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Bohl exponents for time-varying linear differential-algebraic equations*

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Abstract

We study stability of linear time-varying differential-algebraic equations (DAEs). The Bohl exponent is introduced and finiteness of the Bohl exponent is characterized, the equivalence of exponential stability and a negative Bohl exponent is shown and shift properties are derived. We also show that the Bohl exponent is invariant under the set of Bohl transformations. For the class of DAEs which possess a transition matrix introduced in this paper, the Bohl exponent is exploited to characterize boundedness of solutions of a Cauchy problem and robustness of exponential stability.

Keywords: Time-varying linear differential-algebraic equations, transition matrix, Bohl exponent, Bohl transformation, exponential stability, robustness

1 Introduction

We study stability of solutions of time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x} = A(t)x, \tag{1.1}$$

where $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$, $n \in \mathbb{N}$. For brevity, we identify the tuple (E, A) with the DAE (1.1). For robustness analysis we also consider implicit differential equations

$$F(t, x, \dot{x}) = 0, \tag{1.2}$$

where $F \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$.
The functions E, A and F are supposed to be continuous for technical reasons only: the results are also valid if the functions are piecewise continuous, L^∞_{loc}, or distributional. Rather than the systems entries, the solution space is crucial, and here we allow for "right global solutions", see Definition 2.1.

While time-invariant DAEs are well studied, see e.g. the monographs [6, 7, 10] and the textbook [17], stability theory of general time-varying DAEs (1.2) and even linear systems (E, A) is still an active research topic. However, most papers [12, 13, 19, 22, 9] treat only DAEs with tractability index 1 or 2. Kunkel and Mehrmann [18] derive some stability results for time-varying DAEs with existing

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strangeness index (well-defined differentiation index) and Linh and Mehrmann [20] investigate Lyapunov, Bohl and Sacker-Sell spectral intervals for DAEs of this class. For a very good overview of recent DAE theory and its applications see also [20, Sec. 1].

The main focus of stability theory, and therefore of the articles mentioned above, is the asymptotic behaviour of the solutions of the given system. There are two fundamental concepts in the theory of ordinary differential equations (ODEs) to investigate asymptotic behaviour of solutions: the Lyapunov and the Bohl exponent. While the Lyapunov exponent, introduced by Aleksandr M. Lyapunov [21], gives a bound for the exponential growth of the solutions of the system, the Bohl exponent, introduced by Piers Bohl [5], describes the uniform exponential growth of the solutions. It is well-known from ODEs that the Bohl exponent compared to the Lyapunov exponent is the appropriate concept when it comes to time-varying, instead of time-invariant, ODEs. The Bohl exponent has been successfully used to characterize exponential stability and to derive robustness results, see e.g. [11, 14]. For an excellent summary of the history of the development of Lyapunov and Bohl exponents see [11, pp. 146–148].

When it comes to DAEs, the approach to Bohl exponents has, to the author's best knowledge, only been carried out in two contributions [9, 20]. Though [9] generalize several ODE results concerning Bohl exponents to DAEs, they treat only DAEs of index 1. [20] investigate Bohl spectral intervals and Bohl exponents of particular solutions, however the Bohl exponent of the system does not lie in their focus. Both [9] and [20] avoid the problem of a proper definition of the Bohl exponent for DAEs (1.2). [9] define the Bohl exponent via a transition matrix (as it is common for ODEs [15, Def. 3.3.10]), which however is not present for general nonlinear DAEs (1.2). [20] only consider Bohl exponents of particular solutions.

The aim of the present article is to develop a theory of Bohl exponents for DAEs (E, A). To this end we consider the set of right global solutions of (E, A) introduced in [2] to obtain a proper definition of Bohl exponents for DAE systems. The notions of (right global) solutions and exponential stability are introduced in Section 2. In Section 3 we define the Bohl exponent for a solution x and the system (1.2) and derive properties of it. Furthermore, finiteness of the Bohl exponent is characterized, the equivalence of exponential stability and a negative Bohl exponent is shown and shift properties are derived. Lyapunov and Bohl transformations are introduced and it is shown that the Bohl exponent is invariant under Bohl transformations. In Section 4, we focus on the class of DAEs which possess a transition matrix: It is shown that solutions of certain structured perturbations of DAEs can be represented in terms of an integral equation. For this class of systems we derive the main theorems in Section 5: Theorem 5.5 shows that the Bohl exponent is negative if, and only if, the Cauchy problem (5.6) has a bounded solution whenever the inhomogeneity is in class (5.7). Theorem 5.7 characterizes the robustness of exponential stability, more precisely, the behaviour of the Bohl exponent under perturbations is characterized. As it is well-known for ODEs, Theorem 5.7 states that the Bohl exponent does only change little if the perturbation of the system is "sufficiently small". The first approach to robustness of exponential stability of time-varying DAEs has been carried out in [9] - also using the Bohl exponent - however, only for index 1 systems. In the present paper, to the author's best knowledge, the first robustness results for systems of arbitrary index are derived.

The results of the present paper are the generalizations of ODE results in [11, 14, 15, 16] to DAEs.

Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	$:=$	$(0, \infty)$
$\operatorname{im} A$		the image of the matrix $A \in \mathbb{R}^{m \times n}$

$\mathbf{Gl}_n(\mathbb{R})$		the general linear group of degree n, i.e. the set of all invertible $n \times n$ matrices over $\mathbb{R}$
$\|x\|$	$:=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\mathcal{B}_\delta(x^0)$	$:=$	$\{\ x \in \mathbb{R}^n \ \mid \ \|x - x^0\| < \delta \ \}$, the open ball of radius $\delta > 0$ around $x^0 \in \mathbb{R}^n$
$\|A\|$	$:=$	$\sup \{\ \|Ax\| \ \mid \ \|x\| = 1 \ \}$, induced matrix norm of $A \in \mathbb{R}^{n \times m}$
$\mathcal{C}(\mathcal{I}; \mathcal{S})$		the set of continuous functions $f : \mathcal{I} \to \mathcal{S}$ from an open set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\mathcal{C}^k(\mathcal{I}; \mathcal{S})$		the set of k-times continuously differentiable functions $f : \mathcal{I} \to \mathcal{S}$ from an open set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\operatorname{dom} f$		the domain of the function f
$\|f\|_\infty$	$:=$	$\sup \{\ \|f(t)\| \ \mid \ t \in \operatorname{dom} f \ \}$ the infinity norm of the function f
$f\mid_{\mathcal{M}}$		the restriction of the function f on a set $\mathcal{M} \subseteq \operatorname{dom} f$
$f(0+)$	$:=$	$\lim_{t \searrow 0} f(t)$ if this limit exists

2 Stability of solutions

The concept of a solution and its extendability is introduced similarly to ODEs, see for example [1, Sec. 5]. Note that we consider the concept of classical, i.e. continuously differentiable, solutions.

Definition 2.1 (Solutions)**.** A function $x : (\alpha, \omega) \to \mathbb{R}^n$, $0 \le \alpha < \omega$, is called

$$\textit{solution of } (1.2) \quad :\Longleftrightarrow \quad x \in \mathcal{C}^1((\alpha, \omega); \mathbb{R}^n) \text{ and } x \text{ satisfies (1.2) for all } t \in (\alpha, \omega).$$

A solution $\tilde{x} : (\alpha, \tilde{\omega}) \to \mathbb{R}^n$ of (1.2) is called a

$$\textit{(right) extension} \text{ of } x \quad :\Longleftrightarrow \quad \tilde{\omega} \ge \omega \text{ and } x = \tilde{x}\mid_{(\alpha,\omega)}.$$

x is called

$$\begin{aligned} \textit{right maximal} \quad &:\Longleftrightarrow \quad \omega = \tilde{\omega} \text{ for every extension } \tilde{x} : (\alpha, \tilde{\omega}) \to \mathbb{R}^n \text{ of } x, \\ \textit{right global} \quad &:\Longleftrightarrow \quad \omega = \infty, \\ \textit{global} \quad &:\Longleftrightarrow \quad (\alpha, \omega) = \mathbb{R}_+. \end{aligned}$$

◇

Consider the linear system (E, A). Let $(t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n$; then the set of all right maximal solutions of the initial value problem (E, A), $x(t^0) = x^0$ is denoted by

$$\mathcal{S}_{E,A}(t^0, x^0) := \left\{ x : \mathcal{J} \to \mathbb{R}^n \ \middle| \ \begin{array}{l} \mathcal{J} \text{ open interval, } t^0 \in \mathcal{J}, \ x(t^0) = x^0, \\ x(\cdot) \text{ is a right maximal solution of } (E, A) \end{array} \right\},$$

and the set of all right global solutions of (E, A), $x(t^0) = x^0$ by

$$\mathcal{G}_{E,A}(t^0, x^0) := \left\{ \ x(\cdot) \in \mathcal{S}_{E,A}(t^0, x^0) \ \middle| \ x(\cdot) \text{ is right global solution of } (E, A) \ \right\}.$$

For DAEs it is essential to consider the appropriate set of solutions and the corresponding initial values for which these solutions exist. The set $\mathcal{G}_{E,A}(t^0, x^0)$ has proved to be a fundamental solution space of time-varying DAEs (E, A), see [2].

The *set of all pairs of consistent initial values* of $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ and the linear subspace of *initial values which are consistent at time* $t^0 \in \mathbb{R}_+$ is denoted by

$$\begin{aligned} \mathcal{V}_{E,A} &:= \left\{ (t^0,x^0) \in \mathbb{R}_+ \times \mathbb{R}^n \;\middle|\; \exists \text{ (local) sln. } x(\cdot) \text{ of } (E,A):\; t^0 \in \operatorname{dom} x,\; x(t^0) = x^0 \right\}, \\ \mathcal{V}_{E,A}(t^0) &:= \left\{ x^0 \in \mathbb{R}^n \;\middle|\; (t^0,x^0) \in \mathcal{V}_{E,A} \right\}, \end{aligned}$$

resp. Note that if $x : \mathcal{J} \to \mathbb{R}^n$ is a solution of (E,A), then $x(t) \in \mathcal{V}_{E,A}(t)$ for all $t \in \mathcal{J}$.
Possible singular behaviour of right maximal solutions $x : (\alpha,\omega) \to \mathbb{R}^n$ of (E,A) with $\omega < \infty$ is discussed in [2, Sec. 2]: it is shown that DAEs behave very differently compared to ODEs.

Next we define exponential stability of a linear DAE (E,A). Note that usually stability is a property of a particular solution: other existing solutions in a neighborhood of it stay close to it for all time. For linear systems it is sufficient to consider this property only for the trivial solution. However, for DAEs it is at first sight not clear whether this is still true. To this end, it is shown in [2, Thm. 4.3] that also for DAEs (E,A) it suffices to consider the stability properties of the trivial solution. Therefore, we introduce exponential stability of (E,A) as in [2, Defs. 4.1 & 4.4].

Definition 2.2 (Exponential stability)**.** A linear DAE $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ is called

$$\textit{exponentially stable} \quad :\Longleftrightarrow \quad \begin{array}{l} \exists\, \mu, M > 0 \;\forall\, t^0 \in \mathbb{R}_+ \;\exists\, \delta > 0 \;\forall\, x^0 \in \mathcal{B}_\delta(0) \;\forall\, x(\cdot) \in \mathcal{S}_{E,A}(t^0,x^0): \\ {[t^0,\infty)} \subseteq \operatorname{dom} x \;\wedge\; \forall\, t \geq t^0 : \|x(t)\| \leq M e^{-\mu(t-t^0)} \|x^0\|. \end{array}$$

$\diamond$

The local Definition 2.2 (i.e. $\delta > 0$ and $x^0 \in \mathcal{B}_\delta(0)$) seems to be artificial and superfluous for *linear* DAEs. However, we consider general time-varying linear DAEs, where it is not clear whether a condition which holds locally does also hold globally. In Section 5 it is shown that the latter is indeed the case for linear DAEs which possess a transition matrix (cf. Definition 4.1).
Finalizing this section, note that our notion of exponential stability is sometimes referred to as *uniformly exponential stability*, see e.g. [23, Def. 6.5], [15, p. 257] for ODEs. This is due to the fact that M in Definition 2.2 does not depend on the initial time t^0.

3 Bohl exponent and Bohl transformations - general results

We introduce the concept of Bohl exponent for right global solutions of (1.2) and for the system (1.2) itself. Then we concentrate on linear DAEs (E,A) and derive basic properties of the Bohl exponent. We stress that in order to generalize ODE results to DAEs it is crucial to consider the correct (function) spaces and to bear in mind that, in general, there is no transition matrix and associated semi group property.

The notation $\mathcal{B}(\nu,N)$, introduced next for the general nonlinear DAE (1.2) and $\nu \in \mathbb{R}, N \in \mathbb{R}_+$, was first used by P. Bohl [5] for linear ODEs:

$$(1.2) \text{ has property } \mathcal{B}(\nu,N) \quad :\Longleftrightarrow \quad \begin{array}{r} \forall \text{ right global sln. } x : (\alpha,\infty) \to \mathbb{R}^n \text{ of } (1.2) \\ \forall\, t \geq s > \alpha : \;\|x(t)\| \leq N e^{-\nu(t-s)} \|x(s)\|. \end{array}$$

For more information about the applications of this property in (mathematical) history see [11, pp. 146–148].

We continue with the definition of the Bohl exponent for right global solutions of (1.2). This is a straightforward generalization of the corresponding concept for ODEs (see e.g. [11]); [20] generalize this to the class of linear time-varying DAEs (E, A) which have a well-defined differentiation index.

Definition 3.1 (Bohl exponent for solutions)**.** The *Bohl exponent of a right global solution* $x : (\alpha, \infty) \to \mathbb{R}^n$ of (1.2) is defined as

$$k_B(x) := \inf \left\{ \rho \in \mathbb{R} \;\middle|\; \exists\, N_\rho > 0 \;\forall\, t \geq s > \alpha : \; \|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\| \right\}.$$

Note that we use the usual convention $\inf \emptyset = +\infty$. ◇

Remark 3.2 (Calculation of Bohl exponent)**.** On page 169 in [20] it is shown that, as a generalization of the formula in [11, p. 118], the Bohl exponents (as considered in [20]) admit the representation

$$k_B(x) = \limsup_{s,t-s \to \infty} \frac{\ln \|x(t)\| - \ln \|x(s)\|}{t - s}. \tag{3.1}$$

However, formula (3.1) does, in general, not hold true for solutions of systems (1.2), or even linear DAEs $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. Consider the scalar system

$$(t-2)\dot{x} = 2x, \quad t \in \mathbb{R}_+. \tag{3.2}$$

Then $x : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto \begin{cases} (t-2)^2, & 0 < t \leq 2 \\ 0, & t > 2 \end{cases}$ is a non-trivial global solution of (3.2) and $k_B(x) = 0 < \infty$. But obviously, a calculation of $k_B(x)$ via formula (3.1) is impossible. Here $x\,|_{[2,\infty)} = 0$, which is due to the finiteness of the Bohl exponent of x.
If the Bohl exponent is not finite, solutions may vanish only at isolated points. Consider the DAE

$$\sin t\ \dot{x} = \cos t\ x, \quad t \in \mathbb{R}_+. \tag{3.3}$$

The function $x : \mathbb{R}_+ \to \mathbb{R},\ t \mapsto \sin t$, is a non-trivial global solution of (3.3) with $k_B(x) = \infty$. Since $x(k\pi) = 0$ for all $k \in \mathbb{N}$ a calculation of $k_B(x)$ via formula (3.1) is impossible.
Nevertheless the following is easily verified:
If $x : (\alpha, \infty) \to \mathbb{R}^n$ is any right global solution of (1.2) *and $x(t) \neq 0$ for all $t > \alpha$, then $k_B(x)$ may be calculated via formula* (3.1)*.* ◇

Example 3.3. Consider the system

$$(t-1)\dot{x} = (2-t)x, \quad t \in \mathbb{R}_+. \tag{3.4}$$

Any local solution of (3.4) extends uniquely to a global solution and the set of global solutions of (3.4) is given by $\left\{ \mathbb{R}_+ \ni t \mapsto c(t-1)e^{-(t-1)} \;\middle|\; c \in \mathbb{R} \right\}$. Since any global solution of (3.4) is zero at $t = 1$, the Bohl exponent of any non-trivial global solution equals $+\infty$, even though all global solutions tend exponentially to zero. This shows that an exponential decay rate of a solution of a time-varying DAE is not sufficient to deduce a negative or even finite Bohl exponent, as in the case of ODEs. This is due to the possible non-uniqueness of solutions, cf. also Proposition 3.8. ◇

Now, following [11] and using the concept of right global solutions, it is straightforward to define the Bohl exponent for a system (1.2).

Definition 3.4 (Bohl exponent for DAEs)**.** The *Bohl exponent of a system* (1.2) is defined as

$$k_B(1.2) := \inf \left\{ \rho \in \mathbb{R} \;\middle|\; \begin{array}{r} \exists\, N_\rho > 0 \;\forall \text{ right global sln. } x : (\alpha, \infty) \to \mathbb{R}^n \text{ of (1.2)} \\ \forall\, t \geq s > \alpha : \; \|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\| \end{array} \right\}.$$

◇

In the case of a linear DAE $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ we write $k_B(E, A)$ instead of $k_B(1.1)$. We illustrate this definition by an example.

Example 3.5. Consider the nonlinear DAE

$$\begin{array}{rcl} (\dot{x}_1 + x_1)\sin x_1 - x_2 \cos x_1 & = & 0 \\ -(\dot{x}_1 + x_1)\cos x_1 - x_2 \sin x_1 & = & 0. \end{array} \tag{3.5}$$

Simple calculations show that any right global solution $(x_1, x_2) : (\alpha, \infty) \to \mathbb{R}^2$, $\alpha \geq 0$, of (3.5) has the representation:

$$\forall\, t, s > \alpha : \; x_1(t) = e^{-(t-s)} x_1(s) \;\wedge\; x_2(t) = 0.$$

Clearly, system (3.5) has Bohl exponent $k_B(3.5) = -1$. ⋄

Remark 3.6. It is obvious that the Bohl exponent $k_B(x)$ of right global solutions x of (1.2) and the Bohl exponent $k_B(1.2)$ of (1.2) are related in the following way:

$$\forall \text{ right global sln. } x(\cdot) \text{ of } (1.2) : \; k_B(x) \leq k_B(1.2). \tag{3.6}$$

However, equality does not hold: System (3.2) has Bohl exponent $k_B(3.2) = \infty$ since $\mathbb{R}_+ \ni t \mapsto (t-2)^2$ is a global solution of (3.2). However, as shown in Remark 3.2, there are nontrivial global solutions with finite Bohl exponent. ⋄

The Bohl exponent for the nonlinear DAE (1.2) will be used for robustness analysis of linear DAEs at the end of Section 5. In the remainder of this section we will consider linear DAEs $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. The next proposition characterizes finiteness of the Bohl exponent; for ODEs see [15, Prop. 3.3.14]. In contrast to [15] we do not use any transition matrix and its semi group properties.

Proposition 3.7 (Finiteness of the Bohl exponent)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. Then $k_B(E, A) < \infty$ if, and only if,*

$$\sup \left\{ \frac{\|x(t)\|}{\|x^0\|} \;\middle|\; (t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n, \; x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0), \; t \in [t^0, t^0 + 1] \right\} < \infty. \tag{3.7}$$

Here we use the usual conventions $\frac{\alpha}{0} := \infty$ for $\alpha > 0$ and $\frac{0}{0} := 0$; so $x^0 = 0$ in (3.7) is specifically allowed.

Proof: "$\Rightarrow$": Choosing $\rho > \max\{k_B(E, A), 0\}$, there exists $N_\rho > 0$ such that (E, A) has property $\mathcal{B}(-\rho, N_\rho)$. Let $(t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n$, $x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0)$ and $t \in [t^0, t^0 + 1]$. If $x^0 = 0$ then, $\|x(t)\| = 0$ since (E, A) has property $\mathcal{B}(-\rho, N_\rho)$, and hence $\frac{\|x(t)\|}{\|x^0\|} = 0 < \infty$. If $x^0 \neq 0$ then

$$\frac{\|x(t)\|}{\|x^0\|} \overset{\mathcal{B}(-\rho,N_\rho)}{\leq} N_\rho e^{\rho(t-t^0)} \frac{\|x^0\|}{\|x^0\|} \leq N_\rho e^\rho < \infty.$$

"$\Leftarrow$": (3.7) implies existence of $K > 0$ such that

$$\forall\, (t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n \;\forall\, x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0) \;\forall\, t \in [t^0, t^0 + 1] : \; \frac{\|x(t)\|}{\|x^0\|} \leq K. \tag{3.8}$$

Let $x : (\alpha, \infty) \to \mathbb{R}^n$ be a right global solution of (E, A) and $t \geq t^0 > \alpha$. Set $x^0 := x(t^0)$, then $x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0)$. Choose $k \in \mathbb{N}$ such that $t^0 + (k - 1) \leq t < t^0 + k$ and consider three cases:
Case 1: $x^0 = 0$. Then (3.8) implies $x(s) = 0$ for $s \in [t^0, t^0 + 1]$. Furthermore, invoking $x(\cdot) \in$

$\mathcal{G}_{E,A}(t^0+1, x(t^0+1))$, we find $x(s) = 0$ for $s \in [t^0+1, t^0+2]$ and, repeating this argument, $x(s) = 0$ for $s \in [t^0+(k-1), t^0+k]$, whence $x(t) = 0$. This gives

$$\|x(t)\| \le K e^{(\ln K)(t-t^0)} \|x^0\|. \tag{3.9}$$

Case 2: $x^0 \neq 0$ and $x(t-j) = 0$ for some $j \in \{1, \ldots, k-1\}$. Then the same argument as in Case 1 yields $x(t) = 0$ and hence equation (3.9).
Case 3: $x^0 \neq 0$ and $x(t-j) \neq 0$ for all $j \in \{1, \ldots, k-1\}$. Then

$$\begin{aligned}\frac{\|x(t)\|}{\|x^0\|} &= \frac{\|x(t)\|}{\|x^0\|} \prod_{j=1}^{k-1} \frac{\|x(t-j)\|}{\|x(t-j)\|} \\ &= \frac{\|x(t)\|}{\|x(t-1)\|} \cdot \frac{\|x(t-1)\|}{\|x(t-2)\|} \cdots \frac{\|x(t-(k-1))\|}{\|x^0\|} \overset{(3.8)}{\le} K^k \le K e^{(\ln K)(t-t^0)}, \end{aligned} \tag{3.10}$$

which gives (3.9).
Hence $k_B(E,A) \le \ln K < \infty$ and this concludes the proof. □

For the solutions of a DAE (E,A) with finite Bohl exponent $k_B(E,A)$ we have the following uniqueness property.

Proposition 3.8 (Bohl exponent and unique solutions)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ such that $k_B(E,A) < \infty$. If $x_1 : (\alpha_1, \infty) \to \mathbb{R}^n$ and $x_2 : (\alpha_2, \infty) \to \mathbb{R}^n$ both solve the initial value problem (E,A), $x(t^0) = x^0$ for $(t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n$, then $x_1 |_{[t^0,\infty)} = x_2 |_{[t^0,\infty)}$.*

Proof: Since $k_B(E,A) < \infty$ there exist $\rho \in \mathbb{R}, N_\rho > 0$ such that (E,A) has property $\mathcal{B}(-\rho, N_\rho)$. $(x_2 - x_1) : (\max\{\alpha_1, \alpha_2\}, \infty) \to \mathbb{R}^n$ solves the initial value problem (E,A), $x(t^0) = 0$. Therefore,

$$\forall\, t \ge t^0 : \ \|x_2(t) - x_1(t)\| \le N_\rho e^{\rho(t-t^0)} \|x_2(t^0) - x_1(t^0)\| = 0,$$

whence $x_1 |_{[t^0,\infty)} = x_2 |_{[t^0,\infty)}$. □

For linear ODEs it is well-known [15, Thm. 3.3.15] that exponential stability is equivalent to a negative Bohl exponent and to the fact that the transition matrix is bounded in the L^p-norm. This is useful for studying robustness of exponential stability, since the Bohl exponent turns out to be the appropriate tool for robustness analysis, see e.g. [11, 14]. For DAEs we proceed in a similar manner by first stating a DAE-version of [15, Thm. 3.3.15] and then deriving robustness results in Section 5.
The next theorem shows that the Bohl exponent of (E,A) is negative (or $-\infty$) if, and only if, the trivial solution of (E,A) is "essentially" exponentially stable. "Essentially" in the sense that we cannot guarantee that every existing right maximal solution in a neighborhood of the trivial solution is right global, cf. Definition 2.2. But we can guarantee that all right global solutions decay exponentially to zero. In this sense it is a DAE-version of the ODE result [15, Thm. 3.3.15].

Theorem 3.9 (Exponential stability via Bohl exponents)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ such that $k_B(E,A) < \infty$. Then the following statements are equivalent.*

(i) $k_B(E,A) < 0$,

(ii) $\exists\, \mu, M > 0\ \forall\, (t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n\ \forall\, x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0)\ \forall\, t \ge t^0 : \ \|x(t)\| \le M\, e^{-\mu(t-t^0)} \|x^0\|$,

(iii)
$$\forall\, p \in \mathbb{R}_+\ \exists\, c > 0\ \forall\, (t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n\ \forall\, x(\cdot) \in \mathcal{G}_{E,A}(t^0, x^0) : \ \int_{t^0}^{\infty} \|x(t)\|^p \,\mathrm{d}t \le c \|x^0\|^p.$$

Proof: (i) $\Leftrightarrow$ (ii) follows immediately from the definition of Bohl exponents.
(ii) $\Rightarrow$ (iii): For any $p \in \mathbb{R}_+$ the claim holds with $c = \frac{M^p}{p\mu}$.
(iii) $\Rightarrow$ (i): Since $k_B(E,A) < \infty$ there exist $\rho, N_\rho > 0$ such that (E,A) has property $\mathcal{B}(-\rho, N_\rho)$. Let $p \in \mathbb{R}_+$ and $x : (\alpha, \infty) \to \mathbb{R}^n$ be any right global solution of (E,A). Then, for all $t \geq t^0 > \alpha$,

$$\frac{1 - e^{-p\rho(t-t^0)}}{p\rho} \|x(t)\|^p = \int_{t^0}^{t} e^{-p\rho(t-s)} \|x(t)\|^p \, \mathrm{d}s \overset{\mathcal{B}(-\rho,N_\rho)}{\leq} N_\rho^p \int_{t^0}^{t} \|x(s)\|^p \, \mathrm{d}s \overset{\text{Ass.}}{\leq} N_\rho^p c \|x(t^0)\|^p,$$

whence, invoking that $k_B(E,A) < \infty$ implies (3.8) for some $K > 0$ by Proposition 3.7, we obtain, for $\gamma := \max\left\{K, \frac{p\rho N_\rho^p c}{1-e^{-p\rho}}\right\}$,

$$\forall\, t \geq t^0 > \alpha : \ \|x(t)\|^p \leq \gamma \|x(t^0)\|^p. \tag{3.11}$$

Clearly γ does not depend on the choice of the solution $x(\cdot)$. Now we find

$$\forall\, t \geq s > \alpha : \ (t-s)\|x(t)\|^p = \int_s^t \|x(t)\|^p \, \mathrm{d}t^0 \overset{(3.11)}{\leq} \int_s^t \gamma \|x(t^0)\|^p \, \mathrm{d}t^0 \overset{\text{Ass.}}{\leq} \gamma c \|x(s)\|^p,$$

thus having, for $\tau := 2^p \gamma c$,

$$\forall\, s > \alpha : \ \|x(s+\tau)\| \leq \frac{1}{2}\|x(s)\|. \tag{3.12}$$

Now, for $t \geq s > \alpha$, there exists $k \in \mathbb{N}$ such that $s + (k-1)\tau \leq t < s + k\tau$ and hence

$$\|x(t)\| \overset{(3.12)}{\leq} \frac{1}{2}\|x(t-\tau)\| \overset{(3.12)}{\leq} \dots \overset{(3.12)}{\leq} \frac{1}{2^{k-1}}\|x(t-(k-1)\tau)\| \overset{(3.11)}{\leq} \frac{\gamma^{1/p}}{2^{k-1}}\|x(s)\| < 2\gamma^{1/p} e^{-\frac{\ln 2}{\tau}(t-s)}\|x(s)\|,$$

which gives $k_B(E,A) \leq -\frac{\ln 2}{\tau} < 0$. This completes the proof of the theorem. $\square$

Remark 3.10 (Lyapunov function)**.** As a consequence of [2, Thm. 5.2], the existence of a Lyapunov function for (E,A) (see [2, Def. 5.1]) is sufficient for the Bohl exponent $k_B(E,A)$ to be negative (or $-\infty$). $\diamond$

The following proposition states a shift property of the Bohl exponent, i.e. that, roughly speaking, adding a scalar multiple of E to A does shift the Bohl exponent $k_B(E,A)$ by exactly this scalar. It is a generalization of [16, Lem. 3.4], see also [14, Lem. 2.4]; it has been proved for DAEs of index 1 in [9, Rem. 4.9].

Proposition 3.11 (Shift property)**.** *For $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ and $(1,a) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R})^2$ the following statements hold true:*

(i) $k_B(1,a) = \inf\left\{ \omega \in \mathbb{R} \;\middle|\; \exists\, M_\omega > 0\ \forall\, t \geq s > 0 : \ e^{\int_s^t a(\tau)\,\mathrm{d}\tau} \leq M_\omega e^{\omega(t-s)} \right\}$.

(ii) *If $k_B(1,a) < \infty$, then* $k_B(1,a) = \limsup_{s,t-s\to\infty} \frac{\int_s^t a(\tau)\,\mathrm{d}\tau}{t-s}$.

(iii) *If* $k_B(1,a) = \lim_{s,t-s\to\infty} \frac{\int_s^t a(\tau)\,\mathrm{d}\tau}{t-s} < \infty$ *(note that we require the* lim *here instead of the* lim sup*), then*

  (a) $k_B(1,-a) = -k_B(1,a)$ *and*
  (b) $k_B(E, A + aE) = k_B(E,A) + k_B(1,a)$.

Proof: (i), (ii), (iii)(a) are simple statements concerning ODEs, the proofs of which can be found in [16, Sec. 4.3]. We prove (iii)(b). First, we show

$$k_B(E, A + aE) \le k_B(E, A) + k_B(1, a). \tag{3.13}$$

If $k_B(E, A) = \infty$ inequality (3.13) holds trivially. Suppose $k_B(E, A) < \infty$ and let $\rho > k_B(E, A)$, $\omega > k_B(1, a)$. Then there exist $N_\rho, M_\omega > 0$ such that (E, A) has property $\mathcal{B}(-\rho, N_\rho)$ and $(1, a)$ has property $\mathcal{B}(-\omega, M_\omega)$. Let $x : (\alpha, \infty) \to \mathbb{R}^n$ be any right global solution of $(E, A + aE)$ and let $t^0 > \alpha$. Then simple calculations yield that $z : (\alpha, \infty) \to \mathbb{R}^n,\ t \mapsto e^{-\int_{t^0}^t a(\tau)\,\mathrm{d}\tau} x(t)$ is a right global solution of (E, A) and $x(t^0) = z(t^0)$. Therefore, for all $t \ge s > \alpha$,

$$\begin{aligned} \|x(t)\| &= e^{\int_s^t a(\tau)\,\mathrm{d}\tau} e^{\int_{t^0}^s a(\tau)\,\mathrm{d}\tau} \|z(t)\| \\ &\underset{\mathcal{B}(-\omega,M_\omega)}{\overset{\mathcal{B}(-\rho,N_\rho)}{\le}} N_\rho M_\omega e^{(\rho+\omega)(t-s)} e^{\int_{t^0}^s a(\tau)\,\mathrm{d}\tau} \|z(s)\| = N_\rho M_\omega e^{(\rho+\omega)(t-s)} \|x(s)\|, \end{aligned}$$

thus (3.13) holds. Analogously it can be proved that

$$k_B(E, A) \le k_B(E, A + aE) + k_B(1, -a) \overset{\text{(iii)(a)}}{=} k_B(E, A + aE) - k_B(1, a). \qquad \square$$

Remark 3.12 (Shift and exponential stability)**.** Proposition 3.11 yields that any $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ satisfies:

$$\forall\, \alpha \in \mathbb{R} : \ k_B(E, A + \alpha E) = k_B(E, A) + \alpha.$$

If $k_B(E, A) < \infty$, then one may obtain, via a shift $\alpha < -k_B(E, A)$, that $k_B(E, A + \alpha E) < 0$ and therefore, due to Theorem 3.9, system $(E, A + \alpha E)$ is "essentially" exponentially stable in the sense of Theorem 3.9(ii). With this in mind one can say that the DAE has been stabilized. $\diamond$

Now we consider a time-varying coordinate transformation $z(t) = T(t)^{-1}x(t)$, where $T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$. To this end we recall the definition of equivalence of two linear DAEs (E_1, A_1) and (E_2, A_2); see [17, Def. 3.3].

Definition 3.13 (Equivalence of DAEs)**.** The DAEs $(E_1, A_1), (E_2, A_2) \in \mathcal{C}((\tau, \infty); \mathbb{R}^{n\times n})^2$ are called *equivalent* if, and only if, there exists $(S, T) \in \mathcal{C}((\tau, \infty); \mathbf{Gl}_n(\mathbb{R})) \times \mathcal{C}^1((\tau, \infty); \mathbf{Gl}_n(\mathbb{R}))$ such that

$$E_2 = SE_1T\,, \quad A_2 = SA_1T - SE_1\dot{T}\,; \qquad \text{we write} \quad (E_1,\, A_1) \overset{S,T}{\sim} (E_2,\, A_2)\,. \tag{3.14}$$

$\diamond$

We introduce Lyapunov transformations (see for example [23, Def. 6.14] for ODEs) and Bohl transformations (see for example [15, Def. 3.3.16] for ODEs) on the set of all initial values (t, x) for which (E, A) has a right global solution:

$$\mathcal{G}(E, A) \ := \ \{\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \ \mid \mathcal{G}_{E,A}(t, x) \ne \emptyset\ \}\,.$$

We stress that we consider Lyapunov and Bohl transformations on $\mathcal{G}(E, A)$, not on $\mathbb{R}_+ \times \mathbb{R}^n$. The reason is that the set

$$\mathcal{G}(E, A)(t) \ := \ \{\ x \in \mathbb{R}^n \ \mid (t, x) \in \mathcal{G}(E, A)\ \}\ , \qquad t > 0,$$

is a linear subspace of $\mathbb{R}^n$ and if $x : (\alpha, \infty) \to \mathbb{R}^n$ is a right global solution of (E, A), then $x(t) \in \mathcal{G}(E, A)(t)$ for all $t > \alpha$. As seen in [2], the sets $\mathcal{G}(E, A)$ and $\mathcal{G}(E, A)(t)$ are fundamental for the investigation of stability properties of DAEs (E, A). For ODEs (I, A), clearly $\mathcal{G}(I, A)(t) = \mathbb{R}^n$ for all $t \in \mathbb{R}_+$.

Definition 3.14 (Lyapunov and Bohl transformation)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. Then $T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ is called a *Lyapunov transformation on* $\mathcal{G}(E, A)$ if, and only if,

$$\exists\, \ell_1, \ell_2 > 0\ \forall\, (t, x) \in \mathcal{G}(E, A) :\ \ell_1\|x\| \le \|T(t)^{-1}x\| \le \ell_2\|x\|. \tag{3.15}$$

$T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ is called a *Bohl transformation on* $\mathcal{G}(E, A)$ if, and only if,

$$\inf\left\{\ \varepsilon > 0\ \middle|\ \begin{array}{r} \exists\, M_\varepsilon > 0\ \forall\, t, s \in \mathbb{R}_+\ \forall\, x \in \mathcal{G}(E, A)(s)\ \forall\, z \in T(t)^{-1}\mathcal{G}(E, A)(t) : \\ \|T(s)^{-1}x\| \cdot \|T(t)z\| \le M_\varepsilon e^{\varepsilon|t-s|}\|x\| \cdot \|z\| \end{array}\ \right\} = 0. \tag{3.16}$$

◇

Remark 3.15 (Criterion for Bohl transformation)**.** Note that the definition of Bohl transformations on $\mathcal{G}(E, A)$, $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$, differs from the definition of a Bohl transformation in the ODE case, see [15, Def. 3.3.16]. The difference is that in condition (3.16) we take the infimum over all $\varepsilon > 0$ in contrast to that for ODEs (see [15, Def. 3.3.16]) the infimum is taken over all $\varepsilon \in \mathbb{R}$. So, an alternative to condition (3.16) could be

$$\inf\left\{\ \varepsilon \in \mathbb{R}\ \middle|\ \begin{array}{r} \exists\, M_\varepsilon > 0\ \forall\, t, s \in \mathbb{R}_+\ \forall\, x \in \mathcal{G}(E, A)(s)\ \forall\, z \in T(t)^{-1}\mathcal{G}(E, A)(t) : \\ \|T(s)^{-1}x\| \cdot \|T(t)z\| \le M_\varepsilon e^{\varepsilon|t-s|}\|x\| \cdot \|z\| \end{array}\ \right\} = 0. \tag{3.17}$$

If (E, A) is an ODE, then $\mathcal{G}(E, A) = \mathbb{R}_+ \times \mathbb{R}^n$ and it is easily seen that conditions (3.16) and (3.17) are equivalent. But for DAEs it may be that the infimum in (3.17) is $-\infty$, due to the restriction to the set $\mathcal{G}(E, A)$. For instance, consider $(E, A) = (0, I)$. Then $\mathcal{G}(0, I)(t) = \{0\}$ for all $t \in \mathbb{R}_+$ and any state space transformation $T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ would transform the system $0 = x$ into $0 = z$ and hence preserve all properties (in particular the Bohl exponents), but T does not satisfy condition (3.17) since the infimum equals $-\infty$; in fact T satisfies (3.16). ◇

Remark 3.16 (Lyapunov and Bohl transformation)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$.

(i) If (E, A) is an ODE, then $\mathcal{G}(E, A) = \mathbb{R}_+ \times \mathbb{R}^n$ and $T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ is a Lyapunov transformation on $\mathcal{G}(E, A)$ if, and only if, T and T^{-1} are bounded.

(ii) Neither the Lyapunov nor the Bohl transformations on $\mathcal{G}(E, A)$ form a group with respect to pointwise multiplication, since, for instance, the product of two Lyapunov transformations on $\mathcal{G}(E, A)$ is not necessarily a Lyapunov transformation on $\mathcal{G}(E, A)$. This is due to the restriction to the set $\mathcal{G}(E, A)$. However, the following holds true: If

$$(E,\, A)\ \overset{S,T}{\sim}\ (\tilde{E},\, \tilde{A}) \qquad \text{for some } S \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R})),\ T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R})),$$

T is a Lyapunov (Bohl) transformation on $\mathcal{G}(E, A)$, and $\tilde{T} \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ is a Lyapunov (Bohl) transformation on $\mathcal{G}(\tilde{E}, \tilde{A})$, then $T\tilde{T}$ is a Lyapunov (Bohl) transformation on $\mathcal{G}(E, A)$. Concerning Lyapunov transformations this is easily verified. For Bohl transformations this needs a proof: Let $\varepsilon > 0$. Then there exist $M, N > 0$ such that

$$\forall\, t, s \in \mathbb{R}_+\ \forall\, x \in \mathcal{G}(E, A)(s)\ \forall\, z \in T(t)^{-1}\mathcal{G}(E, A)(t) :\ \|T(s)^{-1}x\| \cdot \|T(t)z\| \le Me^{\frac{\varepsilon}{2}|t-s|}\|x\| \cdot \|z\|, \tag{3.18}$$

$$\forall\, t, s \in \mathbb{R}_+\ \forall\, x \in \mathcal{G}(\tilde{E}, \tilde{A})(s)\ \forall\, z \in \tilde{T}(t)^{-1}\mathcal{G}(\tilde{E}, \tilde{A})(t) :\ \|\tilde{T}(s)^{-1}x\| \cdot \|\tilde{T}(t)z\| \le Ne^{\frac{\varepsilon}{2}|t-s|}\|x\| \cdot \|z\|. \tag{3.19}$$

Invoking that $\mathcal{G}(\tilde{E}, \tilde{A})(t) = T(t)^{-1}\mathcal{G}(E, A)(t)$ for all $t \in \mathbb{R}_+$, we have, for all $t, s \in \mathbb{R}_+$ and for all $x \in \mathcal{G}(E, A)(s)$, $z \in (T(t)\tilde{T}(t))^{-1}\mathcal{G}(E, A)(t)$,

$$\begin{aligned} &\|\tilde{T}(s)^{-1}T(s)^{-1}x\| \cdot \|T(t)\tilde{T}(t)z\| \cdot \|\tilde{T}(t)z\| \cdot \|T(s)^{-1}x\| \\ &\qquad\qquad \overset{(3.18),(3.19)}{\le} MNe^{\varepsilon|t-s|}\|T(s)^{-1}x\| \cdot \|z\| \cdot \|x\| \cdot \|\tilde{T}(t)z\|. \end{aligned}$$

If $T(s)^{-1}x = 0$, then $\tilde{T}(s)^{-1}T(s)^{-1}x = 0$ and if $\tilde{T}(t)z = 0$, then $T(t)\tilde{T}(t)z = 0$. If both are unequal to zero we may divide by their norms. In each case we obtain that

$$\|\tilde{T}(s)^{-1}T(s)^{-1}x\| \cdot \|T(t)\tilde{T}(t)z\| \leq MNe^{\varepsilon|t-s|}\|x\| \cdot \|z\|,$$

whence $T\tilde{T}$ is a Bohl transformation on $\mathcal{G}(E, A)$.

(iii) Any Lyapunov transformation on $\mathcal{G}(E, A)$ is a Bohl transformation on $\mathcal{G}(E, A)$. This is easily seen by invoking that (3.15) is equivalent to

$$\exists\, \ell_1, \ell_2 > 0\ \forall\, t \in \mathbb{R}_+\ \forall\, z \in T(t)^{-1}\mathcal{G}(E, A)(t):\ \ell_2^{-1}\|z\| \leq \|T(t)z\| \leq \ell_1^{-1}\|z\|, \tag{3.20}$$

and, for all $\varepsilon > 0$, $t, s \in \mathbb{R}_+, x \in \mathcal{G}(E, A)(s)$ and $z \in T(t)^{-1}\mathcal{G}(E, A)(t)$,

$$\|T(s)^{-1}x\| \cdot \|T(t)z\| \overset{(3.15),(3.20)}{\leq} \frac{\ell_2}{\ell_1}\|x\| \cdot \|z\| \leq \frac{\ell_2}{\ell_1}e^{\varepsilon|t-s|}\|x\| \cdot \|z\|.$$

(iv) There exist Bohl transformations on $\mathcal{G}(E, A)$ which are not Lyapunov transformations on $\mathcal{G}(E, A)$. Consider for example, for any ODE (I, A) and $n = 1$, the state space transformation $T : \mathbb{R}_+ \to \mathbb{R}_+,\ t \mapsto t + 1$. Then, for any $\varepsilon > 0$ and $t, s \in \mathbb{R}_+$,

$$\frac{t+1}{s+1} = \frac{t-s}{s+1} + 1 \leq |t-s| + 1 \leq \begin{cases} \frac{1}{\varepsilon}(\varepsilon|t-s| + \varepsilon) \leq \frac{1}{\varepsilon}(\varepsilon|t-s| + 1) \leq \frac{1}{\varepsilon}e^{\varepsilon|t-s|}, & 0 < \varepsilon \leq 1 \\ e^{|t-s|} \leq e^{\varepsilon|t-s|}, & \varepsilon > 1 \end{cases}$$

and hence T is a Bohl transformation on $\mathcal{G}(I, A) = \mathbb{R}_+ \times \mathbb{R}$, but obviously no Lyapunov transformation. ◇

If (E, A) is transferable into standard canonical form [8, 3], then in [2, Prop. 4.10] it is shown that Lyapunov transformations on $\mathcal{G}(E, A)$ preserve the properties of stability, attractivity and asymptotic stability of a DAE. The next proposition shows that the Bohl exponent, and therefore the property of "essential" exponential stability, is preserved under the larger set of Bohl transformations on $\mathcal{G}(E, A)$.

Proposition 3.17 (Bohl exponent is preserved under Bohl transformation). *Consider* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. *If*

$$(E,\, A) \overset{S,T}{\sim} (\tilde{E},\, \tilde{A}) \qquad \text{for some } S \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R})),\ T \in \mathcal{C}^1(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$$

and T *is a Bohl transformation on* $\mathcal{G}(E, A)$, *then*

$$k_B(E, A) = k_B(\tilde{E}, \tilde{A}).$$

Proof: First observe that $\mathcal{G}(\tilde{E}, \tilde{A})(t) = T(t)^{-1}\mathcal{G}(E, A)(t)$ for all $t \in \mathbb{R}_+$. Hence T^{-1} is a Bohl transformation on $\mathcal{G}(\tilde{E}, \tilde{A})$ and therefore it is sufficient to show that $k_B(E, A) \leq k_B(\tilde{E}, \tilde{A})$.
Case 1: $k_B(\tilde{E}, \tilde{A}) = \infty$. Then clearly $k_B(E, A) \leq k_B(\tilde{E}, \tilde{A})$.
Case 2: $k_B(\tilde{E}, \tilde{A}) < \infty$. Let $\rho > k_B(\tilde{E}, \tilde{A})$ and $\varepsilon > 0$. Then there exists $N_\rho > 0$ such that

$$\forall \text{ right global sln. } z : (\alpha, \infty) \to \mathbb{R}^n \text{ of } (\tilde{E}, \tilde{A})\ \forall\, t \geq s > \alpha:\ \|z(t)\| \leq N_\rho e^{\rho(t-s)}\|z(s)\|, \tag{3.21}$$

and there exists $M_\varepsilon > 0$ such that

$$\forall\, t, s \in \mathbb{R}_+\ \forall\, x \in \mathcal{G}(E, A)(s)\ \forall\, z \in T(t)^{-1}\mathcal{G}(E, A)(t) : \|T(s)^{-1}x\| \cdot \|T(t)z\| \leq M_\varepsilon e^{\varepsilon|t-s|}\|x\| \cdot \|z\|. \tag{3.22}$$

Let $x : (\alpha,\infty) \to \mathbb{R}^n$ be a right global solution of (E, A) and $t \geq s > \alpha$. Then $z := T^{-1}x : (\alpha,\infty) \to \mathbb{R}^n$ is a right global solution of $(\tilde{E}, \tilde{A})$ and

$$\begin{aligned}\|x(t)\| \cdot \|z(s)\| = \|T(t)z(t)\| \cdot \|T(s)^{-1}x(s)\| &\overset{(3.22)}{\leq} M_\varepsilon e^{\varepsilon(t-s)}\|x(s)\| \cdot \|z(t)\| \\ &\overset{(3.21)}{\leq} M_\varepsilon N_\rho e^{(\varepsilon+\rho)(t-s)}\|x(s)\| \cdot \|z(s)\|. \end{aligned} \tag{3.23}$$

If $z(s) = 0$ then, invoking (3.22), $x(t) = x(s) = z(t) = 0$. If $z(s) \neq 0$ we may divide both sides of the inequality (3.23) by $\|z(s)\|$. We obtain

$$\forall\, t \geq s > \alpha :\ \|x(t)\| \leq M_\varepsilon N_\rho e^{(\varepsilon+\rho)(t-s)}\|x(s)\|.$$

This gives $k_B(E, A) \leq k_B(\tilde{E}, \tilde{A})$ and completes the proof. □

4 DAEs which possess a transition matrix

In this section we introduce a subclass of DAEs (E, A) which will be of special interest in due course, especially for robustness analysis. This is the set of all DAEs (E, A) for which, loosely speaking, any consistent initial value problem has a unique solution and there exists a transition matrix for the system.

Definition 4.1 (DAEs which possess a transition matrix)**.** We say that the DAE $(E, A) \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *possesses a transition matrix* $U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ if, and only if, uniqueness holds, i.e.

$$\forall\,(t^0, x^0) \in \mathcal{V}_{E,A}\ \forall \text{sln. } x_1, x_2 \in \mathcal{S}_{E,A}(t^0, x^0) :\ x_1|_{\operatorname{dom} x_1 \cap \operatorname{dom} x_2} = x_2|_{\operatorname{dom} x_1 \cap \operatorname{dom} x_2}\,, \tag{4.1}$$

and

$$\left.\begin{array}{l} \forall\, t \in \mathbb{R}_+ :\ U(t,\cdot), U(\cdot,t) \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n}), \\ \forall\, t, s \in \mathbb{R}_+ :\ E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = A(t)U(t,s), \\ \forall\, t, s, r \in \mathbb{R}_+ :\ U(t,s)U(s,r) = U(s,r), \\ \forall\, t, s \in \mathbb{R}_+ :\ \operatorname{im} U(t,s) \subseteq \mathcal{V}_{E,A}(t)\ \wedge\ \operatorname{im} U(t,t) = \mathcal{V}_{E,A}(t). \end{array}\right\} \tag{4.2}$$

◇

There are several immediate consequences for the transition matrix from Definition 4.1.

Corollary 4.2 (Properties of the transition matrix)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot,\cdot)$. Then the following holds true for all $t \in \mathbb{R}_+$:*

(i) $U(t,t)^2 = U(t,t)$,

(ii) $\forall\, x \in \mathcal{V}_{E,A}(t) :\ U(t,t)x = x$,

(iii) $\forall\, x^0 \in \mathcal{V}_{E,A}(t)\ \forall\, x(\cdot) \in \mathcal{S}_{E,A}(t, x^0)\ \forall\, s \in \operatorname{dom} x :\ x(s) = U(s,t)x^0$. ◇

In particular, the large class of DAEs which are sufficiently smooth and satisfy [17, Hypothesis 3.48] is a subclass of DAEs which possess a transition matrix. Uniqueness of the solution of the consistent initial value problem is shown in [17, Thm. 3.52] and the existence of a transition matrix with the respective properties is shown in [18, Sec. 3.1]. Therefore, also any DAE which is analytically solvable, has a well-defined differentiation index or is transferable into standard canonical form satisfies Definition 4.1, see [17, 8, 3].
However, there are DAEs which are in none of these classes but still possess a transition matrix. To

this end consider [3, Ex. 4.3] with $\mathcal{I} = \mathbb{R}_+$. Then the homogeneous system (E, A) has only the trivial solution and hence Definition 4.1 is trivially satisfied. However, the system is not analytically solvable (and therefore does not satisfy [17, Hypothesis 3.48]) since e.g. for $f(t) = t$ the inhomogeneous equation has no global solution (there is a pole at $t = 1$ as the solution formula clearly indicates).
In particular, this shows that the existence of a transition matrix does not guarantee that variation of constants does work, since the solution formula in [3, Ex. 4.3] shows that only for a small class of inhomogeneities there exists a global solution. Variation of constants is only known to work always in the case of systems transferable into standard canonical form [3, Thm. 3.9].

Although there is no variation of constants formula for DAEs which possess a transition matrix, there is a class of inhomogeneities for which the solutions can be represented via an integral equation. In a way one has to find the appropriate set of inhomogeneities to make variations of constants feasible. In fact the inhomogeneities in this case may also depend on x and $\dot{x}$. In subsequent sections this will be used for robustness analysis.
We allow for inhomogeneities g as follows: Let $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ be such that $\operatorname{im} \Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$. For instance, $t \mapsto U(t,t)$ satisfies this condition. Then, for any $g \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, we consider the "perturbed" DAE

$$E(t)\dot{x} = A(t)x + E(t)\Pi(t)g(t, x(t), \dot{x}(t)). \tag{4.3}$$

The following lemma shows that x solves (4.3) if, and only if, it solves a certain integral equation involving the transition matrix of (E, A).

Lemma 4.3 (Integral equation). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot,\cdot)$, choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im} \Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$ and let $g \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Then $x : (\alpha, \infty) \to \mathbb{R}^n$, $\alpha \geq 0$, is a right global solution of (4.3) if, and only if, $x(\cdot)$ satisfies the integral equation*

$$\forall\, t, t^0 > \alpha : \; x(t) = U(t, t^0)x(t^0) + \int_{t^0}^{t} U(t, s)\Pi(s)g(s, x(s), \dot{x}(s)) \,\mathrm{d}s\,.$$

Proof: "$\Rightarrow$": Fix $t^0 > \alpha$ and consider the functions

$$\begin{aligned} h &: (\alpha, \infty) \to \mathbb{R}^n,\; t \mapsto \; x(t) - \textstyle\int_{t^0}^{t} U(t, s)\Pi(s)g(s, x(s), \dot{x}(s)) \,\mathrm{d}s\,, \\ g &: (\alpha, \infty) \to \mathbb{R}^n,\; t \mapsto \; U(t, t^0)x(t^0). \end{aligned}$$

Then, for all $t > \alpha$,

$$\begin{aligned} E(t)\dot{h}(t) &= E(t)\dot{x}(t) - \int_{t^0}^{t} E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t, s)\Pi(s)g(s, x(s), \dot{x}(s)) \,\mathrm{d}s - E(t)U(t, t)\Pi(t)g(t, x(t), \dot{x}(t)) \\ &\overset{\text{Def. 4.1}}{\underset{\text{Cor. 4.2}}{=}} A(t)x(t) + E(t)\Pi(t)g(t, x(t), \dot{x}(t)) - A(t)\int_{t^0}^{t} U(t, s)\Pi(s)g(s, x(s), \dot{x}(s)) \,\mathrm{d}s - E(t)\Pi(t)g(t, x(t), \dot{x}(t)) \\ &= A(t)h(t), \end{aligned}$$

thus h and g are both right global solutions of (E, A). Therefore, $x(t^0) = h(t^0) \in \mathcal{V}_{E,A}(t^0)$ and Corollary 4.2 gives

$$g(t^0) = U(t^0, t^0)x(t^0) = x(t^0) = h(t^0).$$

Hence h and g are both right global solutions of the initial value problem

$$E(t)\dot{z} = A(t)z, \quad z(t^0) = x(t^0) \in \mathcal{V}_{E,A}(t^0),$$

whence (4.1) yields $h = g$.
"$\Leftarrow$": For fixed $t^0 > \alpha$ differentiation immediately gives, for all $t > \alpha$,

$$\begin{aligned}
&E(t)\dot{x}(t) \\
&\quad = E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,t^0)x(t^0) + \int_{t^0}^{t} E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s)\Pi(s)g(s,x(s),\dot{x}(s))\ \mathrm{d}s\ + E(t)U(t,t)\Pi(t)g(t,x(t),\dot{x}(t)) \\
&\qquad = A(t)U(t,t^0)x(t^0) + A(t)\int_{t^0}^{t} U(t,s)\Pi(s)g(s,x(s),\dot{x}(s))\ \mathrm{d}s\ + E(t)\Pi(t)g(t,x(t),\dot{x}(t)) \\
&\qquad\qquad = A(t)x(t) + E(t)\Pi(t)g(t,x(t),\dot{x}(t)).
\end{aligned}$$

□

We show next that the time-varying subspace $\mathcal{V}_{E,A}(t)$ of a DAE (E,A) which possesses a transition matrix does not have jumps in its dimension.

Corollary 4.4 ($\mathcal{V}_{E,A}(t)$ has constant dimension)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot,\cdot)$. Then there exists $k \in \{0,1,\ldots,n\}$ such that $\dim \mathcal{V}_{E,A}(t) = k$ for all $t \in \mathbb{R}_+$.*

Proof: Fix $s \in \mathbb{R}_+$ and set $k := \dim \mathcal{V}_{E,A}(s)$. We distinguish two cases.
Case 1: $k = 0$. Then $U(s,s) = 0$ and hence

$$U(t,t) \overset{(4.2)}{=} U(t,s)U(s,s)U(s,t) = 0$$

which gives $\dim \mathcal{V}_{E,A}(t) = 0$ for all $t \in \mathbb{R}_+$.
Case 2: $k \geq 1$. Let $\varphi_1,\ldots,\varphi_k \in \mathcal{V}_{E,A}(s)$ be linearly independent. Then, for $i \in \{1,\ldots,k\}$, $t \mapsto U(t,s)\varphi_i$ solves (E,A) due to (4.2) and hence $U(t,s)\varphi_i \in \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$.
Now fix $t \in \mathbb{R}_+$. In order to show that $U(t,s)\varphi_1,\ldots,U(t,s)\varphi_k$ are linearly independent let

$$c_1U(t,s)\varphi_1 + \ldots + c_kU(t,s)\varphi_k = 0$$

for some $c_1,\ldots,c_k \in \mathbb{R}$. Therefore, $U(t,s)y = 0$ for $y := c_1\varphi_1 + \ldots + c_k\varphi_k \in \mathcal{V}_{E,A}(s)$ and hence

$$y = U(s,s)y = U(s,t)U(t,s)y = 0,$$

which gives, by linear independency of $\varphi_1,\ldots,\varphi_k$, that $c_1 = \ldots = c_k = 0$. This proves that $\dim \mathcal{V}_{E,A}(s) \leq \dim \mathcal{V}_{E,A}(t)$ and equality follows since s and t were fixed but arbitrary. □

It remains an open problem as to whether, for the class of DAEs which possess a transition matrix, (4.2) in Definition 4.1 does already imply uniqueness (4.1). Moreover, the integral equation developed in Lemma 4.3 should be further investigated, especially regarding existence and uniqueness of solutions.

5 Bohl exponents for DAEs which possess a transition matrix

In this section we consider Bohl exponents for the class of DAEs which possess a transition matrix as introduced in the previous section. We revisit some results of Section 3 for this class, state a relationship between solutions of a Cauchy problem and negative Bohl exponent, and then derive a robustness result using the Bohl exponent.

It is immediate from Definition 4.1 and Corollary 4.2, that the Bohl exponent of a DAE $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ which possesses a transition matrix $U(\cdot,\cdot)$ has the following representation:

$$k_B(E,A) = \inf\left\{\ \rho \in \mathbb{R}\ \middle|\ \exists\, N_\rho > 0\ \forall\, t \geq s > 0\ \forall\, x \in \mathcal{V}_{E,A}(s):\ \|U(t,s)x\| \leq N_\rho e^{\rho(t-s)}\|x\|\ \right\}. \quad (5.1)$$

Remark 5.1 (Time-invariant linear DAEs)**.** We give a complete characterization of the Bohl exponent of time-invariant linear DAEs $(E, A) \in (\mathbb{R}^{n\times n})^2$. To this end we distinguish the cases that either (E, A) is *regular*, i.e. $\det(sE - A) \not\equiv 0$, or not.
Case 1: (E, A) is regular. In this case we can find invertible matrices $S, T \in \mathbb{R}^{n\times n}$ which transform (E, A) into quasi-Weierstraß form [4]

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

where J is some matrix and N is nilpotent. Note that it is not sufficient to consider the Weierstraß canonical form as in [17, Thm. 2.7], since we require the matrices J, S, T to be real-valued. Now it follows [4] that (E, A) possesses a transition matrix

$$(t, s) \mapsto U(t, s) = T \begin{bmatrix} e^{J(t-s)} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Then formula (5.1) and some simple calculations yield that

$$k_B(E, A) = \max\{ \operatorname{Re}\lambda \mid \lambda \in \sigma(J) \} = \max\{ \operatorname{Re}\lambda \mid \det(\lambda E - A) = 0 \}.$$

This result was already mentioned in [9, Rem. 4.2], but without proof.
Case 2: (E, A) is not regular. In this case we have $\det(\lambda E - A) = 0$ for all $\lambda \in \mathbb{C}$. Therefore, given any $\lambda \in \mathbb{R}$, we find $x_\lambda \in \mathbb{R}^n \setminus \{0\}$ such that $(\lambda E - A)x_\lambda = 0$. Hence, $t \mapsto e^{\lambda t}x_\lambda$ solves the DAE (E, A) for all $\lambda \in \mathbb{R}$, which shows that (E, A) cannot have a finite Bohl exponent, i.e. we obtain

$$k_B(E, A) = \infty.$$

◇

In order to extend Theorem 3.9 to DAEs which possess a transition matrix in such a way that a negative Bohl exponent is equivalent to exponential stability we first need a characterization of exponential stability for this class of DAEs. An immediate characterization, due to Definition 4.1 and Corollary 4.2, is the following: A DAE $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ which possesses a transition matrix $U(\cdot, \cdot)$ is exponentially stable if, and only if,

$$\exists\, \mu, M > 0\ \forall\, (t^0, x^0) \in \mathcal{V}_{E,A}\ \forall\, t \geq t^0 :\ \|U(t, t^0)x^0\| \leq M\, e^{-\mu(t-t^0)}\|x^0\|. \tag{5.2}$$

The proof of this equivalence uses arguments similar to the ones used in Proposition 5.2; we omit it. Note that in particular equations (5.1) and (5.2) show that a DAE (E, A) which possesses a transition matrix is exponentially stable if, and only if, $k_B(E, A) < 0$.
We may also obtain a characterization via replacing x^0 in (5.2) by $\Pi(t^0)$, where Π is continuous and bounded and $\Pi(t^0)$ is a projector on $\mathcal{V}_{E,A}(t^0)$.

Proposition 5.2 (Exponential stability of normal DAEs)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot, \cdot)$ and choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im}\Pi(t) = \mathcal{V}_{E,A}(t)$ and $\Pi(t)^2 = \Pi(t)$ for all $t \in \mathbb{R}_+$. Furthermore, suppose Π is bounded. Then (E, A) is exponentially stable if, and only if,*

$$\exists\, \mu, M > 0\ \forall\, t \geq t^0 > 0 :\ \|U(t, t^0)\Pi(t^0)\| \leq Me^{-\mu(t-t^0)}.$$

Proof: "⇒": Let $t^0 \in \mathbb{R}_+$ and note that Corollary 4.2(iii) implies that any right maximal solution of (E, A) is right global. Thus by Definition 2.2 we have

$$\exists\, \mu, M > 0\ \exists\, \delta = \delta(t^0) > 0\ \forall\, x^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}_{E,A}(t^0)\ \forall\, t \geq t^0 :\ \|U(t, t^0)x^0\| \leq Me^{-\mu(t-t^0)}\|x^0\|. \tag{5.3}$$

Now let $i \in \{1, ..., n\}$. If $\Pi(t^0)e_i \neq 0$, then

$$\begin{aligned} &\forall\, t \geq t^0 : \left\| U(t,t^0) \frac{\delta \Pi(t^0) e_i}{2\|\Pi(t^0)e_i\|} \right\| \leq M e^{-\mu(t-t^0)} \left\| \frac{\delta \Pi(t^0) e_i}{2\|\Pi(t^0)e_i\|} \right\| \\ \implies &\forall\, t \geq t^0 : \left\| U(t,t^0)\Pi(t^0)e_i \right\| \leq M e^{-\mu(t-t^0)} \left\| \Pi(t^0)e_i \right\| \leq M e^{-\mu(t-t^0)} \left\| \Pi(t^0) \right\| . \end{aligned}$$

If $\Pi(t^0)e_i = 0$ the latter inequality trivially holds, hence it is true for all $i \in \{1, ..., n\}$. Using equivalence of norms and boundedness of Π the claim finally follows from some simple calculations.
"$\Leftarrow$": Let $t^0 > 0$. We show that (5.3) holds. Choose $\delta = 1$ and let $x^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}_{E,A}(t^0)$. Since $\Pi(t^0)$ is idempotent we find that $\Pi(t^0)x^0 = x^0$, hence

$$\forall\, t \geq t^0 : \|U(t,t^0)x^0\| = \|U(t,t^0)\Pi(t^0)x^0\| \leq \|U(t,t^0)\Pi(t^0)\| \cdot \|x^0\| \overset{\text{Ass.}}{\leq} M e^{-\mu(t-t^0)} \|x^0\|.$$

This proves the assertion. □

It is clear that the assumption of boundedness of Π in Proposition 5.2 is crucial, since otherwise one may easily construct a counterexample. The assumption of Π being idempotent, i.e. $\Pi(t)^2 = \Pi(t)$ for all $t \in \mathbb{R}_+$, however may possibly be relaxed.
Whereas, having the transition matrix $U(\cdot,\cdot)$ of the DAE (E, A), one may define

$$\Pi : \mathbb{R}_+ \to \mathbb{R}^{n \times n},\ t \mapsto \begin{cases} U(t,t), & \|U(t,t)\| \leq 1 \\ \frac{U(t,t)}{\|U(t,t)\|}, & \|U(t,t)\| > 1 \end{cases},$$

which satisfies all assumptions. Therefore, if ever boundedness or idempotency of Π is required, this is no great restriction in the free choice of Π.

We are now in the position to state a "simplified" version of Theorem 3.9 for DAEs which possess a transition matrix. To this end we use the bounded projector Π and therefore it is clear that the following corollary is a direct generalization of [15, Thm. 3.3.15] to DAEs which possess a transition matrix, since in the case $E = I$ we would have $\Pi = I$ and $U(\cdot,\cdot)$ would be the transition matrix of the ODE (I, A).

Corollary 5.3 (Theorem 3.9 revisited). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times n})^2$ possess a transition matrix $U(\cdot,\cdot)$ and choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times n})$ such that $\operatorname{im} \Pi(t) = \mathcal{V}_{E,A}(t)$ and $\Pi(t)^2 = \Pi(t)$ for all $t \in \mathbb{R}_+$. Furthermore, suppose Π is bounded and $k_B(E, A) < \infty$. Then the following statements are equivalent.*

(i) *$k_B(E, A) < 0$,*

(ii) *(E, A) is exponentially stable,*

(iii)
$$\forall\, p \in \mathbb{R}_+\ \exists\, c > 0\ \forall\, t^0 \in \mathbb{R}_+ : \int_{t^0}^{\infty} \|U(t,t^0)\Pi(t^0)\|^p \,\mathrm{d}t \leq c.$$

Proof: (i)$\Leftrightarrow$(ii): Follows directly from equations (5.1) and (5.2).
(i)$\Leftrightarrow$(iii): By Theorem 3.9 and Corollary 4.2 it follows that $k_B(E, A) < 0$ if, and only if,

$$\forall\, p \in \mathbb{R}_+\ \exists\, c > 0\ \forall\, (t^0, x^0) \in \mathcal{V}_{E,A} : \int_{t^0}^{\infty} \|U(t,t^0)x^0\|^p \,\mathrm{d}t \leq c\|x^0\|^p.$$

Now using the same arguments as in the proof of Proposition 5.2 yields the assertion. □

Note that the matrix-valued function Π is not needed for the equivalence of (i) and (ii) in Corollary 5.3.

In the following we consider Cauchy problems corresponding to the DAE (E, A). In particular, we derive the result that, loosely speaking, the Bohl exponent of (E, A) is negative if, and only if, the Cauchy problem for a special class of bounded inhomogeneities has a bounded solution. For ODEs this is well-known [11, Thm. 5.2] and the class of inhomogeneities is just the set of all bounded continuous functions. However, for DAEs the situation is more subtle, since solutions do not necessarily exist for arbitrary inhomogeneities and variation of constants is, as explained in Section 4, in general not feasible. So one has to find the correct class of inhomogeneities in order to generalize [11, Thm. 5.2] to DAEs.
We call a continuously differentiable function $x : (0, \omega) \to \mathbb{R}^n$, $\omega > 0$, a solution of the *Cauchy problem*

$$E(t)\dot{x} = A(t)x + f(t), \quad x(0+) = x^0, \tag{5.4}$$

where $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$, if, and only if, x satisfies $E(t)\dot{x}(t) = A(t)x(t) + f(t)$ for all $t \in (0, \omega)$ and $x(0+) = x^0$.

In Section 4 we have seen that inhomogeneities of the form $E\Pi f$, where $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is such that $\operatorname{im} \Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$, are the correct concept when it comes to DAEs (E, A) which possess a transition matrix $U(\cdot, \cdot)$. In Theorem 5.5 we show that the Cauchy problem corresponding to the inhomogeneity $E\Pi f$ has a bounded solution whenever Πf is bounded if, and only if, $k_B(E, A) < 0$. This equivalence justifies the consideration of the special perturbations $E\Pi g$ as in (4.3).
In the following lemma we show that the Cauchy problem has a unique global solution for inhomogeneities f as described above. However, we require the additional technical assumption that the limits $(\Pi f)(0+)$ and $U(t, 0+)$ exist for all $t \in \mathbb{R}_+$.

Lemma 5.4 (Unique solution to the Cauchy problem). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot, \cdot)$ and choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im} \Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$. Then, for any $t^0 \in \mathbb{R}_+$ and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$, all right global solutions $x(\cdot)$ of the initial value problem*

$$E(t)\dot{x} = A(t)x + E(t)\Pi(t)f(t), \quad x(t^0) = 0, \tag{5.5}$$

satisfy

$$\forall\, t \geq t^0 : \; x(t) = \int_{t^0}^{t} U(t, s)\Pi(s)f(s) \, \mathrm{d}s\,.$$

If additionally $U(t, 0+)$ exists for all $t \in \mathbb{R}_+$, then, for any $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$ such that $(\Pi f)(0+)$ exists, the Cauchy problem

$$E(t)\dot{x} = A(t)x + E(t)\Pi(t)f(t), \quad x(0+) = 0, \tag{5.6}$$

has the unique global solution

$$x : \mathbb{R}_+ \to \mathbb{R}^n, \; t \mapsto \int_0^t U(t, s)\Pi(s)f(s) \, \mathrm{d}s\,.$$

Proof: Let $t^0 \in \mathbb{R}_+$ and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. Since (E, A) possesses a transition matrix $U(\cdot, \cdot)$ we may apply Lemma 4.3 to conclude that any right global solution $x : (\alpha, \infty) \to \mathbb{R}^n$ of (5.5), $\alpha < t^0$, has the representation

$$\forall\, t > \alpha : \; x(t) = U(t, t^0)x(t^0) + \int_{t^0}^{t} U(t, s)\Pi(s)f(s) \, \mathrm{d}s\,,$$

which, in view of $x(t^0) = 0$, proves the claim.
Considering the Cauchy problem (5.6) we find that, due to Lemma 4.3, any right global solution $x : (0, \infty) \to \mathbb{R}^n$ of (5.6) has the representation

$$\forall\, t, t^0 > 0 : \; x(t) = U(t, t^0)x(t^0) + \int_{t^0}^{t} U(t, s)\Pi(s)f(s) \; \mathrm{d}s\,.$$

Since $\lim_{t^0 \searrow 0} x(t^0) = x(0+) = 0$ and the limits $U(t, 0+), (\Pi f)(0+)$ exist for all $t \in \mathbb{R}_+$, we may take the limit for $t^0 \to 0$ and obtain

$$\forall\, t > 0 : \; x(t) = \int_{0}^{t} U(t, s)\Pi(s)f(s) \; \mathrm{d}s\,.$$

On the other hand, simple calculations show that this is indeed a solution of (5.6), hence the lemma is proved. □

In view of Lemma 5.4 we collect all functions f such that (5.6) has a unique global solution:

$$F_\Pi := \{ \; f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n) \;\mid f \text{ is bounded and } (\Pi f)(0+) = 0 \; \}\,. \tag{5.7}$$

Even for ODEs the result [11, Thm. 5.2] does not hold without further assumptions: the usual assumption is *integral boundedness* of A, see e.g. [11, (5.9)]. However, a careful inspection of the proof of [11, Thm. 5.2] yields that this assumption can be weakened; we formulate it for DAEs which possess a transition matrix $U(\cdot, \cdot)$:

$$\exists\, M > 0 \;\forall\, t^0 \in \mathbb{R}_+ \;\forall\, t \in [t^0, t^0 + 1] : \; \|U(t, t^0)\Pi(t^0)\| \le M. \tag{5.8}$$

The following theorem is the generalization of [11, Thm. 5.2] to DAEs (E, A) which possess a transition matrix. Although the main idea of the proof is as in [11] it is more subtle.

Theorem 5.5 (Cauchy problem)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot, \cdot)$ and choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im}\Pi(t) = \mathcal{V}_{E,A}(t)$ and $\Pi(t)^2 = \Pi(t)$ for all $t \in \mathbb{R}_+$. Suppose that condition (5.8) holds and $U(t, 0+)$ exists for all $t \in \mathbb{R}_+$. Then the unique global solution of the Cauchy problem (5.6) is bounded for every $f \in F_\Pi$ if, and only if, $k_B(E, A) < 0$.*

Proof: "$\Leftarrow$": Let $f \in F_\Pi$. Since (E, A) possesses a transition matrix $U(\cdot, \cdot)$ we may apply Lemma 5.4 to conclude that the unique global solution $x : (0, \infty) \to \mathbb{R}^n$ of the Cauchy problem (5.6) has the representation

$$\forall\, t > 0 : \; x(t) = \int_{0}^{t} U(t, s)\Pi(s)f(s) \; \mathrm{d}s\,. \tag{5.9}$$

Taking norms gives

$$\forall\, t > 0 : \; \|x(t)\| \le \int_{0}^{t} \|U(t, s)\Pi(s)f(s)\| \; \mathrm{d}s\,. \tag{5.10}$$

Since $k_B(E, A) < 0$ there exist $\rho, N_\rho > 0$ such that (E, A) has property $\mathcal{B}(\rho, N_\rho)$ and invoking $\Pi(s)f(s) \in \mathcal{V}_{E,A}(s)$ gives that $t \mapsto U(t, s)\Pi(s)f(s)$ solves (E, A) for all $s \in \mathbb{R}_+$. Therefore,

$$\forall\, t, s \in \mathbb{R}_+ : \; \|U(t, s)\Pi(s)f(s)\| \le N_\rho e^{-\rho(t-s)}\|\Pi(s)f(s)\|.$$

This inequality together with (5.10) implies

$$\forall\, t \in \mathbb{R}_+ : \; \|x(t)\| \le \int_{0}^{t} N_\rho e^{-\rho(t-s)}\|\Pi(s)f(s)\| \; \mathrm{d}s \; \le \|\Pi f\|_\infty \frac{N_\rho}{\rho}.$$

"$\Rightarrow$": *Step 1*: We show that

$$V := \{\ g \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)\ \mid g \text{ is bounded and } \exists\, f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n):\ g = \Pi f\ \}$$

is a Banach space.
Clearly, V is a linear subspace of the Banach space $(\mathfrak{B}, \|\cdot\|_\infty)$, where $\mathfrak{B} = \{\ g \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)\ \mid g \text{ is bounded }\}$. We show that V is a closed subspace of $\mathfrak{B}$. To see this consider the following: for any convergent sequence $g_k \to g$ in V there exist $f_k \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$, $k \in \mathbb{N}$, such that $g_k = \Pi f_k$ and since $\|\Pi f_k - g\|_\infty \to 0$ we also have pointwise convergence, i.e. $\Pi(t) f_k(t) \to g(t)$ for all $t \in \mathbb{R}_+$. This clearly gives $g(t) \in \operatorname{im} \Pi(t)$ and hence $g(t) = \Pi(t) g(t)$ since $\Pi(t)$ is idempotent. But this means $g \in V$ and therefore V is closed. Hence we have that $(V, \|\cdot\|_\infty)$ is a Banach space.

Step 2: We show that, for all $t \in \mathbb{R}_+$,

$$K_t : V \to \mathbb{R}^n,\ g \mapsto \int_0^t U(t,s) g(s)\ \mathrm{d}s$$

is a bounded linear operator and satisfies

$$\exists\, K > 0\ \forall\, g \in V\ \forall\, t \in \mathbb{R}_+:\ \|K_t(g)\| \le K \|g\|_\infty. \tag{5.11}$$

Fix $t \in \mathbb{R}_+$. K_t is bounded since, for all $g \in V$,

$$\left\| \int_0^t U(t,s) g(s)\ \mathrm{d}s \right\| \le \underbrace{\int_0^t \|U(t,s)\|\ \mathrm{d}s}_{=: M_t}\ \|g\|_\infty$$

and the continuity of $s \mapsto U(t,s)$ on the compact set $[0,t]$ implies $M_t < \infty$. Now, for any $g \in V$, there exists $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ such that $g = \Pi f$ and since g is bounded, the Cauchy problem (5.6) corresponding to Πf has a unique global and bounded solution x by assumption, i.e.

$$\forall\, \Pi f \in V\ \exists\, M > 0\ \forall\, t \in \mathbb{R}_+:\ \|K_t(\Pi f)\| = \|x(t)\| \le M,$$

where we used the representation (5.9) of the solution x of the Cauchy problem. Therefore, we have

$$\forall\, g \in V:\ \sup_{t>0} \|K_t(g)\| < \infty.$$

Invoking that V is a Banach space by Step 1, the uniform boundedness principle now gives that

$$\sup_{t>0} \|K_t\| < \infty,$$

and therefore (5.11) holds.

Step 3: Let $t^0 \in \mathbb{R}_+$ and $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ such that Πf is bounded and consider the initial value problem (5.5). By Lemma 5.4 any right global solution x of (5.5) has the representation

$$\forall\, t \ge t^0:\ x(t) = \int_{t^0}^t U(t,s) \Pi(s) f(s)\ \mathrm{d}s \tag{5.12}$$

and indeed x as in (5.12) solves (5.5), i.e. there exists a unique global solution. We show that the solutions are bounded in the following sense:

$$\exists\, K > 0\ \forall\, f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n) \text{ s.t. } \Pi f \text{ is bounded } \forall\, t \ge t^0:\ \|x(t)\| \le K \|\Pi f\|_\infty. \tag{5.13}$$

To this end consider, for $0 < \varepsilon < t^0$, the Cauchy problem

$$E(t)\dot{x}_\varepsilon = A(t)x_\varepsilon + E(t)\Pi(t)f_\varepsilon(t), \quad x_\varepsilon(0) = 0, \tag{5.14}$$

where

$$f_\varepsilon(t) = \begin{cases} 0 & \text{for} \quad 0 < t < t^0 - \varepsilon \\ \frac{1}{\varepsilon} f(t^0)(t - t^0 + \varepsilon) & \text{for} \quad t^0 - \varepsilon \le t < t^0 \\ f(t) & \text{for} \quad t \ge t^0. \end{cases}$$

Note that $f_\varepsilon(0+) = (\Pi f_\varepsilon)(0+) = 0$. Then Lemma 5.4 yields that the unique global solution x_ε of (5.14) has the form

$$\begin{aligned} \forall\, t > 0: \ x_\varepsilon(t) &= \int_0^t U(t,s)\Pi(s)f_\varepsilon(s)\ \mathrm{d}s \\ &= \int_{t^0-\varepsilon}^{t^0} U(t,s)\Pi(s)\left(\frac{1}{\varepsilon}f(t^0)(s - t^0 + \varepsilon)\right)\ \mathrm{d}s + \int_{t^0}^t U(t,s)\Pi(s)f(s)\ \mathrm{d}s\,. \end{aligned} \tag{5.15}$$

Equations (5.15) and (5.11) (see Step 2) now immediately yield that for any $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ such that Πf is bounded, the corresponding solution x_ε of (5.14) satisfies

$$\forall\, t \in \mathbb{R}_+: \ \|x_\varepsilon(t)\| \le K\|\Pi f_\varepsilon\|_\infty. \tag{5.16}$$

Now, for fixed $t \in \mathbb{R}_+$ and $\varepsilon \to 0$ we find that $f_\varepsilon(t) \to f^*(t) = \begin{cases} 0 & \text{for} \quad 0 < t < t^0 \\ f(t) & \text{for} \quad t \ge t^0 \end{cases}$. Hence $x_\varepsilon(t) \to x(t)$ for $t \ge t^0$, where x is as in (5.12) and therefore (5.16) yields that

$$\forall\, t \ge t^0: \ \|x(t)\| \le K\|\Pi f^*\|_\infty \le K\|\Pi f\|_\infty,$$

which proves (5.13).

Step 4: Let $t^0 \in \mathbb{R}_+$. We distinguish two cases.
Case 1: $U(t,t^0)\Pi(t^0) \neq 0$ for all $t \in \mathbb{R}_+$. Then let $y \in \mathbb{R}^n$ and define the continuous functions

$$\chi : \mathbb{R}_+ \to \mathbb{R}, \ t \mapsto \frac{1}{\|U(t,t^0)\Pi(t^0)\|} \quad \text{and} \quad f : \mathbb{R}_+ \to \mathbb{R}^n, \ t \mapsto \frac{U(t,t^0)\Pi(t^0)y}{\|U(t,t^0)\Pi(t^0)\|}.$$

Since $U(t,t^0)\Pi(t^0)y \in \mathcal{V}_{E,A}(t)$ and $\Pi(t)$ is idempotent we find that

$$\|\Pi(t)f(t)\| = \|f(t)\| \le \|y\|$$

for all $t \in \mathbb{R}_+$ and hence Πf is bounded. Then, by (5.12), any solution $x(\cdot)$ of (5.5) satisfies

$$\forall\, t \in \mathbb{R}_+: \ x(t) = \int_{t^0}^t U(t,s)\Pi(s)\frac{U(s,t^0)\Pi(t^0)y}{\|U(s,t^0)\Pi(t^0)\|}\ \mathrm{d}s = U(t,t^0)\Pi(t^0)y \cdot \varphi(t),$$

where

$$\varphi : \mathbb{R}_+ \to \mathbb{R}, \ t \mapsto \int_{t^0}^t \chi(s)\ \mathrm{d}s\,.$$

Clearly $\varphi \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ and $\varphi(t) > 0$ for all $t \ge t^0 + 1$. From (5.13) (see Step 3) we obtain that

$$\forall\, t \ge t^0: \ \|x(t)\| \le K\|\Pi f\| \le K\|y\|,$$

and hence

$$\forall\, t \geq t^0: \ \varphi(t)\|U(t,t^0)\Pi(t^0)y\| \leq K\|y\|. \tag{5.17}$$

We show that

$$\forall\, t \geq t^0+1: \ \frac{1}{K} \leq \frac{\varphi'(t)}{\varphi(t)}. \tag{5.18}$$

Assume that $1/K > \varphi'(\hat{t})/\varphi(\hat{t})$ for some $\hat{t} \geq t^0+1$ and observe that $\varphi'(t) = \chi(t)$ for all $t \in \mathbb{R}_+$. Then, by (5.17) we obtain

$$\forall\, y \in \mathbb{R}^n: \ \frac{\varphi(\hat{t})}{\varphi'(\hat{t})}\|y\| > K\|y\| \overset{(5.17)}{\geq} \varphi(\hat{t})\|U(\hat{t},t^0)\Pi(t^0)y\|$$

and since $\varphi(\hat{t}) > 0$ we have

$$\frac{1}{\varphi'(\hat{t})} > \max_{\substack{y\in\mathbb{R}^n\\ y\neq 0}} \frac{\|U(\hat{t},t^0)\Pi(t^0)y\|}{\|y\|} = \|U(\hat{t},t^0)\Pi(t^0)\| = \frac{1}{\chi(\hat{t})} = \frac{1}{\varphi'(\hat{t})},$$

a contradiction. This proves (5.18). Now integrating (5.18) gives

$$\forall\, t \geq t^0+1: \ \frac{1}{K}(t-t^0-1) = \int_{t^0+1}^t \frac{1}{K}\,\mathrm{d}s \leq \int_{t^0+1}^t \frac{\varphi'(s)}{\varphi(s)}\,\mathrm{d}s = \ln\varphi(t) - \ln\varphi(t^0+1),$$

and applying the exponential yields

$$\forall\, t \geq t^0+1: \ \varphi(t) \geq \varphi(t^0+1)e^{\frac{1}{K}(t-t^0-1)}.$$

Then

$$\frac{1}{\|U(t,t^0)\Pi(t^0)\|} = \chi(t) = \varphi'(t) \overset{(5.18)}{\geq} \frac{\varphi(t)}{K} \geq \frac{\varphi(t^0+1)}{K}e^{\frac{1}{K}(t-t^0-1)}$$

and hence

$$\|U(t,t^0)\Pi(t^0)\| \leq \frac{Ke^{\frac{1}{K}}}{\varphi(t^0+1)}e^{-\frac{1}{K}(t-t^0)}$$

for all $t \geq t^0+1$. Now invoking the assumption (5.8) we find

$$\varphi(t^0+1) = \int_{t^0}^{t^0+1} \frac{1}{\|U(s,t^0)\Pi(t^0)\|}\,\mathrm{d}s \geq \frac{1}{M}$$

and

$$\max_{t\in[t^0,t^0+1]} e^{\frac{1}{K}(t-t^0)}\|U(t,t^0)\Pi(t^0)\| \leq e^{\frac{1}{K}}M.$$

Thus, for

$$\nu := \frac{1}{K}, \quad N := \max\left\{M,\ e^{\frac{1}{K}}M\right\},$$

we obtain

$$\forall\, t \geq t^0: \ \|U(t,t^0)\Pi(t^0)\| \leq Ne^{-\nu(t-t^0)}. \tag{5.19}$$

Case 2: There exists $\hat{t} \in \mathbb{R}_+$ such that $U(\hat{t},t^0)\Pi(t^0) = 0$. Then, for all $t \in \mathbb{R}_+$,

$$U(t,t^0)\Pi(t^0) = U(t,\hat{t})U(\hat{t},t^0)\Pi(t^0) = 0,$$

thus (5.19) does also hold true in this case.

Cases 1 and 2 together and the "$\Leftarrow$"-part of the proof of Proposition 5.2 (which does not need the boundedness of Π) show that (E,A) is exponentially stable and an application of Corollary 5.3 then finally yields that $k_B(E,A) < 0$. $\square$

Remark 5.6 (Cauchy problem and condition (5.8))**.** A careful inspection of the proof of Theorem 5.5 reveals that condition (5.8) is only necessary for the necessity part of the proof. Furthermore, in case of an ODE (I, A), the integral boundedness of A [11, (5.9)] implies condition (5.8). That integral boundedness of A is essential (for the sufficiency part), i.e. cannot be dropped, in the case of an ODE has been shown in [11, p. 131], so it is straightforward that condition (5.8) is essential in the case of DAEs. $\diamond$

In the remainder of this section we investigate robustness of DAEs (E, A) which possess a transition matrix. The appropriate tool for this analysis is, as it turned out for ODEs [11, 14], the Bohl exponent. To this end we consider perturbations of the DAE (E, A) as already introduced in Section 4: Choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im}\Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$. Then, for any $g \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$, we consider the perturbed DAE

$$E(t)\dot{x} = A(t)x + E(t)\Pi(t)g(t, x). \tag{5.20}$$

$E\Pi g$ is a structured perturbation which guarantees "consistency" with the DAE. This becomes clear by considering the equivalent integral equation (Lemma 4.3) and Theorem 5.5, which both justify the term $E\Pi$ in front of g. Note that E and Π are constant invertible matrices if (E, A) is an ODE. Furthermore, the necessity of the term $E\Pi$ concerning stability issues is stressed by the following simple example of an index 1 DAE: The system $0 = x$ is obviously exponentially stable, but the perturbed system $0 = x - \delta$ is unstable for any arbitrary small $\delta > 0$. However, $E\Pi = 0$ in this case and hence any perturbation involving the term $E\Pi$, i.e. any system $0 = x + 0 \cdot g(t, x)$, would still be exponentially stable.

We show in the following that the Bohl exponent of a system (E, A) remains negative under perturbations of the form (5.20), provided g is "sufficiently small" in some sense.

Theorem 5.7 (Stability of Bohl exponents)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot,\cdot)$ and choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im}\Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$. Then for any $\varepsilon > 0$ there exists $\delta > 0$, such that for all $g \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$ and $q \in \mathcal{C}(\mathbb{R}_+; \mathbb{R})$ with the properties*

$$\forall\, (t, x) \in \mathcal{V}_{E,A} : \ \ \|\Pi(t)g(t, x)\| \le q(t)\|x\| \tag{5.21}$$

and

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} q(\tau)\ \mathrm{d}\tau\ < \delta, \tag{5.22}$$

we have

$$k_B(5.20) < k_B(E, A) + \varepsilon.$$

Before we prove Theorem 5.7 we need the following lemma.

Lemma 5.8. *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ possess a transition matrix $U(\cdot,\cdot)$ and choose $\Pi \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $\operatorname{im}\Pi(t) = \mathcal{V}_{E,A}(t)$ for all $t \in \mathbb{R}_+$. Suppose there exist $g \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}^n)$, $q \in \mathcal{C}(\mathbb{R}_+; \mathbb{R})$ such that* (5.21) *holds, $\int_0^{t^0} q(s)\ \mathrm{d}s\ < \infty$ and*

$$\exists\, t^0, s^0 > 0 : \ \sup_{t\ge t^0} \frac{1}{s^0} \int_t^{t+s^0} q(s)\ \mathrm{d}s\ =: M^0 < \infty. \tag{5.23}$$

If (E, A) has property $\mathcal{B}(\nu, N)$ for some $\nu \in \mathbb{R}, N \in \mathbb{R}_+$, then (5.20) *has property $\mathcal{B}(\nu - NM^0, NN^0 e^{s^0 NM^0})$, where $N^0 := e^{N \int_0^{t^0} q(s)\ \mathrm{d}s}$.*

Proof: Let $x : (\alpha, \infty) \to \mathbb{R}^n$ be a right global solution of (5.20) and $s > \alpha$. Then, by Lemma 4.3, x has the representation

$$\forall\, t > \alpha : \; x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)\Pi(\tau)g(\tau, x(\tau)) \,\mathrm{d}\tau\,.$$

Since (E,A) possesses a transition matrix $U(\cdot,\cdot)$ and has property $\mathcal{B}(\nu, N)$ it is immediate that

$$\forall\, t \geq s > 0\; \forall\, x \in \mathcal{V}_{E,A}(s) : \; \|U(t,s)x\| N e^{-\nu(t-s)}\|x\|. \tag{5.24}$$

Now $x(s) \in \mathcal{V}_{E,A}(s)$ (see the proof of Lemma 4.3) and $\Pi(\tau)g(\tau, x(\tau)) \in \mathcal{V}_{E,A}(\tau)$ give that, for all $t > \alpha$,

$$\|x(t)\| \overset{(5.24)}{\leq} N e^{-\nu(t-s)}\|x(s)\| + \int_s^t N e^{-\nu(t-\tau)}\|\Pi(\tau)g(\tau, x(\tau))\| \,\mathrm{d}\tau\,,$$

and hence, invoking (5.21),

$$e^{\nu t}\|x(t)\| \leq N e^{\nu s}\|x(s)\| + \int_s^t N q(\tau) e^{\nu\tau}\|x(\tau)\| \,\mathrm{d}\tau\,.$$

An application of Gronwall's inequality (see e.g. [15, Lem. 2.1.18]) yields

$$\forall\, t > \alpha : \; \|x(t)\| \leq N e^{-\nu(t-s)}\|x(s)\| e^{N\int_s^t q(\tau)\,\mathrm{d}\tau}\,. \tag{5.25}$$

Now we distinguish two cases.
Case 1: $s < t^0$. Let $t \geq s$ and $k \in \mathbb{N}$ such that $s^0(k-1) \leq t - t^0 < s^0 k$. Then

$$\int_{t^0}^t q(\tau)\,\mathrm{d}\tau \overset{(5.23)}{\leq} k s^0 M^0 \leq (t - t^0 + s^0)M^0 \leq (t - s + s^0)M^0, \tag{5.26}$$

and therefore,

$$\begin{aligned}\forall\, t \geq s : \; \|x(t)\| &\overset{(5.25)}{\leq} N e^{-\nu(t-s)}\|x(s)\| e^{N\int_s^{t^0} q(\tau)\,\mathrm{d}\tau} e^{N\int_{t^0}^t q(\tau)\,\mathrm{d}\tau} \\ &\overset{(5.26)}{\leq} N e^{-\nu(t-s)} N^0 e^{N(t-s+s^0)M^0}\|x(s)\| = N N^0 e^{s^0 N M^0} e^{-(\nu - N M^0)(t-s)}\|x(s)\|.\end{aligned}$$

Case 2: $s \geq t^0$. Let $t \geq s$ and $k \in \mathbb{N}$ such that $s^0(k-1) \leq t - s < s^0 k$. Then

$$\int_s^t q(\tau)\,\mathrm{d}\tau \overset{(5.23)}{\leq} k s^0 M^0 \leq (t - s + s^0)M^0, \tag{5.27}$$

and therefore, (5.25) gives

$$\forall\, t \geq s : \; \|x(t)\| \overset{(5.27)}{\leq} N e^{-\nu(t-s)} e^{N(t-s+s^0)M^0}\|x(s)\| \leq N N^0 e^{s^0 N M^0} e^{-(\nu - N M^0)(t-s)}\|x(s)\|. \qquad \square$$

We are now in the position to prove Theorem 5.7.

Proof of Theorem 5.7: (E,A) has property $\mathcal{B}\left(-k_B(E,A) - \frac{\varepsilon}{2}, N_{\frac{\varepsilon}{2}}\right)$, and (5.22) implies existence of $t^0, s^0 > 0$ such that

$$\sup_{t \geq t^0} \frac{1}{s^0}\int_t^{t+s^0} q(s)\,\mathrm{d}s \, < 2\delta.$$

An application of Lemma 5.8 yields that (5.20) has property $\mathcal{B}\left(-k_B(E,A) - \frac{\varepsilon}{2} - 2\delta N_{\frac{\varepsilon}{2}}, \tilde{N}\right)$ for some $\tilde{N} > 0$. Choosing $\delta < \frac{\varepsilon}{4N_{\frac{\varepsilon}{2}}}$ completes the proof of the theorem. $\square$

Remark 5.9 (Bounded Π)**.** If Π in Lemma 5.8 and Theorem 5.7 is bounded, then it is sufficient to assume that g is linearly bounded in the sense that there exists $q \in \mathcal{C}(\mathbb{R}_+; \mathbb{R})$ such that

$$\forall\, (t, x) \in \mathcal{V}_{E,A} : \ \|g(t, x)\| \leq q(t)\|x\|$$

to deduce conditions equivalent to (5.23) and (5.22). $\diamond$

Remark 5.10 (Perturbation of Bohl exponents)**.** Theorem 5.7 characterizes the continuity property of the Bohl exponent. For any given $\varepsilon > 0$ we may choose perturbations of (E, A) so small that the corresponding Bohl exponent does only increase by at most ε.
On the other hand, Theorem 5.7 also characterizes the robustness of exponential stability of the DAE (E, A): if (E, A) is exponentially stable, then any sufficiently small perturbation of (E, A) (i.e. (5.22) is satisfied for sufficiently small $\delta > 0$) is exponentially stable, provided that a negative Bohl exponent is equivalent to exponential stability for the perturbed system. $\diamond$

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