And if the system matrix $A(\cdot) \equiv A \in \mathbb{R}^{n\times n}$ of (1.2) is Hurwitz, then non-negativity of the input $u(\cdot)$ is achieved. Finally, Section 5.6 applies the results of Section 5.5 to control the depth of anesthesia of a three compartment mammillary patient's model.

5.2 Positivity

Volterra-Stieltjes systems (1.2) with constant system matrix, i.e. $A(\cdot) \equiv A \in \mathbb{R}^{n\times n}$, are well understood (see [21]). Positive time-invariant Volterra-Stieltjes systems (1.2) are considered in [68, 70] and time-varying positive systems (1.2) without the Volterra term, i.e. $\eta(\cdot) \equiv 0$, are regarded in [25]. In this section the concept of positivity is investigated and characterized for time-varying Volterra-Stieltjes systems (1.2).

Definition 5.2.1. (Solution)
Consider, for $(A(\cdot), \eta(\cdot), f(\cdot)) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and initial data $\sigma \geq 0$, $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$, the initial value problem

$$\dot{x}(t) = A(t)x(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) + f(t), \qquad \text{for a.a. } t \geq \sigma, \quad x|_{[0,\sigma]} = \phi. \quad (5.3)$$

A function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is said to be a *solution* of the initial value problem (5.3) if, and only if, $x(\cdot)$ is locally absolutely continuous on $[\sigma, \infty)$, $x(\cdot)$ satisfies (5.3) for almost all $t \in [\sigma, \infty)$ and $x|_{[0,\sigma]} = \phi$. If the solution is unique, it is denoted by $x(\cdot; \sigma, \phi, f)$. ◇

Remark 5.2.2.
$M(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ denotes the space of all matrix-valued Borel measures $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ with the norm

$$\|\nu\|_{M(\mathbb{R}_{\geq 0})} := |\nu|(\mathbb{R}_{\geq 0}) := \sup \sum_{i=1}^{k} |\nu(E_i)|,$$

where the supremum is taken over all partitions $\{E_i\}_{i=1,\ldots,k}$ of $\mathbb{R}_{\geq 0}$, $k \in \mathbb{N}$. $|\nu|$ is called the *total variation* of $\nu(\cdot)$. The support of $\nu(\cdot)$ is the complement of the largest open set $E \subset \mathbb{R}_{\geq 0}$ such that $|\nu|(E) = 0$.

Consider the initial value problem (5.3) with constant matrix $A(\cdot) \equiv A \in \mathbb{R}^{n\times n}$. Let

$$\nu_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}, \quad s \mapsto \nu_1(s) := \begin{cases} 0 & , s = 0 \\ A & , s > 0 \end{cases}$$

and define

$$\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}, \quad s \mapsto \nu(s) := \nu_1(s) + \nu_2(s),$$

where $\nu_2(\cdot) \in M(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$. Then, by Definition 5.1.1, the linear Volterra-Stieltjes differential system (5.3) can be written in the form

$$\dot{x}(t) = \int_0^t \mathrm{d}[\nu(\theta)]\, x(t-\theta) + f(t), \qquad \text{for a.a. } t \geq \sigma, \quad x|_{[0,\sigma]} = \phi.$$

◇

Remark 5.2.3. (Shifted initial value problem)
The shift

$$w(t) := x(t+\sigma) \qquad t \geq 0$$

applied to the initial value problem (5.3) converts this into the equivalent initial value problem

$$\dot{w}(t) = \mathcal{A}(t)w(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, w(t-\theta) + F(t), \text{ for a.a. } t \geq 0, \ w(\cdot)|_{[-\sigma,0]} = \phi(\cdot+\sigma), \quad (5.4)$$

where the notation

$$\Big(\tau \;\mapsto\; \mathcal{A}(\tau) := A(\tau+\sigma)\Big) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$$

$$\Big(\tau \;\mapsto\; F(\tau) := \int_{\tau}^{\tau+\sigma} \mathrm{d}[\eta(\theta)]\, \phi(\tau+\sigma-\theta) + f(\tau+\sigma)\Big) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n})$$

is used. This is easily seen since, for almost all $t \geq \sigma$,

$$\begin{aligned}
\dot{w}(t) &= \dot{x}(t+\sigma) = A(t+\sigma)x(t+\sigma) + \int_0^{t+\sigma} \mathrm{d}[\eta(\theta)]\, x(t+\sigma-\theta) + f(t+\sigma) \\
&= A(t+\sigma)x(t+\sigma) + \int_0^{t} \mathrm{d}[\eta(\theta)]\, x(t+\sigma-\theta) + \int_t^{t+\sigma} \mathrm{d}[\eta(\theta)]\, \phi(t+\sigma-\theta) \\
&\quad + f(t+\sigma) \\
&= A(t+\sigma)w(t) + \int_0^{t} \mathrm{d}[\eta(\theta)]\, w(t-\theta) + \int_t^{t+\sigma} \mathrm{d}[\eta(\theta)]\, \phi(t+\sigma-\theta) + f(t+\sigma)
\end{aligned}$$

and

$$w(t) = x(t+\sigma) = \phi(t+\sigma) \qquad \forall\, t \in [-\sigma, 0].$$

$\diamond$

Proposition 5.2.4. ***(Existence and uniqueness of solution)***
For any $\sigma \geq 0$ and any $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$, the initial value problem (5.3) *has a unique solution $x(\cdot;\sigma,\phi,f) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$; the solution depends continuously on the initial values.*

The proof of Proposition 5.2.4 is in Subsection 5.7.1 on page 221. A proof of Proposition 5.2.4 for time-varying systems (5.3) is not available in the literature. The essence of the proof is in formulating the classical existence and uniqueness proof in such a way that [21, Th. 3.6.1] is applicable.

Remark 5.2.5. (Time-invariant systems)
Consider, for $(A, \eta(\cdot)) \in \mathbb{R}^{n\times n} \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, the matrix initial value problem

$$\dot{R}(t) = AR(t) + \int_0^{t} \mathrm{d}[\eta(\theta)]\, R(t-\theta), \quad R(0) = I_n, \qquad \text{for a.a. } t \geq 0. \tag{5.5}$$

(i) By Proposition 5.2.4, there exists a unique solution $R : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ of (5.5); this solution is called the *resolvent* of the linear Volterra-Stieltjes differential system

$$\dot{x}(t) = Ax(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta), \qquad \text{for a.a. } t \geq 0. \tag{5.6}$$

(ii) Consider the time-invariant inhomogeneous system (5.3) (i.e. $A(\cdot) \equiv A$). Then, for any $f(\cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and initial data $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$, $\sigma \geq 0$, there exists a unique solution $x(\cdot; \sigma, \phi, f) : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem (5.3) and, invoking the resolvent $y_{\mathrm{ref}}(\cdot)$ of (5.6), it satisfies, for all $t \geq 0$, the *Variation-of-Constants* formula

$$x(t+\sigma; \sigma, \phi, f) = R(t)\phi(\sigma) + \int_0^t R(t-s)\Big\{ \Big(\int_s^{s+\sigma} \mathrm{d}[\eta(\theta)]\, \phi(s+\sigma-\theta) \Big) + f(s+\sigma) \Big\}\, \mathrm{d}s\,. \tag{5.7}$$

This is seen if (5.3) is written as (5.4) and differentiated.

(iii) Under the same assumptions as in (ii), it follows from (5.7) that, for all $t \geq 0$,

$$\|x(t+\sigma;\sigma,\phi,f)\| \leq \left[\|R(t)\| + \int_0^t \|R(t-s)\|\, \|\eta\|_{\mathrm{Var}([s,s+\sigma])}\, \mathrm{d}s \right] \|\phi\|_{\mathcal{L}^\infty([0,\sigma])} + \int_0^t \|R(t-s)\|\, \|f(s+\sigma)\|\, \mathrm{d}s\,.$$

(iv) If the resolvent $y_{\mathrm{ref}}(\cdot)$ of (5.6) satisfies $y_{\mathrm{ref}}(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, then, together with Remark 5.2.2 and [21, Th. 3.3.5], $\dot{y}_{\mathrm{ref}}(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$; and Lemma 5.7.1 (ii) gives $\lim_{t\to\infty} R(t) = 0$. ◇

Now positive systems are studied.

Definition 5.2.6. (Positivity)
For $(A(\cdot), \eta(\cdot)) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{BV}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, the system

$$\dot{x}(t) = A(t)x(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta), \qquad \text{for a.a. } t \geq 0, \tag{5.8}$$

is said to be *positive* if, and only if, for every non-negative initial data $\sigma \geq 0$, $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$, the unique solution of (5.8) satisfying the initial condition $x|_{[0,\sigma]} = \phi$ is also non-negative, i.e. $x(t;\sigma,\phi,0) \in \mathbb{R}^n_{\geq 0}$ for all $t \geq \sigma$. ◇

Proposition 5.2.7. ***(Positivity of inhomogeneous systems)***
Suppose $A(\cdot) \in (\mathcal{L}^1_{\text{loc}} \cap \mathcal{L}^\infty_{\text{loc}})(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, $A(t)$ is a Metzler matrix for almost all $t \geq 0$ and $\eta(\cdot) \in \mathcal{BV}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ is a non-decreasing function. Then, for any non-negative initial data $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$, $\sigma \geq 0$, and any non-negative inhomogenity $f(\cdot) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n_{\geq 0})$, the unique solution of the initial value problem (5.3) *is non-negative, i.e. $x(\cdot;\sigma,\phi,f) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{\geq 0}$.*

The proof of Proposition 5.2.7 is in Subsection 5.7.1 on page 225.

It has to be noted that Proposition 5.2.4 and 5.2.7 yield that positivity of (5.8) implies monotonicity in the sense that if

$$(\sigma, \phi_k(\cdot), f_k(\cdot)) \in \mathbb{R}_{\geq 0} \times \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0}) \times \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n_{\geq 0}),\ k = 1,2,$$
$$\text{with } \phi_1(\cdot) \leq \phi_2(\cdot) \text{ and } f_1(\cdot) \leq f_2(\cdot),$$

then

$$\forall\, t \geq 0: \quad x(t;\sigma,\phi_1,f_1) \leq x(t;\sigma,\phi_2,f_2).$$

Now, the main result of this section is presented which gives an equivalence to the definition of positivity in terms of the system data. The following theorem characterizes positivity of a system (5.8) in terms of the system entries, namely, $A(t)$ is a Metzler matrix for all $t \geq 0$ and $\eta(\cdot)$ is a non-decreasing function. A check of this two conditions is quite often simpler as the conditions of Definition 5.2.6.

Theorem 5.2.8. ***(Characterization of positivity)***
Let $(A(\cdot), \eta(\cdot)) \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{BV}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$. Then (5.8) *is positive if, and only if, $A(t)$ is a Metzler matrix for all $t \geq 0$ and $\eta(\cdot)$ is a non-decreasing function.*
The implication "$\Leftarrow$" only requires that $A(\cdot) \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$.

The proof of Theorem 5.2.8 is in Subsection 5.7.1 on page 227.

5.3 Stability

In this section, various stability concepts of systems of the form

$$\dot{x}(t) = Ax(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta), \qquad \text{for a.a. } t \geq \sigma, \quad x|_{[0,\sigma]} = \phi, \tag{5.9}$$

where $(A, \eta(\cdot)) \in \mathbb{R}^{n\times n} \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ and $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$, $\sigma \geq 0$, are investigated.

Definition 5.3.1. (Stability concepts for Volterra-Stieltjes systems 5.9)
A system (5.9) – or more precisely, its zero solution – is said to be

1. *stable* $:\Leftrightarrow$

$$\forall\, \varepsilon > 0\; \forall\, \sigma \geq 0\; \exists\, \delta > 0\; \forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n) \text{ with } \|\phi\|_\infty < \delta\; \forall\, t \geq \sigma : \|x(t;\sigma,\phi)\| < \varepsilon,$$

2. *uniformly stable* $:\Leftrightarrow$ stable and $\delta > 0$ can be chosen independently of σ,
3. *attractive* $:\Leftrightarrow$ $\forall\, \sigma \geq 0\; \forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n) : \lim_{t\to\infty} x(t;\sigma,\phi) = 0$,
4. *asymptotically stable* $:\Leftrightarrow$ stable and attractive,
5. *uniformly asymptotically stable* $:\Leftrightarrow$ uniformly stable and

$$\exists\, \delta > 0\; \forall\, \varepsilon > 0\; \exists\, T > 0\; \forall\, \sigma \geq 0\; \forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n),\; \|\phi\|_\infty < \delta\; \forall\, t \geq \sigma + T : \|x(t;\sigma,\phi)\| < \varepsilon,$$

6. *exponentially asymptotically stable* $:\Leftrightarrow$

$$\forall\, \sigma \geq 0\; \exists\, M,\, \lambda > 0 \forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)\; \forall\, t \geq \sigma : \|x(t;\sigma,\phi)\| \leq M e^{-\lambda(t-\sigma)} \|\phi\|_\infty,$$

7. *$\mathcal{L}^p$-stable*, $p \in [1,\infty]$ $:\Leftrightarrow$ $y_{\mathrm{ref}}(\cdot) \in \mathcal{L}^p(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, where $y_{\mathrm{ref}}(\cdot)$ denotes the resolvent of (5.6). $\diamond$

The following Proposition generalizes the results of Proposition 1.1.4 to Volterra-Stieltjes systems, i.e the stability concepts of Definition 1.1.3 (Definition 5.3.1, respectively).

Proposition 5.3.2. ***(Characterization of stability concepts for Volterra-Stieltjes systems 5.9)***
If the system (5.9) *has finite total variation, i.e.* $\int_0^\infty |\,\mathrm{d}\eta(\theta)\,| < \infty$*, then the following stability concepts of* (5.9)

(i) *asymptotic stability*

(ii) $\mathcal{L}^1$*-stability*

(iii) $\mathcal{L}^p$*-stability for all* $p \in [1, \infty]$

(iv) $\forall\, s \in \mathbb{C}_0 : \quad \det\left(sI - A - \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s\theta} \right) \neq 0$

(v) *uniform asymptotic stability*

(vi) $\exists\, M,\ \lambda > 0\ \forall\, t \geq 0 : \quad \|R(t)\| \leq M e^{-\lambda t}$*, where* $y_{\mathrm{ref}}(\cdot)$ *is the resolvent of* (5.6)

(vii) *exponential asymptotic stability*

are related as follows

(i) ⇐ (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) ⇐ (vi) ⇐ (vii).

The proof of Proposition 5.3.2 is in Subsection 5.7.2 on page 230.

If the system (5.9) with $\eta(\cdot) \equiv 0$ is considered (in this case, (5.9) is equivalent to linear systems (1.4)), then the above stability concepts are equivalent (see Proposition 1.1.4). It has to be noted that this does not hold for Volterra-Stieltjes systems (5.9).

Corollary 5.3.3. ***(Stability of inhomogeneous systems)***
Let $(A, \eta(\cdot), f(\cdot)) \in \mathbb{R}^{n\times n} \times \mathcal{BV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$*. Suppose that* $\int_0^\infty |\,\mathrm{d}\eta(\theta)\,| < \infty$ *and the homogeneous part of*

$$\dot{x}(t) = Ax(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) + f(t) \tag{5.10}$$

is uniformly asymptotically stable. Then for any $\phi(\cdot) \in \mathcal{C}([0,\sigma],\mathbb{R}^n)$, $\sigma \geq 0$, *and* $f(\cdot) \in \mathcal{L}^{\infty}_{\text{loc}}(\mathbb{R}_{\geq 0},\mathbb{R}^n)$*, the solution* $x(\cdot;\sigma,\phi,f)$ *of the initial value problem* (5.10)*,* $x|_{[0,\sigma]} = \phi$ *satisfies:*

(i) If $f(\cdot) \in \mathcal{L}^p(\mathbb{R}_{\geq 0},\mathbb{R}^n)$ *for some* $p \in [1,\infty]$*, then* $x(\cdot;\sigma,\phi,f) \in \mathcal{L}^p(\mathbb{R}_{\geq 0},\mathbb{R}^n)$.

(ii) If $\lim_{t\to\infty} f(t) = 0$*, then* $\lim_{t\to\infty} x(t;\sigma,\phi,f) = 0$.

The proof of Corollary 5.3.3 is in Subsection 5.7.2 on page 231.

The following theorem characterizes uniform asymptotic stability and exponential asymptotic stability for positive systems (5.9) which presents explicit criteria.

Theorem 5.3.4. ***(Stability criteria for positive systems)***
If the system (5.9) *is positive and has finite total variation, i.e.* $\int_0^\infty |\,\mathrm{d}\eta(\theta)\,| < \infty$*, then*

(i) (5.9) *is uniformly asymptotically stable* $\iff$ $\mu\left(A + \int_0^\infty \mathrm{d}[\eta(\theta)]\right) < 0$.

(ii) (5.9) *is exponentially asymptotically stable*
$\iff \quad \mu\left(A + \int_0^\infty \mathrm{d}[\eta(\theta)]\right) < 0$ *and* $\left\|\int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{\alpha\theta}\right\| < \infty$ *for some* $\alpha > 0$.

The proof of Theorem 5.3.4 is in Subsection 5.7.2 on page 232.

It has to be noted that Theorem 5.3.4 shows that uniform asymptotic stability does not necessarily imply uniform exponential stability which is so in the non-Volterra case (see Proposition 1.1.4). The reason can only be the positivity assumption such that this implication fails. This is illustrated by an example in the following remark.

Remark 5.3.5. (Necessity of positivity)
In general, positivity of (5.9) cannot be omitted in Theorem 5.3.4. To see this, a non-positive system (5.9) is presented which is not uniformly asymptotically stable but satisfies $\mu\left(A + \int_0^\infty \mathrm{d}[\eta(\theta)]\right) < 0$. Consider

$$\dot{x}(t) = \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) \qquad \text{for a.a. } t \geq 0. \tag{5.11}$$

For $\theta \geq 0$, $\eta(\cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$ is defined by

$$\eta(\theta) := \int_0^\theta b(\tau)\,\mathrm{d}\tau\,, \quad b : \mathbb{R}_{\geq 0} \to \mathbb{R},\; b(\tau) = \begin{cases} \alpha e^{2\tau}, & \tau \in [0,1) \\ \beta e^{2\tau}, & \tau \in [1,2) \\ 0, & \tau \in [2,\infty), \end{cases}$$

where $\alpha := \frac{3e^2}{e^2-1}$ and $\beta := \frac{-4}{e^2-1}$. An easy computation yields that

$$\eta(0) = 0, \qquad \int_0^\infty |\,\mathrm{d}[\eta(\theta)]\,(\theta)| < \infty, \qquad \mu\left(\int_0^\infty \mathrm{d}[\eta(\theta)]\right) = \int_0^\infty \mathrm{d}[\eta(\theta)] = -\frac{e^2}{2} < 0,$$

and, since $\eta(\cdot)$ is not a non-decreasing function, (5.11) is not positive (see Theorem 5.2.8).

It remains to show that (5.11) is not uniformly asymptotically stable. For $s \in \mathbb{C}$ consider the characteristic equation of (5.11)

$$0 = s - \int_0^\infty e^{-s\theta}\,\mathrm{d}[\eta(\theta)] = s - \int_0^\infty e^{-s\theta} b(\theta)\,\mathrm{d}\theta\,.$$

Writing

$$g : \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad t \mapsto g(t) := t - \int_0^\infty e^{-t\theta} b(\theta)\,\mathrm{d}\theta\,,$$

it follows that

$$g(0) = -\int_0^2 b(\theta)\,\mathrm{d}\theta = \frac{e^2}{2} > 0, \quad \text{and} \quad g(2) = 2 - \int_0^2 e^{-2\theta} b(\theta)\,\mathrm{d}\theta = -1 + \frac{1}{e^2-1} < 0,$$

and hence there exists $t^* \in (0,2)$ such that $g(t^*) = 0$. Now Proposition 5.3.2 (iv) implies that (5.11) is not uniformly asymptotically stable. $\diamond$

Now the following remark shows that uniform asymptotic stability does not imply exponential asymptotic stability under the positivity assumption.

Remark 5.3.6. (Uniform asymptotic stability $\not\Rightarrow$ exponential asymptotic stability)
Consider the scalar Volterra-Stieltjes system

$$\dot{x}(t) = -x(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta), \quad \text{for } \eta : \mathbb{R}_{\geq 0} \to \mathbb{R},\ \eta(t) := \int_0^t \frac{1}{2(\tau+1)^2}\, \mathrm{d}\tau \quad (5.12)$$

which is, by Theorem 5.2.8, positive.

(i) First, it is shown that (5.12) is uniformly asymptotically stable.
Seeking a contradiction, assume that (5.12) is not uniformly asymptotically stable. By Proposition 5.3.2 it follows that

$$\exists\, s_0 \in \mathbb{C}_0 : \quad s_0 + 1 - \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s_0\theta} = 0.$$

Then

$$1 \leq |s_0+1| = \left| \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s_0\theta} \right| = \left| \int_0^\infty e^{-s_0\theta} \frac{1}{2(\theta+1)^2}\, \mathrm{d}\theta \right| \leq \int_0^\infty \frac{1}{2(\theta+1)^2}\, \mathrm{d}\theta = \frac{1}{2},$$

yields a contradiction and therefore (5.12) is uniformly asymptotically stable.

(ii) Secondly, it is shown that (5.12) is not exponentially asymptotically stable.
Since

$$\forall\, \gamma > 0\ \exists\, T_0 \geq 0\ \forall\, t \geq T_0 : \quad e^{\gamma t} \frac{1}{2(t+1)^2} > 1,$$

it follows that

$$\forall\, \gamma > 0 : \quad \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{\gamma\theta} = \int_0^\infty e^{\gamma\theta} \frac{1}{2(\theta+1)^2}\, \mathrm{d}\theta \geq \int_0^\infty e^{\gamma\theta} \frac{1}{2(\theta+1)^2}\, \mathrm{d}\theta = \infty.$$

Therefore, (5.12) is not exponentially asymptotically stable, by Theorem 5.3.4 (ii). ◇

5.4 Zero dynamics

Now the concept of (stable) zero dynamics of multi input, multi output Volterra-Stieltjes systems of the form

$$\left.\begin{array}{rcll} \dot{x}(t) &=& Ax(t) + \int\limits_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) + Bu(t), & \text{for a.a. } t \geq 0, \\ y(t) &=& Cx(t) & \end{array}\right\} \tag{5.13}$$

where $(A, B, C, \eta(\cdot)) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathcal{BV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, $u(\cdot) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, is introduced.

Definition 5.4.1. (Zero dynamics)
The *zero dynamics* of the Volterra-Stieltjes system (5.13) are defined as the real vector space of trajectories

$$\mathcal{ZD}(A, B, C, \eta(\cdot)) := \left\{ (x, u, y) \in \mathcal{AC}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \times \mathcal{AC}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \;\middle|\; \begin{array}{l} (x, u, y) \text{ solves} \\ (5.13) \text{ with } y \equiv 0 \end{array} \right\}.$$

The system (5.13) is said to have *stable zero dynamics* if, and only if, for any $(x(\cdot), u(\cdot), y(\cdot)) \in \mathcal{ZD}(A, B, C, \eta(\cdot))$ it holds that $\lim_{t\to\infty} x(t) = 0$ and $x(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. ◇

It has to be noted that the definition of stable zero dynamics coincides with that of Definition 1.1.6 if $\eta(\cdot) \equiv 0$.

Consider the system (5.13) with $\eta(\cdot) \equiv 0$. If $\det CB \neq 0$, then it is well known that the system (5.13) can be transformed into the Byrnes-Isidori form (1.13) (see Proposition 1.1.7). It has to be noted that $CB \neq 0$ implies that the system (5.13) has strict relative degree one (see (1.10)).
It is straightforward to generalize this transformation to Volterra-Stieltjes systems (5.13); the result is stated, the proof is omitted.

Proposition 5.4.2. ***(Byrnes-Isidori form)***
Consider the system (5.13) *which satisfies* $\det CB \neq 0$. *Set*

$$V \in \mathbb{R}^{n\times(n-m)} \textit{ such that } \ker C = \textit{im}\, V \quad \textit{and} \qquad N := (V^\top V)^{-1} V^\top [I_n - B(CB)^{-1}C] \in \mathbb{R}^{(n-m)\times n}. \quad (5.14)$$

Then $T := [B(CB)^{-1}, V]$ *has inverse* $T^{-1} = \begin{bmatrix} C \\ N \end{bmatrix}$ *and* $(y^\top, z^\top)^\top := T^{-1}x$ *converts the initial value problem* (5.13), $x|_{[0,\sigma]} = \phi$ *into, for a.a.* $t \geq \sigma$,

$$\left.\begin{aligned} \dot{y}(t) &= \widehat{A}_1 y(t) + \int_0^t \mathrm{d}[\widehat{\eta}_1(\theta)]\, y(t-\theta) + \widehat{A}_2 z(t) + \int_0^t \mathrm{d}[\widehat{\eta}_2(\theta)]\, z(t-\theta) + CBu(t) \\ \dot{z}(t) &= \widehat{A}_3 y(t) + \int_0^t \mathrm{d}[\widehat{\eta}_3(\theta)]\, y(t-\theta) + \widehat{A}_4 z(t) + \int_0^t \mathrm{d}[\widehat{\eta}_4(\theta)]\, z(t-\theta) \\ y|_{[0,\sigma]} &= C\phi(\cdot), \qquad z|_{[0,\sigma]} = N\phi(\cdot) \end{aligned}\right\} \quad (5.15)$$

where $\widehat{A}_1 = CAB(CB)^{-1} \in \mathbb{R}^{m\times m}$, $\widehat{A}_2 = CAV \in \mathbb{R}^{m\times(n-m)}$, $\widehat{A}_3 = NAB(CB)^{-1} \in \mathbb{R}^{(n-m)\times m}$, $\widehat{A}_4 = NAV \in \mathbb{R}^{(n-m)\times(n-m)}$ *and*

$$\begin{aligned} \widehat{\eta}_1(\cdot) &= C\eta(\cdot)B(CB)^{-1} \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times m}) \\ \widehat{\eta}_2(\cdot) &= C\eta(\cdot)V \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times(n-m)}) \\ \widehat{\eta}_3(\cdot) &= N\eta(\cdot)B(CB)^{-1} \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{(n-m)\times m}) \\ \widehat{\eta}_4(\cdot) &= N\eta(\cdot)V \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{(n-m)\times(n-m)}). \end{aligned}$$

In the latter sections the Byrnes-Isidori form for systems of the form (5.13) is used, where B and C have a special structure. The following remark characterizes the matrices V and N given by (5.14) and the transformation matrix T of Proposition 5.4.2.

Remark 5.4.3. (Special structure of B and C, Byrnes-Isidori form)
Suppose $(B, C) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ satisfy $\det CB \neq 0$ and the matrices have the structure

$$B = [B_1^\top, 0]^\top, \quad C = [C_1, 0], \quad \text{with } B_1, C_1 \in \mathbb{R}^{m\times m}.$$

Then $0 \neq \det CB = \det C_1 B_1$. It has to be noted that for a product of square matrices holds

$$C_1 B_1 \text{ invertible} \quad \Leftrightarrow \quad C_1,\ B_1 \text{ invertible}.$$

Thus B_1 and C_1 are invertible. Then V and N given by (5.14) are of the form

$$V = \begin{pmatrix} 0 \\ I_{n-m} \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0, & N_{n-m} \end{pmatrix}$$

and

$$T = \begin{pmatrix} B(CB)^{-1}, & V \end{pmatrix} = \begin{pmatrix} C_1^{-1} & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

◇

Consider a system (5.13) with $\eta(\cdot) \equiv 0$). Then the definition of stable zero dynamics coincides with that of minimum phase (see [36, Sec. 2.1]). The following proposition generalizes Proposition 1.1.9 to Volterra-Stieltjes systems and gives a characterization of stable zero dynamics.

Proposition 5.4.4. ***(Characterization of stable zero dynamics)***
Suppose (5.13) *satisfies* $\det CB \neq 0$. *Then the following statements are equivalent:*

(i) (5.13) *has stable zero dynamics.*

(ii)

$$\forall\, s \in \mathbb{C}_0 : \quad \det \begin{pmatrix} sI_n - A - \int\limits_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s\theta} & B \\ C & 0 \end{pmatrix} \neq 0 \tag{5.16}$$

(iii) The subsystem

$$\dot z(t) = NAV z(t) + \int\limits_0^t \mathrm{d}[N\eta V(\theta)]\, z(t-\theta), \qquad t \geq 0, \tag{5.17}$$

is uniformly asymptotically stable, where N and V are as in (5.14).

Moreover, if $(x(\cdot), u(\cdot), y(\cdot)) \in \mathcal{ZD}(A, B, C, \eta(\cdot))$*, then* $\lim_{t\to\infty} u(t) = 0$ *and* $u(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$.

The proof of Proposition 5.4.4 is in Subsection 5.7.3 on page 236.

What is with positivity of (5.13) under the transformation (5.15). The answer is affirmative.

Lemma 5.4.5. ***(Special input-output structure ensures positivity of zero dynamics)*** *Suppose the system* (5.13) *satisfies* $\det CB \neq 0$ *and the input-output matrices have the structure*

$$B = [B_1^\top, 0]^\top, \quad C = [C_1, 0], \quad \textit{with } B_1,\, C_1 \in \mathbb{R}^{m\times m}.$$

Then positivity of the homogeneous part of system (5.13) *yields positivity of the subsystem* (5.17).

The proof of Lemma 5.4.5 is in Subsection 5.7.3 on page 237.

Remark 5.4.6. (Necessity of special input-output structure)
The remark uses the notation of Proposition 5.4.2. It is shown that the property of A Metzler is not invariant under the coordinate transformation $(y, z^\top)^\top = T^{-1}x$, $T = [B(CB)^{-1}, V]$, without any special input-output structure.
Consider (5.13) with

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad C := B^\top, \qquad CB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then A is a Metzler matrix and CB is invertible. It is easy to check that

$$V := \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \; T := \begin{bmatrix} 0 & 1/2 & -1 & 0 \\ 0 & 1/2 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & -1 \end{bmatrix}, \; T^{-1}AT = \begin{bmatrix} 2 & 1/2 & -1 & -2 \\ 0 & -1/2 & -3 & 0 \\ 0 & -3/4 & -1/2 & 0 \\ -1/2 & 1/4 & -1/2 & 2 \end{bmatrix}.$$

Neither $T^{-1}AT$ nor $\widehat{A}_4 = \begin{bmatrix} -1/2 & 0 \\ -1/2 & 2 \end{bmatrix}$ is a Metzler matrix. Therefore, in general, Lemma 5.4.5 does not hold true without assuming the special structure of B and C.
◇

The following proposition gives an explicit criterion for stability of the zero dynamics if the system is positive and the input-output matrices have a special structure.

Proposition 5.4.7. ***(Zero dynamics of positive systems, special input-output structure)***
Suppose the system (5.13) *satisfies* $\det CB \neq 0$*, its homogeneous part is positive and the input-output matrices have the structure*

$$B = [B_1^\top, 0]^\top, \quad C = [C_1, 0], \quad \textit{with } B_1,\ C_1 \in \mathbb{R}^{m\times m}.$$

Let V *and* N *be as in* (5.14)*. Then the following statements are equivalent:*

(i) (5.13) *has stable zero dynamics.*

(ii) $\mu\left(NAV + \int_0^\infty \mathrm{d}[N\eta(\theta)V]\right) < 0.$

(iii) $\exists\, p \in \mathbb{R}^{n-m}_{>0} : \quad \left(NAV + \int_0^\infty \mathrm{d}[N\eta(\theta)V]\right) p \in \mathbb{R}^{n-m}_{<0}.$

The proof of Proposition 5.4.7 is in Subsection 5.7.3 on page 238.

The two essential assumptions in Proposition 5.4.7, namely positivity and the special structure of the input-output matrices, are discussed in the following.

Remark 5.4.8. (Necessity of positivity)
If (5.13) with $\eta(\cdot) \equiv 0$ is considered, then Proposition 5.4.7 holds without assuming positivity (see proposition 1.1.9) – the system (5.13) has stable zero dynamics if, and only if, $\sigma(\widehat{A}_4) \subset \mathbb{C}_-$, where the notation as in (5.15) is used.

If Volterra-Stieltjes systems (5.13) are considered, then the assumption of positivity in Proposition 5.4.7 cannot be omitted. To see this consider

$$\left.\begin{aligned} \dot{x}(t) &= \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) + bu(t) \\ y(t) &= cx(t), \qquad\qquad \text{for a.a. } t \geq 0 \end{aligned}\right\} \tag{5.18}$$

with $b^\top := c := [1,\ 0]$ and $\eta(\cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^2)$ defined by

$$\eta(\cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^2), \quad \eta(\theta) := \begin{bmatrix} 0 & 0 \\ 0 & \int_0^\theta \beta(\tau)\,\mathrm{d}\tau \end{bmatrix}$$

$$\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \beta(\tau) := \begin{cases} \alpha e^{2\tau}, & \tau \in [0,1) \\ \beta e^{2\tau}, & \tau \in [1,2) \\ 0, & \tau \in [2,\infty), \end{cases}$$

where $\alpha := \frac{3e^2}{e^2-1}$ and $\beta := \frac{-4}{e^2-1}$. An easy computation yields that (5.18) is equivalent to, by Proposition 5.4.2,

$$\dot{y}(t) = u(t), \quad \dot{z}(t) = \int_0^t \mathrm{d}[\eta_4(\theta)]\, z(t-\theta), \tag{5.19}$$

where $\eta_4(t) = \int_0^t \beta(\tau)\,\mathrm{d}\tau\,,\ t \geq 0$. Example 5.3.5 shows that (5.19) fulfills Proposition 5.4.7 (ii) but is not uniformly asymptotically stable. By Proposition 5.4.4, (5.18) does not have stable zero dynamics. $\diamond$

The following theorem is the main result of this section. It shows that high-gain output feedback stabilizes the system (5.13), provided it is positive, has stable zero dynamics and the input-output matrices are positive diagonal matrices; most importantly, the trajectory of the closed-loop system remains positive.

Theorem 5.4.9. ***(High-gain stabilizability)***
Suppose that the homogeneous part of the system (5.13) *is positive and the input-output matrices have the structure*

$$B = \begin{bmatrix} \text{diag}\,(b_1, ..., b_m) \\ 0_{(n-m)\times m} \end{bmatrix}, \quad C = \begin{bmatrix} \text{diag}\,(c_1, ..., c_m), 0_{m\times(n-m)} \end{bmatrix}$$

with $b_i,\ c_i > 0$ for $i = 1, \ldots, m$. Then the following statements are equivalent:

(i) (5.13) *has stable zero dynamics.*

(ii) There exists $k^* \geq 0$ such that, for all $k_1, \ldots, k_m \geq k^*$, the output feedback

$$u(t) = -\text{diag}\,(k_1, ..., k_m)\, y(t) \tag{5.20}$$

applied to (5.13) *yields, for a.a. $t \geq 0$, a uniform asymptotically stable closed-*

loop system

$$\dot{x}(t) = Ax(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) - B\, \mathrm{diag}\,(k_1, ..., k_m)\, C\, x(t). \tag{5.21}$$

If $\phi(\cdot) \geq 0$*, then the solution of the initial value problem* (5.21)*,* $x|_{[0,\sigma]} = \phi$*,* $\sigma \geq 0$*, is also non-negative, i.e.* $x(t;\sigma,\phi) \in \mathbb{R}^n_{\geq 0}$ *for all* $t \geq \sigma$*.*

The proof of Theorem 5.4.9 is in Subsection 5.7.3 on page 238.

5.5 Funnel control results

The concept of funnel control was introduced in Section 3.3 for the class of linear systems (1.1) with CB positive definite, i.e. for systems (1.1) with strict relative degree one. One may ask the question as to wether the funnel controller (3.19), which is designed for linear systems (1.1) with relative degree one, also works for Volterra-Stieltjes systems (1.2) with CB positive definite. The answer is affirmative in this section. Moreover, if the homogeneous part of (5.13) is positive together with non-negative initial values, then the state $x(\cdot;\sigma,\phi)$ and the output $y(\cdot)$ of (1.2) are non-negative. It will be shown that the same controller is allowed for input constraints in the context of Volterra-Stieltjes systems which is a generalization of the results of Chapter 3.
Furthermore, a small modification of the controller (3.19) ensures non-negativity of the input $u(\cdot)$ and fulfills all other funnel results. These results are a generalization of the results for linear multi input, multi output systems (1.1) to Volterra-Stieltjes systems (1.2).

5.5.1 Funnel control of positive MIMO-systems

In this section funnel control for Volterra-Stieltjes systems (1.2) with constant system matrix $A(\cdot) \equiv A \in \mathbb{R}^{n\times n}$, stable zero dynamics and "diagonal and positive" input and output matrices $B,\ C$ is shown, i.e. the trajectory of the closed-loop system remains bounded. Moreover, if the system (5.13) is positive and the initial values are non-negative, then the state $x(\cdot;\sigma,\phi)$ and the output $y(\cdot)$ are non-negative. This result is fundamental for various generalizations and aspects considered in the following subsections.

Theorem 5.5.1. ***(Funnel control of positive systems)***
Suppose a system (5.13) *has stable zero dynamics (i.e.* (5.16)*), its homogeneous part is positive (i.e.* (5.8)*) and the input-output matrices have the structure*

$$B = [\mathrm{diag}\,(b_1, ..., b_m),\ 0]^\top, \qquad C = [\mathrm{diag}\,(c_1, ..., c_m),\ 0] \tag{5.22}$$

with $b_i,\ c_i > 0$ *for* $i = 1, ..., m$*. Let* $\sigma \geq 0$ *and* $\varphi_i(\cdot) \in \mathcal{S}_1(\sigma)$*,* $i = 1, ..., m$*, and define the performance funnels* $\mathcal{F}(\sigma, \varphi_i)$ *given by* (3.14)*. If the initial function* $\phi(\cdot) \in \mathcal{C}([0, \sigma], \mathbb{R}^n)$ *and the reference signal* $y_{\mathrm{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *are such that*

$$\forall\, i \in \{1, ..., m\}: \quad \varphi_i(\sigma)|c_i\phi_i(\sigma) - y_{\mathrm{ref},i}(\sigma)| < 1. \tag{5.23}$$

Then the application of the funnel controller

$$\boxed{u(t) = -[k_1(t)e_1(t), ..., k_m(t)e_m(t)]^\top,\ k_i(t) = \frac{\varphi_i(t)}{1-\varphi_i(t)\,|e_i(t)|},\ e(t) = Cx(t) - r(t)} \tag{5.24}$$

to (5.13) *yields a closed-loop initial value problem with the following properties.*

(i) Precisely one maximal solution $x : [0, \omega) \to \mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega = \infty$*).*

(ii) The global solution $x(\cdot; \sigma, \phi)$ *is bounded and, for each* $i \in \{1, ..., m\}$*, the input functions* $u_i(\cdot)$ *are bounded and the tracking errors* $e_i(\cdot) = (Cx(\cdot) - y_{\mathrm{ref}}(\cdot))_i$ *evolve within the performance funnels* $\mathcal{F}(\sigma, \varphi_i)$*; more precisely,*

$$\exists\, \varepsilon_0 > 0\ \forall\, i \in \{1, ..., m\}\ \forall\, t \geq \sigma : \quad 1 - \varphi_i(t)|e_i(t)| \geq \varepsilon_0.$$

(iii) The gain functions $k_i(\cdot)$ *are bounded, with* $\|k_i\|_\infty \leq \frac{\|\varphi_i\|_\infty}{\varepsilon_0}$*.*

(iv) If $\phi(\cdot)$ *and* $y_{\mathrm{ref}}(\cdot)$ *are non-negative, i.e.* $\phi(\cdot) \in \mathcal{C}([0, \sigma], \mathbb{R}^n_{\geq 0})$ *and* $y_{\mathrm{ref}}(\cdot)) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m_{\geq 0})$*, then the signals* $x(\cdot; \sigma, \phi)$ *and* $y(\cdot)$ *are non-negative, i.e.*

$$\forall\, t \geq 0 : \quad x(t; \sigma, \phi) \geq 0 \qquad \textit{and} \qquad y(t) \geq 0.$$

The proof of Theorem 5.5.1 is in Subsection 5.7.4 on page 240. It has to be noted that the input $u(\cdot)$ is not necessarily non-negative (see Section 5.6, Figure 5.2).

As in Remark 3.4.8 for linear systems (1.1), the proposed controller (5.24) copes with input and output disturbances $d_u(\cdot)$ and $d_y(\cdot)$, provided that the disturbances belong to the same function class as the reference signal. With reference to (5.24), the control law gets the form

$$u(t) = -\left([k_1(t)e_1(t), ..., k_m(t)e_m(t)]^\top - d_u(t)\right), \qquad e(\cdot) = (y(\cdot) + d_y(\cdot)) - y_{\text{ref}}(\cdot),$$

where $d_u(\cdot),\ d_y(\cdot) \in \mathcal{L}^\infty([\sigma,\infty),\mathbb{R}^m)$. With minor modifications the steps of the proof of Theorem 5.5.1 go through. The details are omitted.

Remark 5.5.2. (Connection to time-varying Volterra-Stieltjes systems)
In Section 5.2 solution theory and positivity for time-varying Volterra-Stieltjes systems (5.3) is introduced. But Theorem 5.5.1 studies time-invariant systems (5.13). A careful injection of the closed-loop system (5.13), (5.24) gives the initial value problem

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta) + f(t), \qquad x|_{[0,\sigma]} = \phi$$

with

$$\mathcal{A}(\cdot) := A - B\,\mathrm{diag}\,(k_1(\cdot), ..., k_m(\cdot))\,C \quad \text{and}$$
$$f(\cdot) := B\,\mathrm{diag}\,(k_1(\cdot), ..., k_m(\cdot))\,y_{\text{ref}}(\cdot) \quad (5.25)$$

which is a time-varying Volterra-Stieltjes system (5.3). Thus, Proposition 5.2.7 shows Assertion (iv) of Theorem 5.5.1. ◇

Theorem 5.5.1 considers only symmetric funnels which is a rather hard assumption, this can be relaxed.

Remark 5.5.3. (Different funnel in each channel and measurement noise)
In many applications more information about the system are known such that non-symmetric funnels in the gains (5.24) make sense. As in Remark 3.4.9 for linear systems (1.1), the control strategy (5.24) can be relaxed by

$$\begin{aligned} u(t) &= -[k_1(t)e_1(t), ..., k_m(t)e_m(t)]^\top \\ k_i(t) &= \min\left\{\frac{\varphi_i^u(t)}{1-\varphi_i^u(t)e_i(t)}, \frac{\varphi_i^\ell(t)}{1+\varphi_i^\ell(t)e_i(t)}\right\} \end{aligned}$$

with $\varphi_i^\ell(\cdot),\ \varphi_i^u(\cdot) \in \mathcal{S}_1(\sigma)$, $i = 1, ..., m$, and the funnel is given by

$$\mathcal{F}(\tau,\varphi_i) := \left\{(t,\eta) \in [\tau,\infty)\times\mathbb{R} \mid -\varphi_i^u(t) < \varphi_i^\ell\varphi_i^u\eta < \varphi_i^\ell(t)\right\}, \qquad (5.26)$$

and determined by $\varphi_i(\cdot) = (\varphi_i^\ell(\cdot), \varphi_i^u(\cdot))$. It has to be noted that, for $(t, \eta) \in \mathcal{F}(\sigma, \varphi)$, $t > 0$ and $i \in \{1, ..., m\}$,

$$-\varphi_i^u(t) < \eta\varphi_i^\ell(t)\varphi_i^u(t) < \varphi_i^\ell(t) \quad \Leftrightarrow \quad -\left(\varphi_i^\ell(t)\right)^{-1} < \eta < (\varphi_i^u(t))^{-1}.$$

For simplicity and since the proof of Theorem 5.5.1 becomes unnecessary technically, Theorem 5.5.1 considers only symmetric funnels. ◇

The following technical lemma will be used to prove Theorem 5.5.1. The lemma shows that the system (5.13) can be described by functional differential equations as long as

(i) the system satisfies $\det CB$ ∅ and

(ii) the input-output matrices have a block structure.

Lemma 5.5.4. ***(Equivalent representation as a functional differential equation)*** *If the system* (5.13) *satisfies* $\det CB \neq 0$ *and the input-output matrices have the structure*

$$B = [B_1^\top, 0]^\top, \quad C = [C_1, 0], \quad \textit{with } B_1,\, C_1 \in \mathbb{R}^{m\times m}, \tag{5.27}$$

then, using the notation as in Proposition 5.4.2, it follows:

(i) (5.13) *is equivalent to the initial value problem*

$$\dot{y}(t) = p(t) + (Ty)(t) + C_1B_1u(t), \qquad y|_{[0,\sigma]} = C\phi(\cdot), \textit{ for a.a. } t \geq \sigma, \tag{5.28}$$

where

$$\begin{aligned}
&T : \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)\\
&y(\cdot) \mapsto (Ty)(\cdot) :=\\
&(CAV,\, C\eta V) * \left(\int_0^{\cdot-\sigma} R_4(\cdot - \sigma - s)[(NAB(CB)^{-1}, N\eta B(CB)^{-1}) * y](s+\sigma)\,\mathrm{d}s \right.\\
&\qquad\qquad \left. +[(CAB(CB)^{-1},\, C\eta B(CB)^{-1}) * y](\cdot) \right)
\end{aligned}$$

and

$$p(\cdot) := (CAV,\, C\eta V) * \Big(R_4(\cdot - \sigma)N\phi(\sigma)$$

$$+\int_0^{\cdot-\sigma} R_4(\cdot-\sigma-s)\left[\int_s^{s+\sigma} \mathrm{d}[N\eta(\theta)V]\, N\phi(s+\sigma-\theta)\right] \mathrm{d}s\Bigg)$$

and $R_4(\cdot)$ *denotes the resolvent of* (5.17).

(ii) *Write* $R_4(\cdot)$ *for the resolvent of* (5.17). *If* (5.13) *has stable zero dynamics and its homogeneous part is positive, then the following inequalities hold*

$$\forall\, t\geq\sigma: \quad \|p(t)\|\leq\widehat{p}\|N\phi\|_\infty \quad \textit{and} \quad \|(Ty)(t)\|_{\mathrm{Op}}\leq\widehat{T}\max_{\tau\in[0,t]}\|y(\tau)\|$$

with

$$\left.\begin{aligned}\widehat{p} &:= \Big[\|CAV\|+\int_0^\infty|\,\mathrm{d}[C\eta V(\theta)]\,|\Big]\ \Big[\|R_4\|_{\mathcal{L}^\infty}+\|R_4\|_{\mathcal{L}^1}\int_0^\infty|\,\mathrm{d}[N\eta V(\theta)]\,|\Big]\\ \widehat{T} &:= \|CAB(CB)^{-1}\|+\int_0^\infty|\,\mathrm{d}[C\eta B(CB)^{-1}(\theta)]\,|\,\|R_4\|_{\mathcal{L}^1}\Big(\|NAB(CB)^{-1}\|\\ &\qquad\times+\int_0^\infty|\,\mathrm{d}[N\eta B(CB)^{-1}(\theta)]\,|\Big)\ \Big(\|CAV\|+\int_0^\infty|\,\mathrm{d}[C\eta V(\theta)]\,|\Big).\end{aligned}\right\} \quad (5.29)$$

The proof of Lemma 5.5.4 is in Subsection 5.7.4 on page 240. The inequalities of Assertion (ii) of the lemma are essential for the proof of Theorem 5.5.1 (see Step 2, 3, 6 of the proof of Theorem 5.5.1) and for the input constrained result in the following section.

5.5.2 Input saturation of positive systems

In many practical applications, the input may be subjected to certain bounds, i.e. there is some maximal bound $\widehat{u}\in\mathbb{R}^m_{>0}$ such that $|u_i(t)|\leq\widehat{u}_i$ is required for all $t\geq\sigma$ and $i=1,...,m$. Now the same class of systems as in Section 5.5.1 is considered and the following theorem shows that funnel control (5.24) is also feasible in the presence of input constraints provided the saturation is larger than a certain feasibility number. The following theorem generalizes the componentwise saturation result (see Theorem 3.4.6) to positive Volterra-Stieltjes systems (5.13).

Theorem 5.5.5. ***(Input constrained funnel control of positive systems)***
Suppose a system (5.13) *has stable zero dynamics (i.e.* (5.16)*), its homogeneous part is positive (i.e.* (5.8)*), the input-output matrices have the structure* (5.27) *and* $C\,B$ *is*

such that (3.34) *holds, i.e.*

$$\det(C\,B) \neq 0, \quad (C\,B)_{ii} > 0,\ i \in \{1,...,m\}, \quad (C\,B)_{ij} \leq 0,\ i,j \in \{1,...,m\},\ i \neq j.$$

Let $\sigma \geq 0$ *and for* $\varphi_i(\cdot) \in \mathcal{G}_1(\sigma)$, $i = 1,...,m$, *adopt the notations of Remark 3.3.1 with* $\varepsilon = 0$, *Remark 3.4.1 and define the performance funnels* $\mathcal{F}(\sigma, \varphi_i)$, $i = 1,...,m$, *given by* (3.14). *If the initial function* $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$ *and the reference signal* $y_{\mathrm{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *are such that*

$$\forall\, i \in \{1,...,m\}: \quad \varphi_i(\sigma)\,|[C\phi(\sigma) - y_{\mathrm{ref}}(\sigma)]_i| < 1 \tag{5.30}$$

and the saturation constraint $\widehat{u} \in \mathbb{R}^m_{>0}$ *such that the feasibility assumption*

$$\forall\, i \in \{1,...,m\}: \quad (CB\widehat{u})_i > \widehat{p}\|N\phi\|_\infty + \widehat{T}[\|\psi\|_\infty + \|y_{\mathrm{ref}}\|_\infty] + \|\dot{y}_{\mathrm{ref}}\|_\infty + \ell_i, \tag{5.31}$$

where $\psi(\cdot) = (\psi_1(\cdot),...,\psi_m(\cdot)) = \left(\frac{1}{\varphi_1(\cdot)},...,\frac{1}{\varphi_m(\cdot)}\right)$, *holds (wherein* $\widehat{p}$ *and* $\widehat{T}$ *are defined as in* (5.29)*), then, for any input disturbance* $d_u(\cdot) \in \mathcal{L}^\infty([\sigma,\infty), \mathbb{R}^m)$, *the application of the feedback strategy*

$$\boxed{\begin{aligned} & u(t) = [u_1(t),...,u_m(t)]^\top \\ & u_i(t) = -\mathit{sat}_{\widehat{u}_i}(k_i(t)e_i(t) - d_{u,i}(t)),\ k_i(t) = \tfrac{1}{\psi_i(t) - |e_i(t)|},\ e_i(t) = y_i(t) - r_i(t) \end{aligned}} \tag{5.32}$$

to (5.13) *yields a closed-loop initial value problem with the following properties.*

(i) There exists precisely one maximal solution $x : [0,\omega) \to \mathbb{R}^n$ *and this solution is global, i.e.* $\omega = \infty$.

(ii) The global solution $x(\cdot;\sigma,\phi)$ *is bounded and, for each* $i \in \{1,...,m\}$, *the tracking errors* $e_i(\cdot) = (Cx(\cdot) - y_{\mathrm{ref}}(\cdot))_i$ *evolve within the performance funnels* $\mathcal{F}(\sigma,\varphi_i)$*; more precisely,*

$$\forall\, i \in \{1,...,m\}\ \forall\, t \geq \sigma: \quad \psi_i(t) - |e_i(t)| \geq \varepsilon_0,$$

where

$$\varepsilon_0 := \frac{1}{2}\min\left\{ \min_{j\in\{1,...,m\}} \lambda_j,\ \min_{j\in\{1,...,m\}} \frac{\lambda_j}{\widehat{u}_j + \|d_u\|_\infty}, \right. \\ \left. 2\min_{j\in\{1,...,m\}}\{\varphi_j(\sigma)^{-1} - |e_j(\sigma)|\} \right\}. \tag{5.33}$$

(iii) The gain functions $k_i(\cdot)$ *are bounded, with* $\|k_i\|_\infty \le \frac{1}{\varepsilon_0}$*,* $i = 1, ..., m$*.*

(iv) Each input $u_i(\cdot)$ *is unsaturated at some time* $\tau_i \ge \sigma$*, i.e.*

$$\forall\, i \in \{1, ..., m\}\ \exists\, \tau_i \ge \sigma :\ |u_i(\tau_i)| < \widehat{u}_i.$$

(v) Let $d_u(\cdot)$ *be zero, i.e.* $d_u(\cdot) \equiv 0$*. If an input* $u_i(\cdot)$ *is unsaturated at some time* $\tau_i \ge \sigma$*, then it remains unsaturated thereafter, i.e.*

$$\forall\, i \in \{1, ..., m\} :\ [\exists\, \tau_i \ge \sigma :\ |u_i(\tau_i)| < \widehat{u}_i]\ \Rightarrow\ [\forall\, t > \tau_i :\ |u_i(t)| < \widehat{u}_i].$$

An input $u_i(\cdot)$ *is globally unsaturated (i.e.,* $|u_i(t)| < \widehat{u}_i$ *for all* $t \ge \sigma$ *and* $i \in \{1, ..., m\}$*) if, and only if,*

$$|(C\phi(\sigma))_i - y_{\mathrm{ref},i}(\sigma)| < \frac{\widehat{u}_i\, \varphi_i(\sigma)^{-1}}{1 + \widehat{u}_i}.$$

(In this case, the input takes the form $u_i(t) = -k_i(t)e_i(t)$*.)*

(vi) If the reference signal $y_{\mathrm{ref}}(\cdot)$ *and the functions* $\varphi_i(\cdot)$*,* $i = 1, ..., m$*, satisfy*

$$\forall\, i \in \{1, ..., m\}\ \forall\, t \ge \sigma :\quad y_{\mathrm{ref},i}(t) - \varphi(t)^{-1} \ge 0 \tag{5.34}$$

and $C\phi(\cdot)$ *is non-negative, i.e.* $C\phi(\cdot) \in \mathcal{C}([0, \sigma], \mathbb{R}^m_{\ge 0})$*, then the output is non-negative, i.e.*

$$\forall\, t \ge 0 :\quad y(t) \ge 0.$$

Moreover, if $N\phi(\cdot)$ *is non-negative, i.e.* $N\phi(\cdot) \in \mathcal{C}([0, \sigma], \mathbb{R}^{n-m}_{\ge 0})$*, and* C *satisfies* (5.22)*, then the signal* $x(\cdot; \sigma, \phi)$ *is non-negative, i.e.*

$$\forall\, t \ge 0 :\quad x(t; \sigma, \phi) \ge 0.$$

(vii) If the input-output matrices satisfy (5.22)*, then* $(\phi(\cdot),\ y_{\mathrm{ref}}(\cdot))$ *are non-negative, i.e.* $(\phi(\cdot), y_{\mathrm{ref}}(\cdot)) \in \mathcal{C}([0, \sigma], \mathbb{R}^n_{\ge 0}) \times \mathcal{W}^{1,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m_{\ge 0})$*, and the input disturbance* $d_u(\cdot)$ *is non-negative, i.e.* $d_u(\cdot) \in \mathcal{L}^\infty([\sigma, \infty), \mathbb{R}^m_{\ge 0})$*, then the signals* $x(\cdot; \sigma, \phi)$ *and* $y(\cdot)$ *are non-negative, i.e.*

$$\forall\, t \ge 0 :\quad x(t; \sigma, \phi) \ge 0 \qquad \textit{and} \qquad y(t) \ge 0.$$

The proof of Theorem 5.5.5 is in Subsection 5.7.4 on page 246.

To give an insight into the above theorem the assumptions are commented and the feasibility relationship (5.31) is interpreted.

Remark 5.5.6. (Comments on Theorem 5.5.5)

(i) As in Chapter 3, only finite funnels are allowed (for details see Remark 3.4.4). As in Remark 5.5.3, different funnels in each channel are possible and the gain functions can be captured by

$$k_i(t) = \frac{1}{\min\{\psi_i^u(t) - e_i(t), \psi_i^l(t) + e_i(t)\}}$$

with $\left(\psi_i^\ell(\cdot), \psi_i^u(\cdot)\right) := \left(\frac{1}{\varphi_i^\ell(\cdot)}, \frac{1}{\varphi_i^u(\cdot)}\right)$ and $\varphi_i^\ell(\cdot),\ \varphi_i^u(\cdot) \in \mathcal{G}_1(\sigma)$.

(ii) Since $\varphi_i(\cdot) \in \mathcal{G}_1(\sigma)$, $i = 1, ..., m$, the condition (5.30) which is equivalent to $|[C\phi(\sigma) - y_{\text{ref}}(\sigma)]_i| < (\varphi_i(\sigma))^{-1}$ is a necessary condition for the control objective and is equivalent to the requirement that $(\sigma, C\phi(\sigma) - y_{\text{ref}}(\sigma)) \in \times ... \times \mathcal{F}(\sigma, \varphi_m)$.

(iii) The feasibility condition (5.31) is a sufficient condition for attainment of the control objective. It quantifies a saturation bound (sufficiently large to ensure performance) in terms of plant data, funnel data, initial data and reference signal data. It has to be noted that the right-hand side of (5.31) depends on $\|\psi\|_\infty$ and so, in view of Remark 3.3.1, the assumption $\varphi_i(\sigma) \neq 0$, $i = 1, ..., m$, is important. The nature of the dependence of the saturation bound on these data is not surprising:

   (a) The uniform asymptotic stable and positive conditions give the inequalities of Lemma 5.5.4 (ii). The parameter $\widehat{p}$ and $\widehat{T}$ of (5.29) quantify the right-hand side of (5.28) and is related to the saturation bound.

   (b) It is to be expected that tracking of large and rapidly varying reference signals $y_{\text{ref}}(\cdot)$ would require control inputs capable of taking sufficiently large values, this is reflected in the dependence of the saturation bound on both $\|r\|_\infty$ and $\|\dot{r}\|_\infty$.

   (c) Transient and asymptotic behavior of the tracking error is influenced by the choice of the funnels $\mathcal{F}(\sigma, \varphi_i)$ determined by the globally Lipschitz functions $\psi_i(\cdot) := \frac{1}{\varphi_i(\cdot)}$, $i = 1, ..., m$. The rapid decay of the transient behavior would be reflected in large Lipschitz constants ℓ_i, appears as an additive term in the saturation bound.

(d) The input disturbance $d_u(\cdot)$ does not have influence on the saturation bound.

(iv) The assumption $\varphi_i(\sigma) \neq 0$ implies that

$$\forall\, t \geq \sigma : \quad k_i(t) = \frac{\varphi_i(t)}{1 - \varphi_i(t)|e_i(t)|} = \frac{1}{\psi_i(t) - |e_i(t)|}$$

with $\psi_i(\cdot) := \frac{1}{\varphi_i(\cdot)}$. It has to be noted that Remark 3.3.1 holds for $\varepsilon = 0$. The second statement of Assertion (ii) of Theorem 5.5.5 implies Assertion (iii).

(v) The high-frequency gain matrix CB satisfies (3.34) and it is required a type of "diagonal dominance" condition (5.31), viz.

$$\sum_{j=1}^{m} (CB)_{ij} \widehat{u}_j - (L + \ell_i) > 0, \qquad i \in \{1, ..., m\},$$

with $L := \widehat{p}\|N\phi\|_\infty + \widehat{T}\,[\|\psi\|_\infty + \|r\|_\infty] + \|\dot{r}\|_\infty \geq 0$. Now consider the discussion of Remark 3.4.7 (d). The feasibility bound depends on all parameters involved in the closed-loop system. In most cases the calculated feasibility condition may be very conservative. If the entries of (5.13) are known, it may be possible to determine a sharper feasibility number.

(vi) Statement (vii) shows that the state $x(\cdot; \sigma, \phi)$ and the output $y(\cdot)$ are non-negative if the input-output matrices satisfy (5.22) and $(\phi(\sigma), y_{\text{ref}}(\cdot))$ are non-negative. This statement corresponds with Assertion (iv) of Theorem 5.5.1.

(vii) Now consider Assertion (vi) of Theorem 5.5.5.

(a) It has to be noted that $\varphi_i(\cdot) > 0$, $i = 1, ..., m$. Hence (5.34) holds if, and only if,

$$y_{\text{ref}}(\cdot) > 0 \qquad \text{and} \qquad \varphi_i(\cdot) \text{ "small enough"}.$$

Then the output $y(\cdot)$ is non-negative if $[C_1,\ 0]\phi(\cdot)$ is non-negative. In contrast to (5.22) the input-output matrices B, C only need a block structure without a positive diagonal form.

(b) If C has the form of (5.22) and $\phi(\cdot)$ satisfies the condition $N\phi(\cdot) \geq 0$, then the state $x(\cdot; \sigma, \phi)$ is non-negative, too.

It has to be noted that it is not assumed that the input matrix B satisfies (5.22). Condition (5.22) can be relaxed. $\diamond$

Remark 5.5.7. (Comparison of Theorem 5.5.1 and 5.5.5)

(i) The block diagonal assumption (5.22) of Theorem 5.5.1 can be relaxed in Theorem 5.5.5. The diagonal condition (5.22) in Theorem 5.5.1 can be weakened to a block shape

$$B = [B_1^\top, 0]^\top, \quad C = [C_1, 0], \quad B_1,\, C_1 \in \mathbb{R}^{m\times m}$$

such that (3.34) and (5.31) hold.

(ii) A careful inspection of Step 2 of the proof of Theorem 5.5.5 reveals that the essential inequality is (for simplicity let $d_u(\cdot) \equiv 0$ and assume that $e_i(\cdot)$ is non-negative)

$$\dot e_i(t) \;\le\; \widehat{p}\,\|z^0\|_\infty + \widehat{T}\,[\|\psi\|_\infty + \|y_{\mathrm{ref}}\|_\infty] + \|\dot y_{\mathrm{ref}}\|_\infty - \sum_{j=1}^{m} \mathrm{sat}_{\widehat{\mathrm{u}}}\,\widehat{u}_j(k_j(t)e_j(t)).$$

which gets, without saturation, the form

$$\dot e_i(t) \le \widehat{p}\,\|z^0\|_\infty + \widehat{T}\,[\|\psi\|_\infty + \|y_{\mathrm{ref}}\|_\infty] + \|\dot y_{\mathrm{ref}}\|_\infty - (C_1B_1)_{ii}\,k_i(t)|e_i(t)| + \sum_{j=1,\,j\neq i}^{m} |(C_1B_1)_{ij}|\,k_j(t)\,|e_j(t)|.$$

The essential property of the proof of Theorem 5.5.1 (see Step 2) is that the above inequality becomes negative under the assumption $\limsup_{t\to\omega} k_i(t) = \infty$. If B_1 and C_1 are not diagonal matrices with positive diagonal elements, then the last sum cannot be canceled by $-(C_1B_1)_{ii}k_i(t)e_i(t)$. This shows that the diagonal structure (5.22) is important for Theorem 5.5.1. ◇

5.5.3 Funnel control with non-negative input

In real life systems, non-negativity of states occurs quite often. Control inputs of drug systems for physiological systems are constrained to be non-negative (see [17, 20, 25]).
In [25] a control law is designed such that the input function $u(\cdot)$ is non-negative on $\mathbb{R}_{\ge 0}$. [25] restricts the class of systems (5.13) to the linear multi input, multi output case (1.1), i.e. $\eta(\cdot) \equiv 0$. The input uses an adaptive control law as in Chapter 2 with an internal model with the drawbacks as in Chapter 2. The feedback is no longer

simple. The following theorem presents a simple control law which uses the funnel controller with a slight modification. This controller overcomes all the drawbacks which are presented in Chapter 2. Then the input $u(\cdot)$ is non-negative and the simulation results are better as in [25].

For the purposes of the proof of Theorem 5.5.9 the following definitions are summarized for later reference (see [55]) and the concept of an ω-limit point (see [53, p. 112]) is used.

Definition 5.5.8.

(i) A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to be *meagre* if $f(\cdot)$ is Lebesgue measurable and

$$\forall\, s > 0 : \quad \lambda(\{t \in \mathbb{R}_{\geq 0} \mid |f(t)| \geq s\}) < \infty.$$

(ii) A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to be *weakly meagre* if $\lim\limits_{n\to\infty} \left(\inf\limits_{t \in I_n} |f(t)| \right) = 0$ for every family $\{I_n \mid n \in \mathbb{N}\}$ of nonempty and pairwise disjoint closed intervals $I_n \subset \mathbb{R}_{\geq 0}$ with $\inf\limits_{n \in \mathbb{N}} \lambda(I_n) > 0$.

(iii) A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *uniformly locally integrable* if $f(\cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and

$$\forall\, \varepsilon > 0\ \exists\, \tau > 0\ \forall\, t \in \mathbb{R}_{\geq 0} : \quad \int_t^{t+\tau} |f(s)|\,\mathrm{d}s \leq \varepsilon.$$

(iv) For a nonempty subset $A \subset \mathbb{R}^n$, $\mathcal{F}(A)$ denotes the class of Carathéodory functions $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ with the property that there exists a uniformly locally integrable function $m(\cdot)$ such that

$$\forall\, (t, \xi) \in \mathbb{R}_{\geq 0} \times A : \quad \|f(t, \xi)\| \leq m(t).$$

(v) Let $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. A point $\xi \in \mathbb{R}^n$ is an *ω-limit point* of $x(\cdot)$ if there exists an unbounded sequence (t_n) in $\mathbb{R}_{\geq 0}$ such that $\lim\limits_{n\to\infty} x(t_n) = \xi$. The (possibly empty) *$\omega$-limit set* of $x(\cdot)$, denoted by $\Omega(x)$, is the set of all ω-limit points of $x(\cdot)$.

Theorem 5.5.9. ***(Funnel control with non-negative input of positive systems)*** *Suppose a system* (5.13) *with* $\eta(\cdot) \equiv 0$ *has positive homogeneous part (i.e.* (5.8)*) and the input-output matrices satisfy* (5.27)*. Furthermore, assume that* $A \in \mathbb{R}^{n\times n}$ *is a Hurwitz matrix. Let* $\sigma \geq 0$ *and* $\varphi_i(\cdot) \in \mathcal{S}_1(\sigma)$*,* $i = 1, ..., m$*. If the initial function* $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$ *and the reference signal* $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *are such that*

$$\forall\, i \in \{1, ..., m\} : \quad 1 + \varphi_i(\sigma)(c_i\phi_i(\sigma) - y_{\text{ref},i}(\sigma)) > 0 \tag{5.35}$$

holds, then the application of the feedback strategy

$$\boxed{\begin{aligned} u(t) &= -[k_1(t)e_1(t), ..., k_m(t)e_m(t)]^\top, \qquad e(t) = y(t) - y_{\text{ref}}(t) \\ k_i(t) &= \begin{cases} 0 & , \text{ if } e_i(t) > 0 \\ \frac{\varphi_i(t)}{1+\varphi_i(t)e_i(t)} & , \text{ if } e_i(t) \leq 0, \end{cases} \end{aligned}} \tag{5.36}$$

to (5.13) *yields a closed-loop initial value problem with the following properties.*

(i) There exists precisely one maximal solution $x : [0, \omega) \to \mathbb{R}^n$ *and this solution is global, i.e.* $\omega = \infty$*.*

(ii) The global solution $x(\cdot; \sigma, \phi)$ *is bounded and, for each* $i \in \{1, ..., m\}$*, the input functions* $u_i(\cdot)$ *are bounded and non-negative, i.e.* $u_i(\cdot) \geq 0$*, and the tracking errors* $e_i(\cdot) = c_i x_i(\cdot) - y_{\text{ref},i}(\cdot)$ *are bounded away from the lower boundary; more precisely,*

$$\exists\, \varepsilon_0 > 0\ \forall\, i \in \{1, ..., m\}\ \forall\, t \geq \sigma : \quad 1 + \varphi_i(t)e_i(t) \geq \varepsilon_0.$$

(iii) The gain functions $k_i(\cdot)$ *are bounded, with* $\|k_i\|_\infty \leq \frac{\|\varphi_i\|_\infty}{\varepsilon_0}$*.*

(iv) If $\phi(\cdot)$ *and* $y_{\text{ref}}(\cdot)$ *are non-negative, i.e.* $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$ *and* $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m_{\geq 0})$*, then the signals* $x(\cdot; \sigma, \phi)$ *and* $y(\cdot)$ *are non-negative, i.e.*

$$\forall\, t \geq 0 : \quad x(t; \sigma, \phi) \geq 0 \qquad \text{and} \qquad y(t) \geq 0.$$

(v) Let

$$M := \left[-\max_{t\geq\sigma} \varphi_1(t)^{-1},\, 0\right] \times ... \times \left[-\max_{t\geq\sigma} \varphi_m(t)^{-1},\, 0\right] \subset \mathbb{R}^m,$$

where the convention $\frac{1}{0} := \infty$ *is used. If*

$$\forall\, i \in \{1, ..., m\} : \quad \lambda(\{t \geq \sigma \mid e_i(t) > 0\}) < \infty, \tag{5.37}$$

where $\lambda(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}_{\geq 0}$ (see [82, Def. 4.8]), then $e(\cdot)$ approaches M, i.e.

$$\lim_{t\to\infty} d_M(e(t)) = 0.$$

(vi) Let $m = 1$ and $y_{\text{ref}}(\cdot) > 0$ on $[\sigma, \infty)$. If $\liminf_{t\to\infty} y_{\text{ref}}(t) > 0$, then it follows that

$$[\exists\, \tau \geq \sigma : \ e(\tau) \geq 0] \quad \Rightarrow \quad [\exists\, t > \tau : \ e(t) < 0].$$

The proof of Theorem 5.5.9 is in Subsection 5.7.4 on page 252.

It has to be noted that the controller in [25] uses monotonically gain functions $k_i(\cdot)$, which are dynamically generated. The advantages of the controller (5.36) are the non-monotone gains $k_i(\cdot)$ and transient behavior of the tracking errors $e_i(\cdot)$ is guaranteed, if the error $e_i(\cdot)$ is not positive.

Remark 5.5.10. (Comments on Theorem 5.5.9)
Let $\eta(\cdot) \equiv 0$ in (5.13). It has to be noted that, in view of the initial values, (5.13) with $\eta(\cdot) \equiv 0$ is not directly equivalent to a system (1.1) if $\sigma > 0$. In view of Remark 5.2.3, a system (5.13) with $\eta(\cdot) \equiv 0$ can be converted into an equivalent initial value problem which has the form (1.1).

(i) The assumption that the homogeneous part of (5.13) is uniformly asymptotically stable is a natural restriction in clinical applications and anesthesia. If the anesthetic is injected and the further input is stopped, then the intuition is that the level of anesthetic into the body decays.

(ii) To track a reference signal, together with (i), the controller (5.36) only needs a lower funnel boundary to guarantee that the input $u(\cdot)$ is non-negative. Loosely speaking, the upper funnel boundary can be interpreted as "∞".

(iii) Since (5.13) has positive homogeneous part and the system matrix A is Hurwitz, (5.13) has stable zero dynamics (see Proposition 5.4.7 and Remark 5.4.8).

(iv) The assumption (5.37) is important for the statement (v). But this condition cannot be checked without the knowledge of the error signal $e(\cdot)$. Thus statement (v) is a nice theoretical result but not practicable for applications. Statement (vi) is a first step to relax this hard assumption.

(v) A careful inspection of Step 6 of the proof of Theorem 5.5.9 shows that the assumptions $m = 1$, $y_{\text{ref}}(\cdot) > 0$ and $\liminf_{t\to\infty} r(t) > 0$ are important for Assertion (vi).

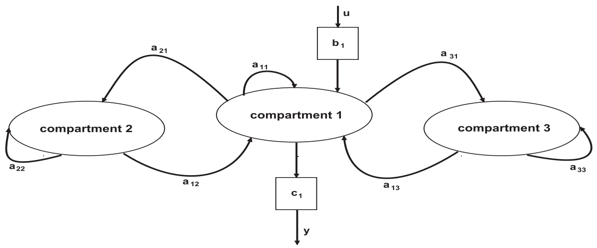


Figure 5.1: Three compartment model for disposition of anesthesia

(vi) The controller (5.36) can be formulated with saturation and a feasibility condition.

5.6 General anesthesia

The potential clinical applications of adaptive control for anesthesia are well known in the literature (see [20]) and adaptive feedback controllers have been suggested (see [25]) to control the concentration of anesthetic. [25] restricts the class of systems (5.13) to the linear multi input, multi output case (1.1), i.e. $\eta(\cdot) \equiv 0$, and applies a high-gain controller with monotonically non-decreasing gain. This has the drawback that the final gain is much too large, hence amplifying noise in the output measurement; also it does not cope with nonlinearities, noise and input saturations.
Furthermore, in [75], the authors assume that the system matrix A is Hurwitz, the tracking signal is non-negative and its dynamic can be generated by a dynamical system.
The funnel controller overcomes these drawbacks and has a non-monotone gain, which contrasts with typical high-gain adaptive control schemes.

In this section the feedback strategies of Theorem 5.5.1, Theorem 5.5.5 and Theorem 5.5.9 are applied to the control of the depth of anesthesia. Propofol is an intravenous anesthetic that has been used for both induction and maintenance of general anesthesia (see [17]). A simple patient model for the disposition of propofol is based on the three compartment mammillary model shown in Figure 5.1 with compartment 1 acting as the central compartment and the remaining two compartments exchanging with the central compartment (see [20]). The three compartment mammillary system provides a model for a patient describing the distribution of propofol into the central compartment and the other various groups of the body. The constant $a_{ii} \leq 0$, in min^{-1}, presents the elimination rate from the i-th compartment, $i = 1, 2, 3$ resp., while the other constants $a_{ij} \geq 0$, $i \neq j$, characterize drug transfer between the i-th and j-th compartment. The non-negative rate constants are uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. A mass balance for the whole compartmental system with the parameters, as presented in [25], yields a single input single output system

$$\left.\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.399 & 0.092 & 0.0048 \\ 0.207 & -0.092 & 0 \\ 0.04 & 0 & -0.0048 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \; x(0) = \begin{pmatrix} 0.045 \\ 0 \\ 0 \end{pmatrix} \\ y(t) &= \begin{bmatrix} 1, & 0, & 0 \end{bmatrix} x(t), \end{aligned}\right\} \tag{5.38}$$

where

$$\begin{aligned} x_i(t) &\;\widehat{=}\; \text{mass in } grams \text{ of anesthetic in compartment } i \text{ at time } t,\ i = 1, 2, 3 \\ u(t) &\;\widehat{=}\; \text{infusion rate in } grams/min \text{ of anesthetic into compartment 1 at time } t \\ y(t) &\;\widehat{=}\; \text{concentration in } grams \text{ of anesthetic in compartment 1 at time } t. \end{aligned}$$

It has to be noted that $y(\cdot) = x_1(\cdot)$ and thus, system (5.38) is in Byrnes-Isidori form (5.15): This example which is a nice practical application is a positive linear systems and thus, the approach of positive linear Volterra-Stieltjes systems is not even utilized.

As noted in [54], the volume in $liters$ of compartment 1 can be approximately calculated by

$$\left(0.159\,\frac{l}{kg}\right) \times \text{(weight in } kg \text{ of the patient)}.$$

The assumption made in [25] is that a $70\,kg$ patient should be treated with propofol concentration levels of $4\,\mu g/ml$; this leads to the desired tracking value

$$4\frac{\mu g}{ml}\,0.159\frac{l}{kg}\,70kg = 4\;0.159\;70\;\frac{1000}{1000000}\frac{g}{l}\frac{l}{kg}kg \;=\; 0.04452\,g \;=\; 44.52\,mg \tag{5.39}$$

and so the reference signal is chosen as the constant signal

$$t \mapsto r(t) := 44.52.$$

During the maintenance stage in general anesthesia, the blood concentration levels of propofol are required to lie between 2.5–6 [$\mu g/ml$] (see [25]) and so, (5.39) yields

$$2.5\frac{\mu g}{ml}\,0.159\frac{l}{kg}\,70kg \;=\; 27.825\,mg \qquad \text{and} \qquad 6\frac{\mu g}{ml}\,0.159\frac{l}{kg}\,70kg \;=\; 66.78\,mg$$

and the control objective is that the output $y(\cdot)$ satisfies

$$\forall\, t \geq 0: \quad y(t) \in [27.825,\, 66.78].$$

In view of Remark 3.3.1 and Remark 5.5.3, non-symmetric Funnel $\mathcal{F}(0,\varphi)$ as in (5.26) is determined by $(\psi^l(\cdot), \psi^u(\cdot))$ given by

$$\begin{aligned}(\psi^l, \ell^l, \lambda^l) &= \big(t \mapsto 3\max\{2e^{-0.2\,t},\ 0.4\},\ 1.2,\ 1.2\big)\\ (\psi^u, \ell^u, \lambda^u) &= \big(t \mapsto 5\max\{2e^{-0.2\,t},\ 0.4\},\ 2,\ 2\big)\end{aligned}$$

and, for (5.24) and $t \geq 0$,

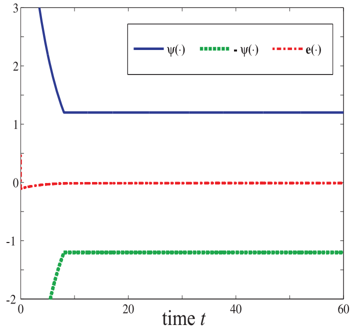
$$k(t) = \frac{1}{\min\{\psi^u(t) - e(t),\ \psi^l(t) + e(t)\}}$$

with $\|\psi^l\|_\infty = 6$, $\psi^l(0) = 6$ and $\|\psi^u\|_\infty = 10$, $\psi^l(0) = 10$. Note that the notation $\psi^i(\cdot) := \frac{1}{\varphi_i(\cdot)}$, $i \in \{\ell, u\}$ is used. This ensures that the propofol concentration $y(\cdot)$ lies between 2.5 and 6 [$\mu g/ml$] and it means exponential decay in the transient phase $[0,T]$, where $T \approx 15$, and constant level 0.3 for $t \geq T$. The simulations depicted in Figure 5.2 confirm the results of Theorem 5.5.1: the tracking error remains uniformly bounded away from the funnel boundary; the gain function $k(\cdot)$ is not monotone and reacts when the error is approaching the funnel boundary. Moreover, the second row shows that the input $u(\cdot)$ is negative at the beginning.

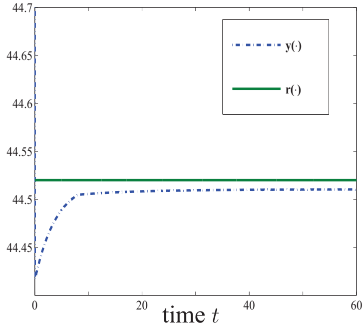
To guarantee that the input $u(\cdot)$ is non-negative the feedback law (5.36) with

$$k(t) = \begin{cases} 0 & ,\ e(t) > 0 \\ \frac{1}{\psi^l(t)+e(t)} & ,\ e(t) \leq 0 \end{cases}$$

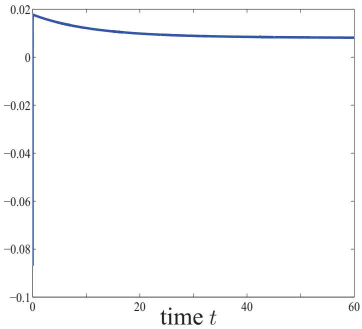
is considered. This ensures that the concentration of propofol $y(\cdot)$ is larger than $2.5\,[\mu g/ml]$. It has to be noted that it cannot be guaranteed that the propofol concentration $y(\cdot)$ is smaller than $6\,[\mu g/ml]$. The parameters are chosen as in [25]. The simulations depicted in Figure 5.3 confirm the results of Theorem 5.5.9: the tracking error is bounded and the input $u(\cdot)$ is non-negative. The gain function $k(\cdot)$ is not monotone. Moreover, the gain in Figure 5.3 satisfies $k(t) \approx 0.85$ for all $t \geq 9$ and



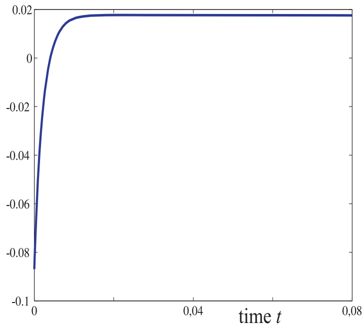
Funnel and tracking error $e(\cdot)$



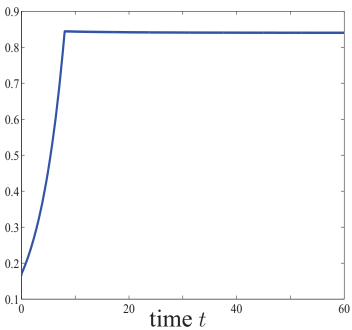
Reference signal $y_{\text{ref}}(\cdot)$ and output $y(\cdot)$



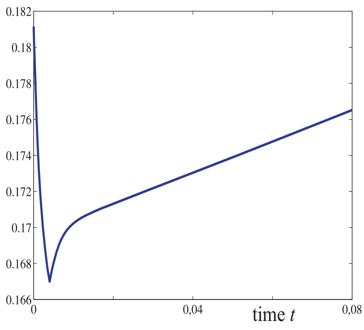
Control $u(\cdot)$



Control $u(\cdot)$ - zoomed



Gain $k(\cdot)$



Gain $k(\cdot)$ - zoomed

Figure 5.2: Control (5.24) applied to the linear system (5.38)

$\|u(\cdot)\| \approx 18$, whereas in [25] the gain satisfies $k(t) \approx 1.1$ for all $t \geq 8$ and $\|u(\cdot)\| \approx 40$. Hence the results of Theorem 5.5.9 are better as in [25]. [75] do not use an adaptive feedback law and thus, the results of [75] cannot be compared with the results of Theorem 5.5.9.

In the presence of measurement noise, the funnel controller also ensures that the error evolves within the funnel. For purposes of illustration, a bounded and chaotic output noise signal $d_y(\cdot)$ is given by

$$t \mapsto d_y(t) = -\frac{50}{9}\xi_1(t),$$

where $\xi_1(\cdot)$ is the first component of the solution of the Lorenz system which is introduced in Remark 2.4.1. Note that $\xi_1(0) = 1$ and easy calculations give $\|d_y\|_\infty \leq 10$, $\|\dot{d}_y\|_\infty \leq 20/3$. It has to be noted that the propofol concentrations $y(\cdot)$ and the noise corrupted output $y(\cdot) + d_y(\cdot)$ lies between 2.5 and 6 $[\mu g/ml]$. Figure 5.4 depicts the behaviour of the closed-loop system of Theorem 5.5.1 with output noise $d_y(\cdot)$.

Moreover, only a maximum mass of propofol can be injected per minute. Therefore the funnel control with input saturation

$$-\operatorname{sat}_{\hat{u}}\left(\frac{e(t)}{\min\{\psi^u(t) - e(t),\, \psi^l(t) + e(t)\}}\right), \qquad e(t) = (y(t) + d_y(t)) - r(t)$$

is appropriate. The closed-loop system (5.38), (5.32) with the function $d_u(\cdot) \equiv 0$. is considered. It is readily verified that

$$\|\widehat{A}_1\|_2 = 0.399, \quad \|\widehat{A}_2\|_2 = 0.0922, \quad \|\widehat{A}_3\|_2 = 0.211, \quad \|e^{\widehat{A}_4 t}\|_2 \leq e^{-0.0048\,t}\ \forall\, t \geq 0.$$

The feasibility condition (5.31) gives, for the reference signal $y_{\text{ref}}(\cdot)$ and output noise $d_y(\cdot)$

$$0.2793 < \widehat{u},$$

and therefore, for $\widehat{u} := 0.2794$, the feasibility condition (5.31) is satisfied according to (5.33). Furthermore a simple calculation gives $\varepsilon_0 = 0.0021$. Figure 5.5 depicts the behaviour of the closed-loop system of Theorem 5.5.5. The simulations confirm the results of Theorem 5.5.5. The simulation shows that $\|u(\cdot)\| \leq 0.06$ which shows that $\widehat{u}$ is conservative.

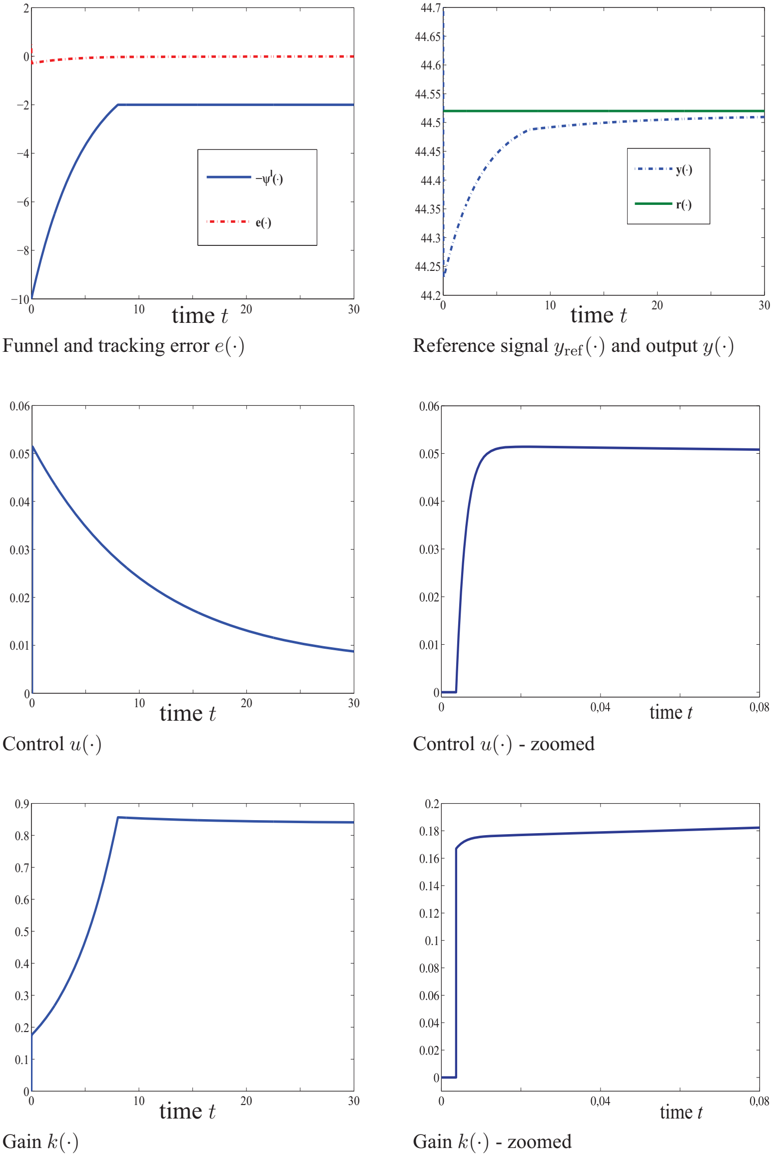
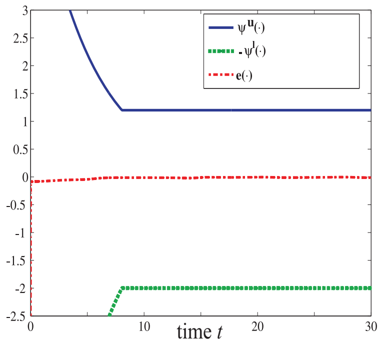
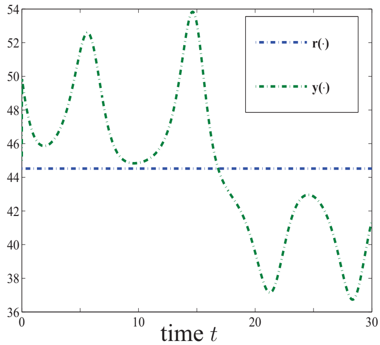


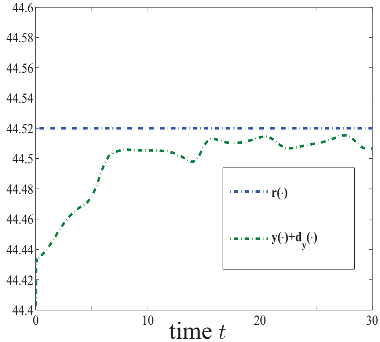
Figure 5.3: Control (5.36) applied to the linear system (5.38)



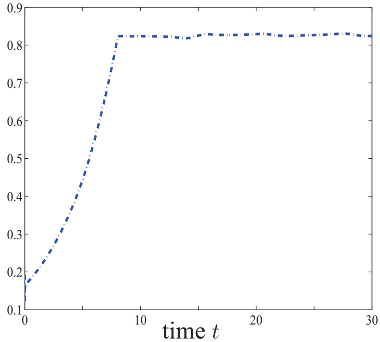
Funnel and tracking error $e(\cdot)$



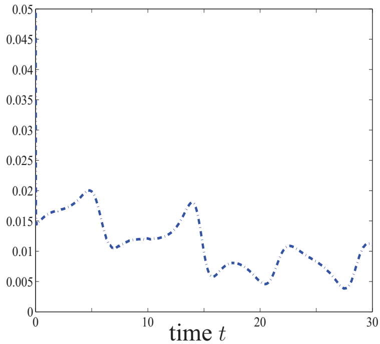
Reference signal $y_{\text{ref}}(\cdot)$ and output $y(\cdot)$



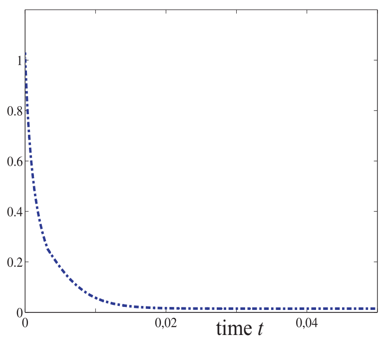
$y_{\text{ref}}(\cdot)$ and output $y(\cdot)$ with noise $d_y(\cdot)$



Gain $k(\cdot)$



Control $u(\cdot)$



Control $u(\cdot)$ - zoomed

Figure 5.4: Control (5.24) with output noise d_y, i.e. $e = (y + d_y) - r$, applied to the linear system (5.38)

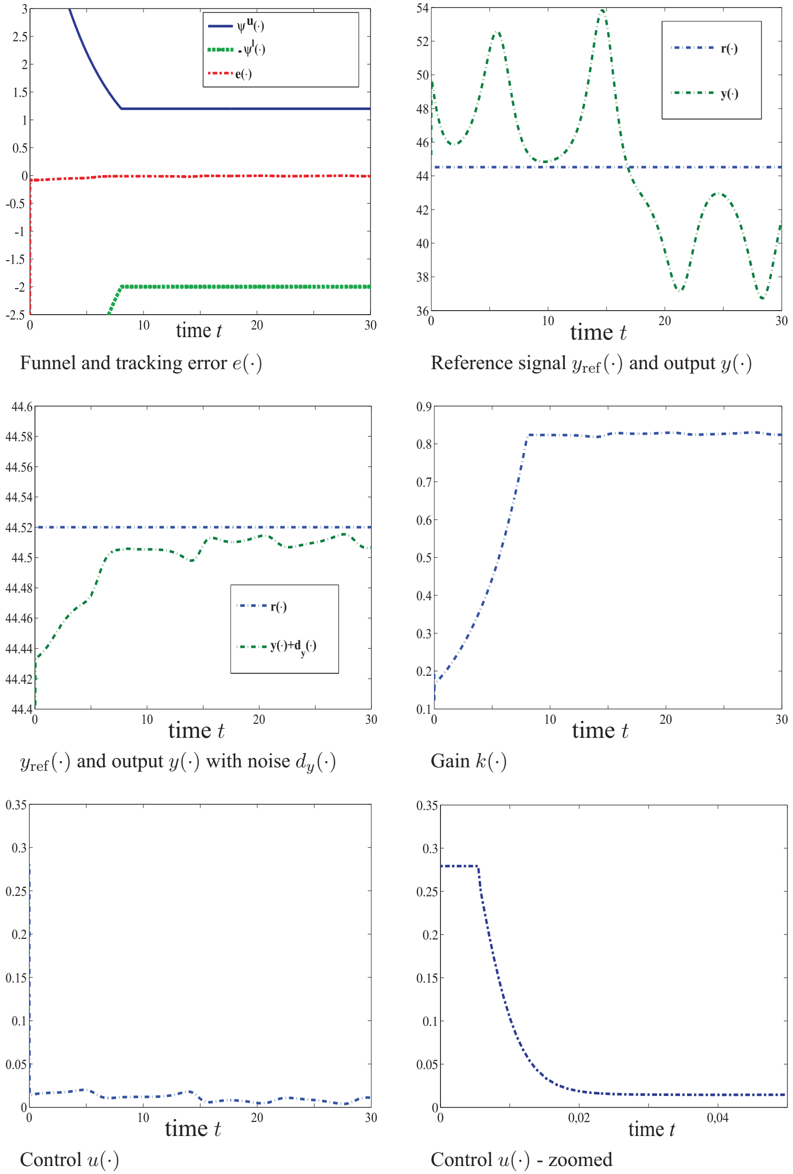


Figure 5.5: Control (5.32) with output noise d_y, i.e. $e = (y + d_y) - r$, applied to the linear system (5.38)

5.7 Proofs

The following lemma gives an overview of some properties of $\mathcal{L}^p$-functions which are necessary for the proofs of Subsection 5.7.1.

Lemma 5.7.1. (On $\mathcal{L}^p$-functions)

(i) If $-\infty < \alpha < \beta < \infty$ and $\eta(\cdot) \in \mathcal{BV}([\alpha,\beta],\mathbb{R}^{m\times n})$, $\phi(\cdot) \in \mathcal{C}([\alpha,\beta],\mathbb{R}^n)$, then

$$\left\| \int_\alpha^\beta \mathrm{d}[\eta(\theta)]\,\phi(\theta) \right\| \le \|\phi(\cdot)\|_{\mathcal{L}^\infty([\alpha,\beta])}\,\|\eta\|_{\mathrm{Var}(\alpha,\beta)}.$$

(ii) If $f(\cdot) \in (\mathcal{AC} \cap \mathcal{L}^1)(\mathbb{R}_{\ge 0},\mathbb{R})$ and $\dot f(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\ge 0},\mathbb{R})$, then $\lim_{t\to\infty} f(t) = 0$.

(iii) If $\eta(\cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times n})$, $\int_0^\infty |\,\mathrm{d}[\eta(\theta)]\,| < \infty$ and $g(\cdot) \in \mathcal{L}^p(\mathbb{R}_{\ge 0},\mathbb{R}^n)$, then

$$\left(t \mapsto \int_0^t \mathrm{d}[\eta(\theta)]\,g(t-\theta) \right) \in \mathcal{L}^p(\mathbb{R}_{\ge 0},\mathbb{R}^n).$$

(iv) If $\eta(\cdot) \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times n})$, $\int_0^\infty |\,\mathrm{d}[\eta(\theta)]\,| < \infty$ and $g(\cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0},\mathbb{R}^n)$ with $\lim_{t\to\infty} g(t) = 0$, then

$$\lim_{t\to\infty} \int_0^t \mathrm{d}[\eta(\theta)]\,g(t-\theta) = 0.$$

Proof: Statement (i) follows immediate from Definition 5.1.1. The proof of Assertion (ii) is given in [36, Lem. 2.1.7]. It remains to show the Assertions (iii) and (iv). Since $\int_0^\infty |\,\mathrm{d}[\eta(\theta)]\,| < \infty$, it follows that $\eta(\cdot) \in \mathcal{BV}(\mathbb{R}_{\ge 0},\mathbb{R}^{n\times n})$, and [21, Th. 3.6.1 (i), (ii)] implies (iii), (iv) which completes the proof of the lemma. □

Next, the following lemma gives a characterization of Metzler matrices if they are asymptotically stable. This characterization is used for the proofs of Subsection 5.7.3.

Lemma 5.7.2. (Characterization of a Metzler matrix)
For a Metzler matrix $A \in \mathbb{R}^{n\times n}$ it holds:

(i) $\quad \operatorname{spec}(A) \subset \mathbb{C}_- \quad \Longleftrightarrow \quad \exists\, p \in \mathbb{R}^n_{>0} : \; Ap \in \mathbb{R}^n_{<0}.$

(ii) $\quad \operatorname{spec}(A) \subset \mathbb{C}_- \quad \Longrightarrow \quad \forall\, i \in \{1, ..., n\}: \; A_{ii} < 0.$

Proof:
STEP 1: *Assertion (i) is shown.*

$\Rightarrow$: Let $q \in \mathbb{R}^n_{<0}$. From [83, Prop. 1] it follows that $A^{-1}q \in \mathbb{R}^n_{\geq 0}$. Choose $\varepsilon > 0$ sufficiently small so that $A\left[A^{-1}q + \varepsilon \mathbf{1}_n\right] = q + \varepsilon A \mathbf{1}_n \in \mathbb{R}^n_{<0}$. Then $p := A^{-1}q + \varepsilon \mathbf{1}_n \in \mathbb{R}^n_{>0}$ and $Ap \in \mathbb{R}^n_{<0}$.

$\Leftarrow$: Let $p \in \mathbb{R}^n_{>0}$ such that $Ap \in \mathbb{R}^n_{<0}$. From [83, Prop. 1] $x \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$ can be chosen in such a way that $A^\top x = \mu(A)x$ and hence $\mu(A)p^\top x = p^\top A^\top x < 0$. Since $p^\top x > 0$, it follows that $\mu(A) < 0$.

STEP 2: *Assertion (ii) is shown.*
Seeking a contradiction, suppose that there exists $i \in \{1, ..., n\}$ such that $A_{ii} \geq 0$. Since A is a stable Metzler matrix, Assertion (i) gives

$$\exists\, p \in \mathbb{R}^n_{>0} : \; Ap \in \mathbb{R}^n_{<0}$$

and thus

$$\forall\, j \in \{1, ..., n\}: \quad A_{ij} p_j \geq 0$$

which arrives at the contradiction

$$0 > (Ap)_i = \sum_{j=1}^{n} A_{ij} p_j \geq 0.$$

This completes the proof. □

5.7.1 Proofs of Section 5.2

Proof of Proposition 5.2.4:
The notation of Remark 5.2.3 is used.
Let $(A(\cdot), \eta(\cdot), f(\cdot)) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)$ for $\sigma \geq 0$. It suffices to investigate the initial value problem (5.4). For an interval $J \subset \mathbb{R}_{\geq 0}$ with $0 \in J$, $D \subset \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$, define the sets

$$\begin{aligned} \mathcal{C}_\alpha(J, \mathbb{R}^n) &:= \{f(\cdot) \in \mathcal{C}(J, \mathbb{R}^n) \mid f(0) = \alpha\} \\ \mathcal{C}_{\alpha,D}(J, \mathbb{R}^n) &:= \{f(\cdot) \in \mathcal{C}_\alpha(J, \mathbb{R}^n) \mid \forall\, t \in J: \; (t, f(t)) \in D\}. \end{aligned}$$

STEP 1: *Existence of a solution* $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *of the initial value problem* (5.4) *is shown.*
Define the functional

$$H : \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n),\ y(\cdot) \mapsto (Hy)(\cdot) := \mathcal{A}(\cdot)y(\cdot) + \int_0^{\cdot} \mathrm{d}[\eta(\theta)]\, y(\cdot - \theta) + F(\cdot).$$

Since [21, Th. 3.6.1] ensures that $\int_0^{\cdot} \mathrm{d}[\eta(\theta)]\, y(\cdot - \theta) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, H is well defined. Hence (5.4) is equivalent to the initial value problem

$$\dot{w}(t) = (Hw)(t), \qquad w(0) = \phi(\sigma). \tag{5.40}$$

STEP 1A: By the definition of H, it is easy to see that H is a causal operator, i.e.

$$\forall\ 0 \leq t_1 < t_2 < \infty\ \forall\, y(\cdot) \in \mathcal{C}([0, t_1], \mathbb{R}^n),\ z(\cdot) \in \mathcal{C}([0, t_2], \mathbb{R}^n) \text{ with } y|_{[0,t_1]} = z|_{[0,t_1]} : \quad (Hy)|_{[0,t_1]} = (Hz)_{[0,t_1]}.$$

STEP 1B: *It is shown that*

$$\begin{aligned} &\forall\, T \geq 0\ \forall\, \varphi(\cdot) \in \mathcal{C}_{\phi(\sigma)}([0, T], \mathbb{R}^n) : \\ &\qquad \text{(i)}\ \ (H\varphi)(\cdot) \in \mathcal{L}^1([0, T], \mathbb{R}^n) \\ &\qquad \text{(ii)}\ \forall\, t \in [0, T] :\ \left(y(\cdot) \mapsto \int_0^t (Hy)(\theta)\, \mathrm{d}\theta \right) \in \mathcal{C}(\mathcal{C}_{\phi(\sigma)}([0, T], \mathbb{R}^n), \mathbb{R}^n). \end{aligned}$$

For fixed $T \geq 0$ let $y(\cdot) \in \mathcal{C}_{\phi(\sigma)}([0, T], \mathbb{R}^n)$. By Lemma 5.7.1 (iii), $t \mapsto \int_0^t \mathrm{d}[\eta(\theta)]\, y(t - \theta) \in \mathcal{L}^1([0, T], \mathbb{R}^n)$. As $\mathcal{A}(\cdot)$, $F(\cdot)$ belong to $\mathcal{L}^1_{\text{loc}}$ and $y(\cdot)$ is continuous, (i) is satisfied.

To prove (ii), let $(y_k)_{k \in \mathbb{N}} \in \mathcal{C}_{\phi(\sigma)}([0, T], \mathbb{R}^n)^{\mathbb{N}}$ with $\lim_{k \to \infty} y_k(\cdot) = y(\cdot)$. Then

$$\forall\, t \in [0, T] : \quad \lim_{k \to \infty} (Hy_k)(t) = (Hy)(t)$$

which gives, for every $k \in \mathbb{N}$ and for all $t \in [0, T]$,

$$\begin{aligned} \|(Hy_k)(t)\|_{\text{Op}} &\leq \|\mathcal{A}(t) y_k(t)\| + \left\| \int_0^t \mathrm{d}[\eta(\theta)]\, y_k(t - \theta) \right\| + \|F(t)\| \\ &\leq \max_{t \in [0,T]} \|\mathcal{A}(t)\| \max_{t \in [0,T]} \|y_k(t)\| + \mathrm{Var}(\eta; 0, T) \max_{t \in [0,T]} \|y_k(t)\| \end{aligned}$$

$$
\begin{aligned}
&\quad + \max_{t\in[0,T]} \|F(t)\| \\
&\leq \left\{ \max_{t\in[0,T]} \|\mathcal{A}(t)\| + \text{Var}(\eta; 0, T) \right\} (\|y_k\|_{L^\infty} + 1) + \max_{t\in[0,T]} \|F(t)\|.
\end{aligned}
$$

The Lebesgue dominated convergence theorem (see [4, Th. X.3.12]) gives

$$
\forall\, t \in [0,T]: \quad \lim_{k\to\infty} \int_0^t (Hy_k)(\theta)\,\mathrm{d}\theta = \int_0^t (Hy)(\theta)\,\mathrm{d}\theta
$$

which completes the proof of Step 1B.

STEP 1C: *For each compact connected set $J \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ with $(0, \phi(\sigma)) \in J$ there exists $g_J(\cdot) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$ such that*

$$
\forall\, T > 0\ \forall\, y(\cdot) \in \mathcal{C}_{\phi(\sigma),J}([0,T], \mathbb{R}^n) \text{ for a.a. } t \in [0,T]: \quad \|(Hy)(t)\| \leq g_J(t).
$$

Let $J = J_1 \times J_2 \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ be a compact connected set containing $(0, \phi(\sigma))$. Since J is compact,

$$
T_J := \max J_1 \qquad \text{and} \qquad \widehat{y}_J := \max_{y_2 \in J_2} \|y_2\|
$$

are well defined. Let $T > 0$ and $y(\cdot) \in \mathcal{C}_{\phi(\sigma),J}([0,T], \mathbb{R}^n)$ be arbitrary but fixed. It has to be noted that $(T, y(T)) \in J$, and thus $t \leq T_J$. It follows, for all $t \in [0,T]$,

$$
\begin{aligned}
\|(Hy)(t)\|_{\text{Op}} &\leq \max_{t\in[0,T]} \|\mathcal{A}(t)\| \max_{t\in[0,T]} \|y(t)\| + \text{Var}(\eta; 0, T) \max_{t\in[0,T]} \|y(t)\| \\
&\quad + \max_{t\in[0,T]} \|F(t)\| \\
&\leq \left\{ \max_{t\in[0,T]} \|\mathcal{A}(t)\| + \text{Var}(\eta; 0, T) \right\} \widehat{y}_J + \max_{t\in[0,T]} \|F(t)\| \\
&=: g_J(t)
\end{aligned}
$$

and therefore Step 1C is completed.

STEP 1D: Apply [21, Th. 12.3.1] to (5.40) and conclude that there exists a solution $w : [0, \omega_0] \to \mathbb{R}^n$ of the initial value problem (5.40) for some $\omega_0 > 0$.

STEP 1E: *Existence of a maximal solution is shown.*
Let $w^* : [0, \omega_*] \to \mathbb{R}^n$, $\omega_* > 0$, a solution of the initial value problem (5.40). Define

$$
\mathcal{S} := \left\{ (\rho, w(\cdot)) \mid \rho \in [\omega_*, \infty],\ w(\cdot) \text{ is a solution of (5.40) on } [0,\rho) \text{ and } w|_{[0,\omega_*)} \equiv w^* \right\}
$$

and let a partial order on $\mathcal{S}$ be given by

$$(\rho_1, w_1(\cdot)) \leq (\rho_2, w_2(\cdot)) \qquad :\Longleftrightarrow \qquad \rho_1 \leq \rho_2 \quad \text{and} \quad w_2|_{[0,\rho_1]} \equiv w_1.$$

Let $\mathcal{S}_1$ be a totally ordered subset of $\mathcal{S}$. Set

$$\rho_{\max} := \sup\left\{\rho \in [\omega_*, \infty] \mid (\rho, w(\cdot)) \in \mathcal{S}_1\right\}.$$

For every $t \in [0, \rho_{\max})$ there exists $(\rho, w(\cdot)) \in \mathcal{S}_1$ such that $t \in [0, \rho)$ and let $w_{\max}(\cdot) := w(\cdot)$. It has to be noted that $w_{\max}(\cdot)$ is independent on choosing $(\rho, w(\cdot)) \in \mathcal{S}_1$ because $\mathcal{S}_1$ is a totally ordered subset of $\mathcal{S}$. Then the Hausdorff maximal principle (see [51, Th. 0.24]) gives that $\mathcal{S}$ contains one maximal element. Hence there exists a maximal solution $w : [0, \omega) \to \mathbb{R}^n$, $\omega \in (0, \infty]$, of the initial value problem (5.40). By [21, Th. 12.3.1], it follows that $\omega = \infty$. This completes the proof of Step 1.

STEP 2: *Uniqueness is shown.*
Assume that $w(\cdot)$ and $v(\cdot)$ are two solutions of (5.40) on $\mathbb{R}_{\geq 0}$. Let $T \geq 0$ arbitrary but fixed. It has to be noted that, for all $t \in [0, T]$,

$$\zeta(t) = \phi(\sigma) + \int_0^t \left\{ \mathcal{A}(\tau)\zeta(\tau) + \int_0^\tau \mathrm{d}[\eta(\theta)]\, \zeta(\tau - \theta) + F(\tau) \right\} \mathrm{d}\tau\,, \qquad \zeta \in \{w, v\},$$

and thus, for all $t \in [0, T]$,

$$\begin{aligned} \|w(t) - v(t)\| \;&\leq\; \max_{\tau \in [0,t]} \|\mathcal{A}(\tau)\| \int_0^t \|w(\tau) - v(\tau)\|\, \mathrm{d}\tau \\ &\qquad\qquad + \mathrm{Var}(\eta; 0, t) \int_0^t \max_{s \in [0,\tau]} \|w(s) - v(s)\|\, \mathrm{d}\tau \\ &\leq\; \left\{ \max_{\tau \in [0,T]} \|\mathcal{A}(\tau)\| + \mathrm{Var}(\eta; 0, T) \right\} \int_0^t \max_{s \in [0,\tau]} \|w(s) - v(s)\|\, \mathrm{d}\tau \end{aligned}$$

which implies that

$$\exists\, m = m(T) > 0\ \forall\, t \in [0, T]: \quad \|w(t) - v(t)\| \leq m \int_0^t \max_{s \in [0,\tau]} \|w(s) - v(s)\|\, \mathrm{d}\tau\,. \tag{5.41}$$

Set

$$z : [0,T] \to \mathbb{R}_{\geq 0}, \qquad t \mapsto z(t) := \max_{s\in[0,t]} \|w(s) - v(s)\|.$$

Then $z(\cdot)$ is continuous on $[0,T]$ and, moreover, (5.41) implies that

$$\forall\, t \in [0,T] : \quad 0 \leq z(t) \leq m \int_0^t z(\tau)\, \mathrm{d}\tau\,,$$

and therefore, by Gronwall's inequality (see [88, Lem. VII.29.VI]),

$$\forall\, t \in [0,T] : \quad z(t) = 0.$$

Since $T \geq 0$ is arbitrary, it follows that $z(t) = 0$ for all $t \geq 0$ and hence $w(\cdot) = v(\cdot)$ on $\mathbb{R}_{\geq 0}$. This proves uniqueness of the solution.

STEP 3: *Continuous dependence*
Finally, it follows from [21, Th. 13.2.3] that the solution depends continuously on the initial values. This completes the proof of the proposition. □

Proof of Proposition 5.2.7:
Let $(A(\cdot), \eta(\cdot), f(\cdot)) \in (\mathcal{L}^1_{\text{loc}} \cap \mathcal{L}^\infty_{\text{loc}})(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{BV}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n_{\geq 0})$, $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$ and $\sigma \geq 0$. Then Proposition 5.2.4 shows that there exists a unique solution $x(\cdot; \sigma, \phi, f) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of the initial value problem (5.3). It remains to show that $x(t) := x(t; \sigma, \phi, f) \geq 0$ for all $t \geq 0$.

STEP 1: *It is shown that for fixed $T > \sigma$ the following implication holds:*

$$\phi(\sigma) \gg 0 \qquad \Rightarrow \qquad \forall\, t \in [0,T] : \quad x(t) \geq 0.$$

It has to be noted that $\phi(\sigma) \gg 0$ is defined as $\phi_i(\sigma) > 0$ for all $i = 1, ..., n$. Since $A(t)$ is a Metzler matrix for almost all $t \geq 0$ and $A(\cdot) \in \mathcal{L}^\infty([0,T], \mathbb{R}^{n\times n})$, it follows

$$\exists\, r > 0 \text{ for a.a. } t \in [0,T] : \quad rI_n + A(t) \in \mathbb{R}^{n\times n}_{\geq 0}.$$

Hence it remains to show that

$$\forall\, t \in [0,T] : \quad z(t) := e^{r(t-\sigma)} x(t) \geq 0.$$

Note that $z(\cdot)$ is the solution of

$$\dot{z}(t) = [rI_n + A(t)]z(t) + \int_0^t \mathrm{d}[\eta(\theta)] \left(e^{r\theta} z(t-\theta)\right) + e^{r(t-\sigma)} f(t),$$
$$z|_{[0,\sigma]} = e^{r(\cdot - \sigma)}\phi(\cdot), \text{ for a.a. } t \in [\sigma, T].$$

Seeking a contradiction, suppose that

$$T_0 := \inf \{t \in [\sigma, T] \mid z(t) \not\gg 0\} \in (\sigma, T].$$

Note that $T_0 > \sigma$ since $\phi(\sigma) \gg 0$. Since $\eta(\cdot)$ is non-decreasing and $z(\tau) \geq 0$ for all $\tau \in [0, T_0]$, it follows that

$$\forall\, t \in [\sigma, T_0] : \quad \int_0^t \mathrm{d}[\eta(\theta)] \left(e^{r\theta} z(t-\theta)\right) \geq 0.$$

Thus,

$$z(T_0) = z(\sigma) + \int_\sigma^{T_0} \dot{z}(\tau)\,\mathrm{d}\tau \; = \phi(\sigma) + \int_\sigma^{T_0} [rI_n + A(\tau)]z(\tau)\,\mathrm{d}\tau$$
$$+ \int_\sigma^{T_0}\int_0^\tau \mathrm{d}[\eta(\theta)]\,(e^{r\theta} z(\tau-\theta))\,\mathrm{d}\tau \; + \int_\sigma^{T_0} e^{r(\tau-\sigma)} f(\tau)\,\mathrm{d}\tau \; \geq \phi(\sigma) \gg 0.$$

This implies, for $\varepsilon > 0$ sufficiently small, that $z(t) \geq 0$ for all $t \in [T_0, T_0 + \varepsilon]$. This conflicts with the definition of T_0 and proves Step 1.

STEP 2: Suppose $\phi(\cdot) \geq 0$. Set

$$\forall\, t \in [0, \sigma]\; \forall\, k \in \mathbb{N} : \quad \phi_k(t) := \phi(t) + k^{-1}\mathbf{1}_n.$$

It follows from Step 1 that

$$\forall\, t \in [0, T]\; \forall\, k \in \mathbb{N} : \quad x(t; \sigma, \phi_k, f) \geq 0.$$

Since the solution depends continuously on the initial values (see Proposition 5.2.4), this gives

$$\forall\, t \in [0, T] : \quad \lim_{k\to\infty} x(t; \sigma, \phi_k, f) = x(t; \sigma, \phi, f) \geq 0.$$

This completes the proof of Step 2 and the proof of the proposition. □

Proof of Theorem 5.2.8:
If $A(\cdot) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ and $f(\cdot) \equiv 0$, then "$\Leftarrow$" follows from Proposition 5.2.7.

It remains to consider a positive system with $A(\cdot) \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, $\sigma \geq 0$, $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$. Let $x(\cdot) := x(\cdot;\sigma,\phi)$ be the solution of (5.8).

STEP 1: *It is shown that if $\eta(\cdot)$ is continuous from the right at σ and the sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ with $t_k \in [\sigma, \sigma + 1/k]$ satisfies* (5.8)*, then*

$$\lim_{k\to\infty} \dot{x}(t_k) = A(\sigma)\phi(\sigma) + \int_0^\sigma \mathrm{d}[\eta(\theta)]\, \phi(\sigma - \theta). \tag{5.42}$$

In fact, it follows that, for all $k \in \mathbb{N}$,

$$\begin{aligned} &\left\| \int_0^{t_k} \mathrm{d}[\eta(\theta)]\, x(t_k - \theta) - \int_0^\sigma \mathrm{d}[\eta(\theta)]\, \phi(\sigma - \theta) \right\| \\ &\quad \leq \left\| \int_0^\sigma \mathrm{d}[\eta(\theta)]\, [x(t_k - \theta) - \phi(\sigma - \theta)] \right\| + \left\| \int_\sigma^{t_k} \mathrm{d}[\eta(\theta)]\, x(t_k - \theta) \right\| \\ &\quad \leq \mathrm{Var}(\eta; 0, \sigma) \sup_{\theta\in[0,\sigma]} \|x(t_k - \theta) - \phi(\sigma - \theta)\| + \mathrm{Var}(\eta; \sigma, t_k) \sup_{\theta\in[\sigma,t_k]} \|x(t_k - \theta)\|\,. \end{aligned}$$

Since $\eta(\cdot)$ is continuous from the right at σ, it follows that

$$\lim_{k\to\infty} \left[\mathrm{Var}(\eta; \sigma, t_k) \sup_{\theta\in[\sigma,t_k]} \|x(t_k - \theta)\| \right] \leq \sup_{t\in[0,\sigma+1]} \|x(t)\| \cdot \lim_{k\to\infty} \mathrm{Var}(\eta; \sigma, \sigma + 1/k) = 0$$

and, as $x(\cdot)$ is uniformly continuous on $[0, \sigma + 1]$, this gives

$$\lim_{k\to\infty} \sup_{\theta\in[0,\sigma]} \|x(t_k - \theta) - \phi(\sigma - \theta)\| = 0.$$

Therefore,

$$\lim_{k\to\infty} \int_0^{t_k} \mathrm{d}[\eta(\theta)]\, x(t_k - \theta) = \int_0^\sigma \mathrm{d}[\eta(\theta)]\, \phi(\sigma - \theta). \tag{5.43}$$

Continuity of $A(\cdot)$ and (5.43) yield

$$\lim_{k\to\infty} \dot{x}(t_k) = \lim_{k\to\infty} \left[A(t_k)x(t_k) + \int_0^{t_k} \mathrm{d}[\eta(\theta)]\, x(t_k - \theta) \right] = A(\sigma)x(\sigma) + \int_0^\sigma \mathrm{d}[\eta(\theta)]\, x(\sigma - \theta)$$

which proves (5.42).

STEP 2: Let $\sigma \geq 0$ and $\phi(\cdot) \in \mathcal{C}([0,\sigma],\mathbb{R}^n_{\geq 0})$ with $\phi(\sigma) = 0$.
For fixed $i \in \{1,2,...,n\}$, it is shown that

$$\forall\, k \in \mathbb{N}\ \exists\, t_k \in [\sigma, \sigma + 1/k] : \quad \text{(5.8) is satisfied at } t = t_k \text{ and } e_i^\top \dot{x}(t_k) \geq 0. \tag{5.44}$$

Since $x(\cdot)$ satisfies (5.8) for almost all $t \in [\sigma, \sigma + 1/k]$, $k \in \mathbb{N}$, there exists a set $N_k \subset [\sigma, \sigma + 1/k]$ such that

$$\text{measure}(N_k) = 0$$

and $x(\cdot)$ satisfies (5.8) for all $t \in [\sigma, \sigma + 1/k]\backslash N_k$, $k \in \mathbb{N}$. Seeking a contradiction to (5.44), suppose that

$$\exists\, k \in \mathbb{N}\ \forall\, t \in [\sigma, \sigma + 1/k]\backslash N_k : \quad e_i^\top \dot{x}(t) < 0. \tag{5.45}$$

Then it follows that, for all $t \in [\sigma, \sigma + 1/k]\backslash N_k$,

$$0 \leq e_i^\top x(t) = e_i^\top \left[x(\sigma) + \int_\sigma^t \dot{x}(\tau)\,\mathrm{d}\tau \right] = e_i^\top \phi(\sigma) + \int_\sigma^t e_i^\top \dot{x}(\tau)\,\mathrm{d}\tau = \int_\sigma^t e_i^\top \dot{x}(\tau)\,\mathrm{d}\tau \leq 0$$

and thus

$$\forall\, t \in [\sigma, \sigma + 1/k]\backslash N_k : \quad e_i^\top x(t) = 0.$$

This contradicts (5.45) and completes Step 2.

STEP 3: *It is shown that $\eta(\cdot)$ is non-decreasing on $\mathbb{R}_{\geq 0}$.*
Let $\sigma \geq 0$ and $(\sigma, \psi(\cdot)) \in \mathbb{R}_{\geq 0} \times \mathcal{C}([0,\sigma],\mathbb{R}_{\geq 0})$ with $\psi(\sigma) = 0$. For arbitrary but fixed $j \in \{1,...,n\}$ define

$$\phi(\cdot) = (\phi_1(\cdot),...,\phi_n(\cdot))^\top \qquad \text{with} \qquad \phi_i(\cdot) := \begin{cases} 0, & i \neq j \\ \psi(\cdot), & i = j. \end{cases}$$

It has to be noted that $\phi(\cdot) \in \mathcal{C}([0,\sigma],\mathbb{R}^n_{\geq 0})$ with $\phi(\sigma) = 0$. Step 2 yields

$$\forall\, k \in \mathbb{N}\ \exists\, t_k \in [\sigma, \sigma + 1/k] : \quad \dot{x}(t_k) = A(t_k)x(t_k) + \int_0^{t_k} \mathrm{d}[\eta(\theta)]\, x(t_k - \theta), \quad e_i^\top \dot{x}(t_k) \geq 0.$$

Let $i \in \{1,...,n\}$ be arbitrary. It has to be noted that $\eta(\cdot)$ is continuous for almost all $t \geq 0$ because $\eta(\cdot)$ is of locally bounded variation on $\mathbb{R}_{\geq 0}$. Assume that $\eta(\cdot)$ is continuous at σ. Step 1 gives, in view of (5.42),

$$0 \leq \lim_{k\to\infty} e_i^\top \dot{x}(t_k) = e_i^\top \int_0^\sigma \mathrm{d}[\eta(\theta)]\, \phi(\sigma - \theta) = \int_0^\sigma \mathrm{d}[\eta_{ij}(\theta)]\, \psi(\sigma - \theta).$$

Thus, the linear functional

$$L : \mathcal{C}([0,\sigma],\mathbb{R}) \to \mathbb{R}, \quad \psi(\cdot) \mapsto L\psi := \int_0^\sigma \mathrm{d}[\eta_{ij}(\theta)]\,\psi(\sigma-\theta)$$

is a positive operator. Then $\eta_{ij}(\cdot)$ is increasing on $[0,\sigma]$ (see [38, Lem. 2.5]). Since σ, j and i are arbitrary, this completes Step 3.

STEP 4: *It is shown that $A(t)$ is a Metzler matrix for every $t \geq 0$.*
Let $\sigma \geq 0$ such that $\eta(\cdot)$ is continuous at σ. For arbitrary but fixed $j \in \{1,...,n\}$ and $m \in \mathbb{N}$ define

$$\phi^{(m)}(\cdot) := \left(\phi_1^{(m)}(\cdot),...,\phi_n^{(m)}(\cdot)\right)^\top$$

with

$$\phi_j^{(m)}(t) := \begin{cases} 0, & t \in [0,\sigma(1-1/m)] \\ \frac{m}{\sigma}t + 1 - m, & t \in (\sigma(1-1/m),\sigma] \end{cases}$$
$$\phi_i^{(m)}(\cdot) \equiv 0, \qquad i \neq j.$$

It has to be noted that $\phi^{(m)}(\cdot) \in \mathcal{C}([0,\sigma],\mathbb{R}^n_{\geq 0})$ and $\phi^{(m)}(\sigma) = e_j$ for every $m \in \mathbb{N}$. Let $i \in \{1,...,n\}\backslash\{j\}$ be arbitrary. Similarly to Step 2 follows that

$$\forall\, k \in \mathbb{N}\,\exists\, t_k^{(m)} \in [\sigma,\sigma+1/k]: \quad e_i^\top\,\dot{x}(t_k^{(m)};\sigma,\phi^{(m)}) \geq 0.$$

Step 1 yields

$$\begin{aligned} 0 \leq \lim_{k\to\infty}\left[e_i^\top \dot{x}(t_k^{(m)};\sigma,\phi^{(m)})\right] &= e_i^\top A(\sigma)\phi^{(m)}(\sigma) + e_i^\top \int_0^\sigma \mathrm{d}[\eta(\theta)]\,\phi^{(m)}(\sigma-\theta) \\ &= e_i^\top A(\sigma)e_j + e_i^\top \int_0^\sigma \mathrm{d}[\eta(\theta)]\,\phi^{(m)}(\sigma-\theta) \qquad \forall\, m \in \mathbb{N}. \end{aligned} \tag{5.46}$$

It remains to show that

$$\lim_{m\to\infty}\left(e_i^\top \int_0^\sigma \mathrm{d}[\eta(\theta)]\,\phi^{(m)}(\sigma-\theta)\right) = 0.$$

With the definition of $\phi^{(m)}(\cdot)$, it follows that

$$\left\| e_i^\top \int_0^\sigma \mathrm{d}[\eta(\theta)]\, \phi^{(m)}(\sigma-\theta) \right\| \leq \left\| \int_{\sigma(1-1/m)}^{\sigma} \mathrm{d}[\eta(\theta)]\, \phi^{(m)}(\sigma-\theta) \right\|$$
$$\leq \mathrm{Var}(\eta;\sigma(1-1/m),\sigma) \sup_{\theta\in[\sigma(1-1/m),\sigma]} \left\| \phi_j^{(m)}(\sigma-\theta) \right\| \leq \mathrm{Var}(\eta;\sigma(1-1/m),\sigma).$$

Let m tends to ∞ in (5.46). Then (5.46) gives

$$\forall\, i,j \in \{1,...,n\},\ i \neq j: \quad e_i^\top A(\sigma) e_j \geq 0.$$

Therefore,

$$\forall\, i,j \in \{1,...,n\},\ i \neq j \text{ for a.a. } t \geq 0: \quad e_i^\top A(t) e_j \geq 0.$$

Moreover, since $A(\cdot)$ is continuous on $\mathbb{R}_{\geq 0}$, this implies that

$$\forall\, i,j \in \{1,...,n\},\ i \neq j\ \forall\, t \geq 0: \quad e_i^\top A(t) e_j \geq 0.$$

This completes Step 4 and the proof of the theorem. □

5.7.2 Proofs of Section 5.3

Proof of Proposition 5.3.2:
As immediate conclusions the following holds:

$$\text{(iv)} \overset{[21, Th.\ 3.3.5]}{\Longleftrightarrow} \text{(ii)} \overset{p=1}{\Longleftarrow} \text{(iii)} \overset{Def\ 5.3.1}{\Longleftarrow} \text{(vi)} \overset{Def\ 5.3.1}{\Longleftarrow} \text{(vii)} \overset{Def\ 5.3.1}{\Longrightarrow} \text{(v)} \overset{[68, Th.\ 3.5]}{\Longleftrightarrow} \text{(iv)}$$

and

$$\text{(v)} \overset{Def\ 5.3.1}{\Longrightarrow} \text{(i)}.$$

It remains to show that

$$\text{(ii)} \Rightarrow \text{(v)} \qquad \text{and} \qquad \text{(ii)} \Rightarrow \text{(iii)}.$$

Since $R(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, Remark 5.2.5 (iv) concludes $\lim_{t\to\infty} R(t) = 0$ and thus, $y_{\mathrm{ref}}(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$.

"(ii) ⇒ (v)": Applying Variation-of-Constants formula (5.7) to (5.9) gives, for all

$t \geq 0$,

$$\begin{aligned}\|x(t+\sigma;\sigma,\phi)\| &\leq \|R(t)\|\|\phi(\sigma)\| + \int_0^t \|R(t-s)\|\|\phi\|_\infty \left(\int_s^{s+\sigma} |\,\mathrm{d}[\eta(\theta)]\,| \right) \mathrm{d}s \quad (5.47)\\ &\leq \sup_{t\geq 0} \left[\|R(t)\| + \int_0^t \|R(\tau)\|\,\mathrm{d}\tau \right] \|\phi\|_\infty \left[1 + \int_0^\infty |\,\mathrm{d}[\eta(\theta)]\,| \right].\end{aligned}$$

This implies, by Definition 5.3.1, that (5.9) is uniformly stable. It has to be noted that $\int_0^\infty |\,\mathrm{d}\eta(\theta)\,| < \infty$ yields

$$\lim_{s\to\infty} \|\eta\|_{\mathrm{Var}([s,s+\sigma])} = 0,$$

and it follows that

$$\lim_{t\to\infty} \int_0^t \|R(t-s)\| \left(\int_s^\infty |\,\mathrm{d}[\eta(\theta)]\,| \right) \mathrm{d}s = 0.$$

Thus, (5.47) shows that (5.9) is uniformly asymptotically stable, by Definition 5.3.1.

"(ii) ⇒ (iii)": As

$$\forall\, p \in (1,\infty): \quad \int_0^\infty \|R(t)\|^p\,\mathrm{d}t \leq \|R\|_\infty^{p-1} \int_0^\infty \|R(t)\|\,\mathrm{d}t < \infty,$$

condition (iii) holds. This completes the proof of the proposition. □

Proof of Corollary 5.3.3:
Since the homogeneous part of (5.10) is uniformly asymptotically stable, it follows, by Proposition 5.3.2 (ii) and Definition 5.3.1, that $R(\cdot) \in \mathcal{L}^p(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ for all $p \in [1,\infty]$ and $\lim_{t\to\infty} x(t;\sigma,0) = 0$.

(i) Let $V := \mathcal{L}^p(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Since $f(\cdot) \in \mathcal{L}^p(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ for some $p \in [1,\infty]$, Remark 5.2.2 and [21, Th. 3.3.9 (iii)] yield the result.

(ii) As $f(\cdot) \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and $\lim_{t\to\infty} f(t) = 0$, it follows that $f(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$. Let $V := \mathcal{L}^\infty_0 := \left\{ f \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \;\middle|\; \lim_{t\to\infty} f(t) = 0 \right\}$. Thus Remark 5.2.2 and

[21, Th. 3.3.9 (iii)] yield the result and this completes the proof of the corollary. □

Proof of Theorem 5.3.4:
The statement of Theorem 5.3.4 (i) is proven in [70, Th. 5.4]. It remains to show Theorem 5.3.4 (ii).

"$\Leftarrow$": By supposition it follows that

$$\exists\, \alpha > 0 : \quad \left\| \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{\alpha\theta} \right\| < \infty. \tag{5.48}$$

STEP 1: *It is shown that*

$$\exists\, K,\ \varepsilon > 0\ \forall\, t \geq 0 : \quad \|R(t)\| \leq K e^{-\varepsilon t}. \tag{5.49}$$

By continuity and (5.48), it is possible to choose $\varepsilon \in (0, \alpha)$ such that

$$\mu\left(A + \varepsilon I_n + \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{\varepsilon\,\theta} \right) < 0 \tag{5.50}$$

holds. It is readily verified that $R(\cdot)$ is the resolvent of (5.9) if, and only if, $R_\varepsilon(\cdot) := e^{\varepsilon\,\cdot} R(\cdot)$ is the resolvent of

$$\dot z(t) = [A + \varepsilon I_n]\, z(t) + \int_0^t \mathrm{d}[\Theta_\varepsilon(\tau)]\ z(t-\tau), \quad t \geq 0, \quad \text{where } \Theta_\varepsilon(\tau) := \int_0^\tau \mathrm{d}[\theta(u)]\ e^{\varepsilon\, u}.$$

Now (5.50) in combination with Theorem 5.3.4 (i) and Proposition 5.3.2 yields that $R_\varepsilon(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$ and Remark 5.2.5 (iv) gives boundedness of $R_\varepsilon(\cdot)$. This proves (5.49).

STEP 2: *It is shown that*

$$\exists\, M > 0\ \forall\, \sigma \geq 0\ \forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n)\ \forall\, t \geq \sigma : \quad \|x(t;\sigma,\phi)\| \leq M e^{-\varepsilon(t-\sigma)} \|\phi\|,$$

where ε is determined in (5.49).
Applying Variation-of-Constants formula (5.7) with $f(\cdot) \equiv 0$ to (5.9), together with

Step 1, shows that it remains to show that

$$\begin{aligned}&\exists\, K_1 > 0\ \forall\, \sigma \ge 0\ \forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma],\mathbb{R}^n)\ \forall\, t \ge \sigma :\\ &\qquad\left\| \int_0^t R(t-s)\Big\{ \int_s^{s+\sigma} \mathrm{d}[\eta(\theta)]\ \phi(s+\sigma-\theta)\Big\}\,\mathrm{d}s \right\| \le K_1 e^{-\varepsilon(t-\sigma)}\|\phi\|. \qquad (5.51)\end{aligned}$$

It has to be noted that (5.48) ensures $\|e^{(\alpha\,\cdot)}\eta(\cdot)\|_{\mathrm{Var}([0,\infty))} < \infty$, and so (5.49) and Lemma 5.7.1 (i) imply that, for all $t \ge 0$,

$$\begin{aligned}&\left\| \int_0^t R(t-s)\left\{ \int_s^{s+\sigma} \mathrm{d}[\eta(\theta)]\ \phi(s+\sigma-\theta)\right\}\ \mathrm{d}s \right\|\\
&\qquad\le \int_0^t K e^{-\varepsilon(t-s)} \left\| \int_s^{s+\sigma} \mathrm{d}[\eta(\theta)]\ \phi(s+\sigma-\theta)\right\|\ \mathrm{d}s\\
&\qquad\le K \int_0^t e^{-\varepsilon(t-s)} \|\eta\|_{\mathrm{Var}([s,s+\sigma])}\ \mathrm{d}s\ \ \|\phi\|\\
&\qquad\le K e^{-\varepsilon t} \int_0^t e^{(\varepsilon-\alpha)s}\ \|e^{\alpha s}\eta\|_{\mathrm{Var}([s,s+\sigma])}\ \mathrm{d}s\ \ \|\phi\|\\
&\qquad\le K e^{-\varepsilon t} \int_0^t e^{-(\alpha-\varepsilon)s}\ \|e^{\alpha\cdot}\eta(\cdot)\|_{\mathrm{Var}([0,\infty))}\ \mathrm{d}s\ \ \|\phi\|\\
&\qquad\le \tfrac{K}{\alpha-\varepsilon}\, e^{-\alpha t}\ \|e^{\alpha\cdot}\eta(\cdot)\|_{\mathrm{Var}([0,\infty))}\ \|\phi\|\end{aligned}$$

which proves (5.51).

"⇒": Since exponential asymptotic stability implies asymptotic stability by definition, it remains, by (i), to show that

$$\int_0^\infty \mathrm{d}[\eta(\theta)]\ e^{\alpha\theta} \in \mathbb{R}^{n\times n} \qquad \text{for some} \quad \alpha > 0. \qquad (5.52)$$

This is proceeded in several steps.

Step 1: Since (5.9) is exponentially asymptotically stable, Proposition 5.3.2 (vi) implies that the Laplace transform $\widehat{y}_{\mathrm{ref}}(\cdot)$ of the resolvent $R(\cdot)$ of (5.5) satisfies:

$$\widehat{R}(\cdot) : \mathbb{C}_{-\beta} \to \mathbb{C}^{n\times n} \qquad \text{is analytic for some } \beta > 0.$$

Writing

$$\Delta(z) := zI_n - A - \int_0^\infty \mathrm{d}[\eta(\theta)]\ e^{-z\theta} \qquad \text{for any } z \in \mathbb{C} \text{ so that } \Delta(z) \text{ exists,}$$

and taking Laplace transform to both sides of (5.5) gives

$$\forall\, z \in \mathbb{C}_0 : \quad \Delta(z)\ \widehat{R}(z) = I_n.$$

Thus, $\det \widehat{R}(0) \neq 0$ and continuity of $z \mapsto \det \widehat{R}(z)$ at $z = 0$ yields

$$\widehat{R}(\cdot)^{-1} : \mathcal{B}_\gamma(0) \to \mathbb{C}^{n\times n} \qquad \text{is analytic for some } \gamma \in (0, \beta).$$

Therefore,

$$V : \mathcal{B}_\gamma(0) \to \mathbb{C}^{n\times n},\ z \mapsto zI_n - A - \widehat{R}(z)^{-1}, \qquad \text{is analytic.}$$

Since

$$\forall\, z \in \mathbb{C}_0 \text{ with } \Re z > 0 : \quad V(z) = zI_n - A - \Delta(z) = \int_0^\infty \mathrm{d}[\eta(\theta)]\ e^{-z\theta},$$

analyticity of $V(\cdot)$ yields

$$\forall\, k \in \mathbb{N}_0\ \forall\, z \in \mathcal{B}_\gamma(0) \cap \mathbb{C}_0 \text{ with } \Re z > 0 : \quad V^{(k)}(z) = (-1)^k \int_0^\infty \mathrm{d}[\eta(\theta)]\ \theta^k\, e^{-z\theta}. \tag{5.53}$$

STEP 2: *It is shown that*

$$\forall\, k \in \mathbb{N}_0 : \quad \int_0^\infty \mathrm{d}[\eta(\theta)]\ \theta^k \in \mathbb{R}^{n\times n}. \tag{5.54}$$

It has to be noted that

$$\forall\, \Theta > 0\ \exists\, \delta > 0\ \forall\, h \in (0,\delta)\ \forall\, \theta \in [0,\Theta] : \quad \frac{1 - e^{-h\theta}}{h} \geq \theta - 1 \tag{5.55}$$

and $V^{(k)}(0)$ exists for all $k \in \mathbb{N}_0$ since $V(\cdot)$ is analytic. Fix canonical basis vectors $e_i,\ e_j \in \mathbb{R}^n$, $\Theta > 0$ and $k \in \mathbb{N}_0$. Then, for all $h \in (0,\delta)$, δ as in (5.55),

$$\frac{V^{(k)}(h) - V^{(k)}(0)}{h} = \int_0^\infty \mathrm{d}[\eta(\theta)]\ \theta^k \frac{e^{-h\theta} - 1}{h}$$

and, for all $\Theta \geq 2$,

$$\left| e_i^\top \frac{V^{(k)}(h) - V^{(k)}(0)}{h} e_j^\top \right| \overset{(5.55)}{\geq} \int_0^\Theta e_i^\top \, \mathrm{d}[\eta(\theta)] \; \left| \theta^k (\theta - 1) \right| e_j^\top$$

$$\geq \int_0^2 e_i^\top \, \mathrm{d}[\eta(\theta)] \; \left| \theta^k (\theta - 1) \right| e_j^\top + \int_2^\Theta e_i^\top \, \mathrm{d}[\eta(\theta)] \; \theta^k e_j^\top$$

and taking limits for $h \to 0$ and $\Theta \to \infty$ yields

$$\left| e_i^\top V^{(k+1)}(0) e_j^\top \right| \geq \int_0^2 e_i^\top \, \mathrm{d}[\eta(\theta)] \; \left| \theta^k (\theta - 1) \right| e_j^\top + e_i^\top \int_2^\infty \mathrm{d}[\eta(\theta)] \; \theta^k \; e_j^\top$$

whence (5.54).

STEP 3: Step 1 and 2 have shown that

$$\forall \, k \in \mathbb{N}_0 : \; V^{(k)}(0) = \lim_{z \to 0+} V^{(k)}(z) \overset{(5.53)}{=} (-1)^k \lim_{z \to 0+} \int_0^\infty \mathrm{d}[\eta(\theta)] \; \theta^k e^{-z\theta}$$

$$\overset{(5.54)}{=} (-1)^k \int_0^\infty \mathrm{d}[\eta(\theta)] \; \theta^k. \quad (5.56)$$

Since the Taylor series expansion about 0 of

$$V(0) = \sum_{k \geq 0} \frac{V^{(k)}(0)}{k!} z^k \quad \text{is absolutely convergent in } \overline{\mathcal{B}_\alpha(0)} \text{ for some } \alpha \in (0, \gamma),$$

it follows that, in view of the increasing property of $\eta(\cdot)$,

$$\int_0^\infty \mathrm{d}[\eta(\theta)] \; e^{\alpha\theta} = \int_0^\infty \mathrm{d}[\eta(\theta)] \; \sum_{k \geq 0} \frac{(\alpha\theta)^k}{k!} = \sum_{k \geq 0} \int_0^\infty \mathrm{d}[\eta(\theta)] \; \frac{(\alpha\theta)^k}{k!}$$

$$= \sum_{k \geq 0} \left[(-1)^k \int_0^\infty \mathrm{d}[\eta(\theta)] \; \theta^k \right] (-1)^k \frac{\alpha^k}{k!} \overset{(5.56)}{=} \sum_{k \geq 0} \frac{(-1)^k V^{(k)}(0)}{k!} \frac{-\alpha^k}{k!}.$$

Since the latter summand is absolutely converging, claim (5.52) follows. This completes the proof. □

5.7.3 Proofs of Section 5.4

Proof of Proposition 5.4.4:
The notation of Proposition 5.4.2 is used.

"(ii) ⇔ (iii)": It holds, for all $s \in \mathbb{C}_0$,

$$\begin{aligned}
&\left|\det\begin{pmatrix} sI_n - A - \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s\theta} & B \\ C & 0 \end{pmatrix}\right| \\
&= \left|\det\left[\begin{pmatrix} T^{-1} & 0 \\ 0 & I_{n-m} \end{pmatrix}\begin{pmatrix} sI_n - A - \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s\theta} & B \\ C & 0 \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & I_{n-m} \end{pmatrix}\right]\right| \\
&= \left|\det\begin{pmatrix} sI_n - T^{-1}AT - \int_0^\infty \mathrm{d}[T^{-1}\eta(\theta)T]\, e^{-s\theta} & T^{-1}B \\ CT & 0 \end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} sI_m - \widehat{A}_1 - \int_0^\infty \mathrm{d}[\widehat{\eta}_1(\theta)]\, e^{-s\theta} & -\widehat{A}_2 - \int_0^\infty \mathrm{d}[\widehat{\eta}_2(\theta)]\, e^{-s\theta} & CB \\ -\widehat{A}_3 - \int_0^\infty \mathrm{d}[\widehat{\eta}_3(\theta)]\, e^{-s\theta} & sI_{n-m} - \widehat{A}_4 - \int_0^\infty \mathrm{d}[\widehat{\eta}_4(\theta)]\, e^{-s\theta} & 0 \\ I_m & 0 & 0 \end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} -\widehat{A}_2 - \int_0^\infty \mathrm{d}[\widehat{\eta}_2(\theta)]\, e^{-s\theta} & CB \\ sI_{n-m} - \widehat{A}_4 - \int_0^\infty \mathrm{d}[\widehat{\eta}_4(\theta)]\, e^{-s\theta} & 0 \end{pmatrix}\right| \\
&= |\det(CB)|\, \left|\det\left(sI_{n-m} - \widehat{A}_4 - \int_0^\infty \mathrm{d}[\widehat{\eta}_4(\theta)]\, e^{-s\theta}\right)\right|
\end{aligned}$$

Now the claim follows from Proposition 5.3.2.

"(i) ⇒ (iii)": Let $y(\cdot) \equiv 0$. Then the second equation of (5.15) with $\sigma = 0$ becomes (5.17) and its unique solution $z(\cdot; 0, z^0)$ belongs to $\mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n-m})$, by Definition 5.4.1. Therefore, the resolvent of (5.17) belongs to $\mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{(n-m)\times(n-m)})$ (see Remark 5.2.5) and thus (5.17) is uniformly asymptotically stable by Proposition 5.3.2.

"(iii) ⇒ (i)": Suppose $y(\cdot) \equiv 0$ in (5.13). Then (5.15) gives

$$u(t) \;=\; -(CB)^{-1}\left[\widehat{A}_2 z(t) + \int_0^t \mathrm{d}[\widehat{\eta}_2(\theta)]\, z(t-\theta)\right], \qquad \text{for a.a. } t \geq 0 \quad (5.57)$$

$$\dot{z}(t) \;=\; \widehat{A}_4 z(t) + \int_0^t \mathrm{d}[\widehat{\eta}_4(\theta)]\, z(t-\theta), \qquad \text{for a.a. } t \geq 0. \quad (5.58)$$

It has to be noted that the homogeneous part of the second equation of (5.15) is equivalent to (5.58). Hence uniform asymptotic stability of (5.17) yields, by Corollary 5.3.3, that $z(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n-m})$ and $\lim_{t\to\infty} z(t) = 0$. Now (i) is immediate from $x = [B(CB)^{-1}, V](y^\top, z^\top)^\top$.

Moreover, since $z(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n-m})$ and $\lim_{t\to\infty} z(t) = 0$, Lemma 5.7.1 (iii), (iv) applied to (5.57) gives

$$u(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \qquad \text{and} \qquad \lim_{t\to\infty} u(t) = 0.$$

This shows the last statement of the proposition and completes the proof. □

Proof of Lemma 5.4.5:
The notation of Proposition 5.4.2 is used.
Consider system (5.13) and let the nominal system data of A and $\eta(\cdot)$ be partitioned as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \qquad \text{and} \qquad \eta(\cdot) = \begin{bmatrix} \eta_1(\cdot) & \eta_2(\cdot) \\ \eta_3(\cdot) & \eta_4(\cdot) \end{bmatrix}, \quad (5.59)$$

where $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times(n-m)}$, $A_3 \in \mathbb{R}^{(n-m)\times m}$, $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $\eta_1(\cdot) \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times m})$, $\eta_2(\cdot) \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times(n-m)})$, $\eta_3^\top(\cdot) \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{(n-m)\times m})$, $\eta_4(\cdot) \in \mathcal{NBV}(\mathbb{R}_{\geq 0}, \mathbb{R}^{(n-m)\times(n-m)})$.

For V, N and T as in (5.14) it follows

$$T = [B(CB)^{-1}, V] = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & I_{n-m} \end{bmatrix}, \qquad N = \begin{bmatrix} 0, & I_{n-m} \end{bmatrix}$$

and the coordinate transformation $(y^\top, z^\top)^\top = T^{-1}x$ leaves A_4 and $\eta_4(\cdot)$ invariant, i.e.

$$A_4 = \widehat{A}_4, \quad \eta_4(\cdot) = \widehat{\eta}_4(\cdot),$$

which gives

$$\begin{aligned} \text{hom. part of (5.13) is positive} &\overset{\text{Th. 5.2.8}}{\Longleftrightarrow} A \text{ is Metzler and } \eta(\cdot) \text{ is non-decreasing} \\ &\Longrightarrow A_4 \text{ is Metzler and } \eta_4(\cdot) \text{ is non-decreasing} \\ &\overset{\text{Th. 5.2.8}}{\Longleftrightarrow} \text{(5.17) is positive} \end{aligned}$$

This completes the proof. □

Proof of Proposition 5.4.7:
The following holds:

$$\begin{aligned} &\text{(5.13) has stable zero dynamics} \\ &\overset{\text{Prop. 5.4.4}}{\Longleftrightarrow} \text{(5.17) is uniformly asymptotically stable} \\ &\overset{\text{Th. 5.3.4}}{\Longleftrightarrow} \mu\left(NAV + \int_0^\infty \mathrm{d}[N\eta(\theta)V]\right) < 0 \\ &\overset{\text{Lem. 5.7.2}}{\Longleftrightarrow} \exists\, p \in \mathbb{R}^{n-m}_{>0} : \left(NAV + \int_0^\infty \mathrm{d}[N\eta(\theta)V]\right) p \in \mathbb{R}^{n-m}_{<0}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5.4.9:
Let A and $\eta(\cdot)$ be partitioned as in (5.59). The spectral abscissa of the generator of the closed-loop system (5.13), (5.20) is given by

$$\widehat{A} := \begin{bmatrix} A_1 - \text{diag}\,(k_1\, b_1\, c_1, ..., k_m\, b_m\, c_m) + \int_0^\infty \mathrm{d}[\eta_1(\theta)] & A_2 + \int_0^\infty \mathrm{d}[\eta_2(\theta)] \\ A_3 + \int_0^\infty \mathrm{d}[\eta_3(\theta)] & A_4 + \int_0^\infty \mathrm{d}[\eta_4(\theta)] \end{bmatrix}. \tag{5.60}$$

"(i) ⇒ (ii)": Proposition 5.4.7 implies that

$$\exists\, p_2 \in \mathbb{R}^{n-m}_{>0} : \quad \left(A_4 + \int_0^\infty \mathrm{d}[\eta_4(\theta)]\right) p_2 \in \mathbb{R}^{n-m}_{<0},$$

and so

$$\exists\, \alpha > 0 : \quad \left[A_3 + \int\limits_0^\infty \mathrm{d}[\eta_3(\theta)]\,,\; A_4 + \int\limits_0^\infty \mathrm{d}[\eta_4(\theta)] \right] \begin{pmatrix} \alpha\, \mathbf{1}_m \\ p_2 \end{pmatrix} \in \mathbb{R}^{n-m}_{<0},$$

whence

$$\exists\, k^* \geq 0 \;\forall\, k_1, \ldots, k_m \geq k^* : \quad \widehat{A} \begin{pmatrix} \alpha\, \mathbf{1}_m \\ p_2 \end{pmatrix} \in \mathbb{R}^n_{<0}.$$

Now Assertion (ii) follows from Theorem 5.3.4 (i).

"(ii) ⇒ (i)": Since (5.13) is positive and uniformly asymptotically stable,

$$\exists \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^n_{>0} : \quad \widehat{A} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^n_{<0},$$

by Theorem 5.3.4 (i) and Lemma 5.7.2, whence Then

$$\left[A_3 + \int\limits_0^\infty \mathrm{d}[\eta_3(\theta)] \right] p_1 + \left[A_4 + \int\limits_0^\infty \mathrm{d}[\eta_4(\theta)] \right] p_2 \in \mathbb{R}^{n-m}_{<0}. \tag{5.61}$$

Since the homogeneous part of (5.13) is positive, Theorem 5.2.8 yields that the matrix in (5.60) is a Metzler matrix, and therefore $A_4 + \int\limits_0^\infty \mathrm{d}[\eta_4(\theta)]$ is a Metzler matrix and $A_3 + \int\limits_0^\infty \mathrm{d}[\eta_3(\theta)] \in \mathbb{R}^{(n-m)\times m}_{\geq 0}$. Thus (5.61) gives $\left[A_4 + \int\limits_0^\infty \mathrm{d}[\eta_4(\theta)] \right] p_2 \in \mathbb{R}^{(n-m)\times(n-m)}_{<0}$. Now Assertion (i) follows from Theorem 5.3.4 (i) and Lemma 5.7.2.

Finally, it is shown that

$$\forall\, \sigma \geq 0 \;\forall\, \phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0}) \;\forall\, t \geq 0 : \quad x(t;\sigma,\phi) \geq 0.$$

By Proposition 5.2.4, there exists a unique solution $x(\cdot) := x(\cdot;\sigma,\phi)$ of the initial value problem

$$\dot{x}(t) = [A - B\, \mathrm{diag}\,(k_1, ..., k_m)\, C]\, x(t) + \int\limits_0^t \mathrm{d}[\eta(\theta)]\, x(t-\theta), \qquad x|_{[0,\sigma]} = \phi, \;\text{ for a.a. } t \geq \sigma.$$

Proposition 5.2.7 with $f(\cdot) \equiv 0$ yields the result and completes the proof. □

5.7.4 Proofs of Section 5.5

Proof of Lemma 5.5.4:

(i) Clearly, (5.15) may be written as

$$\begin{aligned} \dot y(t) &= [(\widehat A_1,\widehat\eta_1) * y](t) + [(\widehat A_2,\widehat\eta_2) * z](t) + CBu(t) \\ \dot z(t) &= [(\widehat A_3,\widehat\eta_3) * y](t) + [(\widehat A_4,\widehat\eta_4) * z](t), \qquad \text{for a.a. } t \ge \sigma \\ y|_{[0,\sigma]} &= C\phi(\cdot), \quad z|_{[0,\sigma]} = N\phi(\cdot) \end{aligned}$$

and (5.7) yields, for almost all $t \ge \sigma$,

$$\begin{aligned} z(t; z^0, y) &= R_4(t-\sigma)N\phi(\sigma) + \int_0^{t-\sigma} R_4(t-\sigma-s)\,[(\widehat A_3,\widehat\eta_3) * y](s+\sigma)\,\mathrm{d}s \\ &\quad + \int_0^{t-\sigma} R_4(t-\sigma-s)\left(\int_s^{s+\sigma} \mathrm{d}[\widehat\eta_4(\theta)]\,N\phi(s+\sigma-\theta)\right)\mathrm{d}s\,, \end{aligned}$$

whence, together with the first equation of (5.15), Assertion (i) follows.

(ii) It is a straightforward calculation to see that p and T as defined in (i) satisfy the bounds in (ii).
This completes the proof of the lemma.

□

Proof of Theorem 5.5.1:
(5.22) implies that $\det CB \neq 0$. The notation of Proposition 5.4.2, Remark 3.3.1 and Lemma 5.5.4 is used.

STEP 1: *It is shown that there exists a maximal solution* $x : [0,\omega) \to \mathbb{R}^n$*,* $\omega \in (\sigma,\infty]$*, of the closed-loop system* (5.13)*,* (5.24).
Some care must be exercised in formulating the initial value problem (5.13), (5.24). In view of Lemma 5.5.4 (i), it remains to show that the closed-loop system (5.28), (5.24) has a maximal solution. Define the relatively open set

$$\mathcal{D} := \left\{(t,\mu) \in [\sigma,\infty) \times \mathbb{R}^m \mid \forall\, i \in \{1,...,m\} :\ (t, \mu_i - y_{\mathrm{ref},i}(t)) \in \mathcal{F}(\sigma,\varphi_i)\right\}.$$

For the operator T, the following holds:

(a) It is easy to see that T is a causal operator.

(b) It is a straightforward calculation to see that T satisfies:
For $t \geq \sigma$ and all $\zeta(\cdot) \in \mathcal{C}([0,t],\mathbb{R}^m)$, there exist $\tau > t$, $\delta > 0$ and $c > 0$ such that, for all $y(\cdot)$, $z(\cdot) \in \mathcal{C}([0,\tau],\mathbb{R}^m)$ with $y|_{[0,t]} = \zeta = z|_{[0,t]}$ and $y(s)$, $z(s) \in \mathcal{B}_\delta(\zeta(t))$ for all $s \in [t,\tau]$,

$$\text{ess-}\sup_{s\in[t,\tau]} \|(Ty)(s) - (Tz)(s)\|_{\mathrm{Op}} \leq c \sup_{s\in[t,\tau]} \|y(s) - z(s)\|.$$

(c) The second inequality of Lemma 5.5.4 (ii) gives that, for all $\delta > 0$, there exists $\Delta > 0$ such that, for all $y(\cdot) \in \mathcal{C}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$,

$$\sup_{t\in\mathbb{R}_{\geq 0}} \|y(t)\| \leq \delta \quad \Rightarrow \quad \|(Ty)(t)\|_{\mathrm{Op}} \leq \Delta \quad \text{for a.a. } t \geq \sigma.$$

This, together with Remark 5.2.3, shows that the operator T satisfies [43, Def. 1]. Moreover, Lemma 5.5.4 (i), (ii) implies that $p(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0},\mathbb{R}^m)$ and thus, (5.28) fulfills [43, Def. 3]. Applying [43, Th. 5] and Remark 5.2.3 to the initial value problem

$$\left.\begin{array}{rcl} \dot{e}(t) &=& p(t) + (T(e+y_{\mathrm{ref}}))(t) - \dot{y}_{\mathrm{ref}}(t) \\ && \quad -C_1B_1\mathrm{diag}\left(\frac{\varphi_1(t)}{1-\varphi_1(t)\,|e_1(t)|},\ldots,\frac{\varphi_m(t)}{1-\varphi_m(t)\,|e_m(t)|}\right)e(t), \text{ for a.a. } t \geq \sigma \\ e|_{[0,\sigma]} &=& C\phi(\cdot) - y_{\mathrm{ref}}(\cdot), \end{array}\right\} \quad (5.62)$$

the system (5.62) has a maximal solution $e : [0,\omega) \to \mathbb{R}^m$, $\sigma < \omega \leq \infty$, and $[\sigma,\omega) \times e([\sigma,\omega)) \subset \mathcal{D}$. Lemma 5.5.4 (i) yields that there exists a maximal solution $x(\cdot;\sigma,\phi)$ of the closed-loop system (5.13), (5.24).

STEP 2: *Some technical notation is introduced.*
The properties of $\mathcal{S}_1(\sigma)$, together with (5.23), give

$$\begin{array}{r} \exists\, \omega_0 \in [\sigma,\sigma+1)\ \forall\, t \in [\sigma,\omega_0]\ \forall\, i \in \{1,\ldots,m\}: \quad |e_i(t)| \leq |e_i(\sigma)| + 1 \quad \wedge \\ 1 - \varphi_i(t)|e_i(t)| \geq \dfrac{1-\varphi_i(\sigma)|e_i(\sigma)|}{2} > 0. \end{array} \quad (5.63)$$

This also shows that $e_i(\cdot)$, $i = 1,\ldots,m$, evolves in the funnel $\mathcal{F}(\sigma,\varphi_i)$ for all $t \in [\sigma,\omega_0]$. Then, in view of Remark 3.3.1, for $\lambda_i := \inf_{t\in[\omega_0,\infty)} \varphi_i(t)^{-1} > 0$, $i = 1,\ldots,m$, it holds $\frac{1}{\varphi_i(\cdot)} \in \mathcal{L}^\infty([\omega_0,\infty),[\lambda_i,\infty))$. For ease of notation the functions $\psi_i(\cdot) := \varphi_i|_{[\omega_0,\infty)}(\cdot)^{-1} \in \mathcal{L}^\infty([\omega_0,\infty),[\lambda_i,\infty))$, $i \in \{1,\ldots,m\}$, are defined. Remark 3.3.1 shows that there exist $\ell_i \geq 0$, $i = 1, ..., m$, such that

$$\forall i \in \{1, ..., m\} \text{ for a.a. } t \geq \omega_0 : \quad |\dot{\psi}_i(t)| \leq \ell_i.$$

Note that $k_i(\cdot) = \frac{\varphi_i(\cdot)}{1-\varphi_i(\cdot)|e_i(\cdot)|} = \frac{1}{\psi_i(\cdot)-|e_i(\cdot)|}$ on $[\omega_0, \omega)$, $i = 1, ..., m$. By (5.22), it is possible to choose some $\widehat{u} \in \mathbb{R}^m_{>0}$ such that

$$\forall\, i \in \{1, ..., m\} : \quad c_i b_i \widehat{u}_i > \widehat{p} \|N\phi\|_\infty + \widehat{T}\left[\|\psi\|_\infty + \|y_{\rm ref}\|_\infty\right] + \|\dot{y}_{\rm ref}\|_\infty + \ell_i, \quad (5.64)$$

where $\psi(\cdot) := (\psi_1(\cdot), ..., \psi_m(\cdot))^\top$.

STEP 3: *It is shown that the tracking error $e(\cdot)$ satisfies, for all $i \in \{1, ..., m\}$ and almost all $t \in [\omega_0, \omega)$,*

$$\operatorname{sgn} e_i(t) \dot{e}_i(t) < -\ell_i + c_i b_i \left[\widehat{u}_i - k_i(t) \operatorname{sgn} e_i(t) e_i(t)\right]. \quad (5.65)$$

Applying (5.28) yields, for almost all $t \in [\omega_0, \omega)$ and all $i \in \{1, ..., m\}$,

$$\begin{aligned}
\operatorname{sgn} e_i(t)\ \dot{e}_i(t) &\overset{(5.28)}{=} \operatorname{sgn} e_i(t) \left[p(t) + (T(e + y_{\rm ref}))(t) - \dot{y}_{\rm ref}(t)\right]_i \\
&\quad - \operatorname{sgn} e_i(t)\ c_i b_i k_i(t) e_i(t) \\
&\overset{(5.29)}{\leq} \widehat{p}\|z^0\|_\infty + \widehat{T}[\|\Psi\|_\infty + \|y_{\rm ref}\|_\infty] + \|\dot{y}_{\rm ref}\|_\infty - c_i b_i k_i(t) \operatorname{sgn} e_i(t)\ e_i(t) \\
&\underset{(5.15)}{\overset{(5.64)}{<}} -\ell_i + c_i b_i \left[\widehat{u}_i - k_i(t) \operatorname{sgn} e_i(t)\ e_i(t)\right],
\end{aligned}$$

and whence (5.65).

STEP 4: *It is shown that*

$$\exists\, \widetilde{\varepsilon}_0 > 0\ \forall\, i \in \{1, ..., m\}\ \forall\, t \in [\omega_0, \omega) : \ \psi_i(t) - |e_i(t)| \geq \widetilde{\varepsilon}_0. \quad (5.66)$$

Let $\widehat{p}$ and $\widehat{T}$ be defined as in (5.29). Write, for $\widehat{u} \in \mathbb{R}^m_{>0}$ as in (5.64),

$$\widetilde{\varepsilon}_0 := \min\left\{ \min_{j \in \{1,...,m\}} \frac{\lambda_j}{2},\ \min_{j \in \{1,...,m\}} \frac{\lambda_j}{2\widehat{u}_j},\ \psi_i(\omega_0) - |e_i(\omega_0)| \right\}. \quad (5.67)$$

Seeking a contradiction, suppose that

$$\exists\, j \in \{1, ..., m\}\ \exists\, t_1 \in [\omega_0, \omega) : \quad \psi_j(t_1) - |e_j(t_1)| < \widetilde{\varepsilon}_0.$$

Since $t \mapsto \psi_j(t) - |e_j(t)|$ is continuous on $[\omega_0, \omega)$, (5.67) ensures that the number

$$t_0 := \max\{t \in [\omega_0, t_1) \mid \psi_j(t) - |e_j(t)| = \widetilde{\varepsilon}_0\}$$

is well defined and it follows that

$$\forall\, t \in [t_0, t_1]: \quad \psi_j(t) - |e_j(t)| \le \widetilde{\varepsilon}_0 \quad \wedge \quad \frac{\min\limits_{i\in\{1,\dots,m\}} \lambda_i}{2} \le \frac{\lambda_j}{2} \overset{(5.67)}{\le} |e_j(t)|$$

and thus

$$\forall\, t \in [t_0, t_1]: \quad k_j(t)|e_j(t)| \;\ge\; \widetilde{\varepsilon}_0^{-1} \frac{\min\limits_{i\in\{1,\dots,m\}} \lambda_i}{2} \overset{(5.67)}{\ge} \widehat{u}_j$$

whence, since $\operatorname{sgn} e_j(\cdot)$ is constant on $[t_0, t_1]$,

$$\forall\, t \in [t_0, t_1]: \quad k_j(t) \operatorname{sgn} e_j(t)\; e_j(t) = k_j(t)|e_j(t)| \ge \widehat{u}_j,$$

and so, in view of (5.65),

$$\operatorname{sgn} e_j(t)\; \dot{e}_j(t) < -\ell_j \qquad \text{for almost all } t \in [t_0, t_1].$$

Integration gives

$$|e_j(t_1)| - |e_j(t_0)| = \int_{t_0}^{t_1} \operatorname{sgn} e_j(\tau)\; \dot{e}_j(\tau)\, \mathrm{d}\tau \;<\; -\ell_j(t_1 - t_0)$$

whence the contradiction

$$\begin{aligned} 0 < \psi_j(t_0) - |e_j(t_0)| - [\psi_j(t_1) - |e_j(t_1)|] &= \psi_j(t_0) - \psi_j(t_1) + [|e_j(t_1)| - |e_j(t_0)|] \\ &< \ell_j(t_1 - t_0) - \ell_j(t_1 - t_0) = 0. \end{aligned}$$

This proves (5.66).

STEP 5: *Assertions (ii) and (iii) are shown and that $x(\cdot)$ is a global solution, i.e. $\omega = \infty$.*
Step 2 and Step 3 guarantee that $(t, e_i(t)) \in \mathcal{F}(\sigma, \varphi_i)$ for all $t \in [\sigma, \omega)$ and $i = 1, \dots, m$. Step 2 ensures that $k_i(\cdot)$, $i = 1, \dots, m$, are uniformly bounded on $[\sigma, \omega_0]$ and Step 3 shows that $\|k_i\|_{\mathcal{L}^\infty([\omega_0,\omega))} \le \frac{1}{\widetilde{\varepsilon}_0}$ and thus it follows that $u_i(\cdot)$ are bounded on $[\sigma, \omega)$. Moreover, Step 2 and Step 3 give

$$\begin{aligned} \forall\, i \in \{1, \dots, m\}\; \forall\, t \in [\sigma, \omega): \quad 1 - \varphi_i(t)|e_i(t)| &\ge \min\left\{ \frac{1 - \varphi_i(\sigma)|e_i(\sigma)|}{2}, \widetilde{\varepsilon}_0 \right\} > 0 \\ \wedge \quad 0 \le k_i(t) &\le \max\left\{ \frac{2\|\varphi_i\|_\infty}{1 - \varphi_i(\sigma)|e_i(\sigma)|}, \frac{1}{\widetilde{\varepsilon}_0} \right\}. \end{aligned}$$

Since $e(\cdot)$ and $y_{\text{ref}}(\cdot)$ are bounded on $[0,\omega)$, so is $y(\cdot)$. By supposition and Proposition 5.4.7, it follows that $\mu\left(\widehat{A}_4 + \int_0^\infty \mathrm{d}[\widehat{\eta}_4(\theta)]\right) < 0$ and so (5.17) is uniformly asymptotically stable. Now Corollary 5.3.3 applied to the second differential equation in (5.15) ensures boundedness of $z(\cdot)$ on $[0,\omega)$. Thus $x(\cdot)$ is bounded on $[0,\omega)$. To establish Assertions (ii), (iii) and $x(\cdot)$ is a global solution, it remains only to show that $\omega = \infty$. Suppose that $\omega < \infty$ and define

$$\mathcal{C} := \{(t,e) \in [\sigma,\infty) \times \mathbb{R}^m \mid \forall\, i \in \{1,...,m\} :\ \varphi_i(t)|e_i| \le 1 - \widetilde{\varepsilon}_0\}.$$

Then $\mathcal{C}$ is a compact subset of $\mathcal{D}$ with the property $(t,e(t)) \in \mathcal{C}$ for all $t \in [\sigma,\omega)$, which contradicts the fact that, by [43, Th. 5], there exist $t' \in [\sigma,\omega)$ such that $(t',e(t')) \notin \mathcal{C}$. Therefore, $\omega = \infty$.

STEP 6: *Uniqueness of the solution* $y : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ *of the initial value problem* (5.28)*,* (5.24) *is shown which proves Assertion (i).*
Assume that $v(\cdot)$ also solves (5.28) on $\mathbb{R}_{\ge 0}$. Let $S \in (\sigma,\infty)$ arbitrary but fixed. Then, for all $t \in [\sigma,S]$ and $\zeta \in \{y,v\}$,

$$\zeta(t) = y^0(\sigma) + \int_\sigma^t \left([p(\tau) + (T\zeta)(\tau)] - \begin{bmatrix} c_1 b_1 \varphi_1(\tau) \frac{\zeta_1(\tau) - y_{\text{ref},1}(\tau)}{1-\varphi_1(\tau)|\zeta_1(\tau)-y_{\text{ref},1}(\tau)|} \\ \vdots \\ c_m b_m \varphi_m(\tau) \frac{\zeta_m(\tau) - y_{\text{ref},m}(\tau)}{1-\varphi_m(\tau)|\zeta_m(\tau)-y_{\text{ref},m}(\tau)|} \end{bmatrix} \right) \mathrm{d}\tau$$

and, by linearity of T and with $\widehat{T}$ as in (5.29) and Lemma 5.5.4 (ii), it follows that, for all $t \in [\sigma,S]$,

$$\begin{aligned} \|y(t) - v(t)\| \le \widehat{T} \int_\sigma^t \max_{s\in[0,\tau]} \|y(s)-v(s)\| \,\mathrm{d}\tau \\ + \max_{i=1,\dots,m}(c_i b_i) \int_\sigma^t \max_{i=1,\dots,m} \max_{s\in[\sigma,\tau]} \Bigg[\Bigg\| \varphi_i(s) \Bigg(\frac{y_i(s) - y_{\text{ref},i}(s)}{1-\varphi_i(s)|y_i(s)-y_{\text{ref},i}(s)|} \\ - \frac{v_i(s) - y_{\text{ref},i}(s)}{1-\varphi_i(s)|v_i(s)-y_{\text{ref},i}(s)|} \Bigg) \Bigg\| \Bigg] \,\mathrm{d}\tau\,. \end{aligned}$$

Note that, by Step 4, $\frac{\varphi_i(\cdot)}{1-\varphi_i(\cdot)|y_i(\cdot)|}$ and $\frac{\varphi_i(\cdot)}{1-\varphi_i(\cdot)|v_i(\cdot)|}$ are bounded and bounded away from zero. A simple calculation gives, together with Assertion (ii), (the argument s is omitted for brevity)

$$\varphi_i \left[\frac{y_i - y_{\text{ref},i}}{1-\varphi_i|y_i - y_{\text{ref},i}|} - \frac{v_i - y_{\text{ref},i}}{1-\varphi_i|v_i - y_{\text{ref},i}|} \right]$$

$$
\begin{aligned}
&= \varphi_i \frac{[1-\varphi_i|v_i - y_{\mathrm{ref},i}|](y_i - y_{\mathrm{ref},i}) - [1-\varphi_i|y_i - y_{\mathrm{ref},i}|](v_i - y_{\mathrm{ref},i})}{[1-\varphi_i|y_i - y_{\mathrm{ref},i}|]\,[1-\varphi_i|v_i - y_{\mathrm{ref},i}|]}\\
&= \varphi_i \frac{[1-\varphi_i|y_i - y_{\mathrm{ref},i}|](y_i - v_i) + \varphi_i(y_i - y_{\mathrm{ref},i})[|y_i - y_{\mathrm{ref},i}| - |v_i - y_{\mathrm{ref},i}|]}{[1-\varphi_i|y_i - y_{\mathrm{ref},i}|]\,[1-\varphi_i|v_i - y_{\mathrm{ref},i}|]}\\
&\leq \varphi_i \frac{[1-\varphi_i|y_i - y_{\mathrm{ref},i}|](y_i - v_i) + \varphi_i(y_i - y_{\mathrm{ref},i})|y_i - v_i|}{[1-\varphi_i|y_i - y_{\mathrm{ref},i}|]\,[1-\varphi_i|v_i - y_{\mathrm{ref},i}|]}\\
&\leq \|\varphi_i\|_\infty \varepsilon_0^{-1} \frac{[1-\varphi_i|y_i - y_{\mathrm{ref},i}|] + \varphi_i(y_i - y_{\mathrm{ref},i})}{1-\varphi_i|y_i - y_{\mathrm{ref},i}|} |y_i - v_i|\\
&\leq \frac{\|\varphi_i\|_\infty}{\varepsilon_0}\left[1 + \frac{\|\varphi_i\|_\infty}{\varepsilon_0}(y_i - y_{\mathrm{ref},i})\right]|y_i - v_i|.
\end{aligned}
$$

Now, by Step 5, boundedness of $y(\cdot)$ and $y_{\mathrm{ref}}(\cdot)$ imply that

$$\exists\, m>0\ \forall\, t\in[0,S]:\quad \|y(t)-v(t)\| \leq m\int_0^t \max_{s\in[0,\tau]}\|y(s)-v(s)\|\,\mathrm{d}\tau\,.$$

The continuous function

$$\gamma:[0,S]\to\mathbb{R}_{\geq 0},\qquad t\mapsto\gamma(t):=\max_{s\in[0,t]}\|y(s)-v(s)\|$$

satisfies

$$\forall\, t\in[0,S]:\quad 0\leq\gamma(t)\leq m\int_0^t\gamma(\tau)\,\mathrm{d}\tau$$

and therefore Gronwall's inequality (see [88, Lem. VII.29.VI]) gives $\gamma(\cdot)\equiv 0$. Since $S\in(\sigma,\infty)$ is arbitrary, it follows that $y(\cdot)=v(\cdot)$ on $\mathbb{R}_{\geq 0}$. This completes the proof of Step 5.

STEP 7: *Assertion (iv) is shown.*
Positivity of the homogeneous part of (5.13) implies, in view of Theorem 5.2.8, that $A\in\mathbb{R}^{n\times n}$ is a Metzler matrix and $\eta(\cdot)$ is a non-decreasing matrix function on $\mathbb{R}_{\geq 0}$. The closed-loop system (5.13), (5.24) may be written as

$$\dot{x}(t) = [A - B\,\mathrm{diag}\,(k_1(t),...,k_m(t))]\,x(t) + \int_0^t \mathrm{d}[\eta(\theta)]\,x(t-\theta) + B\mathrm{diag}\,(k_1(t),...,k_m(t))Cy_{\mathrm{ref}}(t), \quad (5.68)$$

Define

$$A(\cdot) := A - B\,\mathrm{diag}\,(k_1(\cdot),...,k_m(\cdot))\,C \quad\text{and}\quad f(\cdot) := B\,\mathrm{diag}\,(k_1(\cdot),...,k_m(\cdot))\,y_{\mathrm{ref}}(\cdot),$$

which both depend on $x(\cdot)$, and consider the initial value problem

$$\dot{v}(t) = A(t)v(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, v(t-\theta) + f(t), \qquad v|_{[0,\sigma]} = \phi. \tag{5.69}$$

Since $A(t)$ is a Metzler matrix for every $t \geq \sigma$ (see (5.22)) and $f(\cdot) \geq 0$, Proposition 5.2.7 yields that the solution $v(\cdot)$ of (5.69) is non-negative. Finally, by uniqueness of the solutions $x(\cdot)$ and $v(\cdot)$ of (5.68) and (5.69), resp., and identical initial data, they coincide. This proves Assertion (iv) and completes the proof of the theorem. □

Proof of Theorem 5.5.5:
(5.22) implies that $\det CB \neq 0$. The notation of Proposition 5.4.2, Remark 3.3.1, Lemma 5.5.4 and the technical notation of Step 2 of the proof of Theorem 5.5.1 is used. The structure of the proof closely resembles that of Theorem 5.5.1. It is not included a full proof, instead, the essential differences are presented.

STEP 1: *It is shown that there exists a maximal solution $x : [0,\omega) \to \mathbb{R}^n$, $\omega \in (\sigma, \infty]$, of the closed-loop system* (5.13)*,* (5.32)*.*
It remains to show that the closed-loop system (5.28), (5.32) has a maximal solution (see Lemma 5.5.4 (i)). Let $\mathcal{D}$ be defined as in Theorem 5.5.1 (see Step 1). As in Step 1 of the proof of Theorem 5.5.1, the initial value problem

$$\left.\begin{aligned} \dot{e}(t) &= p(t) + (T(e+r))(t) - \dot{r}(t) \\ &\quad -C_1 B_1 \begin{pmatrix} \mathrm{sat}_{\widehat{u}_1}\left(\frac{e_1(t)}{\psi_1(t)-|e_1(t)|} - d_{u,1}(t)\right) \\ \vdots \\ \mathrm{sat}_{\widehat{u}_m}\left(\frac{e_m(t)}{\psi_m(t)-|e_m(t)|} - d_{u,m}(t)\right) \end{pmatrix}, \text{ for a.a. } t \geq \sigma \\ e|_{[0,\sigma]} &= C\phi(\cdot) - y_{\mathrm{ref}}(\cdot), \end{aligned}\right\} \tag{5.70}$$

has a maximal solution $e : [0,\omega) \to \mathbb{R}^m$, $\sigma < \omega \leq \infty$, and $[\sigma,\omega) \times e([\sigma,\omega)) \subset \mathcal{D}$. Lemma 5.5.4 (i) proves Step 1.

STEP 2: *It is shown that the tracking error $e(\cdot)$ satisfies, for all $i \in \{1,...,m\}$ and almost all $t \in [\sigma,\omega)$,*

$$\forall\, i \in \{1,...,m\}: \quad \mathrm{sgn}\, e_i(t)\dot{e}_i(t) < \\ -\ell_i + (CB)_{ii}\left[\widehat{u}_i - \mathrm{sgn}\, e_i(t)\mathrm{sat}_{\widehat{u}_i}(k_i(t)e_i(t) - d_{u,i}(t))\right]. \tag{5.71}$$

It has to be noted that $\varphi_i(\cdot) \in \mathcal{G}_1(\sigma)$, $i = 1, ..., m$, implies that $\omega_0 = \sigma$ in Step 2 of the proof of Theorem 5.5.1 (see Remark 3.3.1). Applying (5.28) to (5.70) yields, for almost all $t \in [\sigma, \omega)$ and all $i \in \{1, ..., m\}$,

$$\begin{aligned}
\operatorname{sgn} e_i(t)\ \dot{e}_i(t) &\overset{(5.28)}{=} \operatorname{sgn} e_i(t) \Big[p(t) + (T(e+r))(t) - \dot{r}(t)\Big]_i - \\
&\qquad \operatorname{sgn} e_i(t) \left[C_1\, B_1 \begin{pmatrix} \mathrm{sat}_{\widehat{u}_1}\left(\frac{e_1(t)}{\psi_1(t)-|e_1(t)|} - d_{u,1}(t)\right) \\ \vdots \\ \mathrm{sat}_{\widehat{u}_m}\left(\frac{e_m(t)}{\psi_m(t)-|e_m(t)|} - d_{u,m}(t)\right) \end{pmatrix} \right]_i \\
&\overset{(5.29)}{\leq} \widehat{p}\, \|z^0\|_\infty + \widehat{T}\, [\|\psi\|_\infty + \|r\|_\infty] + \|\dot{r}\|_\infty - \\
&\qquad \operatorname{sgn} e_i(t) \sum_{j=1}^m (C_1\, B_1)_{ij} \mathrm{sat}_{\widehat{u}_j}(k_j(t) e_j(t) - d_{u,j}(t)) \\
&\underset{(5.15)}{\overset{(5.31)}{<}} -\ell_i + (C_1\, B_1\, \widehat{u})_i - \operatorname{sgn} e_i(t) \sum_{j=1}^m (C_1\, B_1)_{ij} \mathrm{sat}_{\widehat{u}_j}(k_j(t) e_j(t) \\
&\qquad -d_{u,j}(t)) \\
&\overset{(3.34)}{=} -\ell_i + (C_1\, B_1)_{ii} \widehat{u}_i - \sum_{j=1,\ j\neq i}^m |(C_1 B_1)_{ij}| \widehat{u}_j - \\
&\qquad \operatorname{sgn} e_i(t) \sum_{j=1}^m (C_1\, B_1)_{ij} \mathrm{sat}_{\widehat{u}_j}(k_j(t) e_j(t) - d_{u,j}(t)) \\
&\leq -\ell_i + (C_1\, B_1)_{ii} \left[\widehat{u}_i - \operatorname{sgn} e_i(t)\ \mathrm{sat}_{\widehat{u}_i}(k_i(t) e_i(t) - d_{u,j}(t))\right],
\end{aligned}$$

and so, (5.71) follows.

STEP 3: *It is shown that*

$$\forall\, i \in \{1, ..., m\}\ \forall\, t \in [\sigma, \omega):\ \psi_i(t) - |e_i(t)| \geq \varepsilon_0. \tag{5.72}$$

Seeking a contradiction, suppose that

$$\exists\, j \in \{1, ..., m\}\ \exists\, t_1 \in [\sigma, \omega): \quad \psi_j(t_1) - |e_j(t_1)| < \varepsilon_0.$$

Let t_0 be defined as in Step 4 of the proof of Theorem 5.5.1. Then it follows that, for all $t \in [t_0, t_1]$,

$$\psi_j(t) - |e_j(t)| \leq \varepsilon_0 \quad \wedge \quad \frac{\min\limits_{i \in \{1,...,m\}} \lambda_i}{2} \leq \frac{\lambda_j}{2} \overset{(5.33)}{\leq} |e_j(t)|$$

$$\wedge \quad k_j(t)|e_j(t)| \;\geq\; \varepsilon_0^{-1} \frac{\min_{i\in\{1,\dots,m\}} \lambda_i}{2} \overset{(5.33)}{\geq} \widehat{u}_j$$

whence, since $\operatorname{sgn} e_j(\cdot)$ is constant on $[t_0, t_1]$,

$$\forall\, t \in [t_0, t_1] : \quad \operatorname{sgn} e_j(t)\, \mathrm{sat}_{\widehat{u}_j}(k_j(t)e_j(t) - d_{u,j}(t)) = \widehat{u}_j,$$

and so, in view of (5.71),

$$\operatorname{sgn} e_j(t)\; \dot{e}_j(t) < -\ell_j \qquad \text{for almost all } t \in [t_0, t_1].$$

The contradiction follows similarly to Step 4 of the proof of Theorem 5.5.1.

STEP 4: Assertions (i) - (iii) follow as in Step 5 - 6 of the proof of Theorem 5.5.1 and are omitted for brevity.

STEP 5: *Assertion (iv) and (v) are shown.*
Define

$$\forall\, i \in \{1, ..., m\} : \quad \beta_i(\cdot) := k_i(\cdot)e_i(\cdot) - d_{u,i}(\cdot).$$

STEP 5A: *It is shown that for all $i \in \{1, ..., m\}$ the following implication holds:*

$$\begin{aligned} &[\; |e_i(t)| - |e_i(t_0)| < -\ell_i(t - t_0) \text{ for some } t > t_0 \geq \sigma \;] \\ &\qquad\qquad \Rightarrow\; [\; |\beta_i(t)| - |\beta_i(t_0)| < -\ell_i k_i(t)[t - t_0] + 2\|d_u\|_\infty \;]\,. \end{aligned} \tag{5.73}$$

If

$$|e_i(t)| - |e_i(t_0)| < -\ell_i(t - t_0) \qquad \text{for some } t > t_0 \geq \sigma,$$

then

$$\begin{aligned} &|k_i(t)e_i(t) - d_{u,i}(t)| - |k_i(t_0)e_i(t_0) - d_{u,i}(t_0)| \\ &\qquad \leq k_i(t)k_i(t_0)\left[|e_i(t)|\psi_i(t_0) - |e_i(t_0)|\psi_i(t)\right] + |d_{u,i}(t)| + |d_{u,i}(t_0)| \\ &\qquad < k_i(t)k_i(t_0)\left[(|e_i(t_0)| - \ell_i(t - t_0))\psi_i(t_0) - |e_i(t_0)|\psi_i(t)\right] + 2\|d_u\|_\infty \\ &\qquad = k_i(t)k_i(t_0)\left[((\psi_i(t_0) - \psi_i(t))|e_i(t_0)| - \ell_i(t - t_0)\psi_i(t_0)\right] + 2\|d_u\|_\infty \\ &\qquad \leq k_i(t)k_i(t_0)\left[\ell_i(t - t_0)|e_i(t_0)| - \ell_i(t - t_0)\psi_i(t_0)\right] + 2\|d_u\|_\infty \\ &\qquad = -\ell_i k_i(t)[t - t_0] + 2\|d_u\|_\infty, \end{aligned}$$

and so, (5.73) follows.

STEP 5B: Seeking a contradiction to all inputs $u_i(\cdot)$ are unsaturated at some time $\tau_i \geq \sigma$, suppose that

$$\exists\, j \in \{1,...,m\}\ \forall\, t \geq \sigma : \quad |k_j(t)e_j(t) - d_{u,j}(t)| \geq \widehat{u}_j.$$

$\widehat{u}_j > 0$ implies that $\operatorname{sgn}\beta_j(\cdot)$ is constant on $[\sigma, \infty)$. Then $\operatorname{sgn}\beta_j(t)\text{sat}_{\widehat{u}_j}(\beta_j(t)) = \widehat{u}_j$ for all $t \geq \sigma$ which yields with minor modifications of Step 2, for almost all $t \in [\sigma, \omega)$,

$$\operatorname{sgn}\beta_j(t)\dot{e}_j(t) < -\ell_j + (CB)_{jj}\left[\widehat{u}_j - \operatorname{sgn}\beta_j(t)\,\text{sat}_{\widehat{u}_j}(\beta_j(t))\right] = -\ell_j.$$

By integration, this gives the contradiction

$$\forall\, t \geq \sigma : \quad -\|\psi_j\|_\infty \leq -\psi_j(t) \leq \operatorname{sgn}\beta_j(\sigma)\, e_j(t) < \operatorname{sgn}\beta_j(\sigma)\, e_j(\sigma) - \ell_j\, t,$$

and so, Assertion (iv) follows.

STEP 5C: *It is shown that if $d_u(\cdot) \equiv 0$ and an input $u_i(\cdot)$ is unsaturated at some time $\tau_i \geq \sigma$, then it remains unsaturated thereafter.*
Seeking a contradiction, suppose there exist $j \in \{1,...,m\}$ and $t_1 > \tau_j \geq \sigma$ such that

$$|u_j(t_1)| = \widehat{u}_j.$$

Since $\widehat{u}_j - |u_j(\tau_j)| > 0$ for some $\tau_j \geq \sigma$, there exists $\delta \in \left(0,\ \frac{\widehat{u}_j - |u_j(\tau_j)|}{\widehat{u}_j}\right)$ such that

$$(CB\,\widehat{u})_j - \delta(CB)_{jj}\widehat{u}_j \geq \underbrace{\widehat{p}\|N\phi\|_\infty + \widehat{T}\left[\|\psi\|_\infty + \|r\|_\infty\right] + \|\dot{r}\|_\infty}_{=:L} + \ell_j. \tag{5.74}$$

It has to be noted that

$$|k_j(\tau_j)e_j(\tau_j)| = |\beta_j(\tau_j)| = |u_j(\tau_j)| < (1-\delta)\widehat{u}_j$$

which yields that there exists $t_0 \in [\tau_j, t_1)$ such that

$$\begin{aligned}\forall\, t \in [t_0, t_1) : \quad \widehat{u}_j > |u_j(t)| &\geq (1-\delta)\widehat{u}_j \\ &\overset{(3.34)}{\underset{(5.74)}{\geq}} \frac{1}{(CB)_{jj}}\left[L + \ell_j + \sum_{i=1,\ i\neq j}^{m} |(CB)_{ji}|\widehat{u}_i\right].\end{aligned} \tag{5.75}$$

$(1-\delta)\widehat{u}_j > 0$ implies that $\operatorname{sgn}\beta_j(\cdot)$ is constant on $[t_0, t_1]$ and therefore $\operatorname{sgn}\beta_j(t)\, u_j(t) = |u_j(t)|$ for all $t \in [t_0, t_1]$. It has to be noted that $d_u(\cdot) \equiv 0$ implies that $\operatorname{sgn}\beta_j(\cdot) =$

$\operatorname{sgn} e_j(\cdot)$ on $[t_0, t_1]$ and therefore, by invoking (5.75), it follows that

$$\begin{aligned}\operatorname{sgn} e_j(t)\, \dot e_j(t) &\overset{(5.28),(5.29)}{\underset{(5.74)}{\leq}} L - \operatorname{sgn} e_j(t) \sum_{i=1}^{m} (C_1 B_1)_{ji}\, \mathrm{sat}_{\widehat{u}_i}(k_i(t) e_i(t)) \\ &\overset{(3.34)}{\leq} L - (C_1B_1)_{jj} |u_j(t)| + \sum_{i=1,\, i\neq j}^{m} |(C_1 B_1)_{ji}|\, \widehat{u}_i \\ &\overset{(5.75)}{\leq} -\ell_j \qquad \text{for almost all } t \in [t_0, t_1]\end{aligned}$$

which, on integration, yields

$$|e_j(t_1)| \leq |e_j(t_0)| - \ell_j (t_1 - t_0)$$

whence (5.73) gives the contradiction

$$\widehat{u}_j = |u_j(t_1)| = |\beta_j(t_1)| < |\beta_j(t_0)| - \ell_j k_j(t_1)(t_1 - t_0) \overset{(5.75)}{<} \widehat{u}_j.$$

STEP 5D: *Finally, the last claim in Assertion (v) is shown.*
It has to be noted that $|u_i(\sigma)| = \frac{|e_i(\sigma)|}{\psi_i(\sigma) - |e_i(\sigma)|} < \widehat{u}_i$ is equivalent to $|e_i(\sigma)| < \frac{\psi_i(\sigma)\, \widehat{u}_i}{1 + \widehat{u}_i}$ and so the claim follows from Step 5C and setting $\tau_i = \sigma$.

STEP 6: *Assertion (vi) is shown.*
Assertion (ii) ensures that $|e_i(\cdot)| < \psi_i(\cdot)$ on $[\sigma, \infty)$ for all $i = 1, ..., m$ and thus it follows, together with (5.34), that

$$\forall\, i \in \{1, ..., m\}\ \forall\, t \geq \sigma : \quad y_i(t) > r_i(t) - \psi_i(t) \overset{(5.34)}{\geq} 0.$$

This shows, in view of $C\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^m_{\geq 0})$, that $y(\cdot)$ is non-negative on $\mathbb{R}_{\geq 0}$.
The second statement of Assertion (vi) is shown. Consider the system (5.13) and let $x(\cdot) = [x_1^\top(\cdot),\, x_2^\top(\cdot)]^\top$ and the nominal system data A and $\eta(\cdot)$ be partionated as in (5.59). Then it follows that

$$\dot x_2(t) = A_4 x_2(t) + \int_0^t \mathrm{d}[\eta_4(\theta)]\, x_2(t-\theta) + A_3 x_1(t) + \int_0^t \mathrm{d}[\eta_3(\theta)]\, x_1(t-\theta).$$

Positivity of the homogeneous part of (5.13) implies that A is a Metzler matrix and $\eta(\cdot)$ is a non-decreasing matrix function, see Theorem 5.2.8.
Since $C_1 = \mathrm{diag}\,(c_1, ..., c_m)$ with $c_i > 0$ for $i = 1, ..., m$, it follows that $x_1(\cdot) = C_1 y(\cdot)$

is non-negative. Define

$$f(\cdot) := A_3C_1y(\cdot) + \int_0^t \mathrm{d}[\eta_3(\theta)]\, C_1y(t-\theta)$$

which depends on $y(\cdot)$. As $\eta_3(\cdot)$ is a non-decreasing matrix function, $f(\cdot) \geq 0$ on $[0,\infty)$ and Proposition 5.2.7 yields that the solution $x_2(\cdot;\sigma,N\phi)$ is non-negative. In view of Remark 5.4.3, the structure of B and C implies $N\phi(\cdot) = [0\ I_{n-m}]\phi(\cdot)$ which gives

$$\forall\, t \geq 0: \quad x(t;\sigma,\phi) \geq 0.$$

STEP 7: *Assertion (vii) is shown.*
Define, for $i = 1,...,m$,

$$g_i : \mathcal{F}(\sigma,\varphi_i) \to \mathbb{R}, \quad (t,\xi) \mapsto g_i(t,\xi) := \begin{cases} \frac{-\xi}{\psi_i(t)-|\xi|} + d_{u,i}(t) & \text{, if } \frac{-\xi}{\psi_i(t)-|\xi|} + d_{u,i}(t) < \widehat{u}_i \\ \widehat{u}_i & \text{, if } \frac{-\xi}{\psi_i(t)-|\xi|} + d_{u,i}(t) \geq \widehat{u}_i \end{cases}$$

and

$$g : \mathcal{D} \to \mathbb{R}^m, \qquad (t,\xi) \mapsto [g_1(t,\xi_1),...,g_m(t,\xi_m)]^\top$$

which is continuous and locally Lipschitz in ξ. In view of Lemma 5.5.4 (i) and boundedness of $k_i(\cdot)$, $i = 1,...,m$, existence and uniqueness of a solution $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of

$$\dot{v}(t) = Av(t) + \int_0^t \mathrm{d}[\eta(\theta)]\, v(t-\theta) + Bg(t, Cv(t)-r(t)), \quad v|_{[0,\sigma]} = \phi(\cdot) \tag{5.76}$$

follow similarly to Step 1 and Step 6 of the proof of Theorem 5.5.1. Define, for $i = 1,...,m$,

$$h_i : [\sigma,\infty) \to \mathbb{R}, \quad t \mapsto h_i(t) := \begin{cases} k_i(t) & \text{, if } k_i(t)(Cv(t)-r(t))_i - d_{u,i}(t) > -\widehat{u}_i \\ 0 & \text{, if } k_i(t)(Cv(t)-r(t))_i - d_{u,i}(t) \leq -\widehat{u}_i \end{cases}$$

and

$$d_i : [\sigma,\infty) \to \mathbb{R}, \quad t \mapsto d_i(t) := \begin{cases} (d_u(t))_i & \text{, if } k_i(t)(Cv(t)-r(t))_i - d_{u,i}(t) > -\widehat{u}_i \\ \widehat{u}_i & \text{, if } k_i(t)(Cv(t)-r(t))_i - d_{u,i}(t) \leq -\widehat{u}_i. \end{cases}$$

Then, the system (5.76) can be written as

$$\dot{v}(t) = [A - B\,\mathrm{diag}\,(h_1(t),...,h_m(t))\;C]\,v(t) + \int_0^t \mathrm{d}[\eta(\theta)]\,v(t-\theta)$$
$$+ B\left[\mathrm{diag}\,(h_1(t),...,h_m(t))r(t) + (d_1(t),...,d_m(t))^\top\right]. \quad (5.77)$$

Similarly to Step 7 of the proof of Theorem 5.5.1, the solution of (5.77) is non-negative, i.e.

$$\forall\, t \geq 0: \quad v(t;\sigma,\phi) \geq 0.$$

For

$$\widehat{T}:\ \mathcal{C}(\mathbb{R}_{\geq 0},\mathbb{R}^n) \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0},\mathbb{R}^n), \quad x(\cdot) \mapsto (\widehat{T}x)(\cdot) := [(A,\eta) * x](\cdot),$$

the closed-loop initial value problem (5.13), (5.32) satisfy

$$\dot{x}(t) = (\widehat{T}x)(t) - B\begin{bmatrix} \mathrm{sat}_{\widehat{u}_1}(k_1(t)(Cx(t)-r(t))_1 - d_{u,1}(t)) \\ \vdots \\ \mathrm{sat}_{\widehat{u}_m}(k_m(t)(Cx(t)-r(t))_m - d_{u,m}(t)) \end{bmatrix}$$
$$\geq (\widehat{T}x)(t) + Bg(t, Cx(t)-r(t)). \quad (5.78)$$

Define

$$F:\ \mathcal{D}\times\mathbb{R}^n \to \mathbb{R}^n, \quad (t,\mu,\xi) \mapsto F(t,\mu,\xi) := \xi + Bg(t,\mu - r(t))$$

which satisfies

$$\forall\, i \in \{1,...,m\}\ \forall\,(t,\mu,\xi),\,(t,\overline{\mu},\overline{\xi}) \in \mathcal{D}\times\mathbb{R}^n:$$
$$\left[(\mu,\xi)^\top \leq (\overline{\mu},\overline{\xi})^\top \ \wedge\ \mu_i = \overline{\mu}_i,\ \xi_i = \overline{\xi}_i\right] \quad \Rightarrow \quad \left[F_i(t,\mu,\xi) \leq F_i(t,\overline{\mu},\overline{\xi})\right].$$

Applying [88, Prop. III.10.XXII and Rem. III.10.XXII (3)] to (5.78), it follows that

$$\forall\, t \geq 0: \quad x(t;\sigma,\phi) \geq v(t;\sigma,\phi) \geq 0.$$

This completes the proof of the theorem. □

Proof of Theorem 5.5.9:
It has to be noted that $\det(CB) \neq 0$. The notation of Proposition 5.4.2, Remark 3.3.1 and the technical notation of Step 2 of the proof of Theorem 5.5.1 is used.

STEP 1: *It is shown that there exists a maximal solution* $x : [0,\omega) \to \mathbb{R}^n$*,* $\omega \in (\sigma,\infty]$*, of the closed-loop system* (5.13)*,* (5.36).

Define the relatively open set

$$\mathcal{D} := \left\{(t,\eta,\xi) \in [\sigma,\infty) \times \mathbb{R}^m \times \mathbb{R}^{n-m} \mid 1+\varphi_i(t)\eta_i > 0 \text{ for all } i = 1,...,m\right\}$$

and, for all $i \in \{1,...,m\}$,

$$\kappa := (\kappa_1,...,\kappa_m) : \mathcal{D} \to \mathbb{R}^m, \quad (t,\eta_i) \mapsto \kappa_i(t,\eta_i) := \begin{cases} 0 & \text{, if } \eta_i > 0 \\ \frac{\varphi_i(t)}{1+\varphi_i(t)\eta_i} & \text{, if } \eta_i \leq 0. \end{cases}$$

With Proposition 5.4.2 the initial value problem (5.13), (5.36) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} e(t) \\ z(t) \end{pmatrix} = f(t,e(t),z(t)), \qquad \begin{pmatrix} e \\ z \end{pmatrix}\Big|_{[0,\sigma]} = \begin{pmatrix} C\phi(\cdot) - y_{\mathrm{ref}}(\cdot) \\ N\phi(\cdot) \end{pmatrix}, \tag{5.79}$$

where

$$\begin{aligned} &f : \mathcal{D} \to \mathbb{R}^n, \\ &(t,e,z) \mapsto \begin{pmatrix} \widehat{A}_1 e + \widehat{A}_2 z + \widehat{A}_1 r(t) - \dot{r}(t) - C_1 B_1 \left[\kappa_1(t,e_1)e_1,...,\kappa_m(t,e_m)e_m\right]^\top \\ \widehat{A}_3 e + \widehat{A}_4 z + \widehat{A}_3 r(t) \end{pmatrix} \end{aligned}$$

is locally Lipschitz on $\mathcal{D}$. Now, existence and uniqueness of a maximal solution $(e,z) : [0,\omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$, $\sigma < \omega \leq \infty$, of (5.79)follow similarly to Step 1 in the proof of Theorem 3.4.6. The details are omitted which completes the first step.

It has to be noted that the inequality (5.63) holds. This shows that $1 + \varphi_i(t)e_i(t) > 0$ for all $t \in [\sigma,\omega_0]$ and $i \in \{1,...,m\}$ with ω_0 as in Step 2 of the proof of Theorem 5.5.1.

STEP 2: *It is shown that*

$$\exists\, \widetilde{\varepsilon}_0 > 0\ \forall\, i \in \{1,...,m\}\ \forall\, t \in [\omega_0,\omega) : \quad \psi_i(t) + e_i(t) \geq \widetilde{\varepsilon}_0. \tag{5.80}$$

This follows similarly to Step 3 and Step 4 in the proof of Theorem 5.5.1 and is omitted for brevity. It has to be noted that (5.67) implies $\widetilde{\varepsilon}_0 \leq \frac{1}{2} \min_{j\in\{1,...,m\}} \lambda_j \leq \psi_i(t)$ for all $t \in [\omega_0,\omega)$ and $i \in \{1,...,m\}$ and thus (5.80) holds.

STEP 3: *Boundedness of $x(\cdot)$ on $[0,\omega)$ is shown.*
As the homogeneous part of (5.13) is positive, Theorem 5.2.8 yields that A is a Metzler matrix. Now A is a Hurwitz Metzler matrix and [23, Th. 3.3] gives

$\exists\, P := \operatorname{diag}(p_1, ..., p_n)$ positive definite $\exists\, R = R^\top \in \mathbb{R}^{n\times n}$ positive definite :
$$A^\top P + PA = -R.$$

Define
$$V : [\sigma, \omega) \to \mathbb{R}_{\geq 0}, \quad t \mapsto V(t) := x(t)^\top P x(t).$$

For the trajectories of (5.13), (5.36) the derivative of $V(\cdot)$ gives, for all $t \in [\sigma, \omega)$, (the argument t is omitted for brevity)

$$\begin{aligned}
\dot V &= 2x^\top P \dot x \;=\; -x^\top R x - 2\sum_{i=1}^m p_i b_i k_i x_i (c_i x_i - r_i) \\
&\leq -x^\top R x - 2\sum_{i=1}^m p_i b_i c_i k_i x_i^2 + 2\sum_{i=1}^m p_i b_i k_i x_i r_i + \sum_{i=1}^m p_i b_i k_i \left[\sqrt{c_i} x_i - \frac{r_i}{\sqrt{c_i}}\right]^2 \\
&= -x^\top R x - \sum_{i=1}^m p_i b_i c_i k_i x_i^2 + \sum_{i=1}^m p_i b_i k_i \frac{r_i^2}{c_i} \\
&\leq -\gamma V + \delta
\end{aligned}$$

with

$$\begin{aligned}
&0 < \gamma := \|R^{-1}\|^{-1} \, \|P\|^{-1} < \infty \\
&0 < \delta := \sum_{i=1}^m \frac{p_i b_i}{c_i} k_i r_i^2 \leq \sum_{i=1}^m \frac{p_i b_i}{c_i} \|r_i\|_\infty^2 \max\left\{\frac{2\|\varphi_i\|_\infty}{1 + \varphi_i(\sigma) e_i(\sigma)}, \frac{1}{\widetilde{\varepsilon}_0}\right\} < \infty.
\end{aligned}$$

Integration yields

$$\forall\, t \in [\sigma, \omega) : \quad 0 \leq V(t) \leq e^{-\gamma(t-\sigma)} V(\sigma) + \frac{\delta}{\gamma}$$

and thus $x(\cdot) \in \mathcal{L}^\infty([0, \omega), \mathbb{R}^n)$.

STEP 4: *Assertions (i) - (iv) are shown.*
Step 2 and Step 3 guarantee that $(t, e(t), z(t)) \in \mathcal{D}$ for all $t \in [\sigma, \omega)$. Step 2 ensures that $k_i(\cdot)$, $i = 1, ..., m$, are uniformly bounded on $[\sigma, \omega_0]$ and Step 3 shows that $\|k_i\|_{\mathcal{L}^\infty([\omega_0, \omega))} \leq \frac{1}{\widetilde{\varepsilon}_0}$. Moreover, Step 2 and Step 3 give

$$\begin{aligned}
\forall\, i \in \{1, ..., m\}\; \forall\, t \in [\sigma, \omega) : \quad 1 + \varphi_i(t) e_i(t) \geq \min\left\{\frac{1 + \varphi_i(\sigma) e_i(\sigma)}{2}, \widetilde{\varepsilon}_0\right\} \quad \wedge \\
0 \leq k_i(t) \leq \max\left\{\frac{2\|\varphi_i\|_\infty}{1 + \varphi_i(\sigma) e_i(\sigma)}, \frac{1}{\widetilde{\varepsilon}_0}\right\}.
\end{aligned}$$

Boundedness of $x(\cdot)$ (see Step 3) and Proposition 5.4.2 imply boundedness of $y(\cdot)$, $z(\cdot)$ and $e(\cdot)$ and thus $u_i(\cdot) \geq 0$ is bounded on $[\sigma, \omega)$. Step 1 gives that the solution $x(\cdot)$ of the closed-loop initial value problem (5.13), (5.36) is unique on $[0, \omega)$. To establish Assertions (i) - (iii), it remains to show that $\omega = \infty$. Step 3 yields

$$\exists\, \gamma_e,\, \gamma_z > 0\; \forall\, i \in \{1, ..., m\}\; \forall\, t \in [\sigma, \omega] : \quad e_i(t) \leq \gamma_e \quad \wedge \quad \|z(t)\| \leq \gamma_z. \tag{5.81}$$

Suppose that $\omega < \infty$ and define

$$\begin{aligned} \mathcal{C} := \{(t, \eta, \xi) \in [\sigma, \omega] \times \mathbb{R}^m \times \mathbb{R}^{n-m} \mid \forall\, i = 1, ..., m : \\ 1 + \varphi_i(t)\eta_i \geq \varepsilon_0,\, e_i(t) \leq \gamma_e,\, \|\xi\| \leq \gamma_z\}. \end{aligned}$$

Then it follows that $\mathcal{C}$ is a compact subset of $\mathcal{D}$ with the property graph$((e, z)) \subset \mathcal{C}$ which contradicts the fact that the closure of the latter is not a compact subset of $\mathcal{D}$ (see [88, Th. III.10.VI]). Therefore, $\omega = \infty$.
Assertion (iv) follows similarly to Step 7 in the proof of Theorem 5.5.1 and is omitted for brevity.

STEP 5: *Assertion (vi) is shown.*
Since A is a Hurwitz matrix, there exist positive constants $\alpha, \beta > 0$ such that

$$\forall\, t \geq 0 : \quad \|\exp(A\, t)\| \leq \beta e^{-\alpha\, t} \tag{5.82}$$

(see Proposition 1.1.4 (vi)). It has to be noted that $m = 1$ and $y_{\text{ref}}(\cdot) > 0$ on $[\sigma, \infty)$. Seeking a contradiction to Assertion (vi), suppose that

$$\forall\, t > \tau : \quad e(t) \geq 0$$

for some $\tau \geq \sigma$ such that $e(\tau) \geq 0$. Thus

$$\forall\, t > \tau : \quad y(t) \geq r(t) > 0.$$

(5.36) implies $u(t) = -k_1(t)e_1(t) = 0$ for all $t \geq \tau$. It has to be noted that $e(\cdot) = e_1(\cdot)$. Now for any solution $x(\cdot)$ of (5.13), (5.36) on some interval $[\tau, \infty)$ it follows that

$$\forall\, t \geq \tau : \quad x(t) = e^{A(t-\tau)}x(\tau)$$

which, together with (5.82) and $\liminf_{s \to \infty} r(s) > 0$, yields the contradiction

$$\forall\, t \geq \tau : \quad 0 < r(t) \leq y(t) \leq c_1\|x(t)\| \overset{(5.82)}{\leq} c_1\beta e^{-\alpha(t-\tau)}\|x(\tau)\|.$$

STEP 6: *Assertion (v) is shown.*
Boundedness of $e(\cdot)$ and $z(\cdot)$ (see Step 5) implies that

$$\exists\, Z \subset \mathbb{R}^{n-m} \text{ bounded, closed } \forall\, t \geq 0 : \quad z(t) \in Z.$$

Set

$$G := M \times Z \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$$

and define

$$g : G \to \mathbb{R}, \quad (\mu, \eta) \mapsto g(\mu, \eta) := \begin{cases} 0 & \text{, if } (\mu, \eta) \in M \times Z \\ 1 & \text{, else.} \end{cases}$$

It has to be noted that G is a closed subset of $\mathbb{R}^n$ and, by boundedness of $e(\cdot)$ and $z(\cdot)$, $(e, z)([\sigma, \infty)) \subset G$. For the function $g(\cdot)$, the following holds:

(1) The preimage of $0 \in \mathbb{R}^n$ under $g(\cdot)$ is the set

$$\{(\mu, \eta) \in G \mid g(\mu, \eta) = 0\} = M \times Z.$$

(2) *It is shown that the following implication holds, for all* $(\mu, \eta) \in G$,*:*

$$[g(\mu, \eta) \neq 0] \;\Rightarrow\; [\exists\, \delta > 0 : \; \inf\{|g(w_1, w_2)| \mid (w_1, w_2) \in G \cap \mathcal{B}_\delta(\mu, \eta)\} > 0]. \tag{5.83}$$

Let $(\mu, \eta) \in G$ with $g(\mu, \eta) \neq 0$. The definition of $g(\cdot)$ implies

$$\exists\, i \in \{1, ..., m\} : \quad \mu_i \in (0, \infty)$$

and thus

$$\exists\, \delta > 0 : \quad (\mu_i - \delta, \mu_i + \delta) \subset (0, \infty).$$

Therefore, it follows that

$$\inf\{|g(w_1, w_2)| \mid (w_1, w_2) \in G \cap \mathcal{B}_\delta(\mu, \eta)\} = 1 > 0$$

and so, (5.83) holds.

(3) *The right-hand side of* (5.79) *satisfies*

$$\forall\, (\mu, \eta) \in G : \quad [g(\mu, \eta) \neq 0] \;\Rightarrow\; [f(\cdot) \in \mathcal{F}(G \cap \mathcal{B}_\delta(\mu, \eta))], \tag{5.84}$$

where $\delta > 0$ *is chosen as in (2).*

From Definition 5.5.8 (iii) it follows that

$$m(\cdot) \in \mathcal{L}^\infty([\sigma,\infty)) \quad \Rightarrow \quad m(\cdot) \text{ uniformly locally integrable.}$$

Since Z is a bounded, closed set and $y_{\mathrm{ref}}(\cdot)$ and $\dot{y}_{\mathrm{ref}}(\cdot)$ are bounded, it holds

$$\exists\, \gamma > 0\ \forall\, (t,\xi,\zeta) \in [\sigma,\infty) \times (G \cap \mathcal{B}_\delta(\mu,\eta)) : \quad \|f(t,\xi,\zeta)\| \le \gamma.$$

With $t \mapsto m(t) := \gamma$ uniformly locally integrable, (5.84) follows.

(4) *It is shown that* $(g \circ (e,z))(\cdot)$ *is weakly meagre, where* $(e,z)(\cdot)$ *denotes the global solution of* (5.79).
Define

$$h : [\sigma,\infty) \to \mathbb{R}, \quad t \mapsto h(t) := g(e(t), z(t))$$

which is Lebesgue measurable. From Definition 5.5.8 (i), (ii) it follows immediately that a meagre function is weakly meagre. Therefore, it remains to show that $h(\cdot)$ is meagre, i.e.

$$\forall\, s > 0 : \quad \lambda(\{t \ge \sigma \mid |h(t)| \ge s\}) < \infty. \tag{5.85}$$

Recall that $\lambda(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}_{\ge 0}$. If $s > 1$, then $\lambda(\{t \ge \sigma \mid |h(t)| \ge\ s\ >\ 1\,\}) = \lambda(\emptyset) = 0$. For fixed $s \in (0,1]$ it follows

$$\begin{aligned}\{t \ge \sigma \mid |h(t)| \ge s\} &= \{t \ge \sigma \mid g(e(t),z(t)) \ge s\} \\ &= \{t \ge \sigma \mid \exists\, i \in \{1,...,m\} :\ e_i(t) \in (0,\infty)\} \subset \bigcup_{i=1}^{m} \{t \ge \sigma \mid e_i(t) > 0\}\end{aligned}$$

and thus

$$\lambda(\{t \ge \sigma \mid |h(t)| \ge s\}) \le \sum_{i=1}^{m} \lambda(\{t \ge \sigma \mid e_i(t) > 0\}) \overset{(5.37)}{<} \infty$$

and so, (5.85) follows.

Applying [55, Th. 5.4], it follows that $\Omega((e,z)) \subset g^{-1}(0,0) = M \times Z$ and thus $\Omega(e) \subset M$, where $g^{-1}(0,0)$ denotes the preimage of 0 under $g(\cdot)$. Moreover, since $e(\cdot)$ is bounded and continuous, $\Omega(e)$ is nonempty, compact, connected and

$$\lim_{t\to\infty} d_{\Omega(e)}(e(t)) = 0,$$

see [55, Lem. 2.1]. In view of $\Omega(e) \subset M$, the last fact implies that

$$\lim_{t\to\infty} d_M(e(t)) = 0$$

which shows Assertion (v). This completes the proof. □

5.8 Notes and references

Time-invariant Volterra-Stieltjes systems are well understood in [21]. Existence and uniqueness of a solution for time-varying Volterra-Stieltjes systems is presented in this chapter. Positive systems are of great interest (see [7, 19, 25, 56, 76] and their references therein). Positive linear system (1.1) can be regarded as a linear system where the state variables are non-negative for all time (see [19, 56]).
Positive systems are considered for large class of linear systems – positive linear functional differential equations (see [69, 71, 73]), positive linear Volterra integro-differential systems (see [67, 72]) and positive linear Volterra integral systems (see [67]). Further interesting results on positive systems are reachability and controllability, realization of positive systems or adaptive control and feedback control (see the survey [76]).
Positive systems are of great practical importance. The non-negative property occurs quite often in numerous applications and in nature. Positive systems are used to model natural and artificial networks of reservoirs, see [19, Chapt. 2] or [76], and are visible in biology where they are used to describe transportation, accumulation and drainage processes of compounds like hormones, glucose, insulin or metals, see [25]. Moreover, industrial systems which involve chemical reactions, heat exchangers and distillation columns are examples of positive systems, see [19, Part III]. Further applications for population models and economic systems can be found in [56, Chapt. 6] or [7]. The mathematical theory of positive systems is based on the theory of non-negative matrices, see [7, 19].
In Section 5.2 positivity of time-varying Volterra-Stieltjes systems (1.2) is introduced and stability of time-invariant Volterra-Stieltjes systems (5.13) is presented. These generalizes the stability concepts for linear systems (see Appendix, Section 1.1.2) to Volterra-Stieltjes systems. In [5, 66], exponential asymptotic stability of the zero solution of Volterra equations are studied and the purpose of [60] lies on asymptotic stability properties for Volterra integro-differential equations. Section 5.3 generalizes these results to multi input, multi output Volterra-Stieltjes systems.
The main contribution of this thesis is feedback control. In [75, 76, 77], the authors

add the concept of tracking reference signals under disturbances to positive single input, single output and multi input, multi output systems. Section 5.5 introduces the concept of the funnel controller (see Chapter 3) to Volterra-Stieltjes systems. Theorem 5.5.1 shows that the funnel controller (see [43]) works for Volterra-Stieltjes systems and Theorem 5.5.5 generalizes the studies of input saturation of Chapter 3. At least, Theorem 5.5.9 considers non-negative inputs together with the funnel controller. A direct adaptive controller for set-point regulation of positive systems (1.1) with special input output matrices B, C is offered in [25] (and for nonlinear systems see [24]). Moreover, an adaptive control law with non-negative input $u(\cdot)$ is presented, but a careful inspection of the results offers that the existence of a solution of the closed-loop system is unanswered. Furthermore, [25] considers time-varying positive systems (1.2) without the Volterra term, i.e. $\eta(\cdot) \equiv 0$.

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