Additive Set Functions as Linear Functionals

Bachelor thesis
for the acquisition of the academic degree
Bachelor of Science (B. Sc.)
in Mathematics

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Faculty of Mathematics and Informatics

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November 21, 2010

Abstract

We discuss the connection of additive set functions to dual function spaces by means of integrals. Isometrical isomorphisms are presented between: a) bounded contents and the dual of uniformly approximable functions, b) charges on Baire algebras and the duals of bounded, continuous functions over A-topological spaces, c) Hewitt measures on Baire σ-algebras and the duals of continuous functions over A-topological spaces.

Zusammenfassung

Wir behandeln die Verbindung additiver Mengenfunktionen zu dualen Funktionenräumen im Sinne des Integrals. Isometrische Isomorphismen werden vorgestellt zwischen: a) Beschränkten Inhalten und den Dualräumen gleichmäßig approximierbarer Funktionen, b) Ladungen (*charges*) auf Baire Algebren und den Dualräumen beschränkter, stetiger Funktionen A-topologischer Räume, c) Hewitt Maßen auf Baire σ-Algebren und den Dualräumen stetiger Funktionen A-topologischer Räume.

Contents

Preface

In this paper, we consider the connection between classes of additive set functions on the Baire-sets of topological spaces, and the duals of certain function spaces. We limit ourselves to real-valued set functions. However, the reader will find that, most of the results presented can easily be generalized in a straightforward way to complex valued set functions.

The first chapter introduces so called *A-topological spaces*, as generalizations of topological ones, certain function spaces and their duals. The main reason for such a generalization is that, many topologies can not be adequately characterized solely by the continuous functions they allow. Thus, a one to one connection between their Borel sets and their continuous functions is not always possible, and measure theoretic considerations need inevitably be limited to a smaller class of sets, namely the *Baire sets* generated by the *totally closed* ones. In general, the latter only form an A-topology, which gives need to a somewhat special treatment, compared to the standard one deployable for Borel-sets. Furthermore, the mere assumption of an A-topological space, constitutes an interesting generalization of the usual measure theory deployed on topological spaces.

The second and third chapter introduces the reader to the class of bounded, additive set functions on set algebras and set σ-algebras. Three kinds of set functions are introduced: Bounded contents as bounded, additive set functions, charges as bounded, regular contents and measures as σ-additive contents. The resulting content classes all turn out to be Banach spaces, when equipped with the total variation norm. For bounded contents, integration within a certain class of functions is introduced, as a generalization of the Lebesgue integral.

In the fourth chapter, we establish a connection between bounded contents, charges and Hewitt-measures to the duals of uniformly approximable, bounded continuous and continuous functions, by means of the integral. The connections established, all turn out to be isometrical isomorphisms. Key concepts handled were that of regularity and σ-additivity of contents. While the latter is directly connected to the so called σ-continuity of the induced linear functionals, the former turns out to be an essential assumption for injectivity of the relation. As a special case, the Riesz representation theorem is obtained for countably compact, topological spaces.

Most of this treatise is based on the works of Alexandroff[1, 2], Hewitt[10] and Varadarajan[13]. In particular, the treatment of Baire-sets as Borel-sets of an A-topological space, originates from Alexandroff, who himself used the term *abstract space*.

A basic background in measure theory, topology and functional calculus is assumed throughout this paper. The reader is referred to Bogachev[4] for an extensive treatment of σ-additive measures and used notation, Alexandroff[1, 2] for a more thorough treatment of A-topological spaces and charges, Dunford & Schwartz[13] for a self contained treatise of additive set functions, Kelley [11] for the essentials in topology and Conway[5] for an introduction to functional analysis.

Appendix A contains some secondary statements and proofs, which we did not consider to be of importance to the understanding of the subject, but are nonetheless used at some point or the other. Ambiguous symbols and terms, are explained in appendix B.

Chapter 1

A-topologies and function spaces

1.1 A-Topologies

The identification of set functions on set algebras with linear functionals on certain function spaces, requires a strong connection between these functions and the considered set family. It is known for example, that every measure on the Borel sets of a topological space T, induces on the set of real, bounded, continuous functions $\mathcal{C}_b(T)$ a bounded, linear functional. However, the reverse statement is not always true, as the topology, and in a broader sense the Borel sets, is not necessarily fully *described* by the continuous functions. That is to say, the smallest σ-algebra for which all real, continuous functions are measurable, the so called *Baire σ-algebra*[8], may be indeed smaller than the Borel σ-algebra.

It turns out that, by slightly generalizing the notion of a topology to so called A-topologies, the Baire σ-algebra can be described as generated by exactly such an A-topology, which is indeed strongly related to the continuous functions of the space. The origins of such an approach, date back to the works of Alexandroff.

This section is meant as a mere overview of A-topological spaces and introduces the main concepts used thereafter. Many terms are simply generalizations of already known ones for topological spaces. A comparison to topological spaces and other technical subtleties, are handled in appendix sections A.1 and A.2.

1.1.1 Definition: A-topological spaces

Let R be some arbitrary set. We shall call a system of subsets $\mathscr{O} \subseteq \mathscr{P}(R)$ an *A-topology* if:

1. $\mathscr{O}$ contains R and $\emptyset$.

2. $\mathscr{O}$ is closed under finite intersections: For $U_1, U_2 \in \mathscr{O}$ also $U_1 \cap U_2 \in \mathscr{O}$.

3. $\mathscr{O}$ is closed under countable unions: For $U_n \in \mathscr{O}$, $n \in \mathbb{N}$ also $\bigcup_{n=1}^{\infty} U_n \in \mathscr{O}$.

The pair $(R, \mathscr{O})$ is called an *A-topological space*[1]. The sets in $\mathscr{O}$ are called *open*, their complements *closed*. In the following, $\mathscr{O}$ shall always denote the open, $\mathscr{F}$ always the closed sets of the space $(R, \mathscr{O})$, or $(R, \mathscr{F})$ for that matter.

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[1] *Abstract space* by Alexandroff[1]. Note: Topological spaces are A-topological.

1.1.2   Definition: Relative A-topology

Let $(R, \mathscr{O})$ be an A-topological space and $S \subseteq R$. Then

$$S \cap \mathscr{O} := \{S \cap U : U \in \mathscr{O}\} \tag{1.1}$$

defines an A-topology on S and is called the *relative A-topology* or *trace A-topology* on S. Whenever considering subspaces of A-topological spaces as A-topological spaces, we shall assume this A-topology.

1.1.3   Definition: A-subbase & A-base

Let R be some set and $\beta \subseteq \mathscr{P}(R)$. The A-topology *generated by* β is the smallest A-topology $\mathscr{O}$ containing β. It is evident that

$$\mathscr{O} = \bigcap_{\substack{\mathscr{O}' \subseteq \mathscr{P}(R) \\ \beta \subseteq \mathscr{O}' \\ \mathscr{O}' \text{ A-topology}}} \mathscr{O}' = \{O \subseteq R : O \text{ countable union of finite intersections of } \beta\} \quad . \tag{1.2}$$

We say that β is an *A-subbase* of $\mathscr{O}$. If every $O \in \mathscr{O}$ is a countable union of sets in β, we call β an *A-base* of $\mathscr{O}$.

1.1.4   Definition: Continuous functions

Let $(R_1, \mathscr{O}_1)$, $(R_2, \mathscr{O}_2)$ be A-topological spaces. A function $f : R_1 \to R_2$ is called *continuous* if $f^{-1}(O) \in \mathscr{O}_1$ for every $O \in \mathscr{O}_2$.

Note: Compositions of continuous functions are continuous.

1.1.5   Definition: Separating function-system

A function $f : \Omega \to \mathbb{R}$ over some set Ω is said to be *separating* the disjoint sets $A, B \subseteq \Omega$, if

$$f\big|_A = 0 \quad , \quad f\big|_B = 1 \quad , \quad 0 \leq f \leq 1 \quad . \tag{1.3}$$

We say f *perfectly separates* A and B, if

$$A = f^{-1}(0) \quad , \quad B = f^{-1}(1) \quad , \quad 0 \leq f \leq 1 \quad . \tag{1.4}$$

A system Φ of real functions on Ω is said to be (*perfectly*) *separating* the sets of some system $\mathfrak{S} \subseteq \mathscr{P}(\Omega)$, if for disjoint $A, B \in \mathfrak{S}$ there exists an $f \in \Phi$ (perfectly) separating them.

1.1.6   Definition: Kinds of A-topological spaces

An A-topological space $(R, \mathscr{O})$ is called:

- *2nd Countable*, if it has a countable A-base.

- *Compact*, if every open covering of it, contains a finite, open covering.

- *Countably compact*, if every countable, open covering of it contains a finite, open covering.

- *Pseudocompact*, if every real, continuous function is bounded.

- *Hausdorff*, if for each pair $x_1, x_2 \in R$ with $x_1 \neq x_2$, there exist disjoint, open $G_1, G_2 \in \mathscr{O}$ such that $x_i \in G_i$, $i = 1, 2$.

- *Normal*, if for disjoint, closed $F_1, F_2 \subseteq R$ there exist disjoint, open $G_1, G_2 \in \mathscr{O}$ such that $F_i \subseteq G_i$, $i = 1, 2$.

- *Perfectly normal*, if every closed $F \subseteq R$ can be written as $F = f^{-1}(\{0\})$ for some continuous $f \in \mathcal{C}(R)$.

- *Regular*, if for closed $F \subseteq R$ and $x \notin F$, there exist disjoint, open $G_1, G_2 \in \mathscr{O}$ such that $F \subseteq G_1$, $x \in G_2$.

- *Completely regular*, if every closed $F \subseteq R$ and $x \notin F$ can be separated by a continuous function $f \in \mathcal{C}(R)$.

Example: Every metric space, with the topology generated by the metric, is Hausdorff, perfectly normal and completely regular.

1.1.7 Theorem: Separating functions in A-topological spaces

Let $(R, \mathscr{O})$ be an A-topological space. Then:

1. $(R, \mathscr{O})$ is normal iff its real, continuous functions separate its closed sets.

2. $(R, \mathscr{O})$ is perfectly normal iff its real, continuous functions perfectly separate its closed sets.

A proof can be found in Alexandroff[1].

1.1.8 Definition: Regular sequence

Let $(R, \mathscr{F})$ be an A-topological space. We shall call a sequence $(F_n)_{n=1}^{\infty} \subseteq \mathscr{F}$ of closed sets *regular*[13], if $\bigcup_{n=1}^{\infty} F_n = R$ and there exist open sets $(G_n)_{n=1}^{\infty}$ in R such that

$$F_1 \subseteq G_1 \subseteq F_2 \subseteq G_2 \subseteq \dots \quad . \tag{1.5}$$

Regular sequences will be used later on, in characterizing continuous functions and σ-additive set functions. They have the benefit of being describable by single continuous functions (see A.2.4), jet providing with a strong description of the underlying A-topology (see A.2.2).

1.2 Function spaces on A-topological spaces

Within the context of this paper, two concepts are of importance when it comes to the classification of functions with respect to their preimages:

- Continuity, having to do with some A-topology $\mathscr{O}$ on the underlying space R.

- Measurability, having to do with some set-algebra $\mathfrak{S}$, closely linked to set functions on $\mathfrak{S}$.

In case of the set algebra being generated by the A-topology, a connection can be established between the two concepts. One important result is the measurability (and later on, integrability) of every real, bounded, continuous function.

In the following all Banach spaces are assumed to be real or complex, with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denoting the underlying field. On them, we assume the topology induced by the norm.

1.2.1 Definition: Borel algebra & Borel σ-algebra

Let $(R, \mathscr{O})$ be an A-topological space. The *Borel algebra* $\mathscr{B}_\alpha(R) := \alpha(\mathscr{O})$ is the set algebra generated by the A-topology $\mathscr{O}$. The *Borel σ-algebra* $\mathscr{B}_\sigma(R) := \sigma(\mathscr{O})$ is the σ-algebra generated by $\mathscr{O}$.

1.2.2 Definition: Measurable function

Let $\mathfrak{S}, \mathfrak{S}'$ be set-algebras over some sets Ω, Ω' respectively. A function $f : \Omega \to \Omega'$ is called *measurable*, if $f^{-1}(\mathfrak{S}') \subseteq \mathfrak{S}$, that is, if $f^{-1}(S') \in \mathfrak{S}$ for every $S' \in \mathfrak{S}'$.

We shall denote the system of measurable functions by $\mathscr{M}(\mathfrak{S}, \mathfrak{S}')$.

Note: If $\mathfrak{S}' = \alpha(\mathcal{A}')$ for some subsystem $\mathcal{A}' \subseteq \mathscr{P}(\Omega')$, then f is measurable iff $f^{-1}(\mathcal{A}') \subseteq \mathfrak{S}$.

1.2.3 Definition: Simple function

Let $\mathfrak{S}$ be a set-algebra over some set Ω and E a $\mathbb{K}$-Banach space. A function $f : \Omega \to E$ is called *simple*, if it is of the form

$$f = \sum_{i=1}^n f_i \cdot 1_{A_i} \quad , \quad A_1, .., A_n \in \mathfrak{S}, \ f_1, .., f_n \in E, \ n \in \mathbb{N} \quad . \tag{1.6}$$

W.l.o.g. we may assume A_i to be disjoint, f_i pairwise different and $\Omega = \biguplus_{i=1}^n A_i$. We shall denote the linear space of simple functions by $\mathscr{S}(\mathfrak{S}, E)$, or $\mathscr{S}(\mathfrak{S})$ if $E = \mathbb{R}$.

1.2.4 Definition: Bounded & totally bounded function

Let Ω be some set and E a $\mathbb{K}$-Banach space. We shall call a function $f : \Omega \to E$ *totally bounded*, if its image $f(\Omega)$ is totally bounded. The linear space of all totally bounded functions we denote by $\mathcal{B}_t(\Omega, E)$, or $\mathcal{B}_t(\Omega)$ if $E = \mathbb{R}$.

The (in general larger) linear space of bounded functions $f : \Omega \to E$ we shall denote by $\mathcal{B}(\Omega, E)$, or $\mathcal{B}(\Omega)$ if $E = \mathbb{R}$.

On any subspace of $\mathcal{B}(\Omega, E)$ we shall, if not otherwise stated, assume the supremum norm

$$\|f\|_\infty := \sup_{x \in \Omega} \|f(x)\| \quad , \quad f \in \mathcal{B}(\Omega, E) \quad . \tag{1.7}$$

1.2.5 Definition: Uniformly approximable function

Let $\mathfrak{S}$ be a set-algebra over some set Ω and E a $\mathbb{K}$-Banach space. A function $f : \Omega \to E$ is called *uniformly approximable* if there exist simple functions $f_n \in \mathscr{S}(\mathfrak{S}, E)$, such that $\|f - f_n\|_\infty \xrightarrow{n \to \infty} 0$. We shall denote the system of uniformly approximable functions by $\mathscr{U}(\mathfrak{S}, E)$, or $\mathscr{U}(\mathfrak{S})$ if $E = \mathbb{R}$.

1.2.6 Lemma: Permanence properties of uniformly approximable functions

Let $\mathfrak{S}$ be a set algebra over some set Ω, E a $\mathbb{K}$-Banach space and $\mathscr{U}(\mathfrak{S}, E)$ the system of uniformly approximable functions on $(\Omega, \mathfrak{S})$. Then:

1. For $f, g \in \mathscr{U}(\mathfrak{S}, E)$ and $\alpha, \beta \in \mathbb{K}$, also $\alpha f + \beta g \in \mathscr{U}(\mathfrak{S}, E)$.

2. For $f \in \mathscr{U}(\mathfrak{S}, E)$ and $g \in \mathscr{U}(\mathfrak{S}, \mathbb{K})$, also $g \cdot f \in \mathscr{U}(\mathfrak{S}, E)$.

3. For $f \in \mathscr{U}(\mathfrak{S}, E)$, also $\|f\| \in \mathscr{U}(\mathfrak{S}, \mathbb{R})$.

4. For $f \in \mathscr{U}(\mathfrak{S}, E)$ and $A \in \mathfrak{S}$, also $f \cdot 1_A \in \mathscr{U}(\mathfrak{S}, E)$.

5. If F is also a $\mathbb{K}$-Banach space, $\rho : E \to F$ uniformly continuous and $f \in \mathscr{U}(\mathfrak{S}, E)$, then also $\rho \circ f \in \mathscr{U}(\mathfrak{S}, F)$.

Proof

1. Trivial.

2. Recall that both f and g are totally bounded. Let $f_n \in \mathscr{S}(\mathfrak{S}, E)$ and $g_n \in \mathscr{S}(\mathfrak{S}, \mathbb{K})$ such that

$$f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f \quad , \quad g_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} g \quad . \tag{1.8}$$

Note that the sequence $(f_n) \subseteq \mathcal{B}(\Omega, E)$ is bounded. Moreover, $f_n g_n \in \mathscr{S}(\mathfrak{S}, E)$ and

$$\|f \cdot g - f_n \cdot g_n\|_\infty = \|(f - f_n)g + (g - g_n)f_n\|_\infty$$

$$\leq \|(f - f_n)g\|_\infty + \|(g - g_n)f_n\|$$

$$\leq \|f - f_n\|_\infty \cdot \|g\|_\infty + \|g - g_n\|_\infty \cdot \underbrace{\|f_n\|_\infty}_{\substack{\text{uniformly} \\ \text{bounded}}} \quad . \tag{1.9}$$

By assumption (1.8), this shows $f_n g_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} fg$.

3. Trivial, since for $x, y \in E$: $\big| \|x\| - \|y\| \big| \leq \|x - y\|$.

4. If $f_n \in \mathscr{S}(\mathfrak{S}, E)$ are such that $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty}$, then $f_n \cdot 1_A \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f \cdot 1_A$.

5. Let $\varepsilon > 0$ be given and $\delta > 0$ such that $\rho(B_\delta(x)) \subseteq B_\varepsilon(\rho(x))$ for all $x \in E$. Choose $\varphi \in \mathscr{S}(\mathfrak{S}, E)$ such that $\|f - \varphi\|_\infty \leq \delta$. Then $\rho(\varphi) \in \mathscr{S}(\mathfrak{S}, F)$ and $\|\rho(f) - \rho(\varphi)\|_\infty \leq \varepsilon$.

$\square$

1.2.7 Definition: $\mathcal{C}$ and $\mathcal{C}_b$

Let $(R, \mathscr{O})$ be an A-topological space and E a Banach-space. Then $\mathcal{C}(R, \mathscr{O}, E)$, $\mathcal{C}_b(R, \mathscr{O}, E)$ shall denote the systems of continuous and bounded, continuous functions on R into E respectively. In case $E = \mathbb{R}$, we shall omit it. In case $\mathscr{O}$ is understood from the context, we shall omit it.

1.2.8 Lemma: Connection between function spaces

If $\mathfrak{S}$ is a set algebra over some set Ω and E a $\mathbb{K}$-Banach-space, then:

1. All simple functions $f \in \mathscr{S}(\mathfrak{S}, E)$ are uniformly approximable. In fact, $\mathscr{U}(\mathfrak{S}, E)$ is merely the closure of $\mathscr{S}(\mathfrak{S}, E)$ under the $\|\cdot\|_\infty$ norm. As closed subset of the $\mathbb{K}$-Banach space $\mathcal{B}(\Omega, E)$, it is its self a $\mathbb{K}$-Banach space.

2. Every measurable & totally bounded $f : \Omega \to E$ is uniformly approximable, that is:

$$\mathscr{M}(\mathfrak{S}, \mathscr{B}_\alpha(E)) \cap \mathcal{B}_t(\Omega, E) \subseteq \mathscr{U}(\mathfrak{S}, E) \quad . \tag{1.10}$$

3. Every uniformly approximable function $f \in \mathscr{U}(\mathfrak{S}, E)$ is totally bounded.
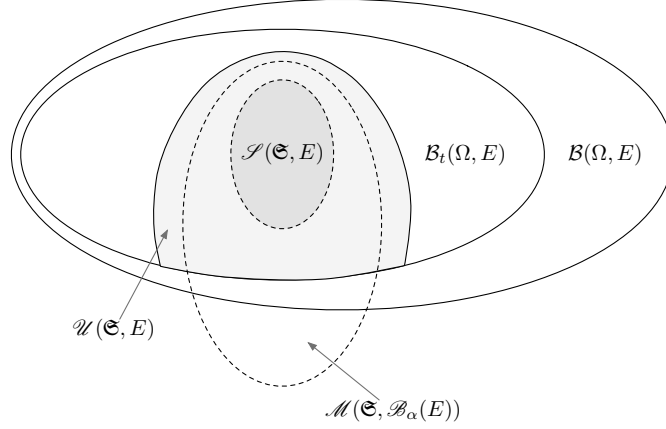
Figure 1.1: Connection between function spaces. Spaces closed with respect to the $\|\cdot\|_\infty$-norm are drawn with a continuous line.

Proof

1. Trivial.

2. For given $\varepsilon > 0$ there exist points $x_1, .., x_n \in E$ such that $f(\Omega) \subseteq \bigcup_{k=1}^n B_\varepsilon(x_k)$. Set

$$A_k := f^{-1}\left[B_\varepsilon(x_k) \setminus \bigcup_{i=1}^{k-1} B_\varepsilon(x_i) \right] \quad , \quad k \in \{1, .., n\} \quad , \tag{1.11}$$

then $A_k \in \mathfrak{S}$ and $\Omega = \biguplus_{k=1}^n A_k$. The simple function $g := \sum_{k=1}^n x_k \cdot 1_{A_k}$ satisfies

$$\|g - f\|_\infty = \sup_{1 \le k \le n} \sup_{\omega \in A_k} \underbrace{\|f(\omega) - x_k\|}_{\le \varepsilon} \le \varepsilon \quad , \tag{1.12}$$

which was to be show.

3. Let $\varepsilon > 0$ be given. Choose a simple $g \in \mathscr{S}(\mathfrak{S}, E)$ such that $\|f - g\|_\infty \le \varepsilon$. Let

$$g = \sum_{i=1}^n g_i \cdot 1_{A_i} \quad , \quad g_i \in E, \ A_i \in \mathfrak{S}, \ \Omega = \biguplus_{i=1}^n A_i \quad . \tag{1.13}$$

Then $f(\Omega) \subseteq \bigcup_{i=1}^n B_\varepsilon(g_i)$, which shows that $f(\Omega)$ is totally bounded.

$\square$

1.2.9 Lemma about continuous functions

Let $(R, \mathscr{O})$ be an A-topological space. Then:

1. A function $f : R \to \mathbb{R}$ is continuous, iff for every $\alpha \in \mathbb{R}$ the sets $\{f \le \alpha\}, \{f \ge \alpha\}$ are closed.

2. Every constant function on R is continuous.

3. The system of continuous functions $\mathcal{C}(R)$ is a linear space.

4. For $f, g \in \mathcal{C}(R)$, also $\max\{f, g\} \in \mathcal{C}(R)$.

5. For $f, g \in \mathcal{C}(R)$, also $f \cdot g \in \mathcal{C}(R)$, and if $g \neq 0$, also $f/g \in \mathcal{C}(R)$.

6. Every bounded, continuous function $f \in \mathcal{C}_b(R)$ is uniformly approximable with respect to $\mathscr{B}_\alpha(R)$.

7. If $f_n \in \mathcal{C}(R)$ are such that $f_n \xrightarrow[\|\cdot\|_\infty]{n\to\infty} f$ for some real function $f : R \to \mathbb{R}$, then also $f \in \mathcal{C}(R)$.

Proof: Statement (1) follows from A.1.1(5), statement (2) is trivial, statements (3), (4) and (5) follow from lemma A.1.2. Statement (6) follows readily from 1.2.8(2).

For statement (7), it suffices by (1) to show $\forall\, \alpha \in \mathbb{R} : \{f \leq \alpha\}$ is closed[2]. Indeed, for $n \in \mathbb{N}$ there exists an $i_n \in \mathbb{N}$ such that

$$f(x) \leq \alpha \quad \Rightarrow \quad f_j(x) \leq \alpha + \frac{1}{n} \quad \forall\, j \geq i_n \tag{1.14}$$

which implies

$$\{f \leq \alpha\} = \underbrace{\bigcap_{n=1}^{\infty} \bigcap_{j=i_n}^{\infty} \left\{ f_j \leq \alpha + \tfrac{1}{n} \right\}}_{\substack{\text{countable intersection} \\ \text{of closed sets}}} \tag{1.15}$$

which was to be shown.

$\square$

1.2.10 Definition: Lattice function space

Let Φ be a real, linear function space over some set Ω. Then Φ is said to be a *lattice* if for every $f, g \in \Phi$, also $\max\{f, g\} \in \Phi$.

Note: The following are equivalent:

1. The space Φ is a lattice.

2. For $f, g \in \Phi$, also $\min\{f, g\} \in \Phi$.

3. For $f \in \Phi$, also $|f| \in \Phi$.

1.2.11 Definition: Ring function space

Let Φ be a $\mathbb{K}$-linear function space over some set Ω. Then Φ is said to be a *ring*, if $1 \in \Phi$ and for $f, g \in \Phi$ also $f \cdot g \in \Phi$.

1.2.12 Definition: Grounded function space

Let Φ be a lattice subspace of the family of real, bounded functions over some set Ω, such that $1 \in \Phi$. Then we call Φ a *grounded* function space.

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[2]Since also $(-f_n) \xrightarrow[\|\cdot\|_\infty]{n\to\infty} (-f)$.

Examples:

- If $\mathfrak{S}$ is a set algebra over Ω, then $\mathscr{S}(\mathfrak{S})$, $\mathscr{U}(\mathfrak{S})$, $\mathcal{B}(\Omega)$ are grounded function spaces and rings.

- If $(R, \mathscr{O})$ is A-topological, then $\mathcal{C}_b(R)$ is a grounded function space. Both $\mathcal{C}_b(R)$ and $\mathcal{C}(R)$ are rings and lattices.

1.3 The Baire-sets

As mentioned in section 1.1, topologies of topological spaces are often larger than needed, for its continuous functions to be indeed continuous. This fact may give the unpleasant feeling that, such a topology somehow *missed the point.* Such topologies fail when it comes to being describable by the continuous functions they allow. Fortunately, on every A-topological space, there exists a minimal A-topology, with respect to which all continuous functions are indeed continuous. This A-topology is perfectly normal and coincides on many spaces (i.e. metric ones) with the intrinsic topology.

In chapter 4, implicit use will be made of this A-topology, simply by assuming a perfectly normal A-topology in the first place.

1.3.1 Theorem: Generation of A-topological spaces

Let Φ be a function space over some set R, such that:

1. Φ is a lattice.

2. $1 \in \Phi$.

3. For $f_n \in \Phi$ with $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$ for some $f : R \to \mathbb{R}$, also $f \in \Phi$.

Then the sets

$$\mathscr{F} := \left\{ f^{-1}(\{0\}) : f \in \Phi \right\} \tag{1.16}$$

form the closed sets of a perfectly normal, A-topological space $(R, \mathscr{F})$, on which all $f \in \Phi$ are continuous. In fact, $\mathscr{F}$ corresponds to the *smallest* A-topology on R, for which all $f \in \Phi$ are continuous[3].

If Φ are the continuous functions with respect to some other A-topology with closed sets $\mathscr{F}_0$, then $\mathscr{F} \subseteq \mathscr{F}_0$ and the continuous functions of $(R, \mathscr{F})$ are exactly Φ.

Proof

Claim: $\mathscr{F}$ defines the closed sets of an A-topology.
Proof: Let $f_n \in \Phi$ and $F_n := f_n^{-1}(\{0\})$. Define $\rho_n \in \Phi$ as

$$\rho_n := \max \left\{ \rho_{n-1}, \ \min \left\{ |f_n|, \tfrac{1}{n^2} \right\} \right\} \quad , \quad \rho_0 := 0 \quad . \tag{1.17}$$

Then ρ_n are non-decreasing and bounded by above by

$$\rho_n \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \quad . \tag{1.18}$$

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[3]That is, if all $f \in \Phi$ are continuous with respect to some other A-topology with closed sets $\mathscr{F}_0$, then $\mathscr{F} \subseteq \mathscr{F}_0$.

Thus $\rho_n \overset{n\to\infty}{\underset{\text{pointwise}}{\longrightarrow}} \rho$ for some $\rho : R \to \mathbb{R}$. Since

$$\|\rho_n - \rho\|_\infty \leq \sum_{k=n+1}^{\infty} \frac{1}{k^2} \overset{n\to\infty}{\longrightarrow} 0 \quad, \tag{1.19}$$

the convergence is uniform: $\rho_n \overset{n\to\infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}}$ and thus $\rho \in \Phi$. But this means

$$\bigcap_{n=1}^{\infty} F_n = \{|f_n| = 0 \ \ \forall\, n \in \mathbb{N}\} = \{\rho = 0\} \in \mathscr{F} \quad. \tag{1.20}$$

Now let $f_1, f_2 \in \Phi$, then

$$\{f_1 = 0\} \cup \{f_2 = 0\} = \{\min\{|f_1|, |f_2|\} = 0\} \in \mathscr{F} \quad. \tag{1.21}$$

Obviously $\emptyset, R \in \mathscr{F}$.

Claim: All $f \in \Phi$ are continuous on $(R, \mathscr{F})$.

Proof: By 1.2.9(1) it suffices to show $\{f \leq \alpha\} \in \mathscr{F} \ \ \forall\, \alpha \in \mathbb{R}, \ f \in \Phi$. But indeed:

$$\{f \leq \alpha\} = \{[\max\{f, \alpha\} - \alpha] = 0\} \in \mathscr{F} \quad. \tag{1.22}$$

In particular, this shows that $(R, \mathscr{F})$ is perfectly normal.

Claim: If $(R, \mathscr{F}_0)$ is an abstract space and $\Phi = \mathcal{C}(R, \mathscr{F}_0)$, then $\mathscr{F} \subseteq \mathscr{F}_0$ and $\mathcal{C}(R, \mathscr{F}) = \mathcal{C}(R, \mathscr{F}_0)$.

Proof: Obviously all $f^{-1}(\{0\})$, $f \in \Phi$ are closed with respect to $\mathscr{F}_0$, so that $\mathscr{F} \subseteq \mathscr{F}_0$. But then

$$\mathcal{C}(R, \mathscr{F}) \subseteq \mathcal{C}(R, \mathscr{F}_0) \quad, \tag{1.23}$$

since $\mathscr{F}$ is coarser than $\mathscr{F}_0$. On the other hand, it was shown that $\Phi \subseteq \mathcal{C}(R, \mathscr{F})$, which implies

$$\mathcal{C}(R, \mathscr{F}) = \mathcal{C}(R, \mathscr{F}_0) \quad. \tag{1.24}$$

$\square$

1.3.2 Definition: Totally closed & Baire-sets

Let $(R, \mathscr{O})$ be an A-topological space. A set $F \subseteq R$ is called *totally closed*, if there exists a continuous $f \in \mathcal{C}(R)$, such that $F = f^{-1}(\{0\})$. Complements of totally closed sets we call *totally open*. If $\mathscr{F}$ and $\mathscr{O}$ are the closed and open sets of R, then $\mathscr{F}^t$ and $\mathscr{O}^t$ shall denote the subsystems of totally closed and totally open sets respectively.

By theorem 1.3.1, $(R, \mathscr{O}^t)$ is a perfectly normal, A-topological space, whose continuous functions are exactly $\mathcal{C}(R, \mathscr{O})$. Clearly, $\mathscr{O}^t = \mathscr{O}$ iff $(R, \mathscr{O})$ is already perfectly normal.

The σ-algebra (algebra) $\sigma(\mathscr{O}^t)$ $(\alpha(\mathscr{O}^t))$ generated by the totally open sets, we call the *Baire σ-algebra (Baire algebra)*[6]. It is a sub-σ-algebra (sub-algebra) of the Borel σ-algebra (Borel algebra), and the smallest σ-algebra (algebra) for which all continuous functions are measurable[4].

Note: If $f : R \to \mathbb{R}$ is continuous, then $g := \arctan \circ f$ is continuous and bounded, whereas $f^{-1}(\{0\}) = g^{-1}(\{0\})$. Thus, in the definition of totally closed sets, $\mathcal{C}(R)$ can be replaced by $\mathcal{C}_b(R)$.

---

[4]With respect to the Borel-algebra on $\mathbb{R}$.

1.3.3 Characterization of totally closed sets

Let $(R, \mathscr{O})$ be an A-topological space and $F \subseteq R$ closed. Then the following are equivalent:

1. F is totally closed.

2. There exists a closed $A \subseteq \mathbb{R}$ and continuous $f \in \mathcal{C}(R)$, such that $F = f^{-1}(A)$.

3. F is an intersection of an enumerable sequence of totally open sets.

4. If R is normal: F is an intersection of an enumerable sequence of open sets.

Proof

1$\Rightarrow$2: Trivial, since $\{0\} \subseteq \mathbb{R}$ is closed.

2$\Rightarrow$1: Since by 1.3.1 f is continuous with respect to the A-topology of totally open sets, $f^{-1}(A)$ is totally closed.

1$\Rightarrow$3: Let $F = f^{-1}(\{0\})$ for some $f \in \mathcal{C}(R)$ and set $U_n := \{|f| < 1/n\}$, then

$$F = \bigcap_{n=1}^{\infty} \underbrace{\left\{|f| < \tfrac{1}{n}\right\}}_{\substack{\text{totally open} \\ \text{by (2)}}} \quad . \tag{1.25}$$

3$\Rightarrow$4: Trivial, since totally open sets are open.

4$\Rightarrow$1: Let $G_n \subseteq R$ be open such that $F = \bigcap_{n=1}^{\infty} G_n$. Let $f_n \in \mathcal{C}(R)$ be such that $0 \leq f_n \leq 1$, $f_n|_F = 0$ and $f_n|_{G_n^c} = 1$ (cmp. 1.1.7(1)). Set

$$f := \sum_{i=1}^{\infty} \frac{f_n}{2^n} \quad , \tag{1.26}$$

then $f : R \to \mathbb{R}$ is the uniform limit of continuous functions, hence its self continuous. Furthermore, F can be written as

$$F = \bigcap_{n=1}^{\infty} \{f_n = 0\} = f^{-1}(\{0\}) \quad , \tag{1.27}$$

which completes the proof.

$\square$

1.3.4 Lemma about vanishing sequences of totally closed sets

Let $(R, \mathscr{O})$ be an A-topological space and $F_n \subseteq R$, $n \in \mathbb{N}$ a sequence of totally closed sets, such that $F_n \downarrow \emptyset$. Then there exists a sequence $G_n \subseteq R$ of totally open sets, such that $F_n \subseteq G_n$ and $G_n \downarrow \emptyset$.

Proof

By 1.3.3, every F_n is an intersection of a sequence $(U_n^i)_{i=1}^{\infty}$ of totally open sets: $F_n = \bigcap_{i=1}^{\infty} U_n^i$. Define

$$G_n := \bigcap_{k=1}^{n} \bigcap_{i=1}^{n} U_k^i \quad , \quad n \in \mathbb{N} \quad . \tag{1.28}$$

Since totally open sets form an A-topology, the sets G_n are also totally open. Obviously $F_n \subseteq G_n$ and $G_n \downarrow \emptyset$.

$\square$

1.4 Linear functionals on function spaces

In the following, linear functionals are considered on lattice function spaces. It will be shown that, every bounded linear functional is decomposable into a so called *positive* and *negative* part. This will be of great importance in chapter 4, where a connection to set-functions on Borel algebras will be sought. It turns out that, positive & negative parts of functionals are closely related to positive & negative parts of the set functions to which they correspond.

As certain concepts such as σ-continuity of linear functionals are interchangeable with this decomposition, several statements regarding them, can be reduced to the case of non-negative ones.

Furthermore, this decomposition will become useful in turning the duals of certain function spaces into normed ones, without the need for a norm in the original function space (see 1.4.4).

1.4.1 Definition: Bounded functionals on function spaces

Let Φ be a $\mathbb{K}$-linear function space over some set Ω. A subsystem $\Gamma \subseteq \Phi$ is called *bounded*, if there exists an $f \in \Phi$ such that

$$\|\gamma(x)\| \leq \|f(x)\| \quad \forall \, x \in \Omega, \, \gamma \in \Gamma \ .$$

A linear functional $L : \Phi \to \mathbb{K}$ is called *bounded*, if it maps bounded sets into bounded sets. The space Φ^* of all bounded, linear functionals on Φ is called the *dual space* of Φ.
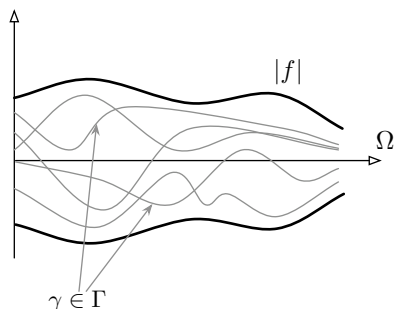
Figure 1.2: On bounded function sets: All functions $\gamma \in \Gamma$ are dominated by one chosen function $f \in \Phi$.

Note:

(i) In case Φ is a lattice space: $\Gamma \subseteq \Phi$ is bounded $\Leftrightarrow \exists \, f, g \in \Phi \ : \ f \leq \gamma \leq g \ \forall \, \gamma \in \Phi$.

(ii) In case Φ is a grounded function space:

- $\Gamma \subseteq \Phi$ is bounded iff it is bounded in the $\|\cdot\|_\infty$-norm, that is, iff there exists $M \geq 0$ such that $\|\gamma\|_\infty \leq M \ \forall \, \gamma \in \Gamma$.
- L is bounded iff it is bounded in the operator-norm, that is, $\displaystyle\sup_{\|f\|_\infty \leq 1} \|Lf\| < \infty$.

1.4.2 Lemma: Bound of linear functionals on lattice function spaces

Let L be a linear functional on a lattice function space Φ, such that $Lf \geq 0$ whenever $0 \leq f \in \Phi$ (we say L is *non-negative*). Then L is bounded.

Proof

Let $\Gamma \subseteq \Phi$ be bounded, that is $|\gamma| \leq |g| \ \forall \ \gamma \in \Gamma$ for some $g \in \Phi$. By assumption also $|g| \in \Phi$ and

$$-|g| \leq \gamma \leq |g| \quad \forall \ \gamma \in \Gamma \ . \tag{1.29}$$

As $L \geq 0$, this implies

$$-L|g| \leq L\gamma \leq L|g| \quad \forall \ \gamma \in \Gamma \ , \tag{1.30}$$

that is, $L(\Gamma)$ is bounded.
$\square$

1.4.3 Theorem: Decomposition of bounded, linear functionals

Let L be a bounded, linear functional on a lattice function-space Φ. Then there exist non-negative, linear functionals L^+, L^- on Φ such that

$$L = L^+ - L^- \tag{1.31}$$

and for each other pair $\widetilde{L}^+, \widetilde{L}^-$ of non-negative, linear functionals with $L = \widetilde{L}^+ - \widetilde{L}^-$, one has

$$\widetilde{L}^+ f \geq L^+ f \ , \ \widetilde{L}^- f \geq L^- f \quad \forall \ 0 \leq f \in \Phi \ . \tag{1.32}$$

The functionals L^+, L^- are of course, due to (1.32), unique. They are called *positive* and *negative* parts of L respectively,

$$|L| := L^+ + L^- \tag{1.33}$$

its *total variation*. In fact, one has for $0 \leq f \in \Phi$ the representation

$$L^+(f) = \sup_{\substack{0 \leq h \leq f \\ h \in \Phi}} Lh \ , \quad L^- f = - \inf_{\substack{0 \leq h \leq f \\ h \in \Phi}} Lh \ , \quad |L|(f) = \sup_{\substack{0 \leq |h| \leq f \\ h \in \Phi}} |Lh| \ . \tag{1.34}$$

Furthermore, for $f \in \Phi$ the inequality

$$|Lf| \leq \big| |L| |f| \big| \tag{1.35}$$

holds[5].

Proof

The proof is inspired by Alexandroff[2]. For $0 \leq f \in \Phi$ define

$$L^+ f := \sup_{0 \leq h \leq f} Lh \ , \quad L^- f := - \inf_{0 \leq h \leq f} Lh \tag{1.36}$$

and for $f \in \Phi$:

$$L^+ f := L^+ f^+ - L^+ f^- \ , \quad L^- f := L^- f^+ - L^- f^- \ . \tag{1.37}$$

Note that L^+, L^- are indeed well defined, as the set $\{h \in \Phi : 0 \leq h \leq f\}$ is bounded.
Claim: L^+ and L^- are non-negative, linear functionals on Φ.

---

[5]Note that $|Lf| \leq \big| |L| f \big|$ is in general not true! See appendix A.3.1 for a counter-example.

Proof: Obviously L^+ and L^- are non-negative and it suffices to show the linearity of L^+. To begin with, let $0 \le f, g \in \Phi$. Then super-additivity follows directly from the definition:

$$L^+(f + g) = \sup_{0 \le p \le f} Lp \ge \sup_{\substack{0 \le p \le f \\ 0 \le q \le g}} L(p + q)$$

$$= \sup_{0 \le p \le f} Lp + \sup_{0 \le q \le g} Lq = L^+ f + L^+ g \tag{1.38}$$

For sub-additivity, it suffices to show that

$$Lh \le L^+ f + L^+ g \quad \forall \, 0 \le h \le (f + g), \ h \in \Phi \ . \tag{1.39}$$

Indeed, let $h \in \Phi$ such that $h \le (f + g)$ and set

$$h_f := \min \{h, f\} \quad , \quad h_g := h - h_f \ . \tag{1.40}$$

Then $0 \le h_f \le f$, $0 \le h_g \le g$ and $(h_f + h_g) = h$, thus

$$Lh = Lh_f + Lh_g \le L^+ f + L^+ g \tag{1.41}$$

and consequently $L^+(f + g) = L^+ f + L^+ g$ for $f, g \ge 0$.
On the other hand, if $h := f - g$, then $h^- + f = h^+ + g$ and by the above

$$L^+ h^- + L^+ f = L^+ h^+ + L^+ g \ , \tag{1.42}$$

which implies

$$L^+ f - L^+ g = L^+ h^+ - L^+ h^- \stackrel{\text{def}}{=} L^+ h \ . \tag{1.43}$$

It is now evident, that L is additive on Φ.
What remains to be shown, is that for $\alpha \in \mathbb{R}$, $f \in \Phi$ one has $L^+(\alpha f) = \alpha \cdot L^+ f$, whereas due to additivity we may assume $0 \le f$. The case $\alpha \ge 0$ is obvious. But so is the case $\alpha < 0$, since

$$L^+(\alpha f) = L^+ \underbrace{(\alpha f)^+}_{0} - L^+ \underbrace{(\alpha f)^-}_{|\alpha| \cdot f} = -|\alpha| \cdot L^+ f = \alpha \cdot L^+ f \ . \tag{1.44}$$

Claim: $L = L^+ - L^-$.
Proof: It suffices to show the claim for non-negative arguments $0 \le f$. Let $\varepsilon > 0$ and choose $0 \le h_1, h_2 \le f$ such that

$$L^+ f \ge Lh_1 \ge L^+ f - \varepsilon \quad , \quad -L^- f \le Lh_2 \le -L^- f + \varepsilon \ . \tag{1.45}$$

In case $L(f - h_1 - h_2) \ge 0$ set $f_1 := f - h_2$, $f_2 := h_2$, in case $L(f - h_1 - h_2) < 0$ set $f_1 := h_1$, $f_2 := f - h_1$. Then in either case:

$$0 \le f_1, f_2 \le f \quad , \quad f_1 + f_2 = f \tag{1.46}$$

and by (1.45)

$$\left| Lf_1 - L^+ f \right| \le \varepsilon \quad , \quad \left| Lf_2 + L^- f \right| \le \varepsilon \ . \tag{1.47}$$

Consequently

$$\left| Lf - (L^+ f - L^- f) \right| \le \left| Lf_1 - L^+ f \right| + \left| Lf_2 + L^- f \right| \le 2\varepsilon \ . \tag{1.48}$$

By arbitrarity of $\varepsilon > 0$ finally

$$Lf = L^+f - L^-f \quad , \quad 0 \leq f \in \Phi \quad . \tag{1.49}$$

Claim: Let L_1, L_2 be non-negative, linear functionals on Φ such that $L = L_1 - L_2$. Then

$$L^+f \leq L_1f \quad , \quad L^-f \leq L_2f \quad \forall\, 0 \leq f \in \Phi \quad . \tag{1.50}$$

Proof: Since L_1, L_2 are non-negative one has

$$Lh = L_1h - L_2h \leq L_1h \leq L_1f \quad \forall\, 0 \leq h \leq f \in \Phi \quad , \tag{1.51}$$

thus

$$L^+f = \sup_{\substack{0 \leq h \leq f \\ h \in \Phi}} Lh \leq L_1f \quad . \tag{1.52}$$

The claim is proven for L^- in a similar way.

Claim: For $0 \leq f \in \Phi$ the relation $|L|\,f = \sup\limits_{0 \leq |h| \leq f} |Lh|$ holds.

Proof: We recall that

$$|L|\,f = L^+f + L^-f \quad , \quad L^+f = \sup_{0 \leq h \leq f} Lh \quad , \quad L^-f = \sup_{0 \leq h \leq f} L(-h) \quad . \tag{1.53}$$

Now let $0 \leq |h| \leq f$ for some $h \in \Phi$, then

$$0 \leq h^+ \leq f \quad , \quad 0 \leq h^- \leq f \quad , \tag{1.54}$$

which implies by construction

$$Lh^+ \leq L^+f \quad , \quad L(-h^-) \leq L^-f \quad . \tag{1.55}$$

But then

$$Lh = Lh^+ - Lh^- \leq L^+f + L^-f = |L|\,f \quad \forall\, 0 \leq |h| \leq f \quad , \tag{1.56}$$

which actually means

$$|Lh| \leq |L|\,f \quad \forall\, 0 \leq |h| \leq f \tag{1.57}$$

or equivalently

$$\sup_{0 \leq |h| \leq f} |Lh| \leq |L|\,f \quad . \tag{1.58}$$

On the other hand, let $0 \leq q \leq f$, $0 \leq p \leq f$ for some $q, p \in \Phi$ and set $h := q - p$. Then $0 \leq |h| \leq f$ and

$$Lq + L(-p) \leq |Lq + L(-p)| = |Lh| \leq \sup_{0 \leq |h'| \leq f} \left|Lh'\right| \tag{1.59}$$

and thus

$$|L|\,f = \sup_{0 \leq q \leq f} Lq + \sup_{0 \leq p \leq f} L(-p) \leq \sup_{0 \leq |h'| \leq f} \left|Lh'\right| \quad , \tag{1.60}$$

which together with (1.58) was to be shown.

Claim: Inequality (1.35) holds for all $f \in \Phi$.

Proof: Assume for a start that $f \geq 0$. Then:

$$|Lf| \overset{(1.33)}{=} \big| \underbrace{L^+f}_{\geq 0} - \underbrace{L^-f}_{\geq 0} \big| \leq |L^+f + L^-f| = \big| |L| f \big| \quad . \tag{1.61}$$

Now, for arbitrary $f \in \Phi$

$$|Lf| = \big| Lf^+ - Lf^- \big| \leq \big| Lf^+ \big| + \big| Lf^- \big|$$

$$\overset{(1.61)}{\leq} \big| \underbrace{|L| f^+}_{\geq 0} \big| + \big| \underbrace{|L| f^-}_{\geq 0} \big| = \big| |L| f^+ + |L| f^- \big| = \big| |L| |f| \big| \quad . \tag{1.62}$$

$\square$

1.4.4 Lemma: Bound of functionals on function rings

Let Φ be a ring, lattice function space over some set Ω and Φ_b the subspace of bounded functions in Φ [6].

1. If $L \in \Phi^*$ is a bounded linear functional, such that $Lf = 0$ for all bounded functions $f \in \Phi_b$, then L is identical to zero.

2. The mapping

$$\|L\| := |L| 1 \overset{(1.34)}{=} \sup_{\substack{h \in \Phi \\ 0 \leq |h| \leq 1}} |Lh| \quad , \quad L \in \Phi^* \tag{1.63}$$

   defines a norm on Φ^*.

3. For every $L \in \Phi^*$ the identity

$$\|L\| = \big\| L^+ \big\| + \big\| L^- \big\| = \big\| |L| \big\| \tag{1.64}$$

   holds.

4. Let Φ_b and Φ_b^* be endowed with the supremum norm and standard operator norm respectively. Then the linear mapping

$$\Phi^* \to \Phi_b^* \quad , \quad L \mapsto L\big|_{\Phi_b} \tag{1.65}$$

   is an isometry between normed spaces. In particular, if all functions $f \in \Phi$ are bounded, then the norm defined in (1.63) coincides with the standard operator norm.

Proof

1. Suppose for a start, that L is non-negative. For $0 \leq f \in \Phi$ and $n \in \mathbb{N}$ the inequality

$$0 \leq f - (f \wedge n) \leq \frac{f^2}{n} \tag{1.66}$$

   always holds. Thus, since L is non-negative:

$$0 \leq L(f - (f \wedge n)) \leq \frac{Lf^2}{n} \overset{n \to \infty}{\longrightarrow} 0 \quad , \tag{1.67}$$

---

[6]Note that Φ_b is a ring and grounded function space.

hence, $Lf = \lim\limits_{n\to\infty} L(f \wedge n) = 0$. By the decomposition $f = f^+ - f^-$ this result extends to arbitrary $f \in \Phi$.

Now let L be arbitrary, then its positive and negative parts L^+, L^- vanish on all bounded functions. Since they are non-negative, they vanish everywhere and so does L.

2. That (1.63) defines indeed a norm, should be clear from part (1) and the representation (1.63).

3. Note that non-negative functionals equal their total variations. Thus

$$\big\| |L| \big\| = \underbrace{\big| |L| \big| 1}_{|L|1 = \|L\|} = L^+ 1 + L^- 1 = \big| L^+ \big| 1 + \big| L^- \big| 1 = \big\| L^+ \big\| + \big\| L^- \big\| \quad . \tag{1.68}$$

4. Trivial.

□

Lemma 1.4.4 allows the duals of certain function spaces lacking a natural norm, to be turned into normed spaces. This norm will later on in chapter 4, prove to be strongly connected with the total variation norm of set functions, to be introduced in 2.1.6.

Identity (1.64) can be formally interpreted as stating that, the positive and negative parts of L are in a sense *perpendicular* to each other. This can be seen as follows: Were the functional norm $\|\cdot\|$ induced by a scalar product, then

$$``\big\langle L^+, L^- \big\rangle" = \frac{1}{4}\left[\big\|L^+ + L^-\big\|^2 - \big\|L^+ - L^-\big\|^2\right] = 0 \tag{1.69}$$

would hold. This is in accordance with the relationship established later on with set functions, whose positive and negative parts *live* on disjoint sets (see A.4.6). Lemma 1.4.5 shows this interesting property for a different class of function spaces.

1.4.5 Lemma: Norm of functionals on grounded function spaces

If L is a bounded, linear functional on the grounded function space Φ, then

$$\|L\| = \big\|L^+\big\| + \big\|L^-\big\| = \big\| |L| \big\| = |L| 1 \tag{1.70}$$

holds.

Proof

Note that the operator norm of any $L \in \Phi^*$ is given by

$$\|L\| = \sup_{0 \le |h| \le 1} |Lh| \overset{(1.34)}{=} |L| 1 \tag{1.71}$$

and in particular, $\|L\| = L1$ if L is non-negative. Consequently:

$$\big\| |L| \big\| = |L| 1 = L^+ 1 + L^- 1 = \big\|L^+\big\| + \big\|L^-\big\| \quad . \tag{1.72}$$

□

1.4.6 Definition: σ-Continuity of functionals

Let L be a linear functional on a real, linear function-space Φ. Then L is called σ-*continuous* if it is continuous with respect to monotone, pointwise convergence of functions, that is, for $f_n \in \Phi$ such that $f_n \downarrow 0$ one has $Lf_n \overset{n\to\infty}{\longrightarrow} 0$.

1.4.7 Lemma: σ-Continuity of positive & negative parts

Let Φ be a lattice function-space and L a bounded, linear functional on Φ. Then the following are equivalent:

1. L is σ-continuous.

2. The positive and negative parts L^+, L^- are σ-continuous.

3. $|L|$ is σ-continuous.

Proof

The proof originates from Varadarajan[13].

1$\Rightarrow$2: As $L^- = L^+ - L$, it suffices to show the σ-continuity of L^+. Suppose L^+ is not σ-continuous, then there exist $\Phi \ni f_n \downarrow 0$ and $\delta > 0$ such that $L^+ f_n > \delta \;\; \forall\, n$. Choose $\widetilde{h}_1 \in \Phi$ such that

$$0 \le \widetilde{h}_1 \le f_1 \quad , \quad L\widetilde{h}_1 > \delta \tag{1.73}$$

(note theorem 1.4.3). Since $f_n \downarrow 0$ one has $\max\left\{f_n, \widetilde{h}_1\right\} \downarrow \widetilde{h}_1$ and thus by assumption

$$L \max\left\{f_n, \widetilde{h}_1\right\} \overset{n\to\infty}{\longrightarrow} L\widetilde{h}_1 > \delta \quad . \tag{1.74}$$

But then for some $n_1 \in \mathbb{N}$ one has

$$L \underbrace{\max\left\{f_{n_1}, \widetilde{h}_1\right\}}_{=: h_1 \in \Phi} > \delta \quad , \tag{1.75}$$

whereas $0 \le f_{n_1} \le h_1 \le f_1$. Repeating the procedure for f_{n_1} instead of f_1, we find $n_1 < n_2 \in \mathbb{N}$ and $h_2 \in \Phi$ such that $0 \le f_{n_2} \le h_2 \le f_{n_1}$ and so on.

We thus obtain a sequence $n_1 < n_2 < \cdots \in \mathbb{N}$ and $(h_k) \in \Phi$ such that

$$f_{n_k} \le h_k \le f_{n_{k-1}} \quad \wedge \quad L h_k > \delta \tag{1.76}$$

In particular $h_k \downarrow 0$, which is a contradiction to the σ-continuity of L.

2$\Rightarrow$3: Follows directly from $|L| = L^+ + L^-$.

3$\Rightarrow$1: Follows directly from inequality (1.35).

$\square$

1.4.8 Lemma: Characterization of σ-continuity of functionals

Let $\Phi \subseteq \mathcal{B}(\Omega)$ be a subspace of the real, bounded functions over some set Ω. Then the following are equivalent:

1. Every bounded, linear functional on Φ is σ-continuous.

2. On Φ, Dini's theorem holds, that is, every pointwise, monotonically convergent to zero sequence of functions in Φ is uniformly convergent.

Proof

The proof is due to Alexandroff[2].

1⇒2: Suppose the contrary. Then there exists a sequence $f_n \in \Phi$ such that $f_n \downarrow 0$, that does not converge uniformly. Then we may w.l.o.g. assume that for some $\varepsilon > 0$, there exist points $(x_n)_{n=1}^\infty \subseteq \Omega$ such that $f_n(x_n) \geq \varepsilon \; \forall \; n \in \mathbb{N}$.

Now let LIM : $l_\infty(\mathbb{R}) \to \mathbb{R}$ be a Banach limit[3, 5], then

$$Lg := \underset{n \to \infty}{\text{LIM}} \; \underbrace{g(x_n)}_{\substack{\in l_\infty \\ \text{since} \\ \|g\|_\infty < \infty}} \quad , \quad g \in \Phi \tag{1.77}$$

defines a bounded, linear functional on Φ by the axioms of a Banach-Limit. But for every $n \in \mathbb{N}$, $f_n(x_k) \geq \varepsilon \; \forall \; k \geq n$, thus $Lf_n \geq \varepsilon \; \forall \; n$, which shows that L is not σ-continuous.

2⇒1: Trivial, since $|Lf_n| \leq \|L\| \cdot \|f_n\|_\infty$ for any sequence of functions $(f_n) \subseteq \Phi$.

$\square$

Chapter 2

Additive set functions

This chapter is meant to be a self-contained introduction to additive set-functions, though commonly known facts about σ-additive set functions are used from time to time. We refer the reader to Bogachev[4] and Halmos[8] for an in depth analysis of the subject. Of particular importance are the concepts of regularity and σ-additivity. Accordingly, we introduce three kinds of set functions: contents, charges and measures. While these are always taken to be real valued, the reader should be made aware of the fact that, contrary to common convention, no assumption is made about their sign.

2.1 Contents

2.1.1 Definition: Content & measure

Let $\mathfrak{S} \subseteq \mathscr{P}(\Omega)$ be a system of subsets of some set Ω. A *set function* on $\mathfrak{S}$ is any real valued function $\mu : \mathfrak{S} \to \mathbb{R}$. A set function $\mu : \mathfrak{S} \to \mathbb{R}$ is called *additive*, if

$$\mu(A \dot\cup B) = \mu(A) + \mu(B) \tag{2.1}$$

for all disjoint $A, B \in \mathfrak{S}$ with $A \dot\cup B \in \mathfrak{S}$. We call μ *σ-additive*, if for disjoint $A_n \in \mathfrak{S}$, $n \in \mathbb{N}$ such that $\biguplus_{n=1}^{\infty} A_n \in \mathfrak{S}$, one has

$$\mu \left(\biguplus_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A) \quad . \tag{2.2}$$

If $\mathfrak{S}$ is a set-algebra and μ additive, we call μ a *content* and $(\Omega, \mathfrak{S}, \mu)$ a *content space*. If $\mathfrak{S}$ is a σ-Algebra and μ is σ-additive, μ is called a *measure* and $(\Omega, \mathfrak{S}, \mu)$ a *measure space*. We call μ *bounded*, if there exists an $M \geq 0$ such that $|\mu(E)| \leq M \ \forall \ E \in \mathfrak{S}$.

The real, linear spaces[1] of all bounded, additive (σ-additive) set-functions on $\mathfrak{S}$ we shall denote by $\mathfrak{M}_\alpha(\mathfrak{S})$ ($\mathfrak{M}_\sigma(\mathfrak{S})$).

2.1.2 Lemma: Bound of measures

Let $(\Omega, \mathfrak{S}, \mu)$ be a measure space. Then μ is bounded.

---

[1]With linear combinations to be understood in a setwise manner.

Proof

This proof is due to Dunford & Schwartz[7]. Adopting their notation, we shall call a set $A \in \mathfrak{S}$ *unbounded*, if

$$\sup_{\substack{B \in \mathfrak{S} \\ B \subseteq A}} \mu(B) = \infty \quad , \tag{2.3}$$

otherwise call it *bounded*.

Claim: Every unbounded set contains an unbounded subset of arbitrarily large measure.
Proof: Suppose there exists an unbounded set $A \in \mathfrak{S}$ and $N \in \mathbb{N}$ such that $\mu(B) \leq N$ for every unbounded subset $B \subseteq A$. Choose some measurable $A_1 \subseteq A$ such that $\mu(A_1) > N$, then A_1 is bounded and thus $A \setminus A_1$ unbounded. Choose some measurable $B_1 \subseteq A \setminus A_1$ such that $\mu(B_1) \geq 1$, then $A_2 := A_1 \dot\cup B_1$ is bounded, since $\mu(A_2) > N$. In a similar way, we can inductively choose a disjoint sequence of measurable $B_1, B_2, .. \subseteq A$ such that $\mu(B_n) \geq 1 \; \forall \; n$. Since μ is σ-additive, this implies $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \infty$, a contradiction!

Claim: Ω is a bounded set.
Proof: Suppose the contrary, then by the previous claim there exists a decreasing sequence of unbounded sets $B_1, B_2, ..$ with increasing measures $\mu(B_n) \uparrow \infty$. Set $B := \bigcap_{n=1}^{\infty} B_n$, then

$$B_n = B \cup \biguplus_{k=n}^{\infty} (B_k \setminus B_{k+1}) \tag{2.4}$$

and thus the series $\sum_{k=1}^{\infty} \mu(B_k \setminus B_{k+1}) = \mu(B_1 \setminus B)$ converges. Hence,

$$\infty = \lim_{n \to \infty} \mu(B_n) \overset{(2.4)}{=} \mu(B) + \underbrace{\lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(B_k \setminus B_{k+1})}_{0} = \mu(B) \tag{2.5}$$

which is a contradiction.
$\square$

2.1.3 Example of a non σ-additive content

In the following, an example is given of a bounded, additive set function μ defined on a σ-algebra, which is not σ-additive[4].
Let LIM be a Banach limit on the linear space of real, bounded sequences l_{∞}. Consider the σ-algebra $\mathscr{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ and define on it the bounded set-function

$$\mu(A) := \mathrm{LIM}\, 1_A \quad , \quad A \subseteq \mathbb{N} \quad , \tag{2.6}$$

where 1_A is the indicator function of A, that is, that sequence whose n-th term is $1_A(n)$. Then μ is additive, since for disjoint $A_1, A_2 \subseteq \mathbb{N}$:

$$1_{A_1 \cup A_2} = 1_{A_1} + 1_{A_2} \quad . \tag{2.7}$$

Clearly, $\mu(\mathbb{N}) = 1$. On the other hand, μ vanishes on all finite sets, since these correspond to sequences converging to 0. Thus, μ is **not** σ-additive.

2.1.4 Theorem: Decomposition of bounded contents

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space. For $A \in \mathfrak{S}$ define

$$|\mu|(A) := \sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| \ : \ n \in \mathbb{N}, \ A_1, .., A_n \in \mathfrak{S}, \ A_i \subseteq A \text{ disjoint} \right\} \tag{2.8}$$

as the *total variation* of μ (not to be confused with its absolute value) and

$$\mu^+ := \frac{1}{2} \left(|\mu| + \mu \right) \ , \ \ \mu^- := \frac{1}{2} \left(|\mu| - \mu \right) \ . \tag{2.9}$$

Then:

1. The set functions $|\mu|, \mu^+, \mu^-$ are non-negative contents on $\mathfrak{S}$.

2. For any $A \in \mathfrak{S}$, one can estimate

$$|\mu|(A) \leq 2 \cdot \sup_{E \in \mathfrak{S}} |\mu(A \cap E)| \ . \tag{2.10}$$

3. If $\widetilde{\mu}^+, \widetilde{\mu}^-$ are also non-negative contents such that $\mu = \widetilde{\mu}^+ - \widetilde{\mu}^-$, then

$$\widetilde{\mu}^+ \geq \mu^+ \ , \ \ \widetilde{\mu}^- \geq \mu^- \ . \tag{2.11}$$

4. μ^+, μ^- can be represented as

$$\mu^+(A) = \sup_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} \mu(E) \ , \ \ \mu^-(A) = - \inf_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} \mu(E) \ . \tag{2.12}$$

The contents μ^+ and μ^- are called the *positive* and *negative* parts of μ. Note that

$$\mu = \mu^+ - \mu^- \ , \ \ |\mu| = \mu^+ + \mu^- \ . \tag{2.13}$$

Proof

This proof is due to Alexandroff[2] and Dunford & Schwartz[7].

1. By part (2) (which shall be proven right after), $|\mu|(A)$ is indeed real valued. By construction (2.9), it suffices to show that μ is a non-negative content (note that $|\mu(A)| \leq |\mu|(A)$, thus μ^+, μ^- will also be non-negative).

   Let $A, B \in \mathfrak{S}$ be disjoint, then any disjoint subsets $\mathfrak{S} \ni A_1, .., A_n \subseteq A$ and $\mathfrak{S} \ni B_1, .., B_m \subseteq B$ are also disjoint subsets of $A \dot\cup B$, thus super-additivity is implied:

$$|\mu|(A \dot\cup B) \geq \mu(A) + \mu(B) \ . \tag{2.14}$$

   On the other hand, let $\mathfrak{S} \ni E_1, .., E_n \subseteq (A \dot\cup B)$ be disjoint, then

$$(A \cap E_1), .., (A \cap E_n) \subseteq A \ , \ \ (B \cap E_1), .., (B \cap E_n) \subseteq B \tag{2.15}$$

   are also disjoint. Thus

$$\sum_{i=1}^{n} |\mu(E_i)| = \sum_{i=1}^{n} |\mu(A \cap E_i) + \mu(B \cap E_i)|$$

$$\leq \sum_{i=1}^{n} |\mu(A \cap E_i)| + \sum_{i=1}^{n} |\mu(B \cap E_i)| \leq |\mu|(A) + |\mu|(B) \tag{2.16}$$

and consequently

$$|\mu|\,(A \dot\cup B) \leq |\mu|\,(A) + |\mu|\,(B) \quad . \tag{2.17}$$

Hence, $|\mu|$ is additive. Non-negativity is evident.

2. Since the restriction of μ to $A \cap \mathfrak{S}$ defines a content on the algebra $A \cap \mathfrak{S}$, we may w.l.o.g. assume $A = \Omega$. Let $|\mu(E)| \leq M \;\; \forall\, E \in \mathfrak{S}$ for some $M \geq 0$. Let $A_1, .., A_n \in \mathfrak{S}$ be disjoint, then

$$\sum_{i=1}^{n} |\mu(A_i)| = \sum_{\substack{1 \leq i \leq n \\ \mu(A_i) \geq 0}} \mu(A_i) - \sum_{\substack{1 \leq i \leq n \\ \mu(A_i) < 0}} \mu(A_i)$$

$$= \mu\bigg(\underbrace{\biguplus_{\mu(A_i) \geq 0} A_i}_{\in [0,M]} \bigg) - \mu\bigg(\underbrace{\biguplus_{\mu(A_i) < 0} A_i}_{\in [-M,0]} \bigg) \in [0, 2M] \quad . \tag{2.18}$$

Thus $|\mu|\,(\Omega) \leq 2M$.

3. It suffices to show that $\widetilde{\mu}^+ \geq \mu^+$. By (2.9) and (2.13), this is equivalent to showing $|\mu| \leq \widetilde{\mu}^+ + \widetilde{\mu}^-$. Indeed, for $A \in \mathfrak{S}$ and disjoint $\mathfrak{S} \ni A_1, .., A_n \subseteq A$:

$$\sum_{i=1}^{n} |\mu(A_i)| = \sum_{i=1}^{n} \left| \widetilde{\mu}^+(A_i) - \widetilde{\mu}^-(A_i) \right| \leq \sum_{i=1}^{n} \big| \underbrace{\widetilde{\mu}^+(A_i)}_{\geq 0} \big| + \sum_{i=1}^{n} \big| \underbrace{\widetilde{\mu}^-(A_i)}_{\geq 0} \big| \leq \widetilde{\mu}^+(A) + \widetilde{\mu}^-(A)$$

$$\tag{2.19}$$

and thus

$$|\mu|\,(A) \leq \widetilde{\mu}^+(A) + \widetilde{\mu}^-(A) \quad . \tag{2.20}$$

4. Since $\mu^- = (-\mu)^+$ and $-\inf\limits_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} \mu(E) = \sup\limits_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} (-\mu)(E)$ it suffices to show the relation for μ^+. On the one hand

$$\mu(E) = \mu^+(E) - \underbrace{\mu^-(E)}_{\geq 0} \leq \mu^+(E) \overset{E \subseteq A}{\leq} \mu^+(A) \quad \forall\, E \subseteq A \tag{2.21}$$

and thus

$$\sup_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} \mu(E) \leq \mu^+(A) \quad . \tag{2.22}$$

Let on the other hand $A_1, .., A_n \subseteq A$ be disjoint and w.l.o.g. $A = \biguplus_{i=1}^{n} A_i$. Then

$$\frac{1}{2}\Big[\overbrace{\sum_{i=1}^{n} \mu(A_i)}^{\mu(A)} + \sum_{i=1}^{n} |\mu(A_i)| \Big] = \sum_{\substack{i=1 \\ \mu(A_i) \geq 0}}^{n} \mu(A_i) = \mu\bigg(\underbrace{\biguplus_{\substack{i=1 \\ \mu(A_i) \geq 0}}^{n} A_i}_{\subseteq A} \bigg) \leq \sup_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} \mu(E) \tag{2.23}$$

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and consequently

$$\frac{1}{2}\big[\mu(A) + |\mu|\,(A)\big] \leq \sup_{\substack{E \in \mathfrak{S} \\ E \subseteq A}} \mu(E) \quad, \tag{2.24}$$

which together with (2.22) was to be shown.

$\square$

The decomposition of contents into their positive and negative parts, presented in 2.1.4, will be of great importance throughout the rest of this paper. As it turns out, two of its central concepts, namely regularity and σ-additivity of contents, are interchangeable with decompositions of this kind. Consequently, many proofs regarding contents will be at first given for non-negative ones, only to be extended to the general case by means of exactly this decomposition.

Its similarity to the decomposition of linear functionals as presented in 1.4.3, will become truly apparent in chapter 4, where it is revealed that there exists a one-to-one connection between the positive and negative parts of functionals and contents.

Interestingly, as the following lemma 2.1.5 and corollary 2.1.6 reveal, the total variation of a content can be interpreted as a *global absolute value* of the mapping, hence the notation $|\mu|$. Furthermore, the positive and negative parts $\mu^{\pm}$ can, similarly to the standard positive and negative pars of real functions, be thought of as actually *living* on disjoint subsets (see A.4.6).

2.1.5   Lemma: Triangle inequality for the total variation

Let $\mathfrak{S}$ be a set algebra on some set Ω and μ, ν bounded contents on $\mathfrak{S}$. Then

$$\big|\,|\mu| - |\nu|\,\big| \leq |\mu + \nu| \leq |\mu| + |\nu| \tag{2.25}$$

holds.

Proof

Right part: Since $\mu + \nu = \overbrace{(\mu^+ + \nu^+)}^{\geq 0} - \overbrace{(\mu^- + \nu^-)}^{\geq 0}$, it follows from 2.1.4(3) that

$$|\mu + \nu| = (\mu + \nu)^+ + (\mu + \nu)^- \leq (\mu^+ + \nu^+) + (\mu^- + \nu^-) = |\mu| + |\nu| \quad. \tag{2.26}$$

Left part: Let $\rho := |\mu| - |\nu|$, $A \in \mathfrak{S}$ and $\varepsilon > 0$. Choose disjoint $A_1, .., A_n \in \mathfrak{S}$ such that $A_i \subseteq A$ and

$$|\rho|\,(A) \leq \sum_{i=1}^{n} |\rho(A_i)| + \varepsilon \quad. \tag{2.27}$$

For $i = 1, .., n$ choose disjoint $A_i^k \in \mathfrak{S}$, $1 \leq k \leq N_i$ such that $A_i^1, .., A_i^{N_i} \subseteq A_i$ and

$$|\mu|\,(A_i) \leq \sum_{k=1}^{N_i} \Big|\mu(A_i^k)\Big| + \frac{\varepsilon}{n} \quad, \quad |\nu|\,(A_i) \leq \sum_{k=1}^{N_i} |\nu(A_i^k)| + \frac{\varepsilon}{n} \quad. \tag{2.28}$$

Then

$$|\rho(A_i)| \leq \frac{2\varepsilon}{n} + \left|\sum_{k=1}^{N_i} \Big|\mu(A_i^k)\Big| - \Big|\nu(A_i^k)\Big|\right| \leq \frac{2\varepsilon}{n} + \sum_{k=1}^{N_i} \left|\,\Big|\mu(A_i^k)\Big| - \Big|\nu(A_i^k)\Big|\,\right|$$

$$\leq \frac{2\varepsilon}{n} + \sum_{k=1}^{N_i} \Big|(\mu + \nu)(A_i^k)\Big| \leq \frac{2\varepsilon}{n} + |\mu + \nu|\,(A_i) \tag{2.29}$$

and together with (2.28):

$$|\rho|(A) \le 3\varepsilon + \sum_{i=1}^{n} |\mu + \nu|(A_i) \le 3\varepsilon + |\mu + \nu|(A) \quad . \tag{2.30}$$

As $\varepsilon > 0$ was arbitrary, it follows $|\rho| \le |\mu + \nu|$.

$\square$

2.1.6 Corollary: The total variation as a norm

Let $\mathfrak{S}$ be a set algebra over some set Ω. The total variation defines a norm $\|\mu\|_t := |\mu|(\Omega)$ on the space $\mathfrak{M}_\alpha(\mathfrak{S})$ of bounded contents on $\mathfrak{S}$. This norm is equivalent to the $\|\cdot\|_\infty$ norm of these set functions over the domain $\mathfrak{S}$. With respect to $\|\cdot\|_t$, the mapping

$$\mathfrak{M}_\alpha(\mathfrak{S}) \to \mathfrak{M}_\alpha(\mathfrak{S}) \quad , \quad \mu \mapsto |\mu| \tag{2.31}$$

is continuous.

Proof

That $\|\mu\|_t := |\mu|(\Omega)$ defines a norm should be clear from the definition of the total variation and the previous lemma 2.1.5. Due to

$$\|\mu\|_\infty = \sup_{A \in \mathfrak{S}} |\mu(A)| \le |\mu|(\Omega) \overset{2.1.4(2)}{\le} 2 \cdot \sup_{A \in \mathfrak{S}} |\mu(A)| = 2 \cdot \|\mu\|_\infty \tag{2.32}$$

the equivalence of the norms is also clear.

Furthermore, if $\mu, \mu_n \in \mathfrak{M}_\alpha(\mathfrak{S})$ are such that, $\mu_n \overset{n\to\infty}{\underset{\|\cdot\|_t}{\longrightarrow}} \mu$, by 2.1.5 this implies $|\mu_n| \overset{n\to\infty}{\underset{\|\cdot\|_t}{\longrightarrow}} |\mu|$.

$\square$

2.1.7 Lemma: Characterization of σ-additivity of contents

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space. Then the following are equivalent:

1. μ is σ-additive.

2. $|\mu|$ is σ-additive.

3. μ is continuous at $\emptyset$, that is, for $(A_n)_{n=1}^\infty \subseteq \mathfrak{S}$ with $A_n \downarrow \emptyset$, one has

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad . \tag{2.33}$$

4. μ is continuous from above, that is, for $(A_n)_{n=1}^\infty \subseteq \mathfrak{S}$ such that $A_n \downarrow A \in \mathfrak{S}$, one has

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) \quad . \tag{2.34}$$

5. μ is continuous from below, that is, for $(A_n)_{n=1}^\infty \subseteq \mathfrak{S}$ such that $A_n \uparrow A \in \mathfrak{S}$, one has

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) \quad . \tag{2.35}$$

Proof

The following proof is taken from Dunford & Schwartz[7].

$1 \Leftrightarrow 3$: Trivial.

$2 \Rightarrow 3$: Follows from the fact $(1 \Rightarrow 3)$ and $|\mu(A_n)| \leq |\mu|(A_n)$.

$1 \Rightarrow 2$: Let $(A_n)_{n \in \mathbb{N}} \subseteq \mathfrak{S}$ be a sequence of disjoint sets and $A := \biguplus_{n=1}^{\infty} A_n \in \mathfrak{S}$. Then, since $|\mu|$ is additive and non-negative on $\mathfrak{S}$, one has

$$|\mu|(A) \geq |\mu| \left(\biguplus_{n=1}^{m} A_n \right) = \sum_{k=1}^{m} |\mu|(A_n) \quad \forall \, m \in \mathbb{N} \tag{2.36}$$

and thus

$$|\mu|(A) \geq \sum_{n=1}^{\infty} |\mu|(A_n) \tag{2.37}$$

On the other hand, if $\mathfrak{S} \ni B_1, .., B_k \subseteq A$ are disjoint, then

$$\sum_{i=1}^{k} |\mu(B_i)| = \sum_{i=1}^{k} \left| \mu \left(\biguplus_{n=1}^{\infty} B_i \cap A_n \right) \right| = \sum_{i=1}^{k} \left| \sum_{n=1}^{\infty} \mu(B_i \cap A_n) \right|$$

$$\leq \sum_{n=1}^{\infty} \sum_{i=1}^{k} |\mu(B_i \cap A_n)| \leq \sum_{n=1}^{\infty} |\mu|(A_n) \quad , \tag{2.38}$$

that is

$$|\mu|(A) \leq \sum_{n=1}^{\infty} |\mu|(A_n) \quad , \tag{2.39}$$

which together with (2.37) was to be shown.

$3 \Leftrightarrow 4$: Trivial.

$4 \Leftrightarrow 5$: Trivial.

$\square$

2.1.8 Lemma: Bounded contents as a Banach space

Let $\mathfrak{S}$ be a set algebra over some set Ω. Then $\mathfrak{M}_\alpha(\mathfrak{S})$ and $\mathfrak{M}_\sigma(\mathfrak{S})$, equipped with the total variation norm $\|\mu\|_t$, are real Banach spaces.

Proof

Linearity is evident, so left to be shown is completeness.

Completeness of $\mathfrak{M}_\alpha$: Suppose $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathfrak{M}_\alpha(\mathfrak{S})$ is Cauchy, then by 2.1.6 it is also Cauchy in the $\|\cdot\|_\infty$ norm. Thus, the μ_n converge uniformly (over $\mathfrak{S}$) to some bounded set function μ, given by

$$\mu(A) := \lim_{n \to \infty} \mu_n(A) \quad , \quad A \in \mathfrak{S} \quad . \tag{2.40}$$

Its additivity follows readily from the additivity of the μ_n. By 2.1.6, μ_n also converge to μ in the total variation norm.

Completeness of $\mathfrak{M}_\sigma$**:** Since $\mathfrak{M}_\sigma(\mathfrak{S}) \subseteq \mathfrak{M}_\alpha(\mathfrak{S})$, it suffices to show that $\mathfrak{M}_\sigma(\mathfrak{S})$ is closed. Let $\mu_n \in \mathfrak{M}_\sigma(\mathfrak{S})$ be such that $\mu_n \xrightarrow[\|\cdot\|_t]{n\to\infty} \mu \in \mathfrak{M}_\alpha(\mathfrak{S})$, or by 2.1.6 equivalently, $\mu_n \xrightarrow[\|\cdot\|_\infty]{n\to\infty} \mu$. By 2.1.7 it suffices to show that, for $(A_n)_{n\in\mathbb{N}} \subseteq \mathfrak{S}$ such that $A_n \downarrow \emptyset$: $\mu(A_n) \xrightarrow{n\to\infty} 0$. But indeed, for $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $\|\mu - \mu_{n_\varepsilon}\|_\infty \leq \varepsilon$ and an $m_\varepsilon \in \mathbb{N}$ such that $|\mu_{n_\varepsilon}(A_m)| \leq \varepsilon \ \forall\, m \geq m_\varepsilon$. Thus:

$$\forall\, m \geq m_\varepsilon \ : \ |\mu(A_m)| \leq \underbrace{|\mu(A_m) - \mu_{n_\varepsilon}(A_m)|}_{\leq \varepsilon} + \underbrace{|\mu_{n_\varepsilon}(A_m)|}_{\leq \varepsilon} \leq 2\varepsilon \ , \tag{2.41}$$

which was to be shown.

$\square$

2.1.9 Theorem: Extensions of σ-additive, bounded set-functions

Let $\mathfrak{S}$ be a set algebra over some set Ω. Then:

1. The mapping

$$\mathfrak{M}_\sigma(\sigma(\mathfrak{S})) \to \mathfrak{M}_\sigma(\mathfrak{S}) \ , \ \ \mu \mapsto \mu\big|_\mathfrak{S} \tag{2.42}$$

   is an isometrical isomorphism between Banach spaces, with respect to the total variation norm.

2. Positive and negative parts of a measure $\mu \in \mathfrak{M}_\sigma(\sigma(\mathfrak{S}))$, are mapped to the positive and negative parts of $\mu\big|_\mathfrak{S}$.

Proof

1. Linearity of the mapping is evident.

   Surjectivity: To begin with, let $0 \leq \mu \in \mathfrak{M}_\sigma(\mathfrak{S})$. Then by Carathéodory, μ can be extended to a non-negative, bounded measure $\widetilde{\mu}$ on $\sigma(\mathfrak{S})$. Now if $\mu \in \mathfrak{M}_\sigma(\mathfrak{S})$ is arbitrary, with positive & negative parts μ^+ & μ^-, let $\widetilde{\mu^+}, \widetilde{\mu^-}$ be non-negative, bounded measures extending them to $\sigma(\mathfrak{S})$. Then $\widetilde{\mu} := \widetilde{\mu^+} - \widetilde{\mu^-}$ extends μ to a bounded measure on $\sigma(\mathfrak{S})$.

   Injectivity: Let $\nu_1, \nu_2 \in \mathfrak{M}_\sigma(\sigma(\mathfrak{S}))$ be such that of $\nu_1\big|_\mathfrak{S} = \nu_2\big|_\mathfrak{S} =: \mu$. As is known, for $A \in \sigma(\mathfrak{S})$ and $\varepsilon > 0$ there exist $(B_n), (C_n) \subseteq \mathfrak{S}$ such that

$$\bigcup_{n=1}^{\infty} B_n =: B \supseteq A \ , \ \ \bigcup_{n=1}^{\infty} C_n =: C \supseteq A \tag{2.43}$$

   and

$$|\nu_1|\,(B \setminus A) \leq \varepsilon \ , \ \ |\nu_2|\,(C \setminus A) \leq \varepsilon \tag{2.44}$$

   (note that $|\nu_1|, |\nu_2|$ are non-negative, bounded measures). W.l.o.g. we assume[2] $B_n = C_n$ and[3] $B_n \uparrow B$. Then the σ-additivity of ν_1, ν_2 implies

$$|\nu_1(A) - \nu_2(A)| \leq \underbrace{|\nu_1(B \setminus A) - \nu_2(B \setminus A)|}_{\leq 2\varepsilon} + |\nu_1(B) - \nu_2(B)|$$

$$\leq 2\varepsilon + \limsup_{n\to\infty} \underbrace{|\nu_1(B_n) - \nu_2(B_n)|}_{0} = 2\varepsilon \ , \tag{2.45}$$

---

[2] Otherwise set $D_{n,m} := B_n \cap C_m \ \in \mathfrak{S}$, then $\bigcup_{n,m\in\mathbb{N}} D_{n,m} = B \cap C \ \supseteq A$ and $|\nu_i|\,(D \setminus A) \leq \varepsilon$.

[3] Otherwise set $\widetilde{B}_n := \bigcup_{k=1}^{n} B_k \ \in \mathfrak{S}$, so that $\widetilde{B}_k \uparrow B$.

whereas in the last step we used the fact that $\nu_1 = \nu_2$ on $\mathfrak{S}$. The arbitrarity of $\varepsilon > 0$ shows $\nu_1(A) = \nu_2(A)$.

Norm-preservation follows directly from part (2) and the fact that $|\mu| = \mu^+ + \mu^-$ for any bounded content μ.

2. It suffices to show, that the total variation $|\widetilde{\mu}|$ of $\widetilde{\mu} \in \mathfrak{M}_\sigma(\sigma(\mathfrak{S}))$ is mapped to the total variation of $\mu\big|_\mathfrak{S}$.

Let μ^+, μ^- be the positive & negative parts of $\mu \in \mathfrak{M}_\sigma(\mathfrak{S})$ and $\widetilde{\mu^+}, \widetilde{\mu^-}$ their extensions to $\mathfrak{M}_\sigma(\sigma(\mathfrak{S}))$. Then $\widetilde{\mu} := \widetilde{\mu^+} - \widetilde{\mu^-}$ is the extension of μ and $\widetilde{|\mu|} := \widetilde{\mu^+} + \widetilde{\mu^-}$ the extension of $|\mu|$. Let $\widetilde{\mu}^+, \widetilde{\mu}^-$ be the positive and negative parts of $\widetilde{\mu}$, then $|\widetilde{\mu}| = \widetilde{\mu}^+ + \widetilde{\mu}^-$ is its total variation. Since

$$\widetilde{\mu}^+ - \widetilde{\mu}^- = \widetilde{\mu} = \widetilde{\mu^+} - \widetilde{\mu^-} \tag{2.46}$$

and $\widetilde{\mu^+}, \widetilde{\mu^-}$ are non-negative, by 2.1.4(3) $\widetilde{\mu}^+ \leq \widetilde{\mu^+}$, $\widetilde{\mu}^- \leq \widetilde{\mu^-}$ and thus $|\widetilde{\mu}| \leq \widetilde{|\mu|}$. On the other hand, for $A \in \mathfrak{S}$:

$$\widetilde{|\mu|}(A) = |\mu|(A) = \sup_{\substack{A_i \subseteq A \\ A_i \in \mathfrak{S} \\ \text{disjoint}}} \sum_{i=1}^n |\mu(A_i)| \leq \sup_{\substack{A_i \subseteq A \\ A_i \in \sigma(\mathfrak{S}) \\ \text{disjoint}}} \sum_{i=1}^n |\widetilde{\mu}(A_i)| = |\widetilde{\mu}|(A) \tag{2.47}$$

that is, $\widetilde{|\mu|}\big|_\mathfrak{S} \leq |\widetilde{\mu}|\big|_\mathfrak{S}$. By Carathéodory this implies $\widetilde{|\mu|} \leq |\widetilde{\mu}|$.

$\square$

2.2 Charges

An important property, some contents on the Borel algebra $\mathscr{B}_\alpha(R)$ of an A-topological $(R, \mathscr{O})$ space share, is that of regularity[2]. Regular measures are distinguished by the fact that, their values can be approximated, up to arbitrary accuracy, by the closed sets of the space. Though a rather intuitively expected feature, since after all, $\mathscr{B}_\alpha(R)$ is generated by those closed sets, it is in no way shared by all contents.

An interesting fact is that, on perfectly normal spaces all measures are indeed regular (theorem 2.2.7). This generalizes what is known for metric spaces, and shows just how strong an assumption σ-additivity is for set functions.

2.2.1 Definition: Regular content & charge

Let $(R, \mathscr{F})$ be an A-topological space and $\mathfrak{S} \subseteq \mathscr{P}(R)$ an algebra containing $\mathscr{F}$. A content μ on $\mathfrak{S}$ is called *regular*, if

$$\forall\, A \in \mathfrak{S},\ \varepsilon > 0 : \exists\, F \in \mathscr{F}\ :\ F \subseteq A\ \wedge\ |\mu(A \setminus F)| \leq \varepsilon\ . \tag{2.48}$$

A bounded, regular content on $\mathfrak{S}$ is called a *charge*[2]. The triplet $(R, \mathfrak{S}, \mu)$ is called a *charge space*. We shall denote by $\mathfrak{M}_\alpha^r(\mathscr{O}, \mathfrak{S})$ or $\mathfrak{M}_\alpha^r(\mathfrak{S})$ the system of all charges and by $\mathfrak{M}_\sigma^r(\mathscr{O}, \mathfrak{S})$ or $\mathfrak{M}_\sigma^r(\mathfrak{S})$ the system of all σ-additive charges on $\mathfrak{S}$.

2.2.2 Examples of non-regular contents

Consider on $\mathscr{P}(\mathbb{N})$ the bounded content μ, defined in 2.1.3.

1. Consider the topology $\mathscr{O}$ on $\mathbb{N}$, comprising the sets

$$O_n := \{2, 4, 6, .., 2n\} \ , \ n \in \mathbb{N}, \ \ O_\infty := \{2, 4, 6, ..\} \ \ , \ \ \emptyset \ , \ \ \mathbb{N} \ . \tag{2.49}$$

Then μ is non regular on the Borel algebra $\alpha(\mathscr{O})$. Indeed, the only closed set contained in O_∞ is $\emptyset$, even though[4] $\mu(O_\infty) = \frac{1}{2}$.

2. Consider the topology $\mathscr{O}$ on $\mathbb{N}$, comprising the sets

$$O_n := \{1, 2, 3, .., n\} \ , \ n \in \mathbb{N}, \ \ , \ \ \emptyset \ , \ \ \mathbb{N} \ . \tag{2.50}$$

Then μ is regular on the Borel algebra $\alpha(\mathscr{O})$ but non regular on the Borel σ-algebra $\sigma(\mathscr{O})$. Indeed, for any set $A \in \alpha(\mathscr{O})$ there exists an $N \in \mathbb{N}$ large enough, so that either

- $\forall \, n \geq N : n \in A$
- or $\forall \, n \geq N : n \notin A$

In the first case, $\mu(A) = 1 = \mu(O_N^c)$ with $O_N^c \subseteq A$ closed. In the second case, $\mu(A) = 0 = \mu(\emptyset)$, with $\emptyset \subseteq A$ closed.

On the other hand, the only closed set contained in $B := \{2, 4, 6, ..\} \in \sigma(\mathscr{O})$ is $\emptyset$, even though $\mu(B) = \frac{1}{2}$. Thus, μ is non-regular on $\sigma(\mathscr{O})$.

2.2.3 Lemma: Linear combinations of non-negative, regular contents

Let $(R, \mathscr{F})$ be an A-topological space and $\mathfrak{S} \subseteq \mathscr{P}(R)$ an algebra containing $\mathscr{F}$. If $\mu_1, .., \mu_n$ are non-negative, regular contents on $\mathfrak{S}$, then so is any linear combination of them.

Proof: It suffices to show the statement for $n = 2$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ (w.l.o.g. $\alpha_i \neq 0$) and $A \in \mathfrak{S}$, $\varepsilon > 0$. Choose $F_1, F_2 \in \mathscr{F}$ such that $F_1, F_2 \subseteq A$ and $\mu_i(A \setminus F_i) \leq \varepsilon / \max\{|\alpha_1|, |\alpha_2|\}$. Then $F := F_1 \cup F_2 \in \mathscr{F}$ and $F \subseteq A$ with $\mu_i(A \setminus F) \leq \varepsilon / \max\{|\alpha_1|, |\alpha_2|\}$. Thus

$$\left| \alpha_1 \mu_1(A \setminus F) + \alpha_2 \mu_2(A \setminus F) \right| \leq \underbrace{|\alpha_1| \cdot \mu_1(A \setminus F)}_{\leq \varepsilon} + \underbrace{|\alpha_2| \cdot \mu_2(A \setminus F)}_{\leq \varepsilon} \leq 2\varepsilon \ , \tag{2.51}$$

which was to be shown.

$\square$

The above statement can in fact be generalized to arbitrary charges, a fact to be proven later in lemma 2.2.5, with the help of the following theorem 2.2.4.

2.2.4 Theorem: Characterization of regular contents

Let $(R, \mathscr{F})$ be an A-topological space, $\mathfrak{S} \subseteq \mathscr{P}(R)$ an algebra containing $\mathscr{F}$ and μ a bounded content on $\mathfrak{S}$. Then the following are equivalent:

1. μ is regular.

2. For $A \in \mathfrak{S}$ and $\varepsilon > 0$ there exists an open $G \supseteq A$ such that $|\mu(G \setminus A)| \leq \varepsilon$.

3. $|\mu|$ is regular.

---

[4]Note that the sequence $(0, 1, 0, 1, 0, ..)$ has Banach limit $\frac{1}{2}$.

4. μ^+, μ^- are regular.

5. μ^+, μ^- satisfy

$$\mu^+(A) = \sup_{\substack{F \subseteq A \\ F \in \mathscr{F}}} \mu(F) \quad, \quad \mu^-(A) = -\inf_{\substack{F \subseteq A \\ F \in \mathscr{F}}} \mu(F) \quad. \tag{2.52}$$

Proof

1⇒2: Let $A \in \mathfrak{S}$, $\varepsilon > 0$ and $\mathscr{F} \ni F \subseteq A^c$ such that $|\mu(A^c \setminus F)| \leq \varepsilon$. Then $A \subseteq F^c$ and

$$|\mu(F^c \setminus A)| = |\mu(A^c \setminus F)| \leq \varepsilon \quad. \tag{2.53}$$

2⇒1: Similar to (1⇒2).

1⇒3: Let $A \in \mathfrak{S}$, $\varepsilon > 0$. Let $\mathfrak{S} \ni A_1, .., A_n \subseteq A$ be disjoint such that

$$\sum_{i=1}^{n} |\mu(A_i)| \geq |\mu|\,(A) - \varepsilon \quad. \tag{2.54}$$

Choose closed $F_i \subseteq A_i$ such that $|\mu(A_i \setminus F_i)| \leq \varepsilon/n$ and set $F := \biguplus_{i=1}^{n} F_i$. Then $F \subseteq A$ is closed and

$$|\mu|\,(A) \overset{(2.54)}{\leq} \varepsilon + \sum_{i=1}^{n} |\mu(A_i \setminus F_i) + \mu(F_i)|$$

$$\leq \varepsilon + \sum_{i=1}^{n} \underbrace{|\mu(A_i \setminus F_i)|}_{\leq \varepsilon/n} + \underbrace{\sum_{i=1}^{n} |\mu(F_i)|}_{\leq |\mu|(F)} \leq |\mu|\,(F) + 2\varepsilon \quad, \tag{2.55}$$

which was to be shown (note that $|\mu|\,(F) \leq |\mu|\,(A)$).

3⇒4: Direct consequence of $0 \leq \mu^+, \mu^- \leq |\mu|$.

4⇒1: Direct consequence of 2.2.3, since $\mu = \mu^+ - \mu^-$.

1⇒5: Let $A \in \mathfrak{S}$ and $\varepsilon > 0$. Choose $E \in \mathfrak{S}$ such that $E \subseteq A$ and $\mu^+(A) - \mu(E) \leq \varepsilon$ (cmp. 2.1.4(4)). Choose closed $F \subseteq E$ such that $|\mu(E \setminus F)| \leq \varepsilon$, then $F \subseteq A$ and

$$\mu^+(A) - \mu(F) = \underbrace{\mu^+(A) - \mu(E)}_{\leq \varepsilon} + \underbrace{\mu(E \setminus F)}_{\leq \varepsilon} \leq 2\varepsilon \quad. \tag{2.56}$$

Together with 2.1.4(4) this shows that μ^+ satisfies (2.52). Since $\mu^- = (-\mu)^+$ and $(-\mu)$ is also regular, μ^- satisfies (2.52) as well.

5⇒4: Let $A \in \mathfrak{S}$ and $F_n \in \mathscr{F}$, $n \in \mathbb{N}$ such that $F_n \subseteq A$ and $\mu(F_n) \uparrow \mu^+(A)$. Then, since

$$\mu(F_n) \leq \mu^+(F_n) \overset{F_n \subseteq A}{\leq} \mu^+(A) \quad, \tag{2.57}$$

it follows $\mu^+(F_n) \uparrow \mu^+(A)$. Since $\mu^- = (-\mu)^+$, the regularity of μ^- follows easily from (2.52).

□

2.2.5   Lemma: Charges as a Banach space

Let $(R, \mathscr{F})$ be an A-topological space and $\mathfrak{S} \subseteq \mathscr{P}(R)$ an algebra containing $\mathscr{F}$. Then the systems $\mathfrak{M}_\alpha^r(\mathfrak{S})$ and $\mathfrak{M}_\sigma^r(\mathfrak{S})$, equipped with the total variation norm, are Banach spaces.

Proof

Since $\mathfrak{M}_\sigma^r(\mathfrak{S}) = (\mathfrak{M}_\sigma(\mathfrak{S}) \cap \mathfrak{M}_\alpha^r(\mathfrak{S})) \subseteq \mathfrak{M}_\alpha(\mathfrak{S})$, by 2.1.8 it suffices to show that $\mathfrak{M}_\alpha^r(\mathfrak{S})$ is a Banach space.

Linearity: Let μ_i^+, μ_i^- be the positive & negative parts of $\mu_i \in \mathfrak{M}_\alpha^r$, $i = 1, 2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then by 2.2.4, these are non-negative charges and by 2.2.3 so is

$$\alpha_1 \mu_1 + \alpha_2 \mu_2 = \alpha_1 \mu_1^+ + \alpha_2 \mu_2^+ - \alpha_1 \mu_1^- - \alpha_2 \mu_2^- \quad . \tag{2.58}$$

Completeness: Since $\mathfrak{M}_\alpha^r(\mathfrak{S}) \subseteq \mathfrak{M}_\alpha(\mathfrak{S})$ and $\mathfrak{M}_\alpha(\mathfrak{S})$ is complete by 2.1.8, it suffices to show that $\mathfrak{M}_\alpha^r(\mathfrak{S})$ is closed in $\mathfrak{M}_\alpha(\mathfrak{S})$. Suppose $\mu_n \in \mathfrak{M}_\alpha^r(\mathfrak{S})$ are such that $\mu_n \xrightarrow[\|\cdot\|_t]{n \to \infty} \mu \in \mathfrak{M}_\alpha(\mathfrak{S})$. For $A \in \mathfrak{S}$ there exist closed $F_n \subseteq A$ such that $|\mu_n(F_n) - \mu_n(A)| \xrightarrow{n \to \infty} 0$. Thus

$$|\mu(F_n) - \mu(A)| \leq \underbrace{|\mu(F_n) - \mu_n(F_n)|}_{\xrightarrow{n \to \infty} 0} + \underbrace{|\mu_n(F_n) - \mu_n(A)|}_{\xrightarrow{n \to \infty} 0} + \underbrace{|\mu_n(A) - \mu(A)|}_{\xrightarrow{n \to \infty} 0} \xrightarrow{n \to \infty} 0 \tag{2.59}$$

and μ is also regular.
$\square$

2.2.6   Lemma: Uniqueness of charges

Let $(R, \mathscr{F})$ be an A-topological space, $\mathfrak{S} \subseteq \mathscr{P}(R)$ an algebra containing $\mathscr{F}$ and μ, ν charges on $\mathfrak{S}$ such that $\mu(F) \geq \nu(F) \ \forall \ F \in \mathscr{F}$. Then $\mu \geq \nu$.

Proof

Let $\mu^+, \mu^-, \nu^+, \nu^-$ be the positive and negative parts of μ and ν respectively, then by 2.2.4 one has

$$\mu^+(A) = \sup_{\mathscr{F} \ni F \subseteq A} \mu(F) \quad , \qquad \qquad \mu^-(A) = - \inf_{\mathscr{F} \ni F \subseteq A} \mu(F) \quad ,$$

$$\nu^+(A) = \sup_{\mathscr{F} \ni F \subseteq A} \nu(F) \quad , \qquad \qquad \nu^-(A) = - \inf_{\mathscr{F} \ni F \subseteq A} \nu(F) \quad , \tag{2.60}$$

for $A \in \mathfrak{S}$. But this implies $\mu^+ \geq \nu^+$ and $\mu^- \leq \nu^-$, which was to be shown.
$\square$

Conclusion: If μ, ν are equal on $\mathscr{F}$, then μ, ν are equal on $\mathfrak{S}$.

2.2.7   Theorem: Regularity of measures

Let $(R, \mathscr{O})$ be a perfectly normal, A-topological space and $\mu \in \mathfrak{M}_\sigma(\mathscr{B}_\sigma(R))$. Then μ is regular.

Proof

This proof is merely a generalization of a well known proof for metric spaces. By 2.1.7 μ^+, μ^- are also σ-additive. By 2.2.4, their regularity would imply the regularity of μ. We may thus

assume $\mu \geq 0$. Define

$$\mathcal{A} := \{A \in \mathscr{B}_\sigma(R) : \mu(A) = \sup\{\mu(F) : F \subseteq A, \ F \text{ closed}\}\}$$

$$\mathcal{O} := \{A \in \mathscr{B}_\sigma(R) : \mu(A) = \inf\{\mu(G) : A \subseteq G, \ G \text{ open}\}\} \tag{2.61}$$

and $\mathfrak{S} := \mathcal{O} \cap \mathcal{A}$. We shall show that $\mathfrak{S} = \mathscr{B}_\sigma(R)$.

Claim: For $A \in \mathcal{A}$ one has $A^c \in \mathcal{O}$ and for $A \in \mathcal{O}$ one has $A^c \in \mathcal{A}$.
Proof: Indeed, from $A \in \mathcal{A}$ follows

$$\mu(A^c) = \mu(R) - \mu(A) = \mu(R) - \underbrace{\sup\{\mu(F) : F \subseteq A, \ F \text{ closed}\}}_{-\inf\{-\mu(F) \ : \ F \subseteq A \text{ closed}\}}$$

$$= \inf\Big\{ \underbrace{\mu(R) - \mu(F)}_{\mu(F^c)} : \underbrace{F \subseteq A}_{\Leftrightarrow \ A^c \subseteq F^c}, \ \underbrace{F \text{ closed}}_{\Leftrightarrow \ F^c \text{ open}} \Big\}$$

$$= \inf\{\mu(F^c) : A^c \subseteq F^c, \ F^c \text{ open}\} \quad . \tag{2.62}$$

Similarly, from $A \in \mathcal{O}$ follows $A^c \in \mathcal{A}$. In particular, this implies that $\mathfrak{S}$ is closed under complementations.

Claim: $\mathfrak{S}$ contains all open sets.
Proof: Let $O \in \mathscr{O}$ be open, then obviously $O \in \mathcal{O}$. By the previous claim, it suffices to show that $O^c \in \mathcal{O}$. By 1.3.3, O^c is the intersection of a decreasing sequence of open sets $G_n \in \mathscr{O}$: $G_n \downarrow O^c$. By 2.1.7 this implies

$$\mu(O^c) = \lim_{n \to \infty} \mu(G_n) = \inf_{n \in \mathbb{N}} \mu(G_n) \geq \inf\{\mu(G) : O^c \subseteq G, \ G \text{ open}\} \geq \mu(O^c) \quad , \tag{2.63}$$

which was to be shown.

Claim: $\mathfrak{S}$ is a σ-algebra.
Proof: Since $\emptyset \in \mathfrak{S}$ and the first claim, it remains to be shown that $\mathfrak{S}$ is closed under countable unions. Let $A_1, A_2, .. \in \mathfrak{S}$ and $A := \bigcup_{n=1}^\infty A_n$. To be shown is that, for $\varepsilon > 0$ there exist closed $F_\varepsilon \subseteq A$ and open $G_\varepsilon \supseteq A$ such that

$$\mu(A) - \mu(F_\varepsilon) \leq \varepsilon \quad \wedge \quad \mu(G_\varepsilon) - \mu(A) \leq \varepsilon \quad . \tag{2.64}$$

Indeed: Since $A_n \in \mathcal{A} \cap \mathcal{O}$, there exist closed $F_n \subseteq A_n$ and open $G_n \supseteq A_n$ satisfying

$$\mu(A_n) - \mu(F_n) \leq \frac{1}{2} \cdot \frac{\varepsilon}{2^n} \quad \wedge \quad \mu(G_n) - \mu(A) \leq \frac{\varepsilon}{2^n} \quad . \tag{2.65}$$

Then the open set $G_\varepsilon := \bigcup_{n \in \mathbb{N}} G_n \supseteq A$ for one, satisfies

$$\mu(G_\varepsilon) - \mu(A) = \mu\left(\bigcup_{n \in \mathbb{N}} G_n \setminus A\right) \leq \mu\left(\bigcup_{n \in \mathbb{N}} G_n \setminus A_n\right) \leq \sum_{n=1}^\infty \underbrace{\mu\left(G_n \setminus A_n\right)}_{\leq \frac{\varepsilon}{2^n}} \leq \varepsilon \quad . \tag{2.66}$$

On the other hand, since $\left(\bigcup_{k=1}^n A_k\right) \underset{n}{\uparrow} A$, by 2.1.7 there exists an $n_\varepsilon \in \mathbb{N}$, such that

$$\mu(A) - \mu\left(\bigcup_{k=1}^{n_\varepsilon} A_k\right) \leq \frac{\varepsilon}{2} \quad . \tag{2.67}$$

Then the closed set

$$F_\varepsilon := \underbrace{\bigcup_{k=1}^{n_\varepsilon} F_n}_{\text{closed}} \subseteq \bigcup_{k=1}^{n_\varepsilon} A_k \subseteq A \ , \tag{2.68}$$

satisfies

$$\mu(A) - \mu(F_\varepsilon) \overset{(2.67)}{\leq} \frac{\varepsilon}{2} + \mu\left(\bigcup_{k=1}^{n_\varepsilon} A_k\right) - \mu(F_\varepsilon)$$

$$\overset{(2.68)}{=} \frac{\varepsilon}{2} + \mu\left(\bigcup_{k=1}^{n_\varepsilon} (A_k \setminus F_\varepsilon)\right) \leq \frac{\varepsilon}{2} + \mu\left(\bigcup_{k=1}^{n_\varepsilon} (A_k \setminus F_k)\right)$$

$$\leq \frac{\varepsilon}{2} + \underbrace{\sum_{k=1}^{n_\varepsilon} \underbrace{\mu(A_k \setminus F_k)}_{\leq \frac{1}{2}\frac{\varepsilon}{2^k}}}_{\leq \frac{\varepsilon}{2}} \leq \varepsilon \ . \tag{2.69}$$

as needed.

Finally, since $\mathfrak{S} \subseteq \mathscr{B}_\sigma(R)$ is a σ-algebra containing all open sets $\mathscr{O}$, the proof is complete.
□

2.2.8 Corollary: Regularity of σ-additive contents

Let $(R, \mathscr{O})$ be a perfectly normal, A-topological space. Then every bounded, σ-additive set function on $\mathscr{B}_\alpha(R)$ is regular.

Proof

By theorem 2.1.9, μ can be extended to a measure on $\mathscr{B}_\sigma(R)$. By theorem 2.2.7, μ is regular.
□

An important consequence of theorem 2.2.7 and corollary 2.2.8, is the fact that all measures on the Baire σ-algebra of an arbitrary A-topological space, are regular! This will prove to be very useful when it comes to showing the uniqueness of measures in the Riesz and Hewitt representation theorems 4.2.9 and 4.3.4 later after.

2.2.9 Lemma: Characterization of σ-additivity of charges

Let $(R, \mathscr{F})$ be an A-topological space, $\mathfrak{S} \subseteq \mathscr{P}(R)$ a set algebra containing $\mathscr{F}$ and μ a charge on $\mathfrak{S}$. Then the following are equivalent:

1. μ is σ-additive.

2. For any vanishing sequence $F_n \downarrow \emptyset$ of closed sets: $\mu(F_n) \overset{n\to\infty}{\longrightarrow} 0$.

3. If R is perfectly normal: For any regular sequence $(F_n) \subseteq \mathscr{F}$: $|\mu|(F_n^c) \overset{n\to\infty}{\longrightarrow} 0$.

Proof

The proof is due to Varadarajan[13].

$1 \Rightarrow 3$: Since $|\mu|$ is also σ-additive and $F_n^c \downarrow \emptyset$, this statement is trivial.

$3 \Rightarrow 2$: Let $F_n \in \mathscr{F}$ be such that $F_n \downarrow \emptyset$, then by A.2.1 there exists a regular sequence $\widetilde{F}_n \subseteq F_n^c$, which implies by assumption

$$|\mu|\,(F_n) \leq |\mu|\,(\widetilde{F}_n^c) \overset{n \to \infty}{\longrightarrow} 0 \quad . \tag{2.70}$$

$2 \Rightarrow 1$: It suffices to show that μ is continuous at $\emptyset$. Let μ^+, μ^- be the positive and negative parts of μ and let $(A_n)_{n \in \mathbb{N}} \in \mathfrak{S}$ be such that $A_n \downarrow \emptyset$ and $\varepsilon > 0$. Choose closed $F_n^+, F_n^- \subseteq A_n$ such that

$$\mu^+(A_n \setminus F_n^+) \leq \frac{\varepsilon}{2^{n+1}} \quad , \quad \mu^-(A_n \setminus F_n^-) \leq \frac{\varepsilon}{2^{n+1}} \tag{2.71}$$

(note the regularity of $\mu^\pm$) and set

$$F_n := \bigcap_{k=1}^{n} \left(F_k^+ \cup F_k^- \right) \quad . \tag{2.72}$$

Then $F_{n+1} \subseteq F_n \subseteq A_n$ and $F_n \downarrow \emptyset$, thus by assumption

$$\mu(F_n) \overset{n \to \infty}{\longrightarrow} 0 \quad . \tag{2.73}$$

On the other hand, (2.71) and (2.72) implies

$$\mu^+ \left(A_n \setminus F_n \right) \leq \frac{\varepsilon}{2} \quad , \quad \mu^- \left(A_n \setminus F_n \right) \leq \frac{\varepsilon}{2} \quad , \tag{2.74}$$

hence

$$|\mu(A_n) - \mu(F_n)| \leq \varepsilon \quad \forall\, n \in \mathbb{N} \quad . \tag{2.75}$$

But together with (2.73) this implies

$$\limsup_{n \to \infty}\ |\mu(A_n)| \leq \varepsilon \quad . \tag{2.76}$$

Finally, the arbitrarity of $\varepsilon > 0$ implies $\lim\limits_{n \to \infty} \mu(A_n) = 0$, which was to be shown.

$\square$

2.2.10 Theorem: Extensions of σ-additive charges

Let $(R, \mathscr{O})$ be an A-topological space, μ a σ-additive charge on $\mathscr{B}_\alpha(R)$. Then its (unique) extension to a bounded measure[5] on $\mathscr{B}_\sigma(R)$ is also a charge.

---

[5]See theorem 2.1.9.

Proof

Let $\widetilde{\mu}$ be a σ-additive, bounded extension of μ on $\mathscr{B}_\sigma(R)$. As is known, for $A \in \mathscr{B}_\sigma(R)$ and $\varepsilon > 0$ there exist $(B_n) \subseteq \mathscr{B}_\alpha(R)$ such that

$$\bigcap_{n=1}^{\infty} B_n =: B \subseteq A \quad , \quad |\widetilde{\mu}|\,(A \setminus B) \leq \varepsilon \quad . \tag{2.77}$$

Choose closed $F_n \subseteq B_n$ such that $|\mu|\,(B_n \setminus F_n) \leq \varepsilon/2^n$ and set $F := \bigcap_{n=1}^{\infty} F_n$. Then $F \subseteq A$ is closed and satisfies:

$$|\widetilde{\mu}|\,(A \setminus F) \leq \underbrace{|\widetilde{\mu}|(A \setminus B)}_{\leq \varepsilon} + |\widetilde{\mu}|(B \setminus F) \leq \varepsilon + |\widetilde{\mu}|\left[\bigcup_{n=1}^{\infty} B \setminus F_n\right]$$

$$\leq \varepsilon + \sum_{n=1}^{\infty} |\widetilde{\mu}|(B \setminus F_n) \leq \varepsilon + \sum_{n=1}^{\infty} |\widetilde{\mu}|\,(B_n \setminus F_n) \leq 2\varepsilon \quad , \tag{2.78}$$

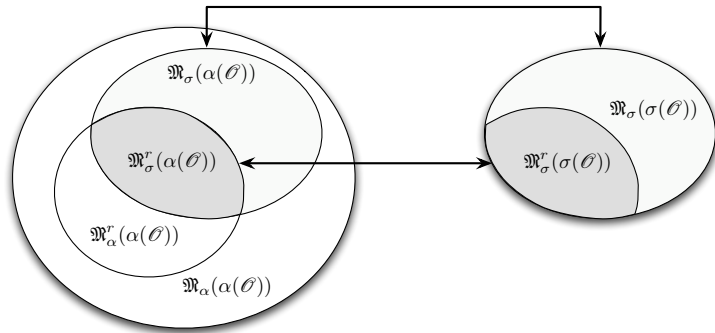which shows the regularity of $\widetilde{\mu}$.
$\square$

Figure 2.1: On extensions of contents and charges: Implications of theorems 2.1.9 and 2.2.10 for charges on an A-topological space $(R, \mathscr{O})$. Double-sided arrows denote isometrical isomorphisms between Banach spaces. Note that by 2.2.8, if R is normal then $\mathfrak{M}_\sigma(\alpha(\mathscr{O})) = \mathfrak{M}_\sigma^r(\alpha(\mathscr{O}))$.

As pointed out in section 1.3, continuous functions do not always completely describe the A-topology of an A-topological space $(R, \mathscr{O})$. In fact, the largest A-topology described by them, is the one of the totally open sets, which instead of the Borel algebra $\alpha(\mathscr{O})$ merely generate the, generally smaller, Baire algebra $\alpha(\mathscr{O}^t)$ on the space.

It will later on be important to distinguish between the two A-topologies and set algebras. In particular, certain charges may exist for the smaller but not for the larger one. On the other hand, any content on $\alpha(\mathscr{O})$ can be restricted to $\alpha(\mathscr{O}^t)$, clearly yielding again, a content. One may ask the question, wether the restriction of a charge in $\mathfrak{M}_\alpha^r(\mathscr{O}, \alpha(\mathscr{O}))$ to $\alpha(\mathscr{O}^t)$ is still regular with respect to the A-topology of totally open sets $\mathscr{O}^t$. Could it be the case that, the totally closed sets of R are so *sparse*, that they can no longer fully approximate the rest of $\alpha(\mathscr{O}^t)$? The following statements aim at providing with a partial answer to this question. A more detailed treatment, will be given in chapter 4 and in particular, theorem 4.2.7.

2.2.11 Lemma: Approximation by totally closed sets

Let $(R, \mathscr{O})$ be an normal, A-topological space and μ a charge on $\alpha(\mathscr{O})$. Let $F \subseteq R$ be closed and $G \subseteq R$ be open, such that $F \subseteq G$. Then there exists a totally closed $\widetilde{F} \subseteq R$ such that

$$F \subseteq \widetilde{F} \subseteq G \quad \wedge \quad |\mu|\,(\widetilde{F} \setminus F) = 0 \quad . \tag{2.79}$$

Proof

By 2.2.4 there exist open sets $G \supseteq G_n \supseteq F$, $n \in \mathbb{N}$ such that $|\mu|\,(G_n \setminus F) \leq \frac{1}{n}$. Choose functions $f_n \in \mathcal{C}_b(R)$ such that $0 \leq f \leq 1$ and $f\big|_F = 0$, $f\big|_{G_n^c} = 1$ and define the totally closed sets $\widetilde{F}_n := \{f_n = 0\}$. Then $F \subseteq \widetilde{F}_n \subseteq G_n$ and thus $|\mu|\,(\widetilde{F}_n \setminus F) \leq \frac{1}{n}$. Set $\widetilde{F} := \bigcap_{n=1}^{\infty} \widetilde{F}_n$, then by 1.3.1, $\widetilde{F}$ is also totally closed. Furthermore, it satisfies

$$F \subseteq \widetilde{F} \subseteq G_n \subseteq G \quad \forall\, n \in \mathbb{N} \tag{2.80}$$

and in particular $|\mu|\,(\widetilde{F} \setminus F) \leq \frac{1}{n}$ $\forall\, n$, which completes the proof.
$\square$

Lemma 2.2.11 in a way poses a contradiction to the idea that continuous functions (i.e. totally closed sets) do not necessarily fully describe the underlying set topology. The fact that totally closed sets are actually so dense to be fitting *in between* closed and open sets, provides with a key to establishing a connection between charges on Baire and Borel algebras, as given in the following theorem 2.2.12.

2.2.12 Theorem: Regularity of restricted charges

Let $(R, \mathscr{O})$ be an normal, A-topological space and μ a charge on the Borel algebra $\alpha(\mathscr{O})$. Then:

1. The restriction μ_t of μ to the Baire algebra $\alpha(\mathscr{O}^t)$, is a charge with respect to the A-topology of totally open sets $\mathscr{O}^t$.

2. If ν is another charge on $\alpha(\mathscr{O})$, such that its restriction ν_t is equal to μ_t, then $\mu = \nu$.

Proof

1. By 2.2.11, for any totally open set $G \subseteq R$ and $\varepsilon > 0$ there exists a totally closed $F \subseteq G$ such that $|\mu|\,(G \setminus F) \leq \varepsilon$. Clearly, also $|\mu_t|\,(G \setminus F) \leq \varepsilon$. By A.4.7, this is a sufficient condition for $|\mu_t|$ (and thus μ_t) to be regular with respect to the A-topology of totally open sets.

2. Let $F \subseteq R$ be closed, then by 2.2.11 there exist totally closed $\widetilde{F}_1, \widetilde{F}_2 \subseteq R$ such that $F \subseteq \widetilde{F}_i$ and $|\mu|\,(\widetilde{F}_1 \setminus F) = 0$, $|\nu|\,(\widetilde{F}_2 \setminus F) = 0$. Set $\widetilde{F} := \widetilde{F}_1 \cap \widetilde{F}_2$, then $\widetilde{F}$ is totally closed and satisfies

$$\mu(F) = \mu(\widetilde{F}) = \nu(\widetilde{F}) = \nu(F) \quad , \tag{2.81}$$

hence, μ and ν are equal on all closed sets. By 2.2.6, this completes the proof.

$\square$

Chapter 3

Integration

3.1 Integration on bounded content spaces

In standard literature, the integral is usually introduced solely for σ-additive set functions, as many facts such as the Lebesgue dominated theorem and Fatou's lemma do not hold for merely additive set functions. Nonetheless, the rather strong requirement of σ-additivity is not needed for an integration theory its self, especially if one restricts the classes of functions to be integrated. In fact, some theorems can only be stated if one also takes into consideration non-σ-additive set functions.

We shall in the following section, introduce an integral for bounded contents. The integrable functions will need to be uniformly approximable with respect to the underlying set algebra. As is shown in chapter 4, bounded contents are anyhow identified with linear functionals on spaces of uniformly approximable functions, so this restriction will not be an issue for us.

Many properties of this integral are already known from Lebesgue integration theory. In case of σ-additive set functions, the integral presented here, coincides with the Lebesgue integral. A thorough elaboration on the latter, is given by Bogachev[4].

3.1.1 Definition: Integral of simple functions

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space and E a $\mathbb{K}$-Banach space. For a simple function

$$f = \sum_{i=1}^{n} f_i \cdot 1_{A_i} \quad , \quad A_i \in \mathfrak{S}, \; f_i \in E, \; i = 1, .., n \tag{3.1}$$

we define the *integral*

$$\int f \; d\mu := \sum_{i=1}^{n} f_i \cdot \mu(A_i) \quad . \tag{3.2}$$

Note:

(i) The integral $\int f \; d\mu$ does indeed not depend on the representation (3.1) of f.

(ii) The integral is linear on simple functions and satisfies

$$\left\| \int f \; d\mu \right\| \leq \int \underbrace{\|f\|}_{\in \mathscr{S}(\mathfrak{S})} \; d\left|\mu\right| \leq \|f\|_{\infty} \cdot \left|\mu\right|(\Omega) \quad \forall \, f \in \mathscr{S}(\mathfrak{S}, E) \quad . \tag{3.3}$$

3.1.2 Lemma: Existence of integral limits

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space, E a $\mathbb{K}$-Banach space, $f \in \mathscr{U}(\mathfrak{S}, E)$ uniformly approximable and $f_n \in \mathscr{S}(\mathfrak{S}, E)$ simple functions, such that $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$. Then the limit

$$\lim_{n \to \infty} \int f_n \, d\mu \tag{3.4}$$

exists and only depends on f.

Proof

By definition 3.1.1 of the integral, it follows easily

$$\left\| \int f_n \, d\mu - \int f_m \, d\mu \right\| = \left\| \int (f_n - f_m) \, d\mu \right\| \leq \int \|f_n - f_m\| \, d\,|\mu|$$

$$\leq \|f_n - f_m\|_\infty \cdot |\mu|\,(\Omega) \overset{n,m \to \infty}{\longrightarrow} 0 \quad, \tag{3.5}$$

that is, $\int f_n \, d\mu$ is Cauchy and thus convergent. Now suppose $g_n \in \mathscr{S}(\mathfrak{S}, E)$ are also such that $g_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$, then similar to above:

$$\limsup_{n \to \infty} \left\| \int f_n \, d\mu - \int g_n \, d\mu \right\| \leq \lim_{n \to \infty} \|f_n - g_n\|_\infty \cdot |\mu|\,(\Omega) = 0 \quad, \tag{3.6}$$

hence, the sequence $\int g_n \, d\mu$ converges to the same limit as $\int f_n \, d\mu$.
$\square$

3.1.3 Definition: Integral of uniformly approximable functions

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space and E a $\mathbb{K}$-Banach space. For a uniformly approximable function $f \in \mathscr{U}(\mathfrak{S}, E)$, with $f_n \in \mathscr{S}(\mathfrak{S}, E)$ such that $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$, we define in view of lemma 3.1.2, the *integral*

$$\int f \, d\mu := \lim_{n \to \infty} \int f_n \, d\mu \quad. \tag{3.7}$$

For any set $A \in \mathfrak{S}$, we define[1]

$$\int_A f \, d\mu := \int 1_A \cdot f \, d\mu \quad. \tag{3.8}$$

See appendix A.5 for more on integrals over subsets.

3.1.4 Theorem: Properties of the integral

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space and E a $\mathbb{K}$-Banach space. Then:

1. The mapping $\int (\cdot) \, d\mu : \mathscr{U}(\mathfrak{S}, E) \to E$, mapping $f \in \mathscr{U}(\mathfrak{S}, E)$ to $\int f \, d\mu$, is a linear one.

---

[1]Note that by 1.2.6, also $f \cdot 1_A \in \mathscr{U}(\mathfrak{S}, E)$.

2. If μ is non-negative, then the mapping $\int(\cdot)\, d\mu : \mathscr{U}(\mathfrak{S}) \to \mathbb{R}$ is non-negative. In particular, if $f \leq g \in \mathscr{U}(\mathfrak{S})$ then

$$\int f\, d\mu \leq \int g\, d\mu \quad . \tag{3.9}$$

3. For $f \in \mathscr{U}(\mathfrak{S}, E)$ the inequality

$$\left\| \int f\, d\mu \right\| \leq \int \|f\|\ d\,|\mu| \leq \|f\|_\infty \cdot |\mu|\,(\Omega) \tag{3.10}$$

holds[2]. In particular, the linear map $\int(\cdot)\, d\mu : \mathscr{U}(\mathfrak{S}, E) \to E$ is bounded with norm

$$\left\| \int (\cdot)\, d\mu \right\| = \|\mu\|_t \quad . \tag{3.11}$$

4. For bounded contents μ, ν on $\mathfrak{S}$, $\alpha, \beta \in \mathbb{R}$ and $f \in \mathscr{U}(\mathfrak{S}, E)$, the identity

$$\int f\, d(\alpha\mu + \beta\nu) = \alpha \int f\, d\mu + \beta \int f\, d\nu \tag{3.12}$$

holds. In particular, the mapping

$$\int : \mathfrak{M}_\alpha(\mathfrak{S}) \to \mathcal{L}\left(\mathscr{U}(\mathfrak{S}, E), E\right) \quad , \quad \int : \mu \mapsto \int(\cdot)\, d\mu \tag{3.13}$$

is a linear isometry between Banach spaces, whereas $\mathfrak{M}_\alpha(\mathfrak{S})$ is equipped with the total variation norm $\|\cdot\|_t$, $\mathcal{L}\left(\mathscr{U}(\mathfrak{S}, E), E\right)$ with the operator-norm.

5. For disjoint $A, B \in \mathfrak{S}$ and $f \in \mathscr{U}(\mathfrak{S}, E)$, it follows

$$\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu \quad . \tag{3.14}$$

In particular if $f \in \mathscr{U}(\mathfrak{S})$, the mapping

$$\mathfrak{S} \to \mathbb{R} \quad , \quad A \mapsto \int_A f\, d\mu \tag{3.15}$$

is a bounded content on $\mathfrak{S}$.

Proof

1. By definition 3.1.1 it is clear that linearity holds for simple functions. Now let $f, g \in \mathscr{U}(\mathfrak{S}, E)$, $\alpha, \beta \in \mathbb{K}$ and $f_n, g_n \in \mathscr{S}(\mathfrak{S}, E)$ be given, such that $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$ and $g_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} g$, then

$$(\alpha \cdot f_n + \beta \cdot g_n) \xrightarrow[\|\cdot\|_\infty]{n \to \infty} (\alpha \cdot f + \beta \cdot g) \quad . \tag{3.16}$$

Consequently:

$$\int (\alpha f + \beta g)\, d\mu = \lim_{n \to \infty} \int (\alpha f_n + \beta g_n)\, d\mu$$

$$= \lim_{n \to \infty} \alpha \int f_n\, d\mu + \lim_{n \to \infty} \beta \int g_n\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu \quad . \tag{3.17}$$

---

[2]Note that $\|f\| \in \mathscr{U}(\mathfrak{S})$.

2. Non-negativity is evident for simple functions. Now let $0 \leq f \in \mathscr{U}(\mathfrak{S})$ and $f_n \in \mathscr{S}(\mathfrak{S})$ be such that $f_n \overset{n \to \infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}} f$. Set $h_n := \max\{0, f_n\}$, then h_n are simple such that $h_n \overset{n \to \infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}} f$, whereas $\int h_n \, d\mu \geq 0$. Hence, also $\lim\limits_{n \to \infty} \int h_n \, d\mu \geq 0$.

3. Inequality (3.10) is evident for simple functions. Now let $f \in \mathscr{U}(\mathfrak{S}, E)$ and $f_n \in \mathscr{S}(\mathfrak{S}, E)$ be such that $f_n \overset{n \to \infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}} f$, then $\|f_n\| \overset{n \to \infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}} \|f\|$. Hence:

$$\left\| \int f \, d\mu \right\| = \left\| \lim_{n \to \infty} \int f_n \, d\mu \right\| = \lim_{n \to \infty} \left\| \int f_n \, d\mu \right\|$$

$$\leq \lim_{n \to \infty} \int \|f_n\| \, d|\mu| = \int \|f\| \, d|\mu| \overset{(2.)}{\leq} \|f\|_\infty \cdot |\mu|(\Omega) \ . \tag{3.18}$$

In particular $\left\| \int (\cdot) \, d\mu \right\| \leq |\mu|(\Omega)$.

Now choose some $x_0 \in E$ such that $\|x_0\| = 1$. For $\varepsilon > 0$ let $A_1, .., A_n \in \mathfrak{S}$ be disjoint, such that

$$\sum_{i=1}^{n} |\mu(A_i)| \geq |\mu|(\Omega) - \varepsilon \tag{3.19}$$

and define

$$f := x_0 \cdot \sum_{i=1}^{n} 1_{A_i} \cdot \operatorname{sgn}(\mu(A_i)) \ . \tag{3.20}$$

Then $\|f\|_\infty \leq 1$ and

$$\left\| \int f \, d\mu \right\| = \underbrace{\|x_0\|}_{1} \cdot \left| \sum_{i=1}^{n} \underbrace{\mu(A_i) \cdot \operatorname{sgn}(\mu(A_i))}_{|\mu(A_i)|} \right| \overset{(3.19)}{\geq} |\mu|(\Omega) - \varepsilon \ , \tag{3.21}$$

which implies $\left\| \int (\cdot) \, d\mu \right\| \geq |\mu|(\Omega) - \varepsilon$. By arbitrarity of $\varepsilon > 0$, $\left\| \int (\cdot) \, d\mu \right\| \geq |\mu|(\Omega)$.

4. Identity (3.12) is easily verified for simple functions. Now let $f \in \mathscr{U}(\mathfrak{S}, E)$ and $f_n \in \mathscr{S}(\mathfrak{S}, E)$ be such that $f_n \overset{n \to \infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}} f$. Then

$$\int f \, d(\alpha\mu + \beta\nu) = \lim_{n \to \infty} \int f_n \, d(\alpha\mu + \beta\nu)$$

$$= \lim_{n \to \infty} \alpha \int f_n \, d\mu + \lim_{n \to \infty} \beta \int f_n \, d\nu = \alpha \int f \, d\mu + \beta \int f \, d\nu \ . \tag{3.22}$$

By 2.1.8, $\mathfrak{M}_\alpha(\mathfrak{S})$ is a real Banach space, and by part (3.), $\int : \mu \mapsto \int (\cdot) \, d\mu$ norm-preserving. Since E is a $\mathbb{K}$-Banach space, so is $\mathcal{L}(\mathscr{U}(\mathfrak{S}, E), E)$.

5. Relation (3.14) follows directly from linearity of the integral and the fact that $1_{A \cup B} = 1_A + 1_B$. That the content in (3.15) is bounded, follows from $\left| \int_A f \, d\mu \right| \leq \|f\|_\infty \cdot |\mu|(\Omega)$.

$\square$

3.1.5 Lemma: Connection of integral to Lebesgue integral

Let $(\Omega, \mathfrak{S}, \mu)$ be a non-negative, bounded content space and $f \in \mathscr{U}(\mathfrak{S})$. Then the identity

$$\int f \, d\mu = \sup_{\substack{g \in \mathscr{S}(\mathfrak{S}) \\ g \leq f}} \int g \, d\mu \tag{3.23}$$

holds.

Proof

For every simple $g \in \mathscr{S}(\mathfrak{S})$ such that $g \leq f$, the non-negativity of μ implies $\int g \, d\mu \leq \int f \, d\mu$. Thus

$$\sup_{\substack{g \in \mathfrak{S} \\ g \leq f}} \int g \, d\mu \leq \int f \, d\mu \quad . \tag{3.24}$$

By A.3.2, there exist simple $g_n \leq f$ such that $g_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$, which by definition of the integral implies equality in (3.24).
$\square$

Interpretation: The integral defined above for uniformly approximable functions with respect to bounded, additive set-functions, is merely a generalization of the Lebesgue integral with respect to bounded, σ-additive set-functions.
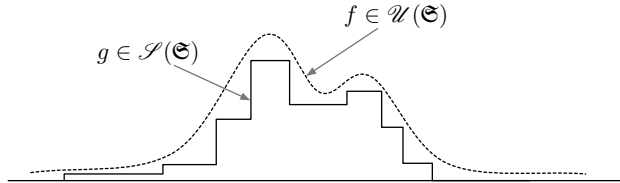

Figure 3.1: On the Lebesgue integral of non-negative functions: Approximation through simple functions from below.

While the total boundedness of the integrated functions may seem to be a strong restriction, it is in a way compensated by allowing certain non-measurable functions to be integrated as well. Actually, the class of uniformly approximable functions turns out to be large enough to both characterize bounded contents and charges (see chapter 4).

3.2 Integration on charge spaces

Due to their regularity, charges are profoundly connected to the A-topology of the underlying space. This leads to a strong connection between integrals of continuous functions and the charges themselves, as the following statements and especially chapter 4 will reveal.

3.2.1 Lemma: Representation of charges through integrals

Let $(R, \mathscr{F})$ be a normal, A-topological space, $\mathfrak{S} \subseteq \mathscr{P}(R)$ a set algebra containing $\mathscr{F}$ and μ a non-negative charge on $\mathfrak{S}$. Then for any closed $F \in \mathscr{F}$, the representation

$$\mu(F) = \inf_{\substack{f \in \mathcal{C}_b(R) \\ 1_F \leq f}} \int f \, d\mu \tag{3.25}$$
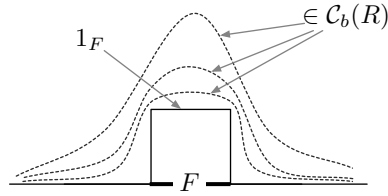
holds.

Figure 3.2: On the integral representation of charges on closed sets.

Proof

Obviously

$$\mu(F) = \int 1_F \, d\mu \leq \inf_{\substack{f \in \mathcal{C}_b(R) \\ 1_F \leq f}} \int f \, d\mu \tag{3.26}$$

holds for every closed set $F \in \mathscr{F}$. Now choose a sequence of closed sets $F_n \in \mathscr{F}$ such that $F_n \subseteq F^c$ and $\mu(F^c \setminus F_n) \overset{n \to \infty}{\longrightarrow} 0$. Choose functions $f_n \in \mathcal{C}_b(R)$ such that $0 \leq f_n \leq 1$ and $f_n\big|_F = 1$, $f_n\big|_{F_n} = 0$. Then $1_F \leq f_n$ and

$$\limsup_{n \to \infty} \int f_n \, d\mu - \mu(F) = \limsup_{n \to \infty} \int\limits_{F^c \setminus F_n} f_n \, d\mu$$

$$\leq \limsup_{n \to \infty} \underbrace{\|f_n\|_\infty}_{\leq 1} \cdot \mu(F^c \setminus F_n) = 0 \quad, \tag{3.27}$$

which together with (3.26) implies equation (3.25).
□

Consequence: For any charge μ (not necessarily non-negative) on $(R, \mathscr{F})$ and closed $F \in \mathscr{F}$, there exist functions $1_F \leq f_n \in \mathcal{C}_b(R)$ such that $\mu(F) = \lim\limits_{n \to \infty} \int f_n \, d\mu$.

3.2.2 Lemma: Uniqueness of charges

Let $(R, \mathscr{F})$ be a normal, A-topological space, $\mathfrak{S} \subseteq \mathscr{P}(R)$ an algebra containing $\mathscr{F}$ and μ, ν charges on $\mathfrak{S}$ such that

$$\int f \, d\mu \geq \int f \, d\nu \quad \forall \, 0 \leq f \in \mathcal{C}_b(R) \quad. \tag{3.28}$$

Then $\mu \geq \nu$.

Proof

By linearity, it suffices to show: Any charge μ on $\mathfrak{S}$ with

$$\int f \, d\mu \geq 0 \quad \forall \, 0 \leq f \in \mathcal{C}_b(R) \tag{3.29}$$

is non-negative. By lemma 2.2.6, for this it suffices to show that $\mu\big|_{\mathscr{F}} \geq 0$. Let $F \in \mathscr{F}$, then by 3.2.1 there exist $0 \leq f_n \in \mathcal{C}_b(R)$ such that $\int f_n \, d\mu \overset{n \to \infty}{\longrightarrow} \mu(F)$. Since $0 \leq \int f_n \, d\mu \, \forall \, n$, the proof is complete.
□

Conclusion: If

$$\int f \, d\mu = \int f \, d\nu \quad \forall \, 0 \leq f \in \mathcal{C}_b(R) \quad , \tag{3.30}$$

then $\mu = \nu$.

The above statements make clear that on normal A-topological spaces, charges, already closely connected to the underlying A-topology, show an equally strong connection to the continuous functions of the space. As seen in the proof of 3.2.1, the requirement of the space being normal, is indeed a very crucial one.

Lemma 3.2.2 will prove to be of great use, when it comes to showing the uniqueness of certain charges in the following chapter 4.

Chapter 4

Set functions as linear functionals on function spaces

As already indicated by theorem 3.1.4, bounded contents naturally induce bounded, linear functionals on the functions they integrate. Depending on the function space and set algebra considered, this relation can actually exist in both ways. That is to say, the dual of the function space in question, is identifiable with a certain class of contents by means of the integral.

One of the most widely known examples, is the Riesz representation theorem, which identifies the dual $\mathcal{C}_b(T)^*$ on a compact topological space T, with the measures on its Baire σ-algebra. As it turns out, this statement can not only be generalized to non-compact spaces, but also to completely different function spaces and content classes.

The set algebra or σ-algebra considered will, in the case of charges and Hewitt-measures, be the Baire algebra and Baire σ-algebra of the underlying topological space. As these are generated by its totally closed sets, which in turn form a perfectly normal A-topology of the space, we will most of the time merely consider the Borel-sets of A-topological spaces. Where needed, we shall then simply assume the A-topology to be normal or perfectly normal for that matter.

All dual function spaces and content classes considered, are treated as normed, linear spaces, equipped with the norms introduced in 1.4.4 and 2.1.6 respectively. The relations between them, are shown to be isometrical isomorphisms, thus revealing their strong, algebraic and geometric connection.

4.1 Contents as linear functionals

4.1.1 Theorem: Contents as linear functionals on $\mathscr{U}$

Let $\mathfrak{S} \subseteq \mathscr{P}(\Omega)$ be a set algebra over some set Ω, $\mathscr{U}(\mathfrak{S})^*$ the dual space of $\mathscr{U}(\mathfrak{S})$. Then:

1. The mapping

$$\smallint : \mathfrak{M}_\alpha(\mathfrak{S}) \to \mathscr{U}(\mathfrak{S})^* \quad , \quad \smallint : \mu \mapsto \int (\cdot) \, d\mu \tag{4.1}$$

   is an isometrical isomorphism between Banach spaces, whereas $\mathfrak{M}_\alpha(\mathfrak{S})$ is equipped with the total variation norm, $\mathscr{U}(\mathfrak{S})^*$ with the operator-norm.

2. In particular, its inverse

$$\smallint^{-1} : \mathscr{U}(\mathfrak{S})^* \to \mathfrak{M}_\alpha(\mathfrak{S}) \quad , \quad \smallint^{-1} : L \mapsto \smallint^{-1}(L) \quad , \tag{4.2}$$

   is given by $\smallint^{-1}(L)(A) = L1_A, \ A \in \mathfrak{S}$.

3. If μ^+, μ^- are the positive & negative parts of μ, then these correspond to the positive & negative parts L_μ^+, L_μ^- of $L_\mu := \int(\cdot) \, d\mu$.
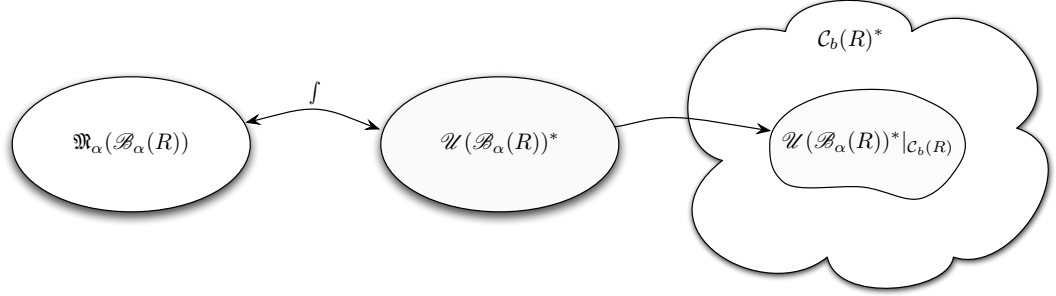

Figure 4.1: Connection between bounded contents and bounded, linear functionals on $\mathcal{C}_b(R)$ & $\mathscr{U}(\mathscr{B}_\alpha(R))$, for an A-topological space $(R, \mathscr{O})$. Note that by 1.2.9(6), any linear functional on $\mathscr{U}(\mathscr{B}_\alpha(R))$ is also a linear functional on $\mathcal{C}_b(R)$.

Proof

1. By 3.1.4 $\int : \mathfrak{M}_\alpha(\mathfrak{S}) \to \mathscr{U}(\mathfrak{S})^*$ is indeed a linear isometry between Banach spaces. Thus, it remains to be shown that it is bijective. Let $L \in \mathscr{U}(\mathfrak{S})^*$. Setting

$$\mu(A) := L1_A \quad , \ A \in \mathfrak{S} \tag{4.3}$$

obviously defines a bounded content on $\mathfrak{S}$, which satisfies

$$\int f \, d\mu = Lf \tag{4.4}$$

for all simple functions $f \in \mathscr{S}(\mathfrak{S})$. For $f \in \mathscr{U}(\mathfrak{S})$ let $f_n \in \mathscr{S}(\mathfrak{S})$ be simple functions, such that $f_n \overset{n \to \infty}{\underset{\|\cdot\|_\infty}{\longrightarrow}} f$. Then $Lf_n \overset{n \to \infty}{\longrightarrow} Lf$ and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} Lf_n = Lf \quad , \tag{4.5}$$

which implies (4.4) for all $f \in \mathscr{U}(\mathfrak{S})$.

2. See proof of part (1.).

3. The functionals $\widetilde{L}_\mu^+ := \int(\cdot) \, d\mu^+$, $\widetilde{L}_\mu^- := \int(\cdot) \, d\mu^-$ are bounded, non-negative and satisfy $\widetilde{L}_\mu^+ - \widetilde{L}_\mu^- = L_\mu$. Thus by 1.4.3:

$$\widetilde{L}_\mu^+ \geq L_\mu^- \quad , \ \ \widetilde{L}_\mu^- \geq L_\mu^- \quad . \tag{4.6}$$

On the other hand, let $\widetilde{\mu}^+, \widetilde{\mu}^-$ be the (non-negative) contents corresponding to L_μ^+, L_μ^- respectively, then $\widetilde{\mu}^+ - \widetilde{\mu}^- = \mu$ and by 2.1.4(3): $\widetilde{\mu}^+ \geq \mu^+$, $\widetilde{\mu}^- \geq \mu^-$. Thus

$$L_\mu^+ \geq \widetilde{L}_\mu^+ \quad , \ \ L_\mu^- \geq \widetilde{L}_\mu^- \quad , \tag{4.7}$$

which together with (4.6) completes the proof.

$\square$

4.1.2 Theorem: σ-additivity of contents and σ-continuity of functionals

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space and $L_\mu := \int (\cdot) \, d\mu$ the linear functional on $\mathscr{U}(\mathfrak{S})$ induced by μ (see theorem 4.1.1). Then the following are equivalent:

1. μ is σ-additive.

2. L_μ is σ-continuous.

3. L_μ satisfies the dominated convergence theorem of Lebesgue[1] on $\mathscr{U}(\mathfrak{S})$.

Proof

3⇒2: Trivial.

2⇒1: Let $(A_n)_{n \in \mathbb{N}} \in \mathfrak{S}$ be such that $A_n \downarrow \emptyset$, then $1_{A_n} \downarrow 0$. Then $\mu(A_n) = L_\mu 1_{A_n} \overset{n \to \infty}{\longrightarrow} 0$, which by 2.1.7 was to be shown.

1⇒3: By 2.1.9, μ can be assumed to be extended to a bounded measure on $\sigma(\mathfrak{S})$. Note that this does not affect the values of integrals on $\mathscr{U}(\mathfrak{S})$. But for bounded, σ-additive set functions, the dominated convergence theorem of Lebesgue is a known fact.

□

Theorem 4.1.1 is the first of three statements to be handled here, relating additive set functions to linear functionals. It is indeed, if not a trivial, a pretty intuitive connection, as linear functionals acting on indicator functions can actually be used as additive set-functions them selfs.

4.2 Charges as linear functionals

The identification of contents with linear functionals presented in the previous section, makes no assumptions about the origin of the considered set algebra and in particular, any possible topological structure of the underlying space. The requirement of the considered functionals, to be defined on such a big function class as the uniformly approximable ones, is often to much of a restriction. On the other hand, any attempt of broadening the class of considered linear functionals (i.e. acting on smaller function spaces), could challenge their representation by additive set functions.

In the following section, we shall present a connection between bounded, linear functionals on the space of bounded, continuous functions $\mathcal{C}_b(R)$ on some A-topological space $(R, \mathscr{O})$ and charges on its Baire algebra $\sigma(\mathscr{O}^t)$ by means of integrals. This theory is mainly due to Alexandroff[2].

As was shown in section 1.3, any A-topological space can be turned into a perfectly normal one, by restricting its A-topology to the totally open sets. As this changes nothing about the function spaces $\mathcal{C}(R)$ & $\mathcal{C}_b(R)$, we shall not consider charges on the generated Baire algebra but on its Borel algebra, while simply assuming $(R, \mathscr{O})$ to be normal (or perfectly normal) in the first place. The obtained contents are then, in the case of more general spaces, defined only on their Baire algebra and regularity is to be understood with respect to the A-topology of totally open sets. The connection between the two content-classes, is clarified in theorem 4.2.7.

Lemma 3.2.1 provides with a representation of the content by means of its integrals, should it be regular. As it turns out, this representation can actually be used to define the charge inducing a given bounded, linear functional on $\mathcal{C}_b(R)$.

---

[1]That is, for every sequence $f, f_n \in \mathscr{U}(\mathfrak{S})$ with $f_n \underset{\text{pointwise}}{\overset{n \to \infty}{\longrightarrow}} f$ and $|f_n| \leq g \; \forall \, n$ for some $g \in \mathscr{U}(\mathfrak{S})$, one has

$$L_\mu f = \lim_{n \to \infty} L_\mu f_n \quad .$$

4.2.1 Definition: Governing functions

Let $(R, \mathscr{F})$ be an A-topological space and $A \in \mathscr{B}_\alpha(\mathscr{F})$. We shall call $\mathcal{G}(A) := \{f \in \mathcal{C}_b(R) : 1_A \leq f\}$ the system of functions *governing* A, and assume on it the direction

$$f \leq g \;\; :\Leftrightarrow \;\; f(x) \leq g(x) \quad \forall \, x \in R \;\; . \tag{4.8}$$

4.2.2 Lemma: Charges induced by non-negative, linear functionals

Let $(R, \mathscr{F})$ be a normal, A-topological space and L a linear, non-negative functional on $\mathcal{C}_b(R)$. Then setting

$$\mu(F) := \inf_{\substack{f \in \mathcal{C}_b(R) \\ 1_F \leq f}} L(f) \tag{4.9}$$

for closed sets $F \in \mathscr{F}$ and

$$\mu(A) := \sup_{\substack{F \in \mathscr{F} \\ F \subseteq A}} \mu(F) \tag{4.10}$$

for any other $A \in \mathscr{B}_\alpha(R) \setminus \mathscr{F}$, defines a non-negative charge on $\mathscr{B}_\alpha(R)$.

Proof
We shall show that μ satisfies the axioms of a charge.

- **Non-negativity:** Since $L\big|_{\mathcal{G}(F)} \geq 0$, (4.9) is well defined and μ is non-negative.

- **Bound:** For every $F \in \mathscr{F}$, $1 \in \mathcal{G}(F)$ and thus $\mu(F) \leq L1$, hence μ is bounded.

- **Monotony on $\mathscr{F}$:** Let $F_1 \subseteq F_2 \in \mathscr{F}$, then since $\mathcal{G}(F_2) \subseteq \mathcal{G}(F_1)$ we have $\mu(F_1) \leq \mu(F_2)$.

- **Regularity:** By construction $\mu(A) = \sup\limits_{\mathscr{F} \ni F \subseteq A} \mu(F)$ for any $A \in \mathscr{B}_\alpha(R) \setminus \mathscr{F}$. But also for closed $\widetilde{F} \in \mathscr{F}$, it is due to the monotony of μ on $\mathscr{F}$ evident that

$$\mu(\widetilde{F}) = \sup_{\mathscr{F} \ni F \subseteq \widetilde{F}} \mu(F) \;\; . \tag{4.11}$$

- **Monotony:** Follows directly from eq. (4.10) and (4.11).

- **Super-additivity on $\mathscr{F}$:** Let $F, \widetilde{F} \in \mathscr{F}$ be disjoint and let $g_n \in \mathcal{G}(F \cup \widetilde{F})$ such that

$$Lg_n \overset{n \to \infty}{\longrightarrow} \mu(F \cup \widetilde{F}) \tag{4.12}$$

and $f \in \mathcal{C}_b(R)$ such that $0 \leq f \leq 1$ and $f\big|_F = 1$, $f\big|_{\widetilde{F}} = 0$. Set

$$f_n := \min\{g_n, f\} \; \in \mathcal{G}(F) \;\; , \;\; \widetilde{f}_n := (g_n - f_n) \; \in \mathcal{G}(\widetilde{F}) \;\; . \tag{4.13}$$

Then $\mu(F) \leq Lf_n$, $\mu(\widetilde{F}) \leq L\widetilde{f}_n$ and thus

$$\mu(F \cup \widetilde{F}) = \lim_{n \to \infty} Lg_n \geq \liminf_{n \to \infty} Lf_n + \liminf_{n \to \infty} L\widetilde{f}_n \geq \mu(F) + \mu(\widetilde{F}) \;\; . \tag{4.14}$$

- **Super-additivity:** Let $A, \widetilde{A} \in \mathscr{B}_\alpha(R)$ be disjoint, then

$$\left\{ F \cup \widetilde{F} : F, \widetilde{F} \in \mathscr{F}, \ F \subseteq A, \ \widetilde{F} \subseteq \widetilde{A} \right\} \subseteq \left\{ F \in \mathscr{F} : F \subseteq A \cup \widetilde{A} \right\} \tag{4.15}$$

and thus

$$\sup_{\substack{F \in \mathscr{F} \\ F \subseteq A \cup \widetilde{A}}} \mu(F) \geq \sup_{\substack{\mathscr{F} \ni F \subseteq A \\ \mathscr{F} \ni \widetilde{F} \subseteq \widetilde{A}}} \underbrace{\mu(F \cup \widetilde{F})}_{\geq \mu(F) + \mu(\widetilde{F})}$$

$$\geq \sup_{\mathscr{F} \ni F \subseteq A} \mu(F) + \sup_{\mathscr{F} \ni \widetilde{F} \subseteq \widetilde{A}} \mu(\widetilde{F}) = \mu(A) + \mu(\widetilde{A}) \ , \tag{4.16}$$

that is,

$$\mu(A \cup \widetilde{A}) \geq \mu(A) + \mu(\widetilde{A}) \ . \tag{4.17}$$

- **Additivity:** By lemma A.4.3 it suffices to show that: If $A, \widetilde{A} \in \mathscr{B}_\alpha(R)$ with $A \subseteq F$, $\widetilde{A} \subseteq F^c$ for some $F \in \mathscr{F}$, then

$$\mu(A \cup \widetilde{A}) = \mu(A) + \mu(\widetilde{A}) \ . \tag{4.18}$$

Actually, by (4.17) and regularity it suffices to show that

$$\mu(H) \leq \mu(A) + \mu(\widetilde{A}) \quad \forall \ \mathscr{F} \ni H \subseteq A \cup \widetilde{A} \ . \tag{4.19}$$

Let $\mathscr{F} \ni H \subseteq A \cup \widetilde{A}$ be chosen, then since $F \cap H \subseteq A$, by monotony it suffices to show

$$\mu(H) \leq \mu(F \cap H) + \mu(\widetilde{A}) \ , \tag{4.20}$$

which by definition (4.9), is equivalent to

$$\mu(H) \leq Lf + \mu(\widetilde{A}) \quad \forall \ f \in \mathcal{G}(F \cap H) \ . \tag{4.21}$$

Let $f \in \mathcal{G}(F \cap H)$, $0 < \varepsilon < 1$ and set

$$G := \{ f \leq 1 - \varepsilon \} \cap H \ \in \mathscr{F} \ . \tag{4.22}$$

Let now $g \in \mathcal{G}(G)$ be arbitrary, then $(f + g)\big|_H \geq 1 - \varepsilon$ and thus

$$\frac{f + g}{1 - \varepsilon} \in \mathcal{G}(H) \ . \tag{4.23}$$

By definition (4.9), this implies

$$(1 - \varepsilon) \cdot \mu(H) \leq L(f + g) = Lf + Lg \tag{4.24}$$

and thus

$$(1 - \varepsilon) \cdot \mu(H) \leq Lf + \mu(G) \ . \tag{4.25}$$

By choice $f\big|_{A \cap H} \geq 1$, which implies $G \cap (A \cap H) = \emptyset$. Together with $H \subseteq A \cup \widetilde{A}$ and (4.22), it follows that $G \subseteq \widetilde{A} \cap H \subseteq \widetilde{A}$. Consequently (4.25) implies

$$(1 - \varepsilon) \cdot \mu(H) \leq Lf + \mu(\widetilde{A}) \quad \forall \ f \in \mathcal{G}(F \cap H), \ \varepsilon > 0 \ . \tag{4.26}$$

But the arbitrariness of $\varepsilon > 0$ connotes (4.21), which was to be shown.

$\square$

4.2.3 Lemma: Compatibility of the created charge

Let $(R, \mathscr{F})$ be a normal, A-topological space and L a non-negative, linear functional on $\mathcal{C}_b(R)$. Then the non-negative charge μ defined in 4.2.2, satisfies

$$\int f \, d\mu = L(f) \quad \forall \, f \in \mathcal{C}_b(R) \quad . \tag{4.27}$$

Proof

Since both sides of eq. (4.27) are linear functionals and obviously $\int 1 \, d\mu = \mu(R) = L1$, it suffices by A.3.3 to show

$$\int f \, d\mu \geq L(f) \quad \forall \, 0 \leq f \in \mathcal{C}_b(R) \quad . \tag{4.28}$$

Since both sides are positive, linear functionals on $\mathcal{C}_b(R)$ we may w.l.o.g. only consider functions $f \in \mathcal{C}_b(R)$ with $0 \leq f < 1$. For given $n \in \mathbb{N}$ define the closed sets

$$F_k := \left\{ f \geq \tfrac{k}{n} \right\} \quad , \quad k = 0, .., n+1 \tag{4.29}$$

and note that

$$\emptyset = F_{n+1} \subseteq F_n \subseteq \ldots F_1 \subseteq F_0 = R \quad . \tag{4.30}$$

Furthermore, define the simple function

$$g_n := \sum_{k=0}^{n} \frac{k}{n} \cdot 1_{F_k \setminus F_{k+1}} = \frac{1}{n} \sum_{k=0}^{n} k \cdot \left[1_{F_k} - 1_{F_{k+1}} \right] \overset{(4.30)}{=} \frac{1}{n} \sum_{k=1}^{n} 1_{F_k} \quad . \tag{4.31}$$

Then

$$F_k \setminus F_{k+1} = \left\{ \tfrac{k}{n} \leq f < \tfrac{k+1}{n} \right\}$$

leads to

$$\|g_n - f\|_\infty \leq \tfrac{1}{n} \quad . \tag{4.32}$$

Choose $1_{F_k} \leq f_k \in \mathcal{C}_b(R)$ such that

$$L(f_k) \leq \mu(F_k) + \frac{1}{n} \quad , \tag{4.33}$$

then

$$\int g_n \, d\mu = \sum_{k=0}^{n} \frac{k}{n} \cdot \mu(F_k \setminus F_{k+1}) = \frac{1}{n} \sum_{k=0}^{n} k \cdot \left[\mu(F_k) - \mu(F_{k+1}) \right]$$

$$\overset{(4.30)}{=} \frac{1}{n} \sum_{k=1}^{n} \mu(F_k) \overset{(4.33)}{\geq} \frac{1}{n} \sum_{k=1}^{n} L(f_k) - \frac{1}{n}$$

$$= L\left[\frac{1}{n} \sum_{k=1}^{n} \underbrace{f_k}_{\geq 1_{F_k}} \right] - \frac{1}{n} \geq Lg_n - \frac{1}{n} \overset{(4.32)}{\geq} Lf - \frac{L1}{n} - \frac{1}{n} \quad . \tag{4.34}$$

Taking the limit on both sides yields

$$\int f \, d\mu \overset{(4.32)}{=} \lim_{n \to \infty} \int g_n \, d\mu \geq L(f) \quad . \tag{4.35}$$

$\square$

Lemmas 4.2.2 and 4.2.3 imply the existence of charges in normal spaces, inducing any non-negative, linear functional on $\mathcal{C}_b(R)$, by means of the integral. The generalization to arbitrary bounded functionals is straightforward and summarized in the following theorem 4.2.4.

4.2.4 Theorem: Charges as linear functionals on $\mathcal{C}_b$ [Alexandroff]

Let $(R, \mathscr{F})$ be a normal, A-topological space. Then:

1. The mapping

$$\int : \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R)) \to \mathcal{C}_b(R)^* \quad , \quad \int : \mu \mapsto \int (\cdot) \, d\mu \tag{4.36}$$

   is an isometrical isomorphism between Banach spaces, whereas $\mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R))$ is equipped with the total variation norm, $\mathcal{C}_b(R)^*$ with the operator-norm.

2. In particular, its inverse

$$\int^{-1} : \mathcal{C}_b(R)^* \to \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R)) \quad , \quad \int^{-1} : L \mapsto \int^{-1}(L) \quad , \tag{4.37}$$

   is characterized by

$$\int^{-1}(L)(F) = \lim_{f \in \mathcal{G}(F)} Lf \quad , \quad F \in \mathscr{F} \quad . \tag{4.38}$$

3. If μ^+, μ^- are the positive & negative parts of $\mu \in \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R))$, then these correspond to the positive & negative parts L_μ^+, L_μ^- of $L_\mu := \int (\cdot) \, d\mu$.

Proof

The proof below is partly due to Alexandroff[2] and Dunford & Schwartz[7].

1. By 2.2.5, $\mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R))$ is indeed a Banach space. By 4.1.1, the mapping $\int : \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R)) \to \mathcal{C}_b(R)^*$ is indeed a linear, bounded one[2]. By 3.2.2 it is injective. Left to be shown is: $\int$ is surjective and preserves norms.

   Let $L \in \mathcal{C}_b(R)^*$ and L^+, L^- be the positive and negative parts of L respectively, then by 4.2.3 there exist non-negative charges $\widetilde{\mu}^+, \widetilde{\mu}^-$ such that $\int (\cdot) \, d\widetilde{\mu}^+ = L^+$, $\int (\cdot) \, d\widetilde{\mu}^- = L^-$. Set $\mu := \widetilde{\mu}^+ - \widetilde{\mu}^-$, then obviously $\int (\cdot) \, d\mu = L$, thus, $\int$ is indeed surjective.

   Norm-preservation shall be proven with the help of part (3). Let μ^+, μ^- be the positive & negative parts of $\mu \in \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R))$ and L_μ^+, L_μ^- the (corresponding) positive & negative parts of $L_\mu := \int (\cdot) \, d\mu$. Then:

$$\|L_\mu\| \overset{1.4.5}{=} \|L_\mu^+\| + \|L_\mu^-\| \overset{1.4.5}{=} L_\mu^+ 1 + L_\mu^- 1 = \mu^+(\Omega) + \mu^-(\Omega) = |\mu|(\Omega) \quad . \tag{4.39}$$

2. Let μ^+, μ^- be the positive & negative parts of the charge $\mu \in \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R))$ and L_{μ^+}, L_{μ^-} the induced functionals. Then $L_\mu := L_{\mu^+} - L_{\mu^-}$ corresponds to μ. Consequently, for any closed $F \in \mathscr{F}$:

$$\mu(F) = \mu^+(F) - \mu^-(F) \overset{3.2.1}{=} \inf_{f \in \mathcal{G}(F)} L_{\mu^+} f - \inf_{f \in \mathcal{G}(F)} L_{\mu^-} f$$

$$= \lim_{f \in \mathcal{G}(F)} L_{\mu^+} f - \lim_{f \in \mathcal{G}(F)} L_{\mu^-} f$$

$$= \lim_{f \in \mathcal{G}(F)} \left[L_{\mu^+} f - L_{\mu^-} f \right] = \lim_{f \in \mathcal{G}(F)} L_\mu f \quad . \tag{4.40}$$

---

[2]Recall that $\mathcal{C}_b(R) \subseteq \mathscr{U}(\mathscr{B}_\alpha(R))$.

3. Let L^+, L^- be the positive & negative parts of $L \in \mathcal{C}_b(R)^*$, corresponding to the non-negative charges $\widetilde{\mu}^+, \widetilde{\mu}^-$ respectively. Left to be shown is: If μ^+, μ^- are the positive and negative parts of $\mu := \widetilde{\mu}^+ - \widetilde{\mu}^-$, then $\widetilde{\mu}^+ = \mu^+$, $\widetilde{\mu}^- = \mu^-$.

   By theorem 2.1.4(3) already

$$\widetilde{\mu}^+ \geq \mu^+ \quad , \quad \widetilde{\mu}^- \geq \mu^- \quad . \tag{4.41}$$

   On the other hand $L = \int(\cdot) \, d\mu^+ - \int(\cdot) \, d\mu^-$, with $\int(\cdot) \, d\mu^+$, $\int(\cdot) \, d\mu^-$ as non-negative, linear functionals on $\mathcal{C}_b(R)$. Thus by theorem 1.4.3

$$\int f \, d\mu^+ \geq \int f \, d\widetilde{\mu}^+ \quad , \quad \int f \, d\mu^- \geq \int f \, d\widetilde{\mu}^- \quad \forall \, 0 \leq f \in \mathcal{C}_b(R) \tag{4.42}$$

   and by lemma 3.2.2, $\mu^+ \geq \widetilde{\mu}^+$, $\mu^- \geq \widetilde{\mu}^-$, which together with (4.41) completes the proof.

   $\square$

Note: Uniqueness of the bounded content μ corresponding to a given bounded, linear functional on $\mathcal{C}_b(R)$, indeed requires the assumption of regularity of μ. For if ν is a non-regular, bounded content (see 2.2.2 for an example), the charge μ corresponding to the induced functional $\int(\cdot) \, d\nu \in \mathcal{C}_b(R)^*$, clearly differs from ν.

4.2.5   Theorem: σ-additivity of charges and σ-continuity of functionals

Let $(R, \mathscr{F})$ be perfectly normal, A-topological space, μ a charge on $\mathscr{B}_\alpha(R)$ and $L_\mu := \int(\cdot) \, d\mu$ the linear functional induced by μ on $\mathcal{C}_b(R)$. Then the following are equivalent:

1. μ is σ-additive.

2. L_μ is σ-continuous.

3. L_μ satisfies the dominated convergence theorem of Lebesgue[3] on $\mathcal{C}_b(R)$.

Proof

3$\Rightarrow$2: Trivial.

2$\Rightarrow$1: By 1.4.7, the positive & negative parts L_μ^+, L_μ^- are also σ-continuous. By 4.2.4, they are induced by μ^+, μ^- and we can thus w.l.o.g. assume $\mu \geq 0$.

   By 2.2.9, it suffices to show that for a decreasing sequence of closed sets $F_n \downarrow \emptyset$, $\mu(F_n) \overset{n \to \infty}{\longrightarrow} 0$. By 1.3.4, there exist open $G_n \supseteq F_n$ such that $G_n \downarrow \emptyset$. By 3.2.1, there exist functions $1_{F_n} \leq f_n' \in \mathcal{C}_b(R)$ such that $|L_\mu f_n' - \mu(F_n)| \overset{n \to \infty}{\longrightarrow} 0$ and by 1.1.7(1) functions $f_n'' \in \mathcal{C}_b(R)$ such that $0 \leq f_n'' \leq 1$ and $f_n''|_{F_n} = 1$, $f_n''|_{G_n^c} = 0$.

   Set $f_n := \min\{f_1', f_1'', .., f_n', f_n''\}$, then $f_n \in \mathcal{C}_b(R)$ is a decreasing sequence such that $f_n \downarrow 0$ (pointwise), $1_{F_n} \leq f_n$ and

$$|L_\mu f_n - \mu(F_n)| \overset{n \to \infty}{\longrightarrow} 0 \quad . \tag{4.43}$$

   By σ-continuity $L_\mu f_n \overset{n \to \infty}{\longrightarrow} 0$, which together with (4.43) implies $\mu(F_n) \overset{n \to \infty}{\longrightarrow} 0$.

---

[3]That is, for every sequence $f, f_n \in \mathcal{C}_b(R)$ with $f_n \overset{n \to \infty}{\underset{\text{pointwise}}{\longrightarrow}} f$ and $|f_n| \leq g \; \forall \, n$ for some $g \in \mathcal{C}_b(R)$, one has

$$L_\mu f = \lim_{n \to \infty} L_\mu f_n \quad .$$

1⇒3: Special case of theorem 4.1.2.

□

Since the totally open sets $\mathscr{O}^t$ of any A-topological space $(R, \mathscr{O})$, turn R into a perfectly normal one, theorem 4.2.4 applies as long as the Baire algebra $\alpha(\mathscr{O}^t)$ is considered instead of the Borel one. Thus, to any A-topological space $(R, \mathscr{O})$ and bounded linear functional $L \in \mathcal{C}_b(R)^*$, corresponds a unique charge in $\mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t))$.

On the other hand, if $(R, \mathscr{O})$ is already normal in the first place, another charge can be found on the Borel algebra $\alpha(\mathscr{O})$ also corresponding to L. This gives rise to the natural question, what the connection is between these two charges, which in the case of R not being perfectly normal, have at least different domains. The fortunately simple answer is provided by lemma 4.2.6 and theorem 4.2.7, falling into line with theorem 2.2.12 stated above.

4.2.6   Lemma: Induced charges on the Borel- & Baire algebra

Let $(R, \mathscr{O})$ be a normal, A-topological space and μ, μ_t charges in $\mathfrak{M}_\alpha^r(\mathscr{O}, \alpha(\mathscr{O}))$ and $\mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t))$ respectively, both inducing the bounded, linear functional $L := \int(\cdot) \, d\mu = \int(\cdot) \, d\mu_t$ on $\mathcal{C}_b(R)$. Then μ_t is the restriction of μ to $\alpha(\mathscr{O}^t)$.

Proof

By theorem 2.2.12, the restriction $\mu\big|_{\alpha(\mathscr{O}^t)}$ is a charge with respect to $\mathscr{O}^t$. Clearly, it satisfies $L = \int(\cdot) \, d\mu\big|_{\alpha(\mathscr{O}^t)}$. By 3.2.2, $\mu\big|_{\alpha(\mathscr{O}^t)} = \mu_t$.

□

4.2.7   Theorem: Charges on the Borel- & Baire algebra

Let $(R, \mathscr{F})$ be a normal, A-topological space. Then:

1. The mapping

$$\mathfrak{M}_\alpha^r(\mathscr{O}, \alpha(\mathscr{O})) \to \mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t)) \quad , \quad \mu \mapsto \mu\big|_{\alpha(\mathscr{O}^t)} \tag{4.44}$$

   is an isometrical isomorphism, with respect to the total variation norm.

2. If $\int : \mathfrak{M}_\alpha^r(\mathscr{O}, \alpha(\mathscr{O})) \to \mathcal{C}_b(R)^*$ and $\int_t : \mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t)) \to \mathcal{C}_b(R)^*$ are the isomorphisms defined in theorem 4.2.4, then (4.44) is equal to $\int_t^{-1} \circ \int$.

3. The restrictions of the positive & negative parts of $\mu \in \mathfrak{M}_\alpha(\mathscr{O}, \alpha(\mathscr{O}))$ to $\alpha(\mathscr{O}^t)$, are the positive & negative parts of $\mu\big|_{\alpha(\mathscr{O}^t)}$.

Proof

1. By theorem 2.2.12, the restriction $\mu\big|_{\alpha(\mathscr{O}^t)}$ of $\mu \in \mathfrak{M}_\alpha^r(\mathscr{O}, \alpha(\mathscr{O}))$ is indeed a charge in $\mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t))$. By the same theorem, the whole linear mapping (4.44) is injective.

   Now let $\mu_t \in \mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t))$, then by theorem 4.2.4 there exists a charge $\mu \in \mathfrak{M}_\alpha^r(\mathscr{O}, \alpha(\mathscr{O}))$ such that $\int(\cdot) \, d\mu = \int(\cdot) \, d\mu_t \in \mathcal{C}_b(R)^*$, whereas

$$|\mu|(R) \overset{4.2.4}{=} \|\textstyle\int(\cdot) \, d\mu_t\| \overset{4.2.4}{=} |\mu_t|(R) \quad . \tag{4.45}$$

   Recall that $\mathcal{C}_b(R)$ is the same for both A-topological spaces $(R, \mathscr{O})$ and $(R, \mathscr{O}^t)$. By 4.2.6, $\mu_t = \mu\big|_{\alpha(\mathscr{O}^t)}$. Thus, the mapping (4.44) is surjective and norm-preserving.

2. Already shown in part (1).

3. By theorem 4.2.4, the mappings $\int, \int_t$ from part (2), map positive & negative parts to positive & negative parts. By part (2), this completes the proof.

$\square$

Depending on the structure of the underlying space, certain further statements can be made about its charges. The strong connection established in theorem 4.2.4 between them and linear functionals on $\mathcal{C}_b(R)$, leads to statements similar to lemma 1.4.8, this time for charges. The resulting richness of characterizations is presented in theorem 4.2.8. A direct consequence will be the known Riesz representation theorem for measures on countably compact spaces, given in Corollary 4.2.9.

4.2.8 Theorem: Characterization of pseudocompact spaces

Let $(R, \mathscr{O})$ be a perfectly normal, A-topological space. Then the following are equivalent:

1. $(R, \mathscr{O})$ is pseudocompact, that is, $\mathcal{C}_b(R) = \mathcal{C}(R)$.

2. For every regular sequence $F_n \uparrow R$, there exists an $n_0 \in \mathbb{N}$ such that $F_{n_0} = R$.

3. $(R, \mathscr{O})$ is countably compact.

4. Every $f \in \mathcal{C}_b(R)$ attains its maximum (and minimum) value.

5. Dini's theorem holds for $\mathcal{C}(R)$, that is: For every sequence $(f_n) \subseteq \mathcal{C}(R)$ with $f_n \downarrow 0$ (pointwise), the convergence is uniform.

6. Dini's theorem holds for $\mathcal{C}_b(R)$, that is: For every sequence $(f_n) \subseteq \mathcal{C}_b(R)$ with $f_n \downarrow 0$ (pointwise), the convergence is uniform.

7. All bounded, linear functionals on $\mathcal{C}_b(R)$ are σ-continuous.

8. All charges on $\mathscr{B}_\alpha(R)$ are σ-additive.

Proof
The following proof is partly taken from Varadarajan[13].

1⇒2: Let $F_n \uparrow R$ be a regular sequence and $f \in \mathcal{C}(R)$ such that $F_n = \{g \leq n\}$ (see A.2.4). By assumption g is bounded, which implies the claim.

2⇒3: To be shown is that, for every decreasing sequence of closed sets $F_n \downarrow \emptyset$, there exists an $n_0 \in \mathbb{N}$ such that $F_{n_0} = \emptyset$. By A.2.1 there exists a regular sequence $\widetilde{F}_n \subseteq F_n^c$. Since for some $n_0 \in \mathbb{N}$ large enough $\widetilde{F}_{n_0} = R$, it follows $F_{n_0} = \emptyset$.

3⇒4: Let $s := \sup f$ and $F_n := \{f \geq s - \frac{1}{n}\}$. Then f attains s if and only if $\bigcap_{n=1}^\infty F_n \neq \emptyset$. Since F_n are non-empty, closed, decreasing and R is countably compact, their intersection is indeed non-empty.

Since $\inf f = -\sup(-f)$, f also attains its minimum.

3⇒5: Let $\varepsilon > 0$ be given and set $F_n := \{f_n \geq \varepsilon\}$. Then since $f_n \downarrow 0$:

$$\emptyset = \{x \in R : f_n(x) \geq \varepsilon \ \forall n\} = \bigcap_{n=1}^\infty F_n \tag{4.46}$$

But R is countable compact, so sooner or later the closed, decreasing F_n will be void, that is $f_n < \varepsilon$ for $n \in \mathbb{N}$ large enough. Thus, since $\varepsilon > 0$ was arbitrary, the convergence $f_n \downarrow 0$ is uniform.

4⇒1: Let $f \in \mathcal{C}(R)$, then $g := \arctan(f) \in \mathcal{C}_b(R)$ and g attains its maximum, that is, $\sup g = g(x)$ for some $x \in R$. But since $\tan : \mathbb{R} \to (-1, 1)$ is continuous and strongly increasing:

$$f(x) = \tan(g(x)) = \sup \tan(g) = \sup f \quad , \tag{4.47}$$

that is, $f \in \mathcal{C}_b(R)$.

5⇒6: Trivial.

6⇔7: Follows directly from 1.4.8.

6⇒2: Suppose $F_n \uparrow R$ is a regular sequence and $F_n \subseteq U_n \subseteq F_{n+1}$ for some open sets U_n, $n \in \mathbb{N}$. Choose functions $f_n \in \mathcal{C}_b(R)$, $0 \leq f_n \leq 1$ such that $f_n\big|_{F_n} = 0$, $f_n\big|_{U_n^c} = 1$. Then $f_n \downarrow 0$ (pointwise). But by assumption, the convergence is uniform and in particular $f_n \leq \frac{1}{2}$ for n large enough. Consequently, $F_n = R$ for n large enough.

7⇔8: Follows from 4.2.4(1) and 4.2.5.

$\square$

4.2.9 Corollary: The Riesz representation theorem

Let $(R, \mathcal{O})$ be an A-topological space and L a bounded, linear functional on the bounded, continuous functions $\mathcal{C}_b(R)$. Then:

1. There exists a unique charge[4] μ on the Baire algebra $\alpha(\mathcal{O}^t)$ of R, such that

$$Lf = \int f \, d\mu \quad \forall \, f \in \mathcal{C}_b(R) \quad . \tag{4.48}$$

2. If $(R, \mathcal{O})$ is countably compact, there exists a unique measure μ on the Baire σ-algebra $\sigma(\mathcal{O}^t)$, such that eq. (4.48) holds. This measure is regular with respect to the A-topology of totally open sets. Its restriction to the Baire algebra of R is exactly the charge from part (1).

Proof

In the following proof, regularity is to be understood with respect to the A-topology of totally open sets.

1. By 1.3.2, the totally open sets $\mathcal{O}^t$ of R turn $(R, \mathcal{O}^t)$ into a perfectly normal, A-topological space, whose continuous, bounded functions are exactly $\mathcal{C}_b(R)$ and whose Borel algebra is exactly $\alpha(\mathcal{O}^t)$. By 4.2.4, there exists a unique charge μ on $\alpha(\mathcal{O}^t)$, such that $L = \int (\cdot) \, d\mu$.

2. Let μ be the charge from part (1). Since by assumption $(R, \mathcal{O})$ is countably compact, by 4.2.8 μ is σ-additive. Thus by 2.1.9 and 2.2.10, μ can be extended to a regular measure on the Baire σ-algebra $\sigma(\mathcal{O}^t)$ and of course still satisfy $L = \int (\cdot) \, d\mu$.

   Now let ν be another measure on $\sigma(\mathcal{O}^t)$, also satisfying $L = \int (\cdot) \, d\nu$. Then by 2.2.7, it is regular. By 3.2.2, $\mu = \nu$.

   $\square$

---

[4]Regular with respect to the A-topology of totally open sets.

4.3 Measures as linear functionals

For measures $\mu \in \mathfrak{M}_\sigma(\mathscr{B}_\sigma(R))$ on the Borel σ-algebra of some A-topological space $(R, \mathscr{O})$, the concept of Lebesgue integration can be extended to include a much wider range of functions[5], such as all continuous ones. The integral operator $\int(\cdot) \, d\mu$ on all integrable (with respect to μ), continuous functions is a bounded one. The question arises, wether the reverse statement holds, that is: If L is a bounded, linear functional on $\mathcal{C}(R)$ and $\mu \in \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R))$ the charge corresponding to the restriction $L\big|_{\mathcal{C}_b(R)}$, is μ σ-additive, as to be extendable to $\mathscr{B}_\sigma(R)$? And if so, does $\int(\cdot) \, d\mu = L$ also hold? These questions are answered by theorem 4.3.4, which identifies the dual $\mathcal{C}(R)^*$ with a subspace of measures on the Baire-σ–algebra, the so called *Hewitt measures*. Hewitt-measures and their connection to linear functionals on $\mathcal{C}(R)$, were studied extensively by Hewitt[10]. As it turns out, these are exactly the measures, for which all continuous functions are essentially bounded.

Recall that by 1.4.4(2) every bounded, linear functional on $\mathcal{C}(R)$ is uniquely determined by its values on $\mathcal{C}_b(R)$. Furthermore, by 1.4.4(2) $\mathcal{C}(R)^*$ is a normed vector space with the norm $\|L\| := |L| \, 1, \; L \in \mathcal{C}(R)^*$ and isometrical to $\mathcal{C}_b(R)^*$.

4.3.1 Definition: Hewitt-measure

Let $(R, \mathscr{O})$ be an A-topological space and $\mathfrak{S} \subseteq \mathscr{P}(R)$ a σ-algebra containing $\mathscr{O}$. A measure μ on $\mathfrak{S}$ shall be called a *Hewitt-measure*, if every continuous function $f \in \mathcal{C}(R)$ is integrable with respect to μ, that is, $\int |f| \, d\,|\mu| < \infty$. We shall denote by $\mathfrak{M}_\sigma^h(\mathfrak{S})$, the system of all Hewitt-measures on $\mathfrak{S}$.

Note: By 2.1.5, $\mathfrak{M}_\sigma^h(\mathfrak{S})$ is a linear subspace of $\mathfrak{M}_\sigma(\mathfrak{S})$. As it turns out though, it is in general neither closed nor open[6] (see 4.3.3 for examples). By theorem 4.3.4, it follows that $\mathcal{C}(R)^*$ is in general not complete!

4.3.2 Theorem: Characterization of Hewitt-measures

Let $(R, \mathscr{O})$ be an A-topological space, $\mathfrak{S} \subseteq \mathscr{P}(R)$ a σ-algebra containing $\mathscr{O}$ and μ a measure on $\mathscr{B}_\sigma(R)$. Then the following are equivalent:

1. The measure μ is Hewitt.

2. Its total variation $|\mu|$ is Hewitt.

3. Its positive and negative parts μ^+, μ^- are Hewitt.

4. For every $f \in \mathcal{C}(R)$, there exists a set $A \in \mathscr{B}_\sigma(R)$ such that $|\mu|\,(A^c) = 0$ and $f\big|_A$ is bounded.

5. If R is perfectly normal: For every regular sequence $F_n \uparrow R$: $\sum_{n=1}^\infty |\mu|\,(F_n^c) < \infty$.

Proof
The following proof is partly due to Varadarajan[13] and Hewitt[10].

1⇔2: Trivial.

2⇒3: Follows from the fact that μ is also Hewitt and μ^-, μ^+ are just linear combinations of μ and $|\mu|$.

---

[5]See for example Bogachev[4].
[6]With respect to the total variation norm.

3⇒2: Trivial, since $\mu = \mu^+ + \mu^-$.

1⇒4: Suppose the contrary, then there exists an $f \in \mathcal{C}(R)$ such that $|\mu|\,(\{f > r\}) > 0$ for every $r \geq 0$. Consequently, one may find integers $a_1 < b_1 < a_2 < .. \in \mathbb{N}$ such that

$$p_n := |\mu|\,\Big(\underbrace{\{a_n \leq f \leq b_n\}}_{A_n}\Big) > 0 \quad . \tag{4.49}$$

with $A_n \in \mathfrak{S}$ as disjoint sets. Define the continuous function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \begin{cases} 0 & : x \leq 0 \\ \frac{1}{p_n} & : a_n \leq x \leq b_n \\ \frac{1}{p_n} + \frac{(x-b_n)}{(a_{n+1}-b_n)}\left(\frac{1}{p_{n+1}} - \frac{1}{p_n}\right) & : b_n < x < a_{n+1} \end{cases} \quad . \tag{4.50}$$

Then $0 \leq g(f) \in \mathcal{C}(R)$ is such that $|\mu|\,(A_n) = p_n$ and $g\big|_{f(A_n)} = 1/p_n$.
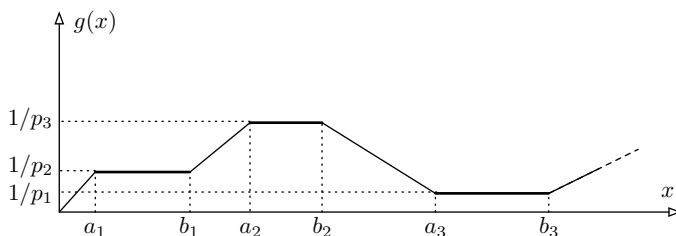
Figure 4.2: On the proof of theorem (4.3.2).

Thus $\int_{A_n} g(f)\,d\,|\mu| = 1$ for every $n \in \mathbb{N}$, which implies $\int g(f)\,d\,|\mu| = \infty$, a contradiction to the initial assumption!

4⇒1: Trivial, since $\int_{A^c} |f|\,d\,|\mu| = 0$ and $\int_A |f|\,d\,|\mu| < \infty$.

2⇒5: Let $L := \int(\cdot)\,d\,|\mu|$ be the non-negative, linear functional on $\mathcal{C}(R)$ induced by $|\mu|$. Let $F_n \uparrow R$ be a regular sequence, then by A.2.4 there exists a $0 \leq g \in \mathcal{C}(R)$ such that $F_n = \{g \leq n^2\}$, which implies

$$Lg \geq \int_{F_n^c} g\,d\,|\mu| \geq n^2 \cdot |\mu|\,(F_n^c) \quad . \tag{4.51}$$

Hence, $|\mu|\,(F_n^c) \leq Lg/n^2$ and the statement is shown.

5⇒2: Let $0 \leq f \in \mathcal{C}(R)$. It suffices to show that, $\int(f \wedge n)\,d\,|\mu|$ is a Cauchy sequence. One can estimate

$$\left|\int(f \wedge (n+1))\,d\,|\mu| - \int(f \wedge n)\,d\,|\mu|\right| \leq \int \underbrace{|(f \wedge (n+1)) - (f \wedge n)|}_{\leq 1_{\{f > n\}}}\,d\,|\mu|$$

$$\leq |\mu|\,(\{f \leq n\}^c) \quad . \tag{4.52}$$

But by A.2.4, $F_n := \{f \leq n\}$ is a regular sequence. Thus $\sum_{n=1}^{\infty} |\mu|\,(F_n^c) < \infty$ and $\int(f \wedge n)\,d\,|\mu|$ is indeed Cauchy.

□

4.3.3 Interesting examples of Hewitt-measures

Consider the topological space $T := (0,1)$ equipped with the standard Borel σ-algebra $\mathscr{B}_\sigma(T)$. Let ν be the uniform distribution on $\mathscr{B}_\sigma(T)$. The following examples show that, the space $\mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(T))$ of Hewitt-measures is neither closed nor open in $\mathfrak{M}_\sigma(\mathscr{B}_\sigma(T))$.

1. The measures μ_n, $n \in \mathbb{N}$ defined on $\mathscr{B}_\sigma(T)$ by the Radon-Nikodym densities[7] $\frac{d\mu_n}{d\lambda} = 1_{\left[\frac{1}{n}, 1-\frac{1}{n}\right]}$ are by 4.3.2 Hewitt, since all continuous functions $f \in \mathcal{C}(T)$ are bounded on the compact sets $\left[\frac{1}{n}, 1-\frac{1}{n}\right]$. Furthermore, they converge in total variation to the uniform distribution λ on $(0,1)$. On the other hand, λ is **not** Hewitt, since $\int x^{-1} \, d\lambda(x) = \infty$.

2. The measures $\mu_n := \lambda/n$, $n \in \mathbb{N}$ are **not** Hewitt, even though they converge in total variation to the zero measure, which is Hewitt.

4.3.4 Theorem: Hewitt-measures as linear functionals on $\mathcal{C}$ [Hewitt]

Let $(R, \mathscr{F})$ be a perfectly normal, A-topological space. Then:

1. The mapping

$$\int : \mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(R)) \to \mathcal{C}(R)^* \quad , \quad \mu \mapsto \int (\cdot) \, d\mu \tag{4.53}$$

is an isometrical isomorphism between normed spaces.

2. If L_μ is the bounded, linear operator corresponding to the Hewitt-measure $\mu \in \mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(R))$, then the restriction[8] $L_\mu\big|_{\mathcal{C}_b(R)}$ corresponds[9] to the restriction of μ to $\mathscr{B}_\alpha(R)$.

3. The positive and negative parts of L_μ correspond to the positive and negative parts of μ.

Proof

The following proof is partly due to Varadarajan[13].

1. Clearly the linear functional $\int (\cdot) \, d\mu$ on $\mathcal{C}(R)$ induced by the Hewitt-measure $\mu \in \mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(R))$ is, in the sense of definition 1.4.1, a bounded one. By 2.2.7, every measure on $\mathscr{B}_\sigma(R)$ is regular. Thus by 3.2.2, the values $\int f \, d\mu$, $f \in \mathcal{C}(R)$ uniquely define it and the linear mapping $\int : \mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(R)) \to \mathcal{C}(R)^*$ is injective.

   We shall now show that, every bounded, linear functional L on $\mathcal{C}(R)$ is induced by some Hewitt-measure on $\mathscr{B}_\sigma(R)$. Let for a start L be non-negative. By 4.2.4, there exists a non-negative charge μ_b on $\mathscr{B}_\alpha(R)$, such that

$$Lf = \int f \, d\mu_b \quad \forall \, f \in \mathcal{C}_b(R) \quad . \tag{4.54}$$

   Now let $F_n \in \mathscr{F}$ be a regular sequence, then by A.2.4 there exists a function $0 \leq g \in \mathcal{C}(R)$ such that $F_n = \{g \leq n^2\}$, $n \in \mathbb{N}$, which implies

$$Lg \geq L \underbrace{\min\{g, n^2\}}_{\in \mathcal{C}_b(R)} \geq \int_{F_n^c} \min\{g, n^2\} \, d\mu_b \geq n^2 \cdot \mu_b(F_n^c) \tag{4.55}$$

---

[7]With respect to the Lebesgue measure λ.

[8]Note that by 1.4.1(ii), $L\big|_{\mathcal{C}_b(R)}$ is bounded in the operator norm.

[9]As described in theorem 4.2.4. Note that by 2.2.8, the restriction $\mu\big|_{\mathscr{B}_\alpha(R)}$ is a charge.

and consequently

$$\mu_b(F_n^c) \leq \frac{Lg}{n^2} \overset{n\to\infty}{\longrightarrow} 0 \quad . \tag{4.56}$$

By 2.2.9, this implies the σ-additivity of μ_b on $\mathscr{B}_\alpha(R)$. Thus by 2.1.9 and 2.2.10, μ_b can be uniquely extended to a non-negative, σ-additive charge μ on $\mathscr{B}_\sigma(R)$. But (4.56) implies by 4.3.2, that this measure μ is Hewitt and induces a bounded, linear functional $\int(\cdot)\, d\mu$ on $\mathcal{C}(R)$. This functional equals L on $\mathcal{C}_b(R)$. Thus, by 1.4.4(1) it equals L on all of $\mathcal{C}(R)$.

Now let L be arbitrary and L^+, L^- its positive and negative parts respectively, with corresponding non-negative Hewitt-measures μ^+, μ^- on $\mathscr{B}_\sigma(R)$. Then $\mu := \mu^+ - \mu^-$ is also a Hewitt-measure on $\mathscr{B}_\sigma(R)$ and satisfies $\int(\cdot)\, d\mu = L$.

As will be shown in part (2), the restriction $L_\mu\big|_{\mathcal{C}_b(R)}$ of a bounded, linear functional $L_\mu \in \mathcal{C}(R)^*$ induced by some Hewitt-measure $\mu \in \mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(R))$, corresponds to the (regular) restriction μ_b of μ on $\mathscr{B}_\alpha(R)$. By 1.4.4(4) $\|L_\mu\| = \big\|L_\mu\big|_{\mathcal{C}_b(R)}\big\|$, by theorem 2.1.9 $\|\mu_b\|_t = \|\mu\|_t$ and by theorem 4.2.4 $\|\mu_b\|_t = \big\|L_\mu\big|_{\mathcal{C}_b(R)}\big\|$. Thus $\|\mu\|_t = \|L_\mu\|$.

2. The statement was already shown for non-negative functionals in the proof of part (1). But theorem 4.3.2 and linearity of both mappings

$$\int : \mathfrak{M}_\sigma^h(\mathscr{B}_\sigma(R)) \to \mathcal{C}(R)^* \quad , \quad \int : \mathfrak{M}_\alpha^r(\mathscr{B}_\alpha(R)) \to \mathcal{C}_b(R)^* \tag{4.57}$$

actually imply the general case.

3. Similar to the proof of 4.2.4(3). Recall that by 2.2.7, all measures $\mu \in \mathfrak{M}_\sigma(\mathscr{B}_\sigma(R))$ are regular and by 3.2.2, non-negative functionals correspond to non-negative measures.

$\square$

4.3.5 Corollary for linear functionals on $\mathcal{C}$ [Hewitt]

Let $(R, \mathscr{O})$ be an A-topological space and L a bounded, linear functional on $\mathcal{C}(R)$. Then there exists a unique measure μ on the Baire σ-algebra $\sigma(\mathscr{O}^t)$, such that

$$\int f\, d\mu = Lf \quad \forall\, f \in \mathcal{C}(R) \quad . \tag{4.58}$$

Proof

By theorem 1.3.1, the totally closed sets of R form a perfectly normal A-topology with same continuous functions $\mathcal{C}(R)$. By theorem 4.3.4, there exists a Hewitt-measure μ on the induced Baire σ-algebra $\sigma(\mathscr{O}^t)$, satisfying eq. (4.58).

$\square$

4.3.6 Corollary: σ-continuity of linear functionals on $\mathcal{C}$

Let $(R, \mathscr{O})$ be an A-topological space. Then every bounded, linear functional on $\mathcal{C}(R)$ is σ-continuous.

Proof

By 4.3.5, every bounded, linear functional on $\mathcal{C}(R)$ is induced by some Hewitt-measure on the Baire σ-algebra $\sigma(\mathscr{O}^t)$. From the Lebesgue dominated convergence theorem for measures, follows the σ-continuity of L.

$\square$

Chapter 5

Summary

As was demonstrated in section 3.1, the theory of integration can be extended to additive set functions (contents) in a straight forward way, provided both the content and the function to be integrated are (totally) bounded. By means of the integral, every bounded content naturally gives rise to a bounded, linear functional on certain function spaces Φ. Interestingly, this interpretation can also be reversed, that is to say, each linear functional can be, provided certain properties of Φ, identified with a content on some special set algebra.

Function spaces considered were the space of uniformly approximable functions $\mathscr{U}(\mathfrak{S})$ over some algebra $\mathfrak{S}$, the space of bounded continuous functions $\mathcal{C}_b(R)$ on some A-topological space R and the space of continuous functions $\mathcal{C}(R)$, depending on the content in question. In particular, isometrical isomorphisms were established between:

1. The bounded contents $\mathfrak{M}_\alpha(\mathfrak{S})$ on some set algebra $\mathfrak{S}$ and the dual space $\mathscr{U}(\mathfrak{S})^*$ (section 4.1, theorem 4.1.1).

2. The bounded, regular contents $\mathfrak{M}_\alpha^r(\mathscr{O}^t, \alpha(\mathscr{O}^t))$ on the Baire algebra $\alpha(\mathscr{O}^t)$ of some A-topological space $(R, \mathscr{O})$ and the dual space $\mathcal{C}_b(R)^*$ (section 4.2, theorem 4.2.4).

3. The Hewitt measures $\mathfrak{M}_\sigma^h(\sigma(\mathscr{O}^t))$ on the Baire σ-algebra $\sigma(\mathscr{O}^t)$ and the dual space $\mathcal{C}(R)^*$ (section 4.3, theorem 4.3.4).
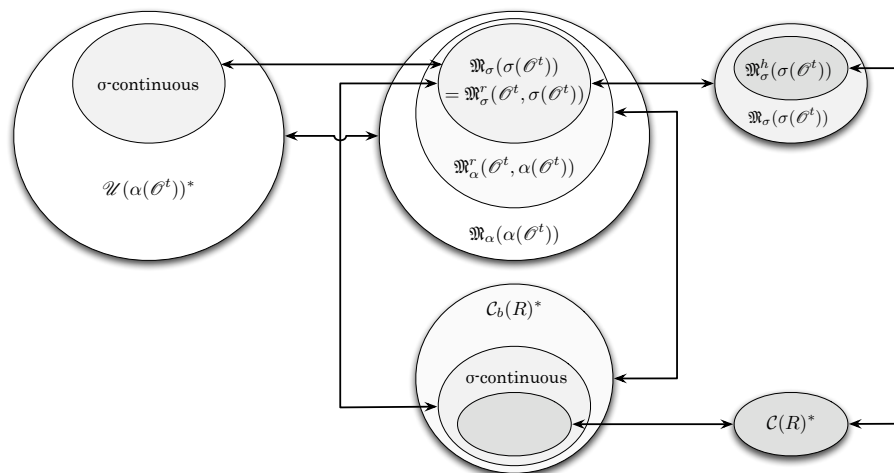
Figure 5.1: On the identification of contents on the Baire algebra $\alpha(\mathscr{O}^t)$ and Baire σ-algebra $\sigma(\mathscr{O}^t)$ of an A-topological space $(R, \mathscr{O})$, with the duals of various function spaces. Double-sided arrows represent isometrical isomorphisms.

Under these identifications, σ-additivity of contents is equivalent to σ-continuity of the respective bounded, linear functionals (theorems 4.1.2 and 4.2.5). As a special case, the Riesz representation theorem 4.2.9 is obtained for countably compact, A-topological spaces. Another side-result was the isometrical identification of charges on the Baire algebra and the Borel algebra on normal spaces (theorem 4.2.7).

In all three cases, the total variations of the contents correspond to the total variations of the functionals induced by them, a fact underlining the strong similarity of the two concepts. Figure (5.1) illustrates a summary of the identifications established in section 1.4 and chapter 4, specialized to the Baire-sets of an A-topological space.

Appendix A

Appendix

This appendix provides with some secondary statements and technical details used in this paper. They are meant to be understood within the respective context of the paper.

A.1   A-topological spaces

A.1.1   Lemma: Characterization of topological spaces

Let $(R, \mathscr{O})$ be an A-topological space.

1. $\beta \subseteq \mathscr{O}$ is an A-subbase $\Leftrightarrow$ The system of all finite intersections of β is an A-base.

2. If some A-subbase $\beta \subseteq \mathscr{O}$ is countable, the space is 2nd-countable.

3. If $(R, \mathscr{O})$ is 2nd-countable, it is topological.

4. If $(R, \mathscr{O})$ is topological, every A-subbase is also a subbase and every A-base a base.

5. If $(R, \mathscr{O})$ is 2nd countable and thus topological, every subbase is an A-subbase and every base is an A-base.

Proof

1. Follows directly from eq. (1.2).

2. If β is countable, so is the system of finite intersections of β, which by claim (1) is an A-base.

3. Let $O_i \in \mathscr{O}$, $i \in I$ be an arbitrary family of open sets and $\beta \subseteq \mathscr{O}$ a countable A-base. Then each of them is union of sets in β, and so is their union $O := \bigcup_{i \in I} O_i$. But since β is countable, O is merely a countable union of open sets, thus $O \in \mathscr{O}$.

4. Follows from the fact that for a given system $\beta \subseteq \mathscr{P}(R)$, generated topologies are generally larger than generated A-topologies.

5. Since bases result from subbases exactly like A-bases result from A-subbases, by claim (1) it suffices to show that every base $\beta \subseteq \mathscr{O}$ is an A-base. Indeed, let $(O_i)_{i \in I} \subseteq \beta$, then it is a known fact about 2nd-countable topological spaces, that $O := \bigcup_{i \in I} O_i$ can also be written as a union of a countable sub-family of $(O_i)_{i \in I}$. Thus every $O \in \mathscr{O}$ is a countable union of sets in β.

$\square$

A.1.2 Lemma about continuous functions and product-topologies

Let $(T_n, \mathscr{O}_n)$, $n \in \mathbb{N}$ be 2nd-countable, topological spaces, $(R, \mathscr{O})$ an A-topological space and $f_n : R \to T_n$ continuous functions. Then

$$f := (f_n)_{n=1}^{\infty} : R \to \underset{n=1}{\overset{\infty}{\times}} T_n =: T \tag{A.1}$$

is continuous with respect to the product-topology of T.

Proof

It is a known fact, that T is also 2nd-countable. Thus by A.1.1(5), to prove the continuity of f, it suffices to show that $f^{-1}(\beta) \subseteq \mathscr{O}$ for some subbase β of T. One such subbase is

$$\beta := \left\{ \underset{n=1}{\overset{\infty}{\times}} O_n : O_n \in \mathscr{O}_n \ \wedge \ O_n = T_n \text{ for all but one } n \right\} \tag{A.2}$$

Indeed, let $O_n \in \mathscr{O}_n$ and $O_n = T_n \ \forall \ n \neq n_0$ for some $n_0 \in \mathbb{N}$, then

$$f^{-1}(\underset{n=1}{\overset{\infty}{\times}} O_n) = f_{n_0}^{-1}(O_{n_0}) \in \mathscr{O} \tag{A.3}$$

which proves the assertion.

$\square$

A.1.3 Characterization of open sets in A-topological spaces

Let $(R, \mathscr{O})$ be an A-topological space and $O \subseteq R$. Then the following are equivalent:

1. O is open.

2. There exists a countable subsystem $\mathscr{O}_0 \subseteq \mathscr{O}$ such that for every $x \in O$ there exists an $U \in \mathscr{O}_0$ such that $x \in U \subseteq O$.

Proof

1$\Rightarrow$2: Trivial, since $\{O\}$ is such a system.

2$\Rightarrow$1: For each $x \in O$ let $x \in U_x \subseteq O$ for some $U_x \in \mathscr{O}_0$. Then

$$O = \underbrace{\bigcup_{x \in O} U_x}_{\substack{\text{countable} \\ \text{union}}} \ , \tag{A.4}$$

hence O is open.

$\square$

A.2 Regular sequences

A.2.1 Lemma: Existence of regular sequences

Let $(R, \mathscr{F})$ be a perfectly normal, A-topological space and $(F_n) \subseteq \mathscr{F}$ a sequence of closed sets such that $F_n \downarrow \emptyset$. Then there exists a regular sequence $\widetilde{F}_n \subseteq F_n^c$.

Proof

The following proof originates from Varadarajan[13]. Choose $f_n \in \mathcal{C}_b(R)$ such that

$$0 \leq f_n \leq 1 \quad , \quad F_n = f_n^{-1}(\{0\}) \tag{A.5}$$

and set

$$\widetilde{f}_n := \max\{f_1, .., f_n\} \in \mathcal{C}_b(R) \quad . \tag{A.6}$$

Then $0 \leq \widetilde{f}_n \leq 1$ and, since (F_n) decreases, $F_n = \widetilde{f}_n^{-1}(\{0\})$. Define

$$\widetilde{F}_n := \left\{ \widetilde{f}_n \geq \tfrac{1}{n} \right\} \quad , \quad \widetilde{G}_n := \left\{ \widetilde{f}_n > \tfrac{1}{n+1} \right\} \quad , \quad n \in \mathbb{N} \tag{A.7}$$

then $\widetilde{F}_n$ are closed, $\widetilde{G}_n$ are open and obviously $\widetilde{F}_n \subseteq F_n^c$. Since $(\widetilde{f}_n)$ increases:

$$\widetilde{F}_n \subseteq \widetilde{G}_n \subseteq \widetilde{F}_{n+1} \quad . \tag{A.8}$$

Now let $x \in R$, then by assumption $x \notin F_{n_0}$ for some $n_0 \in \mathbb{N}$, that is, $\widetilde{f}_{n_0}(x) =: \delta > 0$. Choose $n \geq n_0$ so large that $\delta > 1/n$, then

$$\widetilde{f}_n(x) \overset{n \geq n_0}{\geq} \widetilde{f}_{n_0}(x) \geq \frac{1}{n} \tag{A.9}$$

and thus $x \in \widetilde{F}_n$. This shows $\widetilde{F}_n \uparrow R$.
□

A.2.2 Lemma: Characterization of closed sets through regular sequences

Let $(R, \mathscr{F})$ be an A-topological space and $F_n \uparrow R$ a regular sequence. Then:

1. A set $C \subseteq R$ is closed $\Leftrightarrow C \cap F_n$ is closed (in R) $\forall\, n \in \mathbb{N}$.

2. A function $f : R \to T$ (where T is a topological space) is continuous $\Leftrightarrow f\big|_{F_n} : F_n \to T$ is continuous $\forall\, n \in \mathbb{N}$.

Proof

The proof originates from Varadarajan[13].

1. Direction "$\Rightarrow$" is evident.
   Direction "$\Leftarrow$": Suppose all $C \cap F_n$ are closed. Let $G_n \subseteq R$, $n \in \mathbb{N}$ be open such that $F_n \subseteq G_n \subseteq F_{n+1}$. Let $x \in C^c$, then $x \in G_n$ for some $n \in \mathbb{N}$. Thus

   $$x \in \underbrace{G_n \cap (C \cap F_{n+1})^c}_{\text{open}} \subseteq C^c \quad . \tag{A.10}$$

   This shows: Each point in C^c is contained within some open $U \subseteq C^c$, whereas U can be chosen from the <u>countable</u> system

   $$\{G_n \cap (C \cap F_{n+1})^c : n \in \mathbb{N}\} \quad . \tag{A.11}$$

   By A.1.3, this implies that C^c is open.

2. Direction "$\Rightarrow$" is evident.
   Direction "$\Leftarrow$": To be shown is, that for closed $A \subseteq T$ the origin $f^{-1}(A)$ is closed. By assumption, all $f^{-1}(A) \cap F_n = f\big|_{F_n}^{-1}(A)$ are closed[1] . But by part (1), this is indeed sufficient.

   □

---

[1]Note that, since F_n are closed, a subset $B \subseteq F_n$ is closed in F_n if and only if it is closed in R.

A.2.3 Lemma: Continuous extensions of functions

Let R, S be A-topological spaces, $F \subseteq R$ closed, $G \subseteq R$ open and $f, g : R \to S$ continuous, such that $G \subseteq F$ and

$$f\big|_{F \setminus G} = g\big|_{F \setminus G} \quad . \tag{A.12}$$

Then the function defined by

$$h(x) := \begin{cases} g(x) & : x \in F \\ f(x) & : x \in F^c \end{cases} \tag{A.13}$$
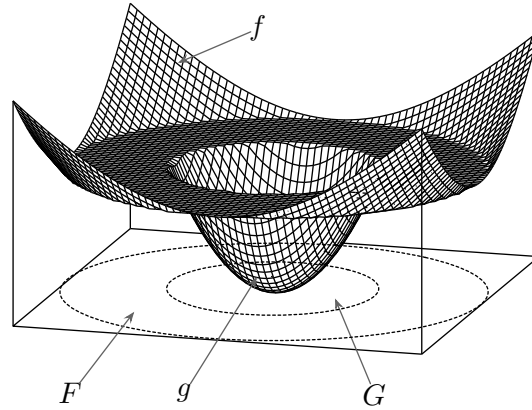
is also continuous.

Figure A.1: On continuous extensions of functions: Functions f, g, matching on the domain $F \setminus G$, can be continuously *attached.*

Proof

Let $A \subseteq S$ be closed, then the origin

$$h^{-1}(A) = \{x \in F : h(x) \in A\} \cup \{x \in G^c : h(x) \in A\}$$

$$= \underbrace{\left[g^{-1}(A) \cap F \right]}_{\text{closed}} \cup \underbrace{\left[f^{-1}(A) \cap G^c \right]}_{\text{closed}} \tag{A.14}$$

is also closed.

□

A.2.4 Lemma: Characterization of regular sequences

Let $(R, \mathscr{F})$ be a perfectly normal, A-topological space, $F_n \subseteq R$, $n \in \mathbb{N}$ and $\varkappa_1 < \varkappa_2 < ... \in \mathbb{R}$ such that $\varkappa_n \uparrow \infty$. Then F_n is a regular sequence, iff for some continuous $f \in \mathcal{C}(R)$:

$$F_n = \{f \leq \varkappa_n\} \quad , \quad n \in \mathbb{N} \quad . \tag{A.15}$$

Proof

The following proof is taken from Varadarajan[13].

Sufficiency of the condition is evident. Now let (F_n) be regular, that is F_n are closed and $F_n \subseteq G_n \subseteq F_{n+1} \uparrow R$ for some open G_n, $n \in \mathbb{N}$.

Claim: There exist functions $f_n \in \mathcal{C}(F_n)$ such that

(i) $f_n\big|_{F_1} = \varkappa_1,\ f_n\big|_{F_2 \setminus G_1} = \varkappa_2,\ ..,\ f_n\big|_{F_n \setminus G_{n-1}} = \varkappa_n$

(ii) $\varkappa_k < f_n\big|_{G_k \setminus F_k} < (\varkappa_{k+1})$ for $k = 1, .., n-1$

(iii) f_n is an extension of f_{n-1}.

> **Proof:** Set $f_1 \equiv \varkappa_1$ on F_1. Now suppose $f_1, .., f_n \in \mathcal{C}(F_n)$ satisfy (i), (ii) and (iii). Since the closed sets $(F_n \setminus G_{n-1})$ and $(F_{n+1} \setminus G_n)$ are disjoint, by perfect normality of R there exists a function $g \in \mathcal{C}(F_{n+1})$ such that $\varkappa_n \le g \le \varkappa_{n+1}$ and
>
> $$F_n \setminus G_{n-1} = g^{-1}(\{\varkappa_n\}) \quad,\quad F_{n+1} \setminus G_n = g^{-1}(\{\varkappa_{n+1}\}) \ . \tag{A.16}$$
>
> Then by A.2.3,
>
> $$f_{n+1}(x) := \begin{cases} f_n(x) & : x \in F_n \\ g(x) & : x \in F_{n+1} \setminus F_n \end{cases} \quad,\quad x \in F_{n+1} \tag{A.17}$$
>
> defines $f_{n+1} \in \mathcal{C}(F_{n+1})$ as an extension of f_n, satisfying (i) and (ii).

Now define $f : R \to \mathbb{R}$ by setting $f\big|_{F_n} = f_n$. By property (iii), f is indeed well defined and by A.2.2(2) continuous. Evidently $F_n = \{f \le \varkappa_n\}$.
$\square$

A.3 Function spaces

A.3.1 An interesting counter-example for linear functionals

The following example shall show that, a bounded linear functional L on a grounded function space Φ, need **not** satisfy the inequality

$$|Lf| \le \big||L|\,f\big| \ ,\quad f \in \Phi \ . \tag{A.18}$$

Consider the space of simple functions $\Phi := \mathscr{S}(\mathscr{B}_\sigma([0,1]))$ on the Borel-σ-algebra of $[0,1]$. Let L be the linear functional induced by the measure $\mu := \delta_0 - \delta_1$ on $\mathscr{B}_\sigma([0,1])$ by means of integration and $f := 1_{\{1\}} - 1_{\{0\}} \in \Phi$. Then

$$|Lf| = \left|\int f\,d\mu\right| = 2 \not\le 0 = \left|\int f\,d(\delta_0 + \delta_1)\right| = \left|\int f\,d\,|\mu|\right| = \big||L|\,f\big| \tag{A.19}$$

shows exactly the claim.

A.3.2 Lemma: Uniform approximability from below

Let $\mathfrak{S}$ be a set algebra over the set Ω and $f \in \mathscr{U}(\mathfrak{S})$ uniformly approximable. Then there exists a non-decreasing sequence $f_n \in \mathscr{S}(\mathfrak{S})$ of simple functions, such that $f_n \le f$ and $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$.

Proof

Let $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{S}(\mathfrak{S})$ be such that $g_n \xrightarrow[\|\cdot\|_\infty]{n\to\infty} f$, whereas

$$g_n = \sum_{i=1}^{N_n} g_n^i \cdot 1_{A_n^i} \quad, \quad \Omega = \biguplus_{i=1}^{N_n} A_n^i \quad, \quad A_n^i \in \mathfrak{S}, \ n \in \mathbb{N} \quad. \tag{A.20}$$

Define

$$h_n := \sum_{i=1}^{N_n} 1_{A_n^i} \cdot \underbrace{\inf f(A_n^i)}_{=:h_n^i} \quad, \quad n \in \mathbb{N} \quad. \tag{A.21}$$

Then

$$\sup_{\omega \in A_n^i} \left| h_n^i - f(\omega) \right| \leq 2 \cdot \sup_{\omega \in A_n^i} \left\| f(\omega) - g_n^i \right\| \leq 2 \cdot \|f - g_n\|_\infty \tag{A.22}$$

and thus

$$\|h_n - f\|_\infty \leq 2 \cdot \|g_n - f\|_\infty \xrightarrow{n\to\infty} 0 \quad. \tag{A.23}$$

Obviously $h_n \leq f$.

Now set $f_n := \max\{h_1, .., h_n\}$, then

$$f_1 \leq f_2 \leq \cdots \leq f \quad, \quad f_n \xrightarrow[\|\cdot\|_\infty]{n\to\infty} f \quad. \tag{A.24}$$

□

A.3.3 Lemma: Uniqueness of linear functionals

Let Φ be a grounded function space and H, L linear functionals on Φ, such that $H1 = L1$ and $Hf \leq Lf$ whenever $0 \leq f \in \Phi$. Then $H = L$.

Proof

Let for a start $0 \leq f \in \Phi$, then $0 \leq \sup f - f =: g \in \Phi$ and thus by assumption

$$Hf \leq Lf = \sup f \cdot \underbrace{L1}_{H1} - \underbrace{Lg}_{\geq Hg} \leq \sup f \cdot H1 - Hg = Hf \quad. \tag{A.25}$$

Since for any other $f \in \Phi$, its positive and negative parts are also in Φ, by linearity it follows that $Lf = Hf$.

□

A.4 Set functions

A.4.1 Definition: μ-set

Let Ω be a set, $\mathfrak{S} \subseteq \mathscr{P}(\Omega)$ a set-algebra and μ a set-function on $\mathfrak{S}$. Then $M \in \mathfrak{S}$ is called a μ-set[7] if

$$\mu(A) = \mu(A \cap M) + \mu(A \cap M^c) \quad \forall A \in \mathfrak{S} \quad. \tag{A.26}$$

We shall write $\mathfrak{S}_\mu$ for the system of all μ-sets in $\mathfrak{S}$.

A.4.2 Lemma: μ-sets as an algebra

Let Ω be a set, $\mathfrak{S} \subseteq \mathscr{P}(\Omega)$ a set-algebra and μ a set-function on $\mathfrak{S}$. Then $\mathfrak{S}_\mu \neq \emptyset$ iff $\mu(\emptyset) = 0$. In this case, $\mathfrak{S}_\mu$ is an algebra, on which μ is additive.

Proof

The following proof is taken from Alexandroff[2]. Obviously $M^c \subseteq \mathfrak{S}_\mu$ for any $M \in \mathfrak{S}_\mu$. Now let $M, N \in \mathfrak{S}_\mu$ and $A \in \mathfrak{S}$, then

$$\mu(A) \stackrel{N \in \mathfrak{S}_\mu}{=} \mu(A \cap N) + \mu(A \cap N^c) \tag{A.27}$$

and

$$\mu(A \cap N) \stackrel{M \in \mathfrak{S}_\mu}{=} \mu(A \cap N \cap M) + \mu(A \cap N \cap M^c) \tag{A.28}$$

and

$$\mu(A \cap (M \cap N)^c) \stackrel{N \in \mathfrak{S}_\mu}{=} \mu((A \cap (M \cap N)^c \cap N) + \mu((A \cap (M \cap N)^c \cap N^c)$$

$$= \mu(A \cap N \cap M^c) + \mu(A \cap N^c) \quad . \tag{A.29}$$

Substituting $\mu(N \cap A)$ from (A.28) and $\mu(A \cap N^c)$ from (A.29) into the RHS of (A.27) yields

$$\mu(A) = \mu(A \cap N \cap M) + \mu(A \cap (N \cap M)^c) \quad , \tag{A.30}$$

that is, $M \cap N \in \mathfrak{S}_\mu$.

Note that $\mu(\emptyset) = 0$ is equivalent to $\emptyset \in \mathfrak{S}_\mu$. As was shown, $\mathfrak{S}_\mu$ is closed under complements and intersections. If $N \in \mathfrak{S}_\mu$ then $\emptyset = N \cap N^c \in \mathfrak{S}_\mu$ and $\mathfrak{S}_\mu$ is an algebra. On the other hand, if $\mu(\emptyset) = 0$ then $\mathfrak{S}_\mu$ is non empty.

Now let $N, M \in \mathfrak{S}_\mu$ be disjoint, then

$$\mu(N \cup M) \stackrel{N \in \mathfrak{S}_\mu}{=} \mu((N \cup M) \cap N) + \mu((N \cup M) \cap N^c) = \mu(N) + \mu(M) \tag{A.31}$$

shows that μ is additive on $\mathfrak{S}_\mu$.
$\square$

A.4.3 Lemma: Sufficient condition for additivity

Let Ω be a set, $\mathscr{F} \subseteq \mathscr{P}(\Omega)$ and $\mathfrak{S} := \alpha(\mathscr{F})$. Let μ be a set function on $\mathfrak{S}$ such that:

$$\mu(A \cup \widetilde{A}) = \mu(A) + \mu(\widetilde{A}) \quad , \tag{A.32}$$

whenever $A, \widetilde{A} \in \mathfrak{S}$ and $F \in \mathscr{F}$ with $A \subseteq F$, $\widetilde{A} \subseteq F^c$. Then μ is additive on $\mathfrak{S}$.

Proof

By lemma A.4.2 it suffices to show that $\mathscr{F} \subseteq \mathfrak{S}_\mu$. Let $F \in \mathscr{F}$ and $A \in \mathfrak{S}$, then

$$A \cap F \subseteq F \quad , \quad A \cap F^c \subseteq F^c \tag{A.33}$$

and thus by assumption

$$\mu(A) = \mu((A \cap F) \cup (A \cap F^c)) = \mu(A \cap F) + \mu(A \cap F^c) \quad , \tag{A.34}$$

which was to be shown.
$\square$

A.4.4   Definition: Outer content

Let $(\Omega, \mathfrak{S}, \mu)$ be a non-negative content space. Then the set function $\mu^* : \mathscr{P}(\Omega) \to \mathbb{R}$, defined by

$$\mu^*(A) := \inf_{\substack{B \in \mathfrak{S} \\ A \subseteq B}} \mu(B) \quad , \quad A \subseteq \Omega \tag{A.35}$$

is called the *outer content, induced by μ*. Note that μ^* is generally not additive!

A.4.5   Lemma: Properties of the outer content

Let $(\Omega, \mathfrak{S}, \mu)$ be a non-negative content space. Then the outer content μ^* induced by μ satisfies:

1. $\mu^*(A) = \mu(A)$ for every $A \in \mathfrak{S}$.

2. Sub-additivity: $\mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$ for all $A, B \subseteq \Omega$.

3. Monotony: $\mu^*(A) \le \mu^*(B)$ for all $A \subseteq B \subseteq \Omega$.

Proof

1. Follows directly from the monotony of μ and the definition of μ^*.

2. Let $A \subseteq \widetilde{A}$ and $B \subseteq \widetilde{B}$ for some $\widetilde{A}, \widetilde{B} \in \mathfrak{S}$. Then $A \cup B \subseteq \widetilde{A} \cup \widetilde{B} \in \mathfrak{S}$ and thus

$$\mu^*(A \cup B) \le \mu(\widetilde{A} \cup \widetilde{B}) \le \mu(\widetilde{A}) + \mu(\widetilde{B}) \quad . \tag{A.36}$$

   Consequently,

$$\mu^*(A \cup B) \le \inf_{A \subseteq \widetilde{A} \in \mathfrak{S}} \mu(\widetilde{A}) + \inf_{B \subseteq \widetilde{B} \in \mathfrak{S}} \mu(\widetilde{B}) = \mu^*(A) + \mu^*(B) \quad . \tag{A.37}$$

3. Follows from property (2).

$\square$

A.4.6   Lemma: Approximate Hahn decomposition

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space and $\varepsilon > 0$. Then there exists a set $H \in \mathfrak{S}$ such that

$$\mu^+(H^c) \le \varepsilon \quad , \quad \mu^-(H) \le \varepsilon \tag{A.38}$$

and in particular

$$\left| \mu^+(A) - \mu(A \cap H) \right| \le 2\varepsilon \quad , \quad \left| \mu^-(A) - \mu(A \cap H^c) \right| \le 2\varepsilon \quad \forall\, A \in \mathfrak{S} \quad . \tag{A.39}$$

Note how this is simply a generalization of the Hahn decomposition theorem for measures. It expresses the notion, that the positive and negative parts of a content, somehow *live* on disjoint regions of the underlying space.

Proof

Choose $H \in \mathfrak{S}$ such that $\mu(H) \geq \mu^+(\Omega) - \varepsilon$. Then clearly

$$\mu^+(H^c) = \mu^+(\Omega) - \underbrace{\mu^+(H)}_{\geq \mu(H)} \leq \mu^+(\Omega) - \mu(H) \leq \varepsilon \quad , \tag{A.40}$$

and since $\mu = \mu^+ - \mu^-$:

$$\mu^-(H) \leq \underbrace{\mu^+(H) - \mu^+(\Omega)}_{\leq 0} + \varepsilon \leq \varepsilon \quad . \tag{A.41}$$

Now let $A \in \mathfrak{S}$, then

$$\left| \mu^+(A) - \mu(A \cap H) \right| \leq \left| \cancel{\mu^+(A \cap H)} + \mu^+(A \cap H^c) - \cancel{\mu^+(A \cap H)} + \mu^-(A \cap H) \right|$$

$$\leq \mu^+(A \cap H^c) + \mu^-(A \cap H) \leq 2\varepsilon \quad . \tag{A.42}$$

The claim for μ^- follows in a similar way.
□

A.4.7 Lemma: Sufficient condition for regularity

Let $(R, \mathscr{O})$ be an A-topological space and μ a non-negative, bounded content on $\mathscr{B}_\alpha(R)$, such that for every open $G \subseteq R$:

$$\mu(G) = \sup_{\substack{F \text{ closed} \\ F \subseteq G}} \mu(F) \quad . \tag{A.43}$$

Then μ is regular.

Proof

The following proof is due to Alexandroff[2]. Let K_0 be the system of all open and closed subsets of $(R, \mathscr{F})$. For $n \in \mathbb{N}_0$, define K_{n+1} as the system of all finite unions of sets in K_n and their complements. We shall by induction show that, for any $A \in K_n$, $n \in \mathbb{N}_0$:

$$\mu(A) = \sup_{\substack{F \text{ closed} \\ F \subseteq A}} \mu(F) \tag{A.44}$$

and

$$\mu(A) = \inf_{\substack{G \text{ open} \\ G \supseteq A}} \mu(G) \quad . \tag{A.45}$$

Since $\mathscr{B}_\alpha(R) = \bigcup_{n=0}^\infty K_n$, this will complete the proof.

It should be noted that, $A \subseteq R$ satisfies (A.44) iff A^c it satisfies A.45. Since all closed sets satisfy (A.44) and by assumption so do all open ones, all sets in K_0 satisfy both (A.44) and (A.45).

Now let (A.44) and (A.45) be satisfied by all sets in K_n. Let $A_1, .., A_n \in K_n$, $A := \bigcup_{i=1}^n A_i$ and $\varepsilon > 0$. Choose closed $F_i \subseteq A_i$ such that $\mu(A_i \setminus F_i) \leq \varepsilon/n$, then $F := \bigcup_{i=1}^n F_i \subseteq A$ is closed and satisfies

$$\mu(A \setminus F) = \mu\left(\bigcup_{i=1}^n A_i \setminus F \right) \leq \mu\left(\bigcup_{i=1}^n A_i \setminus F_i \right) \leq \sum_{i=1}^n \mu(A_i \setminus F_i) \leq \varepsilon \quad . \tag{A.46}$$

Similarly, if $G_i \supseteq A_i$ are open, such that $\mu(G_i \setminus A_i) \leq \varepsilon/n$, then $G := \bigcup_{i=1}^n G_i \supseteq A$ is open and satisfies $\mu(G \setminus A) \leq \varepsilon$. Hence, A (and thus A^c) satisfies (A.44) and (A.45). Consequently, all sets in K_{n+1} satisfy (A.44) and (A.45).
□

A.5 Integration

A.5.1 Lemma: Integration with restricted contents

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space, E a $\mathbb{K}$-Banach space and $A \in \mathfrak{S}$. Define the content $\mu|_A(B) := \mu(A \cap B)$, $B \in \mathfrak{S}$. Then:

$$\int_A f \, d\mu = \int f \, d\mu|_A \quad \forall \, f \in \mathscr{U}(\mathfrak{S}, E) \ . \tag{A.47}$$

Proof

It suffices to show eq. (A.47) for simple functions[2] $f \in \mathscr{S}(\mathfrak{S}, E)$. By linearity of both sides, it suffices to show the equality for indicator functions 1_B, $B \in \mathfrak{S}$. But indeed,

$$\int_A 1_B \, d\mu = \int 1_{A \cap B} \, d\mu = \mu(A \cap B) = \mu|_A(B) = \int 1_B \, d\mu|_A \ . \tag{A.48}$$

$\square$

A.5.2 Lemma: Continuity of the integral

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space, E a Banach space, $f_n \in \mathscr{S}(\mathfrak{S}, E)$ simple functions such that $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f \in \mathscr{U}(\mathfrak{S}, E)$ and $A, A_n \in \mathfrak{S}$ such that $|\mu|(A \Delta A_n) \xrightarrow{n \to \infty} 0$. Then

$$\int_{A_n} f_n \, d\mu \xrightarrow{n \to \infty} \int_A f \, d\mu \ . \tag{A.49}$$

Proof

Using theorem 3.1.4, we may estimate

$$\left\| \int_{A_n} f_n \, d\mu - \int_A f \, d\mu \right\| = \left\| \int_{A \cap A_n} f_n \, d\mu + \int_{A^c \cap A_n} f_n \, d\mu - \int_{A \cap A_n} f \, d\mu - \int_{A \cap A_n^c} f \, d\mu \right\|$$

$$\leq \left\| \int_{A \cap A_n} f_n \, d\mu - \int_{A \cap A_n} f \, d\mu \right\| + \left\| \int_{A^c \cap A_n} f_n \, d\mu \right\| + \left\| \int_{A \cap A_n^c} f \, d\mu \right\|$$

$$\leq \underbrace{\|f_n - f\|_\infty}_{\xrightarrow{n \to \infty} 0} \cdot |\mu|(A \cap A_n) + \underbrace{\|f_n\|_\infty}_{\text{bounded}} \cdot \underbrace{|\mu|(A^c \cap A_n)}_{\xrightarrow{n \to \infty} 0} + \underbrace{\|f\|_\infty}_{< \infty} \cdot \underbrace{|\mu|(A \cap A_n^c)}_{\xrightarrow{n \to \infty} 0} \xrightarrow{n \to \infty} 0 \tag{A.50}$$

which completes the proof.

$\square$

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[2]Note that if $f_n \in \mathscr{S}(\mathfrak{S}, E)$ are such that $f_n \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f$ for some $f \in \mathscr{U}(\mathfrak{S}, E)$, then $f_n \cdot 1_A \xrightarrow[\|\cdot\|_\infty]{n \to \infty} f \cdot 1_A$ for any $A \in \mathfrak{S}$.

A.5.3   Theorem: Characterization of vanishing integrals

Let $(\Omega, \mathfrak{S}, \mu)$ be a bounded content space and $f \in \mathcal{U}(\mathfrak{S})$. Then the following are equivalent:

1. For every $A \in \mathfrak{S}$: $\int_A f \, d\mu = 0$.

2. For every $A \in \mathfrak{S}$: $\int_A f^+ \, d\mu = 0$ and $\int_A f^- \, d\mu = 0$.

3. For every $A \in \mathfrak{S}$: $\int_A |f| \, d\mu = 0$.

4. For every $A \in \mathfrak{S}$: $\int_A f \, d\mu^+ = 0$ and $\int_A f \, d\mu^- = 0$.

5. For every $A \in \mathfrak{S}$: $\int_A f \, d|\mu| = 0$.

6. For every $\varepsilon > 0$: $|\mu|^* \left(\{ |f| \geq \varepsilon \} \right) = 0$, with $|\mu|^*$ being the outer content induced by $|\mu|$.

Proof

1⇒2: Let $A \in \mathfrak{S}$ and $\varepsilon > 0$. Choose simple $g \in \mathscr{S}(\mathfrak{S})$ such that $\|f - g\|_\infty \leq \min\{\varepsilon, \varepsilon/|\mu|(\Omega)\}$ and define $A^+ := \{g \geq 0\} \cap A$. Note that $A^+ \in \mathfrak{S}$, since g is simple. Then

$$\left| \int_A f^+ \, d\mu \right| \leq \underbrace{\left| \int_A g^+ \, d\mu \right|}_{\substack{\int_{A^+} g \, d\mu}} + \underbrace{\|f^+ - g^+\|_\infty \cdot |\mu|(A)}_{\leq \varepsilon} \leq \left| \int_{A^+} g \, d\mu \right| + \varepsilon$$

$$\leq \underbrace{\left| \int_{A^+} f \, d\mu \right|}_{0} + \underbrace{\|f - g\|_\infty \cdot |\mu|(A^+)}_{\leq \varepsilon} + \varepsilon \leq 2\varepsilon \quad . \tag{A.51}$$

Since $\varepsilon > 0$ was arbitrary, $\int_A f^+ \, d\mu = 0$. Similarly, one shows $\int_A f^- \, d\mu = 0$.

2⇒1: Follows from $f = f^+ - f^-$.

2⇒3: Follows from $|f| = f^+ + f^-$.

4⇒1: Follows from $\mu = \mu^+ - \mu^-$.

4⇒5: Follows from $\mu = \mu^+ + \mu^-$.

5⇒1: Due to (1⇒2⇒3), also $\int_A |f| \, d|\mu| = 0$. Due to 3.1.4(3), this implies $\int_A f \, d\mu = 0$.

3⇒4: Since $\left| \int_A f \, d\mu^\pm \right| \leq \int_A |f| \, d\mu^\pm$, we can w.l.o.g. assume $f \geq 0$. Let $\varepsilon > 0$ and set $\delta := \min\{\varepsilon, \varepsilon/\|f\|_\infty\}$. By A.4.6 we may choose an $H \in \mathfrak{S}$, such that $\mu^+(H^c) \leq \delta$ and $\mu^-(H) \leq \delta$. Then

$$\left| \int_A f \, d\mu^+ \right| \leq \underbrace{\left| \int_{A \cap H} f \, d\mu^+ \right|}_{\substack{= \int_{A \cap H} f \, d\mu^- \\ + \int_{A \cap H} f \, d\mu}} + \underbrace{\left| \int_{A \cap H^c} f \, d\mu^+ \right|}_{\leq \|f\|_\infty \cdot \mu^+(H^c) \leq \varepsilon}$$

$$\leq \underbrace{\left| \int_{A \cap H} f \, d\mu^- \right|}_{\leq \|f\|_\infty \cdot \mu^-(H) \leq \varepsilon} + \underbrace{\left| \int_{A \cap H} f \, d\mu \right|}_{0} + \varepsilon \leq 2\varepsilon \quad , \tag{A.52}$$

and by arbitrarity of $\varepsilon > 0$, $\int_A f \ d\mu^+ = 0$. From $\mu^- = \mu^+ - \mu$, also follows $\int_A f \ d\mu^- = 0$.

1⇒6: By all previous claims, we may w.l.o.g. assume $f \geq 0$, $\mu \geq 0$. Suppose $\rho := \mu^*(\{f \geq \varepsilon\}) > 0$. Choose a simple $0 \leq g \in \mathscr{S}(\mathfrak{S})$ such that $\|f - g\|_\infty \leq \min\left\{\frac{\varepsilon}{4}, \frac{\rho\varepsilon}{4\mu(\Omega)}\right\}$. Then the set $F := \left\{g \geq \frac{3\varepsilon}{4}\right\} \in \mathfrak{S}$ contains $\{f \geq \varepsilon\}$ and thus $\mu(F) = \mu^*(F) \geq \rho$. But this implies

$$0 = \left|\int f \ d\mu\right| \geq \underbrace{\left|\int g \ d\mu\right|}_{\geq \mu(F) \cdot \frac{3\varepsilon}{4}} - \underbrace{\|f - g\|_\infty \cdot \mu(\Omega)}_{\leq \frac{\rho\varepsilon}{4}} \geq \frac{2\rho\varepsilon}{4} \quad , \tag{A.53}$$

which is a contradiction!

6⇒1: Since $\left|\int_A f \ d\mu\right| \leq \int |f| \ d|\mu|$, we may w.l.o.g. assume $f \geq 0$, $\mu \geq 0$ and $A = \Omega$. Let $\varepsilon > 0$ and define $\delta := \min\{\varepsilon, \varepsilon/\mu(\Omega)\}$. Now choose $B \supseteq \{f \geq \delta\}$ such that $|\mu|(B) \leq \min\{\varepsilon, \varepsilon/\|f\|_\infty\}$. Then

$$\int f \ d\mu = \int_B f \ d\mu + \int_{B^c} f \ d\mu \leq \underbrace{\|f\|_\infty \cdot \mu(B)}_{\leq \varepsilon} + \underbrace{\delta \cdot \mu(\Omega)}_{\leq \varepsilon} \leq 2\varepsilon \tag{A.54}$$

and by arbitrarity of $\varepsilon > 0$, $\int f \ d\mu = 0$.

$\square$

Appendix B

List of symbols and abbreviations

iff: If and only if.

w.l.o.g.: Without loss of generality.

RHS: Right hand side.

LHS: Left hand side.

$\mathscr{P}(\Omega)$**:** Power-set of some set Ω.

$\mathbb{K}$**:** Field of real or complex numbers.

$B_\varepsilon(x)$**:** Closed sphere with radius ε around the point x.

$B_\varepsilon^o(x)$**:** Open sphere with radius ε around the point x.

λ**:** One-dimensional Lebesgue measure.

LIM: Banach limit on l_∞.

1_A**:** Indicator function 1_A for some set A.

A^c**:** Complement of the set A.

$A_n \downarrow A$**:** Convergence of the decreasing sequence of sets $(A_n)_n$, to the set A.

$A_n \uparrow A$**:** Convergence of the increasing sequence of sets $(A_n)_n$, to the set A.

$f_n \downarrow f$**:** Pointwise convergence of the decreasing function-sequence $(f_n)_n$, to the real function f.

$f_n \uparrow f$**:** Pointwise convergence of the increasing function-sequence $(f_n)_n$, to the real function f.

$f \vee g$**:** Pointwise maximum of the two real functions f, g. Same as $\max\{f, g\}$.

$f \wedge g$**:** Pointwise minimum of the two real functions f, g. Same as $\min\{f, g\}$.

$f\big|_A$**:** Restriction of the function f to the set A.

$\{f \in B\}$**:** Origin of the set B with respect to the function f. Same as $f^{-1}(B)$.

$\|\cdot\|_\infty$**:** Supremum-norm. See 1.2.4.

$\sup f$**:** Supremum of the image of the real function f.

$\inf f$**:** Infimum of the image of the real function f.

f^+: Positive part of a real function f: $f^+ := \max\{f, 0\}$.

f^-: Negative part of a real function f: $f^- := (-f)^+$.

$\mathcal{L}(E, F)$: Space of bounded, linear operators between the normed spaces E, F.

$\|L\|$: Operator norm of the linear operator L.

Φ^*: The space of bounded, linear functionals on the function space Φ.

$(R, \mathscr{O})$: A-topological space with open sets $\mathscr{O}$. See 1.1.1.

$(R, \mathscr{F})$: A-topological space with closed sets $\mathscr{F}$. See 1.1.1.

$\mathscr{O}^t$: Totally open sets of some A-topological space $(R, \mathscr{O})$. See 1.3.3.

$\mathscr{F}^t$: Totally closed sets of some A-topological space $(R, \mathscr{F})$. See 1.3.3.

$\alpha(\mathcal{A})$: Set algebra generated by the set-family $\mathcal{A}$.

$\sigma(\mathcal{A})$: σ-Algebra generated by the set-family $\mathcal{A}$.

$\mathscr{B}_\alpha(R)$: Borel algebra of some A-topological space $(R, \mathscr{O})$. See 1.2.1.

$\mathscr{B}_\sigma(R)$: Borel σ-algebra of some A-topological space $(R, \mathscr{O})$. See 1.2.1.

$\alpha(\mathscr{O}^t)$: Baire algebra of some A-topological space $(R, \mathscr{O})$. See 1.3.2.

$\sigma(\mathscr{O}^t)$: Baire σ-algebra of some A-topological space $(R, \mathscr{O})$. See 1.3.2.

$\mathscr{S}(\mathfrak{S}, E)$: Space of simple functions $f : \Omega \to E$ with respect to some set algebra $\mathfrak{S}$. See 1.2.5.

$\mathscr{U}(\mathfrak{S}, E)$: Space of uniformly approximable functions $f : \Omega \to E$ with respect to some set algebra $\mathfrak{S}$. See 1.2.5.

$\mathcal{B}(\Omega, E)$: Space of bounded functions $f : \Omega \to E$. See 1.2.4.

$\mathcal{B}(\Omega, E)$: Space of totally bounded functions $f : \Omega \to E$. See 1.2.4.

$\mathcal{C}(R, \mathscr{O}, E)$: Space of continuous functions $f : R \to E$ on the A-topological space $(R, \mathscr{O})$. See 1.2.7.

$\mathcal{C}_b(R, \mathscr{O}, E)$: Space of bounded, continuous functions $f : R \to E$ on the A-topological space $(R, \mathscr{O})$. See 1.2.7.

$\mathcal{B}_t(\Omega, E)$: Space of totally bounded functions $f : \Omega \to E$. See 1.2.4.

L^+: Positive part of a linear functional L on a function-space. See 1.4.3.

L^-: Negative part of a linear functional L on a function-space. See 1.4.3.

$|L|$: Total variation of a linear functional L on a function-space. See 1.4.3.

$(\Omega, \mathfrak{S}, \mu)$: Content space over the set Ω, with set algebra $\mathfrak{S}$ and content μ. See 2.1.1.

μ^+: Positive part of content μ. See 2.1.4.

μ^-: Negative part of content μ. See 2.1.4.

$|\mu|$: Total variation of the content μ. See 2.1.4.

μ^*: Outer content induced by the non-negative content μ. See A.4.4.

$\mathfrak{M}_\alpha(\mathfrak{S})$: The space of bounded, additive set functions on the system $\mathfrak{S}$. See 2.1.1.

$\mathfrak{M}_\sigma(\mathfrak{S})$: The space of bounded, σ-additive set functions on the system $\mathfrak{S}$. See 2.1.1.

$\mathfrak{M}_\alpha^r(\mathscr{O}, \mathfrak{S})$: The space of charges on the set algebra $\mathfrak{S}$ over the space $(R, \mathscr{O})$. See 2.2.1.

$\mathfrak{M}_\sigma^r(\mathscr{O}, \mathfrak{S})$: The space of σ-additive charges on the set algebra $\mathfrak{S}$ over the space $(R, \mathscr{O})$. See 2.2.1.

$\mathfrak{M}_\sigma^h(\mathfrak{S})$: The space of Hewitt-measures on the σ-algebra $\mathfrak{S}$. See 4.3.1.

$\|\cdot\|_t$: Total variation norm defined on all bounded contents $\mathfrak{M}_\alpha$. See 2.1.6.

$\int(\cdot)\,d\mu$: The linear operator, mapping some function $f : \Omega \to E$ to the value $\int f\,d\mu$. See 3.1.4.

$\int$: The linear operator, mapping bounded contents to their respective linear mappings $\int(\cdot)\,d\mu$. See 3.1.4 and 4.3.4.

$\mathcal{G}(A)$: The directed system of continuous functions governing the set A. See 4.2.1.

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Acknowledgments

I would like to express my greatest gratitude to my supervisor Prof. Werner Linde, for his guidance and support.