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FREQUENCY COMPENSATION FOR A CLASS OF DAE'S ARISING IN ELECTRICAL CIRCUITS

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Abstract. Structured perturbations of regular pencils of the form $sE - A$, $E, A \in \mathbb{R}^{n\times n}$, $s \in \mathbb{C}$, are considered which model the addition of a capacitance c in an electrical circuit in order to improve the frequency response.

1. Introduction

In this note we consider structured perturbations of regular pencils of the form

$$sE - A, \qquad E, A \in \mathbb{R}^{n\times n},\ s \in \mathbb{C}, \tag{1}$$

which are related to design problems of electrical circuits. Linear electrical networks can be described by DAE's of the form $E\dot{x} = Ax$, with $E, A \in \mathbb{R}^{n\times n}$, where the entries of E are in particular determined by the capacitances of the network. The poles of the transfer function of the corresponding system are given by the eigenvalues of the pencil $sE - A$. To get a desired frequency response, the poles should be properly located in the left open halfplane $\mathbb{C}_-$ with a sufficient distance from the imaginary axis. A location on the rays $\{-t \pm it : t > 0\}$ would imply appropriate properties of the frequency response, see [4],[2]. It is the aim to improve the frequency behavior of the circuit by inserting additional capacitances between certain nodes. This gives, within the model, a structured perturbation of the matrix E by a matrix of the form

$$c(e_i - e_j)(e_i - e_j)^T,$$

where e_i, e_j are the standard unit vectors. The problem is to find the appropriate place, i.e., the node determined by e_i, e_j, and the numerical value of the capacitance $c > 0$, such that the eigenvalues of the perturbed pencil

$$s(E + c(e_i - e_j)(e_i - e_j)^T) - A$$

are in a position to improve the desired properties of the frequency response. It is shown in this note how the parameter c can be determined in terms of a rational function, see relation (6) below, which is

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in particular feasible when the matrix E is singular with a low rank. As a corollary we determine nodes which are not suitable.

2. The generalized Weierstrass form

Consider a pencil of the form (1). It is assumed that the pencil is regular, that is, the characteristic polynomial $P(s) = \det(sE - A)$ does not vanish identically on $\mathbb{C}$. Moreover, it is assumed that the pencil is of index 1, meaning that the degree of the characteristic polynomial $P(s)$ is equal to the rank of E. To be more specific, any regular pencil can be put into the so-called Weierstrass form, see [3], that is, to any regular pencil $sE - A$ with $E, A \in \mathbb{R}^{n\times n}$ there exist invertible matrices $\tilde{S}$ and $\tilde{T}$ in $\mathbb{C}^{n\times n}$ such that the matrices $\tilde{S}E\tilde{T}$ and $\tilde{S}A\tilde{T}$ are of block form

$$\tilde{S}E\tilde{T} = \begin{pmatrix} I_m & 0 \\ 0 & N \end{pmatrix}, \quad \tilde{S}A\tilde{T} = \begin{pmatrix} J & 0 \\ 0 & I_r \end{pmatrix} \tag{2}$$

with identity matrices $I_m \in \mathbb{R}^{m\times m}$ and $I_r \in \mathbb{R}^{r\times r}$ such that $r = n - m$, a nilpotent matrix N of index v, where v is the smallest nonnegative integer such that $N^v = 0$, and a matrix $J \in \mathbb{C}^{m\times m}$ which is in Jordan form. The number v is uniquely determined by the pencil $sE - A$ and it is called the index of the pencil. If E is invertible, then $n = m$ and $v = 0$. Instead of the Weierstrass form, where, in particular, in the calculation of the Jordan matrix J, numerical problems may occur, in this note the generalized Weierstrass form is used, which is numerically stable, see [1].

It is shown in [1] that there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ which transform the pencil into the so-called generalized Weierstrass form such that $\hat{E} = SET$ and $\hat{A} = SAT$ are of the block form

$$\hat{E} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} D & 0 \\ 0 & I_r \end{pmatrix} \tag{3}$$

with some matrix $D \in \mathbb{R}^{m\times m}$.

3. Main result

Assume now that the pencil $sE - A$ undergoes a structured perturbation of the form $scbb^*$ with $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ is a standard unit vector or the difference of two standard unit vectors, that is $b = e_i$ or $b = e_i - e_j$, $i, j \in \{1, \ldots, n\}$. Let $p = Sb$ and $q = T^*b$, then

$$S(sE - A + scbb^*)T = s\hat{E} - \hat{A} + scpq^*, \tag{4}$$

where $\hat{E}$ and $\hat{A}$ are in the generalized Weierstrass form (3).

Note that the relation (4) implies that

$$\det(s\hat{E} - \hat{A} + scpq^*) = \det S \det(sE - A + scbb^*) \det T,$$

and since $\det S$ and $\det T$ are nonzero constants, it follows that the pencils $s\hat{E} - \hat{A} + scpq^*$ and $sE - A + scbb^*$ have the same eigenvalues.

Theorem 3.1. *Assume that the pencil $s\hat{E} - \hat{A}$ is in the generalized Weierstrass form (3). Let $p, q \in \mathbb{R}^n$ be of the form $p = (p_0^T, p_1^T)^T$ and $q = (q_0^T, q_1^T)^T$ with $p_0, q_0 \in \mathbb{R}^m, p_1, q_1 \in \mathbb{R}^r$, then*

$$\begin{aligned}&(-1)^r \det(s\hat{E} - \hat{A} + scpq^*) = \\ &\qquad \det(sI_m - D) + sc(\det(sI_m - D + p_0q_0^*) - \\ &\qquad (1 + q_1^*p_1)\det(sI_m - D)).\end{aligned} \tag{5}$$

The next result shows some particular cases of perturbations of the original pencil. Note that the first item shows which perturbations have no influence on the eigenvalues of the original pencil.

Corollary 3.2. *For a pencil as in Theorem 3.1, with*

$$\mathrm{Adj}(sI_m - D) := (sI_m - D)^{-1}\det(sI_m - D),$$

the following hold.

(1) *If $p_0 = 0$ or $q_0 = 0$, then*

$$(-1)^r \det(s\hat{E} - \hat{A} + scpq^*) = \det(sI_m - D)(1 - scq_1^*p_1).$$

(2) *If $q_1 \perp p_1$, then*

$$(-1)^r \det(s\hat{E} - \hat{A} + scpq^*) = \det(sI_m - D) + scq_0^*\mathrm{Adj}(sI_m - D)p_0.$$

In order to find appropriate perturbations, denote

$$Q(s) = \det(sI_m - D + p_0q_0^*) - (1 + q_1^*p_1)\det(sI_m - D)$$

and

$$P(s) = \det(sI_m - D),$$

then the left hand side of (5) vanishes for some $c > 0$ if and only if $P(s) + scQ(s) = 0$. One has to find values $s \in \{-t \pm it : t > 0\}$ for which

$$c = -\frac{P(s)}{sQ(s)} > 0. \tag{6}$$

For such values, in particular, $\mathrm{Im}\, sQ(s)P(\bar{s})$ has to vanish. Appropriate values can be found by checking the sign changes of $\mathrm{Im}\, sQ(s)P(\bar{s})$ on the ray $\{-t + it : t > 0\}$.

Example 3.3. Assume that a pencil of the form (3) in $\mathbb{R}^{3\times 3}$ is given with $m = 2$, $r = 1$ and

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $a < 0$, $b > 0$. Then

$$P(s) = (s-a)^2 + b^2 = (s-s_0)(s-\bar{s}_0)$$

with $s_0 = a + ib$. Let the perturbation of $\hat{E}$ be of the form $c(e_1 - e_2)(e_1 - e_2)^T$, then $q = p = e_1 - e_2$, and via Corollary 3.2 it can be easily seen that

$$Q(s) = 2(s-a) = (s - s_0 + s - \bar{s}_0).$$

It follows from (6) that

$$\frac{1}{c} = \frac{s}{s_0 - s} + \frac{s}{\bar{s}_0 - s}.$$

A straightforward calculation implies that $\operatorname{Im}(P(s)/sQ(s))$ vanishes on the ray $s \in \{-t + it : t > 0\}$ if and only if $b \geq -a$. Actually, a quadratic equation for the appropriate values of t gives

$$t_{1,2} = -\frac{1}{2a}\left(a^2 + b^2 \pm (b^4 - a^4)^{1/2}\right),$$

and an investigation of $\operatorname{Re}(-P(s)/sQ(s))$ shows that only for

$$s = -t_2 + it_2$$

the value of c in relation (6) is positive.

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